PARAMETRIC EQUATIONS

EXAM QUESTIONS

Question 1 (**)

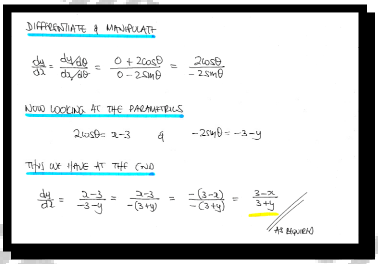
A curve is given parametrically by

$$x = 3 + 2\cos\theta, \quad y = -3 + 2\sin\theta, \quad 0 \le \theta < 2\pi .$$

Show clearly that

$$\frac{dy}{dx} = \frac{3-x}{3+y}.$$

[ ] , proof



Question 2 (**)

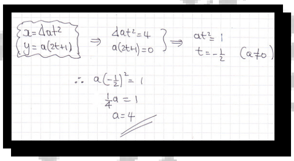
A curve is defined by the following parametric equations

$$x = 4at^2, \quad y = a(2t+1), \quad t \in \mathbb{R},$$

where a is non zero constant.

Given that the curve passes through the point $A(4,0)$, find the value of a.

$\boxed{a = 4}$



Question 3 **(**)**

A curve is defined by the parametric equations

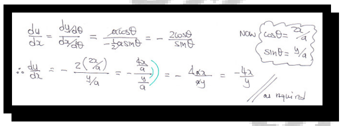
$$x = \frac{1}{2}a\cos\theta, \quad y = a\sin\theta\,, \quad 0 \le \theta < 2\pi\,,$$

where a is a positive constant.

Show **clearly** that

$$\frac{dy}{dx} = -\frac{4x}{y}\,.$$

proof



Question 4 **(**)**

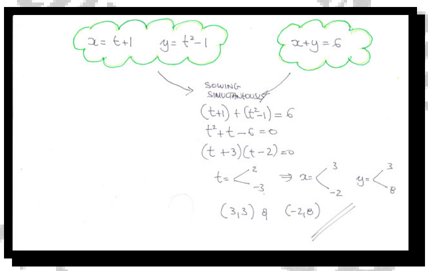
A curve C is given by the parametric equations

$$x = t+1, \quad y = t^2 - 1, \quad t \in \mathbb{R}\,.$$

Determine the coordinates of the points of intersection between C and the straight line with equation

$$x + y = 6\,.$$

$(3,3)$ & $(-2,8)$



Question 5 **(**+)**

A curve is given parametrically by the equations

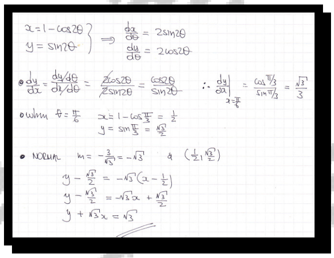
$$x = 1 - \cos 2\theta, \quad y = \sin 2\theta, \quad 0 \le \theta < 2\pi.$$

The point P lies on this curve, and the value of θ at P is $\frac{\pi}{6}$.

Show that an equation of the normal to the curve at P is given by

$$y + \sqrt{3}x = \sqrt{3}.$$

proof



Question 6 **(**+)**

A curve is defined by the parametric equations

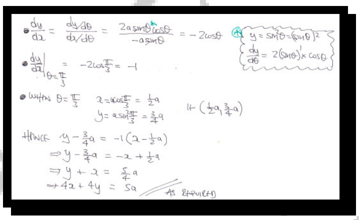
$$x = a\cos\theta, \quad y = a\sin^2\theta, \quad 0 \le \theta < 2\pi,$$

where a is a positive constant.

Show that the equation of the tangent to the curve at the point where $\theta = \frac{\pi}{3}$ is

$$4x + 4y = 5a.$$

proof



Question 7 (**+)

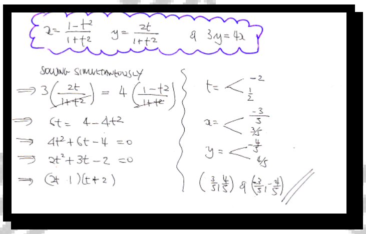
A curve C is given by the parametric equations

$$x = \frac{1-t^2}{1+t^2}, \quad y = \frac{2t}{1+t^2}, \quad t \in \mathbb{R}.$$

Determine the coordinates of the points of intersection between C and the straight line with equation

$$3y = 4x.$$

$\left(-\frac{3}{5}, -\frac{4}{5}\right)$ & $\left(\frac{3}{5}, \frac{4}{5}\right)$



Question 8 (**+)

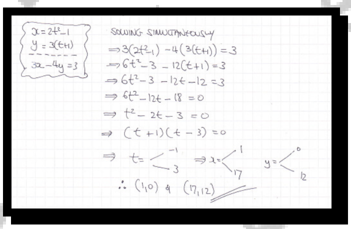
A curve C is given by the parametric equations

$$x = 2t^2 - 1, \quad y = 3(t+1), \quad t \in \mathbb{R}.$$

Determine the coordinates of the points of intersection between C and the straight line with equation

$$3x - 4y = 3.$$

$(17,12)$ & $(1,0)$



Question 9 (**+)

A curve is given parametrically by the equations

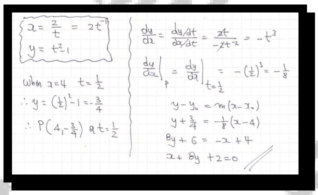
$$x = \frac{2}{t}, \quad y = t^2 - 1, \quad t \in \mathbb{R}, \; t \neq 0.$$

The point $P(4, y)$ lies on this curve.

Show that an equation of the tangent to the curve at P is given by

$$x + 8y + 2 = 0.$$

proof



Question 10 (**+)

A curve C is given parametrically by

$$x = 2t + 1, \quad y = \frac{3}{2t}, \quad t \in \mathbb{R}, \quad t \neq 0.$$

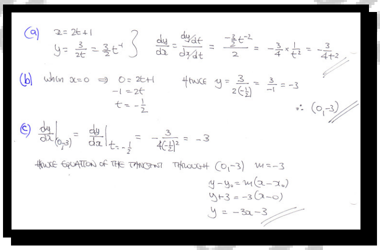
a) Find a simplified expression for $\frac{dy}{dx}$ in terms of t.

The point P is the point where C crosses the y axis.

b) Determine the coordinates of P.

c) Find an equation of the tangent to C at P.

$\boxed{\frac{dy}{dx} = -\frac{3}{4t^2}}$, $\boxed{P(0,-3)}$, $\boxed{y = -3x - 3}$


(a) $x = 2t+1$, $y = \frac{3}{2t} = \frac{3}{2}t^{-1}$ $\Big\}$ $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-\frac{3}{2}t^{-2}}{2} = -\frac{3}{4} \times \frac{1}{t^2} = -\frac{3}{4t^2}$

(b) when $x = 0 \Rightarrow 0 = 2t+1$, $-1 = 2t$, $t = -\frac{1}{2}$ HENCE $y = \frac{3}{2(-\frac{1}{2})} = \frac{3}{-1} = -3$ $\therefore (0,-3)$

(c) $\left.\frac{dy}{dx}\right|_{(0,-3)} = \left.\frac{dy}{dx}\right|_{t=-\frac{1}{2}} = -\frac{3}{4(-\frac{1}{2})^2} = -3$

HENCE EQUATION OF THE TANGENT THROUGH $(0,-3)$ $m = -3$

$y - y_0 = m(x - x_0)$

$y + 3 = -3(x - 0)$

$y = -3x - 3$

Question 11 (**+)

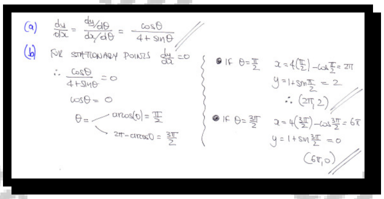
A curve known as a cycloid is given by the parametric equations

$$x = 4\theta - \cos\theta\ ,\quad y = 1 + \sin\theta\ ,\quad 0 \le \theta \le 2\pi\ .$$

a) Find an expression for $\frac{dy}{dx}$, in terms of θ.

b) Determine the exact coordinates of the stationary points of the curve.

$\boxed{\frac{dy}{dx} = \frac{\cos\theta}{4+\sin\theta}}$, $\boxed{(2\pi, 2),\ (6\pi, 0)}$



Question 12 (***)

A curve is given parametrically by

$$x = 4t - 1, \quad y = \frac{5}{2t} + 10, \quad t \in \mathbb{R}, \ t \neq 0.$$

The curve crosses the x axis at the point A.

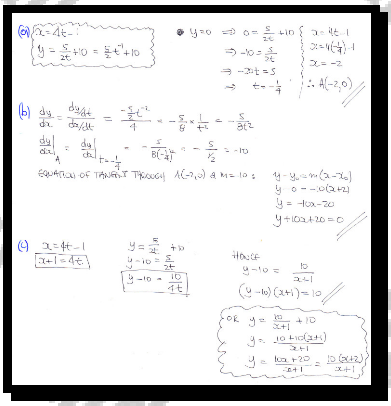
a) Find the coordinates of A.

b) Show that an equation of the tangent to the curve at A is

$$10x + y + 20 = 0.$$

c) Determine a Cartesian equation for the curve.

$\boxed{(-2,0)}$, $\boxed{(x+1)(y-10)=10 \quad \text{or} \quad y = \frac{10(x+2)}{x+1}}$



Question 13 **(***)**

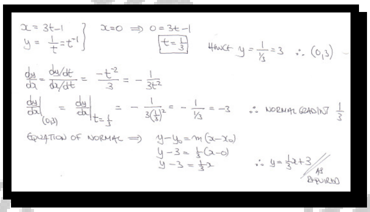
A curve C is given parametrically by

$$x = 3t - 1, \quad y = \frac{1}{t}, \quad t \in \mathbb{R}, \quad t \neq 0.$$

Show that an equation of the normal to C at the point where C crosses the y axis is

$$y = \frac{1}{3}x + 3.$$

proof



Question 14 (***)

A curve C is given by the parametric equations

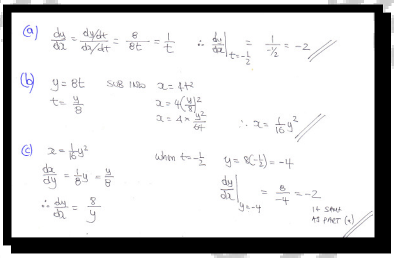
$$x = 4t^2, \quad y = 8t, \quad t \in \mathbb{R}.$$

a) Find the gradient at the point on the curve where $t = -\frac{1}{2}$.

b) Determine a Cartesian equation for C, in the form $x = f(y)$.

c) Use the Cartesian form of C to find $\frac{dy}{dx}$ in terms of y, and use it to verify that the answer obtained in part (a) is correct.

$\boxed{\left.\frac{dy}{dx}\right|_{t=-\frac{1}{2}} = -2}$, $\boxed{x = \frac{1}{16}y^2}$, $\boxed{\frac{dy}{dx} = \frac{8}{y}}$



Question 15 (***)

A curve C is given parametrically by the equations

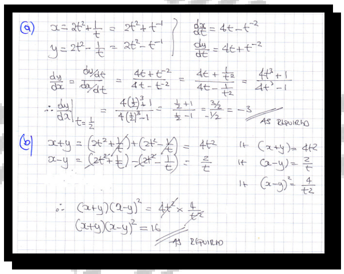
$$x = 2t^2 + \frac{1}{t}, \quad y = 2t^2 - \frac{1}{t}, \quad t \in \mathbb{R}, \ t \neq 0.$$

a) Show that at the point on C where $t = \frac{1}{2}$, the gradient is -3.

b) By considering $(x+y)$ and $(x-y)$, show that a Cartesian equation of C is

$$(x+y)(x-y)^2 = 16.$$

[ ], proof



Question 16 (***)

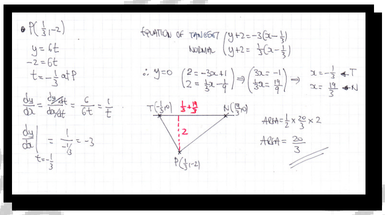
The point $P\left(\frac{1}{3},-2\right)$ lies on the curve with parametric equations

$$x = 3t^2, \quad y = 6t, \quad t \in \mathbb{R} .$$

The tangent and the normal to curve at P meet the x axis at the points T and N, respectively.

Determine the area of the triangle PTN.

$\boxed{\frac{20}{3}}$



Question 17 (***)

A curve C is given parametrically by the equations

$$x = 4t + \frac{1}{t}, \quad y = \frac{3}{2t}, \quad t \in \mathbb{R}, \quad t \neq 0.$$

The point $A(5,6)$ lies on C.

Show clearly that …

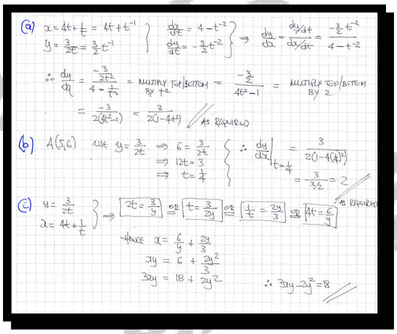
a) … $\dfrac{dy}{dx} = \dfrac{3}{2(1-4t^2)}$.

b) … the gradient at A is 2.

c) … a Cartesian equation of C is

$$3xy - 2y^2 = 18.$$

proof



Question 18 (***)

A curve C is given parametrically by the equations

$$x = t^2 - 8t + 12, \quad y = t - 4, \quad t \in \mathbb{R}.$$

a) Find the coordinates of the points where C crosses the coordinate axes.

The point $P(-3,1)$ lies on C.

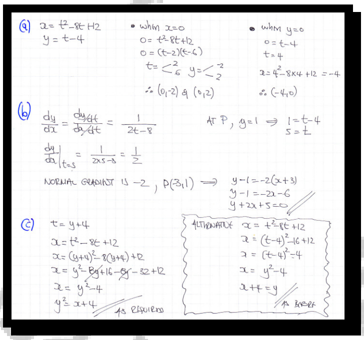
b) Show that the equation of the normal to C at P is

$$y + 2x + 5 = 0.$$

c) Show that a Cartesian equation of C is

$$y^2 = x + 4.$$

C4C, $\boxed{(-4,0),\ (0,-2),\ (0,2)}$


(a) $x = t^2 - 8t + 12$, $y = t - 4$

• when $x = 0$: $0 = t^2 - 8t + 12$, $0 = (t-2)(t-6)$, $t = 2, 6$, $y = -2, 2$ ∴ $(0,-2)$ & $(0,2)$

• when $y = 0$: $0 = t - 4$, $t = 4$, $x = 4^2 - 8 \times 4 + 12 = -4$ ∴ $(-4,0)$

(b) $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1}{2t-8}$

AT P, $y = 1 \Rightarrow 1 = t - 4$, $5 = t$

$\left.\frac{dy}{dx}\right|_{t=5} = \frac{1}{2 \times 5 - 8} = \frac{1}{2}$

NORMAL GRADIENT IS -2, $P(-3,1) \Rightarrow y - 1 = -2(x+3)$, $y - 1 = -2x - 6$, $y + 2x + 5 = 0$

(c) $t = y + 4$

$x = t^2 - 8t + 12$

$x = (y+4)^2 - 8(y+4) + 12$

$x = y^2 + 8y + 16 - 8y - 32 + 12$

$x = y^2 - 4$

$y^2 = x + 4$ AS REQUIRED

ALTERNATIVE: $x = t^2 - 8t + 12$, $x = (t-4)^2 - 16 + 12$, $x = (t-4)^2 - 4$, $x = y^2 - 4$, $x + 4 = y^2$ AS BEFORE

Question 19 (***)

A curve C is given parametrically by the equations

$$x = 5 - 3t, \quad y = 2 + \frac{1}{t}, \quad t \in \mathbb{R}, \quad t \neq 0.$$

The point $A(6,-1)$ lies on C.

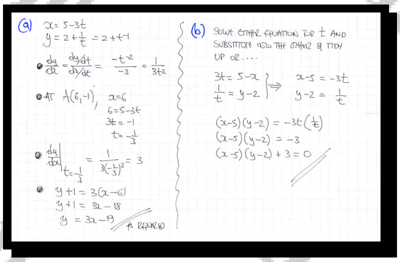
a) Show that the equation of the tangent to C at A is given by

$$y = 3x - 19.$$

b) Show further that a Cartesian equation of C is

$$(x-5)(y-2)+3=0.$$

proof



Question 20 (***)

A curve C is defined by the parametric equations

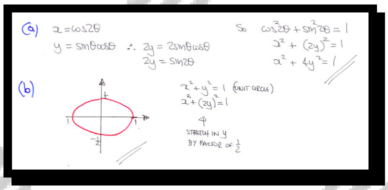
$$x = \cos 2\theta, \quad y = \sin\theta\cos\theta, \quad 0 \le \theta < \pi.$$

a) Show that a Cartesian equation for C is given by

$$x^2 + 4y^2 = 1.$$

b) Sketch the graph of C.

proof



Question 21 (***)

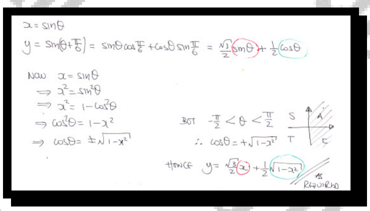
A curve is defined by the parametric equations

$$x = \sin\theta, \quad y = \sin\left(\theta + \frac{\pi}{6}\right), \quad -\frac{\pi}{2} \le \theta < \frac{\pi}{2}.$$

Show that a Cartesian equation of the curve is given by

$$y = \frac{\sqrt{3}}{2}x + \frac{1}{2}\sqrt{1-x^2}.$$

proof



Question 22 (***)

A curve is defined by the parametric equations

$$x = \frac{t+3}{t+1}, \quad y = \frac{2}{t+2}, \quad t \in \mathbb{R},\ t \neq -1,\ t \neq -2.$$

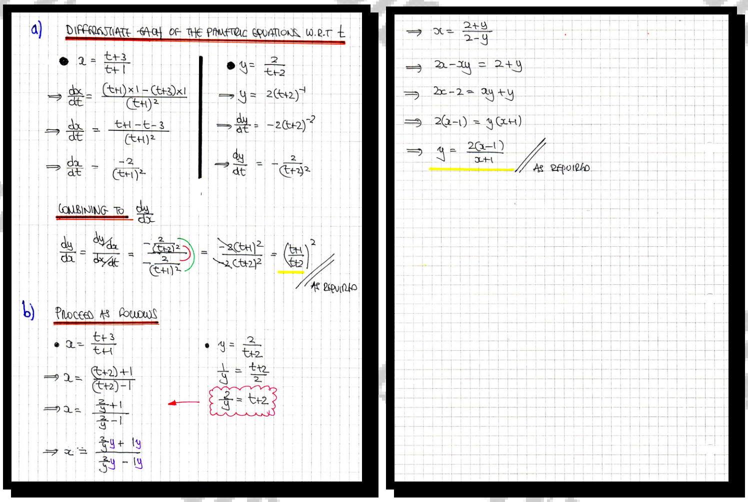
Show, with detailed workings, that …

a) … $\dfrac{dy}{dx} = \left(\dfrac{t+1}{t+2}\right)^2$.

b) … a Cartesian equation for the curve is given by

$$y = \frac{2(x-1)}{x+1}.$$

$\boxed{}$, $\boxed{\text{proof}}$



Question 23 (***)

A curve is defined parametrically by the equations

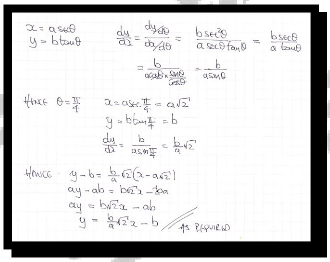
$$x = a\sec\theta\,, \quad y = b\tan\theta\,, \quad 0 < \theta < \frac{\pi}{2},$$

where a and b are positive constants.

Show that an equation of the tangent to the curve at the point where $\theta = \frac{\pi}{4}$ is

$$y = \frac{b}{a}\sqrt{2}x - b\,.$$

proof



Question 24 (***+)

A curve C is defined by the parametric equations

$$x = \cos t, \quad y = \cos 2t \ , \ 0 \le t \le \pi \ .$$

a) Find $\frac{dy}{dx}$ in its simplest form.

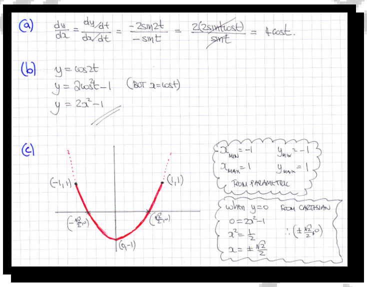
b) Find a Cartesian equation for C.

c) Sketch the graph of C.

The sketch must include

- the coordinates of the endpoints of the graph.
- the coordinates of any points where the graph meets the coordinates axes.

$\boxed{\frac{dy}{dx} = 4\cos t}$, $\boxed{y = 2x^2 - 1}$, $\boxed{(-1,1)(1,1),(0,-1),\left(-\frac{\sqrt{2}}{2},0\right)\left(\frac{\sqrt{2}}{2},0\right)}$



Question 25 (***+)

A curve C is given by the parametric equations

$$x = \frac{3t-2}{t-1}, \quad y = \frac{t^2-2t+2}{t-1}, \quad t \in \mathbb{R}, \quad t \neq 1.$$

a) Show clearly that

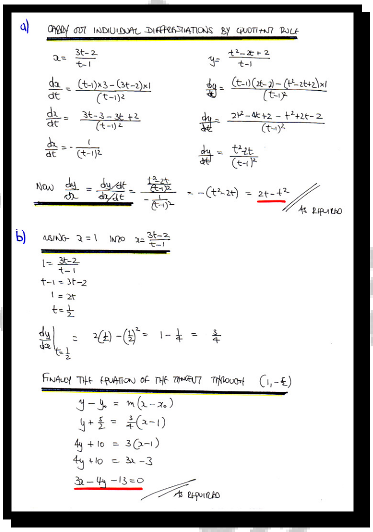
$$\frac{dy}{dx} = 2t - t^2.$$

The point $P\left(1, -\frac{5}{2}\right)$ lies on C.

b) Show that the equation of the tangent to C at the point P is

$$3x - 4y - 13 = 0.$$

[ ], proof


a) CARRY OUT INDIVIDUAL DIFFERENTIATIONS BY QUOTIENT RULE

$x = \frac{3t-2}{t-1}$

$\frac{dx}{dt} = \frac{(t-1)\times 3 - (3t-2)\times 1}{(t-1)^2}$

$\frac{dx}{dt} = \frac{3t-3-3t+2}{(t-1)^2}$

$\frac{dx}{dt} = -\frac{1}{(t-1)^2}$

$y = \frac{t^2-2t+2}{t-1}$

$\frac{dy}{dt} = \frac{(t-1)(2t-2)-(t^2-2t+2)\times 1}{(t-1)^2}$

$\frac{dy}{dt} = \frac{2t^2-4t+2-t^2+2t-2}{(t-1)^2}$

$\frac{dy}{dt} = \frac{t^2-2t}{(t-1)^2}$

NOW $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{t^2-2t}{(t-1)^2}}{-\frac{1}{(t-1)^2}} = -(t^2-2t) = 2t-t^2$ // AS REQUIRED

b) USING $x=1$ INTO $x = \frac{3t-2}{t-1}$

$1 = \frac{3t-2}{t-1}$

$t-1 = 3t-2$

$1 = 2t$

$t = \frac{1}{2}$

$\left.\frac{dy}{dx}\right|_{t=\frac{1}{2}} = 2\left(\frac{1}{2}\right) - \left(\frac{1}{2}\right)^2 = 1 - \frac{1}{4} = \frac{3}{4}$

FINALLY THE EQUATION OF THE TANGENT THROUGH $\left(1, -\frac{5}{2}\right)$

$y - y_0 = m(x - x_0)$

$y + \frac{5}{2} = \frac{3}{4}(x-1)$

$4y + 10 = 3(x-1)$

$4y + 10 = 3x - 3$

$3x - 4y - 13 = 0$ // AS REQUIRED

Question 26 (***+)

The curve C_1 has Cartesian equation

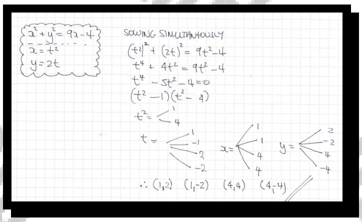
$$x^2 + y^2 = 9x - 4 .$$

The curve C_2 has parametric equations

$$x = t^2, \quad y = 2t \quad , \quad t \in \mathbb{R} .$$

Find the coordinates of the points of intersection of C_1 and C_2.

$$\boxed{(4,4),\ (4,-4),\ (1,2),\ (1,-2)}$$



Question 27 (***+)

A curve has parametric equations

$$x = t^2, \quad y = \frac{6}{t}, \quad t \in \mathbb{R},\ t \neq 0\ .$$

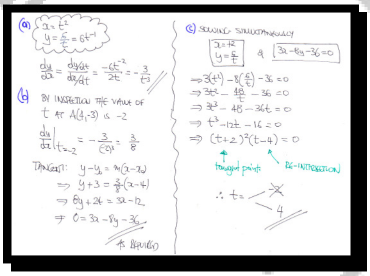
a) Determine a simplified expression for $\frac{dy}{dx}$, in terms of t.

b) Show that an equation of the tangent to the curve at the point $A(4,-3)$ is

$$3x - 8y - 36 = 0\,.$$

c) Find the value of t at the point where the tangent to the curve at A meets the curve again.

$\boxed{\frac{dy}{dx} = -\frac{3}{t^3}}$, $\boxed{t = 4}$



Question 28 (***+)

A curve C is defined by the parametric equations

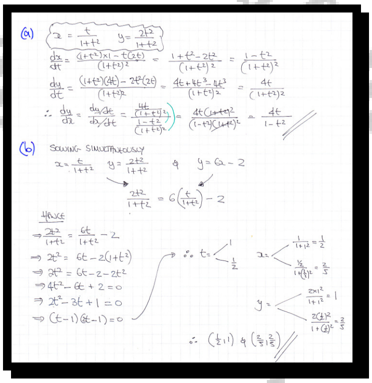
$$x = \frac{t}{1+t^2}, \quad y = \frac{2t^2}{1+t^2}, \quad t \in \mathbb{R}.$$

a) Find a simplified expression for $\frac{dy}{dx}$ in terms of t.

The straight line with equation $y = 6x - 2$ intersects C at the points P and Q.

b) Find the coordinates of P and the coordinates of Q.

$\boxed{}$, $\boxed{\frac{dy}{dx} = \frac{4t}{1-t^2}}$, $\boxed{P\left(\frac{1}{2}, 1\right), \ Q\left(\frac{2}{5}, \frac{2}{5}\right)}$



Question 29 (***+)

A curve C is defined by the parametric equations

$$x = \ln(1+t), \quad y = \ln(1-t) \ , \quad t \in \mathbb{R} \ , \qquad t_1 < t < t_2 \ .$$

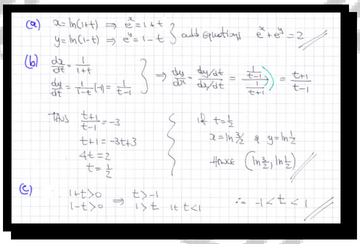
a) Find a Cartesian equation for C.

b) Determine, in terms of natural logarithms, the coordinates of the point on C where the gradient is -3.

The value of t is restricted between t_1 and t_2.

c) Given that the interval between t_1 and t_2 is as large as possible, determine the value of t_1 and the value of t_2.

$\boxed{e^x + e^y = 2}$, $\boxed{\left(\ln\frac{3}{2}, \ln\frac{1}{2}\right)}$, $\boxed{-1 < t < 1}$



Question 30 (***+)

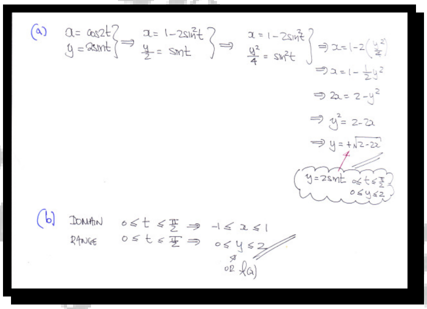
A function relationship is given parametrically by the equations

$$x = \cos 2t, \quad y = 2\sin t, \quad 0 \le t \le \frac{\pi}{2}.$$

a) Find a Cartesian equation for these parametric equations, in the form $y = f(x)$.

b) State the domain and range of this function.

$\boxed{y = \sqrt{2-2x}}$, $\boxed{-1 \le x \le 1}$, $\boxed{0 \le y \le 2}$



Question 31 (***+)

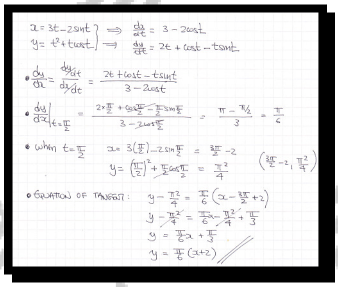
A curve is given parametrically by the equations

$$x = 3t - 2\sin t, \quad y = t^2 + t\cos t \quad , \quad 0 \le t < 2\pi .$$

Show that an equation of the tangent at the point on the curve where $t = \dfrac{\pi}{2}$ is given by

$$y = \frac{\pi}{6}(x+2).$$

proof


$x = 3t - 2\sin t \Rightarrow \frac{dx}{dt} = 3 - 2\cos t$
$y = t^2 + t\cos t \Rightarrow \frac{dy}{dt} = 2t + \cos t - t\sin t$

• $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t + \cos t - t\sin t}{3 - 2\cos t}$

• $\left.\frac{dy}{dx}\right|_{t=\frac{\pi}{2}} = \frac{2\times\frac{\pi}{2} + \cos\frac{\pi}{2} - \frac{\pi}{2}\sin\frac{\pi}{2}}{3 - 2\cos\frac{\pi}{2}} = \frac{\pi - \pi/2}{3} = \frac{\pi}{6}$

• when $t = \frac{\pi}{2}$: $x = 3\left(\frac{\pi}{2}\right) - 2\sin\frac{\pi}{2} = \frac{3\pi}{2} - 2$
$y = \left(\frac{\pi}{2}\right)^2 + \frac{\pi}{2}\cos\frac{\pi}{2} = \frac{\pi^2}{4}$ $\quad \left(\frac{3\pi}{2} - 2, \frac{\pi^2}{4}\right)$

• EQUATION OF TANGENT: $y - \frac{\pi^2}{4} = \frac{\pi}{6}\left(x - \frac{3\pi}{2} + 2\right)$
$y - \frac{\pi^2}{4} = \frac{\pi}{6}x - \frac{\pi^2}{4} + \frac{\pi}{3}$
$y = \frac{\pi}{6}x + \frac{\pi}{3}$
$y = \frac{\pi}{6}(x+2)$

Question 32 (***+)

The point $P(-5,3)$ lies on the curve C with parametric equations

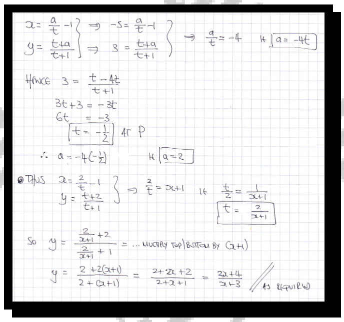
$$x = \frac{a}{t} - 1, \quad y = \frac{t+a}{t+1}, \quad t \in \mathbb{R}, \quad t \neq 0, -1$$

where a is a non zero constant.

