GENERATING FUNCTIONS

PROBABILITY GENERATING FUNCTIONS

(p.g.f.)

Question 1 (**)

The discrete random variable X has probability generating function

$$\mathrm{G}_X(t) = k + \tfrac{1}{5}\left(t^2 + t^3 + t^5\right),$$

where k is a positive constant.

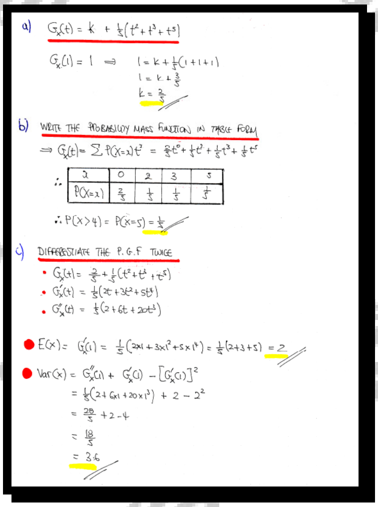
a) State the value of k.

b) Determine the value of $\mathrm{P}(X > 4)$.

c) Use $\mathrm{G}_X(t)$ to calculate the mean and variance of X.

No credit will be given if the mean and variance of X are obtained from calculations based on the probability mass function of X.

$\boxed{k = \frac{2}{5}}$, $\boxed{\mathrm{P}(X > 4) = \frac{1}{5}}$, $\boxed{\mathrm{E}(X) = 2}$, $\boxed{\mathrm{Var}(X) = 2}$



a) $G_X(t) = k + \frac{1}{5}(t^2+t^3+t^5)$

$G_X(1) = 1 \Rightarrow 1 = k + \frac{1}{5}(1+1+1)$

$1 = k + \frac{3}{5}$

$k = \frac{2}{5}$

b) WRITE THE PROBABILITY MASS FUNCTION IN TABLE FORM

$\Rightarrow G_X(t) = \sum P(X=x)t^x = \frac{2}{5}t^0 + \frac{1}{5}t^2 + \frac{1}{5}t^3 + \frac{1}{5}t^5$

$\therefore$

x	0	2	3	5
$P(X=x)$	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

$\therefore P(X>4) = P(X=5) = \frac{1}{5}$

c) DIFFERENTIATE THE P.G.F TWICE

- $G_X(t) = \frac{2}{5} + \frac{1}{5}(t^2+t^3+t^5)$
- $G_X'(t) = \frac{1}{5}(2t+3t^2+5t^4)$
- $G_X''(t) = \frac{1}{5}(2+6t+20t^3)$

- $E(X) = G_X'(1) = \frac{1}{5}(2\times1 + 3\times1^2 + 5\times1^4) = \frac{1}{5}(2+3+5) = 2$
- $\mathrm{Var}(X) = G_X''(1) + G_X'(1) - [G_X'(1)]^2$

$= \frac{1}{5}(2+6\times1+20\times1^3) + 2 - 2^2$

$= \frac{28}{5} + 2 - 4$

$= \frac{18}{5}$

$= 3.6$

Question 2 (**)

The discrete random variable X has probability generating function

$$G_X(t) = k(1+t)^7,$$

where k is a positive constant.

a) State, with justification, the least and greatest value that X can take.

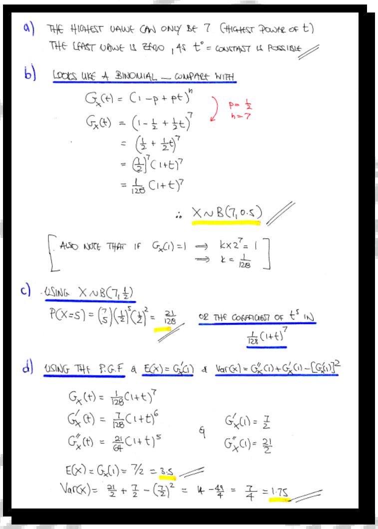
b) State the name of the distribution, fully specifying any parameters.

c) Determine the value of $P(X=5)$.

d) Use $G_X(t)$ to calculate the mean and variance of X.

No credit will be given if the mean and variance of X are obtained by alternative methods.

$\boxed{\text{FS1-R}}$, $\boxed{X_{\min}=0}$, $\boxed{X_{\max}=7}$, $\boxed{X \sim B(7,0.5)}$, $\boxed{P(X=5)=\frac{21}{128}}$, $\boxed{E(X)=3.5}$, $\boxed{\text{Var}(X)=1.75}$



a) THE HIGHEST VALUE CAN ONLY BE 7 (HIGHEST POWER OF t)
THE LEAST VALUE IS ZERO, AS t^0 = CONSTANT IS POSSIBLE

b) LOOKS LIKE A BINOMIAL — COMPARE WITH

$G_X(t) = (1-p+pt)^n$

$p=\frac{1}{2}$, $n=7$

$G_X(t) = \left(1-\frac{1}{2}+\frac{1}{2}t\right)^7 = \left(\frac{1}{2}+\frac{1}{2}t\right)^7 = \left(\frac{1}{2}\right)^7(1+t)^7 = \frac{1}{128}(1+t)^7$

$\therefore X \sim B(7, 0.5)$

[ALSO NOTE THAT IF $G_X(1)=1 \Rightarrow k \times 2^7 = 1 \Rightarrow k=\frac{1}{128}$]

c) USING $X \sim B(7,\frac{1}{2})$

$P(X=5) = \binom{7}{5}\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^2 = \frac{21}{128}$

OR THE COEFFICIENT OF t^5 IN $\frac{1}{128}(1+t)^7$

d) USING THE P.G.F & $E(X)=G_X'(1)$ & $\text{Var}(X) = G_X''(1)+G_X'(1)-[G_X'(1)]^2$

$G_X(t) = \frac{1}{128}(1+t)^7$

$G_X'(t) = \frac{7}{128}(1+t)^6$ & $G_X'(1)=\frac{7}{2}$

$G_X''(t) = \frac{21}{64}(1+t)^5$ & $G_X''(1)=\frac{21}{2}$

$E(X) = G_X'(1) = 7/2 = 3.5$

$\text{Var}(X) = \frac{21}{2}+\frac{7}{2}-\left(\frac{7}{2}\right)^2 = 14-\frac{49}{4} = \frac{7}{4} = 1.75$

Question 3 (**+)

The discrete random variable X has distribution

$$X \sim \mathrm{Po}(\lambda).$$

a) Derive **from first principles** $\mathrm{G}_X(t)$, the probability generating function of X.

b) Use $\mathrm{G}_X(t)$ to show that the mean and variance of X are both λ.

