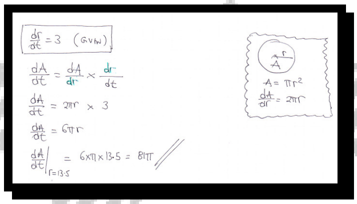
RELATED RATES OF CHANGE

Question 1 (**)

The radius, r cm, of a circle is increasing at the constant rate of $3\ \text{cm}\,\text{s}^{-1}$.

Find the rate at which the area of the circle is increasing when its radius is 13.5 cm.

$\boxed{}$, $\boxed{81\pi \approx 254\ \text{cm}^2\,\text{s}^{-1}}$

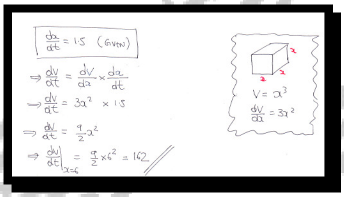


Question 2 (**)

The side length, x cm, of a cube is increasing at the constant rate of $1.5\ \text{cm}\,\text{s}^{-1}$.

Find the rate at which the volume of the cube is increasing when its side is 6 cm.

$\boxed{}$, $\boxed{162\ \text{cm}^3\,\text{s}^{-1}}$



Question 3 (**)

The volume, V cm^3, of a sphere is given by

$$V = \frac{4}{3}\pi r^3,$$

where r is its radius.

The radius of a sphere is increasing at the constant rate of 2.5 cm s^{-1}.

Find the rate at which the volume of the sphere is increasing when its radius is 8 cm.

$\boxed{}$, $\boxed{640\pi \approx 2011 \text{ cm}^3 \text{ s}^{-1}}$

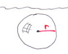
$\boxed{\frac{dr}{dt} = 2.5 \text{ (GIVEN)}}$

$\Rightarrow \frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$

$\Rightarrow \frac{dV}{dt} = 4\pi r^2 \times 2.5$

$\Rightarrow \frac{dV}{dt} = 10\pi r^2$

$\Rightarrow \left.\frac{dV}{dt}\right|_{r=8} = 640\pi \approx 2011$



$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 4\pi r^2$

Question 4 (**)

The surface area, S cm^2, of a sphere is increasing at the constant rate of $512 \text{ cm}^2 \text{ s}^{-1}$.

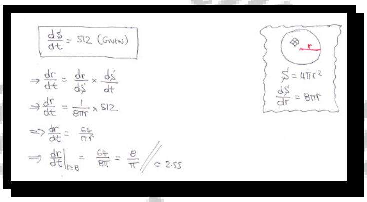
The surface area of a sphere is given by

$$S = 4\pi r^2,$$

where r cm is its radius.

Find the rate at which the radius r of the sphere is increasing, when the sphere's radius has reached 8 cm.

$\boxed{\quad}$, $\boxed{\dfrac{8}{\pi} \approx 2.55 \text{ cm s}^{-1}}$



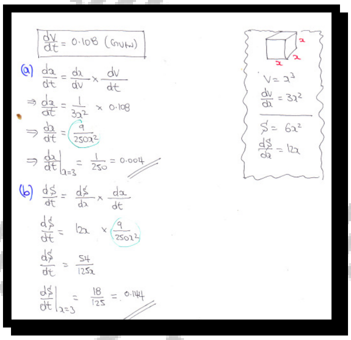
Question 5 (**+)

The volume, V cm^3, of a metallic cube of side length x cm, is increasing at the constant rate of $0.108 \text{ cm}^3\text{s}^{-1}$.

a) Determine the rate at which the side of the cube is increasing when the side length reaches 3 cm.

b) Find the rate at which the surface area of the cube, A cm^2, is increasing when the side length reaches 3 cm.

$\boxed{\frac{1}{250} = 0.004 \text{ cm s}^{-1}}$, $\boxed{\frac{18}{125} = 0.144 \text{ cm}^2\text{s}^{-1}}$

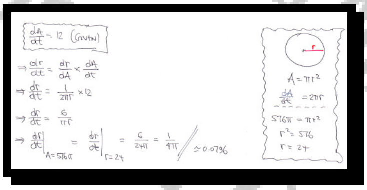


Question 6 (***)

The area, A cm^2, of a circle is increasing at the constant rate of $12 \text{ cm}^2\text{s}^{-1}$.

Find the rate at which the radius, r cm, of the circle is increasing, when the circle's area has reached 576π cm^2.

$\boxed{\frac{1}{4\pi} \approx 0.0796 \text{ cm s}^{-1}}$



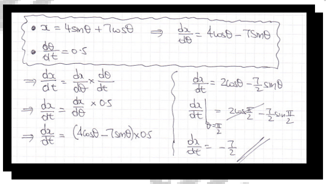
Question 7 (***)

$$x = 4\sin\theta + 7\cos\theta .$$

The value of θ is increasing at the constant rate of 0.5, in suitable units.

Find the rate at which x is changing, when $\theta = \frac{\pi}{2}$.

$\boxed{}$, $\boxed{-\frac{7}{2}}$



- $x = 4\sin\theta + 7\cos\theta \Rightarrow \frac{dx}{d\theta} = 4\cos\theta - 7\sin\theta$
- $\frac{d\theta}{dt} = 0.5$

$\Rightarrow \frac{dx}{dt} = \frac{dx}{d\theta} \times \frac{d\theta}{dt}$

$\Rightarrow \frac{dx}{dt} = \frac{dx}{d\theta} \times 0.5$

$\Rightarrow \frac{dx}{dt} = (4\cos\theta - 7\sin\theta) \times 0.5$

$\frac{dx}{dt} = 2\cos\theta - \frac{7}{2}\sin\theta$

$\left.\frac{dx}{dt}\right|_{\theta=\frac{\pi}{2}} = \cancel{2\cos\frac{\pi}{2}} - \frac{7}{2}\sin\frac{\pi}{2}$

$\frac{dx}{dt} = -\frac{7}{2}$

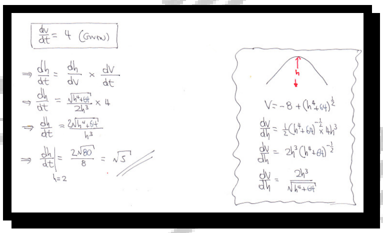
Question 8 (***)

Fine sand is dropping on a horizontal floor at the constant rate of $4 \text{ cm}^3\text{s}^{-1}$ and forms a pile whose volume, $V \text{ cm}^3$, and height, h cm, are connected by the formula

$$V = -8 + \sqrt{h^4 + 64} .$$

Find the rate at which the height of the pile is increasing, when the height of the pile has reached 2 cm.

$\boxed{}$, $\boxed{\sqrt{5} \approx 2.24 \text{ cm s}^{-1}}$



$\boxed{\frac{dV}{dt} = 4 \text{ (Given)}}$

$\Rightarrow \frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$

$\Rightarrow \frac{dh}{dt} = \frac{\sqrt{h^4+64}}{2h^3} \times 4$

$\Rightarrow \frac{dh}{dt} = \frac{2\sqrt{h^4+64}}{h^3}$

$\Rightarrow \left.\frac{dh}{dt}\right|_{h=2} = \frac{2\sqrt{80}}{8} = \sqrt{5}$

$V = -8 + (h^4+64)^{\frac{1}{2}}$

$\frac{dV}{dh} = \frac{1}{2}(h^4+64)^{-\frac{1}{2}} \times 4h^3$

$\frac{dV}{dh} = 2h^3(h^4+64)^{-\frac{1}{2}}$

$\frac{dV}{dh} = \frac{2h^3}{\sqrt{h^4+64}}$

Question 9 (***)

An oil spillage on the surface of the sea remains circular at all times.

The radius of the spillage, r km, is increasing at the constant rate of 0.5 km h^{-1}.

a) Find the rate at which the area of the spillage, $A \text{ km}^2$, is increasing, when the circle's radius has reached 10 km.

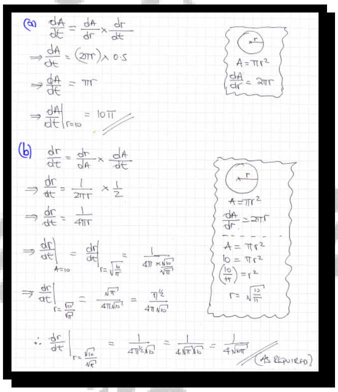
A different oil spillage on the surface of the sea also remains circular at all times.

The area of this spillage, $A \text{ km}^2$, is increasing at the rate of $0.5 \text{ km}^2 \text{h}^{-1}$.

b) Show that when the area of the spillage has reached 10 km^2, the rate at which the radius r of the spillage is increasing is

$$\frac{1}{4\sqrt{10\pi}} \text{ km h}^{-1}.$$

$$\boxed{10\pi \approx 31.4 \text{ km}^2 \text{h}^{-1}}$$


(a) $\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$

$\Rightarrow \frac{dA}{dt} = (2\pi r) \times 0.5$

$\Rightarrow \frac{dA}{dt} = \pi r$

$\Rightarrow \left.\frac{dA}{dt}\right|_{r=10} = 10\pi$

$A = \pi r^2$

$\frac{dA}{dr} = 2\pi r$

(b) $\frac{dr}{dt} = \frac{dr}{dA} \times \frac{dA}{dt}$

$\Rightarrow \frac{dr}{dt} = \frac{1}{2\pi r} \times \frac{1}{2}$

$\Rightarrow \frac{dr}{dt} = \frac{1}{4\pi r}$

$\Rightarrow \left.\frac{dr}{dt}\right|_{A=10} = \left.\frac{dr}{dt}\right|_{r=\sqrt{\frac{10}{\pi}}} = \frac{1}{4\pi \times \frac{\sqrt{10}}{\sqrt{\pi}}}$

$\Rightarrow \left.\frac{dr}{dt}\right|_{r=\frac{\sqrt{10}}{\sqrt{\pi}}} = \frac{\sqrt{\pi}}{4\pi\sqrt{10}} = \frac{\pi^{\frac{1}{2}}}{4\pi\sqrt{10}}$

$\therefore \left.\frac{dr}{dt}\right|_{r=\frac{\sqrt{10}}{\sqrt{\pi}}} = \frac{1}{4\pi^{\frac{1}{2}}\sqrt{10}} = \frac{1}{4\sqrt{\pi}\sqrt{10}} = \frac{1}{4\sqrt{10\pi}}$ (AS REQUIRED)

$A = \pi r^2$

$\frac{dA}{dr} = 2\pi r$

$A = \pi r^2$

$10 = \pi r^2$

$\left(\frac{10}{\pi}\right) = r^2$

$r = \sqrt{\frac{10}{\pi}}$

Question 10 (***)

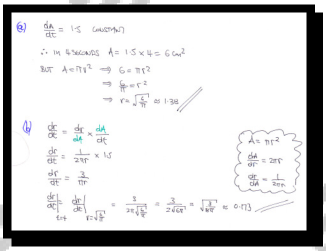
Liquid dye is poured onto a large flat cloth and forms a circular stain, the area of which grows at a steady rate of $1.5 \text{ cm}^2\text{s}^{-1}$.

Calculate, correct to three significant figures, …

a) … the radius, in cm, of the stain 4 seconds after it started forming.

b) … the rate, in cm s^{-1}, of increase of the radius of the stain after 4 seconds.

$\boxed{}$, $\boxed{r = \sqrt{\frac{6}{\pi}} \approx 1.38 \text{ cm}}$, $\boxed{\sqrt{\frac{3}{32\pi}} \approx 0.173 \text{ cm s}^{-1}}$



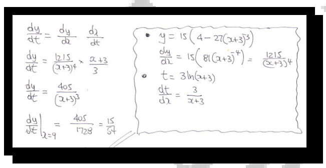
Question 11 (***)

The variables y, x and t are related by the equations

$$y = 15\left(4 - \frac{27}{(x+3)^3}\right) \quad \text{and} \quad \ln(x+3) = \frac{1}{3}t, \; x > -3.$$

Find the value of $\frac{dy}{dt}$, when $x = 9$.

$\boxed{}$, $\boxed{\frac{dy}{dt} = \frac{15}{64}}$



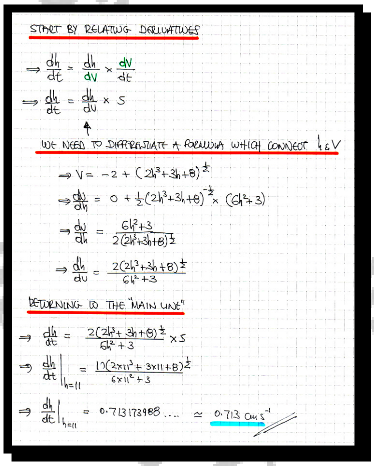
Question 12 (***)

Fine sand is dropping on a horizontal floor at the constant rate of $5 \text{ cm}^3\text{s}^{-1}$ and forms a pile whose volume, V cm^3, and height, h cm, are connected by the formula

$$V = -2 + \sqrt{2h^3 + 3h + 8}\,.$$

Find the rate at which the height of the pile is increasing, when the height of the pile has reached 11 cm.

