HYPOTHESIS TESTING

errors

Question 1 (**)

The discrete random variable X has binomial distribution.

A random sample of 25 observations of X is used to test the hypotheses

$$\mathrm{H}_0 : \; p = 0.25 \qquad \text{against} \qquad \mathrm{H}_1 : p > 0.25 ,$$

at the 5% level of significance.

a) Find size of the test.

b) Given further that $p = 0.45$ determine the power of the test.

$\boxed{}$, $\boxed{0.0297}$, $\boxed{0.6157}$

a) $X \sim B(25, 0.25)$ / DETERMINE THE CRITICAL REGION

$H_0 : p = 0.25$
$H_1 : p > 0.25$, p THE PROPORTION IN THE ENTIRE POPULATION

WORKING AT THE BINOMIAL TABLES

$P(X \geqslant 10) = 1 - P(X \leqslant 9) = 1 - 0.9287 = 0.0713 > 0.05$
$P(X \geqslant 11) = 1 - P(X \leqslant 10) = 1 - 0.9703 = 0.0297 < 0.05$

$C.R = \{11, 12, 13, \ldots, 25\}$

SIZE OF TEST $=$ P(TYPE I ERROR) $=$ ACTUAL SIGNIFICANCE $= 0.0297$

b) POWER OF A TEST $= 1 -$ P(TYPE II ERROR)

"REJECT H_1 WHEN H_1 IS TRUE"

$P(X \leqslant 10)$ FROM $B(25, 0.45)$

... tables ...

$= 1 - 0.3843$

$= 0.6157$

Question 2 (**)

Bags of cement are filled by factory machinery. It is known from past records that the weights of these bags are Normally distributed with a standard deviation of 400 grams.

The bags of cement have an advertised weight of 50 kg.

At the start of each month a random sample of 25 bags are weighed, in order to test the functionality of the machinery.

The test is carried at the 5% level of significance, for

$$\text{H}_0: \ \mu = 50 \text{ kg} \qquad \text{versus} \qquad \text{H}_1: \ \mu \neq 50 \text{ kg}.$$

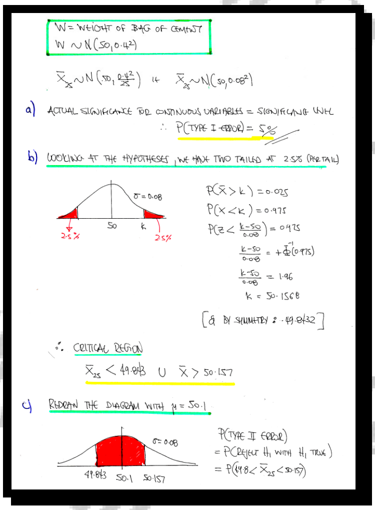
For this test…

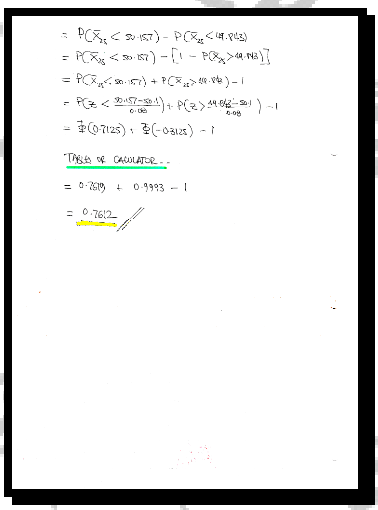
a) … state the probability of a Type I error.

b) … find the critical region, correct to the nearest gram.

c) … determine the probability of a Type II error, if the mean weight has in fact changed to 50.1 kg.

$\boxed{0.05}$, $\boxed{(49.843, 50.157)}$, $\boxed{0.7612}$





Question 3 (**+)

Bags of flour have a nominal weight of 2 kg.

