POISSON DISTRIBUTION

INTRODUCTORY CALCULATIONS

Question 1

Accidents occur on a certain stretch of motorway at the rate of three per month.

Find the probability that on a given month there will be …

a) … no accidents.

b) … one accident.

c) … two accidents.

d) … three accidents.

e) … four accidents.

$\boxed{0.0498}$, $\boxed{0.1494}$, $\boxed{0.2240}$, $\boxed{0.2240}$, $\boxed{0.1680}$

X = accidents per month

$X \sim Po(3)$

(a) $P(X=0) = \dfrac{e^{-3} \times 3^0}{0!} = 0.0498$

(b) $P(X=1) = \dfrac{e^{-3} \times 3^1}{1!} = 0.1494$

(c) $P(X=2) = \dfrac{e^{-3} \times 3^2}{2!} = 0.2240$

(d) $P(X=3) = \dfrac{e^{-3} \times 3^3}{3!} = 0.2240$

(e) $P(X=4) = \dfrac{e^{-3} \times 3^4}{4!} = 0.1680$

Question 2

Cretan Airlines services which arrive late to Athens Airport on a typical week can be modelled by a Poisson distribution with mean of 4.5 .

a) Determine the probability that on a given week there will be …

i. … four late arrivals.

ii. … less than four late arrivals.

iii. … at least seven late arrivals.

b) Determine the probability that on a given two week period there will be between eight and thirteen (inclusive) late arrivals.

$\boxed{0.1898}$, $\boxed{0.3423}$, $\boxed{0.1689}$, $\boxed{0.6022}$

X = late arrivals per week
$X \sim Po(4.5)$

(a) (I) $P(X=4) = \dfrac{e^{-4.5} \times 4.5^4}{4!} = 0.1898$

(II) $P(X<4) = P(X \leqslant 3) = \dots \text{tables} \dots = 0.3423$

(III) $P(X \geqslant 7) = 1 - P(X \leqslant 6) = \dots \text{tables} \dots$
$1 - 0.8311 = 0.1689$

Y = late arrivals per two week periods
$Y \sim Po(9)$

(b) $P(8 \leqslant Y \leqslant 13) = P(Y \leqslant 13) - P(Y \leqslant 7) = \text{tables}$
$= 0.9261 - 0.3239$
$= 0.6022$

Question 3

Sheep are randomly scattered in a field which is divided into equal size squares.

There are three sheep on average on each square.

Determine the probability that on a given square there will be …

a) … no more than five sheep.

b) … more than two but no more than seven sheep.

$\boxed{0.9161}$, $\boxed{0.5649}$

X = sheep per square
$X \sim Po(3)$

(a) $P(X \leq 5) = \dots \text{tables} \dots = 0.9161$

(b) $P(3 < X \leq 7) = P(3 \leq X \leq 7) = P(X \leq 7) - P(X \leq 2)$
$= \dots \text{tables} \dots$
$= 0.9881 - 0.4232 = 0.5649$

Question 4

Post boxes in London are randomly spaced out and on average five post boxes are found per square mile.

a) Determine the probability that on a certain square mile there will be …

i. … exactly seven post boxes.

ii. … less than eight but no less than four post boxes.

The number of post boxes in a smaller area of $\frac{1}{4}$ of a square mile, are counted.

b) Find the probability that this smaller area will contain post boxes.

$\boxed{0.1044}$, $\boxed{0.6016}$, $\boxed{0.7135}$

X = post boxes per square mile.
$X \sim Po(5)$

(a) (I) $P(X=7) = \frac{e^{-5} \times 5^7}{7!} = 0.1044$

(II) $P(4 \leqslant X < 8) = P(4 \leqslant X \leqslant 7) = P(X \leqslant 7) - P(X \leqslant 3)$
$= \text{tables} \ldots\ldots =$
$= 0.8666 - 0.2650 = 0.6016$

(b) Y = post boxes per ¼ square mile
$Y \sim Po(1.25)$

$P(X \geqslant 1) = 1 - P(X=0) = 1 - \frac{e^{-1.25} \times 1.25^0}{0!}$
$= 1 - 0.2865 = 0.7135$

EXAM QUESTIONS

Question 1 (**)

The water from a lake is tested and is found to contain on average three bacteria per litre of water. A sample of 250 ml is collected from the lake.

a) Determine the probability that the 250 ml water sample will contain …

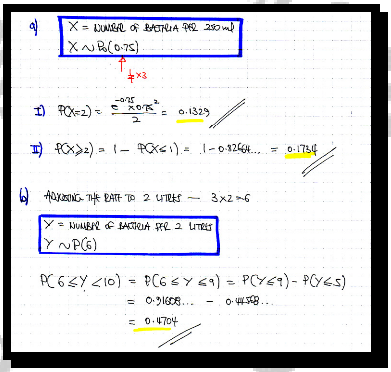
i. … exactly two bacteria.

ii. … at least two bacteria.

A larger sample of two litres of water is collected from the lake.

b) Find the probability that this larger sample will contain less than ten but no less than six bacteria.

$\boxed{}$, $\boxed{0.1329}$, $\boxed{0.1734}$, $\boxed{0.4704}$



a) $\boxed{\begin{array}{l} X = \text{NUMBER OF BACTERIA PER } 250\text{ ml} \\ X \sim Po(0.75) \end{array}}$

$\frac{1}{4} \times 3$

I) $P(X=2) = \frac{e^{-0.75} \times 0.75^2}{2} = 0.1329$ //

II) $P(X \geqslant 2) = 1 - P(X \leqslant 1) = 1 - 0.82664... = 0.1734$ //

b) ADJUSTING THE RATE TO 2 LITRES — $3 \times 2 = 6$

$\boxed{\begin{array}{l} Y = \text{NUMBER OF BACTERIA PER 2 LITRES} \\ Y \sim P(6) \end{array}}$

$P(6 \leqslant Y < 10) = P(6 \leqslant Y \leqslant 9) = P(Y \leqslant 9) - P(Y \leqslant 5)$

$= 0.91608... - 0.44568...$

$= 0.4704$ //

Question 2 **(***)**

A discrete random variable X has Poisson distribution with mean λ.

Given that $\mathrm{P}(X=8)=\mathrm{P}(X=9)$, determine the value of $\mathrm{P}(4<X\le 10)$.

$\boxed{}$, $\boxed{0.6510}$

$X \sim Po(\lambda)$

$\Rightarrow P(X=8) = P(X=9)$

$\Rightarrow \dfrac{e^{-\lambda} \times \lambda^8}{8!} = \dfrac{e^{-\lambda} \times \lambda^9}{9!}$ DIVIDE THROUGH BY $e^{-\lambda} \neq 0$

$\Rightarrow \dfrac{\lambda^8}{8!} = \dfrac{\lambda^9}{9!}$

$\Rightarrow \dfrac{9!}{8!} = \dfrac{\lambda^9}{\lambda^8}$

$\Rightarrow 9 = \lambda$

NOW WE CAN EVALUATE $P(4<X\leq 10)$

$P(4<X\leq 10) = P(5\leq X\leq 10)$

$= P(X\leq 10) - P(X\leq 4)$

$= 0.7060\ldots - 0.05496\ldots$

$= 0.6510$

Question 3 (***+)

A shop sells a particular make of smart phone. It is assumed that the daily sales of this type of phone is a Poisson variable with mean 3.

a) Find the probability, giving the answers in terms of e, that on a particular day the shop sells …

- **i.** … exactly 3 smart phones.
- **ii.** … at least 4 smart phones.

It is further given that in a particular day at least 4 smart phones were sold.

b) Show that the probability that exactly 7 smart phones were actually sold that day is given by

$$\frac{243}{k\left(\mathrm{e}^{3}-13\right)},$$

where k is an integer to be found.