$\boxed{}$, $\boxed{\approx 0.713 \text{ cm s}^{-1}}$


START BY RELATING DERIVATIVES

$$\Rightarrow \frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$$

$$\Rightarrow \frac{dh}{dt} = \frac{dh}{dV} \times 5$$

WE NEED TO DIFFERENTIATE A FORMULA WHICH CONNECT h & V

$$\Rightarrow V = -2 + (2h^3+3h+8)^{\frac{1}{2}}$$

$$\Rightarrow \frac{dV}{dh} = 0 + \tfrac{1}{2}(2h^3+3h+8)^{-\frac{1}{2}} \times (6h^2+3)$$

$$\Rightarrow \frac{dV}{dh} = \frac{6h^2+3}{2(2h^3+3h+8)^{\frac{1}{2}}}$$

$$\Rightarrow \frac{dh}{dV} = \frac{2(2h^3+3h+8)^{\frac{1}{2}}}{6h^2+3}$$

RETURNING TO THE "MAIN LINE"

$$\Rightarrow \frac{dh}{dt} = \frac{2(2h^3+3h+8)^{\frac{1}{2}}}{6h^2+3} \times 5$$

$$\Rightarrow \left.\frac{dh}{dt}\right|_{h=11} = \frac{10(2\times11^3+3\times11+8)^{\frac{1}{2}}}{6\times11^2+3}$$

$$\Rightarrow \left.\frac{dh}{dt}\right|_{h=11} = 0.713173988\ldots \simeq 0.713 \text{ cm s}^{-1}$$

Question 13 (***+)

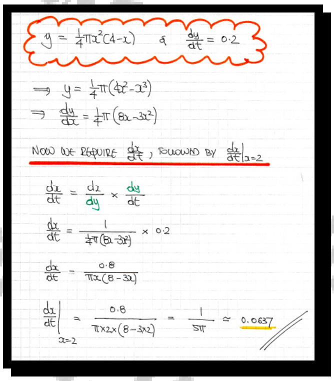
Two variables x and y are related by

$$y = \frac{1}{4}\pi x^2 (4 - x).$$

The variable y is changing with time t, at the constant rate of 0.2, in suitable units.

Find the rate at which x is changing with respect to t, when $x = 2$.

$$\boxed{\frac{1}{5\pi} \approx 0.0637}$$



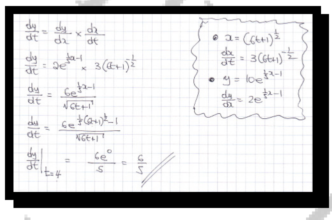
Question 14 (***+)

The variables y, x and t are related by the equations

$$y = 10e^{\frac{1}{5}x - 1} \quad \text{and} \quad x = \sqrt{6t + 1}, \quad t \geq 0.$$

Find the value of $\frac{dy}{dt}$, when $t = 4$.

$$\boxed{\left.\frac{dy}{dt}\right|_{t=4} = \frac{6}{5}}$$



Question 15 (****)

Liquid is pouring into a container at the constant rate of $30 \text{ cm}^3\text{s}^{-1}$.

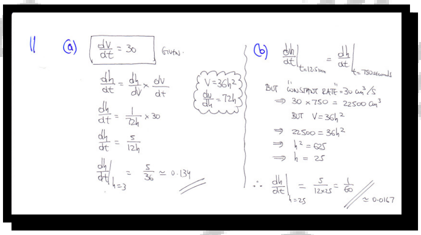
The container is initially empty and when the height of the liquid in the container is h cm the volume of the liquid, V cm^3, is given by

$$V = 36h^2 .$$

a) Find the rate at which the height of the liquid in the container is rising when the height of the liquid reaches 3 cm.

b) Determine the rate at which the height of the liquid in the container is rising 12.5 minutes after the liquid started pouring in.

$\boxed{}$, $\boxed{\frac{5}{36} = 0.139 \text{ cm s}^{-1}}$, $\boxed{\frac{1}{60} = 0.0167 \text{ cm s}^{-1}}$



Question 16 (****)

The radius R of a circle, in cm, at time t seconds is given by

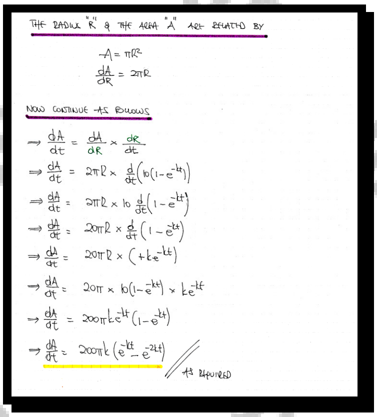
$$R = 10\left(1 - \mathrm{e}^{-kt}\right),$$

where k is a positive constant and $t > 0$.

Show that if A is the area of the circle, in cm^2, then

$$\frac{dA}{dt} = 200\pi k\left(\mathrm{e}^{-kt} - \mathrm{e}^{-2kt}\right).$$

proof



THE RADIUS "R" & THE AREA "A" ARE RELATED BY

$$A = \pi R^2$$
$$\frac{dA}{dR} = 2\pi R$$

NOW CONTINUE AS FOLLOWS

$$\Rightarrow \frac{dA}{dt} = \frac{dA}{dR} \times \frac{dR}{dt}$$
$$\Rightarrow \frac{dA}{dt} = 2\pi R \times \frac{d}{dt}\left(10\left(1 - e^{-kt}\right)\right)$$
$$\Rightarrow \frac{dA}{dt} = 2\pi R \times 10 \frac{d}{dt}\left(1 - e^{-kt}\right)$$
$$\Rightarrow \frac{dA}{dt} = 20\pi R \times \frac{d}{dt}\left(1 - e^{-kt}\right)$$
$$\Rightarrow \frac{dA}{dt} = 20\pi R \times \left(+ke^{-kt}\right)$$
$$\Rightarrow \frac{dA}{dt} = 20\pi \times 10\left(1 - e^{-kt}\right) \times ke^{-kt}$$
$$\Rightarrow \frac{dA}{dt} = 200\pi k e^{-kt}\left(1 - e^{-kt}\right)$$
$$\Rightarrow \frac{dA}{dt} = 200\pi k\left(e^{-kt} - e^{-2kt}\right)$$

// AS REQUIRED

Question 17 (****)

The volume of water, V cm^3, in a container is given by the formula

$$V = \sqrt{3x^2 + 2x^3}\ ,$$

where x is the depth of the water in cm.

a) Find the value of $\frac{dV}{dx}$ when $x = 11$.

It is further given that the volume of the water in the container is increasing at the constant rate of 14.4 cm^{3}s^{-1}

b) Determine the rate at which the depth of the water in the container is increasing when the depth has reached 11 cm.

$\boxed{}$, $\boxed{7.2 \text{ cm}^3\,\text{s}^{-1}}$, $\boxed{2 \text{ cm}\,\text{s}^{-1}}$

(a) $V = (3x^2 + 2x^3)^{\frac{1}{2}}$

$\frac{dV}{dx} = \frac{1}{2}(3x^2+2x^3)^{-\frac{1}{2}}(6x+6x^2) = 3x(1+x)(3x^2+2x^3)^{-\frac{1}{2}}$

$\frac{dV}{dx} = \frac{3x(1+x)}{\sqrt{3x^2+2x^3}}$

$\left.\frac{dV}{dx}\right|_{x=11} = \frac{33 \times 12}{\sqrt{3025}} = \frac{36}{5} = 7.2$

(b) $\frac{dx}{dt} = \frac{dx}{dV} \times \frac{dV}{dt}$

$\frac{dx}{dt} = \frac{dx}{dV} \times 14.4$

$\left.\frac{dx}{dt}\right|_{x=11} = \left.\frac{dx}{dV}\right|_{x=11} \times 14.4$

$\left.\frac{dx}{dt}\right|_{x=11} = \frac{1}{7.2} \times 14.4 = 2 \text{ cms}^{-1}$

Question 18 (****)

Oil leaking from a damaged tanker is forming a circular oil spillage on the surface of the sea, whose area is increasing at the constant rate of $360 \text{ m}^2 \text{ s}^{-1}$.

We may assume that the spillage is of negligible thickness.

a) Find the rate at which the radius of the oil spillage is increasing when the radius of the spillage reaches 100 m.

b) Determine the rate at which the radius of the oil spillage is increasing 1 minute after it started forming.

$\boxed{\text{MP2-A}}$, $\boxed{\frac{9}{5\pi} \approx 0.573 \text{ ms}^{-1}}$, $\boxed{\sqrt{\frac{3}{2\pi}} \approx 0.691 \text{ ms}^{-1}}$

a) $\frac{dA}{dt} = +360$ (GIVEN)

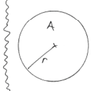
$\Rightarrow \frac{dr}{dt} = \frac{dr}{dA} \times \frac{dA}{dt}$

$\Rightarrow \frac{dr}{dt} = \frac{1}{2\pi r} \times 360$

$\Rightarrow \frac{dr}{dt} = \frac{180}{\pi r}$

$\Rightarrow \left.\frac{dr}{dt}\right|_{r=100} = \frac{180}{100\pi} = \frac{9}{5\pi}$

$\simeq 0.573 \text{ ms}^{-1}$



$A = \pi r^2$

$\frac{dA}{dr} = 2\pi r$

$\frac{dr}{dA} = \frac{1}{2\pi r}$

b) "CONSTANT RATE" OF 360 m^2 PER SECOND

$\Rightarrow$ $t=0$ $A=0$

$t=1$ $A=360$

$t=2$ $A=360\times 2$

$\vdots$

$t=60$ $A=360\times 60 = 21600$

USING $A = \pi r^2$

$\Rightarrow \pi r^2 = 21600$

$\Rightarrow r = 82.918\ldots$

FINALLY

$\left.\frac{dr}{dt}\right|_{t=1\text{ min}} = \left.\frac{dr}{dt}\right|_{t=60\text{ s}} = \left.\frac{dr}{dt}\right|_{r=82.918..} = \frac{180}{\pi \times 82.918..} \approx 0.691 \text{ ms}^{-1}$

Question 19 (****)

A bubble is formed and its volume is increasing at the constant rate of $300 \text{ cm}^3\text{s}^{-1}$.

The shape of the bubble remains spherical at all times.

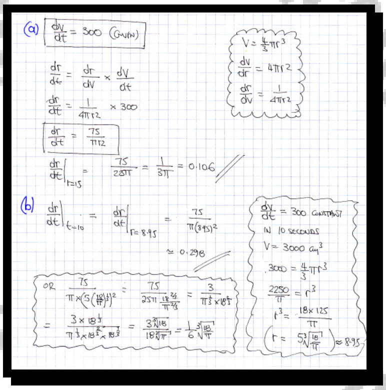
Find the rate at which the radius of the bubble is increasing …

a) … when the radius of the bubble reaches 15 cm.

b) … ten seconds after the bubble was first formed.

$\left[\text{volume of a sphere of radius } r \text{ is given by } \frac{4}{3}\pi r^3 \right]$

$\boxed{}$, $\boxed{\frac{1}{3\pi} \approx 0.106 \text{ cm s}^{-1}}$, $\boxed{\sqrt[3]{\frac{1}{12\pi}} \approx 0.298 \text{ cm s}^{-1}}$



Question 20 (****)

The shape of a bubble remains spherical at all times.

A bubble is formed and its radius is increasing at the constant rate of 0.2 cm s^{-1}.

a) Find the rate at which the volume of the bubble is increasing when the radius of the bubble reaches 8 cm.

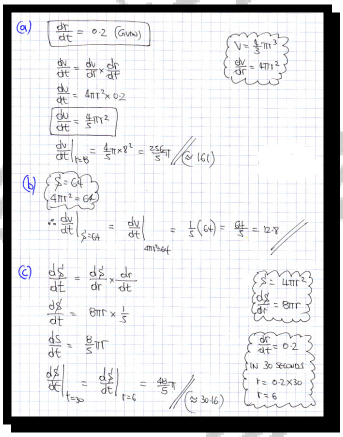
b) Determine the rate at which the volume of the bubble is increasing when the surface area of the bubble reaches 64 cm^2.

c) Calculate the rate at which the surface area of the bubble is increasing 30 seconds after the bubble was first formed.

$\left[\text{surface area of a sphere of radius } r \text{ is given by } 4\pi r^2\right]$

$\left[\text{volume of a sphere of radius } r \text{ is given by } \frac{4}{3}\pi r^3\right]$

$\boxed{\frac{256\pi}{5} \approx 161 \text{ cm}^3 \text{ s}^{-1}}$, $\boxed{\frac{64}{5} = 12.8 \text{ cm}^3 \text{ s}^{-1}}$, $\boxed{\frac{48\pi}{5} \approx 30.16 \text{ cm}^2 \text{ s}^{-1}}$


(a) $\boxed{\frac{dr}{dt} = 0.2 \text{ (GIVEN)}}$

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 4\pi r^2$

$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$

$\frac{dV}{dt} = 4\pi r^2 \times 0.2$

$\boxed{\frac{dV}{dt} = \frac{4}{5}\pi r^2}$

$\left.\frac{dV}{dt}\right|_{r=8} = \frac{4}{5}\pi \times 8^2 = \frac{256}{5}\pi \quad (\approx 161)$

(b) $S = 64$

$4\pi r^2 = 64$

$\therefore \left.\frac{dV}{dt}\right|_{S=64} = \left.\frac{dV}{dt}\right|_{4\pi r^2=64} = \frac{1}{5}(64) = \frac{64}{5} = 12.8$

(c) $\frac{dS}{dt} = \frac{dS}{dr} \times \frac{dr}{dt}$

$S = 4\pi r^2$

$\frac{dS}{dr} = 8\pi r$

$\frac{dS}{dt} = 8\pi r \times \frac{1}{5}$

$\frac{dS}{dt} = \frac{8}{5}\pi r$

$\frac{dr}{dt} = 0.2$

IN 30 SECONDS

$r = 0.2 \times 30$

$r = 6$

$\left.\frac{dS}{dt}\right|_{t=30} = \left.\frac{dS}{dt}\right|_{r=6} = \frac{48}{5}\pi \quad (\approx 30.16)$

Question 21 (****)

A bubble is formed and its volume is increasing at the constant rate of $20 \text{ cm}^3\text{s}^{-1}$.

The shape of the bubble remains spherical at all times.

Find the rate at which the radius of the bubble is increasing …

a) … when the radius of the bubble reaches 5 cm.

b) ... when the volume of the bubble reaches 300 cm^3.

c) … ten seconds after the bubble was first formed.