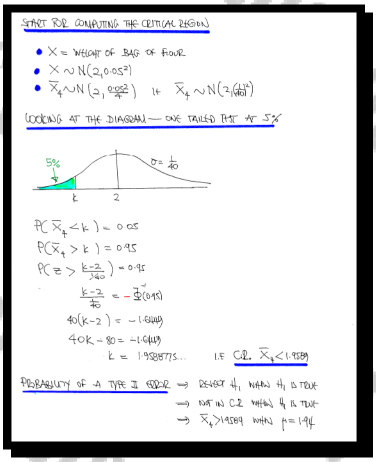
It is known from past records that the weights of these bags are Normally distributed with a standard deviation of 50 grams.

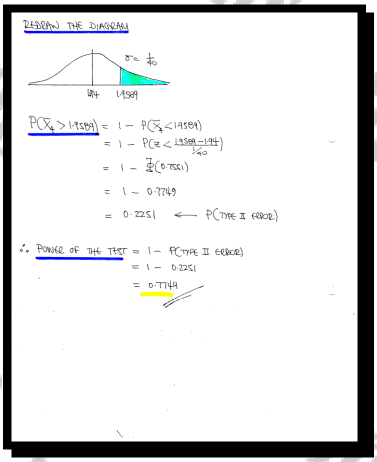
A random sample of 4 bags are weighed, in order to carry out a test at the 5% level of significance for the following hypotheses.

$$H_0: \mu = 2 \text{ kg} \qquad \text{versus} \qquad H_1: \mu < 2 \text{ kg},$$

Given further that the mean weight has in fact changed to 1.94 kg, calculate the power of this test.

$\boxed{}$, $\boxed{0.7744}$





Question 4 (***)

The discrete random variable X has Poisson distribution.

A critical region is determined, to test the hypotheses

$$H_0: \lambda = 8 \qquad \text{against} \qquad H_1: \lambda \neq 8,$$

so that the probability of rejection in each tail is as close as possible to 5%.

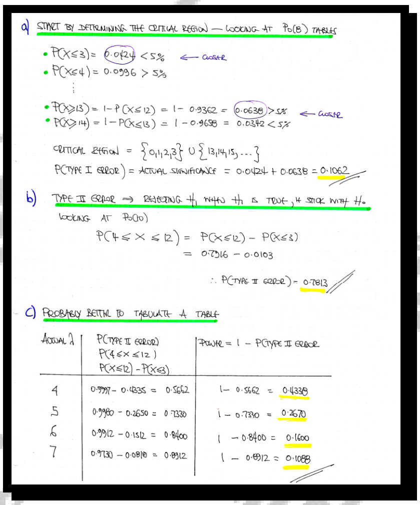
a) Find the probability of a Type I error occurring.

b) If $\lambda = 10$, determine the probability of a Type II error occurring.

c) Determine the power function, for integer values of λ, $4 \leq \lambda \leq 7$.

$\boxed{}$, $\boxed{0.1062}$, $\boxed{0.7813}$,

$\lambda = 4$	0.4338
$\lambda = 5$	0.2670
$\lambda = 6$	0.1600
$\lambda = 7$	0.1088



a) START BY DETERMINING THE CRITICAL REGION — LOOKING AT Po(8) TABLES

- $P(X \leq 3) = 0.0424 < 5\%$ ← CLOSER
- $P(X \leq 4) = 0.0996 > 5\%$

⋮

- $P(X \geq 13) = 1 - P(X \leq 12) = 1 - 0.9362 = 0.0638 > 5\%$ ← CLOSER
- $P(X \geq 14) = 1 - P(X \leq 13) = 1 - 0.9658 = 0.0342 < 5\%$

CRITICAL REGION $= \{0,1,2,3\} \cup \{13,14,15,\dots\}$

P(TYPE I ERROR) = ACTUAL SIGNIFICANCE $= 0.0424 + 0.0638 = 0.1062$

b) TYPE II ERROR ⇒ REJECTING H_1 WHEN H_1 IS TRUE, IE STICK WITH H_0

LOOKING AT Po(10)

$$P(4 \leq X \leq 12) = P(X \leq 12) - P(X \leq 3)$$
$$= 0.7916 - 0.0103$$

$\therefore$ P(TYPE II ERROR) $= 0.7813$

c) PROBABLY BETTER TO TABULATE A TABLE

ACTUAL λ	P(TYPE II ERROR) $P(4 \leq X \leq 12)$ $P(X \leq 12) - P(X \leq 3)$	POWER = 1 − P(TYPE II ERROR
4	$0.9997 - 0.4335 = 0.5662$	$1 - 0.5662 = 0.4338$
5	$0.9980 - 0.2650 = 0.7330$	$1 - 0.7330 = 0.2670$
6	$0.9912 - 0.1512 = 0.8400$	$1 - 0.8400 = 0.1600$
7	$0.9730 - 0.0818 = 0.8912$	$1 - 0.8912 = 0.1088$

Question 5 (***+)

A train operator claims that the proportion p, of their trains running late, is 0.1.

