Question

It is known that the velocity, in ms^{-1}, of surface water waves depends on

- The density of the water, ρ, in kg m^{-3}.
- The wavelength of the waves, λ, in m.
- The gravitational acceleration, g, in ms^{-2}.
- The surface tension of the water, T, in kg s^{-2}.

It is further given that for shallow water surface waves the velocity is independent of g and for deep water surface waves the velocity is independent of T.

Use dimensional analysis to show that

$$T = k\left[\frac{T}{g\rho\lambda^2}\right]^{\delta}\sqrt{\lambda g},$$

where k and δ are constants, and hence deduce expressions for the velocity of shallow water surface waves and deep water surface waves.

$$\boxed{T = k\sqrt{\lambda g}}, \quad \boxed{T = k\sqrt{\frac{T}{\rho\lambda}}}$$

AS THE SPEED DEPENDS, ON WAVELENGTH λ (m)
ON GRAVITY g (m s^{-2})
ON WATER DENSITY ρ (kg m^{-3})
ON SURFACE TENSION T (kg s^{-2})

$$\boxed{V = k\lambda^{\alpha}g^{\beta}\rho^{\gamma}T^{\delta}}$$

CONSIDERING DIMENSIONS BASED ON THE ABOVE UNITS

$$[V] = [k\lambda^{\alpha}g^{\beta}\rho^{\gamma}T^{\delta}]$$
$$[V] = [k][\lambda]^{\alpha}[g]^{\beta}[\rho]^{\gamma}[T]^{\delta}$$
$$L^{1}T^{-1} = 1 \times L^{\alpha}(LT^{-2})^{\beta}(ML^{-3})^{\gamma}(MT^{-2})^{\delta}$$
$$L^{1}T^{-1} = L^{\alpha} \times L^{\beta}T^{-2\beta} \times M^{\gamma}L^{-3\gamma} \times M^{\delta}T^{-2\delta}$$
$$L^{1}T^{-1} = L^{\alpha+\beta-3\gamma}\,T^{-2\beta-2\delta}\,M^{\gamma+\delta}$$

COMPARING WE OBTAIN

$$\left.\begin{array}{r} \alpha+\beta-3\gamma = 1 \\ -2\beta-2\delta = -1 \\ \gamma+\delta = 0 \end{array}\right\} \Rightarrow \delta = -\gamma \quad \text{OR} \quad \gamma = -\delta$$

$$\Rightarrow 2\beta = 1 - 2\delta$$
$$\Rightarrow \beta = \tfrac{1}{2} - \delta$$

$$\Rightarrow \alpha + (\tfrac{1}{2} - \delta) - 3(-\delta) = 1$$
$$\Rightarrow \alpha + \tfrac{1}{2} - \delta + 3\delta = 1$$
$$\Rightarrow \alpha = \tfrac{1}{2} - 2\delta$$

HENCE WE HAVE A FORMULA

$$V = k\lambda^{\alpha}g^{\beta}\rho^{\gamma}T^{\delta}$$
$$V = k\lambda^{\frac{1}{2}-2\delta}g^{\frac{1}{2}-\delta}\rho^{-\delta}T^{\delta}$$
$$V = k(\lambda g)^{\frac{1}{2}}\left(\frac{T}{\rho g\lambda^2}\right)^{\delta}$$

NOW FOR "DEEP WATER WAVES", THE VELOCITY IS INDEPENDENT OF SURFACE TENSION

LOOKING AT $V = k\lambda^{\alpha}g^{\beta}\rho^{\gamma}T^{\delta}$, $\delta = 0$

$$\therefore V = k_1\sqrt{\lambda g}$$

AND FOR "SHALLOW WATER WAVES", THE VELOCITY IS INDEPENDENT OF GRAVITY

LOOKING AT $V = k\lambda^{\alpha}g^{\beta}\rho^{\gamma}T^{\delta}$, $\beta = 0$
$\therefore \delta = \frac{1}{2}$

$$\therefore V = k\lambda^{\frac{1}{2}}\cancel{g^{\frac{1}{2}}}\frac{T^{\frac{1}{2}}}{\rho^{\frac{1}{2}}\cancel{g^{\frac{1}{2}}}\lambda}$$
$$V = \frac{kT^{\frac{1}{2}}}{\rho^{\frac{1}{2}}\lambda^{\frac{1}{2}}}$$
$$V = k\sqrt{\frac{T}{\rho\lambda}}$$