$\left[\text{volume of a sphere of radius } r \text{ is given by } \frac{4}{3}\pi r^3\right]$

$\boxed{}$, $\boxed{\frac{1}{5\pi} \approx 0.0637 \text{ cm s}^{-1}}$, $\boxed{\sqrt[3]{\frac{1}{405\pi}} \approx 0.0923 \text{ cm s}^{-1}}$, $\boxed{\sqrt[3]{\frac{1}{180\pi}} \approx 0.121 \text{ cm s}^{-1}}$

a) START BY RELATING DERIVATIVES

$\Rightarrow \frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$

$\Rightarrow \frac{dr}{dt} = \frac{1}{4\pi r^2} \times 20$ ← GIVEN

$\Rightarrow \boxed{\frac{dr}{dt} = \frac{5}{\pi r^2}}$

VOLUME OF A SPHERE

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 4\pi r^2$

$\frac{dr}{dV} = \frac{1}{4\pi r^2}$

$\Rightarrow \left.\frac{dr}{dt}\right|_{r=5} = \frac{5}{\pi \times 5^2} = \frac{1}{5\pi} \approx 0.0637 \text{ cm s}^{-1}$

b) WE NEED TO FIND THE RADIUS OF THE BUBBLE WHEN ITS VOLUME REACHES 300 cm^3

$\Rightarrow V = \frac{4}{3}\pi r^3$

$\Rightarrow 300 = \frac{4}{3}\pi r^3$

$\Rightarrow \frac{225}{\pi} = r^3$

$\Rightarrow r = \sqrt[3]{\frac{225}{\pi}} \approx 4.1528\ldots$

$\Rightarrow \left.\frac{dr}{dt}\right|_{r=4.1528\ldots} = \frac{5}{\pi \times (4.1528\ldots)^2} \approx 0.0923 \text{ cm s}^{-1}$

c) WE NEED TO CHANGE THE "TIME" INTO "VOLUME" FIRST

"... CONSTANT RATE OF 20 cm^3 PER SECOND ..."

$\therefore$ IN 10 SECONDS THE VOLUME WILL BE $20 \times 10 = 200 \text{ cm}^3$

PROCEED AS IN PART (b)

$\Rightarrow V = \frac{4}{3}\pi r^3$

$\Rightarrow 200 = \frac{4}{3}\pi r^3$

$\Rightarrow \frac{150}{\pi} = r^3$

$\Rightarrow r = \sqrt[3]{\frac{150}{\pi}} \approx 3.6278\ldots$

$\Rightarrow \left.\frac{dr}{dt}\right|_{r=3.6278\ldots} = \frac{5}{\pi(3.6278\ldots)^2} \approx 0.121 \text{ cm s}^{-1}$

Question 22 **(****)**

A cube has side length x cm, surface area A cm^2 and volume V cm^3.

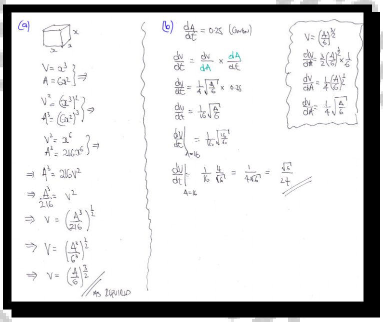
a) Show clearly that

$$V = \left(\frac{A}{6}\right)^{\frac{3}{2}}.$$

The surface area of the cube is increasing at the constant rate of $0.25\ \text{cm}^2\,\text{s}^{-1}$.

b) Find, in terms of surds, the rate at which the volume of the cube is increasing when its surface area has reached $16\ \text{cm}^2$.

$$\boxed{\frac{1}{24}\sqrt{6}}$$



Question 23 (****)

Air is pumped into a balloon at the constant rate of $15 \text{ cm}^3 \text{ s}^{-1}$.

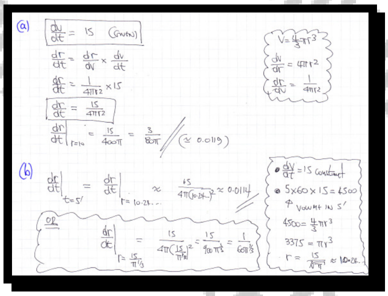
The shape of the balloon remains spherical at all times.

a) Find the rate at which the radius of the balloon is increasing when its radius has reached 10 cm .

b) If the balloon is initially empty, find the rate at which its radius is increasing 5 minutes after the air started being pumped in.

$\left[\text{volume of a sphere of radius } r \text{ is given by } \frac{4}{3}\pi r^3 \right]$

$\boxed{\frac{3}{80\pi} \approx 0.0119 \text{ cm s}^{-1}}$, $\boxed{\frac{1}{60\sqrt[3]{\pi}} \approx 0.0114 \text{ cm s}^{-1}}$



Question 24 (****)

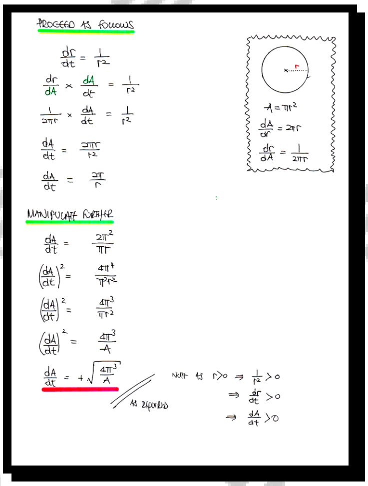
The radius r of a circle is changing so that

$$\frac{dr}{dt} = \frac{1}{r^2}.$$

Show that the rate at which the area of the circle A changes satisfies the equation

$$\frac{dA}{dt} = \sqrt{\frac{4\pi^3}{A}}.$$

$\boxed{}$, $\boxed{\text{proof}}$


PROCEED AS FOLLOWS

$$\frac{dr}{dt} = \frac{1}{r^2}$$
$$\frac{dr}{dA} \times \frac{dA}{dt} = \frac{1}{r^2}$$
$$\frac{1}{2\pi r} \times \frac{dA}{dt} = \frac{1}{r^2}$$
$$\frac{dA}{dt} = \frac{2\pi r}{r^2}$$
$$\frac{dA}{dt} = \frac{2\pi}{r}$$

r

$$A = \pi r^2$$
$$\frac{dA}{dr} = 2\pi r$$
$$\frac{dr}{dA} = \frac{1}{2\pi r}$$

MANIPULATE FURTHER

$$\frac{dA}{dt} = \frac{2\pi^2}{\pi r}$$
$$\left(\frac{dA}{dt}\right)^2 = \frac{4\pi^4}{\pi^2 r^2}$$
$$\left(\frac{dA}{dt}\right)^2 = \frac{4\pi^3}{\pi r^2}$$
$$\left(\frac{dA}{dt}\right)^2 = \frac{4\pi^3}{A}$$
$$\frac{dA}{dt} = +\sqrt{\frac{4\pi^3}{A}}$$

AS REQUIRED

NOTE AS $r>0 \Rightarrow \frac{1}{r^2} > 0$
$\Rightarrow \frac{dr}{dt} > 0$
$\Rightarrow \frac{dA}{dt} > 0$

Question 25 (****)

A particle is moving on the curve with equation

$$x = y^3 + y + 1, \quad y \in \mathbb{R}.$$

The particle has coordinates (x, y) at time t.

When the y coordinate of the particle is 5 the rate at which the y coordinate is changing with respect to time t is $\frac{1}{4}$.

Find the rate at which the x coordinate of the particle changes with time, at that instant.

V, N, $\boxed{\frac{dx}{dt} = 19}$

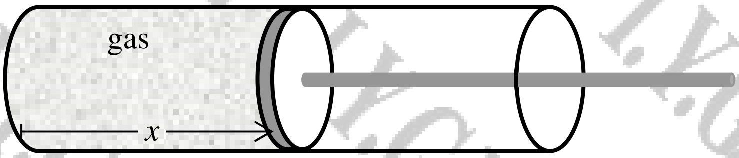
AS THE EQUATION IS GIVEN AS $x = f(y)$, DIFFERENTIATE W.R.T x

$$\Rightarrow x = y^3 + y + 1$$
$$\Rightarrow \frac{dx}{dy} = 3y^2 + 1$$

INTRODUCE TIME t, WITH $\frac{dy}{dt} = \frac{1}{4}$, AT $y = 5$

$$\Rightarrow \frac{dx/dt}{dy/dt} = 3y^2 + 1$$
$$\Rightarrow \frac{\frac{dx}{dt}}{\frac{1}{4}} = 3 \times 5^2 + 1$$
$$\Rightarrow \frac{dx}{dt} = \frac{1}{4} \times 76$$
$$\Rightarrow \frac{dx}{dt} = 19$$

Question 26 (****)



A piston can slide inside a combustion cylinder which is closed at one end.

The cylinder is filled with gas whose pressure P, in suitable units, is given by

$$P = \frac{60}{x}, \quad x \neq 0$$

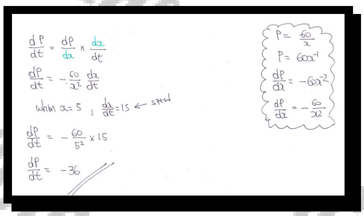
where x is the distance, in cm, of the piston from the closed end.

At a given instant

- the distance of the piston from the closed end is 5 cm.
- its speed is 15 cm s^{-1}, moving away from the closed end.

Determine the rate at which the pressure of the gas is changing at that given instant.

$\boxed{}$, $\boxed{\frac{dP}{dt} = -36}$



Question 27 (****)

The volume of the water, V m^3, in a container satisfies the equation

$$V = x^3 \mathrm{e}^{-x^2},$$

where x m is the depth of the water in the container.

Find the rate of increase of the volume of the water in the container when its depth is 0.5 m and is rising at the rate of 0.01 m s^{-1}.

$\boxed{}$, $\boxed{\frac{1}{160}\mathrm{e}^{-\frac{1}{4}} \approx 0.00487 \text{ m}^3\text{s}^{-1}}$

RELATING V, x AND t AS FOLLOWS

$$V = x^3 e^{-x^2}$$

$$\frac{dV}{dx} = (3x^2) \times e^{-x^2} + x^3 \times e^{-x^2}(-2x)$$

$$\frac{dV}{dx} = 3x^2 e^{-x^2} - 2x^4 e^{-x^2}$$

NOW WE HAVE

$$\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt}$$

$$\frac{dV}{dt} = \left(3x^2 e^{-x^2} - 2x^4 e^{-x^2}\right)\frac{dx}{dt}$$

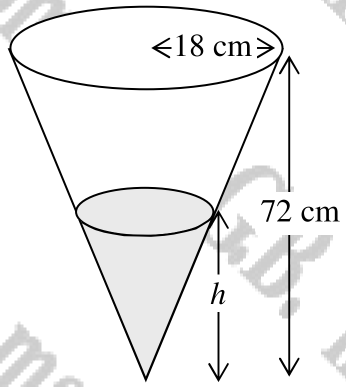
WHEN $x = \frac{1}{2}$ & $\frac{dx}{dt} = 0.01$

$$\frac{dV}{dt} = \left(3 \times \frac{1}{4} e^{-\frac{1}{4}} - 2 \times \frac{1}{16} e^{-\frac{1}{4}}\right) \times 0.01$$

$$\frac{dV}{dt} = \left(\frac{3}{4} e^{-\frac{1}{4}} - \frac{1}{8} e^{-\frac{1}{4}}\right) \times \frac{1}{100}$$

$$\frac{dV}{dt} = \frac{5}{800} e^{-\frac{1}{4}} \approx 0.00487 \text{ m}^3\text{s}^{-1}$$

Question 28 (****)



Flowers at a florists' are stored in vases which are in the shape of hollow inverted right circular cones with height 72 cm and radius 18 cm.

One such vase is initially empty and placed, with its axis vertical, under a tap where the water is flowing into the vase at the constant rate of $6\pi \text{ cm}^3 \text{ s}^{-1}$.

a) Show that the volume, $V \text{ cm}^3$, of the water in the vase is given by

$$V = \frac{1}{48}\pi h^3,$$

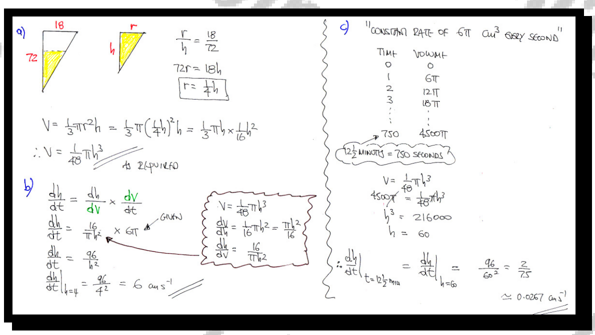
where, h cm, is the height of the water in the vase.

b) Find the rate at which h is rising when $h = 4 \text{ cm}$.

c) Determine the rate at which h is rising 12.5 **minutes** after the vase was placed under the tap.

$\left[\text{volume of a cone of radius } r \text{ and height } h \text{ is given by } \tfrac{1}{3}\pi r^2 h\right]$

$\boxed{}$, $\boxed{6 \text{ cm s}^{-1}}$, $\boxed{\frac{2}{75} \approx 0.0267 \text{ cm s}^{-1}}$



Question 29 (****)

The surface area A, of a metallic cube of side length x, is increasing at the constant rate of $0.45 \text{ cm}^2 \text{ s}^{-1}$.

Find the rate at which the volume of the cube is increasing, when the cube's side length is 8 cm.