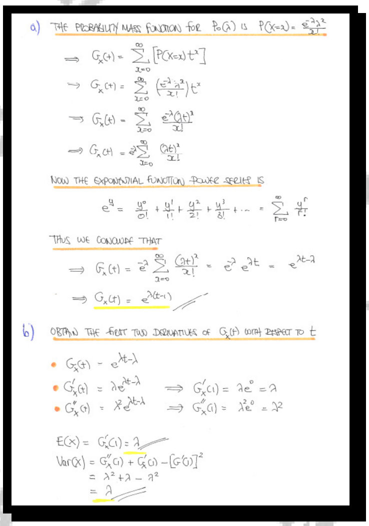
Another discrete random variable Y has distribution

$$Y \sim \mathrm{Po}(\mu).$$

c) Use a method involving generating functions to show that

$$\mathrm{E}(X+Y) = \lambda + \mu.$$

$\boxed{}$, $\boxed{\mathrm{G}_X(t) = \mathrm{e}^{\lambda(t-1)}}$



a) THE PROBABILITY MASS FUNCTION FOR $Po(\lambda)$ IS $P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}$

$\Longrightarrow G_X(t) = \sum_{x=0}^{\infty} \left[P(X=x)t^x\right]$

$\Longrightarrow G_X(t) = \sum_{x=0}^{\infty} \left(\frac{e^{-\lambda}\lambda^x}{x!}\right)t^x$

$\Longrightarrow G_X(t) = \sum_{x=0}^{\infty} \frac{e^{-\lambda}(\lambda t)^x}{x!}$

$\Longrightarrow G_X(t) = e^{-\lambda}\sum_{x=0}^{\infty} \frac{(\lambda t)^x}{x!}$

NOW THE EXPONENTIAL FUNCTION POWER SERIES IS

$e^y = \frac{y^0}{0!} + \frac{y^1}{1!} + \frac{y^2}{2!} + \frac{y^3}{3!} + \dots = \sum_{r=0}^{\infty} \frac{y^r}{r!}$

THUS WE CONCLUDE THAT

$\Longrightarrow G_X(t) = e^{-\lambda}\sum_{x=0}^{\infty} \frac{(\lambda t)^x}{x!} = e^{-\lambda}e^{\lambda t} = e^{\lambda t - \lambda}$

$\Longrightarrow G_X(t) = e^{\lambda(t-1)}$

b) OBTAIN THE FIRST TWO DERIVATIVES OF $G_X(t)$ WITH RESPECT TO t

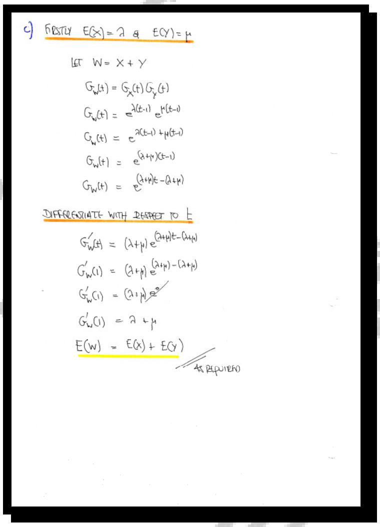
- $G_X(t) = e^{\lambda t - \lambda}$
- $G_X'(t) = \lambda e^{\lambda t - \lambda} \Longrightarrow G_X'(1) = \lambda e^0 = \lambda$
- $G_X''(t) = \lambda^2 e^{\lambda t - \lambda} \Longrightarrow G_X''(1) = \lambda^2 e^0 = \lambda^2$

$E(X) = G_X'(1) = \lambda$

$Var(X) = G_X''(1) + G_X'(1) - [G'(1)]^2$

$= \lambda^2 + \lambda - \lambda^2$

$= \lambda$



c) FIRSTLY $E(X) = \lambda$ & $E(Y) = \mu$

LET $W = X + Y$

$G_W(t) = G_X(t)G_Y(t)$

$G_W(t) = e^{\lambda(t-1)}e^{\mu(t-1)}$

$G_W(t) = e^{\lambda(t-1)+\mu(t-1)}$

$G_W(t) = e^{(\lambda+\mu)(t-1)}$

$G_W(t) = e^{(\lambda+\mu)t-(\lambda+\mu)}$

DIFFERENTIATE WITH RESPECT TO t

$G_W'(t) = (\lambda+\mu)e^{(\lambda+\mu)t-(\lambda+\mu)}$

$G_W'(1) = (\lambda+\mu)e^{(\lambda+\mu)-(\lambda+\mu)}$

$G_W'(1) = (\lambda+\mu)\cancel{e^0}$

$G_W'(1) = \lambda + \mu$

$E(W) = E(X) + E(Y)$

AS REQUIRED

Question 4 (+)**

The discrete random variable X has distribution

$$X \sim \mathrm{B}(n, p).$$

a) Derive **from first principles** $\mathrm{G}_X(t)$, the probability generating function of X.

b) Use $\mathrm{G}_X(t)$ to calculate the mean and variance of X.