Show that a Cartesian equation of C is

$$y = \frac{2x+4}{x+3}.$$

[ ], [proof]



$x = \frac{a}{t} - 1 \Rightarrow -5 = \frac{a}{t} - 1$, $y = \frac{t+a}{t+1} \Rightarrow 3 = \frac{t+a}{t+1}$ $\Rightarrow \frac{a}{t} = -4 \;\; \therefore \; a = -4t$

HENCE $3 = \frac{t - 4t}{t+1}$

$3t + 3 = -3t$

$6t = -3$

$t = -\frac{1}{2}$ AT P

$\therefore \; a = -4\left(-\frac{1}{2}\right) \;\; \therefore \; a = 2$

THUS $x = \frac{2}{t} - 1$, $y = \frac{t+2}{t+1}$ $\Rightarrow \frac{2}{t} = x + 1 \;\; \therefore \; \frac{t}{2} = \frac{1}{x+1}$, $t = \frac{2}{x+1}$

SO $y = \dfrac{\frac{2}{x+1} + 2}{\frac{2}{x+1} + 1} = \ldots$ MULTIPLY TOP/BOTTOM BY $(x+1)$

$y = \frac{2 + 2(x+1)}{2 + (x+1)} = \frac{2 + 2x + 2}{2 + x + 1} = \frac{2x+4}{x+3}$ // AS REQUIRED

Question 33 (***+)

The curve C has parametric equations

$$x = \sin\theta, \ y = 3 - 2\cos 2\theta, \ 0 \le \theta \le \frac{\pi}{2}.$$

a) Express $\frac{dy}{dx}$ in terms of θ.

b) Explain why…

… no point on C has negative gradient.

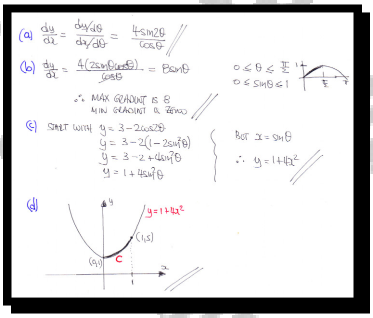
… the maximum gradient on C is 8.

c) Show that C satisfies the Cartesian equation

$$y = 1 + 4x^2.$$

d) Show by means of a single sketch how the graph of $y = 1 + 4x^2$ and the graph of C are related.

$$\boxed{\frac{dy}{dx} = \frac{4\sin 2\theta}{\cos\theta} = 8\sin\theta}$$



Question 34 (***+)

The curve C has parametric equations

$$x = \cos\theta, \quad y = \sin 2\theta, \quad 0 \le \theta < 2\pi.$$

The point P lies on C where $\theta = \frac{\pi}{6}$.

a) Find the gradient at P.

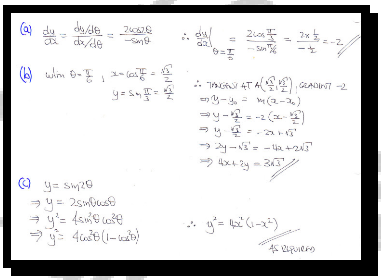
b) Hence show that the equation of the tangent at P is

$$2y + 4x = 3\sqrt{3}.$$

c) Show that a Cartesian equation of C is

$$y^2 = 4x^2\left(1 - x^2\right).$$

$$\left.\frac{dy}{dx}\right|_P = -2$$



(a) $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2\cos 2\theta}{-\sin\theta}$ $\therefore \left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{6}} = \frac{2\cos\frac{\pi}{3}}{-\sin\frac{\pi}{6}} = \frac{2 \times \frac{1}{2}}{-\frac{1}{2}} = -2$

(b) with $\theta = \frac{\pi}{6}$, $x = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$, $y = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$\therefore$ TANGENT AT $A\left(\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right)$, GRADIENT -2

$\Rightarrow y - y_0 = m(x - x_0)$

$\Rightarrow y - \frac{\sqrt{3}}{2} = -2\left(x - \frac{\sqrt{3}}{2}\right)$

$\Rightarrow y - \frac{\sqrt{3}}{2} = -2x + \sqrt{3}$

$\Rightarrow 2y - \sqrt{3} = -4x + 2\sqrt{3}$

$\Rightarrow 4x + 2y = 3\sqrt{3}$

(c) $y = \sin 2\theta$

$\Rightarrow y = 2\sin\theta\cos\theta$

$\Rightarrow y^2 = 4\sin^2\theta\cos^2\theta$

$\Rightarrow y^2 = 4\cos^2\theta(1 - \cos^2\theta)$

$\therefore y^2 = 4x^2(1 - x^2)$

AS REQUIRED

Question 35 **(***+)**

The point $P\left(a,\sqrt{2}\right)$ lies on the curve C with parametric equations

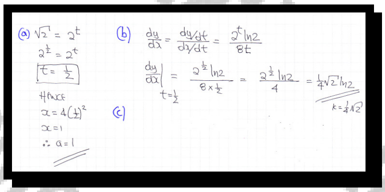
$$x = 4t^2\,, \qquad y = 2^t\,,\ t \in \mathbb{R}$$

where a is a constant.

a) Determine the value of a.

b) Show that the gradient at P is $k\ln 2$, where k is a constant to be found.

$\boxed{a=1}$, $\boxed{\frac{1}{4}\sqrt{2}\ln 2}$



Question 36 (***+)

A curve C is defined parametrically by

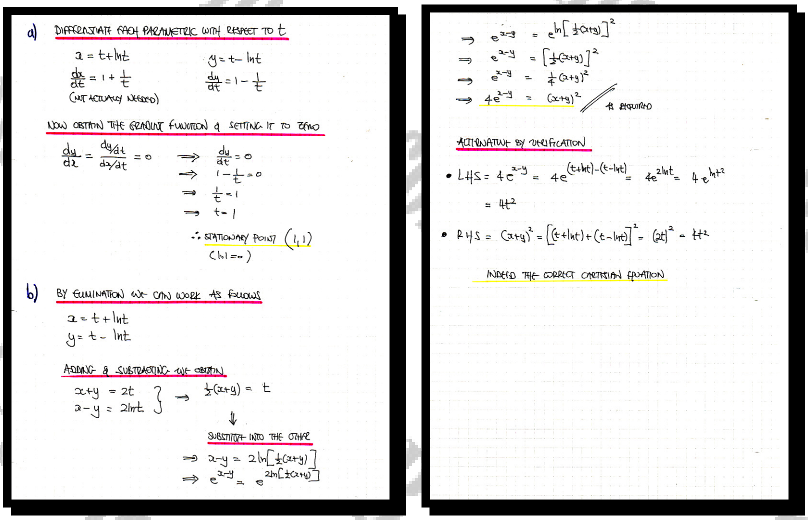
$$x = t + \ln t, \quad y = t - \ln t, \quad t > 0.$$

a) Find the coordinates of the turning point of C.

b) Show that a Cartesian equation for C is

$$4e^{x-y} = (x+y)^2.$$

$\boxed{}$, $\boxed{(1,1)}$


a) DIFFERENTIATE EACH PARAMETRIC WITH RESPECT TO t

$x = t + \ln t$ $\quad y = t - \ln t$

$\frac{dx}{dt} = 1 + \frac{1}{t}$ (NOT ACTUALLY NEEDED) $\quad \frac{dy}{dt} = 1 - \frac{1}{t}$

NOW OBTAIN THE GRADIENT FUNCTION & SETTING IT TO ZERO

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = 0 \Rightarrow \frac{dy}{dt} = 0 \Rightarrow 1 - \frac{1}{t} = 0 \Rightarrow \frac{1}{t} = 1 \Rightarrow t = 1$

$\therefore$ STATIONARY POINT $(1,1)$ $(\ln 1 = 0)$

b) BY ELIMINATION WE CAN WORK AS FOLLOWS

$x = t + \ln t$

$y = t - \ln t$

ADDING & SUBTRACTING WE OBTAIN

$x + y = 2t$, $x - y = 2\ln t$ $\Rightarrow \frac{1}{2}(x+y) = t$

SUBSTITUTE INTO THE OTHER

$\Rightarrow x - y = 2\ln\left[\frac{1}{2}(x+y)\right]$

$\Rightarrow e^{x-y} = e^{2\ln\left[\frac{1}{2}(x+y)\right]}$

$\Rightarrow e^{x-y} = e^{\ln\left[\frac{1}{2}(x+y)\right]^2}$

$\Rightarrow e^{x-y} = \left[\frac{1}{2}(x+y)\right]^2$

$\Rightarrow e^{x-y} = \frac{1}{4}(x+y)^2$

$\Rightarrow 4e^{x-y} = (x+y)^2$ // AS REQUIRED

ALTERNATIVE BY VERIFICATION

• LHS $= 4e^{x-y} = 4e^{(t+\ln t)-(t-\ln t)} = 4e^{2\ln t} = 4e^{\ln t^2} = 4t^2$

• RHS $= (x+y)^2 = \left[(t+\ln t)+(t-\ln t)\right]^2 = (2t)^2 = 4t^2$

INDEED THE CORRECT CARTESIAN EQUATION

Question 37 **(***+)**

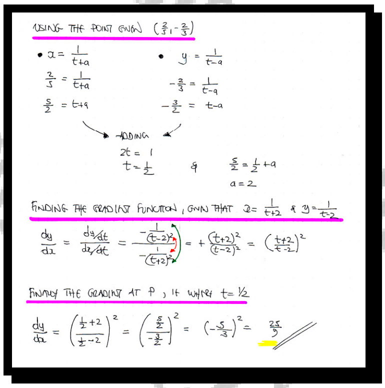
The point $P\left(\frac{2}{5},-\frac{2}{3}\right)$ lies on the curve C with parametric equations

$$x=\frac{1}{t+a}, \quad y=\frac{1}{t-a}, \quad t\in\mathbb{R}, \quad t\neq\pm a ,$$

where a is a non zero constant.

Show that the gradient at P is $\frac{25}{9}$.

[ ] , [proof]



Question 38 (***+)

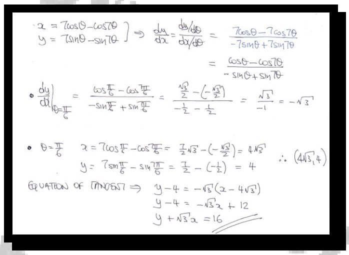
A curve C is given by the parametric equations

$$x = 7\cos\theta - \cos 7\theta, \quad y = 7\sin\theta - \sin 7\theta, \quad 0 \le \theta < 2\pi .$$

Show that the equation of the tangent to C at the point where $\theta = \frac{\pi}{6}$ is

$$y + \sqrt{3}x = 16 .$$

proof



Question 39 (***+)

A curve C is given parametrically by

$$x = \frac{1}{t}, \quad y = t^2, \quad t \in \mathbb{R}, \quad t \neq 0.$$

The point P lies on C at the point where $t = 1$.

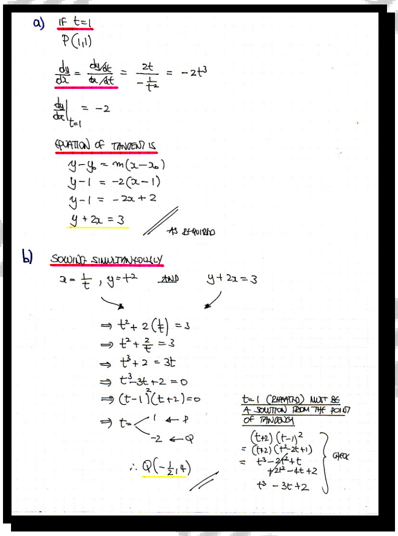
a) Show that an equation of the tangent to C at P is

$$y + 2x = 3.$$

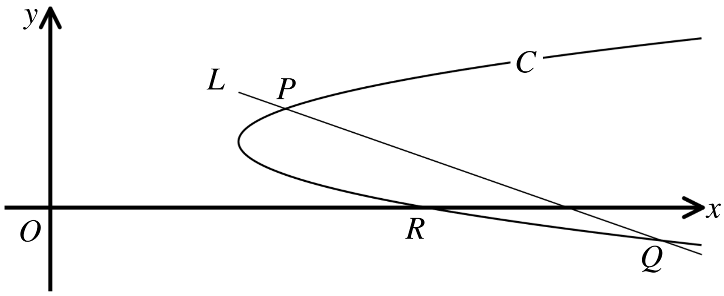
The tangent to C at P meets the curve again at the point Q.

b) Determine the coordinates of Q.

$\boxed{}$, $\boxed{Q\left(-\frac{1}{2}, 4\right)}$



Question 40 (***+)



The figure above shows the curve C with parametric equations

$$x = t^2 + 4, \quad y = 2t + 4 \ , \quad t \in \mathbb{R} .$$

The curve crosses the x axis at the point R.

a) Find the coordinates of R.

The point $P(5,6)$ lies on C. The straight line L is a normal to C at P.

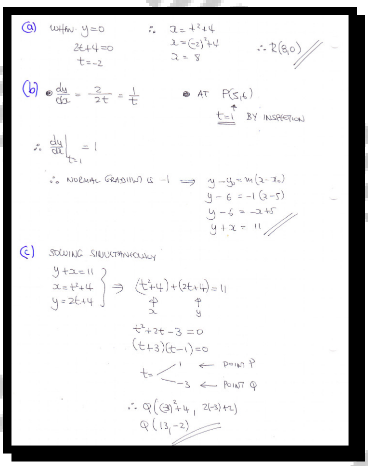
b) Show that an equation of L is

$$x + y = 11 .$$

The normal L meets C again, at the point Q.

c) Find the coordinates of Q.

$\boxed{R(8,0)}$, $\boxed{Q(13,-2)}$



Question 41 (***+)

A curve is given parametrically by

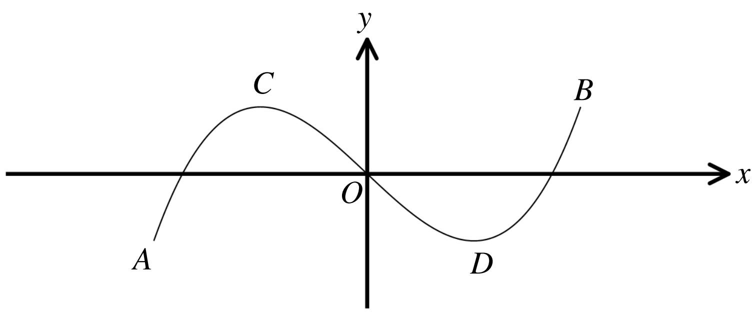
$$x = \cos t\,,\quad y = \cos 3t\quad,\quad 0 \le t < 2\pi\,.$$

a) By writing $\cos 3t$ as $\cos(2t+t)$, prove the trigonometric identity

$$\cos 3t \equiv 4\cos^3 t - 3\cos t\,.$$

b) Hence state a Cartesian equation for the curve.

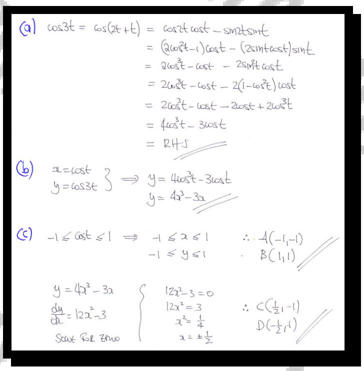
The figure below shows a sketch of the curve.



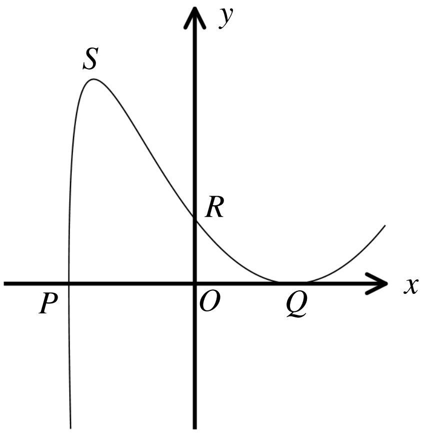
The points A and B are the endpoints of the graph and the points C and D are stationary points.

c) Determine the coordinates of A, B, C and D.

$\boxed{y = 4x^3 - 3x}$, $\boxed{A(-1,-1)}$, $\boxed{B(1,1)}$, $\boxed{C\left(-\frac{1}{2},1\right)}$, $\boxed{C\left(\frac{1}{2},-1\right)}$



Question 42 (***+)



The figure above shows part of the curve with parametric equations

$$x = t^2 - 9, \quad y = t(4-t)^2, \quad t \in \mathbb{R}.$$

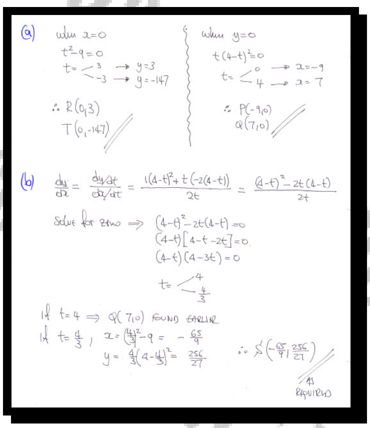
The curve meets the x axis at the points P and Q, and the y axis at the points R and T. The point T is not shown in the figure.

a) Find the coordinates of the points P, Q, R and T.

The point S is a stationary point of the curve.

b) Show that the coordinates of S are $\left(-\frac{65}{9}, \frac{256}{27}\right)$.

$P(-9,0),\ Q(7,0),\ R(0,3),\ T(0,-147)$



Question 43 **(***+)**

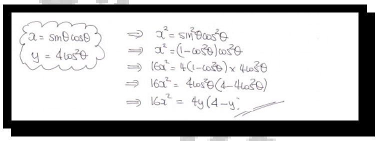
A parametric relationship is given by

$$x = \sin\theta\cos\theta, \ \ y = 4\cos^2\theta, \ \ 0 \le \theta < 2\pi .$$

Show that a Cartesian equation for this relationship is

$$16x^2 = y(4-y).$$

proof



$x = \sin\theta\cos\theta$, $y = 4\cos^2\theta$

$\Rightarrow x^2 = \sin^2\theta\cos^2\theta$

$\Rightarrow x^2 = (1-\cos^2\theta)\cos^2\theta$

$\Rightarrow 16x^2 = 4(1-\cos^2\theta)\times 4\cos^2\theta$

$\Rightarrow 16x^2 = 4\cos^2\theta(4-4\cos^2\theta)$

$\Rightarrow 16x^2 = 4y(4-y)$

Question 44 **(***+)**

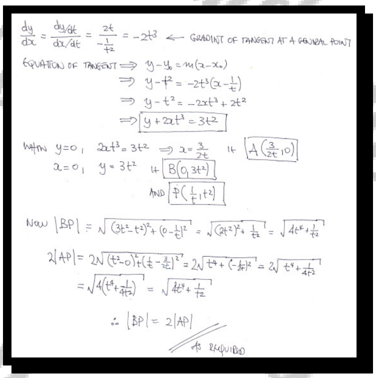
A curve is given parametrically by the equations

$$x = \frac{1}{t}, \quad y = t^2, \ t \neq 0 .$$

The tangent to the curve at the point P meets the x axis at the point A and the y axis at the point B.

Show that for all possible coordinates of P, $|BP| = 2|AP|$.

proof



$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{-\frac{1}{t^2}} = -2t^3$ ← GRADIENT OF TANGENT AT A GENERAL POINT

EQUATION OF TANGENT $\Rightarrow y - y_0 = m(x - x_0)$

$\Rightarrow y - t^2 = -2t^3\left(x - \frac{1}{t}\right)$

$\Rightarrow y - t^2 = -2xt^3 + 2t^2$

$\Rightarrow y + 2xt^3 = 3t^2$

WHEN $y = 0$, $2xt^3 = 3t^2 \Rightarrow x = \frac{3}{2t}$ ∴ $A\left(\frac{3}{2t}, 0\right)$

$x = 0$, $y = 3t^2$ ∴ $B(0, 3t^2)$

AND $P\left(\frac{1}{t}, t^2\right)$

NOW $|BP| = \sqrt{(3t^2-t^2)^2 + \left(0-\frac{1}{t}\right)^2} = \sqrt{(2t^2)^2 + \frac{1}{t^2}} = \sqrt{4t^4 + \frac{1}{t^2}}$

$2|AP| = 2\sqrt{(t^2-0)^2 + \left(\frac{1}{t} - \frac{3}{2t}\right)^2} = 2\sqrt{t^4 + \left(-\frac{1}{2t}\right)^2} = 2\sqrt{t^4 + \frac{1}{4t^2}}$

$= \sqrt{4\left(t^4 + \frac{1}{4t^2}\right)} = \sqrt{4t^4 + \frac{1}{t^2}}$

$\therefore |BP| = 2|AP|$

AS REQUIRED

Question 45 (***+)

The curve C is given parametrically by the equations

$$x = 2t^2 - 1, \quad y = 3t^3 + 4, \quad t \in \mathbb{R}.$$

a) Show that a Cartesian equation of C is

$$8(y-4)^2 = 9(x+1)^3.$$

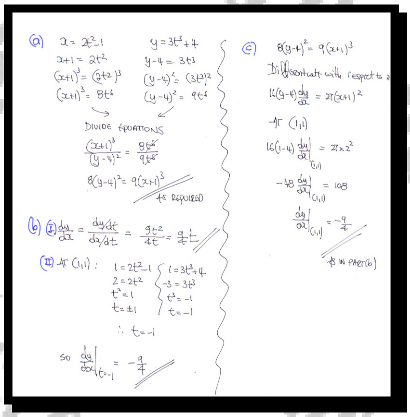
b) Find …

i. … an expression for $\frac{dy}{dx}$ in terms of t.

ii. … the gradient at the point on C with coordinates $(1,1)$.

c) By differentiating the Cartesian equation of C implicitly, verify that the gradient at the point with coordinates $(1,1)$ is the same as that of part (b) (ii)

$$\boxed{\frac{dy}{dx} = \frac{9}{4}t}, \quad \boxed{\left.\frac{dy}{dx}\right|_{(1,1)} = -\frac{9}{4}}$$


(a) $x = 2t^2 - 1$, $y = 3t^3 + 4$

$x + 1 = 2t^2$, $y - 4 = 3t^3$

$(x+1)^3 = (2t^2)^3$, $(y-4)^2 = (3t^3)^2$

$(x+1)^3 = 8t^6$, $(y-4)^2 = 9t^6$

DIVIDE EQUATIONS

$\frac{(x+1)^3}{(y-4)^2} = \frac{8t^6}{9t^6}$

$8(y-4)^2 = 9(x+1)^3$ // AS REQUIRED

(b) (i) $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{9t^2}{4t} = \frac{9}{4}t$

(ii) AT $(1,1)$: $1 = 2t^2 - 1$, $2 = 2t^2$, $t^2 = 1$, $t = \pm 1$; $1 = 3t^3 + 4$, $-3 = 3t^3$, $t^3 = -1$, $t = -1$

$\therefore t = -1$

SO $\left.\frac{dy}{dx}\right|_{t=-1} = -\frac{9}{4}$

(c) $8(y-4)^2 = 9(x+1)^3$

Differentiate with respect to x

$16(y-4)\frac{dy}{dx} = 27(x+1)^2$

AT $(1,1)$

$16(1-4)\left.\frac{dy}{dx}\right|_{(1,1)} = 27 \times 2^2$

$-48\left.\frac{dy}{dx}\right|_{(1,1)} = 108$

$\left.\frac{dy}{dx}\right|_{(1,1)} = -\frac{9}{4}$

AS IN PART (b)

Question 46 (***+)

The curve C is given parametrically by the equations

$$x = \cos t\,,\ \ y = 2\sin t\,,\ \ 0 \le t < 2\pi\,.$$

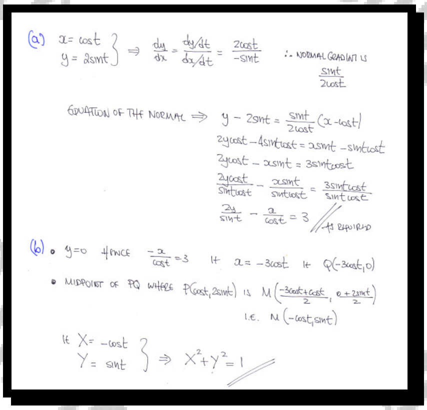
a) Show that an equation of the normal to C at the general point $P(\cos t, 2\sin t)$ can be written as

$$\frac{2y}{\sin t} - \frac{x}{\cos t} = 3\,.$$

The normal to C at P meets the x axis at the point Q. The midpoint of PQ is M.

b) Find the equation of the locus of M as t varies.

$$\boxed{x^2 + y^2 = 1}$$


(a) $x = \cos t$, $y = 2\sin t$ $\Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2\cos t}{-\sin t}$ ∴ NORMAL GRADIENT IS $\frac{\sin t}{2\cos t}$

EQUATION OF THE NORMAL $\Rightarrow$ $y - 2\sin t = \frac{\sin t}{2\cos t}(x - \cos t)$

$2y\cos t - 4\sin t\cos t = x\sin t - \sin t\cos t$

$2y\cos t - x\sin t = 3\sin t\cos t$

$\frac{2y\cos t}{\sin t\cos t} - \frac{x\sin t}{\sin t\cos t} = \frac{3\sin t\cos t}{\sin t\cos t}$

$\frac{2y}{\sin t} - \frac{x}{\cos t} = 3$ // AS REQUIRED

(b) • $y = 0$ HENCE $\frac{-x}{\cos t} = 3$ IE $x = -3\cos t$ IE $Q(-3\cos t, 0)$

• MIDPOINT OF PQ WHERE $P(\cos t, 2\sin t)$ IS $M\left(\frac{-3\cos t + \cos t}{2}, \frac{0 + 2\sin t}{2}\right)$

I.E. $M(-\cos t, \sin t)$

IE $X = -\cos t$, $Y = \sin t$ $\Rightarrow X^2 + Y^2 = 1$

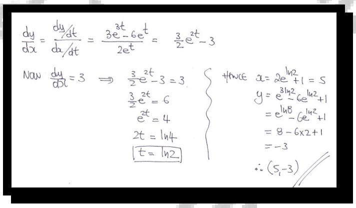
Question 47 (***+)

The curve C is given parametrically by the equations

$$x = 2e^{t} + 1, \quad y = e^{3t} - 6e^{t} + 1, \qquad t \in \mathbb{R}.$$

Determine the coordinates of the point on C with $\frac{dy}{dx} = 3$.

$\boxed{(5,-3)}$



Question 48 (***+)

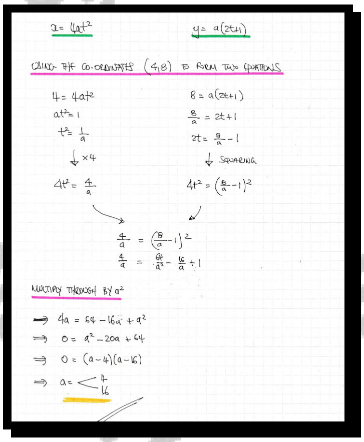
A curve is defined by the following parametric equations

$$x = 4at^2, \quad y = a(2t+1), \quad t \in \mathbb{R},$$

where a is non zero constant.

Given that the curve passes through the point $A(4,8)$, find the possible values of a.

$\boxed{a = 4 \ \cup \ a = 16}$



Question 49 (***+)

A curve is defined by the parametric equations

$$x = t^2 + t,\ \ y = 2t - 1,\ \ t \in \mathbb{R}.$$

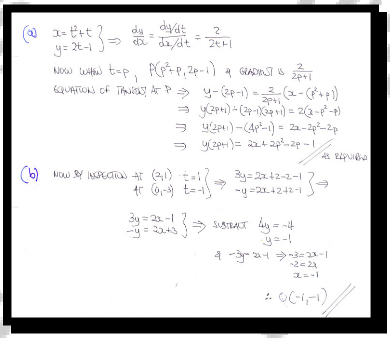
a) Show that an equation of the tangent to the curve at the point P where $t = p$ can be written as

$$y(2p+1) = 2x + 2p^2 - 2p - 1.$$

The tangents to curve at the points $(2,1)$ and $(0,-3)$ meet at the point Q.

b) Find the coordinates of Q.