$\boxed{\quad\quad}$, $\boxed{\frac{9}{2}\mathrm{e}^{-3} \approx 0.2240}$, $\boxed{1-13\mathrm{e}^{-3} \approx 0.3528}$, $\boxed{k=560}$

$\boxed{\begin{array}{l} X = \text{NUMBER OF PHONES SOLD PER DAY} \\ X \sim Po(3) \end{array}}$

a) I) $P(X=3) = \frac{e^{-3} \times 3^3}{3!} = \frac{e^{-3} \times 27}{6} = \frac{9}{2}e^{-3} = \frac{9}{2e^3}$

II) $P(X \geqslant 4) = 1 - P(X \leqslant 3) = 1 - P(X=0,1,2,3)$

$= 1 - \left[\frac{e^{-3}\times 3^0}{0!} + \frac{e^{-3}\times 3^1}{1!} + \frac{e^{-3}\times 3^2}{2!} + \frac{e^{-3}\times 3^3}{3!}\right]$

$= 1 - \left[e^{-3} + 3e^{-3} + \frac{9}{2}e^{-3} + \frac{9}{2}e^{-3}\right]$

$= 1 - 13e^{-3}$

b) WE REQUIRE $P(X=7 \mid X \geqslant 4)$

$\frac{P(X=7)}{P(X \geqslant 4)} = \frac{\frac{e^{-3}\times 3^7}{7!}}{1-13e^{-3}} = \frac{e^{-3}\times 3^7}{7!} \div (1-13e^{-3})$

$= \frac{243}{560}e^{-3} \times \frac{1}{1-13e^{-3}} = \frac{243}{560e^{3}} \times \frac{1}{1-13e^{-3}}$

$= \frac{243}{560(e^{3}-13e^{-3}e^{3})} = \frac{243}{560(e^{3}-13e^{0})} = \frac{243}{560(e^{3}-13)}$

AS REQUIRED $(k=560)$

Question 4 (***+)

The number of houses sold by an estate agent follows a Poisson distribution, with a mean of 2.5 houses per week.

a) Find the probability that in the next four weeks the estate agent sells …

i. … exactly 4 houses

ii. … more than 6 houses.

The estate agent monitors the house sales in periods of 4 weeks.

b) Find the probability that in the next twelve of those four week periods there are exactly 7 four week periods in which more than 6 houses are sold.

$\boxed{}$, $\boxed{0.0189}$, $\boxed{0.8699}$, $\boxed{0.0111}$

a) RATE GIVEN IS 2.5 PER WEEK

X = HOUSE SALES PER 4 WEEK PERIOD
$X \sim Po(10)$

I) $P(X=4) = \frac{e^{-10} \times 10^4}{4!} = 0.0189$

II) $P(X>6) = P(X \geq 7) = 1 - P(X \leq 6) = 1 - 0.1301 = 0.8699$

b) Y = NUMBER OF 4 WEEK PERIODS WITH AT LEAST 7 SALES EACH
$Y \sim B(12, 0.8699)$

$P(Y=7) = \binom{12}{7}(0.8699)^7(0.1301)^5 = 0.0111$

Question 5 (***+)

The number of errors per page typed by Lena is assumed to follow a Poisson distribution with a mean of 0.45.

a) State two conditions, for a Poisson distribution to be a suitable model for the number of errors per page, typed by Lena.

A page typed by Lena is picked at random.

b) Calculate the probability that this page will contain exactly 2 errors.

c) Calculate the probability that this page will contain at least 2 errors.

Another 20 pages typed by Lena are picked at random.

d) Determine the least integer k such that the probability of having k or more typing errors, in these 20 pages typed by Lena, is less than 1% .

Finally, 320 pages typed by Lena are picked at random.

e) Use a distributional approximation to find the probability of having less than 125 typing errors, in these 320 pages typed by Lena.

$\boxed{X}$, $\boxed{\approx 0.0646}$, $\boxed{\approx 0.0754}$, $\boxed{k = 18}$, $\boxed{\approx 0.052}$

a)
- NUMBER OF ERRORS IN ONE PAGE IS INDEPENDENT OF THE NUMBER OF ERRORS IN ANOTHER PAGE
- RATE OF ERRORS IS UNIFORM & CONSTANT

$\boxed{X = \text{NUMBER OF ERRORS PER PAGE (LENA)} \quad X \sim Po(0.45)}$

b) $P(X=2) = \dfrac{e^{-0.45} \times 0.45^2}{2!} = 0.0646$

c) $P(X \geqslant 2) = 1 - P(X \leqslant 1) = 1 - 0.92456 = 0.0754$

d) $\boxed{Y = \text{NUMBER OF ERRORS PER 20 PAGES} \quad Y \sim Po(9)}$

$9 = 0.45 \times 20$

$\Rightarrow P(Y \geqslant k) < 0.01$

$\Rightarrow 1 - P(Y \leqslant k-1) < 0.01$

$\Rightarrow -P(Y \leqslant k-1) < -0.99$

$\Rightarrow P(Y \leqslant k-1) > 0.99$

BY TABLES OR CALCULATOR

$P(Y \leqslant 16) = 0.9889 < 0.99$

$P(Y \leqslant 17) = 0.9947 > 0.99$

$\therefore k - 1 = 17$

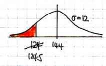
$k = 18$

e) ADJUST RATE TO $0.45 \times 320 = 144$

$\boxed{W = \text{NUMBER OF ERRORS PER 320 PAGES} \quad W \sim Po(144)}$

APPROXIMATE BY NORMAL

$V \sim N(144, 144)$



$P(W < 125)$

$= P(W \leqslant 124)$

$= P(V < 124.5)$

$= 1 - P(V > 124.5)$

$= 1 - P\left(Z > \frac{124.5 - 144}{12}\right)$

$= 1 - \Phi(-1.625)$

$= 1 - 0.9479$

$= 0.0521$

Question 6 (***+)

In a survey, along a certain coastline, plastic objects were found at a constant average rate of 250 per km.

a) Determine the probability that a 10 m length of this coastline will contain more than 4 plastic objects.

A similar survey, along the same coastline, drinks cans were found at a constant average rate of 160 per km.

b) Calculate the probability that a 30 m length of this coastline will contain exactly 5 drinks cans.

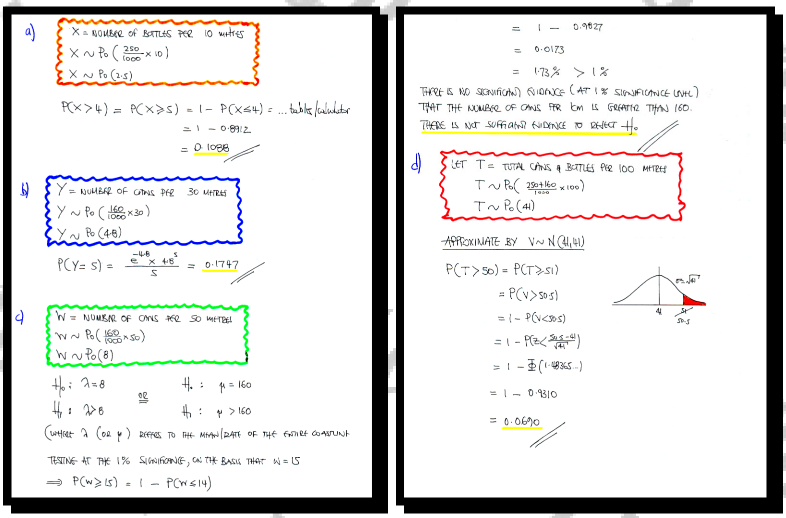
The local authority believes that in this coastline the average rate of drinks cans is higher than 160 per km.

c) Test, at the 1% level of significance, the local authority's belief.

Let T represent the total number of plastic objects and drinks cans per 100 m of the above mentioned coastline.

d) Find the approximate value of $\mathrm{P}(T > 50)$.