A test, based on a random sample of 10 trains, is to be used to investigate the train operator's claim.

The hypothesis that $p = 0.1$ is to be rejected, if more than 2 trains arrive late.

a) Find the size of this test.

b) Show that the power function of this test is given by

$$1-(1-p)^8\left(36p^2+8p+1\right).$$

c) Plot the power function of part **(b)** in a suitably labelled set of axes, using values from $p = 0.2$ to $p = 0.7$, with a step of 0.1.

$\boxed{}$, $\boxed{0.0702}$

a) X = NUMBER OF LATE TRAINS

$X \sim B(10, 0.1)$

- $H_0: p = 0.1$ p = IS THE PROPORTION OF LATE TRAINS IN GENERAL
- $H_1: p > 0.1$

THE CRITICAL REGION IS $\{3, 4, 5, 6, \dots, 10\}$

↑ REJECT H_0

SIZE = P(TYPE I ERROR) = ACTUAL SIGNIFICANCE

$= P(X \geqslant 3)$

$= 1 - P(X \leqslant 2)$

$= 1 - 0.9298$

$= 0.0702$

b) POWER = 1 − P(TYPE II ERROR)

THIS OCCURS IF WE REJECT H_1, WHEN H_1 IS TRUE, IE $X = 0, 1, 2$ WITH $p > 0.1$

POWER FUNCTION $= 1 - P(X = 0, 1, 2)$

$$= 1 - \left[\binom{10}{0}p^0(1-p)^{10} + \binom{10}{1}p^1(1-p)^9 + \binom{10}{2}p^2(1-p)^8\right]$$

$$= 1 - \left[(1-p)^{10} + 10p(1-p)^9 + 45p^2(1-p)^8\right]$$

$$= 1 - (1-p)^8\left[(1-p)^2 + 10p(1-p) + 45p^2\right]$$

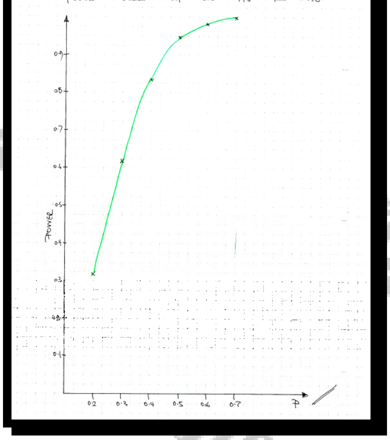
$$= 1 - (1-p)^8\left[1 - 2p + p^2 + 10p - 10p^2 + 45p^2\right]$$

$$= 1 - (1-p)^8\left[36p^2 + 8p + 1\right]$$

AS REQUIRED

c) GENERATING SOME VALUES, SAY 3 DECIMAL PLACES

p	0.2	0.3	0.4	0.5	0.6	0.7
POWER	0.322	0.617	0.833	0.945	0.988	0.998



Question 6 (****)

It is thought that a proportion p, of the tiles produced in a factory, have minor flaws.

A quality test, based on a random sample of 10 tiles from the daily production is devised, in order to test if p is greater than 0.1.

The hypothesis that $p = 0.1$ is rejected, if more than 4 tiles with flaws are found in the sample. The table below shows the power function for this test.

p	0.15	0.2	0.25	0.3	0.35	0.4
power	0.0099	0.0328	k	0.1503	0.2485	0.3669

a) Find the size of this test.

b) Determine the value of k, in the above table.

A second test, based on a random sample of 20 tiles from the daily production is proposed, in order to test if p is greater than 0.1.