$\boxed{}$, $\boxed{\mathrm{G}_X(t) = (1-p+pt)^n}$, $\boxed{\mathrm{E}(X) = np}$, $\boxed{\mathrm{Var}(X) = np(1-p)}$

a) START WITH THE PROBABILITY MASS FUNCTION OF $B(n,p)$

$$P(X=x) = \binom{n}{x} p^x (1-p)^{n-x}$$

NOW NOTE THAT $(A+B)^n = \sum_{r=0}^{n} \left[\binom{n}{r} A^r B^{n-r} \right]$

BY THE DEFINITION OF P.G.F

$$G_X(t) = \sum_{x=0}^{n} \left[P(X=x)\, t^x \right] = \sum_{x=0}^{n} \left[\binom{n}{x} p^x (1-p)^{n-x} t^x \right]$$

$$= \sum_{x=0}^{n} \left[\binom{n}{x} (pt)^x (1-p)^{n-x} \right]$$

COMPARE WITH

$$\sum_{r=0}^{n} \left[\binom{n}{r} A^r B^{n-r} \right] = (A+B)^n$$

$$= \left[pt + (1-p) \right]^n$$

$$= (1-p+pt)^n$$

b) NOW $E(X) = G_X'(1)$

$$\Rightarrow \frac{d}{dt}(1-p+pt)^n = n(1-p+pt)^{n-1} \times p = np(1-p+pt)^{n-1}$$

LET $t=1$ GIVES $E(X) = np(1-p+p)^{n-1} = np \times 1^{n-1} = np$

DIFFERENTIATE WITH RESPECT TO t, ONCE MORE

$$\Rightarrow \frac{d}{dt}\left[np(1-p+pt)^{n-1} \right] = np(n-1)(1-p+pt)^{n-2} \times p$$

$$= n(n-1)p^2(1-p+pt)^{n-2}$$

LET $t=1$ GIVES

$$n(n-1)p^2(1-p+p)^{n-2} = n(n-1)p^2$$

THUS SO FAR WE HAVE $G_X'(1) = np$ & $G_X''(1) = n(n-1)p^2$

$$\Rightarrow \mathrm{Var}(X) = G_X''(1) + G_X'(1) - [G_X'(1)]^2$$

$$\Rightarrow \mathrm{Var}(X) = n(n-1)p^2 + np - (np)^2$$

$$\Rightarrow \mathrm{Var}(X) = (n^2-n)p^2 + np - n^2p^2$$

$$\Rightarrow \mathrm{Var}(X) = \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2}$$

$$\Rightarrow \mathrm{Var}(X) = np - np^2$$

$$\Rightarrow \mathrm{Var}(X) = np(1-p)$$

Question 5 (***)

The discrete random variable X has probability generating function

$$\mathrm{G}_X(t) = k\left(1+3t+3t^2+t^3\right)^3,$$

where k is a positive constant.

a) State the name of the distribution, fully specifying any parameters.

b) Use $\mathrm{G}_X(t)$ to calculate the mean and variance of X.

No credit will be given if the mean and variance of X are obtained by alternative methods.

$\boxed{}$, $\boxed{X \sim \mathrm{B}(9,0.5)}$, $\boxed{\mathrm{E}(X)=4.5}$, $\boxed{\mathrm{Var}(X)=2.25}$

a) LOOKING AT THE EXPRESSION

$(1+t)^3 = 1+3t+3t^2+t^3$

HENCE WE HAVE

$G_X(t) = k(1+3t+3t^2+t^3)^3$

$G_X(t) = k[(1+t)^3]^3$

$G_X(t) = k(1+t)^9$

COMPARE WITH THE BINOMIAL $(9,\frac{1}{2})$ WITH $G_X(t) = (1-\frac{1}{2}+\frac{1}{2}t)^9$

$G_X(t) = (\frac{1}{2}+\frac{1}{2}t)^9 = \frac{1}{2^9}(1+t)^9 = \frac{1}{512}(1+t)^9$ IF $k=\frac{1}{512}$

$\therefore X \sim B(9,\frac{1}{2})$

b) USING THE RESULTS $E(X) = G'_X(1)$ & $Var(X) = G''_X(1) + G'_X(1) - [G'_X(1)]^2$

USING $G_X(t) = \frac{1}{512}(1+t)^9$

$G'_X(t) = \frac{9}{512}(1+t)^8$ $\quad G'_X(1) = \frac{9}{2}$

$G''_X(t) = \frac{9}{64}(1+t)^7$ $\quad G''_X(1) = 18$

$\therefore E(X) = G'_X(1) = \frac{9}{2}$

$Var(X) = G''_X(1) + G'_X(1) - [G'_X(1)]^2$

$Var(X) = 18 + \frac{9}{2} - (\frac{9}{2})^2$

$\therefore Var(X) = 2.25$

Question 6 (***)

The discrete random variable X has probability generating function

$$\mathrm{G}_X(t) = k\left(1+2t+3t^2\right)^2,$$

where k is a positive constant.

a) State, with justification, the least and greatest value that X can take.

b) Determine the value of $\mathrm{P}(X=2)$.

c) Use $\mathrm{G}_X(t)$ to calculate the mean and variance of X.

No credit will be given if the mean and variance of X are obtained by alternative methods.

d) Find a probability generating function for $3X-2$.

Give the answer in the form $k\left[f(t)\right]^2$

$\boxed{}$, $\boxed{X_{\text{min}}=0}$, $\boxed{X_{\text{max}}=4}$, $\boxed{\mathrm{P}(X=2)=\frac{5}{18}}$, $\boxed{\mathrm{E}(X)=\frac{8}{3}}$, $\boxed{\mathrm{Var}(X)=\frac{10}{9}}$, $\boxed{\frac{1}{36}\left(t^{-1}+2t^2+3t^5\right)^2}$