$\boxed{}$, $\boxed{0.1088}$, $\boxed{0.1747}$, $\boxed{\text{not significant, } 1.73\% > 1\%}$, $\boxed{0.0690}$


a) X = NUMBER OF BOTTLES PER 10 METRES
$X \sim Po\left(\frac{250}{1000} \times 10\right)$
$X \sim Po(2.5)$

$P(X>4) = P(X \geq 5) = 1 - P(X \leq 4) = \dots$ tables/calculator
$= 1 - 0.8912$
$= 0.1088$

b) Y = NUMBER OF CANS PER 30 METRES
$Y \sim Po\left(\frac{160}{1000} \times 30\right)$
$Y \sim Po(4.8)$

$P(Y=5) = \frac{e^{-4.8} \times 4.8^5}{5!} = 0.1747$

c) W = NUMBER OF CANS PER 50 METRES
$W \sim Po\left(\frac{160}{1000} \times 50\right)$
$W \sim Po(8)$

$H_0 : \lambda = 8$ OR $H_0 : \mu = 160$
$H_1 : \lambda > 8$ $H_1 : \mu > 160$

(WHERE λ (OR μ) REFERS TO THE MEAN/RATE OF THE ENTIRE COASTLINE

TESTING AT THE 1% SIGNIFICANCE, ON THE BASIS THAT $W = 15$
$\Rightarrow P(W \geq 15) = 1 - P(W \leq 14)$
$= 1 - 0.9827$
$= 0.0173$
$= 1.73\% > 1\%$

THERE IS NO SIGNIFICANT EVIDENCE (AT 1% SIGNIFICANCE LEVEL) THAT THE NUMBER OF CANS PER km IS GREATER THAN 160.
THERE IS NOT SUFFICIENT EVIDENCE TO REJECT H_0

d) LET T = TOTAL CANS & BOTTLES PER 100 METRES
$T \sim Po\left(\frac{250+160}{1000} \times 100\right)$
$T \sim Po(41)$

APPROXIMATE BY $V \sim N(41, 41)$

$P(T>50) = P(T \geq 51)$
$= P(V > 50.5)$
$= 1 - P(V < 50.5)$
$= 1 - P\left(Z < \frac{50.5 - 41}{\sqrt{41}}\right)$
$= 1 - \Phi(1.48365\dots)$
$= 1 - 0.9310$
$= 0.0690$

Question 7 (***+)

A bakery sells chocolate birthday cakes through the internet.

Orders for chocolate cakes are random and arrive at the constant rate of 4.5 per day.

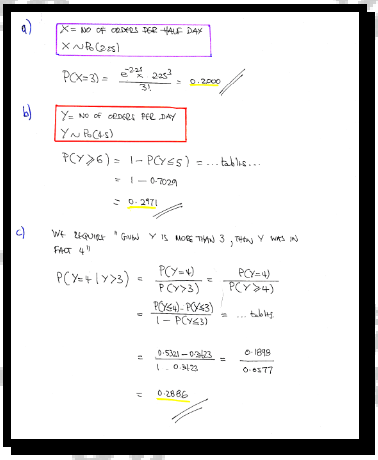
At the start of any given day, the bakery produces 6 chocolate birthday cakes and produces no more until the day after.

a) Find the probability that by the middle of a working day the bakery would have sold half the chocolate birthday cakes it produced for that day.

b) Calculate the probability that by the end of the working day the bakery would have sold all the chocolate birthday cakes it produced for the day.

c) Find the probability that if by the end of the day more than half the chocolate birthday cakes are sold, then exactly 4 were actually sold.

$\boxed{}$, $\boxed{0.1382}$, $\boxed{0.9093}$, $\boxed{\dfrac{5}{16}}$


a) $\boxed{\begin{array}{l} X = \text{NO OF ORDERS PER HALF DAY} \\ X \sim Po(2.25) \end{array}}$

$P(X=3) = \dfrac{e^{-2.25} \times 2.25^3}{3!} = 0.2000$

b) $\boxed{\begin{array}{l} Y = \text{NO OF ORDERS PER DAY} \\ Y \sim Po(4.5) \end{array}}$

$P(Y \geqslant 6) = 1 - P(Y \leqslant 5) = \ldots \text{tables} \ldots$
$= 1 - 0.7029$
$= 0.2971$

c) WE REQUIRE "GIVEN Y IS MORE THAN 3, THEN Y WAS IN FACT 4"

$P(Y=4 \mid Y>3) = \dfrac{P(Y=4)}{P(Y>3)} = \dfrac{P(Y=4)}{P(Y \geqslant 4)}$
$= \dfrac{P(Y \leqslant 4) - P(Y \leqslant 3)}{1 - P(Y \leqslant 3)} = \ldots \text{tables}$
$= \dfrac{0.5321 - 0.3423}{1 - 0.3423} = \dfrac{0.1898}{0.6577}$
$= 0.2886$

Question 8 (***+)

The number of typing errors per page in the first edition of a textbook follows a Poisson distribution with mean 0.7.

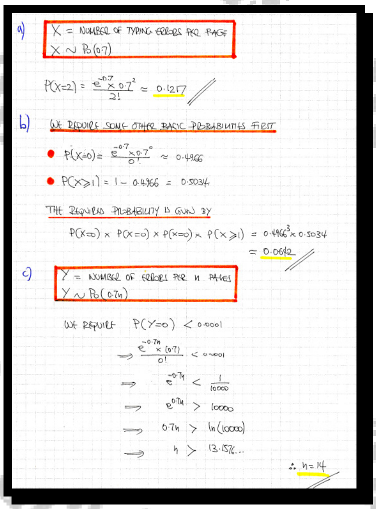
a) Determine the probability that a randomly chosen page in the first edition of this textbook contains exactly 2 typing errors.

The textbook is proof-read page by page, starting with page one.

b) Calculate the probability that the fourth page of the textbook is the first page to contain typing errors.

c) If n pages are examined find the smallest value of n, so that the probability of these n pages **not** containing errors is less than 0.0001.

$\boxed{}$, $\boxed{\approx 0.1217}$, $\boxed{\approx 0.0642}$, $\boxed{n = 14}$



a) $\boxed{\begin{array}{l} X = \text{NUMBER OF TYPING ERRORS PER PAGE} \\ X \sim Po(0.7) \end{array}}$

$P(X=2) = \dfrac{e^{-0.7} \times 0.7^2}{2!} \simeq 0.1217$ //

b) WE REQUIRE SOME OTHER BASIC PROBABILITIES FIRST

- $P(X=0) = \dfrac{e^{-0.7} \times 0.7^0}{0!} \simeq 0.4966$
- $P(X \geqslant 1) = 1 - 0.4966 = 0.5034$

THE REQUIRED PROBABILITY IS GIVEN BY

$P(X=0) \times P(X=0) \times P(X=0) \times P(X \geqslant 1) = 0.4966^3 \times 0.5034$
$\simeq 0.0642$ //

c) $\boxed{\begin{array}{l} Y = \text{NUMBER OF ERRORS PER } n \text{ PAGES} \\ Y \sim Po(0.7n) \end{array}}$

WE REQUIRE $P(Y=0) < 0.0001$

$\Rightarrow \dfrac{e^{-0.7n} \times (0.7)}{0!} < 0.0001$

$\Rightarrow e^{-0.7n} < \dfrac{1}{10000}$

$\Rightarrow e^{0.7n} > 10000$

$\Rightarrow 0.7n > \ln(10000)$

$\Rightarrow n > 13.157...$

$\therefore n = 14$ //

Question 9 (****)

The daily number of breakdowns of taxis of a certain taxi firm can be modelled by a Poisson distribution with mean 1.5.

a) Find the probability that on a given day there will be exactly 2 breakdowns.

b) Determine the probability that in a seven day week there will be exactly 3 days without a breakdown.