$\boxed{}$, $\boxed{0.9 \text{ cm}^3 \text{ s}^{-1}}$

START BY RELATING DERIVATIVES

$$\Rightarrow \frac{dV}{dt} = \frac{dV}{dA} \times \frac{dA}{dt}$$

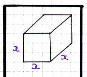
$$\Rightarrow \frac{dV}{dt} = \frac{dV}{dA} \times 0.45$$

$$\Rightarrow \frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dA} \times 0.45$$

$$\Rightarrow \frac{dV}{dt} = 3x^{\cancel{2}} \times \frac{1}{12\cancel{x}} \times 0.45$$

$$\Rightarrow \frac{dV}{dt} = \frac{9x}{80}$$

$$\Rightarrow \left.\frac{dV}{dt}\right|_{x=8} = \frac{9 \times 8}{80} = 0.9 \text{ cm}^3 \text{s}^{-1}$$



- $V = x^3$
  $\frac{dV}{dx} = 3x^2$
- $A = 6x^2$
  $\frac{dA}{dx} = 12x$
  $\frac{dx}{dA} = \frac{1}{12x}$

ALTERNATIVE METHOD

$$\Rightarrow \frac{dV}{dt} = \frac{dV}{dA} \times \frac{dA}{dt}$$

$$\Rightarrow \frac{dV}{dt} = \frac{dV}{dA} \times 0.45$$

$$\Rightarrow \frac{dV}{dt} = \frac{\sqrt{6}}{36} \times \frac{3}{2} A^{\frac{1}{2}} \times 0.45$$

$$\Rightarrow \frac{dV}{dt} = \frac{\sqrt{6}}{24} A^{\frac{1}{2}} \times 0.45$$

$$\Rightarrow \frac{dV}{dt} = \frac{\sqrt{6}}{24} (6x^2)^{\frac{1}{2}} \times 0.45$$

$$\Rightarrow \frac{dV}{dt} = \frac{\sqrt{6}}{24} \times \sqrt{6}x \times 0.45$$

$$\Rightarrow \frac{dV}{dt} = \frac{1}{4}x \times 0.45$$

$$\Rightarrow \frac{dV}{dt} = \frac{9x}{80}$$

$$\Rightarrow \left.\frac{dV}{dt}\right|_{x=8} = 0.9 \text{ cm}^3 \text{s}^{-1}$$

AS BEFORE

OBTAIN A RELATIONSHIP BETWEEN V & A

$V = x^3$, $A = 6x^2$ $\Rightarrow$ $V^2 = x^6$, $A^3 = 216x^6$

DIVIDE THE EQUATIONS

$\frac{A^3}{V^2} = 216$

$A^3 = 216V^2$

$V = +\left(\frac{A^3}{216}\right)^{\frac{1}{2}} = \frac{\sqrt{6}}{36} A^{\frac{3}{2}}$

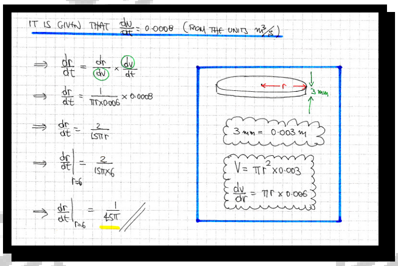
Question 30 (****)

After a road accident, fuel is leaking from a tanker onto a flat section of the motorway forming a circle of thickness 3 mm.

Petrol is leaking at a steady rate of $0.0008 \text{ m}^3\text{s}^{-1}$

Find, in terms of π, the rate at which the radius of the circle of petrol is increasing at the instant when the radius has reached 6 m.

$\boxed{}$, $\boxed{\dfrac{1}{45\pi}}$



Question 31 (****)

A particle is moving on the curve with equation

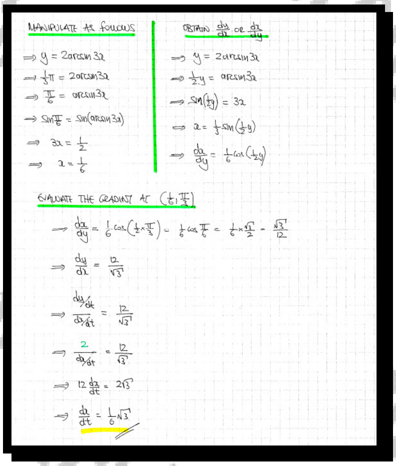
$$y = 2\arcsin 3x, \quad -\tfrac{1}{3} \le x \le \tfrac{1}{3}.$$

The particle has coordinates (x, y) at time t.

When the y coordinate of the particle is $\tfrac{1}{3}\pi$ the rate at which the y coordinate is changing with time t is 2.

Find the rate at which the x coordinate of the particle changes with time, at that instant.

$\boxed{}$, $\boxed{\dfrac{dx}{dt} = \dfrac{1}{6}\sqrt{3}}$



MANIPULATE AS FOLLOWS

$\Rightarrow y = 2\arcsin 3x$

$\Rightarrow \frac{1}{3}\pi = 2\arcsin 3x$

$\Rightarrow \frac{\pi}{6} = \arcsin 3x$

$\Rightarrow \sin\frac{\pi}{6} = \sin(\arcsin 3x)$

$\Rightarrow 3x = \frac{1}{2}$

$\Rightarrow x = \frac{1}{6}$

OBTAIN $\frac{dy}{dx}$ OR $\frac{dx}{dy}$

$\Rightarrow y = 2\arcsin 3x$

$\Rightarrow \frac{1}{2}y = \arcsin 3x$

$\Rightarrow \sin\left(\frac{1}{2}y\right) = 3x$

$\Rightarrow x = \frac{1}{3}\sin\left(\frac{1}{2}y\right)$

$\Rightarrow \frac{dx}{dy} = \frac{1}{6}\cos\left(\frac{1}{2}y\right)$

EVALUATE THE GRADIENT AT $\left(\frac{1}{6}, \frac{\pi}{3}\right)$

$\Rightarrow \frac{dx}{dy} = \frac{1}{6}\cos\left(\frac{1}{2}\times\frac{\pi}{3}\right) = \frac{1}{6}\cos\frac{\pi}{6} = \frac{1}{6}\times\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{12}$

$\Rightarrow \frac{dy}{dx} = \frac{12}{\sqrt{3}}$

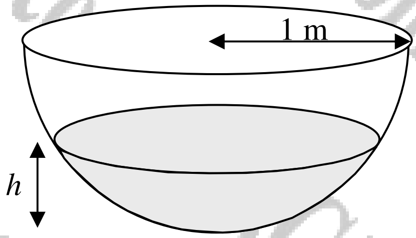
$\Rightarrow \frac{dy/dt}{dx/dt} = \frac{12}{\sqrt{3}}$

$\Rightarrow \frac{2}{dx/dt} = \frac{12}{\sqrt{3}}$

$\Rightarrow 12\frac{dx}{dt} = 2\sqrt{3}$

$\Rightarrow \frac{dx}{dt} = \frac{1}{6}\sqrt{3}$

Question 32 (****)



A tank for storing water is in the shape of a hollow inverted hemisphere with a radius of one metre.

It can be shown by calculus that when the depth of the water in the tank is h m, its volume, V m^3, is given by the formula

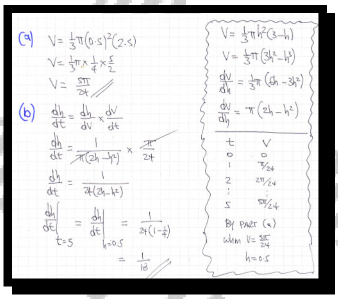
$$V = \frac{1}{3}\pi h^2 (3-h).$$

a) Find the volume of the water in the tank when $h = 0.5$.

The tank is initially empty and water then begins to pour in at the constant rate of $\frac{\pi}{24}$ m^3 per hour.

b) Determine the rate at which the height of the water is increasing 5 hours later.

$\boxed{}$, $\boxed{V = \frac{5\pi}{24} \text{ m}^3}$, $\boxed{\frac{1}{18} = 0.0556 \text{ m h}^{-1}}$



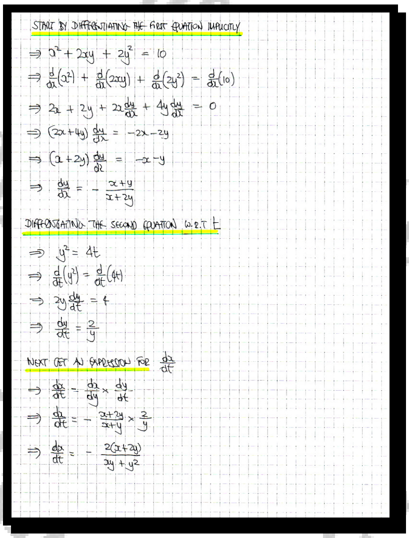
Question 33 (****+)

The variables y, x and t are related by the equations

$$x^2 + 2xy + 2y^2 = 10 \qquad \text{and} \qquad y^2 = 4t, \; t \geq 0.$$

Find the possible values of $\frac{dx}{dt}$, when $t = \frac{1}{4}$.

$$\boxed{}, \quad \boxed{\left.\frac{dy}{dt}\right|_{t=\frac{1}{4}} = \left\{-\frac{8}{3}, -\frac{4}{3}, \frac{4}{3}, \frac{8}{3}\right\}}$$



START BY DIFFERENTIATING THE FIRST EQUATION IMPLICITLY

$\Rightarrow x^2 + 2xy + 2y^2 = 10$

$\Rightarrow \frac{d}{dx}(x^2) + \frac{d}{dx}(2xy) + \frac{d}{dx}(2y^2) = \frac{d}{dx}(10)$

$\Rightarrow 2x + 2y + 2x\frac{dy}{dx} + 4y\frac{dy}{dx} = 0$

$\Rightarrow (2x+4y)\frac{dy}{dx} = -2x-2y$

$\Rightarrow (x+2y)\frac{dy}{dx} = -x-y$

$\Rightarrow \frac{dy}{dx} = -\frac{x+y}{x+2y}$

DIFFERENTIATING THE SECOND EQUATION W.R.T t

$\Rightarrow y^2 = 4t$

$\Rightarrow \frac{d}{dt}(y^2) = \frac{d}{dt}(4t)$

$\Rightarrow 2y\frac{dy}{dt} = 4$

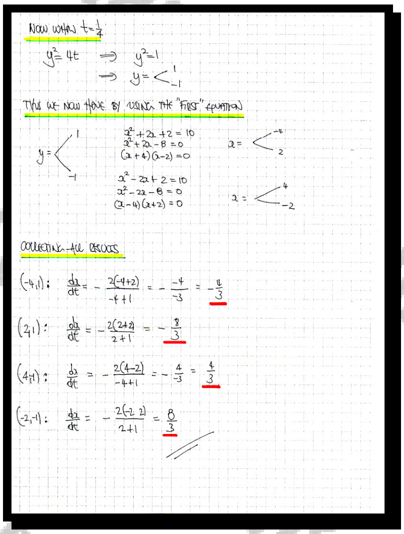
$\Rightarrow \frac{dy}{dt} = \frac{2}{y}$

NEXT GET AN EXPRESSION FOR $\frac{dx}{dt}$

$\Rightarrow \frac{dx}{dt} = \frac{dx}{dy} \times \frac{dy}{dt}$

$\Rightarrow \frac{dx}{dt} = -\frac{x+2y}{x+y} \times \frac{2}{y}$

$\Rightarrow \frac{dx}{dt} = -\frac{2(x+2y)}{xy+y^2}$



NOW WHEN $t = \frac{1}{4}$

$y^2 = 4t \Rightarrow y^2 = 1$

$\Rightarrow y = \begin{cases} 1 \\ -1 \end{cases}$

THUS WE NOW HAVE BY USING THE "FIRST" EQUATION

$y = 1$: $x^2 + 2x + 2 = 10$, $x^2 + 2x - 8 = 0$, $(x+4)(x-2) = 0$, $x = \begin{cases} -4 \\ 2 \end{cases}$

$y = -1$: $x^2 - 2x + 2 = 10$, $x^2 - 2x - 8 = 0$, $(x-4)(x+2) = 0$, $x = \begin{cases} 4 \\ -2 \end{cases}$

COLLECTING ALL RESULTS

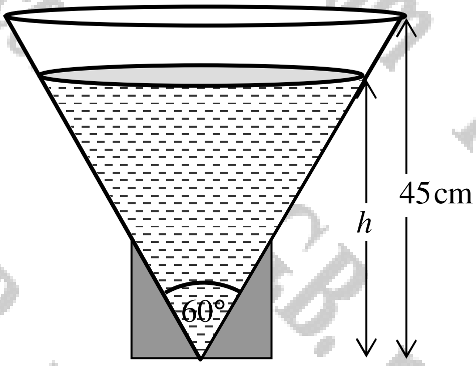
$(-4,1)$: $\frac{dx}{dt} = -\frac{2(-4+2)}{-4+1} = -\frac{-4}{-3} = -\frac{4}{3}$

$(2,1)$: $\frac{dx}{dt} = -\frac{2(2+2)}{2+1} = -\frac{8}{3}$

$(4,-1)$: $\frac{dx}{dt} = -\frac{2(4-2)}{-4+1} = -\frac{4}{-3} = \frac{4}{3}$

$(-2,-1)$: $\frac{dx}{dt} = -\frac{2(-2-2)}{2+1} = \frac{8}{3}$

Question 34 (****+)



A container, in the shape of a hollow inverted cone, is filled up the water.

The height of the container is 45 cm and the angle between the sides of the cone, when viewed as a cross section, is $60°$.

a) Show that the volume, V cm^3, of the water in the container is given by

$$V = \frac{1}{9}\pi h^3,$$

where h cm is the height of the water in the container.

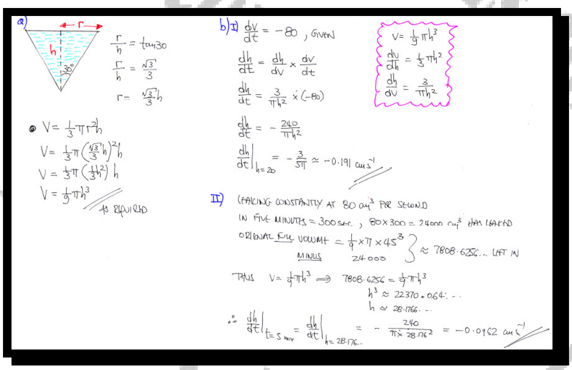
The container is filled up with water to the rim and then the water is allowed to leak from a small hole at the bottom of the cone, at the **constant** rate of $80 \text{ cm}^3\text{s}^{-1}$.

b) Determine the rate at which the height of the water is decreasing …

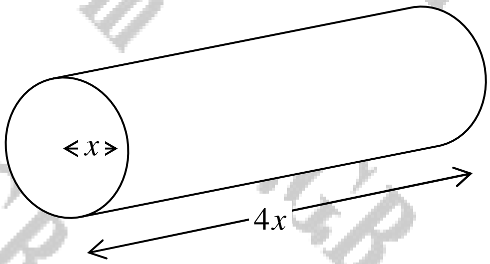
i. … when the height of the water is 20 cm.

ii. … five minutes after the leaking started.