The hypothesis that $p = 0.1$ is rejected if more than 9 tiles with flaws are found in the sample.

c) Find the probability of a Type I error occurring, when using the second test.

The table below shows the power function for the second test.

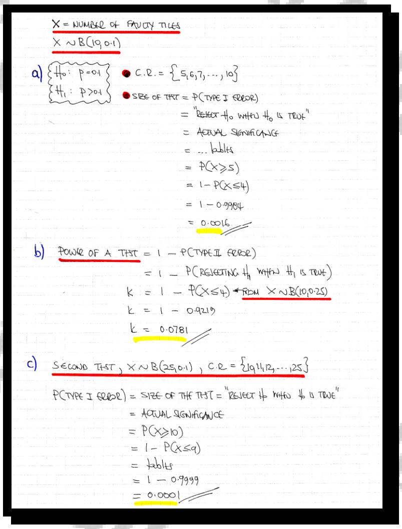
p	0.15	0.2	0.25	0.3	0.35	0.4
power	0.0021	0.0173	0.0713	0.1894	0.3697	0.5754

d) Plot the graphs of the two power functions in the same set of axes.

e) Using these graphs, state the value of p, when the two graphs meet, briefly explaining the significance if p is smaller or greater than this value.

$\boxed{}$, $\boxed{0.0016}$, $\boxed{0.0781}$, $\boxed{0.0001}$, $\boxed{p \approx 0.27}$

[solution overleaf]



X = NUMBER OF FAULTY TILES

$X \sim B(10, 0.1)$

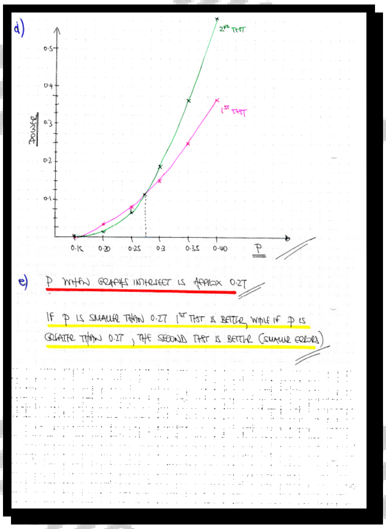
a) $H_0 : p = 0.1$
$H_1 : p > 0.1$

- $C.R. = \{5, 6, 7, \ldots, 10\}$
- SIZE OF TEST = P(TYPE I ERROR)
= "REJECT H_0 WHEN H_0 IS TRUE"
= ACTUAL SIGNIFICANCE
= ... tables
$= P(X \geqslant 5)$
$= 1 - P(X \leqslant 4)$
$= 1 - 0.9984$
$= 0.0016$

b) POWER OF A TEST = 1 − P(TYPE II ERROR)
= 1 − P(REJECTING H_1 WHEN H_1 IS TRUE)
$k = 1 - P(X \leqslant 4)$ ← FROM $X \sim B(10, 0.25)$
$k = 1 - 0.9219$
$k = 0.0781$

c) SECOND TEST, $X \sim B(25, 0.1)$, $C.R = \{10, 11, 12, \ldots, 25\}$

P(TYPE I ERROR) = SIZE OF THE TEST = "REJECT H_0 WHEN H_0 IS TRUE"
= ACTUAL SIGNIFICANCE
$= P(X \geqslant 10)$
$= 1 - P(X \leqslant 9)$
= tables
$= 1 - 0.9999$
$= 0.0001$



d)

e) P WHEN GRAPHS INTERSECT IS APPROX 0.27

IF P IS SMALLER THAN 0.27 1ST TEST IS BETTER, WHILE IF P IS GREATER THAN 0.27, THE SECOND TEST IS BETTER (SMALLER ERRORS)

Question 7 **(****)**

A private maths tutor has some of his session cancelled randomly, at the constant rate of 8 sessions per month.

a) Find the probability that, in a given month, …

i. … exactly 8 sessions are cancelled.

ii. … exactly 4 sessions are cancelled in each half of the month.