a) MIN IS 0 & MAXIMUM VALUE IS 4, AS THESE ARE THE LOWEST & HIGHEST VALUES OF THE POWERS OF t, WHEN FULLY EXPANDED

b) USING $G_X(1)=1$

$k(1+2+3)^2=1$

$36k=1$

$k=\frac{1}{36}$

EXPANDING FULLY

$G_X(t)=\frac{1}{36}\left[1+4t^2+9t^4+4t+12t^3+6t^2\right]$

$=\frac{1}{36}\left[1+4t+10t^2+12t^3+9t^4\right]$

$\therefore P(X=2)=\frac{10}{36}=\frac{5}{18}$

c) DIFFERENTIATE $G_X(t)$ WITH RESPECT TO t

$G_X(t)=\frac{1}{36}(1+4t+10t^2+12t^3+9t^4)$

$G_X'(t)=\frac{1}{36}(4+20t+36t^2+36t^3)$

$G_X'(1)=\frac{1}{36}(4+20+36+36)=\frac{8}{3}$

$\therefore E(X)=\frac{8}{3}$

DIFFERENTIATE ONCE MORE

$G_X'(t)=\frac{1}{36}(4+20t+36t^2+36t^3)$

$G_X''(t)=\frac{1}{36}(20+72t+108t^2)$

$G_X''(1)=\frac{1}{36}(20+72+108)=\frac{50}{9}$

$\therefore \mathrm{Var}(X)=G_X''(1)+G_X'(1)-\left[G_X'(1)\right]^2$

$\mathrm{Var}(X)=\frac{50}{9}+\frac{8}{3}-\left(\frac{8}{3}\right)^2$

$\mathrm{Var}(X)=\frac{10}{9}$

FINALLY WE HAVE IF $Y=3X-2$

$G_X(t)=\frac{1}{36}(1+2t+3t^2)^2$

$G_Y(t)=t^{-2}\times\frac{1}{36}\left[1+2(t^3)+3(t^3)^2\right]^2$

IF $Y=aX+b$ $\quad G_Y(t)=t^b G_X(t^a)$

$G_Y(t)=\frac{t^{-2}}{36}\left[1+2t^3+3t^6\right]^2$

$G_Y(t)=\frac{(t^{-1})^2}{36}(1+2t^3+36t^6)^2$

$G_Y(t)=\frac{1}{36}(t^{-1}+2t^2+36t^5)^2$

Question 7 (***)

The discrete random variable X has probability generating function

$$\mathrm{G}_X(t) = k\left[t^2\left(2t^2+3\right)+\left(t+1\right)^4\right],$$

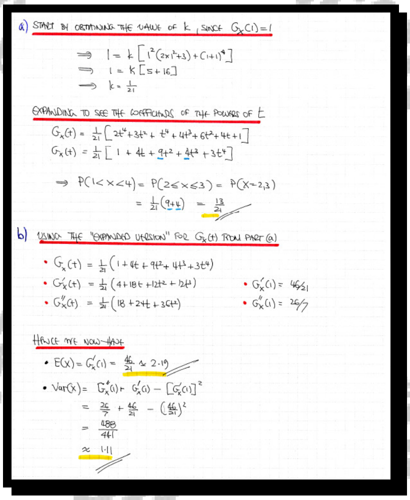
where k is a positive constant.

a) Determine the value of $\mathrm{P}(1 < X < 4)$.

b) Use $\mathrm{G}_X(t)$ to calculate the mean and variance of X.

No credit will be given if the mean and variance of X are obtained from calculations based on the probability mass function of X.

$\boxed{}$, $\boxed{\mathrm{P}(1<X<4)=\frac{13}{21}}$, $\boxed{\mathrm{E}(X)=\frac{46}{21}\approx 2.19}$, $\boxed{\mathrm{Var}(X)=\frac{488}{441}\approx 1.11}$



a) START BY OBTAINING THE VALUE OF k, SINCE $G_X(1)=1$

$\Rightarrow \ 1 = k\left[1^2(2\times1^2+3)+(1+1)^4\right]$
$\Rightarrow \ 1 = k[5+16]$
$\Rightarrow \ k = \frac{1}{21}$

EXPANDING TO SEE THE COEFFICIENTS OF THE POWERS OF t

$G_X(t) = \frac{1}{21}\left[2t^4+3t^2+t^4+4t^3+6t^2+4t+1\right]$
$G_X(t) = \frac{1}{21}\left[1+4t+9t^2+4t^3+3t^4\right]$

$\Rightarrow \ P(1<X<4) = P(2\leqslant X\leqslant 3) = P(X=2,3)$
$= \frac{1}{21}(9+4) = \frac{13}{21}$

b) USING THE "EXPANDED VERSION" FOR $G_X(t)$ FROM PART (a)

- $G_X(t) = \frac{1}{21}\left(1+4t+9t^2+4t^3+3t^4\right)$
- $G'_X(t) = \frac{1}{21}\left(4+18t+12t^2+12t^3\right)$ • $G'_X(1) = 46/21$
- $G''_X(t) = \frac{1}{21}\left(18+24t+36t^2\right)$ • $G''_X(1) = 26/7$

HENCE WE NOW HAVE

- $E(X) = G'_X(1) = \frac{46}{21} \approx 2.19$
- $\mathrm{Var}(X) = G''_X(1) + G'_X(1) - \left[G'_X(1)\right]^2$
$= \frac{26}{7} + \frac{46}{21} - \left(\frac{46}{21}\right)^2$
$= \frac{488}{441}$
≈ 1.11

Question 8 (***)

The discrete random variables, X and Y, are independent of one another and have respective probability generating functions

$$\mathrm{G}_X(t) = \alpha(2+t)^4 \quad \text{and} \quad \mathrm{G}_Y(t) = \beta(2+t)^5,$$

where α and β are a positive constants.

a) Use $\mathrm{G}_X(t)$ to calculate the mean and variance of X.

b) Find a probability generating function for $X+Y$ and use it to find the mean and variance of $X+Y$.

$\boxed{}$, $\boxed{\mathrm{E}(X)=\frac{4}{3}}$, $\boxed{\mathrm{Var}(X)=\frac{8}{9}}$, $\boxed{\frac{1}{19683}(2+t)^9}$, $\boxed{\mathrm{E}(X)=3}$, $\boxed{\mathrm{Var}(X)=2}$