$\boxed{}$, $\boxed{-\frac{3}{5\pi} \approx -0.191 \text{ cm s}^{-1}}$, $\boxed{\approx -0.0962 \text{ cm s}^{-1}}$



Question 35 (****+)



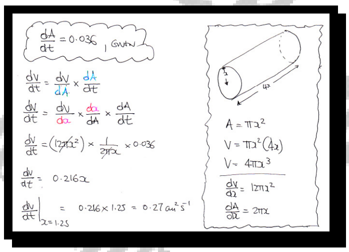
A metal bolt is in the shape of a right circular cylinder, with radius x cm and length $4x$ cm.

The bolt is heated so that the area of its circular cross section is expanding at the constant rate of $0.036 \text{ cm}^2 \text{ s}^{-1}$.

Find the rate at which the volume of the bolt is increasing, when the radius of the bolt has reached 1.25 cm.

(*You may assume that the bolt is expanding uniformly when heated.*)

$\boxed{}$, $\boxed{0.27 \text{ cm}^3 \text{ s}^{-1}}$



Question 36 (****+)

A solid right circular cone has radius x cm and perpendicular height $6x$ cm.

The cone is heated so that the area of its circular base is expanding at the constant rate of $0.25 \text{ cm}^2 \text{ s}^{-1}$.

Find the rate at which the volume of the cone is increasing, when the radius of the base of the cone has reached 2.5 cm.

(*You may assume that the bolt is expanding uniformly when heated*)

$\left[\text{volume of a cone of radius } r \text{ and height } h \text{ is given by } \frac{1}{3}\pi r^2 h\right]$

$\boxed{}$, $\boxed{1.875 \text{ cm}^3 \text{ s}^{-1}}$

WORK WITH 3 DERIVATIVES AS FOLLOWS

$\frac{dV}{dt} = \frac{dV}{dA} \times \frac{dA}{dt}$

$\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dA} \times \frac{dA}{dt}$

$\frac{dV}{dt} = \cancel{6\pi} x^2 \times \frac{1}{\cancel{2\pi} x} \times 0.25$

$\frac{dV}{dt} = \frac{3}{4}x$

$\left.\frac{dV}{dt}\right|_{x=2.5} = \frac{3}{4} \times \frac{5}{2} = \frac{15}{8} = 1.875 \text{ cm}^3\text{s}^{-1}$

BASE AREA

$A = \pi x^2$

$\frac{dA}{dx} = 2\pi x$

TOTAL VOLUME

$V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3}\pi x^2(6x)$

$V = 2\pi x^3$

$\frac{dV}{dx} = 6\pi x^2$

ALTERNATIVE APPROACH

$\frac{dV}{dt} = \frac{dV}{dA} \times \frac{dA}{dt}$

$\frac{dV}{dt} = \frac{6A^2}{\pi V} \times 0.25$

$\frac{dV}{dt} = \frac{6(\pi x^2)^2}{\pi(2\pi x^3)} \times \frac{1}{4}$

$\frac{dV}{dt} = \frac{6\pi^2 x^4}{8\pi^2 x^3} = \frac{3}{4}x$

$\left.\frac{dV}{dt}\right|_{x=2.5} = \frac{3}{4} \times 2.5 = 1.875 \text{ cm}^3\text{s}^{-1}$

$V^2 = \frac{4}{\pi}A^3$

$2V\frac{dV}{dA} = \frac{12}{\pi}A^2$

$\frac{dV}{dA} = \frac{6A^2}{\pi V}$

- $A = \pi x^2$
- $V = 2\pi x^3$ (AS ABOVE)
- $A^3 = \pi^3 x^6$
- $V^2 = 4\pi^2 x^6$

DIVIDING

$\frac{V^2}{A^3} = \frac{4\pi^2 x^6}{\pi^3 x^6}$

$\frac{V^2}{A^3} = \frac{4}{\pi}$

$V^2 = \frac{4}{\pi}A^3$

Question 37 (****+)

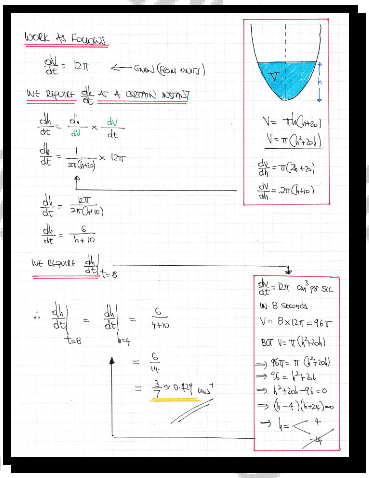
Liquid is pouring into a container at the constant rate of $12\pi \text{ cm}^3\text{s}^{-1}$.

The container is initially empty and when the height of the liquid in the container is h cm the volume of the liquid, V cm^3, is given by

$$V = \pi h(h+20).$$

Determine the rate at which the height of the liquid in the container is rising 8 seconds after the liquid started pouring in.

$\boxed{\dfrac{3}{7} = 0.429 \text{ cm s}^{-1}}$



Question 38 (****+)

Liquid is pouring into a container at the constant rate of 2π cm^3s^{-1}.

The container initially contains some liquid and when the height of the liquid in the container is h cm the volume of the liquid, V cm^3, is given by

$$V = \pi\left(h^2 + 5h - 16\right), \quad h \geq 6 .$$

Determine the rate at which the height of the liquid in the container is rising 30 seconds after the liquid started pouring in.

V, $\boxed{}$, $\boxed{\dfrac{2}{23} = 0.087 \text{ cm s}^{-1}}$

LOOKING AT THE VOLUME WHICH IS STRICTLY INCREASING FOR $h \geqslant 6$

$\Rightarrow h = 6 \qquad V = \pi(36 + 30 - 16) = 50\pi \leftarrow$ ALREADY IN THE CONTAINER

NOW $\frac{dV}{dt} = 2\pi$ = CONSTANT

$\Rightarrow$ 2π cm^3 EVERY SECOND

$\Rightarrow$ $2\pi \times 30$ cm^3 AFTER 30 SECONDS

$\Rightarrow$ 60π cm^3 EXTRA GONE IN

TOTAL LIQUID AFTER 30 SECONDS MUST BE $60\pi + 50\pi = 110\pi$

USING $V = \pi[h^2 + 5h - 16]$ WITH $V = 110\pi$ TO FIND h

$$\Rightarrow 110\pi = \pi(h^2 + 5h - 16)$$
$$\Rightarrow 110 = h^2 + 5h - 16$$
$$\Rightarrow h^2 + 5h - 126 = 0$$
$$\Rightarrow (h + 14)(h - 9)$$
$$\Rightarrow h = \cancel{-14},\ 9$$

FINALLY WE HAVE

$$V = \pi(h^2 + 5h - 16)$$
$$\frac{dV}{dh} = \pi[2h + 5]$$
$$\frac{dh}{dV} = \frac{1}{\pi(2h+5)}$$

$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$$
$$\frac{dh}{dt} = \frac{1}{\pi(2h+5)} \times 2\pi$$
$$\frac{dh}{dt} = \frac{2}{2h+5}$$
$$\left.\frac{dh}{dt}\right|_{t=30} = \left.\frac{dh}{dt}\right|_{h=9} = \frac{2}{2\times 9+5} = \frac{2}{23} \approx 0.087 \text{ cm s}^{-1}$$

Question 39 (****+)

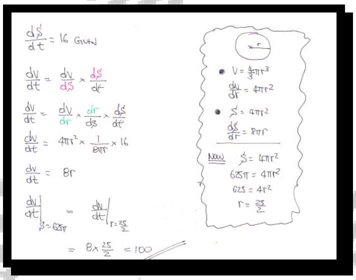
The surface area S of a sphere is increasing at the constant rate of $16 \text{ cm}^2 \text{s}^{-1}$.

Find the rate at which the volume V of the sphere is increasing, when the sphere's surface area is $625\pi \text{ cm}^2$.

$\left[\text{surface area of a sphere of radius } r \text{ is given by } 4\pi r^2\right]$

$\left[\text{volume of a sphere of radius } r \text{ is given by } \frac{4}{3}\pi r^3\right]$

$\boxed{}$, $\boxed{100 \text{ cm}^3\text{s}^{-1}}$



Question 40 (****+)

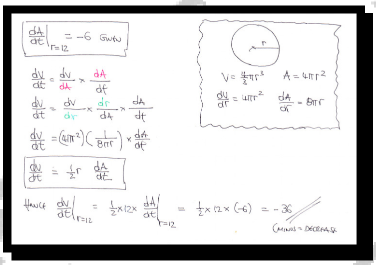
The surface area of a sphere is decreasing at the rate of $6 \text{ cm}^2 \text{s}^{-1}$ at the instant when its radius is 12 cm.

Find the rate at which the volume of the sphere is decreasing at that instant.

$\left[\text{surface area of a sphere of radius } r \text{ is given by } 4\pi r^2\right]$

$\left[\text{volume of a sphere of radius } r \text{ is given by } \frac{4}{3}\pi r^3\right]$

$\boxed{}$, $\boxed{36 \text{ cm}^3\text{s}^{-1}}$



Question 41 (****+)

The variables x, y, z and t are related by the equations

$$z = \sqrt{t^3 + 8t^{\frac{1}{2}} + 1} \qquad y = \frac{1}{(x+3)^2} \qquad \ln(x+3)^3 = \frac{1}{3}z,$$

where $x > -3$ and $x \geq 0$.

Find the value of z, when $t = 4$ and hence determine the value of $\frac{dy}{dt}$, when $y = e^{-2}$.

$\boxed{\quad\quad}$, $\boxed{-\frac{50}{81e^2} \approx -0.0835}$

START BY EVALUATING z AT $t=4$

$z(4) = \sqrt{4^3 + 8 \times 4^{\frac{1}{2}} + 1} = \sqrt{64 + 16 + 1} = \sqrt{81} = 9$

NEXT FORM A CHAIN OF RELATED DERIVATIVES

$\Rightarrow \frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dz} \times \frac{dz}{dt}$

$y = \frac{1}{(x+3)^2}$

$\frac{dy}{dx} = -\frac{2}{(x+3)^3}$

$z = (t^3 + 8t^{\frac{1}{2}} + 1)^{\frac{1}{2}}$

$\frac{dz}{dt} = \frac{1}{2}(t^3 + 8t^{\frac{1}{2}} + 1)^{-\frac{1}{2}} \times (3t^2 + 4t^{-\frac{1}{2}})$

$\frac{dz}{dt} = \frac{3t^2 + 4t^{-\frac{1}{2}}}{2(t^3 + 8t^{\frac{1}{2}} + 1)^{\frac{1}{2}}}$

$\ln(x+3)^3 = \frac{1}{3}z$

$3\ln(x+3) = \frac{1}{3}z$

$z = 9\ln(x+3)$

$\frac{dz}{dx} = \frac{9}{x+3}$

$\frac{dx}{dz} = \frac{x+3}{9}$

$\Rightarrow \frac{dy}{dt} = -\frac{\cancel{2}}{(x+3)^{\cancel{3}\,2}} \times \frac{\cancel{x+3}}{9} \times \frac{3t^2 + 4t^{-\frac{1}{2}}}{\cancel{2}\sqrt{t^3 + 8\sqrt{t} + 1}}$

$\Rightarrow \frac{dy}{dt} = -\frac{1}{9(x+3)^2} \times \frac{3t^2 + 4t^{-\frac{1}{2}}}{\sqrt{t^3 + 8\sqrt{t} + 1}}$

$\Rightarrow \frac{dy}{dt} = -\frac{1}{9}y \times \frac{3t^2 + 4t^{-\frac{1}{2}}}{z}$

$\Rightarrow \frac{dy}{dt} = -\frac{y(3t^2 + 4t^{-\frac{1}{2}})}{9z}$

NEXT WE USE $y = -e^{-2}$

• $y = e^{-2} \Rightarrow y = \frac{1}{e^2}$

$\Rightarrow \frac{1}{(x+3)^2} = \frac{1}{e^2}$

$\Rightarrow (x+3)^2 = e^2$

$\Rightarrow x+3 = \begin{cases} e \\ -e \end{cases}$

$\Rightarrow x = \begin{cases} e-3 \\ \cancel{-e-3} \end{cases} \quad x > -3$

• $\ln(x+3)^3 = \frac{1}{3}z \Rightarrow z = 9\ln(x+3)$ ← FROM EARLIER

$\Rightarrow z = 9\ln(e-3+3)$

$\Rightarrow z = 9$

$\therefore$ WHEN $y = e^{-2}$, $x = e-3$, $z = 9$ & $t = 4$

FINALLY WE HAVE

$\left.\frac{dy}{dt}\right|_{y=e^{-2}} = -\frac{e^{-2}(3 \times 4^2 + 4 \times \ )}{9 \times 9}$

$\left.\frac{dy}{dt}\right|_{y=e^{-2}} = -\frac{50}{81e^2} \approx -0.0835$

Question 42 (****+)

The variables x, y and t satisfy

$$\frac{dx}{dt} = kx \qquad \text{and} \qquad 2\left(x^2 + y^2\right) = 5xy,$$

where k is a non zero constant.

