RELATED RANDOM VARIABLES

Question 1 (***)

The continuous random variable X is uniformly distributed in the interval $[1,16]$.

The continuous random variable Y is related to X by the equation $Y = \sqrt{X}$.

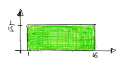
Determine the probability density function of Y.

$$\boxed{f(y) = \begin{cases} \frac{2}{15}y & 1 < y < 4 \\ 0 & \text{otherwise} \end{cases}}$$

START BY OBTAINING THE C.D.F. OF THE UNIFORM DISTRIBUTION

$X \sim U[1,16]$

$$f(x) = \begin{cases} \frac{1}{15} & 1 \leqslant x \leqslant 16 \\ 0 & \text{OTHERWISE} \end{cases}$$



$$F(x) = \int_1^x \frac{1}{15}\,dx = \left[\frac{1}{15}x\right]_1^x = \frac{1}{15}x - \frac{1}{15}$$

NOW WE PROCEED FROM THE C.D.F OF Y

$$G(y) = P(Y < y)$$

$\leqslant$ IS NOT NEEDED AS BOTH X & Y ARE CONTINUOUS

$$= P(\sqrt{X} < y)$$
$$= P(X < y^2)$$
$$= F(y^2)$$
$$= \frac{1}{15}y^2 - \frac{1}{15}$$

THUS THE P.D.F CAN BE FOUND BY DIFFERENTIATION

$$g(y) = \frac{d}{dy}G(y) = \frac{d}{dy}\left[\frac{1}{15}y^2 - \frac{1}{15}\right] = \frac{2}{15}y$$

$$\therefore g(y) = \begin{cases} \frac{2}{15}y & 1 \leqslant y \leqslant 4 \\ 0 & \text{OTHERWISE} \end{cases}$$

Question 2 (***)

The continuous random variable X is uniformly distributed in the interval $[2,6]$.

The continuous random variable Y is related to X by the equation $Y = \dfrac{1}{X}$.

Determine the probability density function of Y.

$$\boxed{f(y) = \begin{cases} \dfrac{1}{4y^2} & \dfrac{1}{6} < y < \dfrac{1}{2} \\ 0 & \text{otherwise} \end{cases}}$$

WORKING WITH THE CUMULATIVE DISTRIBUTION FUNCTION

$F(y) = P(Y \le y)$

$= P\left(\frac{1}{X} \le y\right)$

$= P\left(X \ge \frac{1}{y}\right)$

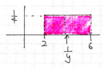
BUT X IS UNIFORMLY DISTRIBUTED IN $[2,6]$

$= 1 - P\left(X \le \frac{1}{y}\right)$

$= 1 - \left(\frac{1}{y} - 2\right) \times \frac{1}{4}$

$= 1 - \frac{1}{4y} + \frac{1}{2}$

$= \frac{3}{2} - \frac{1}{4y}$



DIFFERENTIATING WE HAVE

$F(y) = \frac{3}{2} - \frac{1}{4y}$

$f(y) = \frac{1}{4y^2}$

THUS WE HAVE

$$f(y) = \begin{cases} \frac{1}{4y^2} & \frac{1}{6} < y < \frac{1}{2} \\ 0 & \text{OTHERWISE} \end{cases}$$

Question 3 (***+)

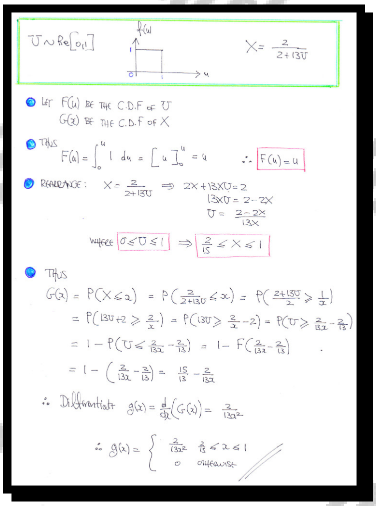
The continuous random variable U is uniformly distributed in $[0,1]$.

The continuous random variable X satisfies

$$X = \frac{2}{2+13U}.$$

Determine the probability density function of X.

$$\boxed{}, \quad \boxed{f(x) = \begin{cases} \dfrac{2}{13x^2} & \dfrac{2}{15} < x < 1 \\ 0 & \text{otherwise} \end{cases}}$$



$\boxed{U \sim Re[0,1]}$ $\qquad f(u)$ $\qquad X = \frac{2}{2+13U}$

- LET $F(u)$ BE THE C.D.F OF U, $G(x)$ BE THE C.D.F OF X
- THUS $F(u) = \int_0^u 1 \, du = [u]_0^u = u \qquad \therefore \boxed{F(u) = u}$
- REARRANGE: $X = \frac{2}{2+13U} \Rightarrow 2X + 13XU = 2$
  $13XU = 2 - 2X$
  $U = \frac{2-2X}{13X}$
  WHERE $\boxed{0 \leq U \leq 1} \Rightarrow \boxed{\frac{2}{15} \leq X \leq 1}$
- THUS
  $G(x) = P(X \leq x) = P\left(\frac{2}{2+13U} \leq x\right) = P\left(\frac{2+13U}{2} \geq \frac{1}{x}\right)$
  $= P\left(13U + 2 \geq \frac{2}{x}\right) = P\left(13U \geq \frac{2}{x} - 2\right) = P\left(U \geq \frac{2}{13x} - \frac{2}{13}\right)$
  $= 1 - P\left(U \leq \frac{2}{13x} - \frac{2}{13}\right) = 1 - F\left(\frac{2}{13x} - \frac{2}{13}\right)$
  $= 1 - \left(\frac{2}{13x} - \frac{2}{13}\right) = \frac{15}{13} - \frac{2}{13x}$

$\therefore$ Differentiate $g(x) = \frac{d}{dx}(G(x)) = \frac{2}{13x^2}$

$$\therefore g(x) = \begin{cases} \frac{2}{13x^2} & \frac{2}{15} \leq x \leq 1 \\ 0 & \text{OTHERWISE} \end{cases}$$