$\boxed{}$, $\boxed{Q(-1,-1)}$



(a) $x = t^2 + t$, $y = 2t - 1$ $\Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2}{2t+1}$

NOW WHEN $t = p$, $P(p^2+p, 2p-1)$ & GRADIENT IS $\frac{2}{2p+1}$

EQUATION OF TANGENT AT P $\Rightarrow y - (2p-1) = \frac{2}{2p+1}(x - (p^2+p))$

$\Rightarrow y(2p+1) - (2p-1)(2p+1) = 2(x - p^2 - p)$

$\Rightarrow y(2p+1) - (4p^2 - 1) = 2x - 2p^2 - 2p$

$\Rightarrow y(2p+1) = 2x + 2p^2 - 2p - 1$ // AS REQUIRED

(b) NOW BY INSPECTION AT $(2,1)$ $t = 1$, AT $(0,-3)$ $t = -1$ $\Rightarrow$ $3y = 2x + 2 - 2 - 1$, $-y = 2x + 2 + 2 - 1$ $\Rightarrow$

$3y = 2x - 1$, $-y = 2x + 3$ $\Rightarrow$ SUBTRACT $4y = -4$, $y = -1$

& $-3y = 2x - 1 \Rightarrow -3 = 2x - 1$, $-2 = 2x$, $x = -1$

$\therefore Q(-1,-1)$ //

Question 50 (****)

A curve C is given by the parametric equations

$$x = \sec\theta, \quad y = \ln(1+\cos 2\theta), \quad 0 \le \theta < \frac{\pi}{2}.$$

a) Show clearly that

$$\frac{dy}{dx} = -2\cos\theta.$$

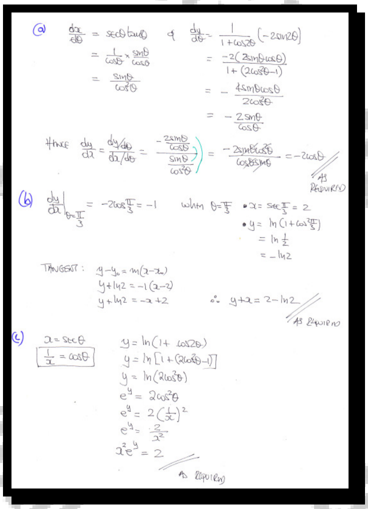
The straight line L is a tangent to C at the point where $\theta = \frac{\pi}{3}$.

b) Find an equation for L, giving the answer in the form $y + x = k$, where k is an exact constant to be found.

c) Show that a Cartesian equation of C is

$$x^2 e^y = 2.$$

$$\boxed{y + x = 2 - \ln 2}$$



Question 51 (****)

A curve C is given by the parametric equations

$$x = \cos 2\theta, \quad y = 2\sin^3\theta, \quad 0 \le \theta < 2\pi \,.$$

a) Show clearly that

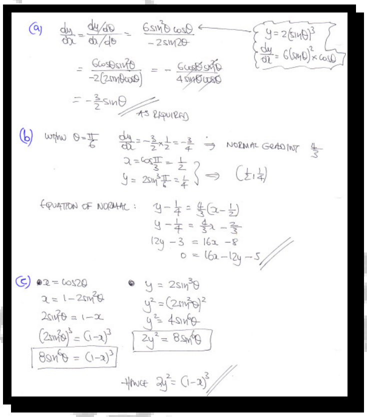
$$\frac{dy}{dx} = -\frac{3}{2}\sin\theta \,.$$

b) Find an equation of the normal to C at the point where $\theta = \frac{\pi}{6}$.

c) Show that a Cartesian equation of C is

$$2y^2 = (1-x)^3 \,.$$

$$\boxed{16x - 12y - 5 = 0}$$



(a) $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{6\sin^2\theta\cos\theta}{-2\sin 2\theta}$ ← $y = 2(\sin\theta)^3$, $\frac{dy}{d\theta} = 6(\sin\theta)^2 \times \cos\theta$

$= \frac{6\cos\theta\sin^2\theta}{-2(2\sin\theta\cos\theta)} = -\frac{6\cos\theta\sin^2\theta}{4\sin\theta\cos\theta}$

$= -\frac{3}{2}\sin\theta$ AS REQUIRED

(b) WHEN $\theta = \frac{\pi}{6}$ $\frac{dy}{dx} = -\frac{3}{2}\times\frac{1}{2} = -\frac{3}{4}$ ⇒ NORMAL GRADIENT $\frac{4}{3}$

$x = \cos\frac{\pi}{3} = \frac{1}{2}$, $y = 2\sin^3\frac{\pi}{6} = \frac{1}{4}$ ⇒ $\left(\frac{1}{2}, \frac{1}{4}\right)$

EQUATION OF NORMAL: $y - \frac{1}{4} = \frac{4}{3}\left(x - \frac{1}{2}\right)$

$y - \frac{1}{4} = \frac{4}{3}x - \frac{2}{3}$

$12y - 3 = 16x - 8$

$0 = 16x - 12y - 5$

(c) • $x = \cos 2\theta$ • $y = 2\sin^3\theta$

$x = 1 - 2\sin^2\theta$ $y^2 = (2\sin^3\theta)^2$

$2\sin^2\theta = 1 - x$ $y^2 = 4\sin^6\theta$

$(2\sin^2\theta)^3 = (1-x)^3$ $\boxed{2y^2 = 8\sin^6\theta}$

$\boxed{8\sin^6\theta = (1-x)^3}$

HENCE $2y^2 = (1-x)^3$

Question 52 (****)

A curve C is given by the parametric equations

$$x = 2\cos\theta + \sin 2\theta, \quad y = \cos\theta - 2\sin 2\theta, \quad 0 \le \theta < 2\pi .$$

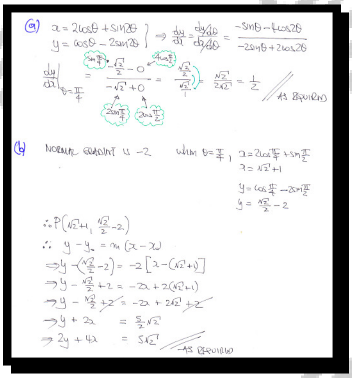
The point P lies on C where $\theta = \dfrac{\pi}{4}$.

a) Show that the gradient at P is $\dfrac{1}{2}$.

b) Show that an equation of the normal to C at P is

$$4x + 2y = 5\sqrt{2} .$$

proof



Question 53 (****)

The curve C has parametric equations

$$x = \sin 2\theta, \quad y = 2\cos^2\theta, \quad 0 \le \theta < 2\pi .$$

a) Show clearly that

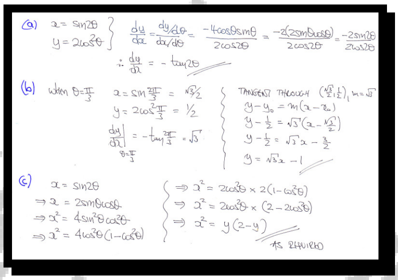
$$\frac{dy}{dx} = -\tan 2\theta .$$

b) Find an equation of the tangent to C, at the point where $\theta = \frac{\pi}{3}$.

c) Show that a Cartesian equation of C is

$$x^2 = y(2-y).$$

$$\boxed{y = \sqrt{3}x - 1}$$



(a) $x = \sin 2\theta$, $y = 2\cos^2\theta$ $\quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-4\cos\theta\sin\theta}{2\cos 2\theta} = \frac{-2(2\sin\theta\cos\theta)}{2\cos 2\theta} = \frac{-2\sin 2\theta}{2\cos 2\theta}$

$\therefore \frac{dy}{dx} = -\tan 2\theta$

(b) when $\theta = \frac{\pi}{3}$ $\quad x = \sin\frac{2\pi}{3} = \sqrt{3}/2$, $\quad y = 2\cos^2\frac{\pi}{3} = 1/2$, $\quad \left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{3}} = -\tan\frac{2\pi}{3} = \sqrt{3}$

TANGENT THROUGH $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, $m = \sqrt{3}$

$y - y_0 = m(x - x_0)$

$y - \frac{1}{2} = \sqrt{3}\left(x - \frac{\sqrt{3}}{2}\right)$

$y - \frac{1}{2} = \sqrt{3}x - \frac{3}{2}$

$y = \sqrt{3}x - 1$

(c) $x = \sin 2\theta$

$\Rightarrow x = 2\sin\theta\cos\theta$

$\Rightarrow x^2 = 4\sin^2\theta\cos^2\theta$

$\Rightarrow x^2 = 4\cos^2\theta(1-\cos^2\theta)$

$\Rightarrow x^2 = 2\cos^2\theta \times 2(1-\cos^2\theta)$

$\Rightarrow x^2 = 2\cos^2\theta \times (2-2\cos^2\theta)$

$\Rightarrow x^2 = y(2-y)$

AS REQUIRED

Question 54 (****)

A curve C is given parametrically by

$$x = \frac{1}{t} + \frac{1}{t^2}, \quad y = \frac{1}{t} - \frac{1}{t^2}, \quad t \in \mathbb{R}, \quad t \neq 0.$$

Show clearly that …

a) … $\dfrac{dy}{dx} = \dfrac{t-2}{t+2}$.

b) … an equation of the tangent to C at the point where $t = \frac{1}{2}$ is

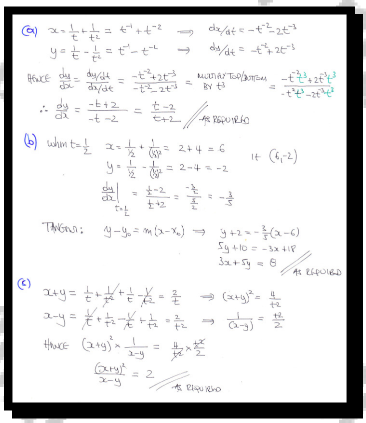
$$3x + 5y = 8.$$

c) … a Cartesian equation of C is

$$\frac{(x+y)^2}{x-y} = 2.$$

You may find considering $(x+y)$ and $(x-y)$ useful in this part.

proof



Question 55 (****)

A curve C is given parametrically by

$$x = \tan\theta, \quad y = \sin 2\theta, \quad 0 \le \theta < 2\pi.$$

a) Find the gradient at the point on C where $\theta = \frac{\pi}{6}$.

b) Show that

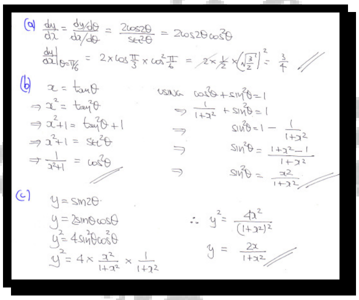
$$\cos^2\theta = \frac{1}{x^2+1},$$

and find a similar expression for $\sin^2\theta$.

c) Hence find a Cartesian equation of C in the form

$$y = f(x).$$

$$\boxed{\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{6}} = \frac{3}{4}}, \quad \boxed{\sin^2\theta = \frac{x^2}{x^2+1}}, \quad \boxed{y = \frac{2x}{x^2+1}}$$


(a) $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2\cos 2\theta}{\sec^2\theta} = 2\cos 2\theta\cos^2\theta$

$\left.\frac{dy}{dx}\right|_{\theta=\pi/6} = 2\times\cos\frac{\pi}{3}\times\cos^2\frac{\pi}{6} = 2\times\frac{1}{2}\times\left(\sqrt{\frac{3}{2}}\right)^2 = \frac{3}{4}$

(b) $x = \tan\theta$

$\Rightarrow x^2 = \tan^2\theta$

$\Rightarrow x^2+1 = \tan^2\theta+1$

$\Rightarrow x^2+1 = \sec^2\theta$

$\Rightarrow \frac{1}{x^2+1} = \cos^2\theta$

USING $\cos^2\theta+\sin^2\theta=1$

$\Rightarrow \frac{1}{1+x^2}+\sin^2\theta=1$

$\Rightarrow \sin^2\theta = 1-\frac{1}{1+x^2}$

$\Rightarrow \sin^2\theta = \frac{1+x^2-1}{1+x^2}$

$\Rightarrow \sin^2\theta = \frac{x^2}{1+x^2}$

(c) $y = \sin 2\theta$

$y = 2\sin\theta\cos\theta$

$y^2 = 4\sin^2\theta\cos^2\theta$

$y^2 = 4\times\frac{x^2}{1+x^2}\times\frac{1}{1+x^2}$

$\therefore y^2 = \frac{4x^2}{(1+x^2)^2}$

$y = \frac{2x}{1+x^2}$

Question 56 (****)

A curve C is given parametrically by the equations

$$x = 4t^2 + t, \quad y = \frac{1}{2}t^2 + 2t^3, \quad t \in \mathbb{R}.$$

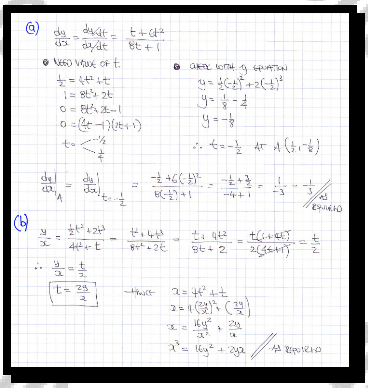
The point $A\left(\frac{1}{2}, -\frac{1}{8}\right)$ lies on C.

a) Show that the gradient at A is $-\frac{1}{3}$.

b) By considering $\frac{y}{x}$, or otherwise, show that a Cartesian equation of C is

$$x^3 = 16y^2 + 2xy.$$

proof



Question 57 **(****)**

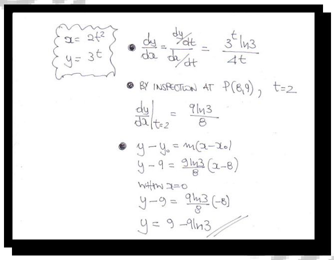
The point $P(8,9)$ lies on the curve C with parametric equations

$$x = 2t^2, \quad y = 3^t, \quad t \in \mathbb{R}.$$

The tangent to C at P meets the y axis at the point Q.

Determine the exact y coordinate of Q.

$\boxed{9 - 9\ln 3}$



Question 58 (****)

The curve C is given parametrically by

$$x = 1 - 3t, \quad y = \frac{t+6}{t+2}, \quad t \in \mathbb{R}.$$

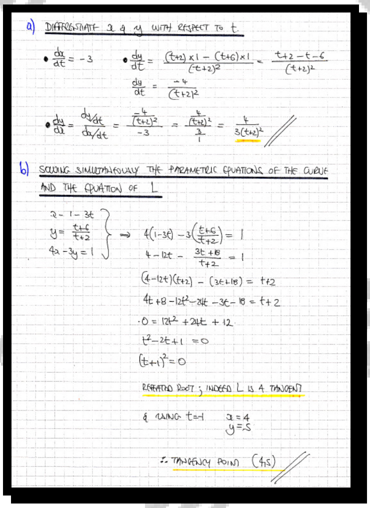
a) Find a simplified expression for $\frac{dy}{dx}$, in terms of t.

b) Show that the straight line L with equation

$$4x - 3y = 1$$

is a tangent to C, and determine the coordinates of the point of tangency between L and C.

$\boxed{}$, $\boxed{\frac{dy}{dx} = \frac{4}{3(t+2)^2}}$, $\boxed{(4,5)}$



a) DIFFERENTIATE x & y WITH RESPECT TO t

- $\frac{dx}{dt} = -3$
- $\frac{dy}{dt} = \frac{(t+2)\times 1 - (t+6)\times 1}{(t+2)^2} = \frac{t+2-t-6}{(t+2)^2}$

$\frac{dy}{dt} = \frac{-4}{(t+2)^2}$

- $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{-4}{(t+2)^2}}{-3} = \frac{\frac{4}{(t+2)^2}}{\frac{3}{1}} = \frac{4}{3(t+2)^2}$

b) SOLVING SIMULTANEOUSLY THE PARAMETRIC EQUATIONS OF THE CURVE AND THE EQUATION OF L

$x = 1 - 3t$
$y = \frac{t+6}{t+2}$
$4x - 3y = 1$

$\Rightarrow 4(1-3t) - 3\left(\frac{t+6}{t+2}\right) = 1$

$4 - 12t - \frac{3t+18}{t+2} = 1$

$(4-12t)(t+2) - (3t+18) = t+2$

$4t + 8 - 12t^2 - 24t - 3t - 18 = t + 2$

$0 = 12t^2 + 24t + 12$

$t^2 - 2t + 1 = 0$

$(t+1)^2 = 0$

REPEATED ROOT; INDEED L IS A TANGENT

& USING $t=-1$ $\quad x = 4$, $y = 5$

$\therefore$ TANGENCY POINT $(4,5)$

Question 59 (****)

A curve C is defined by the parametric equations:

$$x = \tan\theta, \quad y = \sin 2\theta \ , \quad -\frac{\pi}{2} \le \theta < \frac{\pi}{2}$$

a) State the range of C.

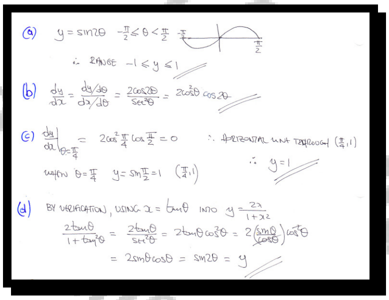
b) Find an expression for $\frac{dy}{dx}$ in terms of θ.

c) Find an equation of the tangent to the curve where $\theta = \frac{\pi}{4}$.

d) Show, or verify, that a Cartesian equation for C is

$$y = \frac{2x}{1+x^2}.$$

$\boxed{-1 \le y \le 1}$, $\boxed{\frac{dy}{dx} = \frac{2\cos 2\theta}{\sec^2\theta}}$, $\boxed{y = 1}$



(a) $y = \sin 2\theta \quad -\frac{\pi}{2} \le \theta < \frac{\pi}{2}$

$\therefore$ RANGE $-1 \le y \le 1$

(b) $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2\cos 2\theta}{\sec^2\theta} = 2\cos^2\theta \cos 2\theta$

(c) $\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{4}} = 2\cos^2\frac{\pi}{4}\cos\frac{\pi}{2} = 0 \quad \therefore$ HORIZONTAL LINE THROUGH $(\frac{\pi}{4}, 1)$

WHEN $\theta = \frac{\pi}{4} \quad y = \sin\frac{\pi}{2} = 1 \quad (\frac{\pi}{4}, 1) \quad \therefore y = 1$

(d) BY VERIFICATION, USING $x = \tan\theta$ INTO $y = \frac{2x}{1+x^2}$

$\frac{2\tan\theta}{1+\tan^2\theta} = \frac{2\tan\theta}{\sec^2\theta} = 2\tan\theta\cos^2\theta = 2\left(\frac{\sin\theta}{\cos\theta}\right)\cos^2\theta$

$= 2\sin\theta\cos\theta = \sin 2\theta = y$

Question 60 (****)

A curve C is traced by the parametric equations

$$x = t^2 - t, \quad y = \frac{at}{1-t}, \quad t \in \mathbb{R}, \; t \neq 1.$$

a) Find an expression for $\frac{dy}{dx}$ in terms of the parameter t and the constant a.

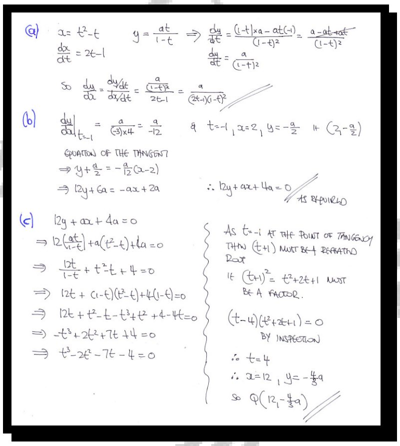
b) Show that an equation of the tangent to C at the point where $t = -1$ is

$$12y + ax + 4a = 0.$$

This tangent meets the curve again at the point Q.

c) Determine the coordinates of Q in terms of a.

$$\boxed{\frac{dy}{dx} = \frac{a}{(2t-1)(1-t)^2}}, \quad \boxed{Q\left(12, -\frac{4}{3}a\right)}$$


(a) $x = t^2 - t$, $\frac{dx}{dt} = 2t-1$; $y = \frac{at}{1-t} \Rightarrow \frac{dy}{dt} = \frac{(1-t)\times a - at(-1)}{(1-t)^2} = \frac{a - at + at}{(1-t)^2}$, $\frac{dy}{dt} = \frac{a}{(1-t)^2}$

So $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{a}{(1-t)^2}}{2t-1} = \frac{a}{(2t-1)(1-t)^2}$

(b) $\left.\frac{dy}{dx}\right|_{t=-1} = \frac{a}{(-3)\times 4} = \frac{a}{-12}$ & $t=-1$, $x=2$, $y=-\frac{a}{2}$ i.e. $(2, -\frac{a}{2})$

EQUATION OF THE TANGENT

$\Rightarrow y + \frac{a}{2} = -\frac{a}{12}(x-2)$

$\Rightarrow 12y + 6a = -ax + 2a$

$\therefore 12y + ax + 4a = 0$ AS REQUIRED

(c) $12y + ax + 4a = 0$

$\Rightarrow 12\left(\frac{at}{1-t}\right) + a(t^2-t) + 4a = 0$

$\Rightarrow \frac{12t}{1-t} + t^2 - t + 4 = 0$

$\Rightarrow 12t + (1-t)(t^2-t) + 4(1-t) = 0$

$\Rightarrow 12t + t^2 - t - t^3 + t^2 + 4 - 4t = 0$

$\Rightarrow -t^3 + 2t^2 + 7t + 4 = 0$

$\Rightarrow t^3 - 2t^2 - 7t - 4 = 0$

As $t=-1$ at the point of tangency then $(t+1)$ must be a repeated root, i.e. $(t+1)^2 = t^2+2t+1$ must be a factor.

$(t-4)(t^2+2t+1) = 0$ by inspection

$\therefore t = 4$

$\therefore x = 12$, $y = -\frac{4}{3}a$

So $Q(12, -\frac{4}{3}a)$

Question 61 (****)

The curve C has parametric equations

$$x = 2\tan\theta, \quad y = 2\cos^2\theta, \quad 0 \le \theta < \frac{\pi}{2}.$$

a) Show clearly that

$$\frac{dy}{dx} = -2\sin\theta\cos^3\theta.$$

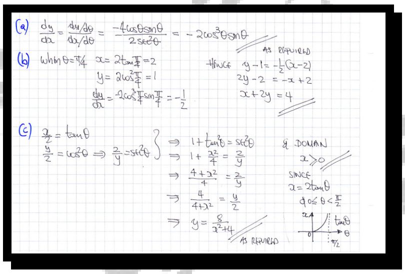
b) Find an equation of tangent to C, at the point where $\theta = \frac{\pi}{4}$.

c) Show that a Cartesian equation of C is

$$y = \frac{8}{x^2+4},$$

and state its domain.

$\boxed{x + 2y = 4}$, $\boxed{x \ge 0}$


(a) $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-4\cos\theta\sin\theta}{2\sec^2\theta} = -2\cos^3\theta\sin\theta$ AS REQUIRED

(b) when $\theta = \frac{\pi}{4}$ $x = 2\tan\frac{\pi}{4} = 2$, $y = 2\cos^2\frac{\pi}{4} = 1$, $\frac{dy}{dx} = -2\cos^3\frac{\pi}{4}\sin\frac{\pi}{4} = -\frac{1}{2}$

HENCE $y - 1 = -\frac{1}{2}(x-2)$, $2y - 2 = -x + 2$, $x + 2y = 4$

(c) $\frac{x}{2} = \tan\theta$, $\frac{y}{2} = \cos^2\theta \Rightarrow \frac{2}{y} = \sec^2\theta$ $\Rightarrow 1 + \tan^2\theta = \sec^2\theta$ $\Rightarrow 1 + \frac{x^2}{4} = \frac{2}{y}$ $\Rightarrow \frac{4+x^2}{4} = \frac{2}{y}$ $\Rightarrow \frac{4}{4+x^2} = \frac{y}{2}$ $\Rightarrow y = \frac{8}{x^2+4}$ AS REQUIRED

& DOMAIN $x \ge 0$

SINCE $x = 2\tan\theta$ & $0 \le \theta < \frac{\pi}{2}$

Question 62 (****)

The point $P(20,60)$ lies on a curve with parametric equations

$$x = 2at\,,\ \ y = 8at - at^2\,,\ \ t \in \mathbb{R}\,,\ \ t \geq 0\,,$$

where a is a non zero constant.

a) Find the value of a.

b) Determine a Cartesian equation of the curve.

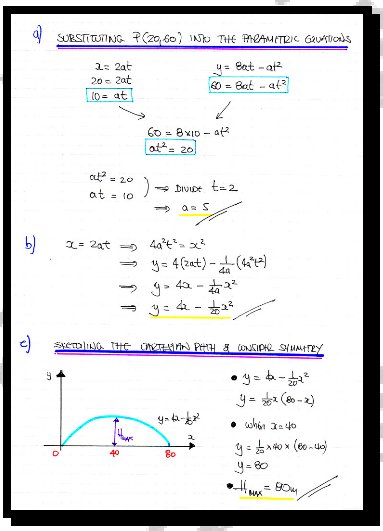
The above set of parametric equations represents the path of a golf ball, t seconds after it was struck from a fixed point on the ground, O.

The horizontal distance from O is x metres and the vertical distance above the ground level is y metres.

The ball hits the lowest point of a TV airship, which was recording the golf tournament from the air.

c) Assuming that the ground is level and horizontal, find the greatest possible height of the airship from the ground.

$\boxed{a=5}$, $\boxed{y = 4x - \frac{1}{20}x^2}$, $\boxed{80}$



Question 63 (****)

A curve C is defined by the parametric equations

$$x = 2t - 1 \quad y = \frac{4}{t}, \; t \in \mathbb{R}, \; t \neq 0.$$

The curve C meets the y axis at the point A.

a) Determine the coordinates of A.

b) Show that an equation of the normal to C at A is given by

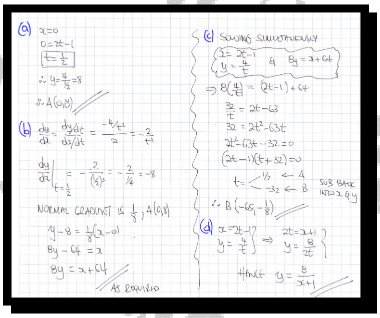
$$8y = x + 64.$$

This normal meets C again at the point B.

c) Calculate the coordinates of B.

d) Find a Cartesian equation for C.

$\boxed{A(0,8)}$, $\boxed{B\left(-65, -\frac{1}{8}\right)}$, $\boxed{y = \frac{8}{x+1}}$


(a) $x=0$
$0=2t-1$
$t=\frac{1}{2}$
$\therefore y=\frac{4}{1/2}=8$
$\therefore A(0,8)$

(b) $\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{-4/t^2}{2}=-\frac{2}{t^2}$
$\left.\frac{dy}{dx}\right|_{t=\frac{1}{2}}=-\frac{2}{(1/2)^2}=-\frac{2}{1/4}=-8$
NORMAL GRADIENT IS $\frac{1}{8}$, $A(0,8)$
$y-8=\frac{1}{8}(x-0)$
$8y-64=x$
$8y=x+64$
AS REQUIRED

(c) SOLVING SIMULTANEOUSLY
$x=2t-1$, $y=\frac{4}{t}$ & $8y=x+64$
$\Rightarrow 8\left(\frac{4}{t}\right)=(2t-1)+64$
$\frac{32}{t}=2t-63$
$32=2t^2-63t$
$2t^2-63t-32=0$
$(2t-1)(t+32)=0$
$t = \frac{1}{2} \leftarrow A$, $t = -32 \leftarrow B$ SUB BACK INTO x & y
$\therefore B\left(-65,-\frac{1}{8}\right)$

(d) $x=2t-1$, $y=\frac{4}{t}$ $\Rightarrow$ $2t=x+1$, $y=\frac{8}{2t}$
HENCE $y=\frac{8}{x+1}$

Question 64 (****)

A curve C is given parametrically by the equations

$$x = 6\ln t - 3t^2, \quad y = 2t^3 - 36t + 6\ln t, \quad t \in \mathbb{R}, \quad t > t_0.$$

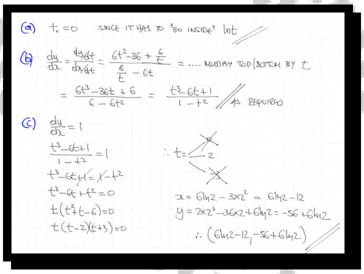
a) State the smallest possible value that t_0 can take .

b) Show that

$$\frac{dy}{dx} = \frac{t^3 - 6t + 1}{1 - t^2}.$$

c) Find the exact coordinates of the only point on C where the gradient is 1.

$\boxed{t_0 = 0}$, $\boxed{(-12 + 6\ln 2, -56 + 6\ln 2)}$


(a) $t_0 = 0$ SINCE IT HAS TO "GO INSIDE" $\ln t$

(b) $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{6t^2 - 36 + \frac{6}{t}}{\frac{6}{t} - 6t} = \ldots$ MULTIPLY TOP/BOTTOM BY t

$= \frac{6t^3 - 36t + 6}{6 - 6t^2} = \frac{t^3 - 6t + 1}{1 - t^2}$ AS REQUIRED

(c) $\frac{dy}{dx} = 1$

$\frac{t^3 - 6t + 1}{1 - t^2} = 1$

$t^3 - 6t + 1 = 1 - t^2$

$t^3 - 6t + t^2 = 0$

$t(t^2 + t - 6) = 0$

$t(t-2)(t+3) = 0$

$\therefore t = 0$ (rejected), 2, -3 (rejected)

$x = 6\ln 2 - 3 \times 2^2 = 6\ln 2 - 12$

$y = 2 \times 2^3 - 36 \times 2 + 6\ln 2 = -56 + 6\ln 2$

$\therefore (6\ln 2 - 12, -56 + 6\ln 2)$

Question 65 (****)

A curve C is defined by the parametric equations

$$x = 2t + 4 \ , \quad y = t^3 - 4t + 1 \, , \quad t \in \mathbb{R} \, .$$

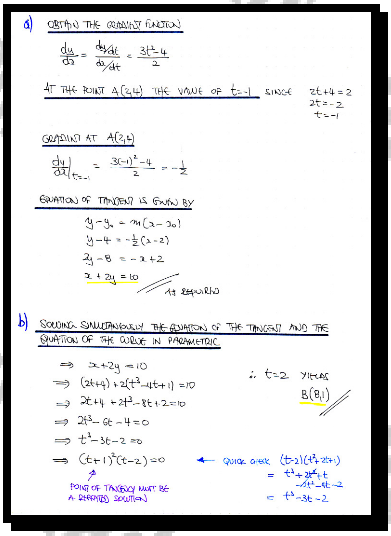
a) Show that an equation of the tangent to the curve at $A(2,4)$ is

$$2y + x = 10 \, .$$

The tangent to C at A re-intersects C at the point B.

b) Determine the coordinates of B.

SYN-C , $B(8,1)$



Question 66 (****)

A curve is given parametrically by the equations

$$x = 4 - t^2, \ y = 1 - t, \ t \in \mathbb{R}.$$

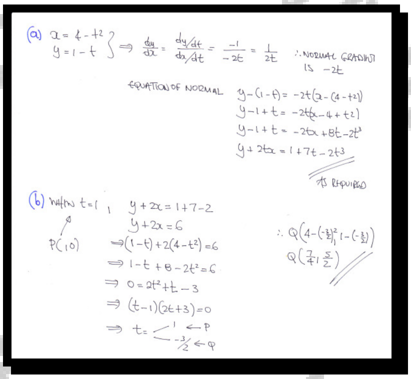
a) Show that an equation of the normal at a general point on the curve is

$$y + 2tx = 1 + 7t - 2t^3.$$

The normal to curve at $P(3,0)$ meets the curve again at the point Q.

b) Find the coordinates of Q.

$Q\left(\frac{7}{4}, \frac{5}{2}\right)$


(a) $x = 4 - t^2$, $y = 1 - t$ $\Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-1}{-2t} = \frac{1}{2t}$ ∴ NORMAL GRADIENT IS $-2t$

EQUATION OF NORMAL $y - (1-t) = -2t(x - (4 - t^2))$

$y - 1 + t = -2t(x - 4 + t^2)$

$y - 1 + t = -2tx + 8t - 2t^3$

$y + 2tx = 1 + 7t - 2t^3$

AS REQUIRED

(b) WHEN $t = 1$, $y + 2x = 1 + 7 - 2$

$P(3,0)$ $y + 2x = 6$

$\Rightarrow (1-t) + 2(4 - t^2) = 6$

$\Rightarrow 1 - t + 8 - 2t^2 = 6$

$\Rightarrow 0 = 2t^2 + t - 3$

$\Rightarrow (t-1)(2t+3) = 0$

$\Rightarrow t = 1 \leftarrow P$, $t = -\frac{3}{2} \leftarrow Q$

∴ $Q\left(4 - \left(-\frac{3}{2}\right)^2, 1 - \left(-\frac{3}{2}\right)\right)$

$Q\left(\frac{7}{4}, \frac{5}{2}\right)$

Question 67 **(****)**

A curve is given by the parametric equations

$$x = \tan^2 t \ , \quad y = \sqrt{2}\sin t \, , \quad 0 \le t < \frac{\pi}{2} .$$

a) Find an expression for $\frac{dy}{dx}$ in terms of t.