Find, in terms of k, the possible values of $\frac{dy}{dt}$ when $x = 2$.

$$\boxed{}, \quad \boxed{\frac{dy}{dt} = k \ \cup \ \frac{dy}{dt} = 4k}$$

WE ARE GIVEN THAT

$$\frac{dx}{dt} = kx \qquad \text{AND} \qquad 2(x^2+y^2) = 5xy$$

FIRSTLY DIFFERENTIATE THE IMPLICIT RELATIONSHIP W.R.T x.

$$\Rightarrow 2\left(2x + 2y\frac{dy}{dx}\right) = 5y + 5x\frac{dy}{dx}$$

$$\Rightarrow 4x + 4y\frac{dy}{dx} = 5y + 5x\frac{dy}{dx}$$

$$\Rightarrow (4y - 5x)\frac{dy}{dx} = 5y - 4x$$

$$\Rightarrow \frac{dy}{dx} = \frac{5y-4x}{4y-5x}$$

NOW WE HAVE

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt} = \frac{5y-4x}{4y-5x} \times kx$$

NOW WHEN $x=2$

$$\Rightarrow 2(2^2+y^2) = 5\times 2\times y$$

$$\Rightarrow 2(4+y^2) = 10y$$

$$\Rightarrow 4+y^2 = 5y$$

$$\Rightarrow y^2 - 5y + 4 = 0$$

$$\Rightarrow (y-4)(y-1)$$

$$\Rightarrow y = \begin{cases} 1 \\ 4 \end{cases}$$

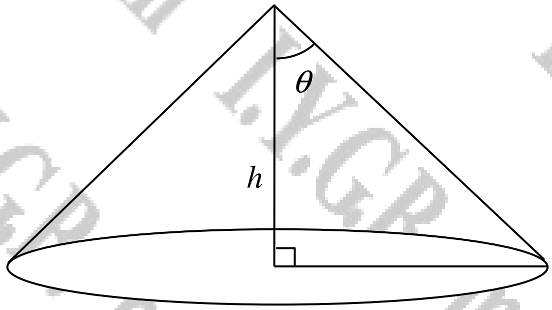
FINALLY WE HAVE

$$\left.\frac{dy}{dt}\right|_{(2,1)} = \frac{5-8}{4-10} \times (k\times 2) = 2k \times \frac{-3}{-6}$$

$$\left.\frac{dy}{dt}\right|_{(2,4)} = \frac{20-8}{16-10} \times (k\times 2) = \frac{12}{6} \times 2k$$

$$\therefore \frac{dy}{dt} = \begin{cases} k \\ 4k \end{cases}$$

Question 43 (*****)



Fine sand starts falling onto a horizontal floor at the constant rate of $50 \text{ cm}^3\text{s}^{-1}$.

A heap is formed in the shape of a right circular cone of height h cm.

The angle θ, where $\tan\theta = 3$, is formed between the vertical height and the slant height of the cone, as shown in the figure above.

Show that when $t = 60$

$$\frac{dh}{dt} = \frac{1}{18}\pi^{-\frac{1}{3}}.$$

where t is the time in seconds since the sand started falling.

$\left[\text{volume of a cone of radius } r \text{ and height } h \text{ is given by } \frac{1}{3}\pi r^2 h\right]$

$\boxed{\text{proof}}$

START BY CONNECTING DERIVATIVES

$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$$

$$\frac{dh}{dt} = \frac{dh}{dV} \times 50$$

$$\frac{dh}{dt} = \frac{1}{9\pi h^2} \times 50$$

$$\frac{dh}{dt} = \frac{50}{9\pi h^2}$$

NEXT WE ARE TOLD THAT

... CONSTANT RATE OF 50 cm^3 PER SECOND...

... IN 60 SECONDS...

$V = 50 \times 60$

$V = 3000 \text{ cm}^3$

$\tan\theta = 3$

$\frac{r}{h} = 3$

$r = 3h$

VOLUME OF CONE

$V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3}\pi (3h)^2 h$

$V = 3\pi h^3$

DIFFERENTIATE W.R.T h

$\frac{dV}{dh} = 9\pi h^2$

$\frac{dh}{dV} = \frac{1}{9\pi h^2}$

CONVERT THIS VOLUME INTO h

$\Rightarrow V = 3\pi h^3$

$\Rightarrow 3000 = 3\pi h^3$

$\Rightarrow h^3 = \frac{1000}{\pi}$

$\Rightarrow h = \frac{10}{\pi^{1/3}}$

FINALLY WE HAVE

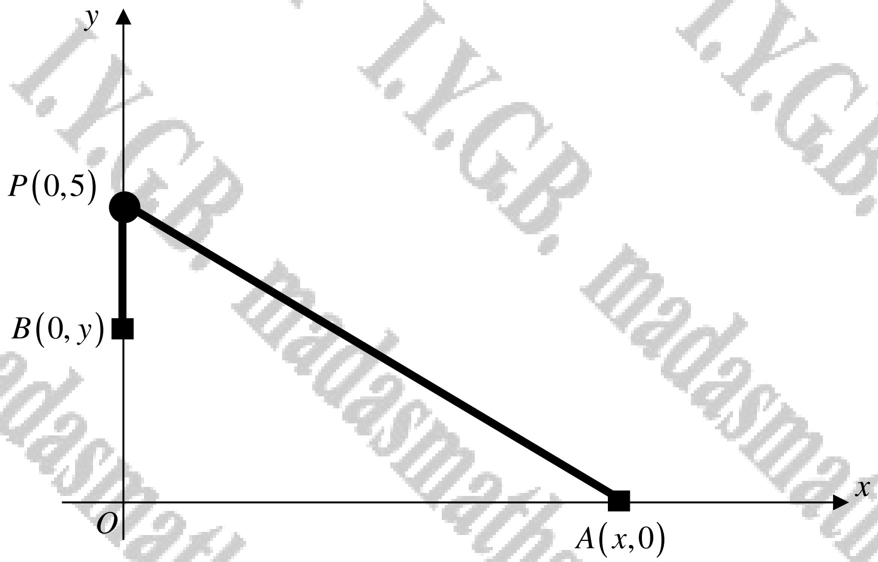
$\Rightarrow \left.\frac{dh}{dt}\right|_{t=60} = \left.\frac{dh}{dt}\right|_{V=3000} = \left.\frac{dh}{dt}\right|_{h=\frac{10}{\pi^{1/3}}}$

$\Rightarrow \left.\frac{dh}{dt}\right|_{t=60} = \frac{50}{9\pi\left(\frac{10}{\pi^{1/3}}\right)^2} = \frac{50}{900\pi^{1/3}}$

$\Rightarrow \left.\frac{dh}{dt}\right|_{t=60} = \frac{1}{18\pi^{\frac{1}{3}}}$

$\Rightarrow \left.\frac{dh}{dt}\right|_{t=60} = \frac{1}{18}\pi^{-\frac{1}{3}}$ // AS REQUIRED

Question 44 (*****)



Two particles, A and B, can move on the positive x axis and positive y axis respectively. They are connected with a rope which remains taut at all times.

Particle A has coordinates $(x,0)$ metres, where $x \geq 0$ and particle B has coordinates $(0, y)$ metres, where $0 \leq y \leq 5$.

The rope connecting the two particles has a length of 15 metres and passes over a small fixed pulley located at $P(0,5)$ metres.

a) Show that

$$\frac{dy}{dx} = \frac{x}{y+10}.$$

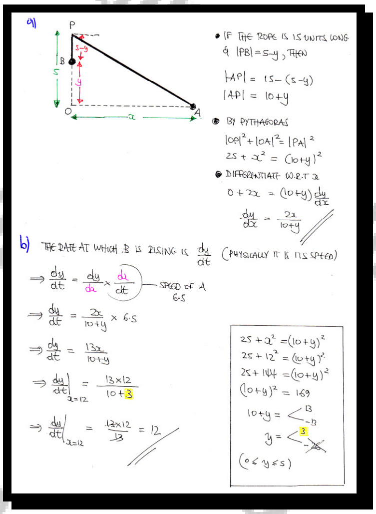
[continues overleaf]

[continued from overleaf]

At a given instant the particle A is at the point with coordinates $(12,0)$ metres and moving away from O with a speed of 6.5 metres per second.

b) Find the rate at which the particle B is rising at that instant.

$\boxed{}$, $\boxed{6 \text{ ms}^{-1}}$



• IF THE ROPE IS 15 UNITS LONG & $|PB| = 5-y$, THEN

$|AP| = 15-(5-y)$

$|AP| = 10+y$

• BY PYTHAGORAS

$|OP|^2 + |OA|^2 = |PA|^2$

$25 + x^2 = (10+y)^2$

• DIFFERENTIATE W.R.T x

$0 + 2x = (10+y)\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{2x}{10+y}$

b) THE RATE AT WHICH B IS RISING IS $\frac{dy}{dt}$ (PHYSICALLY IT IS ITS SPEED)

$\Rightarrow \frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$ — SPEED OF A 6.5

$\Rightarrow \frac{dy}{dt} = \frac{2x}{10+y} \times 6.5$

$\Rightarrow \frac{dy}{dt} = \frac{13x}{10+y}$

$\Rightarrow \left.\frac{dy}{dt}\right|_{x=12} = \frac{13 \times 12}{10+3}$

$\Rightarrow \left.\frac{dy}{dt}\right|_{x=12} = \frac{\cancel{13} \times 12}{\cancel{13}} = 12$

$25 + x^2 = (10+y)^2$

$25 + 12^2 = (10+y)^2$

$25 + 144 = (10+y)^2$

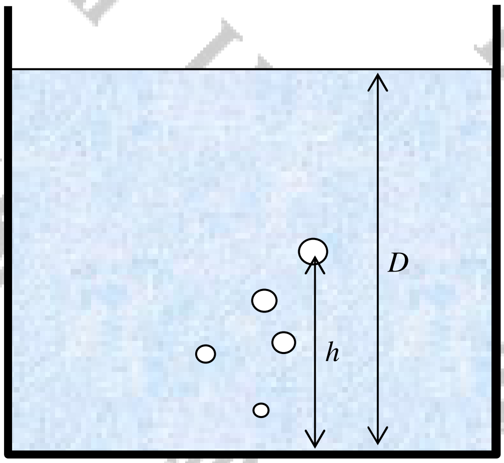
$(10+y)^2 = 169$

$10 + y = \begin{cases} 13 \\ -13 \end{cases}$

$y = \begin{cases} 3 \\ \cancel{-26} \end{cases}$

$(0 \leqslant y \leqslant 5)$

Question 45 (*****)



An air bubble, rising in a water tank, increases in volume as the pressure of the fluid around it decreases. It is assumed that the shape of the bubble remains spherical at all times.

It is further assumed the volume V cm^3 of an air bubble satisfies the equation

$$V = \frac{k}{D-h},$$

where h cm is the height of the bubble from the bottom of the tank, D cm is the depth of the water in the tank, and k is a positive constant.

The tank is filled up with water to a depth of 800 cm.

A bubble with a volume of 8 cm^3 is created in the water tank at a height of 350 cm from the bottom of the tank.

[continues overleaf]

[continued from overleaf]

Show that by the time the bubble has risen by 50 cm, …

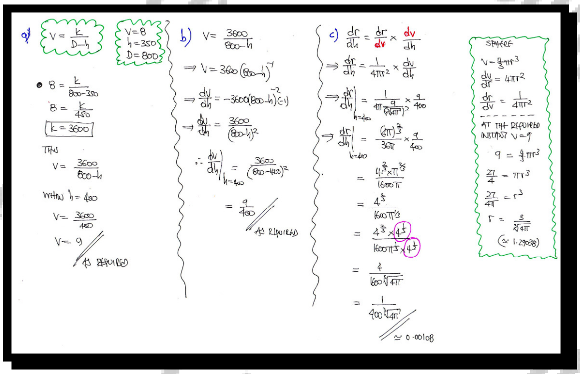
a) … the volume of the bubble increases to 9 cm^3

b) … the volume of the bubble increases at the rate of $\frac{9}{400} \text{ cm}^3$ per cm risen.

c) … the rate at which the radius of the bubble is increasing is

$$\frac{1}{400\sqrt[3]{4\pi}} \text{ cm per cm risen.}$$

$\boxed{\text{proof}}$



Question 46 (*****)

A metallic component is in the shape of a right circular cone, with radius $4x$ cm and height $3x$ cm.

The metallic component is heated so that the area of its curved face is expanding at a rate inversely proportional to x.

a) Show that volume of the metallic component is increasing at a constant rate.

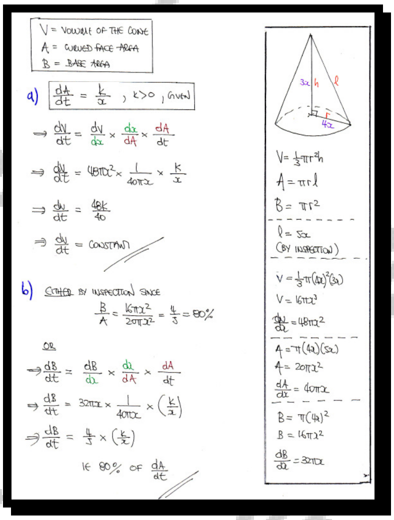
b) Find the percentage rate of increase of the base area relative to the curved face area of the metallic component.