$\boxed{}$, $\boxed{0.2510}$, $\boxed{0.1416}$

a) $\boxed{\begin{array}{l} X = \text{No of breakdowns per day} \\ X \sim Po(1.5) \end{array}}$

$P(X=2) = \dfrac{e^{-1.5} \times 1.5^2}{2!} = 0.2510$

b) FIRSTLY FIND THE $P(X=0)$

$P(X=0) = \dfrac{e^{-1.5} \times 1.5^0}{0!} = 0.22313\ldots$

REMODEL AS FOLLOWS

$\boxed{\begin{array}{l} Y = \text{number of days, without a breakdown} \\ Y \sim B(7, 0.22313) \end{array}}$

$P(Y=3) = \binom{7}{3}(0.22313)^3(0.77687)^4 = 0.1416$

Question 10 (****)

A large office block is illuminated by light tubes which when they fail they are replaced by the block's caretaker.

The mean number of tubes that fail on a particular weekday, Monday to Friday, is 1.

The mean number of tubes that fail on a random two day weekend, is 0.5.

a) Find the probability that …

i. … exactly 4 light tubes fail on a particular Wednesday.

ii. … more than 2 light tubes fail on a particular weekend.

iii. … less than 4 light tubes fail on a particular complete, 7 day week.

The caretaker looks at his stock one Monday morning.

He wants to have the probability of running out of light tubes before the next Monday morning, less than 1%.

b) Calculate the smallest number of tubes he must have in stock.

$\boxed{}$, $\boxed{\approx 0.0153}$, $\boxed{\approx 0.0144}$, $\boxed{\approx 0.2017}$, $\boxed{12}$

a) START BY DEFINING DISTRIBUTIONS

I) $\boxed{X = \text{no of tube fails per weekday} \quad X \sim Po(1)}$

II) $\boxed{Y = \text{no of tube fails per weekend} \quad Y \sim Po(0.5)}$

$P(X=4) = \dfrac{e^{-1} \times 1^4}{4!} = \underline{0.0153}$

$P(Y>2) = P(Y \geqslant 3)$
$= 1 - P(Y \leqslant 2)$
… tables …
$= 1 - 0.9856$
$= \underline{0.0144}$

III) $\boxed{X+Y \sim Po(5 \times 1 + 0.5) \quad X+Y \sim Po(5.5)}$

$P(X+Y < 4) = P(X+Y \leqslant 3) = \text{… tables …} = \underline{0.2017}$

b) USING $X+Y \sim Po(5.5)$

$\Rightarrow P(X+Y > n) < 1\%$
$\Rightarrow P(X+Y \geqslant n+1) < 0.01$
$\Rightarrow 1 - P(X+Y) \leqslant n) < 0.01$
$\Rightarrow -P(X+Y \leqslant n) < -0.99$
$\Rightarrow P(X+Y \leqslant n) > 0.99$

LOOKING AT THE TABLES OF $Po(5.5)$

$\Rightarrow \underline{n = 12}$

Question 11 (****)

A car showroom salesman receives on average one call every 15 minutes.

a) Assuming a Poisson model, determine the probability that the salesman will receive exactly 6 calls between 9 a.m. and 10 a.m.

One morning the salesman received 10 calls between 9 a.m. and 11 a.m.

b) Assuming the same Poisson model, determine the probability that the salesman received exactly 6 calls between 9 a.m. and 10 a.m. on that morning.

$\boxed{}$, $\boxed{0.1042}$, $\boxed{\frac{105}{512} \approx 0.2051}$

a) ADJUSTING THE RATE – 1 CALL PER 15 MINUTES
4 CALLS PER 60 MINUTES

$$\boxed{\begin{array}{l} X = \text{NUMBER OF CALLS PER HOUR} \\ X \sim Po(4) \end{array}}$$

$$P(X=6) = \frac{e^{-4} \times 4^6}{6!} = 0.1042$$

b) ADJUST THE RATE TO 2 HOURS

$$\boxed{\begin{array}{l} Y = \text{NUMBER OF CALLS PER 2 HOURS} \\ Y \sim Po(8) \end{array}}$$

$$P(Y=10) = \frac{e^{-8} \times 8^{10}}{10!} = 0.992615....$$

WE REQUIRE

$$P(X=6 \mid Y=10) = \frac{P(X=6 \cap Y=10)}{P(Y=10)}$$

6 CALLS BETWEEN 9 & 10

4 CALLS BETWEEN 10 & 11

$$= \frac{P(X=6) \cap P(X=4)}{P(Y=10)}$$

10 CALLS BETWEEN 9 & 11

$$= \frac{\frac{e^{-4} \times 4^6}{6!} \times \frac{e^{-4} \times 4^4}{4!}}{\frac{e^{-8} \times 8^{10}}{10!}}$$

$$= \frac{\cancel{e^{-8}} \times \frac{4^{10}}{6!4!}}{\cancel{e^{-8}} \times \frac{8^{10}}{10!}} = \frac{10! \times 4^{10}}{8^{10} \times 6! \times 4!}$$

$$= \frac{105}{512} \simeq 0.2051$$

Question 12 (****)

A village post office opens on Wednesdays at 10.00.

After that time, customers arrive at this post office at the constant rate of 4 customers every ten minutes.

a) Find the probability that more than 3 but less than 8 customers will arrive at this post office between 10.00 and 10.10.

b) Find the probability that at least 10 customers will arrive at this post office between 10.00 and 10.20.

The time period between 10.00 and 10.20 is split into four 5 minute intervals.

c) Determine the probability that in only two of these four 5 minute intervals, there will be customers arriving at this post office.

$\boxed{\text{FS}}$, $\boxed{0.5154}$, $\boxed{0.2834}$, $\boxed{0.082}$

a) RATE GIVEN AS 4 CUSTOMERS PER 10 MINUTES

$\boxed{\begin{array}{l} X = \text{CUSTOMERS PER 10 MINUTES} \\ X \sim Po(4) \end{array}}$

$P(3 < X < 8) = P(4 \leq X \leq 7) = P(X \leq 7) - P(X \leq 3)$

$= 0.948866... - 0.433470... = 0.5154$

b) ADJUST THE RATE

$\boxed{\begin{array}{l} Y = \text{CUSTOMERS PER 20 MINUTES} \\ Y \sim Po(8) \end{array}}$

$P(X \geq 10) = 1 - P(X \leq 9) = ... = 1 - 0.71662...$

$= 0.2834$

c) ADJUSTING THE RATE YET AGAIN

$\boxed{\begin{array}{l} W = \text{CUSTOMERS PER 5 MINUTES} \\ W \sim Po(2) \end{array}}$

$P(W = 0) = \dfrac{e^{-2} \times 2^0}{0!} = 0.135335...$

$\boxed{\begin{array}{l} V = \text{A FIVE MINUTE INTERVAL WITH CUSTOMERS} \\ V \sim B(4, 0.86466..) \end{array}}$ $\leftarrow 1 - 0.13533...$

$P(V = 2) = \binom{4}{2}(0.86466)^2(0.135335)^2 = 0.082161...$

$\simeq 0.082$

Question 13 (****)

During rush hour commuters arrive at a busy train station at the steady rate of 7 commuters every 30 seconds.

a) Calculate the probability that in a random 30 second interval there will be more than 5 but no more than 11 commuters arriving at this station.