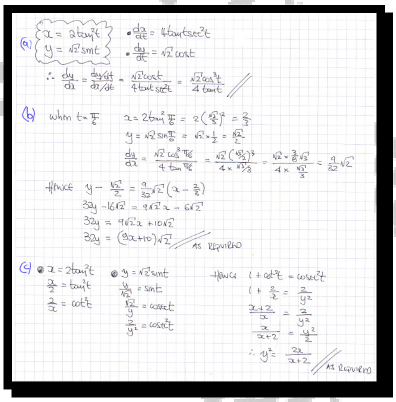
b) Show that an equation of the tangent to the curve at the point where $t = \frac{\pi}{6}$, is

$$32y = (9x + 10)\sqrt{2} .$$

c) Show that a Cartesian equation of the curve is

$$y^2 = \frac{2x}{x+2} .$$

$$\boxed{\frac{dy}{dx} = \frac{\sqrt{2}\cos^3 t}{4\tan t}}$$



Question 68 (****)

A curve C is given parametrically by

$$x = (t+2)^2, \quad y = t^3 + 2, \quad t \in \mathbb{R}.$$

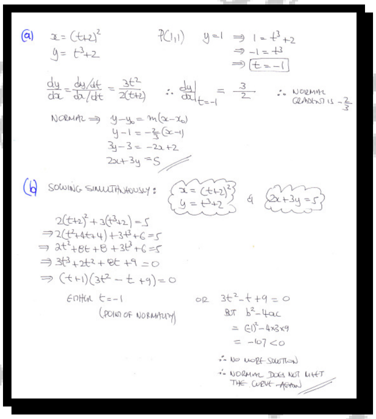
The point $P(1,1)$ lies on C.

a) Show that the equation of the normal to C at P is

$$3y + 2x = 5.$$

b) Show further that the normal to C at P does not meet C again.

proof


(a) $x = (t+2)^2$, $y = t^3 + 2$ P(1,1) $y = 1 \Rightarrow 1 = t^3 + 2 \Rightarrow -1 = t^3 \Rightarrow t = -1$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2}{2(t+2)}$ ∴ $\left.\frac{dy}{dx}\right|_{t=-1} = \frac{3}{2}$ ∴ NORMAL GRADIENT IS $-\frac{2}{3}$

NORMAL ⇒ $y - y_0 = m(x - x_0)$

$y - 1 = -\frac{2}{3}(x-1)$

$3y - 3 = -2x + 2$

$2x + 3y = 5$

(b) SOLVING SIMULTANEOUSLY: $x = (t+2)^2$, $y = t^3 + 2$ & $2x + 3y = 5$

$2(t+2)^2 + 3(t^3+2) = 5$

$\Rightarrow 2(t^2+4t+4) + 3t^3 + 6 = 5$

$\Rightarrow 2t^2 + 8t + 8 + 3t^3 + 6 = 5$

$\Rightarrow 3t^3 + 2t^2 + 8t + 9 = 0$

$\Rightarrow (t+1)(3t^2 - t + 9) = 0$

EITHER $t = -1$ (POINT OF NORMALITY) OR $3t^2 - t + 9 = 0$

BUT $b^2 - 4ac = (-1)^2 - 4 \times 3 \times 9 = -107 < 0$

∴ NO MORE SOLUTION

∴ NORMAL DOES NOT MEET THE CURVE AGAIN

Question 69 (****)

A curve C is given by the parametric equations

$$x = t^3 - 9t, \quad y = \frac{1}{2}t^2, \quad t \in \mathbb{R}.$$

The point $P(10,2)$ lies on C.

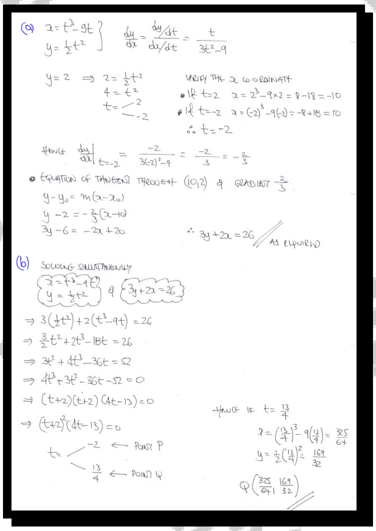
a) Show that the equation of the tangent to C at P is

$$3y + 2x = 26.$$

The tangent to C at P crosses C again at the point Q.

b) Find as exact fractions the coordinates of Q.

$$\boxed{Q\left(\frac{325}{64}, \frac{169}{32}\right)}$$



(a) $x = t^3 - 9t$, $y = \frac{1}{2}t^2$ $\quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{t}{3t^2-9}$

$y = 2 \Rightarrow 2 = \frac{1}{2}t^2$, $4 = t^2$, $t = 2$ or -2

VERIFY THE x CO-ORDINATE
- If $t=2$: $x = 2^3 - 9\times2 = 8 - 18 = -10$
- If $t=-2$: $x = (-2)^3 - 9(-2) = -8 + 18 = 10$

$\therefore t = -2$

HENCE $\left.\frac{dy}{dx}\right|_{t=-2} = \frac{-2}{3(-2)^2-9} = \frac{-2}{3} = -\frac{2}{3}$

- EQUATION OF TANGENT THROUGH $(10,2)$ & GRADIENT $-\frac{2}{3}$

$y - y_0 = m(x - x_0)$

$y - 2 = -\frac{2}{3}(x - 10)$

$3y - 6 = -2x + 20$

$\therefore 3y + 2x = 26$ AS REQUIRED

(b) SOLVING SIMULTANEOUSLY

$x = t^3 - 9t$, $y = \frac{1}{2}t^2$ & $3y + 2x = 26$

$\Rightarrow 3\left(\frac{1}{2}t^2\right) + 2(t^3 - 9t) = 26$

$\Rightarrow \frac{3}{2}t^2 + 2t^3 - 18t = 26$

$\Rightarrow 3t^2 + 4t^3 - 36t = 52$

$\Rightarrow 4t^3 + 3t^2 - 36t - 52 = 0$

$\Rightarrow (t+2)(t+2)(4t-13) = 0$

$\Rightarrow (t+2)^2(4t-13) = 0$

$t = -2$ ← POINT P, $t = \frac{13}{4}$ ← POINT Q

HENCE IF $t = \frac{13}{4}$

$x = \left(\frac{13}{4}\right)^3 - 9\left(\frac{13}{4}\right) = \frac{325}{64}$

$y = \frac{1}{2}\left(\frac{13}{4}\right)^2 = \frac{169}{32}$

$Q\left(\frac{325}{64}, \frac{169}{32}\right)$

Question 70 (****)

A curve C is given by the parametric equations

$$x = 2t - \frac{1}{2t}, \quad y = 2t + \frac{1}{2t} + 2, \quad t \in \mathbb{R}, \quad t \neq 0.$$

a) Show that

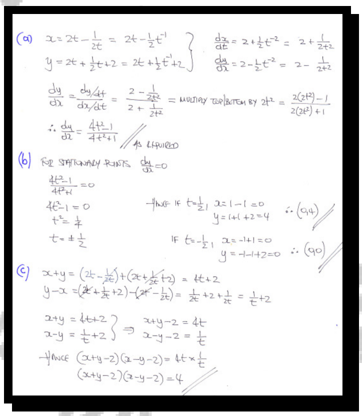
$$\frac{dy}{dx} = \frac{4t^2 - 1}{4t^2 + 1}.$$

b) Hence find the coordinates of the stationary points of the curve.

c) Show that a Cartesian equation of the curve is

$$(y + x - 2)(y - x - 2) = 4.$$

$\boxed{(0,0), \ (0,4)}$


(a) $x = 2t - \frac{1}{2t} = 2t - \frac{1}{2}t^{-1}$, $y = 2t + \frac{1}{2}t + 2 = 2t + \frac{1}{2}t^{-1} + 2$ ⟹ $\frac{dx}{dt} = 2 + \frac{1}{2}t^{-2} = 2 + \frac{1}{2t^2}$, $\frac{dy}{dx} = 2 - \frac{1}{2}t^{-2} = 2 - \frac{1}{2t^2}$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2 - \frac{1}{2t^2}}{2 + \frac{1}{2t^2}}$ = MULTIPLY TOP/BOTTOM BY $2t^2$ = $\frac{2(2t^2) - 1}{2(2t^2) + 1}$

$\therefore \frac{dy}{dx} = \frac{4t^2 - 1}{4t^2 + 1}$ // AS REQUIRED

(b) FOR STATIONARY POINTS $\frac{dy}{dx} = 0$

$\frac{4t^2 - 1}{4t^2 + 1} = 0$

$4t^2 - 1 = 0$

$t^2 = \frac{1}{4}$

$t = \pm\frac{1}{2}$

HENCE IF $t = \frac{1}{2}$, $x = 1 - 1 = 0$, $y = 1 + 1 + 2 = 4$ $\therefore (0,4)$ //

IF $t = -\frac{1}{2}$, $x = -1 + 1 = 0$, $y = -1 - 1 + 2 = 0$ $\therefore (0,0)$ //

(c) $x + y = \left(2t - \frac{1}{2t}\right) + \left(2t + \frac{1}{2t} + 2\right) = 4t + 2$

$y - x = \left(2t + \frac{1}{2t} + 2\right) - \left(2t - \frac{1}{2t}\right) = \frac{1}{2t} + 2 + \frac{1}{2t} = \frac{1}{t} + 2$

$x + y = 4t + 2$, $x - y = \frac{1}{t} + 2$ ⟹ $x + y - 2 = 4t$, $x - y - 2 = \frac{1}{t}$

HENCE $(x+y-2)(x-y-2) = 4t \times \frac{1}{t}$

$(x+y-2)(x-y-2) = 4$ //

Question 71 **(****)**

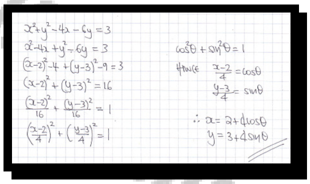
A circle has Cartesian equation

$$x^2 + y^2 - 4x - 6y = 3 .$$

Determine a set of parametric equations for this circle in the form

$$x = a + p\cos\theta , \quad y = b + p\cos\theta , \quad 0 \le \theta < 2\pi .$$

$$\boxed{x = 2 + 4\cos\theta, \quad y = 3 + 4\cos\theta}$$



Question 72 (****)

A curve C is given by the parametric equations

$$x = 3\cos 2\theta \ , \quad y = -2 + 4\sin\theta \ , \quad 0 \le \theta < 2\pi \ .$$

a) Show that a Cartesian equation of the curve is

$$3y^2 + 12y + 8x = 12 \ .$$

The point P lies on C, where $\sin\theta = \frac{1}{3}$.

b) Show that an equation of the normal to C at P is

$$y = x - 3 \ .$$

The normal to C at P meets C again at the point Q.

c) Find the coordinates of Q.

d) State the domain and range of C, and given further that C is not a closed curve describe the position of the point Q on the curve.

$\boxed{Q(-3,-6)}$, $\boxed{-3 \le x \le 3}$, $\boxed{-6 \le y \le 2}$, $\boxed{Q \text{ is an endpoint of } C}$

Question 73 (****)

A curve is given by the parametric equations

$$x = \cos t \ , \quad y = \sin 2t \ , \quad 0 \le t < 2\pi \ .$$

a) Find a Cartesian equation of the curve, giving the answer in the form

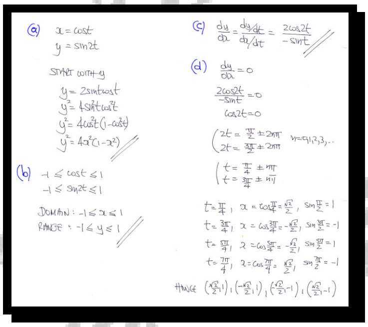
$$y^2 = f(x) \ .$$

b) State the domain and range of the curve.

c) Find an expression for $\frac{dy}{dx}$ in terms of t.

d) Hence, find the coordinates of the 4 stationary points of the curve.

$\boxed{y^2 = 4x^2(1-x^2)}$, $\boxed{-1 \le x \le 1}$, $\boxed{-1 \le y \le 1}$, $\boxed{\frac{dy}{dx} = -\frac{2\cos 2t}{\sin t}}$,
$\boxed{\left(\frac{\sqrt{2}}{2},1\right), \left(-\frac{\sqrt{2}}{2},1\right), \left(-\frac{\sqrt{2}}{2},-1\right), \left(\frac{\sqrt{2}}{2},-1\right)}$



Question 74 (****)

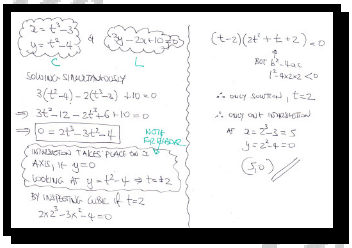
A curve C is defined by the parametric equations

$$x = t^3 - 3, \quad y = t^2 - 4 \ , \quad t \in \mathbb{R} .$$

The straight line L with equation $3y - 2x + 10 = 0$ intersects with C.

Show that L and C intersect at a single point on the x axis, stating its coordinates.

$\boxed{(5,0)}$



Question 75 (****)

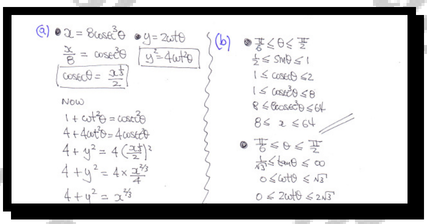
A curve C is defined by the parametric equations

$$x = 8\operatorname{cosec}^3\theta \ , \quad y = 2\cot\theta \, , \quad \frac{\pi}{6} \le \theta \le \frac{\pi}{2} .$$

a) Find a Cartesian equation for C, in the form $y = f(x)$.

b) Determine the range of values of x and the range of values of y, which the graph of C can achieve.

$\boxed{y = \sqrt{x^{\frac{2}{3}} - 4}}$, $\boxed{8 \le x \le 64}$, $\boxed{0 \le y \le 2\sqrt{3}}$



Question 76 (****)

A curve C is defined by the parametric equations

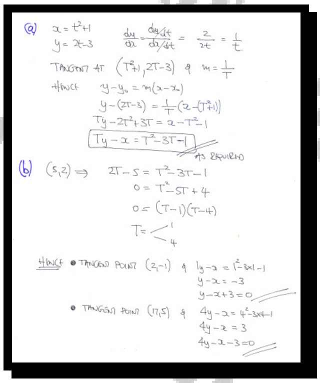
$$x = t^2 + 1 \ , \ y = 2t - 3 \, , \ t \in \mathbb{R} \, .$$

a) Show that the equation of the tangent to C, at the point where $t = T$, is given by

$$Ty - x = T^2 - 3T - 1 \, .$$

b) Find the equations of the two tangents to C, passing through the point $(5,2)$ and deduce the coordinates of their corresponding points of tangency.

$\boxed{y - x + 3 = 0, \ (2,-1)}$, $\boxed{4y - x - 3 = 0, \ (17,5)}$


(a) $x = t^2+1$, $y = 2t-3$ $\quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2}{2t} = \frac{1}{t}$

TANGENT AT $(T^2+1, 2T-3)$ & $m = \frac{1}{T}$

HENCE $y - y_0 = m(x - x_0)$

$y - (2T-3) = \frac{1}{T}(x - (T^2+1))$

$Ty - 2T^2 + 3T = x - T^2 - 1$

$\boxed{Ty - x = T^2 - 3T - 1}$ AS REQUIRED

(b) $(5,2) \Rightarrow$ $2T - 5 = T^2 - 3T - 1$

$0 = T^2 - 5T + 4$

$0 = (T-1)(T-4)$

$T = 1, 4$

HENCE • TANGENT POINT $(2,-1)$ & $1y - x = 1^2 - 3\times1 - 1$, $y - x = -3$, $y - x + 3 = 0$

• TANGENT POINT $(17,5)$ & $4y - x = 4^2 - 3\times4 - 1$, $4y - x = 3$, $4y - x - 3 = 0$

Question 77 (****)

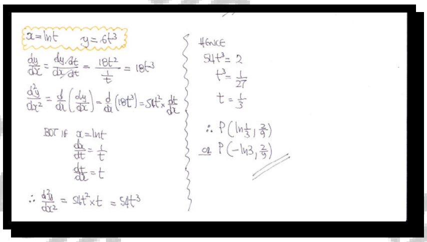
A curve C is defined by the parametric equations

$$x = \ln t \ , \quad y = 6t^3 , \quad t > 0 .$$

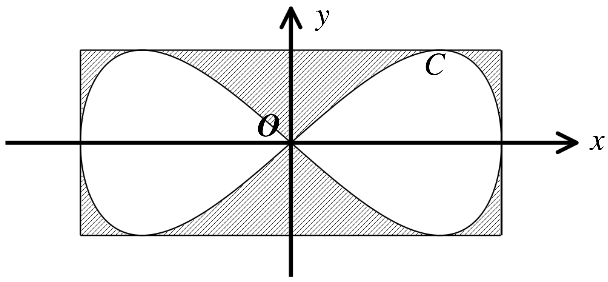
The point P lies on C, so that $\dfrac{d^2y}{dx^2} = 2$ at P.

Determine the exact coordinates of P.

$P\left(-\ln 3, \frac{2}{9}\right)$



Question 78 (****)



The figure above shows the curve C known as the "lemniscate of Bernoulli", defined by the parametric equations

$$x = 3\sin\theta, \quad y = 2\sin 2\theta, \quad 0 \le \theta \le 2\pi .$$

The curve is symmetrical in the x axis and in the y axis.

a) Show that a Cartesian equation of C is

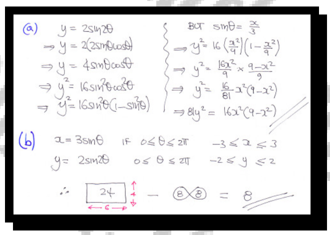
$$81y^2 = 16x^2\left(9 - x^2\right).$$

In the figure above, the curve C is shown bounded by a rectangle whose sides are tangents to the curve parallel to the coordinate axes.

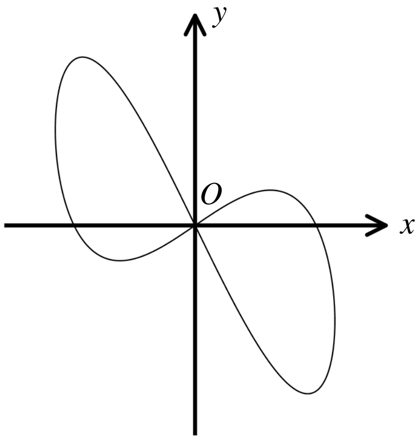
The shaded region represents the points within the rectangle but outside C.

b) Given that the area of one loop of C is 8 square units, find the area of the shaded region.

$\boxed{\text{area} = 8}$



Question 79 **(****)**



The figure above shows the curve C with parametric equations

$$x = \cos\theta,\quad y = \sin 2\theta - \cos\theta,\quad 0 \le \theta < 2\pi.$$

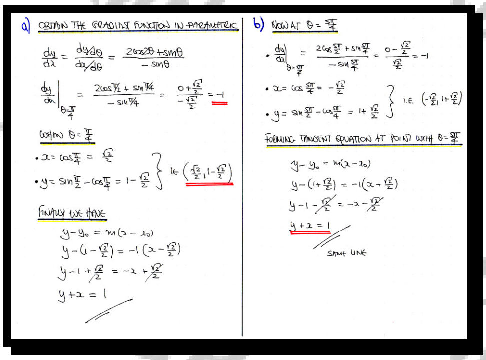
a) Find an equation of the tangent to C at the point where $\theta = \dfrac{\pi}{4}$.

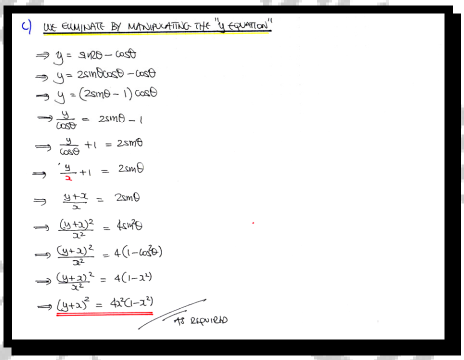
b) Show that the tangent to C at the point where $\theta = \dfrac{5\pi}{4}$ is the same line as the tangent to C at the point where $\theta = \dfrac{\pi}{4}$.

c) Show further that a Cartesian equation of the curve is

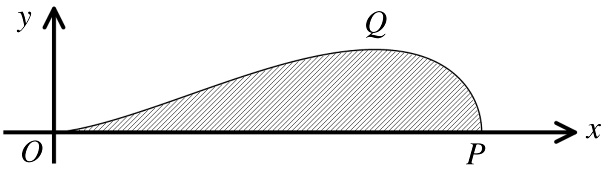
$$4x^2\left(1-x^2\right) = (x+y)^2.$$

MP2-L , $\boxed{x+y=1}$





Question 80 (****)



The figure above shows the curve C with parametric equations

$$x = 2 + 2\sin\theta, \quad y = 2\cos\theta + \sin 2\theta, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}.$$

The curve meets the x axis at the origin O and at the point P. The point Q is the stationary point of C.

a) Find an expression for $\frac{dy}{dx}$ in terms of θ.

b) Hence find the exact coordinates of Q.

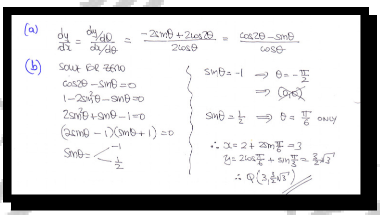
c) Show that the Cartesian equation of C can be written as

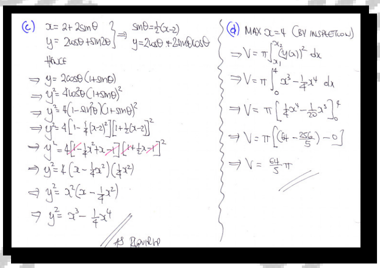
$$y^2 = x^3 - \frac{1}{4}x^4.$$

The finite region bounded by C and the x axis is rotated by 2π radians about the x axis to form a solid of revolution S.

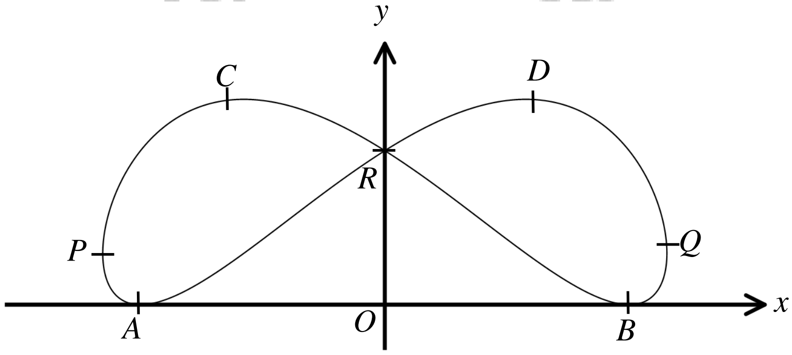
d) Find the exact volume of S.

$\boxed{\frac{dy}{dx} = \frac{\cos 2\theta - \sin\theta}{\cos\theta}}$, $\boxed{Q\left(3, \frac{3}{2}\sqrt{3}\right)}$, $\boxed{V = \frac{64}{5}\pi}$





Question 81 (****)



The figure above shows the curve with parametric equations

$$x = \sin\left(t + \frac{\pi}{6}\right), \quad y = 1 + \cos 2t, \quad 0 \le t < 2\pi .$$

The curve meets the coordinate axes at the points A, B and R.

a) Find an expression for $\frac{dy}{dx}$ in terms of t.

b) Determine the coordinates of the points A, B and R.

At the points C and D the tangent to the curve is parallel to the x axis, and at the points P and Q the tangent to the curve is parallel to the y axis.

c) Find the coordinates of C and D.

d) State the x coordinates of P and Q.

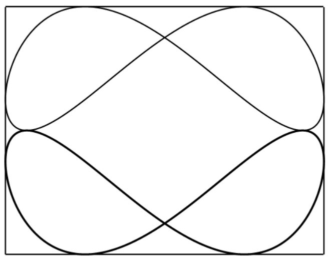
[continues overleaf]

[continues from previous page]

The curve is reflected in the x axis to form the design of a window.

The resulting design fits snugly inside a rectangle.

The sides of this rectangle are tangents to the curve and its reflection, parallel to the coordinate axes. This is shown in the figure below.

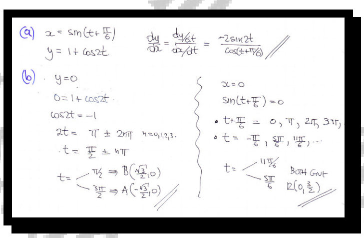


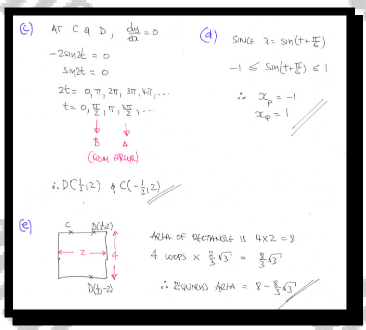
It is given that the area on one of the four loops of the curve is $\frac{2}{3}\sqrt{3}$ square units.

e) Find the exact area of the region which lies within the rectangle but not inside the four loops of the design.

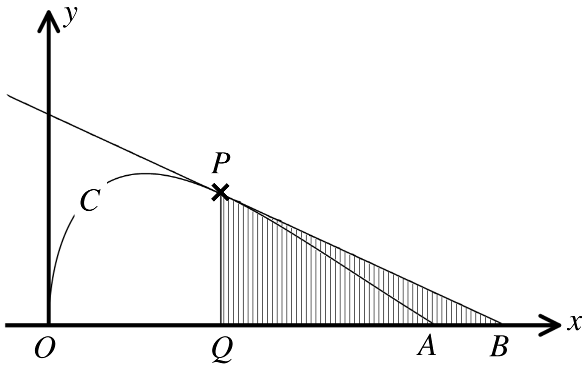
$\boxed{\frac{dy}{dx} = -\frac{2\sin 2t}{\cos\left(t+\frac{\pi}{6}\right)}}$ $\boxed{A\left(-\frac{\sqrt{3}}{2},0\right), B\left(\frac{\sqrt{3}}{2},0\right), R\left(1,\frac{3}{2}\right)}$, $\boxed{C\left(-\frac{1}{2},2\right), D\left(\frac{1}{2},2\right)}$,

$\boxed{x_P = -1,\ x_Q = 1}$, $\boxed{\text{area} = 8 - \frac{8}{3}\sqrt{3}}$





Question 82 (****)



The figure above shows the curve C with parametric equations

$$x = t^2, \quad y = \sin t, \quad 0 \le t \le \pi .$$

The curve crosses the x axis at the origin O and at the point A.

a) Find the coordinates of A.

The point P lies on C where $t = \dfrac{2\pi}{3}$. The line T is a tangent to C at the point P.

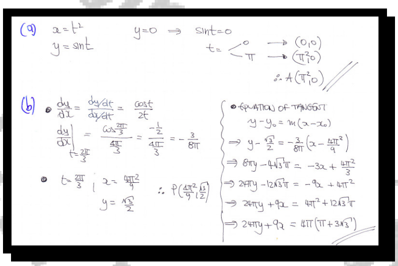
b) Show that the equation of T can be written as

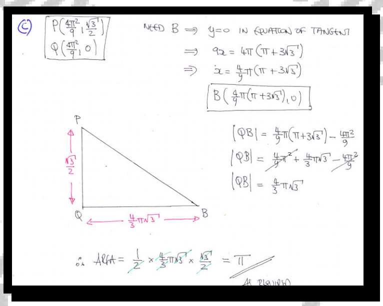
$$24\pi y + 9x = 4\pi\left(\pi + 3\sqrt{3}\right).$$

The point Q lies on the x axis, so that PQ is parallel to the y axis. The point B is the point where T crosses the x axis.

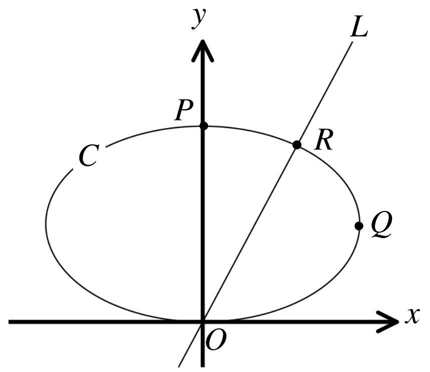
c) Show that the area of the triangle PBQ is π square units.

$\boxed{A\left(\pi^2, 0\right)}$, $\boxed{\text{area} = \pi}$





Question 83 (****)



The figure above shows a curve C and a straight line L, meeting at the origin and at the point R. The points P and Q are such so the tangent to C at those points is horizontal and vertical, respectively.

The curve C has parametric equations

$$x = 2\sqrt{2}\sin 2t, \quad y = 1 - \cos 2t \ , \ 0 \le t < \pi \ ,$$

and the straight line L has equation $y = x$.

a) Find the coordinates of P and Q.

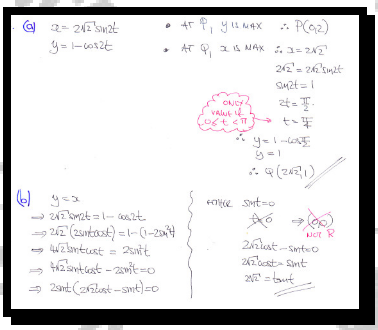
b) Show that at R, $\tan t = 2\sqrt{2}$.

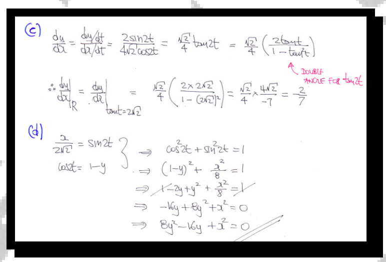
c) Hence determine the exact value of the gradient at R.

d) Show that a Cartesian equation for C is

$$8y^2 - 16y + x^2 = 0.$$

$\boxed{P(0,2), Q\left(2\sqrt{2},1\right)}$, $\boxed{-\frac{2}{7}}$





Question 84 (****)

A curve C is defined by the parametric equations

$$x = \sin^2\theta \ , \ y = \sin 2\theta \quad 0 \le \theta < \pi \ .$$

a) Show that

$$\frac{dy}{dx} = 2\cot 2\theta \ .$$

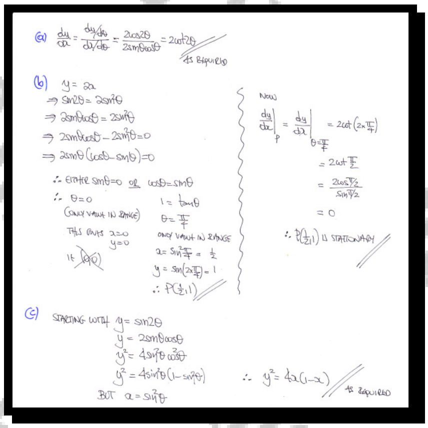
The straight line with equation $y = 2x$ intersects C, at the origin and at the point P.

b) Find the coordinates of P, and show further that P is a stationary point of C.

c) Show further that a Cartesian equation of C is

$$y^2 = 4x(1-x) \ .$$

$$\boxed{P\left(\frac{1}{2},1\right)}$$


(a) $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2\cos 2\theta}{2\sin\theta\cos\theta} = 2\cot 2\theta$ // AS REQUIRED

(b) $y = 2x$
$\Rightarrow \sin 2\theta = 2\sin^2\theta$
$\Rightarrow 2\sin\theta\cos\theta = 2\sin^2\theta$
$\Rightarrow 2\sin\theta\cos\theta - 2\sin^2\theta = 0$
$\Rightarrow 2\sin\theta(\cos\theta - \sin\theta) = 0$

$\therefore$ EITHER $\sin\theta = 0$ OR $\cos\theta = \sin\theta$
$\therefore \theta = 0$ (ONLY VALUE IN RANGE) THIS GIVES $x=0$, $y=0$ IE $(0,0)$
$1 = \tan\theta$, $\theta = \frac{\pi}{4}$ ONLY VALUE IN RANGE
$x = \sin^2\frac{\pi}{4} = \frac{1}{2}$
$y = \sin(2\times\frac{\pi}{4}) = 1$
$\therefore P(\frac{1}{2},1)$ //

NOW
$\left.\frac{dy}{dx}\right|_P = \left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{4}} = 2\cot(2\times\frac{\pi}{4}) = 2\cot\frac{\pi}{2} = \frac{2\cos\pi/2}{\sin\pi/2} = 0$

$\therefore P(\frac{1}{2},1)$ IS STATIONARY //

(c) STARTING WITH $y = \sin 2\theta$
$y = 2\sin\theta\cos\theta$
$y^2 = 4\sin^2\theta\cos^2\theta$
$y^2 = 4\sin^2\theta(1-\sin^2\theta)$
BUT $x = \sin^2\theta$
$\therefore y^2 = 4x(1-x)$ // AS REQUIRED

Question 85 (****)

The curve C is given parametrically by

$$x = \cos t + \sin t - 2, \quad y = \sin 2t, \quad 0 \le t < 2\pi .$$

a) By using appropriate trigonometric identities, show that a Cartesian equation for C is given by

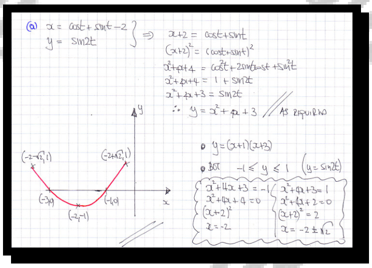
$$y = x^2 + 4x + 3 .$$

b) Sketch the part of C which corresponds to the above parametric equations.