(*You may assume that the metallic component is expanding uniformly when heated.*)

[surface area of the curved face of a cone of radius r and slant height l, is given by πrl]

[volume of a cone of radius r and height h, $\frac{1}{3}\pi r^2 h$]

$\boxed{}$, $\boxed{80\%}$



V = VOLUME OF THE CONE
A = CURVED FACE AREA
B = BASE AREA

a) $\boxed{\frac{dA}{dt} = \frac{k}{x}, \ k>0, \text{ GIVEN}}$

$\Rightarrow \frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dA} \times \frac{dA}{dt}$

$\Rightarrow \frac{dV}{dt} = 48\pi x^2 \times \frac{1}{40\pi x} \times \frac{k}{x}$

$\Rightarrow \frac{dV}{dt} = \frac{48k}{40}$

$\Rightarrow \frac{dV}{dt} = \text{CONSTANT}$

b) EITHER BY INSPECTION SINCE

$\frac{B}{A} = \frac{16\pi x^2}{20\pi x^2} = \frac{4}{5} = 80\%$

OR

$\Rightarrow \frac{dB}{dt} = \frac{dB}{dx} \times \frac{dx}{dA} \times \frac{dA}{dt}$

$\Rightarrow \frac{dB}{dt} = 32\pi x \times \frac{1}{40\pi x} \times \left(\frac{k}{x}\right)$

$\Rightarrow \frac{dB}{dt} = \frac{4}{5} \times \left(\frac{k}{x}\right)$

IE 80% OF $\frac{dA}{dt}$

$V = \frac{1}{3}\pi r^2 h$

$A = \pi r l$

$B = \pi r^2$

$l = 5x$ (BY INSPECTION)

$V = \frac{1}{3}\pi (4x)^2 (3x)$

$V = 16\pi x^3$

$\frac{dV}{dx} = 48\pi x^2$

$A = \pi (4x)(5x)$

$A = 20\pi x^2$

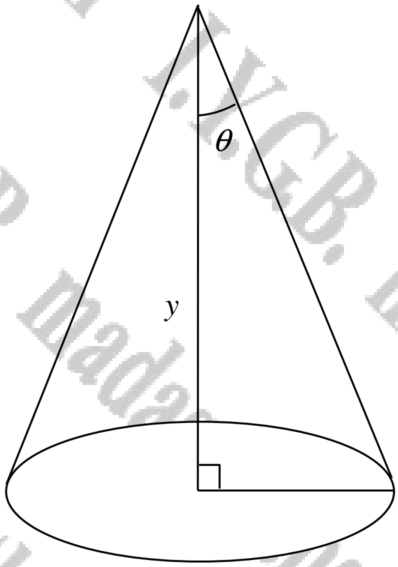
$\frac{dA}{dx} = 40\pi x$

$B = \pi (4x)^2$

$B = 16\pi x^2$

$\frac{dB}{dx} = 32\pi x$

Question 47 (*****)



Fine sand starts falling onto a horizontal floor at the constant rate of $3.2\ \text{cm}^3\text{s}^{-1}$.

A heap is formed in the shape of a right circular cone of height y cm, where t is the time in seconds since the sand started falling. The angle θ between the vertical height and the slant height of the cone is such so that $\tan\theta = \frac{1}{\sqrt{3}}$, as shown in the figure.

a) Show clearly that

$$y^3 = \frac{144t}{5\pi}.$$

[continues overleaf]

[continued from overleaf]

The curved surface area of the heap is A cm^2.

b) Show further that when $t = 60$, …

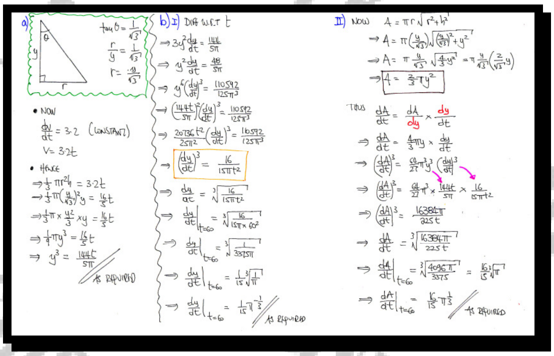
i. … $\dfrac{dy}{dt} = \dfrac{1}{15}\pi^{-\frac{1}{3}}$.

ii. … $\dfrac{dA}{dt} = \dfrac{16}{15}\pi^{\frac{1}{3}}$.

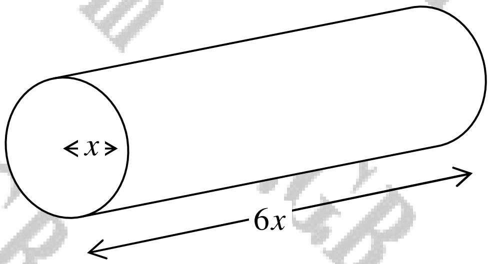
You may assume that the volume V and curved surface area A of a right circular cone of radius r and height h are given by

$$V = \frac{1}{3}\pi r^2 h \qquad \text{and} \qquad A = \pi r\sqrt{r^2 + h^2}\ .$$

$\boxed{}$, $\boxed{\text{proof}}$



Question 48 (*****)



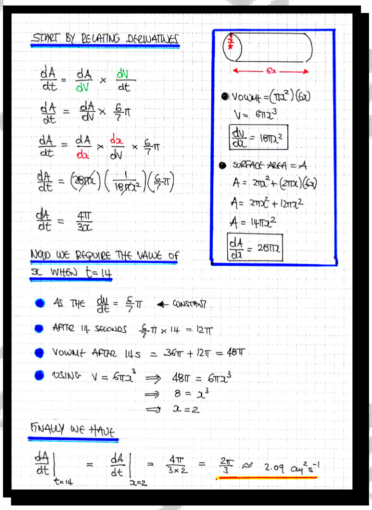
A solid machine component, made of metal, is in the shape of a right circular cylinder, with radius x cm and length $6x$ cm.

The component is heated so that it is expanding at the constant rate of $\frac{6}{7}\pi \text{ cm}^3\text{s}^{-1}$.

Given that the initial volume of the component was $36\pi \text{ cm}^3$, find the rate at which the surface area of the component is increasing 14 s after the heating started.

You may assume that the shape of the component is mathematically similar to its original shape at all times.

$\boxed{}$, $\boxed{\frac{2}{3}\pi \approx 2.09 \text{ cm}^2\text{s}^{-1}}$



START BY RELATING DERIVATIVES

$$\frac{dA}{dt} = \frac{dA}{dV} \times \frac{dV}{dt}$$

$$\frac{dA}{dt} = \frac{dA}{dV} \times \frac{6}{7}\pi$$

$$\frac{dA}{dt} = \frac{dA}{dx} \times \frac{dx}{dV} \times \frac{6}{7}\pi$$

$$\frac{dA}{dt} = (28\pi x)\left(\frac{1}{18\pi x^2}\right)\left(\frac{6}{7}\pi\right)$$

$$\frac{dA}{dt} = \frac{4\pi}{3x}$$

6x

- VOLUME $= (\pi x^2)(6x)$

$V = 6\pi x^3$

$\boxed{\frac{dV}{dx} = 18\pi x^2}$

- SURFACE AREA $= A$

$A = 2\pi x^2 + (2\pi x)(6x)$

$A = 2\pi x^2 + 12\pi x^2$

$A = 14\pi x^2$

$\boxed{\frac{dA}{dx} = 28\pi x}$

NOW WE REQUIRE THE VALUE OF x WHEN $t = 14$

- AS THE $\frac{dV}{dt} = \frac{6}{7}\pi$ ← CONSTANT
- AFTER 14 SECONDS $\frac{6}{7}\pi \times 14 = 12\pi$
- VOLUME AFTER 14 s $= 36\pi + 12\pi = 48\pi$
- USING $V = 6\pi x^3 \Rightarrow 48\pi = 6\pi x^3$

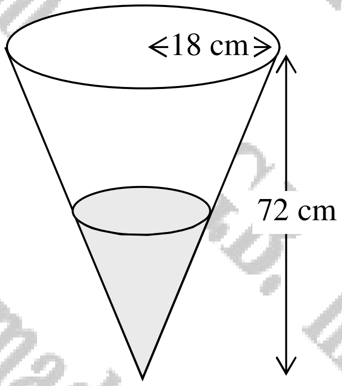
$\Rightarrow 8 = x^3$

$\Rightarrow x = 2$

FINALLY WE HAVE

$$\left.\frac{dA}{dt}\right|_{t=14} = \left.\frac{dA}{dt}\right|_{x=2} = \frac{4\pi}{3\times 2} = \frac{2\pi}{3} \approx 2.09 \text{ cm}^2\text{s}^{-1}$$

Question 49 (*****)



A container is in the shape of hollow inverted right circular cone of height 72 cm and radius 18 cm.

The container, which is initially empty, is placed, with its axis vertical, under a tap where water is flowing in at the constant rate of k cm^3 s^{-1}.

The rate at which the height of the water in the container is rising 12.5 **minutes** after it was placed under the tap is $\frac{2}{75}$ cm s^{-1}.

Determine the value of k.

$\left[\text{volume of a cone of radius } r \text{ and height } h \text{ is given by } \frac{1}{3}\pi r^2 h\right]$

$\boxed{}$, $\boxed{k = 6\pi}$

START BY CONNECTING DERIVATIVES

$\Rightarrow \frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$,

WHERE h IS THE HEIGHT OF THE WATER IN THE CONE AND V ITS VOLUME

$\Rightarrow \frac{dh}{dt} = \frac{dh}{dV} \times k$

WE NEED $V = f(h)$ TO FIND $\frac{dV}{dh}$

$\Rightarrow V = \frac{1}{3}\pi r^2 h$

$\Rightarrow V = \frac{1}{3}\pi\left(\frac{1}{4}h\right)^2 h$

$\Rightarrow V = \frac{1}{48}\pi h^3$

$\Rightarrow \frac{dV}{dh} = \frac{1}{16}\pi h^2$

$\boxed{\frac{dh}{dV} = \frac{16}{\pi h^2}}$

$\frac{r}{h} = \frac{18}{72}$

$r = \frac{1}{4}h$

RETURNING TO THE "MAIN" LINE

$\Rightarrow \frac{dh}{dt} = \frac{16}{\pi h^2} \times k$

$\Rightarrow \frac{dh}{dt} = \frac{16k}{\pi h^2}$

NEXT CONVERT THE TIME INTO HEIGHT h

... CONSTANT RATE OF k cm^3 PER SECOND ...

... $12\frac{1}{2}$ MINUTES = 750 SECONDS

... $V = 750k$ cm^3

... $\frac{1}{48}\pi h^3 = 750k$

... $\pi h^3 = 36000k$

FINALLY WE ARE GIVEN THAT $\left.\frac{dh}{dt}\right|_{t=60} = \frac{2}{75}$

... $\frac{2}{75} = \frac{16k}{\pi h^2}$

... $\pi h^2 = 600k$

SOLVING SIMULTANEOUSLY

$\left.\begin{array}{l} \pi h^3 = 36000k \\ \pi h^2 = 600k \end{array}\right\} \Rightarrow$ DIVIDING $h = 60$

$\Rightarrow \pi h^2 = 600k$

$\Rightarrow \pi \times 60^2 = 600k$

$\Rightarrow \pi \times 60 = 10k$

$\Rightarrow k = 6\pi$

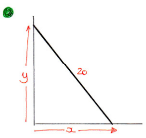
Question 50 (*****)

An extended ladder AB, of length 20 m, has one end A on level horizontal ground and the other end B resting against a vertical wall.

The end A begins to slip away from the wall with constant speed 0.3 ms^{-1}, and the end B slips down the wall.

Determine the speed of the end B, when B has reached a height of 12 m above the ground.

$\boxed{}$, $\boxed{0.4\text{ ms}^{-1}}$



$\boxed{x^2+y^2=20}$

Diff w.r.t t

$2x\frac{dx}{dt}+2y\frac{dy}{dt}=0$

$y\frac{dy}{dt}=-x\frac{dx}{dt}$

$\frac{dy}{dt}=-\frac{x}{y}\frac{dx}{dt}$

NOW FROM $x^2+y^2=20^2$

$x^2+12^2=20^2$

$x^2+144=400$

$x^2=256$

$x=16$

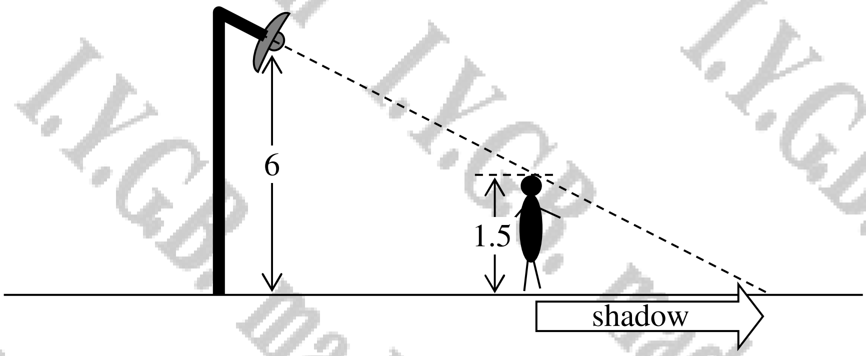
THUS THE REQUIRED RATE IS WHEN $x=16$, $y=12$, $\frac{dx}{dt}=0.3\text{ ms}^{-1}$

$\frac{dy}{dt}=-\frac{16}{12}\times 0.3$

$\frac{dy}{dt}=-\frac{4}{3}\times\frac{3}{10}$

$\frac{dy}{dt}=-0.4\text{ ms}^{-1}$

Question 51 (*****)

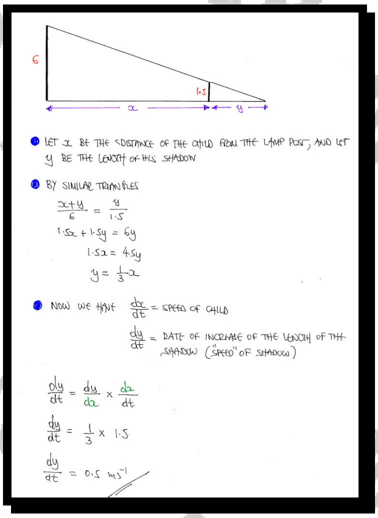


The light bulb in a lamp-post stands 6 m high.

A boy, of height 1.5 m, is walking in a straight line away from the lamp-post at constant speed of 1.5 ms^{-1}.

Determine the rate at which the length of its shadow is increasing.

$\boxed{0.5 \text{ ms}^{-1}}$



- LET x BE THE DISTANCE OF THE CHILD FROM THE LAMP POST, AND LET y BE THE LENGTH OF HIS SHADOW
- BY SIMILAR TRIANGLES

$$\frac{x+y}{6} = \frac{y}{1.5}$$
$$1.5x + 1.5y = 6y$$
$$1.5x = 4.5y$$
$$y = \frac{1}{3}x$$

- NOW WE HAVE $\frac{dx}{dt}$ = SPEED OF CHILD

  $\frac{dy}{dt}$ = RATE OF INCREASE OF THE LENGTH OF THE SHADOW ("SPEED" OF SHADOW)

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$
$$\frac{dy}{dt} = \frac{1}{3} \times 1.5$$
$$\frac{dy}{dt} = 0.5 \text{ ms}^{-1}$$

Question 52 (*****)

The variables x, y and w are related by the equations

$$y = xy + 1 \qquad \text{and} \qquad w = x^3 + wx \, .$$

At a certain instant the rate of change of y with respect to t is increasing at the constant rate of 2, in suitable units.