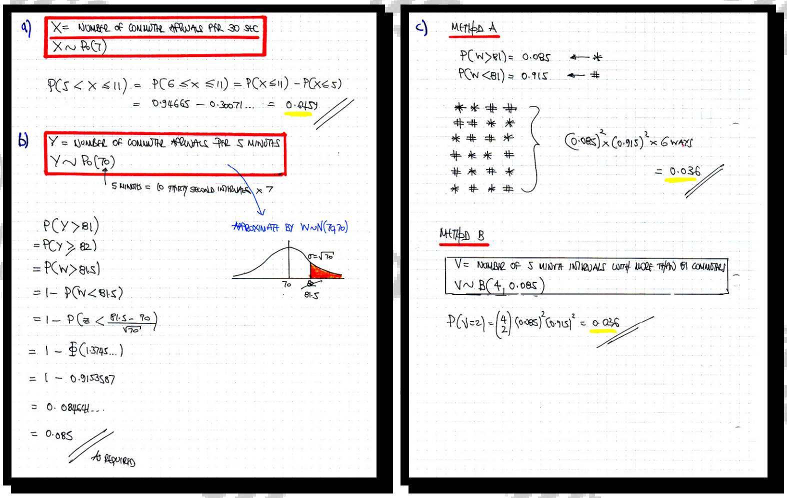
The next 5 minutes are monitored.

b) Show that the probability that more than 81 commuters will arrive at this station is approximately 0.085.

The next 20 minutes are divided into 4 intervals of 5 minutes.

c) Determine the probability that more than 81 commuters will arrive at this station in 2 of these intervals of 5 minutes.

$\boxed{}$, $\boxed{0.6460}$, $\boxed{0.036}$



Question 14 (****)

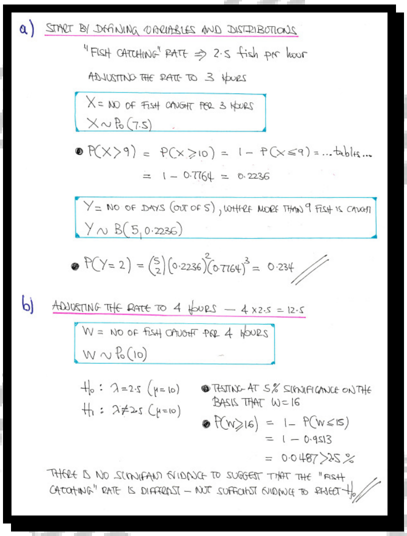
Since his retirement, Fred goes fishing Monday to Friday, for 3 hours on each of these 5 days. The number of fish he catches every hour follows a Poisson distribution with mean 2.5.

a) Find the probability that Fred catches more than 9 fish on exactly 2 of the days, in a given 5 day fishing week.

Fred buys a new type of bait and decides to test whether there is any difference to the rate at which he catches fish. He tries his new bait by going fishing on a Sunday and ends up catching 14 fish in 4 hours.

b) Carry out a significance test, at the 5% level, stating your hypotheses clearly.

$\boxed{}$, $\boxed{0.234}$, $\boxed{\text{not significant evidence, } 4.87\% > 2.5\%}$


a) START BY DEFINING VARIABLES AND DISTRIBUTIONS

"FISH CATCHING" RATE $\Rightarrow$ 2·5 fish per hour

ADJUSTING THE RATE TO 3 HOURS

X = NO OF FISH CAUGHT PER 3 HOURS
$X \sim Po(7.5)$

- $P(X>9) = P(X \geqslant 10) = 1 - P(X \leqslant 9) = \ldots \text{tables} \ldots$
$= 1 - 0.7764 = 0.2236$

Y = NO OF DAYS (OUT OF 5), WHERE MORE THAN 9 FISH IS CAUGHT
$Y \sim B(5, 0.2236)$

- $P(Y=2) = \binom{5}{2}(0.2236)^2(0.7764)^3 = 0.234$

b) ADJUSTING THE RATE TO 4 HOURS — $4 \times 2.5 = 12.5$

W = NO OF FISH CAUGHT PER 4 HOURS
$W \sim Po(10)$

$H_0: \lambda = 2.5 \ (\mu = 10)$
$H_1: \lambda \neq 2.5 \ (\mu = 10)$

- TESTING AT 5% SIGNIFICANCE ON THE BASIS THAT $W = 16$
- $P(W \geqslant 16) = 1 - P(W \leqslant 15)$
$= 1 - 0.9513$
$= 0.0487 > 2.5\%$

THERE IS NO SIGNIFICANT EVIDENCE TO SUGGEST THAT THE "FISH CATCHING" RATE IS DIFFERENT – NOT SUFFICIENT EVIDENCE TO REJECT H_0

Question 15 (****)

The number of disconnections experienced by a company server follows a Poisson distribution with a rate of one disconnection every four hours.

a) Find the smallest value of n, such that the probability that there is at least n disconnections in a 4 hour period is less than 0.01.

b) Determine the smallest value of hours h, such that the probability that there is **no** disconnections in h hours is less than 0.02.

c) Find the probability that in 3 consecutive 4 hour periods, there will be **only one** 4 hour period **without** any disconnections.

d) Calculate the probability that the number of disconnections in 3 consecutive 4 hour periods, will be equal to the expected number of disconnections in 3 consecutive 4 hour periods.

On a particular day there were 4 disconnections in a period of 8 hours.

e) Calculate the probability that there were more disconnections in the first 4 hours than in the last 4 hours.

$\boxed{}$, $\boxed{n=4}$, $\boxed{h=16}$, $\boxed{\approx 0.4410}$, $\boxed{\approx 0.2240}$, $\boxed{\frac{11}{32}}$

a) X = NUMBER OF DISCONNECTIONS PER 4 HOURS
$X \sim Po(1)$

$P(X \geqslant n) < 0.01$
$1 - P(X \leqslant n-1) < 0.01$
$-P(X \leqslant n-1) < -0.99$
$P(X \leqslant n-1) < 0.99$

LOOKING AT TABLES OF $Po(1)$
- $P(X \leqslant 3) = 0.9810$
- $P(X \leqslant 4) = 0.9963$

$n-1 = 3$
$n = 4$

b) Y = NUMBER OF DISCONNECTIONS PER h HOURS
$Y \sim Po(\frac{1}{4} \times h)$

$P(Y=0) < 0.02 \Rightarrow \frac{e^{-\frac{1}{4}h} \times (\frac{1}{4}h)^0}{0!} < 0.02$
$\Rightarrow e^{-\frac{1}{4}h} < \frac{1}{50}$
$\Rightarrow e^{\frac{1}{4}h} > 50$
$\Rightarrow \frac{1}{4}h > \ln(50)$
$\Rightarrow h > 4\ln 50 \approx 15.64$
$\Rightarrow h = 16$

c) USING $X \sim Po(1)$

- $P(X=0) = \frac{e^{-1} \times 1^0}{0!} = \frac{1}{e}$
- $P(X \geqslant 1) = 1 - \frac{1}{e}$

$\therefore$ REQUIRED PROBABILITY $= \frac{1}{e} \times (1-\frac{1}{e}) \times (1-\frac{1}{e}) \times 3$ WAYS
$= \frac{3}{e}(1 - \frac{2}{e} + \frac{1}{e^2})$
$\simeq 0.4410$

d) W = NUMBER OF DISCONNECTIONS PER 12 HOURS
$W \sim Po(3)$

- $E(W) = 3$
- $P(W=3) = \frac{e^{-3} \times 3^3}{3!} = \frac{9}{2}e^{-3} \simeq 0.2240$

e) V = NUMBER OF DISCONNECTIONS PER 8 HOURS
$V \sim Po(2)$

$P(V=4) = \frac{e^{-2} \times 2^4}{4!} = \frac{2}{3e^2}$

SO NOW WE HAVE

EITHER 4 & 0 $\Rightarrow P(X=4) \times P(X=0)$
OR 3 & 1 $\Rightarrow P(X=3) \times P(X=1)$