The sketch must include

- the coordinates of any points where C meets the coordinate axes.
- the exact coordinates of the endpoints of C .

graph



Question 86 (****)

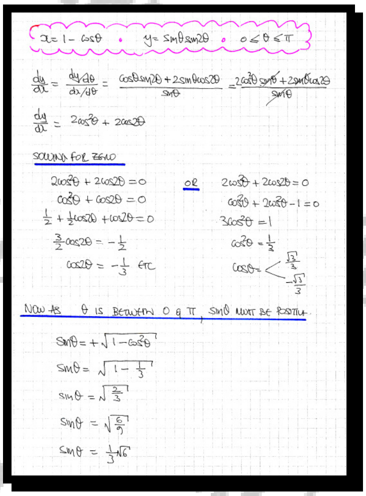
A curve has parametric equations

$$x = 1 - \cos\theta \,, \qquad y = \sin\theta \sin 2\theta \quad , \quad 0 \le \theta \le \pi \,.$$

Determine in exact form the coordinates of the stationary points of the curve.

No credit will be given for methods involving a Cartesian form of this curve.

$$\boxed{}\,, \quad \boxed{\left(\frac{3-\sqrt{3}}{3}, \frac{4\sqrt{3}}{9}\right) \cup \left(\frac{3+\sqrt{3}}{3}, -\frac{4\sqrt{3}}{9}\right)}$$



$x = 1 - \cos\theta$ • $y = \sin\theta\sin 2\theta$ • $0 \le \theta \le \pi$

$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cos\theta\sin 2\theta + 2\sin\theta\cos 2\theta}{\sin\theta} = \frac{2\cos^2\theta\sin\theta + 2\sin\theta\cos 2\theta}{\sin\theta}$

$\frac{dy}{dx} = 2\cos^2\theta + 2\cos 2\theta$

SOLVING FOR ZERO

$2\cos^2\theta + 2\cos 2\theta = 0$

$\cos^2\theta + \cos 2\theta = 0$

$\frac{1}{2} + \frac{1}{2}\cos 2\theta + \cos 2\theta = 0$

$\frac{3}{2}\cos 2\theta = -\frac{1}{2}$

$\cos 2\theta = -\frac{1}{3}$ ETC

OR

$2\cos^2\theta + 2\cos 2\theta = 0$

$\cos^2\theta + 2\cos^2\theta - 1 = 0$

$3\cos^2\theta = 1$

$\cos^2\theta = \frac{1}{3}$

$\cos\theta = \frac{\sqrt{3}}{3}$ or $-\frac{\sqrt{3}}{3}$

NOW AS θ IS BETWEEN 0 & π, $\sin\theta$ MUST BE POSITIVE

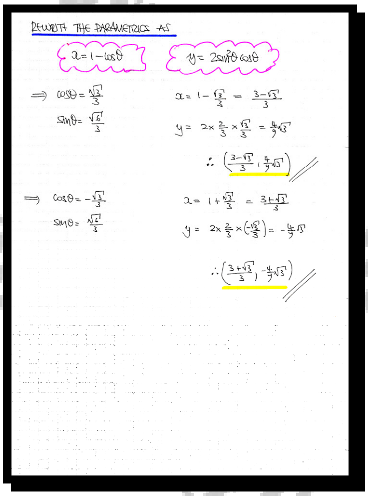
$\sin\theta = +\sqrt{1-\cos^2\theta}$

$\sin\theta = \sqrt{1-\frac{1}{3}}$

$\sin\theta = \sqrt{\frac{2}{3}}$

$\sin\theta = \sqrt{\frac{6}{9}}$

$\sin\theta = \frac{1}{3}\sqrt{6}$



REWRITE THE PARAMETRICS AS

$x = 1 - \cos\theta$ $\quad$ $y = 2\sin^2\theta\cos\theta$

$\Rightarrow \cos\theta = \frac{\sqrt{3}}{3}$, $\sin\theta = \frac{\sqrt{6}}{3}$

$x = 1 - \frac{\sqrt{3}}{3} = \frac{3-\sqrt{3}}{3}$

$y = 2 \times \frac{2}{3} \times \frac{\sqrt{3}}{3} = \frac{4}{9}\sqrt{3}$

$\therefore \left(\frac{3-\sqrt{3}}{3}, \frac{4}{9}\sqrt{3}\right)$

$\Rightarrow \cos\theta = -\frac{\sqrt{3}}{3}$, $\sin\theta = \frac{\sqrt{6}}{3}$

$x = 1 + \frac{\sqrt{3}}{3} = \frac{3+\sqrt{3}}{3}$

$y = 2 \times \frac{2}{3} \times \left(-\frac{\sqrt{3}}{3}\right) = -\frac{4}{9}\sqrt{3}$

$\therefore \left(\frac{3+\sqrt{3}}{3}, -\frac{4}{9}\sqrt{3}\right)$

Question 87 (****)

A curve is given parametrically by the equations

$$x = 3\cos t, \quad y = 4\sin t, \quad 0 \le t \le 2\pi .$$

a) Show that the equation of the tangent to the curve at the point where $t = \theta$ is

$$3y\sin\theta + 4x\cos\theta = 12 .$$

The tangent to the curve at the point where $t = \theta$ meets the y axis at the point $P(0,8)$ and the x axis at the point Q.

b) Find the exact area of the triangle POQ, where O is the origin.

MP2-Q, $8\sqrt{3}$

a) START BY OBTAINING THE GRADIENT FUNCTION IN PARAMETRIC

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4\cos t}{-3\sin t} = -\frac{4\cos t}{3\sin t}$$

$$\left.\frac{dy}{dx}\right|_{t=\theta} = -\frac{4\cos\theta}{3\sin\theta} \quad \text{AT POINT } (3\cos\theta, 4\sin\theta)$$

EQUATION OF TANGENT IS GIVEN BY

$$\Rightarrow y - y_0 = m(x - x_0)$$

$$\Rightarrow y - 4\sin\theta = -\frac{4\cos\theta}{3\sin\theta}(x - 3\cos\theta)$$

$$\Rightarrow 3y\sin\theta - 12\sin^2\theta = -4x\cos\theta + 12\cos^2\theta$$

$$\Rightarrow 3y\sin\theta + 4x\cos\theta = 12\cos^2\theta + 12\sin^2\theta$$

$$\Rightarrow 3y\sin\theta + 4x\cos\theta = 12(\cos^2\theta + \sin^2\theta)$$

$$\Rightarrow 3y\sin\theta + 4x\cos\theta = 12$$

AS REQUIRED

b) SETTING $x=0$ IN THE EQUATION OF THE TANGENT YIELDS $P(0,8)$

$$\Rightarrow 3 \times 8\sin\theta + 0 = 12$$

$$\sin\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{6}, \ \frac{5\pi}{6}$$

EQUATION OF TANGENT BECOMES

$$3y\sin\frac{\pi}{6} + 4x\cos\frac{\pi}{6} = 12 \qquad \text{OR} \qquad 3y\sin\frac{5\pi}{6} + 4x\cos\frac{5\pi}{6} = 12$$

$$\frac{3}{2}y + 2\sqrt{3}x = 12 \qquad \qquad \frac{3}{2}y - 2\sqrt{3}x = 12$$

SOLVING EACH EQUATION FOR $y=0$ TO OBTAIN Q

$$2\sqrt{3}x = 12 \qquad \text{OR} \qquad -2\sqrt{3}x = 12$$

$$x = \frac{6}{\sqrt{3}} \qquad \qquad x = -\frac{6}{\sqrt{3}}$$

$$\therefore x = \pm 2\sqrt{3} \quad \& \quad Q(\pm 2\sqrt{3}, 0)$$

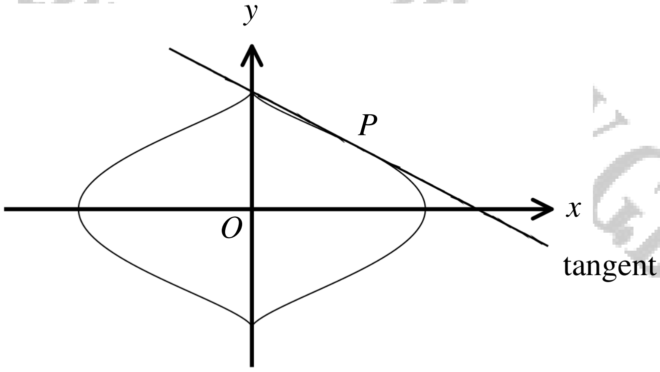
FINALLY AREA CAN BE FOUND

$$\text{AREA} = \frac{1}{2} \times 8 \times |\pm 2\sqrt{3}|$$

$$\text{AREA} = 4 \times 2\sqrt{3}$$

$$\text{AREA} = 8\sqrt{3}$$

Question 88 (****)



The figure above shows the curve C with parametric equations

$$x = a\cos^3\theta \ , \ y = b\sin\theta \, , \ \ 0 \le \theta < 2\pi \, ,$$

where a and b are positive constants.

The point P lies on C, where $\theta = \dfrac{\pi}{6}$.

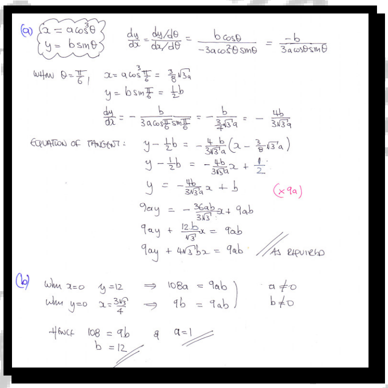
a) Show that an equation of the tangent to C at P is

$$9ay + 4bx\sqrt{3} = 9ab \, .$$

The tangent to C at P crosses the coordinate axes at $(0,12)$ and $\left(\dfrac{3\sqrt{3}}{4}, 0\right)$.

b) Find the value of a and the value of b.

$a = 1, \ b = 12$


(a) $x = a\cos^3\theta$, $y = b\sin\theta$; $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{b\cos\theta}{-3a\cos^2\theta\sin\theta} = \frac{-b}{3a\cos\theta\sin\theta}$

When $\theta = \frac{\pi}{6}$, $x = a\cos^3\frac{\pi}{6} = \frac{3}{8}\sqrt{3}a$, $y = b\sin\frac{\pi}{6} = \frac{1}{2}b$

$\frac{dy}{dx} = -\frac{b}{3a\cos\frac{\pi}{6}\sin\frac{\pi}{6}} = -\frac{b}{\frac{3}{4}\sqrt{3}a} = -\frac{4b}{3\sqrt{3}a}$

EQUATION OF TANGENT: $y - \frac{1}{2}b = -\frac{4b}{3\sqrt{3}a}\left(x - \frac{3}{8}\sqrt{3}a\right)$

$y - \frac{1}{2}b = -\frac{4b}{3\sqrt{3}a}x + \frac{1}{2}b$

$y = -\frac{4b}{3\sqrt{3}a}x + b$ ($\times 9a$)

$9ay = -\frac{36ab}{3\sqrt{3}a}x + 9ab$

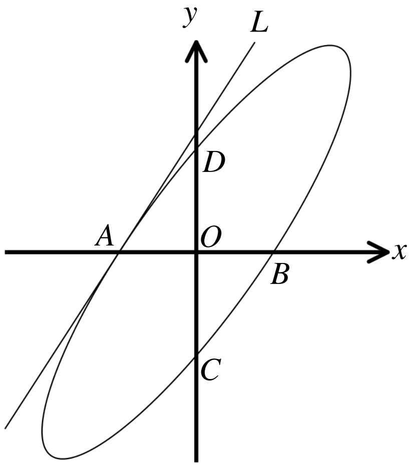
$9ay + \frac{12b}{\sqrt{3}}x = 9ab$

$9ay + 4\sqrt{3}bx = 9ab$ // AS REQUIRED

(b) when $x=0$, $y=12$ ⟹ $108a = 9ab$; when $y=0$, $x = \frac{3\sqrt{3}}{4}$ ⟹ $9b = 9ab$; $a \ne 0$, $b \ne 0$

HENCE $108 = 9b$, $b = 12$ & $a = 1$

Question 89 (****)



The figure above shows an ellipse with parametric equations

$$x = 2\cos\theta \quad y = 6\sin\left(\theta + \frac{\pi}{3}\right), \quad 0 \le \theta < 2\pi .$$

The curve meets the coordinate axes at the points A, B, C and D.

a) Determine the coordinates of the points A, B, C and D.

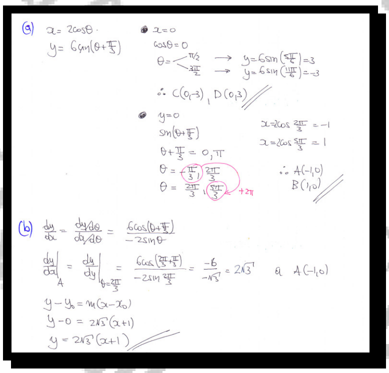
The straight line L is the tangent to the ellipse at the point A.

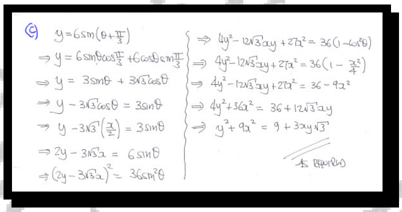
b) Find an equation of L.

c) Show that a Cartesian equation of the ellipse is

$$y^2 + 9x^2 = 9 + 3xy\sqrt{3} .$$

$\boxed{A(-1,0),\ B(1,0),\ C(-3,0),\ D(-3,0)}$, $\boxed{y = 2\sqrt{3}(x+1)}$





Question 90 (****)

A curve C is given parametrically by the equations

$$x = \sin^2\theta,\ \ y = 6\sin\theta - \sin^3\theta,\ \ \frac{\pi}{2} < \theta < \frac{\pi}{2}.$$

a) Find an expression for $\frac{dy}{dx}$, in terms of $\sin\theta$.

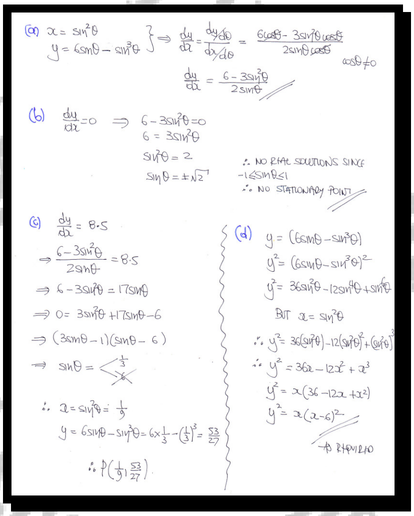
b) Hence show that C has no stationary points.

c) Determine the exact coordinates of the point on C, where the gradient is $8\frac{1}{2}$.

d) Show that a Cartesian equation of C is

$$y^2 = x(x-6)^2.$$

$\boxed{\frac{dy}{dx} = \frac{6-3\sin^2\theta}{2\sin\theta}}$, $\boxed{P\left(\frac{1}{9}, \frac{53}{27}\right)}$


(a) $x = \sin^2\theta$, $y = 6\sin\theta - \sin^3\theta$ $\Rightarrow \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{6\cos\theta - 3\sin^2\theta\cos\theta}{2\sin\theta\cos\theta}$ $\cos\theta \neq 0$

$\frac{dy}{dx} = \frac{6-3\sin^2\theta}{2\sin\theta}$

(b) $\frac{dy}{dx} = 0 \Rightarrow 6 - 3\sin^2\theta = 0$, $6 = 3\sin^2\theta$, $\sin^2\theta = 2$, $\sin\theta = \pm\sqrt{2}$

∴ NO REAL SOLUTIONS SINCE $-1 \leq \sin\theta \leq 1$
∴ NO STATIONARY POINT

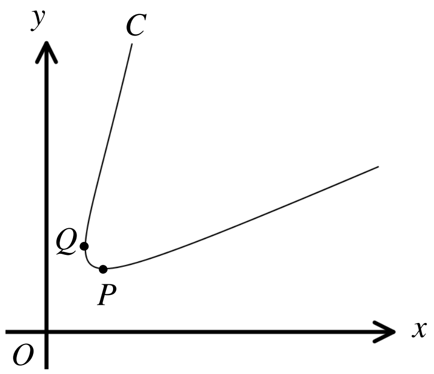
(c) $\frac{dy}{dx} = 8.5$
$\Rightarrow \frac{6-3\sin^2\theta}{2\sin\theta} = 8.5$
$\Rightarrow 6 - 3\sin^2\theta = 17\sin\theta$
$\Rightarrow 0 = 3\sin^2\theta + 17\sin\theta - 6$
$\Rightarrow (3\sin\theta - 1)(\sin\theta + 6)$
$\Rightarrow \sin\theta = \frac{1}{3}$ ~~-6~~

∴ $x = \sin^2\theta = \frac{1}{9}$
$y = 6\sin\theta - \sin^3\theta = 6\times\frac{1}{3} - \left(\frac{1}{3}\right)^3 = \frac{53}{27}$

∴ $P\left(\frac{1}{9}, \frac{53}{27}\right)$

(d) $y = (6\sin\theta - \sin^3\theta)$
$y^2 = (6\sin\theta - \sin^3\theta)^2$
$y^2 = 36\sin^2\theta - 12\sin^4\theta + \sin^6\theta$
BUT $x = \sin^2\theta$
∴ $y^2 = 36(\sin^2\theta) - 12(\sin^2\theta)^2 + (\sin^2\theta)^3$
∴ $y^2 = 36x - 12x^2 + x^3$
$y^2 = x(36 - 12x + x^2)$
$y^2 = x(x-6)^2$
— AS REQUIRED

Question 91 (****)



The figure above shows a curve C with parametric equations

$$x = \frac{t^2}{t-1}, \quad y = \frac{t^3}{t-1}, \quad t \in \mathbb{R}, \ t > 1.$$

The points P and Q lie on C so that the tangents to the curve at those points are horizontal and vertical respectively.

a) Show that

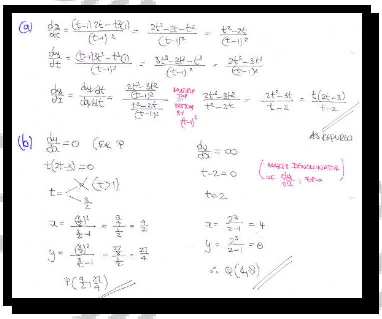
$$\frac{dy}{dx} = \frac{t(2t-3)}{t-2}.$$

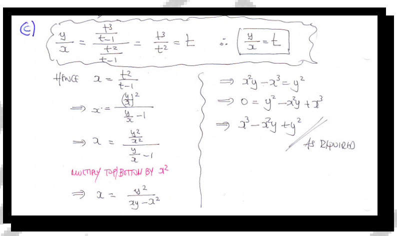
b) Find the coordinates of P and Q.

c) Show further that a Cartesian equation for C is

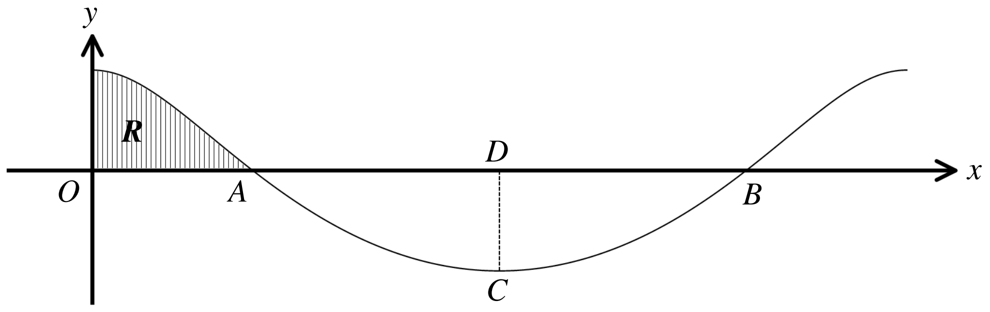
$$y^2 - yx^2 + x^3 = 0.$$

$P\left(\frac{9}{2}, \frac{27}{4}\right), \ Q(4,8)$





Question 92 (****)



The figure above shows the curve defined by the parametric equations

$$x = 4\theta - \sin\theta, \quad y = 2\cos\theta \;, \text{ for } \; 0 \le \theta < 2\pi \,.$$

The curve crosses the x axis at points A and B.

The point C is the minimum point on the curve and CD is perpendicular to the x axis and a line of symmetry for the curve.

a) Find the exact coordinates of A, B and C.

b) Show that an equation of the tangent to the curve at the point A is given by

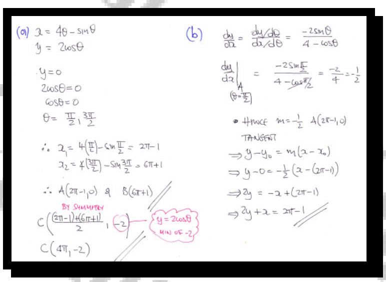
$$x + 2y = 2\pi - 1 \,.$$

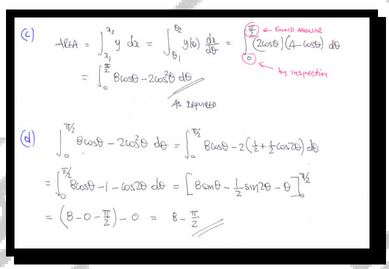
c) Show that the area of the region R bounded by the curve and the coordinate axes is given by

$$\int_0^{\frac{\pi}{2}} 8\cos\theta - 2\cos^2\theta \; d\theta \,.$$

d) Find an exact value for this integral.

$\boxed{A(2\pi-1,0)}$, $\boxed{B(6\pi+1,0)}$, $\boxed{C(4\pi,-2)}$, $\boxed{8-\frac{\pi}{2}}$





Question 93 (****)

The curve C is given parametrically by the equations

$$x = \cos^3 t,\ \ y = \sin^3 t,\ \ 0 < t < \frac{\pi}{2}.$$

a) Show that an equation of the normal to C at the point where $t = \theta$ is

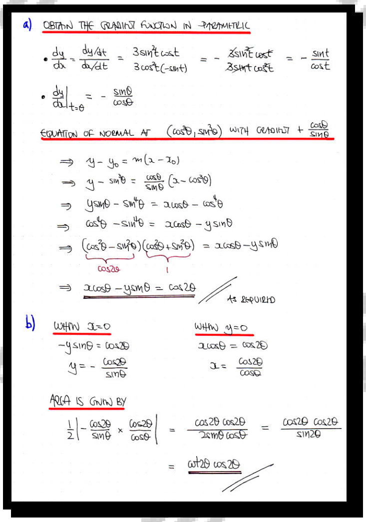
$$x\cos\theta - y\sin\theta = \cos 2\theta .$$

The normal to C at the point where $t = \theta$ meets the coordinate axes at the points A and B.

b) Given that O is the origin, show further that the area of the triangle AOB is

$$\cos 2\theta \cot 2\theta .$$

[ ] , [proof]


a) OBTAIN THE GRADIENT FUNCTION IN PARAMETRIC

- $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3\sin^2 t\cos t}{3\cos^2 t(-\sin t)} = -\frac{3\sin^2 t\cos t}{3\sin t\cos^2 t} = -\frac{\sin t}{\cos t}$
- $\left.\frac{dy}{dx}\right|_{t=\theta} = -\frac{\sin\theta}{\cos\theta}$

EQUATION OF NORMAL AT $(\cos^3\theta, \sin^3\theta)$ WITH GRADIENT $+\frac{\cos\theta}{\sin\theta}$

$\Rightarrow y - y_0 = m(x - x_0)$

$\Rightarrow y - \sin^3\theta = \frac{\cos\theta}{\sin\theta}(x - \cos^3\theta)$

$\Rightarrow y\sin\theta - \sin^4\theta = x\cos\theta - \cos^4\theta$

$\Rightarrow \cos^4\theta - \sin^4\theta = x\cos\theta - y\sin\theta$

$\Rightarrow \underbrace{(\cos^2\theta - \sin^2\theta)}_{\cos 2\theta}\underbrace{(\cos^2\theta + \sin^2\theta)}_{1} = x\cos\theta - y\sin\theta$

$\Rightarrow x\cos\theta - y\sin\theta = \cos 2\theta$ // AS REQUIRED

b) WHEN $x = 0$

$-y\sin\theta = \cos 2\theta$

$y = -\frac{\cos 2\theta}{\sin\theta}$

WHEN $y = 0$

$x\cos\theta = \cos 2\theta$

$x = \frac{\cos 2\theta}{\cos\theta}$

AREA IS GIVEN BY

$\frac{1}{2}\left|-\frac{\cos 2\theta}{\sin\theta} \times \frac{\cos 2\theta}{\cos\theta}\right| = \frac{\cos 2\theta\cos 2\theta}{2\sin\theta\cos\theta} = \frac{\cos 2\theta\cos 2\theta}{\sin 2\theta}$

$= \cot 2\theta\cos 2\theta$ //

Question 94 (****)

The curve C is given parametrically by the equations

$$x = 3t, \quad y = \frac{3}{t}, \quad t \neq 0.$$

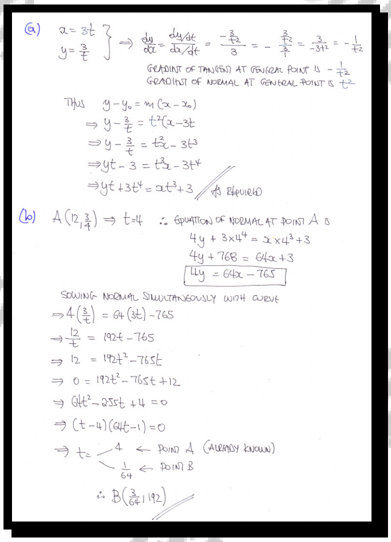
a) Show that an equation of the normal to C at the point with parameter t is

$$yt + 3t^4 = xt^3 + 3.$$

The point $A\left(12, \frac{3}{4}\right)$ lies on C. The normal at $A\left(12, \frac{3}{4}\right)$ meets the curve again at B.

b) Determine the coordinates of B.

$$\boxed{B\left(\frac{3}{64}, 192\right)}$$


(a) $x = 3t$, $y = \frac{3}{t}$ $\Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-\frac{3}{t^2}}{3} = -\frac{\frac{3}{t^2}}{\frac{3}{1}} = \frac{3}{-3t^2} = -\frac{1}{t^2}$

GRADIENT OF TANGENT AT GENERAL POINT IS $-\frac{1}{t^2}$

GRADIENT OF NORMAL AT GENERAL POINT IS t^2

THUS $y - y_0 = m(x - x_0)$

$\Rightarrow y - \frac{3}{t} = t^2(x - 3t)$

$\Rightarrow y - \frac{3}{t} = t^2x - 3t^3$

$\Rightarrow yt - 3 = t^3x - 3t^4$

$\Rightarrow yt + 3t^4 = xt^3 + 3$ // AS REQUIRED

(b) $A\left(12, \frac{3}{4}\right) \Rightarrow t = 4$ $\therefore$ EQUATION OF NORMAL AT POINT A IS

$4y + 3 \times 4^4 = x \times 4^3 + 3$

$4y + 768 = 64x + 3$

$\boxed{4y = 64x - 765}$

SOLVING NORMAL SIMULTANEOUSLY WITH CURVE

$\Rightarrow 4\left(\frac{3}{t}\right) = 64(3t) - 765$

$\Rightarrow \frac{12}{t} = 192t - 765$

$\Rightarrow 12 = 192t^2 - 765t$

$\Rightarrow 0 = 192t^2 - 765t + 12$

$\Rightarrow 64t^2 - 255t + 4 = 0$

$\Rightarrow (t - 4)(64t - 1) = 0$

$\Rightarrow t = 4$ ← POINT A (ALREADY KNOWN)

$t = \frac{1}{64}$ ← POINT B

$\therefore B\left(\frac{3}{64}, 192\right)$ //

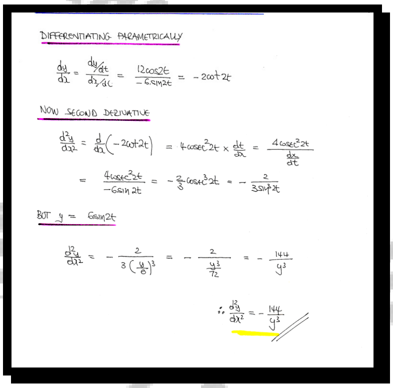
Question 95 (****)

A curve is defined parametrically by the equations

$$x = 3\cos 2t, \quad y = 6\sin 2t, \quad 0 \le t < 2\pi.$$

Express $\dfrac{d^2y}{dx^2}$ in terms of y.

$\boxed{}$, $\boxed{\dfrac{d^2y}{dx^2} = -\dfrac{144}{y^3}}$



DIFFERENTIATING PARAMETRICALLY

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{12\cos 2t}{-6\sin 2t} = -2\cot 2t$$

NOW SECOND DERIVATIVE

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(-2\cot 2t\right) = 4\operatorname{cosec}^2 2t \times \frac{dt}{dx} = \frac{4\operatorname{cosec}^2 2t}{\frac{dx}{dt}}$$

$$= \frac{4\operatorname{cosec}^2 2t}{-6\sin 2t} = -\frac{2}{3}\operatorname{cosec}^3 2t = -\frac{2}{3\sin^3 2t}$$

BUT $y = 6\sin 2t$

$$\frac{d^2y}{dx^2} = -\frac{2}{3\left(\frac{y}{6}\right)^3} = -\frac{2}{\frac{y^3}{72}} = -\frac{144}{y^3}$$

$$\therefore \frac{d^2y}{dx^2} = -\frac{144}{y^3}$$

Question 96 (****+)

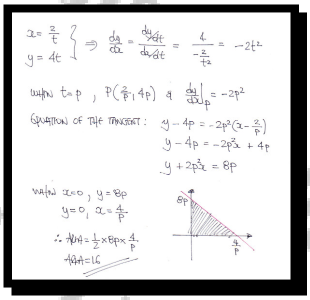
The curve C is given by the parametric equations

$$x = \frac{2}{t}, \quad y = 4t \ , \ t > 0 .$$

The tangent to the C at the point P where $t = p$, meets the coordinate axes at the points A and B.

Show that the area of the triangle OAB, where O is the origin, is independent of p, and state that area.

$\boxed{\text{area} = 16}$



Question 97 (****+)

The curve C is given parametrically by the equations

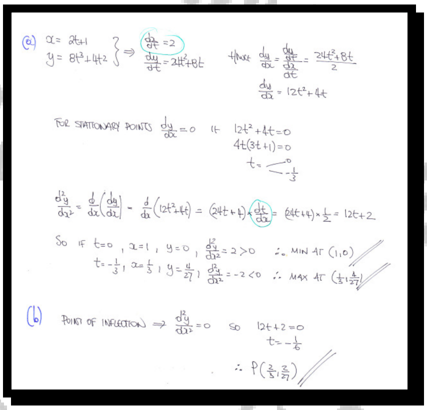
$$x = 2t + 1,\ y = 8t^3 + 4t^2,\ t \in \mathbb{R}.$$

a) Find the coordinates of the stationary points of C, and determine their nature.