a) START BY FINDING THE VALUE OF α

$G_X(1)=1 \implies \alpha(2+1)^4=1$

$\alpha=\frac{1}{81}$

NOW DIFFERENTIATE $G_X(t)$ WITH RESPECT TO t, TWICE

$G_X(t)=\frac{1}{81}(2+t)^4$

$G_X'(t)=\frac{4}{81}(2+t)^3$

$G_X''(t)=\frac{4}{27}(2+t)^2$

NOW WE HAVE

$E(X)=G_X'(1)=\frac{4}{81}\times 3^3=\frac{4}{3}$

$\mathrm{Var}(X)=G_X''(1)+G_X'(1)-(G_X'(1))^2$

$=\frac{4}{27}\times 3^2+\left(\frac{4}{3}\right)-\left(\frac{4}{3}\right)^2$

$=\frac{8}{9}$

b) FIRSTLY FIND THE CONSTANT β

$G_Y(t)=\beta(2+t)^5 \implies \beta(2+1)^5=1$

$\implies \beta\times 243=1$

$\implies \beta=\frac{1}{243}$

NOW $G_{X+Y}(t)=(G_X(t))\times(G_Y(t))$

$\implies G_{X+Y}(t)=\frac{1}{81}(2+t)^4\times\frac{1}{243}(2+t)^5$

$=\frac{1}{19683}(2+t)^9$

DIFFERENTIATE WITH RESPECT TO t, TWICE

$\implies G_{X+Y}(t)=\frac{1}{19683}(2+t)^9$

$\implies G_{X+Y}'(t)=\frac{1}{2187}(2+t)^8$

$\implies G_{X+Y}''(t)=\frac{8}{2187}(2+t)^7$

HENCE WE NOW HAVE

$E(X)=G_X'(1)=\frac{1}{2187}\times(2+1)^8=3$

$\mathrm{Var}(X)=G_X''(1)+G_X'(1)-[G_X'(1)]^2$

$\mathrm{Var}(X)=\frac{8}{2187}\times 3^7+3-3^2$

$\mathrm{Var}(X)=8+3-9$

$\mathrm{Var}(X)=2$

Question 9 (***)

A fair six-sided die is rolled repeatedly until a 6 is obtained.

The discrete random variable X represents the number of rolls required until the first 6 is obtained.

a) Derive **from first principles** $\mathrm{G}_X(t)$, the probability generating function of X.

Give the answer in its simplest form.

b) Use $\mathrm{G}_X(t)$ to calculate the mean and variance of X.

No credit will be given if the mean and variance of X are obtained by alternative methods.

$\boxed{}$, $\boxed{\mathrm{G}_X(t)=\dfrac{t}{6-5t}}$, $\boxed{\mathrm{E}(X)=6}$, $\boxed{\mathrm{Var}(X)=30}$

THIS IS CLEARLY A GEOMETRIC DISTRIBUTION, $X \sim \mathrm{Geo}\left(\frac{1}{6}\right)$

$$\Rightarrow P(X=x) = \left(\frac{5}{6}\right)^{x-1}\left(\frac{1}{6}\right)$$

$$\Rightarrow \sum_{x=1}^{\infty} P(X=x)t^x = G_X(t)$$

$$\Rightarrow \sum_{x=1}^{\infty}\left[\frac{1}{6}\left(\frac{5}{6}\right)^{x-1}t^x\right] = G_X(t)$$

$$\Rightarrow G_X(t) = \frac{1}{6}\sum_{x=1}^{\infty}\left[t\left(\frac{5}{6}t\right)^{x-1}\right]$$

NOW ASSUMING THAT THE PARAMETER t IS SMALL ENOUGH FOR THE SERIES TO CONVERGE

$$\Rightarrow G_X(t) = \frac{1}{6}t\sum_{x=1}^{\infty}\left(\frac{5}{6}t\right)^{x-1} \quad \leftarrow \text{THIS IS A G.P. WITH } a=1,\ r=\frac{5}{6}t$$

$$\Rightarrow G_X(t) = \frac{1}{6}t\left[\frac{1}{1-\frac{5}{6}t}\right] \quad \leftarrow S_\infty = \frac{a}{1-r}$$

$$\Rightarrow G_X(t) = \frac{1}{6}t\left(\frac{6}{6-5t}\right)$$

$$\Rightarrow G_X(t) = \frac{t}{6-5t}$$

b) DIFFERENTIATE & USE STANDARD RESULTS

$$G_X'(x) = \frac{(6-5t)-t(-5)}{(6-5t)^2} \quad \text{(QUOTIENT RULE)}$$

$$G_X'(x) = \frac{6}{(6-5t)^2} = 6(6-5t)^{-2}$$

$$G_X''(x) = 60(6-5t)^{-3} = \frac{60}{(6-5t)^3}$$

$$G'(1) = 6$$

$$G''(1) = 60$$

FINALLY WE HAVE

$$E(X) = G_X'(1) = 6$$

$$\mathrm{Var}(X) = G_X''(1) + G_X'(1) - \left(G_X'(1)\right)^2$$

$$\mathrm{Var}(X) = 60 + 6 - 6^2$$

$$\mathrm{Var}(X) = 30$$

Question 10 (***+)

$G_X(t)$ is the probability generating function of a discrete random variable X.

$$G_X(t) = t^4 + \frac{2 + 6t + 9t^2 + 9t^3 + kt^4}{2e^3},$$

where k is a non zero constant.

a) Determine the value of k.

b) Calculate $P\left[X > E(X)\right]$, correct to four decimal places.

c) Find the value of $\mathrm{Var}(X)$, correct to two decimal places.

$\boxed{k = -26}$, $\boxed{\approx 0.5768}$, $\boxed{\approx 1.52}$

a) USING THE FACT $G_X(1) = 1$

$\Rightarrow 1 + \frac{2+6+9+9+k}{2e^2} = 1$

$\Rightarrow \frac{26+k}{2e^2} = 1$

$\Rightarrow k = -26$

b) FIND $E(X)$ FIRST, NOTING $X = 0,1,2,3,4$ ONLY

$E(X) = G'(1)$

$\Rightarrow G_X'(t) = 4t^3 + \frac{6 + 18t + 27t^2 - 104t^3}{2e^3}$

$\Rightarrow E(X) = 4 + \frac{-53}{2e^3} = 2.6806\ldots$

$\Rightarrow P(X > E(X)) = P(X > 2.68\ldots) = P(X = 3, 4)$

$= \underbrace{\frac{9}{2e^3}}_{\text{coeff of } t^3} + \underbrace{\left(1 - \frac{26}{2e^3}\right)}_{\text{coeff of } t^4}$

$= 1 - \frac{17}{2e^3}$

$\simeq 0.5768$

c) DIFFERENTIATE AGAIN $G_X'(t)$

$G_X'(t) = 4t^3 + \frac{6 + 18t + 27t^2 - 104t^3}{2e^3}$

$G_X''(t) = 12t^2 + \frac{18 + 54t - 312t^2}{2e^3}$

$G_X''(1) = 12 - \frac{120}{e^3}$

THUS THE VARIANCE IS

$G_X''(1) + G_X'(1) - [G_X'(1)]^2$

$= \left(12 - \frac{120}{e^3}\right) + \left(4 - \frac{53}{2e^3}\right) - \left(4 - \frac{53}{2e^3}\right)^2$

$= 1.520349\ldots$

$\simeq 1.52$

Question 11 (***+)

The discrete random variable X has probability mass function

$$\mathrm{P}(X=x)=\frac{x^2}{k}, \quad x=1,2,3,4,5$$

where k is a positive constant.