HENCE THE REQUIRED PROBABILITY IS

$$\frac{P(X=4)P(X=0) + P(X=3)P(X=1)}{P(V=4)}$$
$$= \frac{\frac{e^{-1} \times 1^4}{4!} \times \frac{e^{-1} \times 1^0}{0!} + \frac{e^{-1} \times 1^3}{3!} \times \frac{e^{-1} \times 1^1}{1!}}{\frac{2}{3e^2}}$$
$$= \frac{\frac{1}{16e} \times \frac{1}{e} + \frac{1}{6e} \times \frac{1}{e}}{\frac{2}{3e^2}} = \frac{\frac{1}{16e^2} + \frac{1}{6e^2}}{\frac{2}{3e^2}} = \frac{\frac{1}{16} + \frac{1}{6}}{\frac{2}{3}}$$
$$= \frac{3+8}{32} = \frac{11}{32}$$

Question 16 (****)

The number of customers X entering a shop follows a Poisson distribution with mean $0.2t$, where t represents a time interval of t minutes.

a) Find the probability that exactly 4 customers enter this shop between 2 p.m. and 2.20 p.m.

b) If $\mathrm{P}(X=4) \approx 0.14142$, use a suitable approach to find the value of t.

c) Use a **binomial approximation** with $n=80$, to estimate the probability that between 2.20 p.m. and 2.35 p.m, exactly 3 customers enter this shop.

d) Use a **direct** Poisson calculation to estimate the percentage error of the answer of part (**c**).

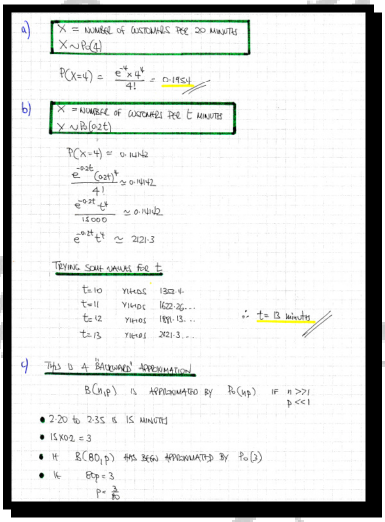
A new discrete random variable Y is defined as

$$Y = aX + b,$$

where a and b are positive constants.

e) If the mean of X is 4 and the mean and variance of Y are both 36, explain with suitable calculations why Y cannot be Poisson distributed.

$\boxed{}$, $\boxed{\approx 0.1954}$, $\boxed{t=13}$, $\boxed{\approx 0.2284}$, $\boxed{\approx 1.96\%}$



a) $\boxed{X = \text{NUMBER OF CUSTOMERS PER 20 MINUTES} \quad X \sim Po(4)}$

$P(X=4) = \frac{e^{-4} \times 4^4}{4!} = 0.1954$

b) $\boxed{X = \text{NUMBER OF CUSTOMERS PER } t \text{ MINUTES} \quad X \sim Po(0.2t)}$

$P(X=4) = 0.14142$

$\frac{e^{-0.2t}(0.2t)^4}{4!} \simeq 0.14142$

$\frac{e^{-0.2t}t^4}{15000} \simeq 0.14142$

$e^{-0.2t}t^4 \simeq 2121.3$

TRYING SOME VALUES FOR t

$t=10$ YIELDS 1353.4

$t=11$ YIELDS 1622.26...

$t=12$ YIELDS 1881.13...

$t=13$ YIELDS 2121.3...

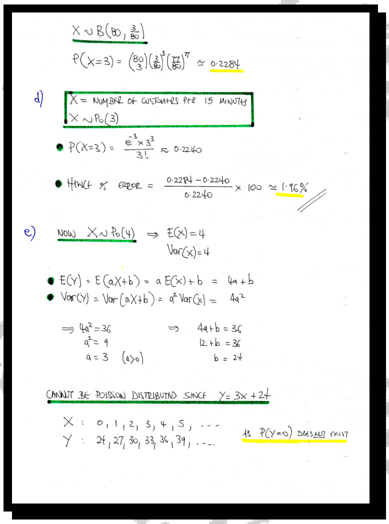
$\therefore t = 13$ minutes

c) THIS IS A "BACKWARD" APPROXIMATION

$B(n,p)$ IS APPROXIMATED BY $Po(np)$ IF $n \gg 1$, $p \ll 1$

- 2.20 to 2.35 IS 15 MINUTES
- $15 \times 0.2 = 3$
- IF $B(80,p)$ HAS BEEN APPROXIMATED BY $Po(3)$
- IF $80p = 3$

  $p = \frac{3}{80}$



$X \sim B\left(80, \frac{3}{80}\right)$

$P(X=3) = \binom{80}{3}\left(\frac{3}{80}\right)^3\left(\frac{77}{80}\right)^{77} \simeq 0.2284$

d) $\boxed{X = \text{NUMBER OF CUSTOMERS PER 15 MINUTES} \quad X \sim Po(3)}$

- $P(X=3) = \frac{e^{-3} \times 3^3}{3!} \approx 0.2240$
- HENCE % ERROR $= \frac{0.2284 - 0.2240}{0.2240} \times 100 \simeq 1.96\%$

e) NOW $X \sim Po(4) \Rightarrow E(X) = 4$, $Var(X) = 4$

- $E(Y) = E(aX+b) = aE(X) + b = 4a + b$
- $Var(Y) = Var(aX+b) = a^2 Var(X) = 4a^2$

$\Rightarrow 4a^2 = 36$, $a^2 = 9$, $a = 3$ $(a>0)$ $\quad \Rightarrow 4a + b = 36$, $12 + b = 36$, $b = 24$

CANNOT BE POISSON DISTRIBUTED SINCE $Y = 3X + 24$

X : 0, 1, 2, 3, 4, 5, ...

Y : 24, 27, 30, 33, 36, 39, ...

AS $P(Y=0)$ DOES NOT EXIST

Question 17 (****)

The discrete random variable X follows a Poisson distribution with mean 2, and another discrete random variable Y also follows a Poisson distribution with mean 3.

It is further given that X and Y are independent from one another.

a) Find the value of $\text{Var}(XY)$.

b) Determine $\text{P}(XY = 4)$

$\boxed{}$, $\boxed{\text{Var}(XY) = 36}$, $\boxed{\text{P}(XY = 4) \approx 0.1196}$

a) $X \sim Po(2)$ $Y \sim Po(3)$

WORK AS FOLLOWS

$E(X) = 2$ $E(Y) = 3$

$\text{Var}(X) = 2$ $\text{Var}(Y) = 3$

CALCULATE THE CORRESPONDING VALUES OF $E(X^2)$ & $E(Y^2)$

$\text{Var}(X) = E(X^2) - [E(X)]^2$ $\text{Var}(Y) = E(Y^2) - [E(Y)]^2$

$2 = E(X^2) - 2^2$ $3 = E(Y^2) - 3^2$

$E(X^2) = 6$ $E(Y^2) = 12$

NOW WE HAVE

$E(XY) = E(X)E(Y) = 2 \times 3 = 6$

$E(X^2Y^2) = E(X^2)E(Y^2) = 6 \times 12 = 72$

THUS THE VARIANCE CAN NOW BE FOUND

$\text{Var}(XY) = E((XY)^2) - [E(XY)]^2$

$\text{Var}(XY) = E(X^2Y^2) - [E(XY)]^2$

$\text{Var}(XY) = 72 - 6^2$

$\text{Var}(XY) = 72 - 36$

$\text{Var}(XY) = 36$

b) CALCULATING THE PROBABILITY

$$P(XY=4) = P(X=1)P(Y=4) + P(X=4)P(Y=1) + P(X=2)P(Y=2)$$

$$= \frac{e^{-2} \times 2^1}{1!} \times \frac{e^{-3} \times 3^4}{4!} + \frac{e^{-2} \times 2^4}{4!} \times \frac{e^{-3} \times 3^1}{1!} + \frac{e^{-2} \times 2^2}{2!} \times \frac{e^{-3} \times 3^2}{2!}$$

$$= e^{-5}\left[2 \times \frac{81}{24} + \frac{16 \times 3}{24} + \frac{4}{2} \times \frac{9}{2}\right]$$

$$= e^{-5}\left[\frac{162}{24} + \frac{48}{24} + 9\right]$$

$$= e^{-5} \times \frac{71}{4}$$

$$\simeq 0.1196$$

Question 18 (****)