It is further given that C has a single point of inflection at P.

b) Determine the coordinates of P.

$\boxed{\text{min}(1,0)}$, $\boxed{\text{max}\left(\frac{1}{3},\frac{4}{27}\right)}$, $\boxed{P\left(\frac{1}{3},\frac{2}{27}\right)}$


(a) $x = 2t+1$, $y = 8t^3+4t^2$ $\Rightarrow$ $\frac{dx}{dt} = 2$, $\frac{dy}{dt} = 24t^2+8t$

Hence $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{24t^2+8t}{2}$

$\frac{dy}{dx} = 12t^2+4t$

FOR STATIONARY POINTS $\frac{dy}{dx} = 0$ i.e. $12t^2+4t=0$

$4t(3t+1)=0$

$t = 0,\ -\frac{1}{3}$

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(12t^2+4t) = (24t+4)\times\frac{dt}{dx} = (24t+4)\times\frac{1}{2} = 12t+2$

So if $t=0$, $x=1$, $y=0$, $\frac{d^2y}{dx^2} = 2>0$ $\therefore$ MIN AT $(1,0)$

$t=-\frac{1}{3}$, $x=\frac{1}{3}$, $y=\frac{4}{27}$, $\frac{d^2y}{dx^2} = -2<0$ $\therefore$ MAX AT $\left(\frac{1}{3},\frac{4}{27}\right)$

(b) POINT OF INFLECTION $\Rightarrow \frac{d^2y}{dx^2} = 0$ so $12t+2=0$

$t = -\frac{1}{6}$

$\therefore P\left(\frac{2}{3},\frac{2}{27}\right)$

Question 98 (****+)

The curve C is given by the parametric equations

$$x = 3at, \quad y = at^3, \quad t \in \mathbb{R},$$

where a is a positive constant.

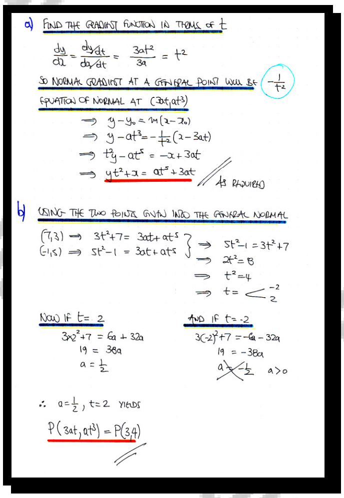
a) Show that an equation of the normal to C at the general point $\left(3at, at^3\right)$ is

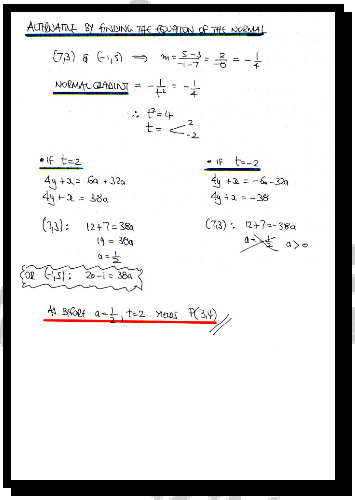
$$yt^2 + x = 3at + at^5 .$$

The normal to C at some point P, passes through the points with coordinates $(7,3)$ and $(-1,5)$.

b) Determine the coordinates of P.

$\boxed{}$, $\boxed{P(3,4)}$





Question 99 (****+)

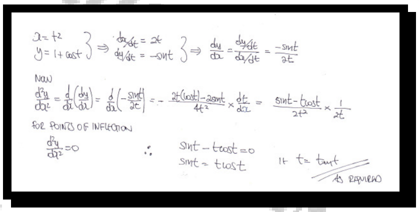
The curve C is given parametrically by the equations

$$x = t^2, \ y = 1 + \cos t, \ t \in \mathbb{R}.$$

Show that the value of t at any points of inflection of C is a solution of the equation

$$t = \tan t.$$

proof



$$\left.\begin{array}{l} x = t^2 \\ y = 1 + \cos t \end{array}\right\} \Rightarrow \left.\begin{array}{l} dx/dt = 2t \\ dy/dt = -\sin t \end{array}\right\} \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-\sin t}{2t}$$

NOW

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(-\frac{\sin t}{2t}\right) = -\frac{2t(\cos t) - 2\sin t}{4t^2} \times \frac{dt}{dx} = \frac{\sin t - t\cos t}{2t^2} \times \frac{1}{2t}$$

FOR POINTS OF INFLECTION

$$\frac{d^2y}{dx^2} = 0 \quad \therefore \quad \sin t - t\cos t = 0, \quad \sin t = t\cos t, \quad \text{i.e. } t = \tan t$$

AS REQUIRED

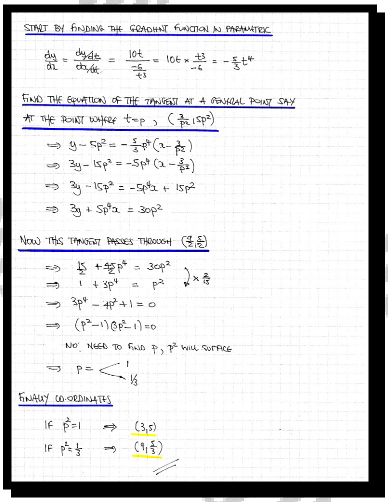
Question 100 (****+)

A curve has parametric equations

$$x = \frac{3}{t^2}, \qquad y = 5t^2 \ , \quad t > 0 .$$

If the tangent to the curve at the point P passes through the point with coordinates $\left(\frac{9}{2}, \frac{5}{2}\right)$, determine the possible coordinates of P.

$(3,5) \cup \left(9, \frac{5}{3}\right)$



START BY FINDING THE GRADIENT FUNCTION IN PARAMETRIC

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{10t}{-\frac{6}{t^3}} = 10t \times \frac{t^3}{-6} = -\frac{5}{3}t^4$$

FIND THE EQUATION OF THE TANGENT AT A GENERAL POINT SAY AT THE POINT WHERE $t = p$, $\left(\frac{3}{p^2}, 5p^2\right)$

$$\Rightarrow y - 5p^2 = -\frac{5}{3}p^4\left(x - \frac{3}{p^2}\right)$$
$$\Rightarrow 3y - 15p^2 = -5p^4\left(x - \frac{3}{p^2}\right)$$
$$\Rightarrow 3y - 15p^2 = -5p^4x + 15p^2$$
$$\Rightarrow 3y + 5p^4x = 30p^2$$

NOW THIS TANGENT PASSES THROUGH $\left(\frac{9}{2}, \frac{5}{2}\right)$

$$\Rightarrow \frac{15}{2} + \frac{45}{2}p^4 = 30p^2 \quad \times \frac{2}{15}$$
$$\Rightarrow 1 + 3p^4 = p^2$$
$$\Rightarrow 3p^4 - 4p^2 + 1 = 0$$
$$\Rightarrow (p^2 - 1)(3p^2 - 1) = 0$$

NO NEED TO FIND p, p^2 WILL SUFFICE

$$\Rightarrow p^2 = 1 \text{ or } \tfrac{1}{3}$$

FINALLY CO-ORDINATES

IF $p^2 = 1 \Rightarrow (3, 5)$

IF $p^2 = \frac{1}{3} \Rightarrow \left(9, \frac{5}{3}\right)$

Question 101 (****+)

A curve is given parametrically by the equations

$$x = 3\sin 2\theta, \quad y = 4\cos 2\theta, \quad 0 \le \theta \le 2\pi$$

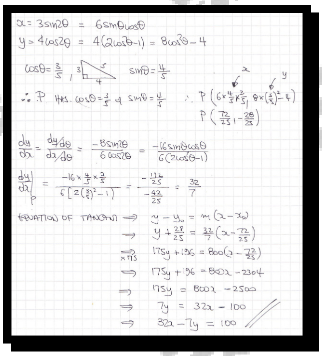
The point P lies on the curve so that

$$\cos\theta = \frac{3}{5}, \; 0 \le \theta \le \frac{\pi}{2}.$$

Show that an equation of the tangent at P is

$$32x - 7y = 100$$

proof



Question 102 (****+)

$$y = a^x, \ a > 0, \ x \in \mathbb{R}$$

a) Show clearly that

$$\frac{dy}{dx} = a^x \ln a .$$

A curve C is given by the parametric equations

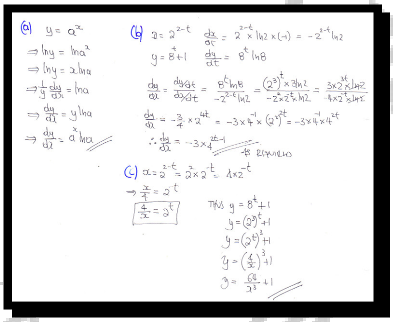
$$x = 2^{2-t}, \quad y = 8^t + 1, \quad t \in \mathbb{R}.$$

b) Show that for points on C,

$$\frac{dy}{dx} = -3 \times 4^{2t-1}.$$

c) Find in simplified form a Cartesian equation for C.

$$\boxed{y = \frac{64}{x^3} + 1}$$



Question 103 (****+)

A curve C is given parametrically by

$$x = \frac{4\cos t}{1+4\sin^2 t}, \quad y = \frac{4\sin 2t}{1+4\sin^2 t}, \quad t \in \mathbb{R}.$$

Show that …

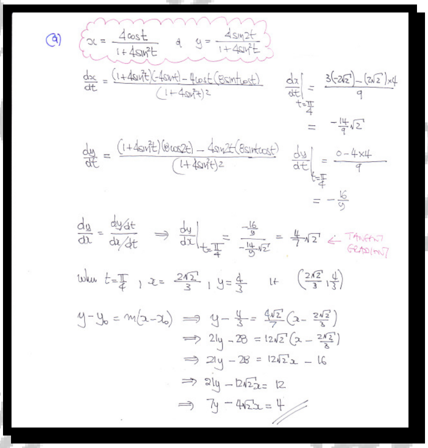
a) … an equation of the tangent at the point where $t = \dfrac{\pi}{4}$ is

$$7y - 4\sqrt{2}x = 4.$$

b) … a Cartesian equation of C is

$$\left(x^2+y^2\right)^2 = 4\left(4x^2 - y^2\right).$$

proof



(a) $x = \dfrac{4\cos t}{1+4\sin^2 t}$ & $y = \dfrac{4\sin 2t}{1+4\sin^2 t}$

$\dfrac{dx}{dt} = \dfrac{(1+4\sin^2 t)(-4\sin t) - 4\cos t(8\sin t\cos t)}{(1+4\sin^2 t)^2}$ $\qquad \left.\dfrac{dx}{dt}\right|_{t=\frac{\pi}{4}} = \dfrac{3(-2\sqrt{2}) - (2\sqrt{2})\times 4}{9} = -\dfrac{14}{9}\sqrt{2}$

$\dfrac{dy}{dt} = \dfrac{(1+4\sin^2 t)(8\cos 2t) - 4\sin 2t(8\sin t\cos t)}{(1+4\sin^2 t)^2}$ $\qquad \left.\dfrac{dy}{dt}\right|_{t=\frac{\pi}{4}} = \dfrac{0-4\times 4}{9} = -\dfrac{16}{9}$

$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} \Rightarrow \left.\dfrac{dy}{dx}\right|_{t=\frac{\pi}{4}} = \dfrac{-\frac{16}{9}}{-\frac{14}{9}\sqrt{2}} = \dfrac{4}{7}\sqrt{2}$ ← TANGENT GRADIENT

when $t=\frac{\pi}{4}$, $x = \frac{2\sqrt{2}}{3}$, $y = \frac{4}{3}$ i.e. $\left(\frac{2\sqrt{2}}{3}, \frac{4}{3}\right)$

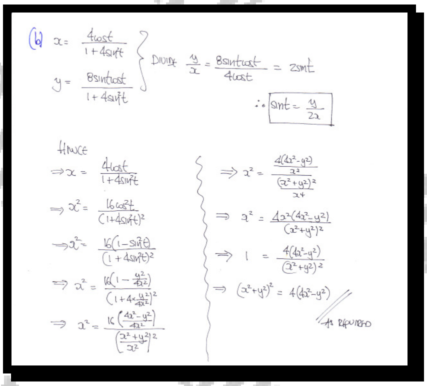
$y - y_0 = m(x-x_0) \Rightarrow y - \frac{4}{3} = \frac{4\sqrt{2}}{7}\left(x - \frac{2\sqrt{2}}{3}\right)$

$\Rightarrow 21y - 28 = 12\sqrt{2}\left(x - \frac{2\sqrt{2}}{3}\right)$

$\Rightarrow 21y - 28 = 12\sqrt{2}x - 16$

$\Rightarrow 21y - 12\sqrt{2}x = 12$

$\Rightarrow 7y - 4\sqrt{2}x = 4$



(b) $x = \dfrac{4\cos t}{1+4\sin^2 t}$, $y = \dfrac{8\sin t\cos t}{1+4\sin^2 t}$ DIVIDE $\dfrac{y}{x} = \dfrac{8\sin t\cos t}{4\cos t} = 2\sin t$

$\therefore \sin t = \dfrac{y}{2x}$

HENCE

$\Rightarrow x = \dfrac{4\cos t}{1+4\sin^2 t}$

$\Rightarrow x^2 = \dfrac{16\cos^2 t}{(1+4\sin^2 t)^2}$

$\Rightarrow x^2 = \dfrac{16(1-\sin^2\theta)}{(1+4\sin^2 t)^2}$

$\Rightarrow x^2 = \dfrac{16\left(1-\frac{y^2}{4x^2}\right)}{\left(1+4\times\frac{y^2}{4x^2}\right)^2}$

$\Rightarrow x^2 = \dfrac{16\left(\frac{4x^2-y^2}{4x^2}\right)}{\left(\frac{x^2+y^2}{x^2}\right)^2}$

$\Rightarrow x^2 = \dfrac{\frac{4(4x^2-y^2)}{x^2}}{\frac{(x^2+y^2)^2}{x^4}}$

$\Rightarrow x^2 = \dfrac{4x^2(4x^2-y^2)}{(x^2+y^2)^2}$

$\Rightarrow 1 = \dfrac{4(4x^2-y^2)}{(x^2+y^2)^2}$

$\Rightarrow (x^2+y^2)^2 = 4(4x^2-y^2)$

AS REQUIRED

Question 104 (****+)

A curve is defined by the parametric equations

$$x = 2\cos t\,,\ \ y = 4\sin t\,,\ \ 0 \le t \le \frac{\pi}{2}.$$

a) Show that an equation of the tangent to the curve at the point P where $t = \theta$ can be written as

$$y\sin\theta + 2x\cos\theta = 4\,.$$

The tangent to curve at P meets the coordinate axes at the points A and B.

The triangle OAB, where O is the origin, has the least possible area.

b) Find the coordinates of P.

$\boxed{}$, $\boxed{P\left(\sqrt{2}, 2\sqrt{2}\right)}$

a) $x = 2\cos t \quad y = 4\sin t \quad 0 \le t \le \frac{\pi}{2}$

- $\frac{dx}{dt} = -2\sin t$ • $\frac{dy}{dt} = 4\cos t$

$\Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4\cos t}{-2\sin t} = -2\cot t$

$\Rightarrow \left.\frac{dy}{dx}\right|_{t=\theta} = -2\cot\theta$

EQUATION OF TANGENT THROUGH $(2\cos\theta, 4\sin\theta)$

$\Rightarrow y - 4\sin\theta = -2\cot\theta\,(x - 2\cos\theta)$

$\Rightarrow y - 4\sin\theta = -\frac{2\cos\theta}{\sin\theta}(x - 2\cos\theta)$

$\Rightarrow y\sin\theta - 4\sin^2\theta = (-2\cos\theta)x + 4\cos^2\theta$

$\Rightarrow y\sin\theta + 2x\cos\theta = 4\cos^2\theta + 4\sin^2\theta$

$\Rightarrow y\sin\theta + 2x\cos\theta = 4(\cos^2\theta + \sin^2\theta)$

$\Rightarrow y\sin\theta + 2x\cos\theta = 4$ // AS REQUIRED

b) WHEN $x = 0$, $y\sin\theta = 4$, $y = \frac{4}{\sin\theta}$ $\quad A\left(0, \frac{4}{\sin\theta}\right)$

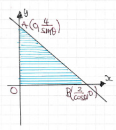
WHEN $y = 0$, $2x\cos\theta = 4$, $x = \frac{2}{\cos\theta}$ $\quad B\left(\frac{2}{\cos\theta}, 0\right)$

AREA OF $\overset{\triangle}{OAB} = \frac{1}{2}|OA||OB|$

$= \frac{1}{2} \times \frac{4}{\sin\theta} \times \frac{2}{\cos\theta}$

$= \frac{8}{2\sin\theta\cos\theta}$

$= \frac{8}{\sin 2\theta}$



AS $0 \le \theta \le \frac{\pi}{2}$ $\Rightarrow$ $\sin 2\theta$ LIES BETWEEN 0 & 1

$\Rightarrow \frac{1}{\sin 2\theta}$ IS AT LEAST 1

$\Rightarrow \frac{8}{\sin 2\theta}$ IS AT LEAST 8

$\Rightarrow$ MINIMUM AREA OF 8 UNITS

$\Rightarrow$ WHICH OCCURS WHEN $\theta = \frac{\pi}{4}$

$\Rightarrow P(\sqrt{2}, 2\sqrt{2})$ //

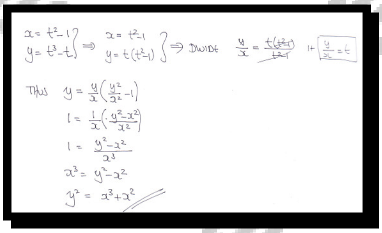
Question 105 (****+)

A curve C is given parametrically by the equations

$$x = t^2 - 1, \quad y = t^3 - t, \quad t \in \mathbb{R}.$$

Find a Cartesian equation C, in the form $y^2 = f(x)$.

$$\boxed{y^2 = x^3 + x^2}$$



Question 106 (****+)

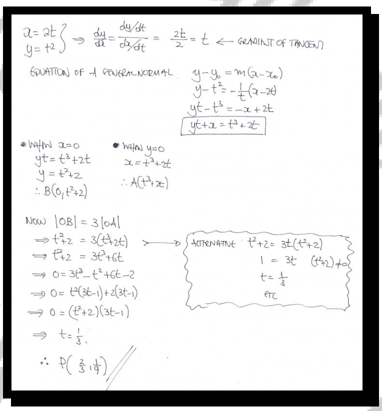
A curve is given parametrically by the equations

$$x = 2t, \quad y = t^2, \quad t \in \mathbb{R}.$$

The normal to the curve at the point P meets the x axis at the point A and the y axis at the point B.

Given that $|OB| = 3|OA|$, where O is the origin, determine the coordinates of P.

$$\boxed{P\left(\frac{2}{3}, \frac{1}{9}\right)}$$



Question 107 (****+)

A curve is given parametrically by the equations

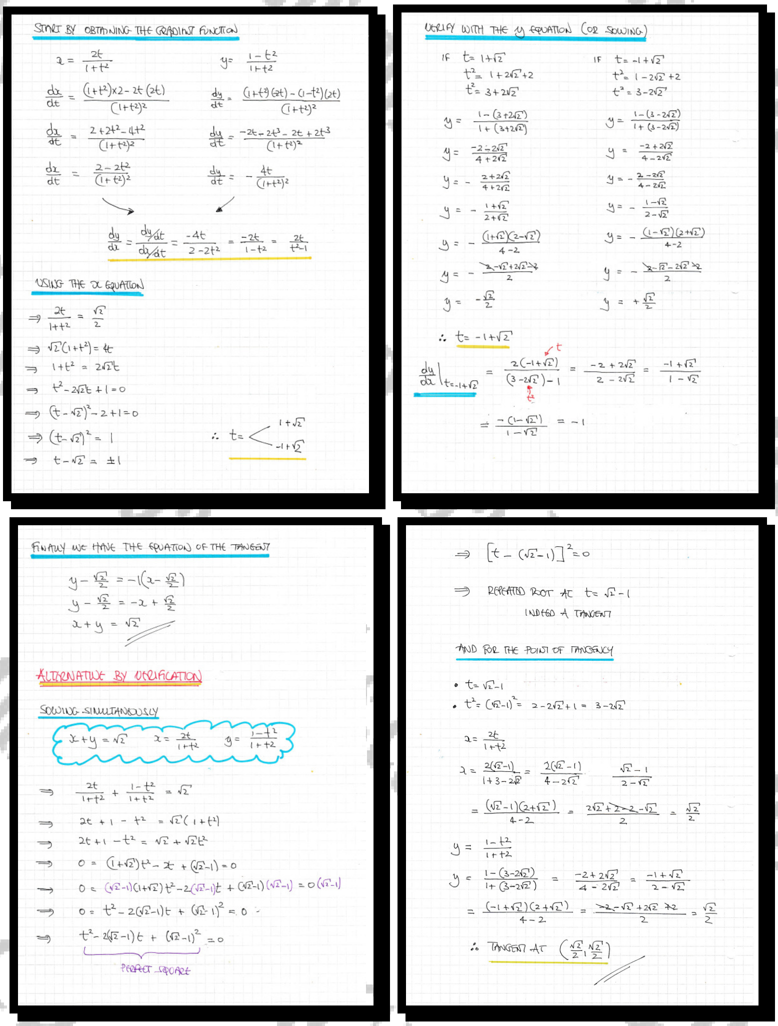
$$x = \frac{2t}{1+t^2}, \quad y = \frac{1-t^2}{1+t^2}, \quad t \in \mathbb{R}.$$

The point $P\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ lies on this curve.

Show that an equation of the tangent at the point P is given by

$$x + y = \sqrt{2}.$$

proof



START BY OBTAINING THE GRADIENT FUNCTION

$x = \frac{2t}{1+t^2}$ $\qquad$ $y = \frac{1-t^2}{1+t^2}$

$\frac{dx}{dt} = \frac{(1+t^2)\times 2 - 2t(2t)}{(1+t^2)^2}$ $\qquad$ $\frac{dy}{dt} = \frac{(1+t^2)(-2t) - (1-t^2)(2t)}{(1+t^2)^2}$

$\frac{dx}{dt} = \frac{2+2t^2-4t^2}{(1+t^2)^2}$ $\qquad$ $\frac{dy}{dt} = \frac{-2t-2t^3-2t+2t^3}{(1+t^2)^2}$

$\frac{dx}{dt} = \frac{2-2t^2}{(1+t^2)^2}$ $\qquad$ $\frac{dy}{dt} = -\frac{4t}{(1+t^2)^2}$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-4t}{2-2t^2} = \frac{-2t}{1-t^2} = \frac{2t}{t^2-1}$

USING THE x EQUATION

$\Rightarrow \frac{2t}{1+t^2} = \frac{\sqrt{2}}{2}$

$\Rightarrow \sqrt{2}(1+t^2) = 4t$

$\Rightarrow 1+t^2 = 2\sqrt{2}t$

$\Rightarrow t^2 - 2\sqrt{2}t + 1 = 0$

$\Rightarrow (t-\sqrt{2})^2 - 2 + 1 = 0$

$\Rightarrow (t-\sqrt{2})^2 = 1$

$\Rightarrow t - \sqrt{2} = \pm 1$ $\qquad$ $\therefore t = 1+\sqrt{2}$ or $-1+\sqrt{2}$

VERIFY WITH THE y EQUATION (OR SOLVING)

IF $t = 1+\sqrt{2}$, $t^2 = 1+2\sqrt{2}+2$, $t^2 = 3+2\sqrt{2}$

$y = \frac{1-(3+2\sqrt{2})}{1+(3+2\sqrt{2})}$, $y = \frac{-2-2\sqrt{2}}{4+2\sqrt{2}}$, $y = -\frac{2+2\sqrt{2}}{4+2\sqrt{2}}$, $y = -\frac{1+\sqrt{2}}{2+\sqrt{2}}$, $y = -\frac{(1+\sqrt{2})(2-\sqrt{2})}{4-2}$, $y = -\frac{2-\sqrt{2}+2\sqrt{2}-2}{2}$, $y = -\frac{\sqrt{2}}{2}$

IF $t = -1+\sqrt{2}$, $t^2 = 1-2\sqrt{2}+2$, $t^2 = 3-2\sqrt{2}$

$y = \frac{1-(3-2\sqrt{2})}{1+(3-2\sqrt{2})}$, $y = \frac{-2+2\sqrt{2}}{4-2\sqrt{2}}$, $y = -\frac{2-2\sqrt{2}}{4-2\sqrt{2}}$, $y = -\frac{1-\sqrt{2}}{2-\sqrt{2}}$, $y = -\frac{(1-\sqrt{2})(2+\sqrt{2})}{4-2}$, $y = -\frac{2-\sqrt{2}-2\sqrt{2}-2}{2}$, $y = +\frac{\sqrt{2}}{2}$

$\therefore t = -1+\sqrt{2}$

$\left.\frac{dy}{dx}\right|_{t=-1+\sqrt{2}} = \frac{2(-1+\sqrt{2})}{(3-2\sqrt{2})-1} = \frac{-2+2\sqrt{2}}{2-2\sqrt{2}} = \frac{-1+\sqrt{2}}{1-\sqrt{2}} = \frac{-(1-\sqrt{2})}{1-\sqrt{2}} = -1$

FINALLY WE HAVE THE EQUATION OF THE TANGENT

$y - \frac{\sqrt{2}}{2} = -1\left(x - \frac{\sqrt{2}}{2}\right)$

$y - \frac{\sqrt{2}}{2} = -x + \frac{\sqrt{2}}{2}$

$x + y = \sqrt{2}$

ALTERNATIVE BY VERIFICATION

SOLVING SIMULTANEOUSLY

$x + y = \sqrt{2}$ $\qquad$ $x = \frac{2t}{1+t^2}$ $\qquad$ $y = \frac{1-t^2}{1+t^2}$

$\Rightarrow \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2} = \sqrt{2}$

$\Rightarrow 2t + 1 - t^2 = \sqrt{2}(1+t^2)$

$\Rightarrow 2t + 1 - t^2 = \sqrt{2} + \sqrt{2}t^2$

$\Rightarrow 0 = (1+\sqrt{2})t^2 - 2t + (\sqrt{2}-1) = 0$

$\Rightarrow 0 = (\sqrt{2}-1)(1+\sqrt{2})t^2 - 2(\sqrt{2}-1)t + (\sqrt{2}-1)(\sqrt{2}-1) = 0(\sqrt{2}-1)$

$\Rightarrow 0 = t^2 - 2(\sqrt{2}-1)t + (\sqrt{2}-1)^2 = 0$

$\Rightarrow t^2 - 2(\sqrt{2}-1)t + (\sqrt{2}-1)^2 = 0$ — PERFECT SQUARE

$\Rightarrow [t - (\sqrt{2}-1)]^2 = 0$

$\Rightarrow$ REPEATED ROOT AT $t = \sqrt{2}-1$

INDEED A TANGENT

AND FOR THE POINT OF TANGENCY

- $t = \sqrt{2}-1$
- $t^2 = (\sqrt{2}-1)^2 = 2 - 2\sqrt{2} + 1 = 3 - 2\sqrt{2}$

$x = \frac{2t}{1+t^2}$

$x = \frac{2(\sqrt{2}-1)}{1+3-2\sqrt{2}} = \frac{2(\sqrt{2}-1)}{4-2\sqrt{2}} = \frac{\sqrt{2}-1}{2-\sqrt{2}}$

$= \frac{(\sqrt{2}-1)(2+\sqrt{2})}{4-2} = \frac{2\sqrt{2}+2-2-\sqrt{2}}{2} = \frac{\sqrt{2}}{2}$

$y = \frac{1-t^2}{1+t^2}$

$y = \frac{1-(3-2\sqrt{2})}{1+(3-2\sqrt{2})} = \frac{-2+2\sqrt{2}}{4-2\sqrt{2}} = \frac{-1+\sqrt{2}}{2-\sqrt{2}}$

$= \frac{(-1+\sqrt{2})(2+\sqrt{2})}{4-2} = \frac{-2-\sqrt{2}+2\sqrt{2}+2}{2} = \frac{\sqrt{2}}{2}$

$\therefore$ TANGENT AT $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

Question 108 **(****+)**

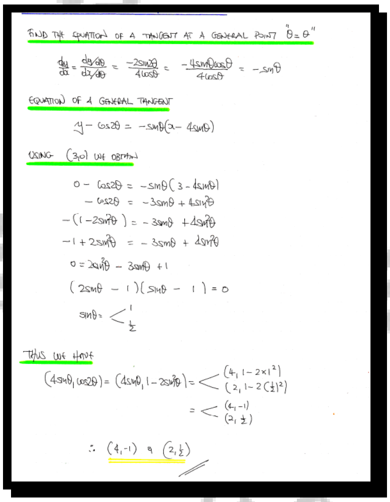
A curve is given parametrically by the equations

$$x = 4\sin\theta, \; y = \cos 2\theta, \; 0 \le \theta < \pi.$$

The tangent to the curve at the point P meets the x axis at the point $(3,0)$.

Determine the possible coordinates of P.