The probability generating function of X is $\mathrm{G}_X(t)$.

a) Find an expression for $\mathrm{G}_X(t)$, in terms of k.

Give the answer in simplified sigma $[\Sigma]$, notation.

b) Find $\frac{d^2}{dt^2}\left[\mathrm{G}_X(t)\right]$, in terms of k.

Give the answer in simplified sigma $[\Sigma]$, notation.

c) State the value of k.

d) Use $\mathrm{G}_X(t)$ and its derivatives to calculate the value of $\mathrm{Var}(X)$.

No credit will be given if the value of $\mathrm{Var}(X)$ *is obtained from calculations based on the probability mass function of* X.

$\boxed{}$, $\boxed{\mathrm{G}_X(t)=\frac{1}{k}\sum_{r=1}^{5}\left(r^2t^r\right)}$, $\boxed{\mathrm{G}''_X(t)=\frac{1}{k}\sum_{r=2}^{5}\left[r^3(r-1)t^{r-2}\right]}$,

$\boxed{\mathrm{Var}(X)=\frac{644}{605}\approx 1.06}$

a) NO REAL NEED TO WRITE THE PROBABILITY DISTRIBUTION IN A TABLE

$$G_X(t)=\sum P(X=x)t^x$$
$$=\frac{1^2}{k}t^1+\frac{2^2}{k}t^2+\frac{3^2}{k}t^3+\frac{4^2}{k}t^4+\frac{5^2}{k}t^5$$
$$=\frac{1}{k}\left[t+4t^2+9t^3+16t^4+25t^5\right]$$
$$\therefore G_X(t)=\frac{1}{k}\sum_{r=1}^{5}(r^2t^r)$$

b) EASIER TO DIFFERENTIATE IN SIGMA NOTATION — NOTE THAT ALL THE DIFFERENTIATIONS ARE WITH RESPECT TO t

$$\Longrightarrow \frac{d}{dt}\left(G_X(t)\right)=G'_X(t)=\frac{1}{k}\sum_{r=1}^{5}\left[r^2(rt^{r-1})\right]=\frac{1}{k}\sum_{r=1}^{5}(r^3t^{r-1})$$
$$\Longrightarrow \frac{d^2}{dt^2}\left(G_X(t)\right)=\frac{d}{dt}\left[\frac{1}{k}\sum_{r=1}^{5}(r^3t^{r-1})\right]=\frac{1}{k}\sum_{r=1}^{5}\left[r^3(r-1)t^{r-2}\right]$$

AS THE FIRST TERM IS ZERO $(r=1)$ WE MAY START FROM $r=2$

$$\therefore G''_X(t)=\frac{1}{k}\sum_{r=2}^{5}\left[r^3(r-1)t^{r-2}\right]$$

c) USING THE FACT $G_X(1)=1$

$$\Longrightarrow G_X(t)=\frac{1}{k}\sum_{r=1}^{5}(r^2t^r)=\frac{1}{k}\left(1^2\times t^1+2^2\times t^2+3^2\times t^3+4^2\times t^4+5^2\times t^5\right)$$
$$\Longrightarrow 1=\frac{1}{k}\left(1^2+2^2+3^2+4^2+5^2\right)$$
$$\Longrightarrow 1=\frac{1}{k}(1+4+9+16+25)$$
$$\Longrightarrow k=55$$

EASIER TO HAVE USED THE EXPANDED FORM FROM ABOVE

$G_X(t)=\frac{1}{k}\left(t+4t^2+9t^3+16t^4+25t^5\right)$

d) USING ALL THE PREVIOUS RESULTS WITH $k=55$

- $G'_X(t)=\frac{1}{k}\sum_{r=1}^{5}(r^3t^{r-1})$

$G'_X(1)=\frac{1}{k}\sum_{r=1}^{5}r^3$ $\quad (1^{r-1}=1)$

$G'_X(1)=\frac{1}{55}(1+8+27+64+125)$

$\boxed{G'_X(1)=\frac{45}{11}}$

- $G''_X(t)=\frac{1}{k}\sum_{r=2}^{5}\left[r^3(r-1)t^{r-2}\right]$

$G''_X(t)=\frac{1}{k}\sum_{r=2}^{5}\left[r^3(r-1)\right]$ $\quad (1^{r-2}=1)$

$G''_X(1)=\frac{1}{55}\left[2^3\times1+3^3\times2+4^3\times3+5^3\times4\right]$

$G''_X(1)=\frac{1}{55}(8+54+192+500)$

$\boxed{G''_X(1)=\frac{754}{11}}$

FINALLY WE HAVE

$$\mathrm{Var}(X)=G''_X(1)+G'_X(1)-\left[G'_X(1)\right]^2$$
$$=\frac{754}{11}+\frac{45}{11}-\left(\frac{45}{11}\right)^2$$
$$=\frac{644}{605}\approx 1.06$$

Question 12 (****)

The discrete random variable X has probability generating function $\mathrm{G}_X(t)$.

a) Show, with detailed workings, that …

i. … $\mathrm{E}(X) = \mathrm{G}'_X(1)$.

ii. … $\mathrm{Var}(X) = \mathrm{G}''_X(1) + \mathrm{G}'_X(1) - \left[\mathrm{G}'_X(1)\right]^2$.

It is further given that

$$\mathrm{G}_X(t) = \frac{kt}{1-0.2t^2},$$

where k is a positive constant.

b) Determine the value of $\mathrm{P}(6 \le X \le 8)$.

c) Use $\mathrm{G}_X(t)$ to calculate the mean and variance of X.