The discrete random variables X and Y are independent from one another and are defined as

$$X \sim \mathrm{B}(16, 0.25) \quad \text{and} \quad Y \sim \mathrm{Po}(2).$$

a) Find the value of $\mathrm{Var}(XY)$.

b) Determine $\mathrm{P}(XY = 3)$

$\boxed{\mathrm{Var}(XY) = 50}$, $\boxed{\mathrm{P}(XY = 3) \approx 0.0659}$

a) WORK AS FOLLOWS

- $X \sim B(16, 0.25)$
- $E(X) = np = 16 \times 0.25 = 4$
- $\mathrm{Var}(X) = np(1-p) = 4 \times 0.75 = 3$

- $Y \sim Po(2)$
- $E(Y) = 2$
- $\mathrm{Var}(Y) = 2$

NOW WE HAVE

- $E(X^2) = \mathrm{Var}(X) + [E(X)]^2$
- $E(X^2) = 3 + 4^2 = 19$

- $E(Y^2) = \mathrm{Var}(Y) + [E(Y)]^2$
- $E(Y^2) = 2 + 2^2 = 6$

$E(XY) = E(X)E(Y) = 4 \times 2 = 8$

$E((XY)^2) = E(X^2Y^2) = E(X^2)E(Y^2) = 19 \times 6 = 114$

FINALLY WE HAVE

$\Rightarrow \mathrm{Var}(XY) = E((XY)^2) - [E(XY)]^2$

$\Rightarrow \mathrm{Var}(XY) = 114 - 8^2$

$\Rightarrow \mathrm{Var}(XY) = 50$

b) $P(XY=3) = P(X=1)P(Y=3) + P(X=3)P(Y=1)$

$= \binom{16}{1}(0.25)^1(0.75^{15}) \times \frac{e^{-2} \times 2^3}{3!} + \binom{16}{3}(0.25)^3(0.75)^{13} \times \frac{e^{-2} \times 2^1}{1!}$

$= \frac{16}{3}e^{-2}(0.75)^{15} + \frac{35}{2}e^{-2}(0.75)^{13}$

$= 0.009646\ldots + 0.0562659\ldots$

$= 0.0659$

Question 19 (****+)

It is known that during the first hour of trading, customers arrive at a garden centre at the rate of 3 customers every 10 minutes.

a) State two conditions, so that a Poisson distribution could be used to model the number of arrivals in the garden centre.

b) State one reason as to why the Poisson model might not be suitable.

Using a suitable Poisson model, and ignoring the answer to part **(b)**, determine the probability that...

c) … exactly 4 customers arrive in the first 10 minutes of trading.

d) … exactly 8 customers arrive in the first 20 minutes of trading.

e) … exactly 4 customers arrive in the first 10 minutes of trading and a further 4 customers arrive in the next 10 minutes.

f) … the **first** customer will arrive 7 minutes after opening.

The first hour after opening is subdivided into 6 equal 10 minute intervals.

Calculate the probability that...

g) … exactly 4 customers arrive in **none** of these 10 minute intervals.

h) … there will be a **single** 10 minute interval, where **no** customers arrive to the garden centre.

The garden centre claims that 98% of its seeds will germinate. It was found that in a random sample of 125 seeds, 117 seeds germinated.

i) Use a suitable approximation, to test at the 1% level of significance, whether the garden centre overstates thr germination proportion of its seeds. You must state your hypotheses clearly in this part.

$\boxed{}$, $\boxed{0.1680}$, $\boxed{0.1033}$, $\boxed{0.0282}$, $\boxed{0.1225}$, $\boxed{0.3316}$, $\boxed{0.2314}$, $\boxed{\text{significant evidence, } 0.42\% < 1\%}$

[solution overleaf]

a)
- CUSTOMERS ARRIVE INDEPENDENTLY OF ONE ANOTHER
- CUSTOMERS ARRIVE AT A UNIFORM CONSTANT RATE PER UNIT TIME

b) CUSTOMERS ARE UNLIKELY TO ARRIVE "SINGLY", AS GARDEN CENTRES TEND TO BE VISITED BY COUPLES OR FAMILIES

c) X = NO OF CUSTOMER ARRIVALS PER 10 MINUTES

$X \sim Po(3)$

$$P(X=4) = \frac{e^{-3} \times 3^4}{4!} = 0.1680$$

d) Y = NO OF CUSTOMER ARRIVALS PER 20 MINUTES

$Y \sim Po(6)$

$$P(Y=8) = \frac{e^{-6} \times 6^8}{8!} = 0.1033$$

e) USING THE RESULT OF PART (c)

$$P(X=4) \times P(X=4) = \left(\frac{e^{-3} \times 3^4}{4!}\right)^2 = 0.0282$$

f) EITHER

V = NO OF CUSTOMERS PER MINUTE

$V \sim Po(0.3)$

$$P(V=0) = \frac{e^{-0.3} \times 0.3^0}{0!} = e^{-0.3}$$

$\therefore$ REQUIRED PROBABILITY

$$(e^{-0.3})^7 = e^{-2.1} \approx 0.122$$

OR

W = NO OF CUSTOMERS PER 7 MINUTES

$W \sim Po(2.1)$

$$P(W=0) = \frac{e^{-2.1} \times 2.1^0}{0!} \approx 0.1225$$

g) MODEL AS FOLLOWS, USING VARIABLE X, AS PREVIOUSLY DEFINED

$$P(X=4) = \frac{e^{-3} \times 3^4}{4!}$$

$$\text{REQUIRED PROBABILITY} = \left(1 - \frac{e^{-3} \times 3^4}{4!}\right)^6 = 0.3316$$

h) USING X AGAIN

$$P(X=0) = \frac{e^{-3} \times 0^3}{0!} = e^{-3}$$

REQUIRED PROBABILITY IS $(1-e^{-3})^5 \times e^{-3} \times 6$ WAYS

OR DEFINE A BINOMIAL

U = NO OF INTERVALS WITHOUT CUSTOMERS

$U \sim B(6, e^{-3})$

$$P(U=1) = \binom{6}{1} \times (e^{-3})^1 \times (1-e^{-3})^5 \approx 0.2314$$

i) G = NO OF GERMINATING SEEDS

G' = NO OF NON GERMINATING SEEDS

$G \sim B(125, 0.98)$ AND $G' \sim B(\overset{n}{125}, \overset{p}{0.02})$

$\downarrow$ APPROX BY POISSON

$G' \sim Po(\overset{np}{2.5})$

- TO TEST THE HYPOTHESIS WITH G $\quad P(G \leqslant 117)$ (117 GERMINATED)
- TO TEST THE HYPOTHESIS WITH G' $\quad P(G' \geqslant 8)$ (8 DIDN'T GERMINATE)

$$\left.\begin{array}{l} H_0: p = 0.02 \\ H_1: p > 0.02 \end{array}\right\}$$ WHERE p IS THE PROPORTION OF NON GERMINATING SEEDS IN THE ENTIRE POPULATION

- $P(G' \geqslant 8) = 1 - P(G' \leqslant 7)$

$$= 1 - 0.9958$$
$$= 0.0042$$
$$= 0.42\%$$
$$< 1\%$$

THERE IS SIGNIFICANT EVIDENCE THAT THE GARDEN CENTRE OVERSTATES THE GERMINATION PROPORTION OF ITS SEEDS – SUFFICIENT EVIDENCE TO REJECT H_0

NOTE IT IS FINE TO PUT HYPOTHESES AS

$$\left.\begin{array}{l} p = 0.98 \\ p < 0.98 \end{array}\right\}$$ WHERE p IS THE PROPORTION OF GERMINATING SEEDS

& THE REST IDENTICAL AS ABOVE

Question 20 (****+)

The number of customer complaints received by a company is thought to follow a Poisson distribution with a mean of 1.8 complaints per day.