$\left(2,\frac{1}{2}\right)$ or $(4,-1)$



FIND THE EQUATION OF A TANGENT AT A GENERAL POINT "$\theta = \theta$"

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-2\sin 2\theta}{4\cos\theta} = \frac{-4\sin\theta\cos\theta}{4\cos\theta} = -\sin\theta$$

EQUATION OF A GENERAL TANGENT

$$y - \cos 2\theta = -\sin\theta(x - 4\sin\theta)$$

USING $(3,0)$ WE OBTAIN

$$0 - \cos 2\theta = -\sin\theta(3 - 4\sin\theta)$$
$$-\cos 2\theta = -3\sin\theta + 4\sin^2\theta$$
$$-(1 - 2\sin^2\theta) = -3\sin\theta + 4\sin^2\theta$$
$$-1 + 2\sin^2\theta = -3\sin\theta + 4\sin^2\theta$$
$$0 = 2\sin^2\theta - 3\sin\theta + 1$$
$$(2\sin\theta - 1)(\sin\theta - 1) = 0$$
$$\sin\theta = \begin{cases} 1 \\ \frac{1}{2} \end{cases}$$

THUS WE HAVE

$$(4\sin\theta, \cos 2\theta) = (4\sin\theta, 1 - 2\sin^2\theta) = \begin{cases} (4, 1 - 2\times 1^2) \\ (2, 1 - 2(\frac{1}{2})^2) \end{cases} = \begin{cases} (4,-1) \\ (2,\frac{1}{2}) \end{cases}$$

$\therefore (4,-1)$ & $(2,\frac{1}{2})$

Question 109 **(****+)**

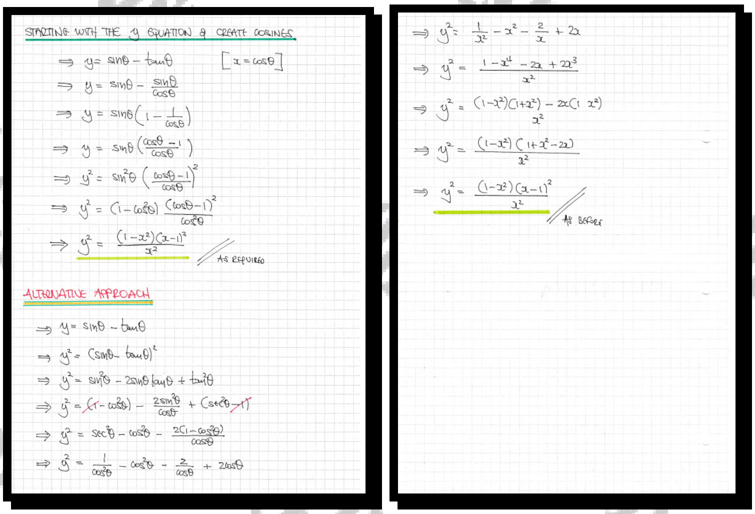
A curve is defined by the parametric equations

$$x = \cos\theta, \quad y = \sin\theta - \tan\theta, \qquad 0 \le \theta < 2\pi .$$

Show that a Cartesian equation of the curve is given by

$$y^2 = \frac{(x-1)^2\left(1-x^2\right)}{x^2}.$$

[ ], [proof]


STARTING WITH THE y EQUATION & CREATE COSINES

$\Rightarrow y = \sin\theta - \tan\theta \qquad [x = \cos\theta]$

$\Rightarrow y = \sin\theta - \frac{\sin\theta}{\cos\theta}$

$\Rightarrow y = \sin\theta\left(1 - \frac{1}{\cos\theta}\right)$

$\Rightarrow y = \sin\theta\left(\frac{\cos\theta - 1}{\cos\theta}\right)$

$\Rightarrow y^2 = \sin^2\theta\left(\frac{\cos\theta - 1}{\cos\theta}\right)^2$

$\Rightarrow y^2 = (1-\cos^2\theta)\frac{(\cos\theta-1)^2}{\cos^2\theta}$

$\Rightarrow y^2 = \frac{(1-x^2)(x-1)^2}{x^2}$ // AS REQUIRED

ALTERNATIVE APPROACH

$\Rightarrow y = \sin\theta - \tan\theta$

$\Rightarrow y^2 = (\sin\theta - \tan\theta)^2$

$\Rightarrow y^2 = \sin^2\theta - 2\sin\theta\tan\theta + \tan^2\theta$

$\Rightarrow y^2 = (1-\cos^2\theta) - \frac{2\sin^2\theta}{\cos\theta} + (\sec^2\theta - 1)$

$\Rightarrow y^2 = \sec^2\theta - \cos^2\theta - \frac{2(1-\cos^2\theta)}{\cos\theta}$

$\Rightarrow y^2 = \frac{1}{\cos^2\theta} - \cos^2\theta - \frac{2}{\cos\theta} + 2\cos\theta$

$\Rightarrow y^2 = \frac{1}{x^2} - x^2 - \frac{2}{x} + 2x$

$\Rightarrow y^2 = \frac{1 - x^4 - 2x + 2x^3}{x^2}$

$\Rightarrow y^2 = \frac{(1-x^2)(1+x^2) - 2x(1-x^2)}{x^2}$

$\Rightarrow y^2 = \frac{(1-x^2)(1+x^2-2x)}{x^2}$

$\Rightarrow y^2 = \frac{(1-x^2)(x-1)^2}{x^2}$ // AS BEFORE

Question 110 (****+)

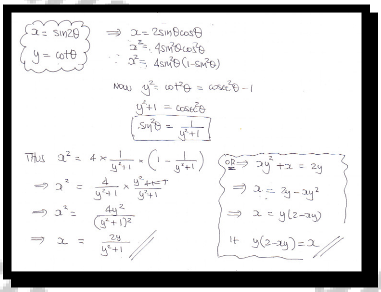
A parametric relationship is given by

$$x = \sin 2\theta, \quad y = \cot\theta, \quad 0 < \theta < \pi.$$

Show that a Cartesian equation for this relationship is

$$y(2 - xy) = x.$$

proof



Question 111 (****+)

A curve has parametric equations

$$x = 3 - t, \quad y = t^2 - 1, \quad t \in \mathbb{R}.$$

a) Find, in terms of t, the gradient of the normal at any point on the curve.

The distinct points P and Q lie on the curve where $t = p$ and $t = q$, respectively.

b) Show that the gradient of the straight line segment PQ is $-(p+q)$.

The straight line segment PQ is a normal to the curve at P.

c) Show further that

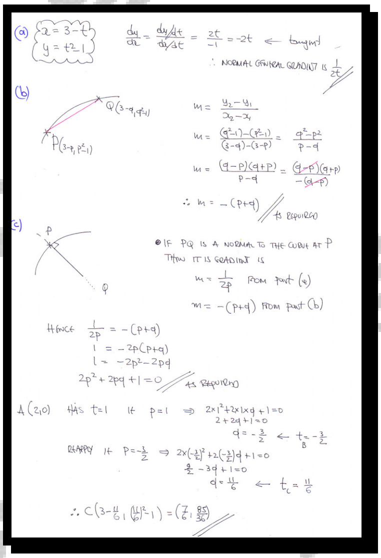
$$2p^2 + 2pq + 1 = 0.$$

The point $A(2,0)$ lies on the curve.

The normal to the curve at A meets the curve again at B. The normal to the curve at B meets the curve again at C.

d) Find the exact coordinates of C.

$$\boxed{\frac{dy}{dx}_{(\text{normal})} = \frac{1}{2t}}, \quad \boxed{C\left(\frac{7}{6}, \frac{85}{36}\right)}$$



(a) $x = 3 - t$, $y = t^2 - 1$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{-1} = -2t \leftarrow$ tangent

$\therefore$ NORMAL GENERAL GRADIENT IS $\frac{1}{2t}$

(b) $Q(3-q, q^2-1)$, $P(3-p, p^2-1)$

$m = \frac{y_2 - y_1}{x_2 - x_1}$

$m = \frac{(q^2-1)-(p^2-1)}{(3-q)-(3-p)} = \frac{q^2-p^2}{p-q}$

$m = \frac{(q-p)(q+p)}{p-q} = \frac{(q-p)(q+p)}{-(q-p)}$

$\therefore m = -(p+q)$ AS REQUIRED

(c) • IF PQ IS A NORMAL TO THE CURVE AT P THEN ITS GRADIENT IS

$m = \frac{1}{2p}$ FROM PART (a)

$m = -(p+q)$ FROM PART (b)

HENCE $\frac{1}{2p} = -(p+q)$

$1 = -2p(p+q)$

$1 = -2p^2 - 2pq$

$2p^2 + 2pq + 1 = 0$ AS REQUIRED

$A(2,0)$ HAS $t=1$ IF $p=1 \Rightarrow 2\times1^2 + 2\times1\times q + 1 = 0$

$2 + 2q + 1 = 0$

$q = -\frac{3}{2} \leftarrow t_B = -\frac{3}{2}$

REAPPLY IF $p = -\frac{3}{2} \Rightarrow 2\times\left(-\frac{3}{2}\right)^2 + 2\left(-\frac{3}{2}\right)q + 1 = 0$

$\frac{9}{2} - 3q + 1 = 0$

$q = \frac{11}{6} \leftarrow t_C = \frac{11}{6}$

$\therefore C\left(3 - \frac{11}{6}, \left(\frac{11}{6}\right)^2 - 1\right) = \left(\frac{7}{6}, \frac{85}{36}\right)$

Question 112 (****+)

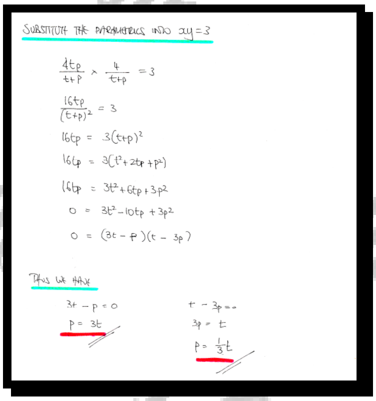
The curve with equation $xy = 3$ is traced by the following parametric equations

$$x = \frac{4tp}{t+p}, \quad y = \frac{4}{t+p}, \quad t, p \in \mathbb{R}, \; t \neq p$$

where t and p are parameters.

Find the relationship between t and p, giving the answer in the form $p = f(t)$.

MP2-W, $p = 3t$ or $p = \frac{1}{3}t$


SUBSTITUTE THE PARAMETRICS INTO $xy = 3$

$$\frac{4tp}{t+p} \times \frac{4}{t+p} = 3$$

$$\frac{16tp}{(t+p)^2} = 3$$

$$16tp = 3(t+p)^2$$

$$16tp = 3(t^2 + 2tp + p^2)$$

$$16tp = 3t^2 + 6tp + 3p^2$$

$$0 = 3t^2 - 10tp + 3p^2$$

$$0 = (3t - p)(t - 3p)$$

THUS WE HAVE

$3t - p = 0$, $p = 3t$

$t - 3p = 0$, $3p = t$, $p = \frac{1}{3}t$

Question 113 (****+)

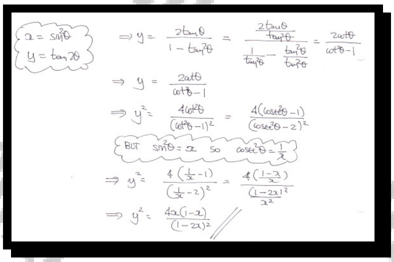
A parametric relationship is given by

$$x = \sin^2\theta, \quad y = \tan 2\theta, \; 0 \le \theta < \frac{\pi}{4}.$$

Show that a Cartesian equation for this relationship is

$$y^2 = \frac{4x(1-x)}{(1-2x)^2}.$$

proof



Question 114 (****+)

A curve C is given by the parametric equations

$$x = 2\cos 2t, \quad y = 5\sin t, \quad -\frac{\pi}{2} \le t \le \frac{\pi}{2}.$$

The point $P\left(1, \frac{5}{2}\right)$ lies on C.

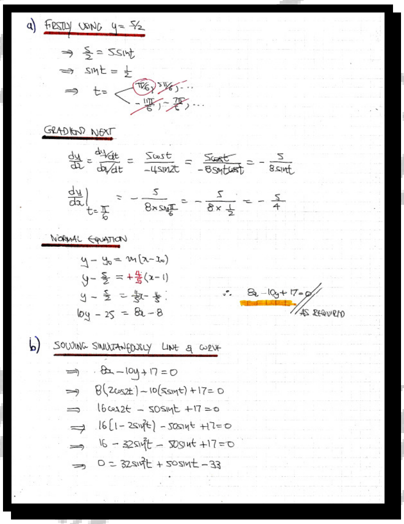
a) Find the value of the gradient at P, and hence, show that an equation of the normal to C at P is

$$8x - 10y + 17 = 0.$$

The normal at P meets C again at the point Q.

b) Show that the y coordinate of Q is $-\frac{165}{16}$.

$\left.\frac{dy}{dx}\right|_P = -\frac{5}{4}$



a) FIRSTLY USING $y = 5/2$

$\Rightarrow \frac{5}{2} = 5\sin t$

$\Rightarrow \sin t = \frac{1}{2}$

$\Rightarrow t = \frac{\pi}{6}, \frac{5\pi}{6}, \dots$ / $-\frac{11\pi}{6}, -\frac{7\pi}{6}, \dots$

GRADIENT NEXT

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{5\cos t}{-4\sin 2t} = \frac{5\cos t}{-8\sin t\cos t} = -\frac{5}{8\sin t}$

$\left.\frac{dy}{dx}\right|_{t=\frac{\pi}{6}} = -\frac{5}{8\times\sin\frac{\pi}{6}} = -\frac{5}{8\times\frac{1}{2}} = -\frac{5}{4}$

NORMAL EQUATION

$y - y_0 = m(x - x_0)$

$y - \frac{5}{2} = +\frac{4}{5}(x-1)$

$y - \frac{5}{2} = \frac{4}{5}x - \frac{4}{5}$

$10y - 25 = 8x - 8$

$\therefore 8x - 10y + 17 = 0$ AS REQUIRED

b) SOLVING SIMULTANEOUSLY LINE & CURVE

$\Rightarrow 8x - 10y + 17 = 0$

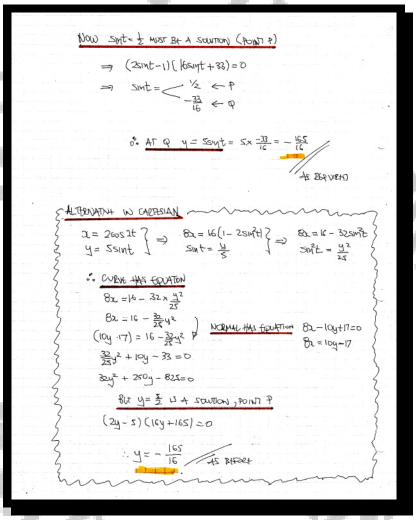
$\Rightarrow 8(2\cos 2t) - 10(5\sin t) + 17 = 0$

$\Rightarrow 16\cos 2t - 50\sin t + 17 = 0$

$\Rightarrow 16(1 - 2\sin^2 t) - 50\sin t + 17 = 0$

$\Rightarrow 16 - 32\sin^2 t - 50\sin t + 17 = 0$

$\Rightarrow 0 = 32\sin^2 t + 50\sin t - 33$



NOW $\sin t = \frac{1}{2}$ MUST BE A SOLUTION (POINT P)

$\Rightarrow (2\sin t - 1)(16\sin t + 33) = 0$

$\Rightarrow \sin t = \frac{1}{2} \leftarrow P$, $\; -\frac{33}{16} \leftarrow Q$

$\therefore$ AT Q $y = 5\sin t = 5\times\frac{-33}{16} = -\frac{165}{16}$ AS REQUIRED

ALTERNATIVE IN CARTESIAN

$x = 2\cos 2t,\ y = 5\sin t \Rightarrow 8x = 16(1 - 2\sin^2 t),\ \sin t = \frac{y}{5} \Rightarrow 8x = 16 - 32\sin^2 t,\ \sin^2 t = \frac{y^2}{25}$

$\therefore$ CURVE HAS EQUATION

$8x = 16 - 32\times\frac{y^2}{25}$

$8x = 16 - \frac{32}{25}y^2$

NORMAL HAS EQUATION $8x - 10y + 17 = 0$, $8x = 10y - 17$

$(10y - 17) = 16 - \frac{32}{25}y^2$

$\frac{32}{25}y^2 + 10y - 33 = 0$

$32y^2 + 250y - 825 = 0$

BUT $y = \frac{5}{2}$ IS A SOLUTION, POINT P

$(2y - 5)(16y + 165) = 0$

$\therefore y = -\frac{165}{16}$ AS BEFORE

Question 115 (****+)

A curve C is defined by the parametric equations

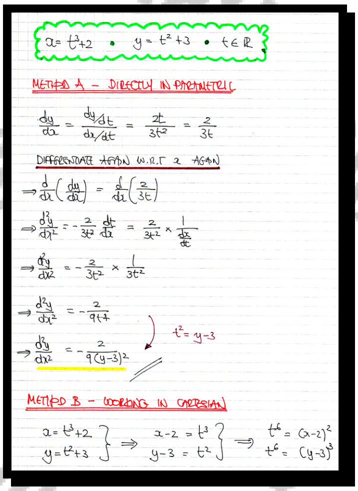
$$x = t^3 + 2 \ , \ y = t^2 + 3 \, , \ t \in \mathbb{R} \, .$$

Show clearly that

$$\frac{d^2 y}{dx^2} = f(y) \, ,$$

where f must by explicitly stated.

proof



$x = t^3+2 \ \bullet \ y = t^2+3 \ \bullet \ t \in \mathbb{R}$

METHOD A – DIRECTLY IN PARAMETRIC

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{3t^2} = \frac{2}{3t}$

DIFFERENTIATE AGAIN W.R.T x AGAIN

$\Rightarrow \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{2}{3t}\right)$

$\Rightarrow \frac{d^2y}{dx^2} = -\frac{2}{3t^2}\frac{dt}{dx} = \frac{2}{3t^2} \times \frac{1}{\frac{dx}{dt}}$

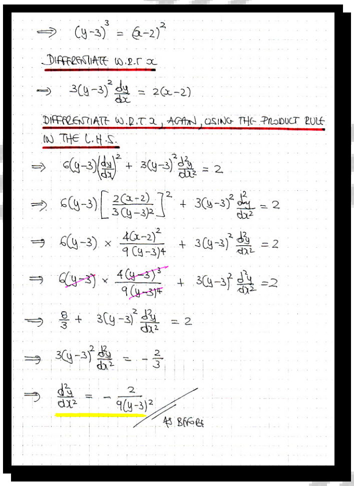
$\Rightarrow \frac{d^2y}{dx^2} = -\frac{2}{3t^2} \times \frac{1}{3t^2}$

$\Rightarrow \frac{d^2y}{dx^2} = -\frac{2}{9t^4}$ $\quad t^2 = y-3$

$\Rightarrow \frac{d^2y}{dx^2} = -\frac{2}{9(y-3)^2}$

METHOD B – WORKING IN CARTESIAN

$\left.\begin{array}{l} x = t^3+2 \\ y = t^2+3 \end{array}\right\} \Rightarrow \left.\begin{array}{l} x-2 = t^3 \\ y-3 = t^2 \end{array}\right\} \Rightarrow \begin{array}{l} t^6 = (x-2)^2 \\ t^6 = (y-3)^3 \end{array}$



$\Rightarrow (y-3)^3 = (x-2)^2$

DIFFERENTIATE W.R.T x

$\Rightarrow 3(y-3)^2\frac{dy}{dx} = 2(x-2)$

DIFFERENTIATE W.R.T x, AGAIN, USING THE PRODUCT RULE IN THE L.H.S.

$\Rightarrow 6(y-3)\left(\frac{dy}{dx}\right)^2 + 3(y-3)^2\frac{d^2y}{dx^2} = 2$

$\Rightarrow 6(y-3)\left[\frac{2(x-2)}{3(y-3)^2}\right]^2 + 3(y-3)^2\frac{d^2y}{dx^2} = 2$

$\Rightarrow 6(y-3) \times \frac{4(x-2)^2}{9(y-3)^4} + 3(y-3)^2\frac{d^2y}{dx^2} = 2$

$\Rightarrow 6(y-3) \times \frac{4(y-3)^3}{9(y-3)^4} + 3(y-3)^2\frac{d^2y}{dx^2} = 2$

$\Rightarrow \frac{8}{3} + 3(y-3)^2\frac{d^2y}{dx^2} = 2$

$\Rightarrow 3(y-3)^2\frac{d^2y}{dx^2} = -\frac{2}{3}$

$\Rightarrow \frac{d^2y}{dx^2} = -\frac{2}{9(y-3)^2}$ AS BEFORE

Question 116 **(****+)**

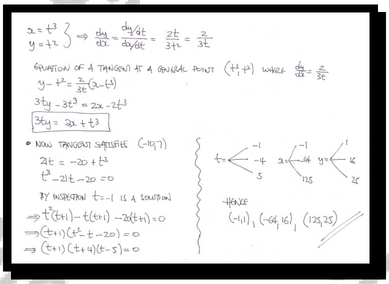
A curve C is defined parametrically by the equations

$$x = t^3 \ , \ y = t^2 , \ t \in \mathbb{R} .$$

The tangent to C at point P passes through the point with coordinates $(-10, 7)$.

Find the possible coordinates of P.

$\boxed{(-1,1), \ (-64,16), \ (125,25)}$



$x = t^3$, $y = t^2$ $\Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{3t^2} = \frac{2}{3t}$

EQUATION OF A TANGENT AT A GENERAL POINT (t^3, t^2) WHERE $\frac{dy}{dx} = \frac{2}{3t}$

$y - t^2 = \frac{2}{3t}(x - t^3)$

$3ty - 3t^3 = 2x - 2t^3$

$\boxed{3ty = 2x + t^3}$

• NOW TANGENT SATISFIES $(-10, 7)$

$21t = -20 + t^3$

$t^3 - 21t - 20 = 0$

BY INSPECTION $t = -1$ IS A SOLUTION

$\Rightarrow t^2(t+1) - t(t+1) - 20(t+1) = 0$

$\Rightarrow (t+1)(t^2 - t - 20) = 0$

$\Rightarrow (t+1)(t+4)(t-5) = 0$

$t = -1, -4, 5$; $x = -1, -64, 125$; $y = 1, 16, 25$

HENCE

$(-1,1), (-64,16), (125,25)$

Question 117 (**+)**

A curve C is defined by the parametric equations

$$x = \cos\theta + (\theta + \varphi)\sin\theta \ , \ \ y = \sin\theta - (\theta + \varphi)\cos\theta \, ,$$

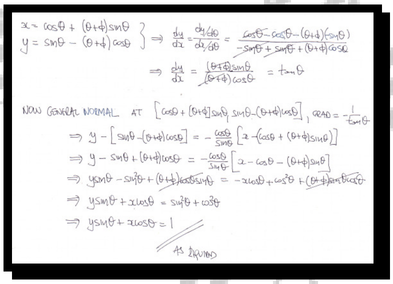
where φ is a constant and θ is a parameter, such that

$$0 < \theta < \frac{\pi}{2}, \quad 0 < \varphi < \frac{\pi}{2} \quad \text{and} \quad \theta + \varphi \neq 0 \, .$$

Show that the equation of a normal to C at the point with parameter θ is given by

$$y\sin\theta + x\cos\theta = 1$$

proof



$x = \cos\theta + (\theta+\phi)\sin\theta$, $y = \sin\theta - (\theta+\phi)\cos\theta$ $\Rightarrow \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cos\theta - \cos\theta - (\theta+\phi)(-\sin\theta)}{-\sin\theta + \sin\theta + (\theta+\phi)\cos\theta}$

$\Rightarrow \frac{dy}{dx} = \frac{(\theta+\phi)\sin\theta}{(\theta+\phi)\cos\theta} = \tan\theta$

NOW GENERAL NORMAL AT $[\cos\theta + (\theta+\phi)\sin\theta, \sin\theta - (\theta+\phi)\cos\theta]$, GRAD $= -\frac{1}{\tan\theta}$

$\Rightarrow y - [\sin\theta - (\theta+\phi)\cos\theta] = -\frac{\cos\theta}{\sin\theta}[x - (\cos\theta + (\theta+\phi)\sin\theta)]$

$\Rightarrow y - \sin\theta + (\theta+\phi)\cos\theta = -\frac{\cos\theta}{\sin\theta}[x - \cos\theta - (\theta+\phi)\sin\theta]$

$\Rightarrow y\sin\theta - \sin^2\theta + (\theta+\phi)\cos\theta\sin\theta = -x\cos\theta + \cos^2\theta + (\theta+\phi)\sin\theta\cos\theta$

$\Rightarrow y\sin\theta + x\cos\theta = \sin^2\theta + \cos^2\theta$

$\Rightarrow y\sin\theta + x\cos\theta = 1$

AS REQUIRED

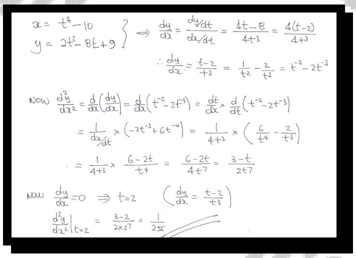
Question 118 (****+)

A curve C is defined parametrically by the equations

$$x = t^4, \quad y = 2t^2 - 8t + 9 \ , \ t \in \mathbb{R} .$$

Find the value of $\frac{d^2y}{dx^2}$ at the stationary point of C.

$$\boxed{\frac{1}{256}}$$



Question 119 (****+)

The curve C is given parametrically by

$$x = \frac{1}{2}\left(1+t^2\right), \quad y = t^3, \quad t \in \mathbb{R}.$$

a) Show that an equation of the tangent to the curve at the point P where $t = p$ is

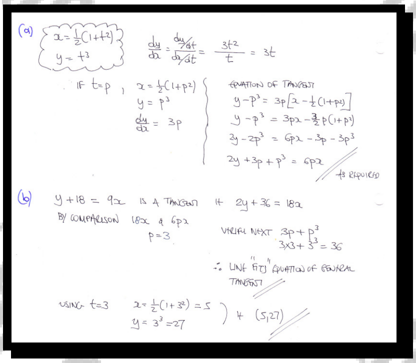
$$2y + 3p + p^3 = 6px.$$

b) Show further that the straight line with equation

$$y = 9x - 18$$

is a tangent to C and determine the coordinates of the point of tangency.

$\boxed{(5,27)}$



(a) $x = \frac{1}{2}(1+t^2)$, $y = t^3$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2}{t} = 3t$

IF $t = p$, $x = \frac{1}{2}(1+p^2)$, $y = p^3$, $\frac{dy}{dx} = 3p$

EQUATION OF TANGENT

$y - p^3 = 3p\left[x - \frac{1}{2}(1+p^2)\right]$

$y - p^3 = 3px - \frac{3}{2}p(1+p^2)$

$2y - 2p^3 = 6px - 3p - 3p^3$

$2y + 3p + p^3 = 6px$ AS REQUIRED

(b) $y + 18 = 9x$ IS A TANGENT IF $2y + 36 = 18x$

BY COMPARISON $18x$ & $6px$

$p = 3$

VERIFY NEXT $3p + p^3$

$3 \times 3 + 3^3 = 36$

∴ LINE "FITS" EQUATION OF GENERAL TANGENT

USING $t = 3$: $x = \frac{1}{2}(1+3^2) = 5$, $y = 3^3 = 27$) ∴ $(5,27)$

Question 120 (****+)

A curve C is given by the parametric equations

$$x = \cos t, \quad y = \cos 2t \ , \quad -\pi \le t \le \pi .$$

The point P lies on C, where $t = \dfrac{\pi}{3}$.

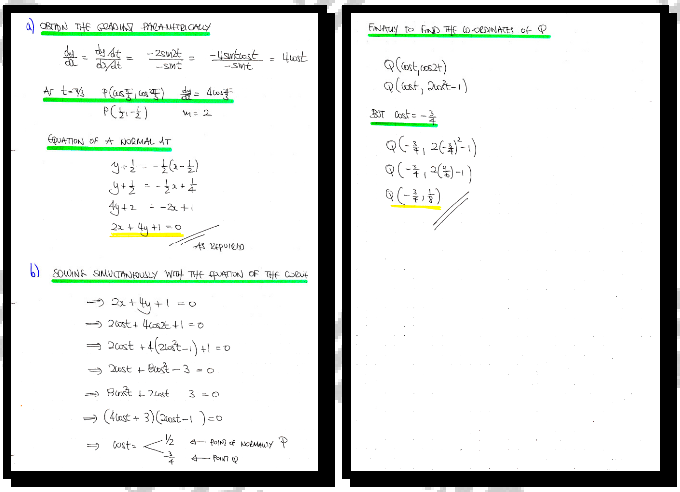
a) Show that an equation of the normal to C at P is

$$2x + 4y + 1 = 0 .$$

The normal at P meets C again at the point Q.

b) Determine, by showing a clear detailed method, the exact coordinates of Q.

$\boxed{}$, $\boxed{Q\left(-\frac{3}{4}, \frac{1}{8}\right)}$


a) OBTAIN THE GRADIENT PARAMETRICALLY

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-2\sin 2t}{-\sin t} = \frac{-4\sin t\cos t}{-\sin t} = 4\cos t$

AT $t = \pi/3$ $P\left(\cos\frac{\pi}{3}, \cos\frac{2\pi}{3}\right)$ $\frac{dy}{dx} = 4\cos\frac{\pi}{3}$

$P\left(\frac{1}{2}, -\frac{1}{2}\right)$ $m = 2$

EQUATION OF A NORMAL AT

$y + \frac{1}{2} = -\frac{1}{2}\left(x - \frac{1}{2}\right)$

$y + \frac{1}{2} = -\frac{1}{2}x + \frac{1}{4}$

$4y + 2 = -2x + 1$

$2x + 4y + 1 = 0$ // AS REQUIRED

b) SOLVING SIMULTANEOUSLY WITH THE EQUATION OF THE CURVE

$\Rightarrow 2x + 4y + 1 = 0$

$\Rightarrow 2\cos t + 4\cos 2t + 1 = 0$

$\Rightarrow 2\cos t + 4(2\cos^2 t - 1) + 1 = 0$

$\Rightarrow 2\cos t + 8\cos^2 t - 3 = 0$

$\Rightarrow 8\cos^2 t + 2\cos t - 3 = 0$

$\Rightarrow (4\cos t + 3)(2\cos t - 1) = 0$

$\Rightarrow \cos t = \frac{1}{2}$ ← POINT OF NORMALITY P; $\cos t = -\frac{3}{4}$ ← POINT Q

FINALLY TO FIND THE CO-ORDINATES OF Q

$Q(\cos t, \cos 2t)$

$Q(\cos t, 2\cos^2 t - 1)$

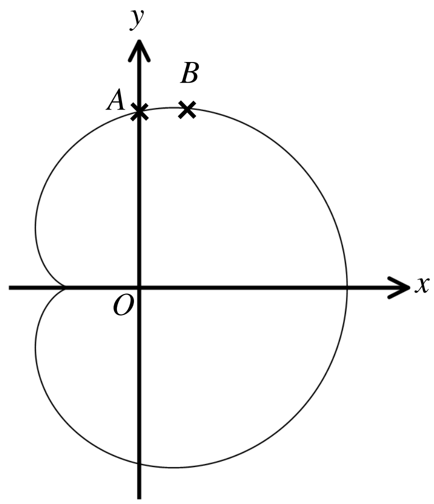
BUT $\cos t = -\frac{3}{4}$

$Q\left(-\frac{3}{4}, 2\left(-\frac{3}{4}\right)^2 - 1\right)$

$Q\left(-\frac{3}{4}, 2\left(\frac{9}{16}\right) - 1\right)$

$Q\left(-\frac{3}{4}, \frac{1}{8}\right)$ //

Question 121 (****+)



The figure above shows a curve known as a Cardioid. The curve crosses the y axis at the point A and the point B is the highest point of the curve.

The parametric equations of this Cardioid are

$$x = 4\cos\theta + 2\cos 2\theta, \quad y = 4\sin\theta + 2\sin 2\theta, \quad 0 \le \theta < 2\pi .$$

a) Find a simplified expression for $\frac{dy}{dx}$, in terms of θ.

b) Hence show that the coordinates of B are $\left(1, 3\sqrt{3}\right)$.

c) Find the exact value of $\cos\theta$ at A.