In a random period of 2 months, 17 sessions were cancelled.

b) Show that the probability of **at least** 8 sessions being cancelled **in each** of these 2 months, is exactly $\left(\frac{1}{2}\right)^{16} \times C_8^{17}$.

The tutor claims that the rate of his cancelled sessions have decreased, because in the most recent month he had 4 cancelled sessions.

c) Stating your hypotheses clearly, …

i. … test at the 5% level of significance the tutor's claim.

ii. … state the critical region for this test of part (**ii**).

iii. … state the probability of a Type I error occurring, if the critical region of part (**ii**) is to be used.

iv. … determine the power of the test if the rate of the cancelled sessions is in fact 6 per month.

$\boxed{\quad}$, $\boxed{0.1396}$, $\boxed{0.0382}$, $\boxed{\text{not significant, } 9.96\% > 5\%}$, $\boxed{\text{C.R.} = \{0,1,2,3\}}$, $\boxed{0.0424}$, $\boxed{0.1512}$

c) I $X \sim Po(8)$ WITH HYPOTHESES

- $H_0 : \lambda = 8$
- $H_1 : \lambda < 8$ $\quad\lambda$ THE MONTHLY RATE OF CANCELLATIONS IN GENERAL

$P(X \leqslant 4) = \ldots \text{tables} \ldots = 0.0996 = 9.96\% > 5\%$

THERE IS NO SIGNIFICANT EVIDENCE TO SUPPORT THE TUTOR'S CLAIM – INSUFFICIENT EVIDENCE TO REJECT H_0

II) LOOKING AT TABLES AGAIN

$$\left.\begin{array}{l} P(X \leqslant 4) = 0.0996 > 5\% \\ P(X \leqslant 3) = 0.0424 < 5\% \end{array}\right\} \therefore \text{C.R.} = \{0,1,2,3\}$$

III) P(TYPE I ERROR) = ACTUAL SIGNIFICANCE = 0.0424

IV) "POWER OF THE TEST" $= 1 - \text{P(TYPE II ERROR)}$

$= 1 - \text{P(REJECTING } H_1 \text{ WHEN } H_1 \text{ IS TRUE)}$

$= 1 - P(X \geqslant 4)$ ← FROM $Po(6)$

$= 1 - [1 - P(X \leqslant 3)]$

$= P(X \leqslant 3)$ ← FROM $Po(6)$

$= 0.1512$

a) ASSUME A POISSON MODEL

- X = NUMBER OF CANCELLED SESSIONS PER MONTH
- $X \sim Po(8)$

I) $P(X=8) = \dfrac{e^{-8} \times 8^8}{8!} = 0.1396$

II) ADJUST THE RATE

- Y = NUMBER OF CANCELLED SESSIONS PER ½ MONTH
- $Y \sim Po(4)$

REQUIRED PROBABILITY $= [P(Y=4)]^2 = \left(\dfrac{e^{-4} \times 4^4}{4!}\right)^2 = 0.0382$

b) ADJUST THE RATE TO TWO MONTHS

- W = NUMBER OF CANCELLED SESSIONS PER TWO MONTHS
- $W \sim Po(16)$

$P(W=17) = \dfrac{e^{-16} \times 16^{17}}{17!}$

"AT LEAST 8 IN EACH OF THE TWO MONTHS", CAN ONLY HAPPEN AS $P(X=8) \times P(X=9)$ OR THE OTHER WAY ROUND

∴ REQUIRED PROBABILITY IS

$$\frac{2 \times P(X=8) \times P(X=9)}{P(W=17)} = \frac{2 \times \frac{e^{-8} \times 8^8}{8!} \times \frac{e^{-8} \times 8^9}{9!}}{\frac{e^{-16} \times 16^{17}}{17!}}$$

$$= \frac{2 \times \cancel{e^{-16}} \times 8^{17}}{8!\,9!} \times \frac{17!}{\cancel{e^{-16}} \times 16^{17}} = 2 \times \left(\frac{17!}{8!\,9!}\right) \times \left(\frac{8}{16}\right)^{17}$$

$$= 2 \times \binom{17}{8} \times \left(\frac{1}{2}\right)^{17} = C_8^{17}\left(\frac{1}{2}\right)^{16}$$