In a randomly chosen 5 day week, the probability that there will be at least n customer complaints is 12.42%.

a) Determine the value of n.

b) Use a distributional approximation to find the probability that in a period of 20 working days there fewer than 30 customer complaints.

A week of 5 working days is called a "bad week" if at least n customer complaints are received, where n is the value found in part **(a)**.

c) Use a distributional approximation to find the probability that in 40 randomly chosen weeks more than 2 are "bad".

$\boxed{n=13}$, $\boxed{\approx 0.140}$, $\boxed{\approx 0.889}$

(a) ADJUST RATE $5 \times 1.8 = 9$

X = COMPLAINTS PER 5 DAYS
$X \sim Po(9)$

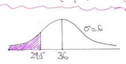
$P(X \geqslant n) = 12.42\%$
$\Rightarrow 1 - P(X \leqslant n-1) = 0.1242$
$\Rightarrow -P(X \leqslant n-1) = -0.8758$
$\Rightarrow P(X \leqslant n-1) = 0.8758$
LOOKING AT $Po(9)$ TABLES
$\Rightarrow n-1 = 12$
$\Rightarrow n = 13$

(b) ADJUST RATE $20 \times 1.8 = 36$

Y = COMPLAINTS PER 20 DAYS
$Y \sim Po(36)$

APPROXIMATE BY NORMAL
$W \sim N(36, 36)$

$P(Y<30) = P(Y \leqslant 29) = P(W<29.5)$
$= 1 - P(W>29.5)$
$= 1 - P\left(z > \frac{29.5-36}{6}\right)$
$= 1 - \Phi(-1.0833)$
$= 1 - 0.8599$
$= 0.140$



(c) REMODEL BY BINOMIAL

V = NO OF "BAD" WEEKS
$V \sim B(40, 0.1242)$

$P(V>2) = P(V \geqslant 3)$
$= 1 - P(V \leqslant 2)$
$= 1 - P(V=0,1,2)$
$$= 1 - \left\{ \binom{40}{0}(0.1242)^0(0.8758)^{40} + \binom{40}{1}(0.1242)^1(0.8758)^{39} + \binom{40}{2}(0.1242)^2(0.8758)^{38} \right\}$$
$= 1 - 0.1111$
$= 0.889$

Question 21 (****+)

The number of car caught speeding per day, by a fixed police camera, is thought to follow a Poisson distribution with mean 1.2.

a) Find the probability that on a given 7 day week exactly 8 cars will be caught by this police camera.

A car has just been caught by the police camera.

b) Determine the probability that the period that elapses before another car gets caught is less than 48 hours.

In fact 5 cars were caught in the next 48 hours.

c) Calculate the probability that 2 cars were caught in the first 24 hours and 3 cars were caught in the next 24 hours.

$\boxed{}$, $\boxed{0.1382}$, $\boxed{0.9093}$, $\boxed{\frac{5}{16}}$

a) RATE GIVEN AS 1.2 CARS CAUGHT PER DAY

ADJUST THE RATE TO 7 DAYS — $7 \times 1.2 = 8.4$

$\boxed{\begin{array}{l} X = \text{NO OF CARS CAUGHT PER WEEK} \\ X \sim Po(8.4) \end{array}}$

$$P(X=8) = \frac{e^{-8.4} \times 8.4^8}{8!} = 0.1382$$

b) ADJUST THE RATE TO 2 DAYS

$\boxed{\begin{array}{l} Y = \text{NO OF CARS PER TWO DAYS} \\ Y \sim Po(2.4) \end{array}}$

$$P(Y \geq 1) = 1 - P(Y=0) = 1 - \frac{e^{-2.4} \times 2.4^0}{0!} = 0.9093$$

c) $\boxed{\begin{array}{l} W = \text{NO OF CARS CAUGHT PER DAY} \\ W \sim Po(1.2) \end{array}}$

- $P(W=2) = \frac{e^{-1.2} \times 1.2^2}{2!}$
- $P(W=3) = \frac{e^{-1.2} \times 1.2^3}{3!}$
- $P(Y=5) = \frac{e^{-2.4} \times 2.4^5}{5!}$

$$\therefore \frac{P(W=2) \cap P(W=3)}{P(Y=5)} = \frac{\frac{e^{-1.2} \times 1.2^2}{2!} \times \frac{e^{-1.2} \times 1.2^3}{3!}}{\frac{e^{-2.4} \times 2.4^5}{5!}} = \frac{\cancel{e^{-2.4}} \times \frac{1.2^5}{12}}{\cancel{e^{-2.4}} \times \frac{2.4^5}{120}} = \frac{1.2^5 \times 120}{2.4^5 \times 12} = 10 \times \frac{1.2^5}{2.4^5} = \frac{5}{16}$$

$$\simeq 0.3125$$

Question 22 (*****)

The number of supernovae observed in a certain part of the sky in a 10 period can be modelled by a Poisson distribution with mean 1.

The probability that exactly 6 supernovae are observed in this part of the sky in a period of x years is 0.1128, correct to 4 decimal places.

Determine the possible values of x, correct to the nearest integer.

$\boxed{}$, $\boxed{x = 42,\ 83}$

- DEFINE VARIABLES & DISTRIBUTION

1 SUPERNOVA EVERY 10 YEARS

$\frac{1}{10}$ SUPERNOVA EVERY YEAR

$\frac{x}{10}$ SUPERNOVA EVERY x YEAR

$\boxed{\begin{array}{l} X = \text{NO OF SUPERNOVAE PER } x \text{ YEARS} \\ X \sim Po\left(\frac{x}{10}\right) \end{array}}$

- FORMING AN EQUATION

$$P(X=6) = 0.1128$$

$$\frac{e^{-\frac{x}{10}} \times \left(\frac{x}{10}\right)^6}{6!} = 0.1128$$

$$e^{-\frac{1}{10}x} \times \left(\frac{1}{10}\right)^6 \times x^6 = 0.1128 \times 6!$$

$$x^6 e^{-\frac{1}{10}x} = 0.1128 \times 720 \times 10^6$$

$$x^6 e^{-\frac{1}{10}x} = 81,216,000$$

- TRIAL AND IMPROVEMENT FOR DIFFERENT VALUES OF x

$x = 50 \quad 50^6 \times e^{-5} = 105280421$

$x = 60 \quad 60^6 \times e^{-6} = 115\,645\,661$ ← GOING UP IMPLIES THAT DEFINITELY IT WILL "COME DOWN" AS THE NEGATIVE EXPONENTIAL WILL EVENTUALLY TAKE IT TO 0

$x = 70 \quad 70^6 \times e^{-7} = 107\,282\,001$

$x = 80 \quad 80^6 \times e^{-8} = 87\,939\,515$

$x = 81 \quad 81^6 \times e^{-8.1} = 85\,728\,418$

$x = 82 \quad 82^6 \times e^{-8.2} = 83\,496\,517$

$x = 83 \quad 83^6 \times e^{-8.3} = 81\,250\,184$

$\therefore \underline{\underline{x = 83}}$

- ALSO LOOKING FOR LOW VALUES (BELOW 50)

$x = 40 \quad 40^6 \times e^{-4} = 75\,020\,857$

$x = 41 \quad 41^6 \times e^{-4.1} = 78\,721\,935$

$x = 42 \quad 42^6 \times e^{-4.2} = 82\,311\,197$

$\therefore \underline{\underline{x = 42}}$