At the same instant the rate of change of w with respect to t is decreasing at the constant rate of 8, also in suitable units.

Determine the value of w at that instant.

$\boxed{}$, $\boxed{x = 2}$

- WE ARE GIVEN TWO RELATIONSHIPS

$$y = xy + 1 \qquad \& \qquad w = x^3 + wx$$

- REQUIRED TO FIND THE VALUE OF W AT THE INSTANT

$$\frac{dw}{dt} = -8 \ \text{(DECREASING)} \ , \quad \frac{dy}{dt} = +2 \ \text{(INCREASING)}$$

- START BY REARRANGING THE RELATIONSHIPS & DIFFERENTIATING

$$\Rightarrow y - xy = 1$$
$$\Rightarrow y(1-x) = 1$$
$$\Rightarrow y = \frac{1}{1-x}$$
$$\Rightarrow \frac{dy}{dx} = \frac{1}{(1-x)^2}$$

$$\Rightarrow w - wx = x^3$$
$$\Rightarrow w(1-x) = x^3$$
$$\Rightarrow w = \frac{x^3}{1-x}$$
$$\Rightarrow \frac{dw}{dx} = \frac{(1-x)(3x^2) - x^3(-1)}{(1-x)^2}$$
$$\Rightarrow \frac{dw}{dx} = \frac{3x^2 - 3x^3 + x^3}{(1-x)^2}$$

- FORMING A DIFFERENTIAL EQUATION

$$\Rightarrow \frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dw} \times \frac{dw}{dt}$$
$$\Rightarrow 2 = \frac{1}{\cancel{(1-x)^2}} \times \frac{\cancel{(1-x)^2}}{3x^2 - 2x^3} \times (-8)$$
$$\Rightarrow 2 = \frac{-8}{3x^2 - 2x^3}$$
$$\Rightarrow 1 = \frac{-4}{3x^2 - 2x^3}$$
$$\Rightarrow 3x^2 - 2x^3 = -4$$
$$\Rightarrow 0 = 2x^3 - 3x^2 - 4$$

- BY INSPECTION $x=2$ IS A SOLUTION $(16 - 12 - 4 = 0)$

$$\Rightarrow 2x^2(x-2) + x(x-2) + 2(x-2) = 0$$
$$\Rightarrow (x-2)(2x^2 + x + 2) = 0$$

$$b^2 - 4ac = 1^2 - 4 \times 2 \times 2 < 0$$

$$\Rightarrow x = 2 \text{ IS THE ONLY SOLUTION}$$

- FINALLY USING $W = x^3 + Wx$

$$W = 8 + 2W$$
$$W = -8$$

Question 53 (*****)

Liquid is pouring into a container which initially contains 8.1 litres of liquid.

When the height of the liquid in the container is h cm, the volume of the liquid, V cm^3, is given by

$$V = 36h^2.$$

The rate at which the water is pouring into the container is $2t$ cm^{3}s^{-1}, where t s is the time since the liquid started pouring in.

Determine the rate at which the height of the liquid in the container is rising 2 minutes after the liquid started pouring in.

$\left[1 \text{ litre} = 1000\text{cm}^3 \right]$

$\boxed{}$, $\boxed{\frac{2}{15} = 0.133 \text{ cm s}^{-1}}$

STARTING WITH $\frac{dV}{dt} = 2t$, NOTING $8.1\ell = 8100 \text{ cm}^3$ & 2 MINUTES = 120 s

$$\Rightarrow dV = 2t\,dt$$
$$\Rightarrow \int_{V=8100}^{V} 1\,dV = \int_{t=0}^{t=120} 2t\,dt$$
$$\Rightarrow \left[V \right]_{8100}^{V} = \left[t^2 \right]_0^{120}$$
$$\Rightarrow V - 8100 = 120^2 - 0$$
$$\Rightarrow V = 14400 + 8100$$
$$\Rightarrow V = 22500$$

NEXT WE HAVE

$$\Rightarrow \frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$$
$$\Rightarrow \frac{dh}{dt} = \frac{1}{72h} \times 2t$$
$$\Rightarrow \boxed{\frac{dh}{dt} = \frac{t}{36h}}$$

$V = 36h^2$

$\frac{dV}{dh} = 72h$

$\frac{dh}{dV} = \frac{1}{72h}$

WHEN $V = 22500$, $V = 36h^2$

$22500 = 36h^2$

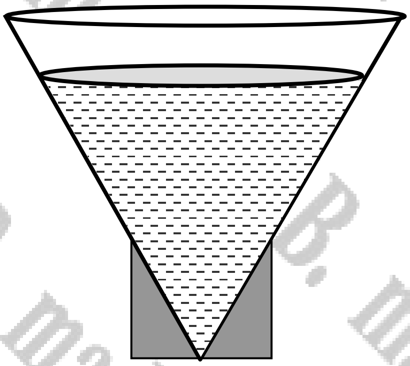
$h^2 = 625$

$h = 25$

FINALLY WE HAVE

$$\left. \frac{dh}{dt} \right|_{\substack{t=120 \\ h=25}} = \frac{120}{36 \times 25} = \frac{2}{15} \approx 0.133 \text{ cm s}^{-1}$$

Question 54 (*****)



A container is in the shape of a hollow inverted right circular cone, whose ratio of its base radius to its height is $\pi : 1$.

The container is initially empty when water is begins to flow in at the constant rate k.

At time t, the area of the circular surface of the water in the cone is A.

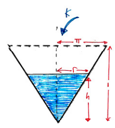
Show that at time $t = T$, the rate at which A is changing is

$$2\pi \sqrt[3]{f(k,T)},$$

where $f(k,T)$ is an expression to be found.

$\boxed{}$, $\boxed{f(k,T) = \frac{k^2}{3T}}$

START WITH A DIAGRAM TO OBTAIN THE VOLUME OF THE WATER IN THE CONTAINER



$\frac{r}{h} = \frac{\pi}{1}$ (SIMILAR TRIANGLES)

$r = \pi h$

$\Rightarrow V = \frac{1}{3}\pi r^2 h$

$\Rightarrow V = \frac{1}{3}\pi r^2 \left(\frac{r}{\pi}\right)$

$\Rightarrow V = \frac{1}{3} r^3$

NOW BY THE CHAIN RULE WE CONNECT DERIVATIVES

$\Rightarrow \frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dV} \times \frac{dV}{dt}$

$\Rightarrow \frac{dA}{dt} = 2\pi r \times \frac{1}{r^2} \times k$

$\Rightarrow \frac{dA}{dt} = \frac{2\pi k}{r}$

- $A = \pi r^2$
  $\frac{dA}{dr} = 2\pi r$
- $V = \frac{1}{3} r^3$
  $\frac{dV}{dr} = r^2$
  $\frac{dr}{dV} = \frac{1}{r^2}$

NEXT WE NEED TO RELATE THE TIME TO THE RADIUS r

$\frac{dV}{dt} = k \Rightarrow V = kT$

$\Rightarrow \frac{1}{3} r^3 = kT$

$\Rightarrow r^3 = 3kT$

$\Rightarrow r = (3kT)^{\frac{1}{3}}$

FINALLY WE OBTAIN

$$\left.\frac{dA}{dt}\right|_{t=T} = \left.\frac{dA}{dt}\right|_{r=(3kT)^{\frac{1}{3}}} = \frac{2\pi k}{(3kT)^{\frac{1}{3}}} = 2\pi \frac{(k^3)^{\frac{1}{3}}}{(3kT)^{\frac{1}{3}}}$$

$$= 2\pi \left(\frac{k^3}{3kT}\right)^{\frac{1}{3}} = 2\pi \sqrt[3]{\frac{k^2}{3T}}$$

Question 55 (*****)

Fine magnetised iron fillings are falling onto a horizontal surface forming a heap in the shape of a right circular cone of height $7x$ cm and radius x cm.

The area of the curved surface of the conical heap is increasing at the constant rate of $k\ \text{cm}^2\text{s}^{-1}$, $k > 0$.

Determine the value of k, given further that when $x = 5$ the volume of the heap is increasing at the rate of $24.5\ \text{cm}^3\text{s}^{-1}$.

You may assume that the volume V and curved surface area A of a right circular cone of radius r and height h are given by

$$V = \frac{1}{3}\pi r^2 h \qquad \text{and} \qquad A = \pi r\sqrt{r^2 + h^2}.$$

$\boxed{}$, $\boxed{k = 7\sqrt{2}}$

- START WITH SOME FORMULAE AUXILIARIES

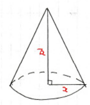
$V = \frac{1}{3}\pi r^2 h$ $\qquad$ $A = \pi r\sqrt{r^2+h^2}$

$V = \frac{1}{3}\pi x^2(7x)$ $\qquad$ $A = \pi x\sqrt{x^2+(7x)^2}$

$V = \frac{7}{3}\pi x^3$ $\qquad$ $A = \pi x\sqrt{50x^2}$

$A = \pi x^2 \times 5\sqrt{2}$

$A = 5\sqrt{2}\pi x^2$



- DIFFERENTIATE THESE TWO EXPRESSIONS WITH RESPECT TO x

$\frac{dV}{dx} = 7\pi x^2$ $\qquad$ $\frac{dA}{dx} = 10\sqrt{2}\pi x$

- NOW FORMING AN EXPRESSION CONNECTING THE DIFFERENT RATES

$\Rightarrow \frac{dV}{dt} = \frac{dV}{dA} \times \frac{dA}{dt}$

$\Rightarrow \frac{dV}{dt} = \frac{dV}{dA} \times k$

$\Rightarrow \frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dA} \times k$

$\Rightarrow \frac{dV}{dt} = (7\cancel{\pi x}x) \times \left(\frac{1}{10\sqrt{2}\cancel{\pi x}}\right) \times k$

$\Rightarrow \frac{dV}{dt} = \frac{7kx}{10\sqrt{2}}$

- FINALLY WHEN $x = 5$, $\frac{dV}{dt} = 24.5$

$\Rightarrow 24.5 = \frac{7k \times 5}{10\sqrt{2}}$

$\Rightarrow 245\sqrt{2} = 7k \times 5$ $\quad \div 5$

$\Rightarrow 49\sqrt{2} = 7k$

$\Rightarrow k = 7\sqrt{2}$

Question 56 (*****)

The point P lies on the curve given parametrically as

$$x = t^2, \quad y = t^2 - t, \quad t \in \mathbb{R}.$$

The tangent to the curve at P meets the y axis at the point A and the straight line with equation $y = x$ at the point B.

P is moving along the curve so that its x coordinate is increasing at the constant rate of 15 units of distance per unit time.

Determine the rate at which the area of the triangle OAB is increasing at the instant when the coordinates of P are $(36,30)$.

$\boxed{\text{SP-C}}$, $\boxed{45}$

START BY FINDING THE EQUATION OF THE TANGENT AT A GENERAL POINT P ON THE CURVE, IE WHERE $t=p$, SO $P(p^2, p^2-p)$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t-1}{2t}$$

$$\left.\frac{dy}{dx}\right|_{t=p} = \frac{2p-1}{2p}$$

EQUATION OF THE TANGENT AT P IS GIVEN BY

$$y - (p^2-p) = \frac{2p-1}{2p}(x - p^2)$$

WHEN $x=0$

$$\Rightarrow y - p^2 + p = \frac{2p-1}{2p}(-p^2)$$

$$\Rightarrow y - p^2 + p = \frac{2p-1}{2}(-p)$$

$$\Rightarrow y = p^2 - p - p^2 + \tfrac{1}{2}p$$

$$\Rightarrow y = -\tfrac{1}{2}p \qquad \therefore A(0, -\tfrac{1}{2}p)$$

WHEN $y=x$

$$\Rightarrow x - p^2 + p = \frac{2p-1}{2p}(x - p^2)$$

$$\Rightarrow 2px - 2p^3 + 2p^2 = (2p-1)(x-p^2)$$

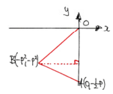
$$\Rightarrow 2px - 2p^3 + 2p^2 = (2p-1)x - p^2(2p-1)$$

$$\Rightarrow p^2(2p-1) - 2p^3 + 2p^2 = (2p-1)x - 2px$$

$$\Rightarrow \cancel{2p^3} - p^2 - \cancel{2p^3} + 2p^2 = \cancel{2px} - x - \cancel{2px}$$

$$\Rightarrow x = -p^2 \qquad \therefore B(-p^2, -p^2)$$

QUICK SKETCH, TAKING $p>0$ WITHOUT LOSS OF GENERALITY



AREA OF THE TRIANGLE IS

$$A(p) = \tfrac{1}{2}|-p^2||-\tfrac{1}{2}p|$$

$$A(p) = \tfrac{1}{4}p^3$$

NOW $\boxed{P(p^2, p^2-1) \text{ I.E } x = p^2 \;\&\; \frac{dx}{dT} = 20 \quad (T = \text{TIME})}$

SO WE HAVE

$$\frac{dA}{dT} = \frac{dA}{dp} \times \frac{dp}{dx} \times \frac{dx}{dT}$$

$$\frac{dA}{dT} = \left(\tfrac{3}{4}p^2\right)\left(\frac{1}{2p}\right) \times 20$$

$$\frac{dA}{dT} = \frac{15}{2}p$$

$$\left.\frac{dA}{dT}\right|_{(36,30)} = \left.\frac{dA}{dT}\right|_{p=6} = \frac{15}{2} \times 6 = 45 \text{ UNITS}^2/\text{UNIT TIME}$$