$\boxed{}$, $\boxed{\mathrm{P}(6 \le X \le 8) = 0.0064}$, $\boxed{\mathrm{E}(X) = 1.5}$, $\boxed{\mathrm{Var}(X) = 1.25}$

a) LET US NOTE THAT $E(f(x)) = \sum P(X=x) f(x)$

$\Rightarrow G_X(t) = \sum\left[P(X=x)t^x\right] = E(t^X)$ (as above)

I) DIFFERENTIATE WITH RESPECT TO t

$\Rightarrow G'_X(t) = \sum\left[\left[xP(X=x)\right]t^{x-1}\right] = E(Xt^{X-1})$

$\Rightarrow G'_X(1) = \sum\left[xP(X=x)\right] = E(X)$ $\{1^{x-1} = 1\}$

$\therefore E(X) = G'_X(1)$ AS REQUIRED

II) DIFFERENTIATE WITH RESPECT TO t ONCE MORE

$\Rightarrow G'_X(t) = \sum\left[xP(X=x)t^{x-1}\right] = E(Xt^{X-1})$

$\Rightarrow G''_X(t) = \sum\left[x(x-1)P(X=x)t^{x-1}\right] = E(X(X-1)t^{X-1})$

$\Rightarrow G''_X(t) = \sum\left[x^2P(X=x)t^{x-1} - xP(X=x)t^{x-1}\right]$

$\Rightarrow G''_X(t) = \sum\left[x^2P(X=x)t^{x-1}\right] - \sum\left[xP(X=x)t^{x-1}\right]$

$\Rightarrow G''_X(t) = E(X^2t^{X-1}) - E(Xt^{X-1})$

EVALUATE AT $t=1$ GIVES

$G''_X(1) = E(X^2) - E(X)$

BUT $\mathrm{Var}(X) = E(X^2) - \left[E(X)\right]^2$

$\Rightarrow \mathrm{Var}(X) = \left[G''_X(1) + E(X)\right] - (E(X))^2$

$\Rightarrow \mathrm{Var}(X) = G''_X(1) + E(X) - (E(X))^2$

$\Rightarrow \mathrm{Var}(X) = G''_X(1) + G'_X(1) - (G'_X(1))^2$ AS REQUIRED

b) FIRSTLY FIND k

$G_X(1) = 1 \Rightarrow \frac{k}{1-0.2} = 1$

$\Rightarrow k = 0.8$

$G_X(t) = \frac{0.8t}{1-0.2t^2} = 0.8t(1-0.2t^2)^{-1}$

$= 0.8t(1 + 0.2t^2 + 0.04t^4 + 0.008t^6 + 0.0016t^8 + \ldots)$

$= 0.8t + 0.16t^3 + 0.032t^5 + 0.0064t^7 + 0.00128t^9 + \ldots$

$P(X=1)$ $P(X=3)$ $P(X=5)$ $P(X=7)$ $P(X=9)$

$\therefore P(6 \le X \le 8) = P(X=7) = 0.0064$

c) DIFFERENTIATE $G_X(t)$ WITH RESPECT TO t, TWICE

$G_X(t) = \frac{0.8t}{1-0.2t^2} = \frac{4t}{5-t^2}$

$G'_X(t) = \frac{4(5-t^2) - 4t(-2t)}{(5-t^2)^2} = \frac{20-4t^2+8t^2}{(5-t^2)^2} = \frac{4t^2+20}{(5-t^2)^2}$

$G''_X(t) = \frac{(5-t^2)^{\cancel{2}}(8t) - (4t^2+20)\times(-4t)\cancel{(5-t^2)}}{(5-t^2)^{\cancel{4}3}} = \frac{8t(5-t^2) + 4t(4t^2+20)}{(5-t^2)^3}$

EVALUATE AT $t=1$

$G'_X(1) = \frac{3}{2}$ $G''_X(1) = 2$

USING PART (a)

$E(X) = \frac{3}{2}$

$\mathrm{Var}(X) = 2 + \frac{3}{2} - \left(\frac{3}{2}\right)^2 = \frac{5}{4}$

Question 13 (*****)

The discrete random variable X has probability mass function

$$\mathrm{P}(X = x) = \frac{x^2}{k}, \quad x = 1, 2, 3, 4, 5, \ldots, N$$

where k is a positive constant.

The probability generating function of X is $\mathrm{G}_X(t)$.

a) Find an expression for $\mathrm{G}_X(t)$, in terms of k and N.

Give the answer in simplified sigma $[\Sigma]$, notation.

b) Find the value of k in terms of N.

c) Use $\mathrm{G}_X(t)$ to show that

$$\mathrm{E}(X) = \frac{3N(N+1)}{2(2N+1)}.$$

d) Show further that the value of $\mathrm{Var}(X)$ is

$$\frac{3N^4 + 6N^3 - N^2 - 4N - 4}{20(2N+1)^2}.$$

You may assume without proof $\displaystyle\sum_{r=1}^{n} r^4 = \frac{1}{30}n(n+1)\left(6n^3 + 9n^2 + n - 1\right)$

$\boxed{\text{FS}}$, $\boxed{\mathrm{G}_X(t) = \frac{1}{k}\sum_{x=1}^{N}\left(x^2 t^x\right)}$, $\boxed{k = \frac{1}{6}N(N+1)(2N+1)}$

[solution overleaf]

a) START FROM THE DEFINITION OF $G(t)$

$$G_X(t) = \sum\left[P(X=x)\,t^x\right] = \sum_{x=1}^{N} \frac{x^2}{k} t^x = \frac{1}{k}\sum_{x=1}^{N}\left(x^2 t^x\right)$$

b) USING THE FACT $G_X(1) = 1$; NOTE THAT x IS AN INTEGER

$$\frac{1}{k}\sum_{x=1}^{N} x^2 1^x = 1 \qquad \left\{1^x = 1\right\}$$

$$\frac{1}{k}\sum_{x=1}^{N} x^2 = 1$$

$$\frac{1}{k}\left(\frac{1}{6}N(N+1)(2N+1)\right) = 1$$

$$k = \frac{1}{6}N(N+1)(2N+1)$$

c) $E(X) = G_X'(1)$, WHERE DASH ' IMPLIES DIFFERENTIATION W.R.T t

$$\Rightarrow G_X(t) = \frac{1}{k}\sum_{x=1}^{N}\left(x^2 t^x\right)$$

$$\Rightarrow G_X'(t) = \frac{1}{k}\sum_{x=1}^{N}\left[x^2\left(x t^{x-1}\right)\right]$$

$$\Rightarrow G_X'(t) = \frac{1}{k}\sum_{x=1}^{N}\left(x^3 t^{x-1}\right)$$

$$\Rightarrow G_X'(1) = \frac{1}{k}\sum_{x=1}^{N} x^3 \qquad \left\{1^{x-1} = 1\right\}$$

$$\Rightarrow G_X'(1) = \frac{1}{\frac{1}{6}N(N+1)(2N+1)} \times \frac{1}{4}N^2(N+1)^2$$ ← STANDARD RESULT FOR THE SUM OF CUBES $n \in \mathbb{N}$

$$\Rightarrow G_X'(1) = \frac{3}{2} \times \frac{N^{\cancel{2}}(N+1)^{\cancel{2}}}{\cancel{N}\cancel{(N+1)}(2N+1)}$$

$$\Rightarrow G_X'(1) = \frac{3N(N+1)}{2(2N+1)}$$

$$\therefore E(X) = \frac{3N(N+1)}{2(2N+1)}$$

AS REQUIRED

d) NOW USING THE FACT $\text{Var}(X) = G_X''(1) + G_X'(1) - \left[G_X'(1)\right]^2$

$$G_X'(t) = \frac{1}{k}\sum_{x=1}^{N}\left[x^3 t^{x-1}\right]$$

$$G_X''(t) = \frac{1}{k}\sum_{x=1}^{N}\left[x^3(x-1)t^{x-2}\right]$$

$$G_X''(1) = \frac{1}{k}\sum_{x=1}^{N}\left[x^3(x-1)\right] \qquad \left\{t^{x-2} = 1, \text{ IF } x = 1\right\}$$

$$G_X''(1) = \frac{1}{k}\sum_{x=2}^{N}\left[x^3(x-1)\right]$$ ← AS THE FIRST TERM IS ZERO

NOW MANIPULATE USING STANDARD SUMMATION RESULTS — BETTER TO LEAVE EXPRESSIONS IN TERMS OF k UNTIL THE VERY END

$$\text{Var}(X) = \frac{1}{k}\sum_{x=2}^{N}\left[x^3(x-1)\right] + \underbrace{\frac{1}{k}\sum_{x=1}^{N} x^3}_{G_X'(1)} - \underbrace{\left[\frac{1}{k}\sum_{x=1}^{N} x^3\right]^2}_{\left[G_X'(1)\right]^2}$$

$$\text{Var}(X) = \frac{1}{k}\sum_{x=2}^{N} x^4 - \frac{1}{k}\sum_{x=2}^{N} x^3 + \frac{1}{k}\sum_{x=1}^{N} x^3 - \frac{1}{k^2}\left[\sum_{x=1}^{N} x^3\right]^2$$

CAREFUL AS SOME SUMMATIONS DO NOT START FROM 1 !!

$$\text{Var}(X) = \left[\frac{1}{k}\sum_{x=1}^{N} x^4 - \frac{1^4}{k}\right] - \left[\frac{1}{k}\sum_{x=1}^{N} x^3 - \frac{1^3}{k}\right] + \frac{1}{k}\sum_{x=1}^{N} x^3 - \frac{1}{k^2}\left[\sum_{x=1}^{N} x^3\right]^2$$

TIDYING UP AS 4 TERMS CANCEL

$$\text{Var}(X) = \frac{1}{k}\sum_{x=1}^{N} x^4 - \cancel{\frac{1}{k}} - \cancel{\frac{1}{k}\sum_{x=1}^{N} x^3} + \cancel{\frac{1}{k}} + \cancel{\frac{1}{k}\sum_{x=1}^{N} x^3} - \frac{1}{k^2}\left[\sum_{x=1}^{N} x^3\right]^2$$

$$\Rightarrow \text{Var}(X) = \frac{1}{k}\sum_{x=1}^{N} x^4 - \frac{1}{k^2}\left[\sum_{x=1}^{N} x^3\right]^2$$

USING THE RESULT GIVEN FOR THE SUM OF x^4

$$\Rightarrow \text{Var}(X) = \frac{1}{\frac{1}{6}N(N+1)(2N+1)} \times \frac{1}{30}N(N+1)(6N^3+9N^2+N-1) - \frac{\overbrace{\left[\frac{1}{4}N^2(N+1)\right]^2}^{\left(\sum x^3\right)^2}}{\underbrace{\left[\frac{1}{6}N(N+1)(2N+1)\right]^2}_{k^2}}$$

$$= \frac{\cancel{N}\cancel{(N+1)}(6N^3+9N^2+N-1)}{5\cancel{N}\cancel{(N+1)}(2N+1)} - \frac{\frac{1}{16}N^{\cancel{4}\,2}(N+1)^{\cancel{4}\,2}}{\frac{1}{36}\cancel{N^2}\cancel{(N+1)^2}(2N+1)^2}$$

$$= \frac{6N^3+9N^2+N-1}{5(2N+1)} - \frac{9N^2(N+1)^2}{4(2N+1)^2}$$

"ADDING" THE FRACTIONS

$$\Rightarrow \text{Var}(X) = \frac{4(2N+1)(6N^3+9N^2+N-1) - 45N^2(N+1)^2}{20(2N+1)^2}$$

TIDY THE NUMERATOR, AS NUMERATOR ALREADY MATCHES THE "SHOW"

$$\left\{\begin{matrix} 48N^4 + 72N^3 + 8N^2 - 8N \\ 24N^3 + 36N^2 + 4N - 4 \end{matrix}\right\} - 45N^2(N^2+2N+1)$$

$$= 48N^4 + 96N^3 + 44N^2 - 4N - 4 - 45N^4 - 90N^3 - 45N^2$$

$$= 3N^4 + 6N^3 - N^2 - 4N - 4$$

$$\therefore \text{Var}(X) = \frac{3N^4+6N^3-N^2-4N-4}{20(2N+1)^2}$$

AS REQUIRED

NOTE !!

IN THE QUESTION $\sum_{x=1}^{N} x^3(x-1)$ OF COURSE STARTS FROM $x=2$ BUT TRULY THERE IS NO NEED TO DO ALL THIS AS $\sum_{x=1}^{N} x^3(x-1) = \sum_{x=1}^{N} x^4 - \sum_{x=1}^{N} x^3$ — I APPROACH IT IN A LENGTHY MANNER TO SHOW SERIES MANIPULATIONS

MOMENT GENERATING FUNCTIONS

(m.g.f.)