[continues overleaf]

[continued from overleaf]

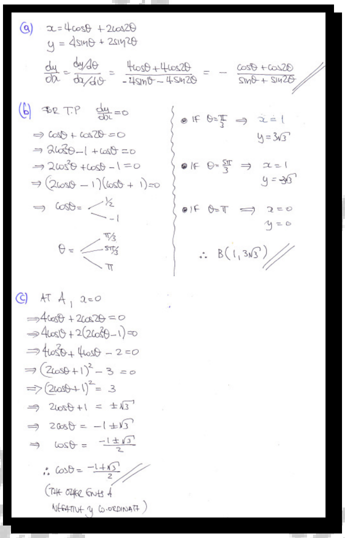
The distance of a point $P(x, y)$ from the origin is $\sqrt{x^2 + y^2}$.

d) Show that for points that lie on this cardioid

$$x^2 + y^2 = 20 + 16\cos\theta,$$

and use this result to find the shortest and longest distance of any point on the cardioid from the origin.

$$\boxed{\frac{dy}{dx} = -\frac{\cos\theta + \cos 2\theta}{\sin\theta + \sin 2\theta}}, \quad \boxed{\cos\theta = \frac{-1+\sqrt{3}}{2}}, \quad \boxed{|OP|_{\min} = 2}, \quad \boxed{|OP|_{\max} = 6}$$



(a) $x = 4\cos\theta + 2\cos 2\theta$

$y = 4\sin\theta + 2\sin 2\theta$

$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{4\cos\theta + 4\cos 2\theta}{-4\sin\theta - 4\sin 2\theta} = -\frac{\cos\theta + \cos 2\theta}{\sin\theta + \sin 2\theta}$

(b) FOR T.P $\frac{dy}{dx} = 0$

$\Rightarrow \cos\theta + \cos 2\theta = 0$

$\Rightarrow 2\cos^2\theta - 1 + \cos\theta = 0$

$\Rightarrow 2\cos^2\theta + \cos\theta - 1 = 0$

$\Rightarrow (2\cos\theta - 1)(\cos\theta + 1) = 0$

$\Rightarrow \cos\theta = \frac{1}{2}, -1$

$\theta = \pi/3, 5\pi/3, \pi$

• IF $\theta = \frac{\pi}{3} \Rightarrow x = 1, y = 3\sqrt{3}$

• IF $\theta = \frac{5\pi}{3} \Rightarrow x = 1, y = -3\sqrt{3}$

• IF $\theta = \pi \Rightarrow x = 0, y = 0$

$\therefore B(1, 3\sqrt{3})$

(c) AT A, $x = 0$

$\Rightarrow 4\cos\theta + 2\cos 2\theta = 0$

$\Rightarrow 4\cos\theta + 2(2\cos^2\theta - 1) = 0$

$\Rightarrow 4\cos^2\theta + 4\cos\theta - 2 = 0$

$\Rightarrow (2\cos\theta + 1)^2 - 3 = 0$

$\Rightarrow (2\cos\theta + 1)^2 = 3$

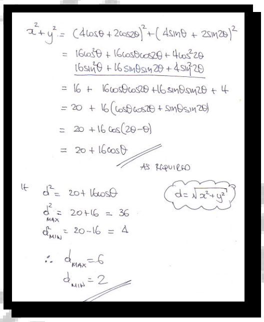
$\Rightarrow 2\cos\theta + 1 = \pm\sqrt{3}$

$\Rightarrow 2\cos\theta = -1 \pm \sqrt{3}$

$\Rightarrow \cos\theta = \frac{-1 \pm \sqrt{3}}{2}$

$\therefore \cos\theta = \frac{-1+\sqrt{3}}{2}$

(THE OTHER GIVES A NEGATIVE y CO-ORDINATE)



$x^2 + y^2 = (4\cos\theta + 2\cos 2\theta)^2 + (4\sin\theta + 2\sin 2\theta)^2$

$= 16\cos^2\theta + 16\cos\theta\cos 2\theta + 4\cos^2 2\theta$
$\;\;\; 16\sin^2\theta + 16\sin\theta\sin 2\theta + 4\sin^2 2\theta$

$= 16 + 16\cos\theta\cos 2\theta + 16\sin\theta\sin 2\theta + 4$

$= 20 + 16(\cos\theta\cos 2\theta + \sin\theta\sin 2\theta)$

$= 20 + 16\cos(2\theta - \theta)$

$= 20 + 16\cos\theta$ AS REQUIRED

IF $d^2 = 20 + 16\cos\theta$ $\quad d = \sqrt{x^2+y^2}$

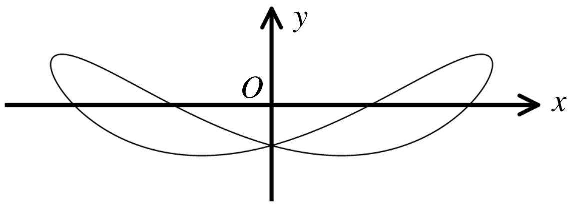
$d^2_{MAX} = 20 + 16 = 36$

$d^2_{MIN} = 20 - 16 = 4$

$\therefore d_{MAX} = 6$

$d_{MIN} = 2$

Question 122 (****+)



The figure above shows the curve C given parametrically by the equations

$$x = \cos t + 2\sin t, \quad y = \sin 2t, \quad 0 \le t < 2\pi .$$

a) Find the coordinates of the points where C crosses the x axis.

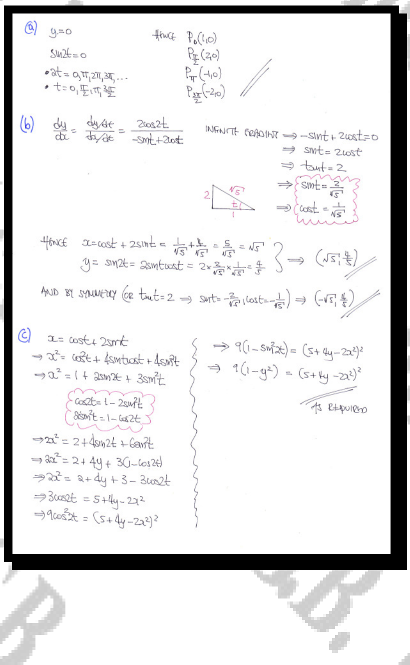
There are two points on C where the tangent to C is parallel to the y axis.

b) Determine the exact coordinates of these two points.

c) Show that a Cartesian equation of C is

$$9\left(1 - y^2\right) = \left(5 + 4y - 2x^2\right)^2 .$$

$\boxed{(-2,0),\ (-1,0)\ ,(1,0),\ (2,0)}$, $\boxed{\left(-\sqrt{5},\frac{4}{5}\right),\left(\sqrt{5},\frac{4}{5}\right)}$



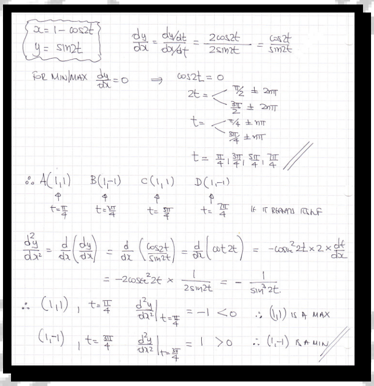
Question 123 (****+)

A curve given parametrically by the equations

$$x = 1 - \cos 2t \ , \quad y = \sin 2t \ , \quad 0 \le t < 2\pi$$

Find the turning points of the curve and use $\frac{d^2y}{dx^2}$ to determine their nature.

$\boxed{\max(1,1), \ \min(1,-1)}$



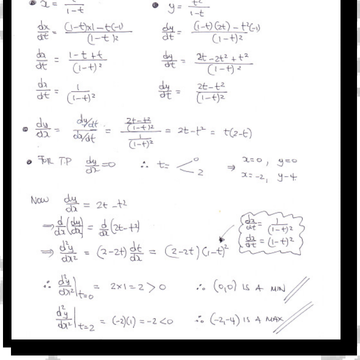
Question 124 (****+)

For the curve given parametrically by

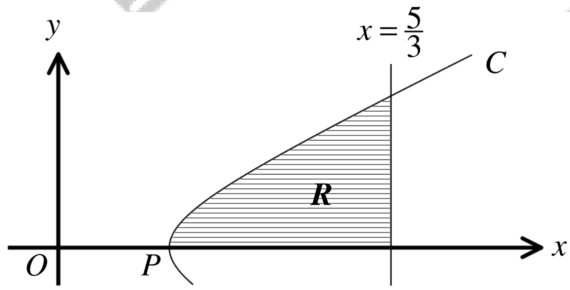
$$x = \frac{t}{1-t} \ , \quad y = \frac{t^2}{1-t} \ , \quad t \in \mathbb{R} \ , \ t \neq 1$$

find the coordinates of the turning points and determine their nature.

$\boxed{\max(-2,-4), \ \min(0,0)}$



Question 125 (****+)



The figure above shows part of the curve C with parametric equations

$$x = t + \frac{1}{4t}, \quad y = t - \frac{1}{4t}, \quad t > 0.$$

The curve crosses the x axis at P.

a) Determine the coordinates of P.

b) Show that the gradient at any point on C is given by

$$\frac{dy}{dx} = \frac{4t^2 + 1}{4t^2 - 1}.$$

c) By considering $x + y$ and $x - y$, or otherwise, find a Cartesian equation for C.

[continues overleaf]

[continued from overleaf]

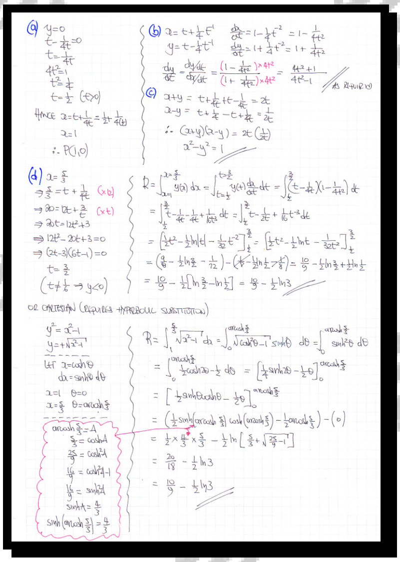
The finite region R bounded by C, the line $x=\frac{5}{3}$ and the x axis is shown shaded in the figure.

d) Show that the area of R is given by

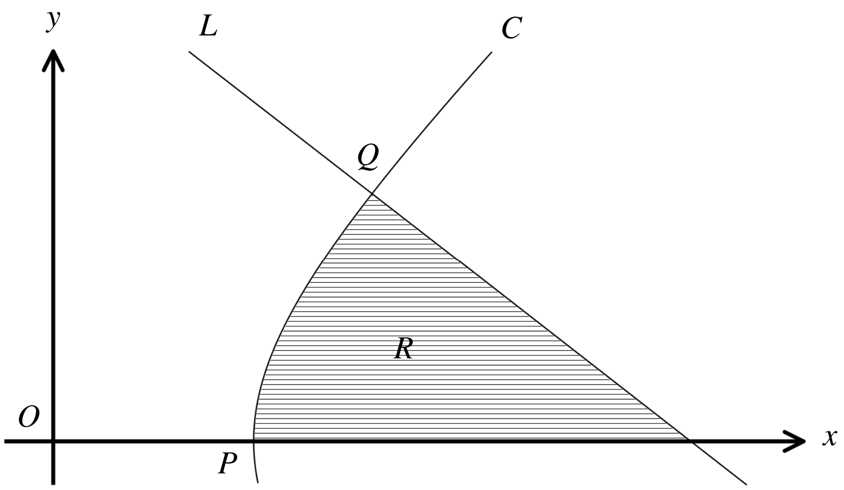
$$\int_{\frac{1}{2}}^{\frac{3}{2}} \left(t-\frac{1}{4t}\right)\left(1-\frac{1}{4t^2}\right)\, dt .$$

e) Hence calculate an exact value for the area of R.

$\boxed{P(1,0)}$, $\boxed{x^2-y^2=1}$, $\boxed{\text{Area}=\frac{10}{9}-\frac{1}{2}\ln 3}$



Question 126 (****+)



The figure above shows part of the curve C with parametric equations

$$x = 2t + \frac{1}{t}, \quad y = 2t - \frac{1}{t} \quad , \quad t > 0.$$

The curve crosses the x axis at the point P and the L is a normal to C at the point Q, where $t = 2$.

a) Determine the exact coordinates of P.

b) Show that the gradient at any point on C is given by

$$\frac{dy}{dx} = \frac{2t^2 + 1}{2t^2 - 1}.$$

[continues overleaf]

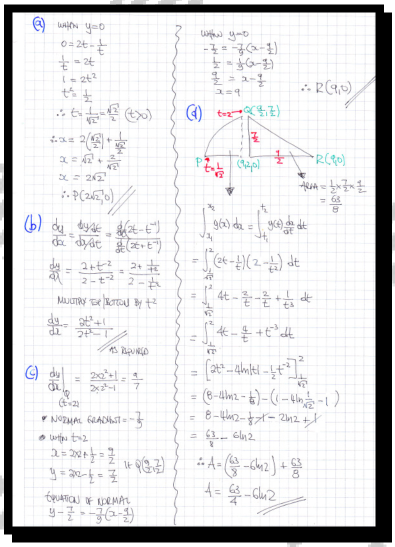
[continued from overleaf]

The normal L crosses the x axis at R. The region bounded by C, by L and the x axis, shown shaded in the figure, has area A.

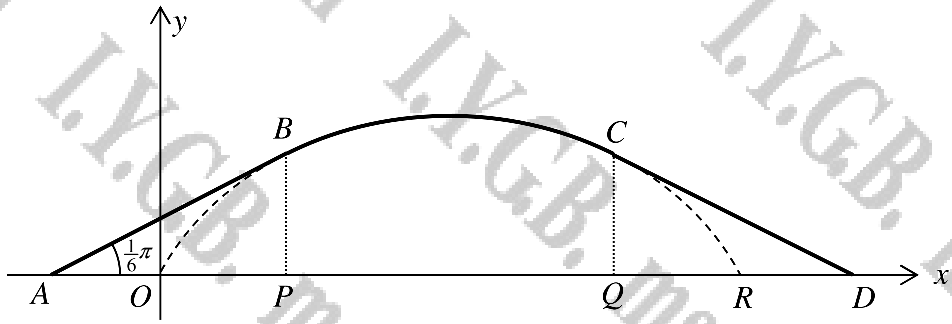
c) Find the coordinates of R.

d) Calculate an exact value for A.

$\boxed{P\left(2\sqrt{2},0\right)}$, $\boxed{R(9,0)}$, $\boxed{A=\frac{63}{4}-6\ln 2}$



Question 127 **(****+)**



The figure above shows a **symmetrical** design for a suspension bridge arch $ABCD$.

The curve $OBCR$ is a cycloid with parametric equations

$$x = 6(2t - \sin 2t), \quad y = 6(1 - \cos 2t), \quad 0 \le t \le \pi .$$

a) Show clearly that

$$\frac{dy}{dx} = \cot t .$$

b) Find the in exact form the length of OR.

c) Determine the maximum height of the arch.

[continues overleaf]

[continued from overleaf]

The arch design consists of the curved part BC and the straight lines AB and CD.

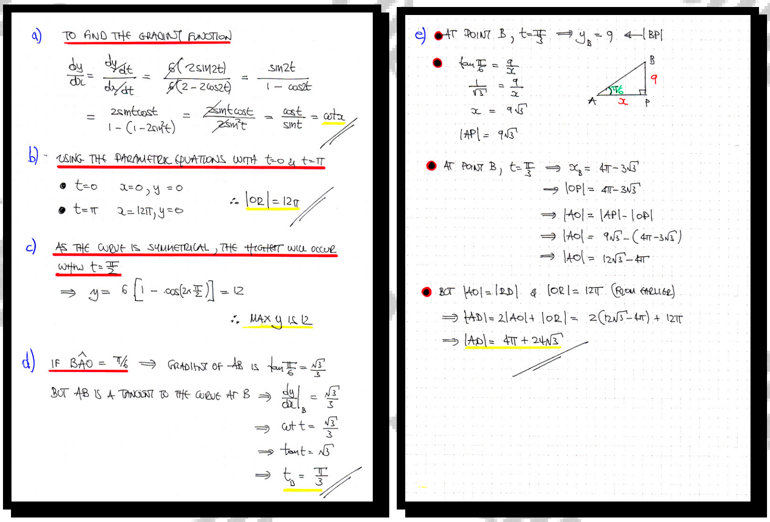
The straight lines AB and CD are tangents to the cycloid at the points B and C.

The angle BAO is $\frac{\pi}{6}$.

d) Find the value of t at B, by considering the gradient at that point.

e) Find, in exact form, the length of the straight line AD.

$\boxed{\quad}$, $\boxed{|OR| = 12\pi}$, $\boxed{y_{max} = 12}$, $\boxed{t_B = \frac{\pi}{3}}$, $\boxed{|AD| = 4\pi + 24\sqrt{3}}$


a) TO FIND THE GRADIENT FUNCTION

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{6(2\sin 2t)}{6(2-2\cos 2t)} = \frac{\sin 2t}{1-\cos 2t}$

$= \frac{2\sin t\cos t}{1-(1-2\sin^2 t)} = \frac{2\sin t\cos t}{2\sin^2 t} = \frac{\cos t}{\sin t} = \cot t$

b) USING THE PARAMETRIC EQUATIONS WITH $t=0$ & $t=\pi$

- $t=0$ $x=0, y=0$
- $t=\pi$ $x=12\pi, y=0$

$\therefore |OR| = 12\pi$

c) AS THE CURVE IS SYMMETRICAL, THE HIGHEST WILL OCCUR WHEN $t=\frac{\pi}{2}$

$\Rightarrow y = 6\left[1-\cos\left(2\times\frac{\pi}{2}\right)\right] = 12$

$\therefore$ MAX y IS 12

d) IF $B\hat{A}O = \pi/6 \Rightarrow$ GRADIENT OF AB IS $\tan\frac{\pi}{6} = \frac{\sqrt{3}}{3}$

BUT AB IS A TANGENT TO THE CURVE AT B $\Rightarrow \left.\frac{dy}{dx}\right|_B = \frac{\sqrt{3}}{3}$

$\Rightarrow \cot t = \frac{\sqrt{3}}{3}$

$\Rightarrow \tan t = \sqrt{3}$

$\Rightarrow t_B = \frac{\pi}{3}$

e)
- AT POINT B, $t=\frac{\pi}{3} \Rightarrow y_B = 9 \leftarrow |BP|$
- $\tan\frac{\pi}{6} = \frac{9}{x}$

$\frac{1}{\sqrt{3}} = \frac{9}{x}$

$x = 9\sqrt{3}$

$|AP| = 9\sqrt{3}$

- AT POINT B, $t=\frac{\pi}{3} \Rightarrow x_B = 4\pi - 3\sqrt{3}$

$\Rightarrow |OP| = 4\pi - 3\sqrt{3}$

$\Rightarrow |AO| = |AP| - |OP|$

$\Rightarrow |AO| = 9\sqrt{3} - (4\pi - 3\sqrt{3})$

$\Rightarrow |AO| = 12\sqrt{3} - 4\pi$

- BUT $|AO| = |RD|$ & $|OR| = 12\pi$ (FROM EARLIER)

$\Rightarrow |AD| = 2|AO| + |OR| = 2(12\sqrt{3} - 4\pi) + 12\pi$

$\Rightarrow |AD| = 4\pi + 24\sqrt{3}$

Question 128 (****+)

A curve is given parametrically by the equations

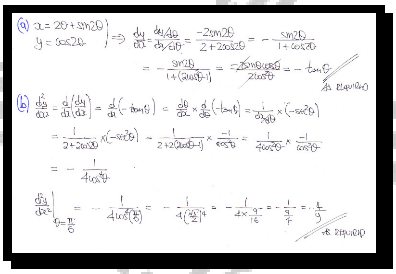
$$x = 2\theta + \sin 2\theta, \quad y = \cos 2\theta, \quad 0 \le \theta < \pi .$$

Show that …

a) … $\dfrac{dy}{dx} = -\tan\theta$.

b) … the value of $\dfrac{d^2y}{dx^2}$ evaluated at the point where $\theta = \dfrac{\pi}{6}$ is $-\dfrac{4}{9}$.

proof


(a) $x = 2\theta + \sin 2\theta$, $y = \cos 2\theta$ $\Rightarrow \dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{-2\sin 2\theta}{2+2\cos 2\theta} = -\dfrac{\sin 2\theta}{1+\cos 2\theta}$

$= -\dfrac{\sin 2\theta}{1+(2\cos^2\theta - 1)} = \dfrac{-2\sin\theta\cos\theta}{2\cos^2\theta} = -\tan\theta$ // AS REQUIRED

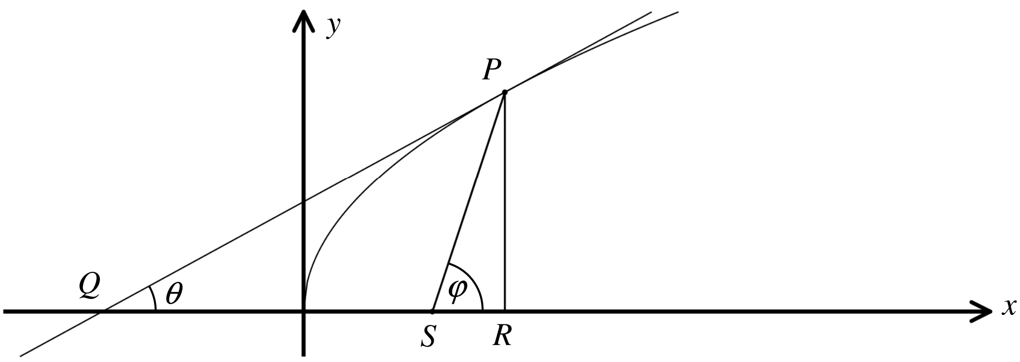
(b) $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d}{dx}(-\tan\theta) = \dfrac{d\theta}{dx} \times \dfrac{d}{d\theta}(-\tan\theta) = \dfrac{1}{dx/d\theta} \times (-\sec^2\theta)$

$= \dfrac{1}{2+2\cos 2\theta} \times (-\sec^2\theta) = \dfrac{1}{2+2(2\cos^2\theta-1)} \times \dfrac{-1}{\cos^2\theta} = \dfrac{1}{4\cos^2\theta} \times \dfrac{-1}{\cos^2\theta}$

$= -\dfrac{1}{4\cos^4\theta}$

$\left.\dfrac{d^2y}{dx^2}\right|_{\theta=\frac{\pi}{6}} = -\dfrac{1}{4\cos^4\left(\frac{\pi}{6}\right)} = -\dfrac{1}{4\left(\frac{\sqrt{3}}{2}\right)^4} = -\dfrac{1}{4\times\frac{9}{16}} = -\dfrac{1}{\frac{9}{4}} = -\dfrac{4}{9}$ // AS REQUIRED

Question 129 (****+)



The figure above shows the curve C with parametric equations

$$x = t^2, \ y = 2t, \ t \in \mathbb{R}, \ t \geq 0.$$

The point P lies on C, where $t = p$. The point R lies on the x axis so that PR is parallel to the y axis. The tangent to C at the point P meets the x axis at the point Q, so that the angle $\measuredangle PQR = \theta$.

a) Find the coordinates of Q in terms of p.

b) By considering the triangle PQR, show $\tan\theta = \dfrac{1}{p}$.

The point S has coordinates $(1,0)$ and $\measuredangle PSR = \varphi$.

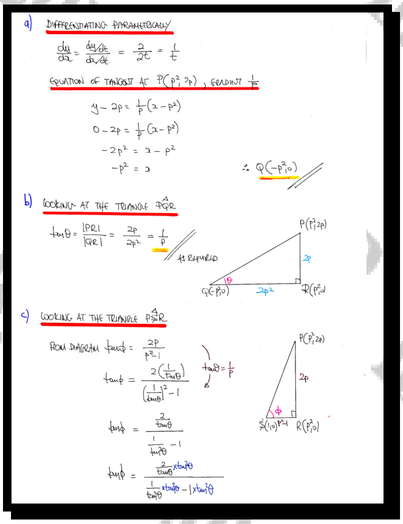
c) Find an expression for $\tan\varphi$ in terms of p and hence show that $\varphi = 2\theta$.

d) Deduce that $|SP| = |SQ|$.

$\boxed{}$, $\boxed{Q(-p^2,0)}$ $\boxed{\tan\varphi = \dfrac{2p}{p^2-1}}$

[solution overleaf]

[question overleaf]



a) DIFFERENTIATING PARAMETRICALLY

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2}{2t} = \frac{1}{t}$$

EQUATION OF TANGENT AT $P(p^2, 2p)$, GRADIENT $\frac{1}{p}$

$$y - 2p = \frac{1}{p}(x - p^2)$$
$$0 - 2p = \frac{1}{p}(x - p^2)$$
$$-2p^2 = x - p^2$$
$$-p^2 = x$$

$\therefore Q(-p^2, 0)$

b) LOOKING AT THE TRIANGLE $P\hat{Q}R$

$$\tan\theta = \frac{|PR|}{|QR|} = \frac{2p}{2p^2} = \frac{1}{p}$$ AS REQUIRED

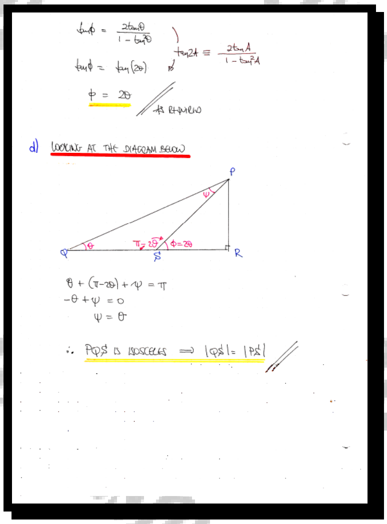
c) LOOKING AT THE TRIANGLE $P\hat{S}R$

FROM DIAGRAM $\tan\phi = \frac{2p}{p^2 - 1}$

$$\tan\phi = \frac{2\left(\frac{1}{\tan\theta}\right)}{\left(\frac{1}{\tan\theta}\right)^2 - 1}$$ ($\tan\theta = \frac{1}{p}$)

$$\tan\phi = \frac{\frac{2}{\tan\theta}}{\frac{1}{\tan^2\theta} - 1}$$

$$\tan\phi = \frac{\frac{2}{\tan\theta} \times \tan^2\theta}{\frac{1}{\tan^2\theta} \times \tan^2\theta - 1 \times \tan^2\theta}$$



$$\tan\phi = \frac{2\tan\theta}{1 - \tan^2\theta}$$ ($\tan 2A \equiv \frac{2\tan A}{1 - \tan^2 A}$)

$$\tan\phi = \tan(2\theta)$$

$$\phi = 2\theta$$ AS REQUIRED

d) LOOKING AT THE DIAGRAM BELOW

$$\theta + (\pi - 2\theta) + \psi = \pi$$
$$-\theta + \psi = 0$$
$$\psi = \theta$$

$\therefore$ PQS IS ISOSCELES $\Rightarrow |QS| = |PS|$

Question 130 (****+)

A curve C is given by the parametric equations

$$x = \tan\theta - \sec\theta, \quad y = \cot\theta - \operatorname{cosec}\theta, \quad 0 < \theta < \frac{\pi}{2}.$$

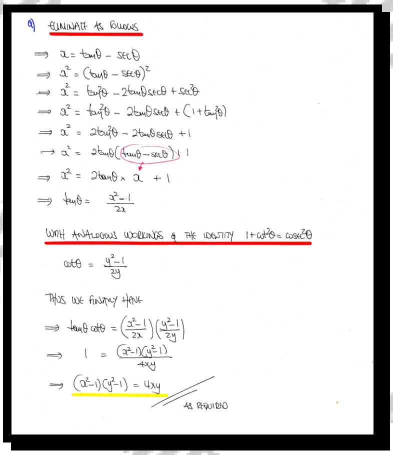
Show clearly that …

a) … a Cartesian equation of C is

$$(x^2 - 1)(y^2 - 1) = 4xy.$$

b) … $\dfrac{dy}{dx} = \dfrac{1 - y^2}{2x}$.

[ ] , [proof]



a) ELIMINATE AS FOLLOWS

$\Rightarrow x = \tan\theta - \sec\theta$

$\Rightarrow x^2 = (\tan\theta - \sec\theta)^2$

$\Rightarrow x^2 = \tan^2\theta - 2\tan\theta\sec\theta + \sec^2\theta$

$\Rightarrow x^2 = \tan^2\theta - 2\tan\theta\sec\theta + (1 + \tan^2\theta)$

$\Rightarrow x^2 = 2\tan^2\theta - 2\tan\theta\sec\theta + 1$

$\Rightarrow x^2 = 2\tan\theta(\tan\theta - \sec\theta) + 1$

$\Rightarrow x^2 = 2\tan\theta \times x + 1$

$\Rightarrow \tan\theta = \dfrac{x^2 - 1}{2x}$

WITH ANALOGOUS WORKINGS & THE IDENTITY $1 + \cot^2\theta = \operatorname{cosec}^2\theta$

$\cot\theta = \dfrac{y^2 - 1}{2y}$

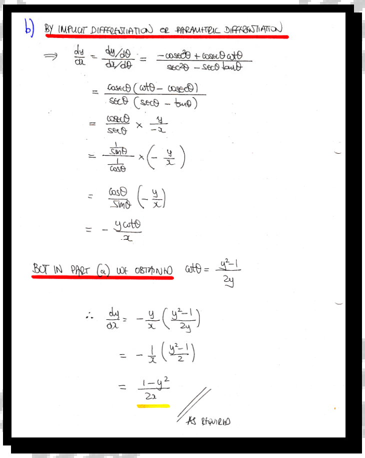
THUS WE FINALLY HAVE

$\Rightarrow \tan\theta\cot\theta = \left(\dfrac{x^2 - 1}{2x}\right)\left(\dfrac{y^2 - 1}{2y}\right)$

$\Rightarrow 1 = \dfrac{(x^2 - 1)(y^2 - 1)}{4xy}$

$\Rightarrow (x^2 - 1)(y^2 - 1) = 4xy$

AS REQUIRED



b) BY IMPLICIT DIFFERENTIATION OR PARAMETRIC DIFFERENTIATION

$\Rightarrow \dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{-\operatorname{cosec}^2\theta + \operatorname{cosec}\theta\cot\theta}{\sec^2\theta - \sec\theta\tan\theta}$

$= \dfrac{\operatorname{cosec}\theta(\cot\theta - \operatorname{cosec}\theta)}{\sec\theta(\sec\theta - \tan\theta)}$

$= \dfrac{\operatorname{cosec}\theta}{\sec\theta} \times \dfrac{y}{-x}$

$= \dfrac{\frac{1}{\sin\theta}}{\frac{1}{\cos\theta}} \times \left(-\dfrac{y}{x}\right)$

$= \dfrac{\cos\theta}{\sin\theta}\left(-\dfrac{y}{x}\right)$

$= -\dfrac{y\cot\theta}{x}$

BUT IN PART (a) WE OBTAINED $\cot\theta = \dfrac{y^2 - 1}{2y}$

$\therefore \dfrac{dy}{dx} = -\dfrac{y}{x}\left(\dfrac{y^2 - 1}{2y}\right)$

$= -\dfrac{1}{x}\left(\dfrac{y^2 - 1}{2}\right)$

$= \dfrac{1 - y^2}{2x}$

AS REQUIRED

Question 131 (****+)

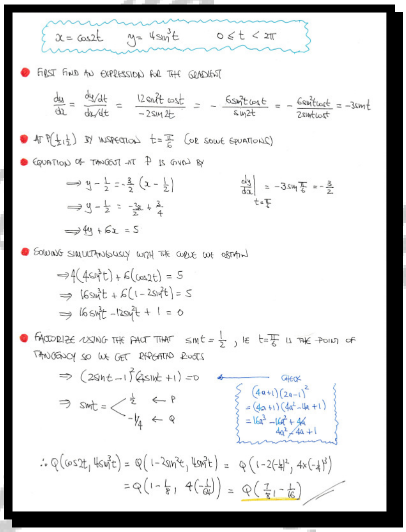
The point $P\left(\frac{1}{2},\frac{1}{2}\right)$ lies on the curve given parametrically as

$$x=\cos 2t\,,\qquad y=4\sin^3 t\,,\quad 0\le t<2\pi\,.$$

The tangent to the curve at P meets the curve again at the point Q.

Determine the exact coordinates of Q.

$\boxed{}\,,\ \boxed{\left(\frac{7}{8},-\frac{1}{16}\right)}$



Question 132 (****+)

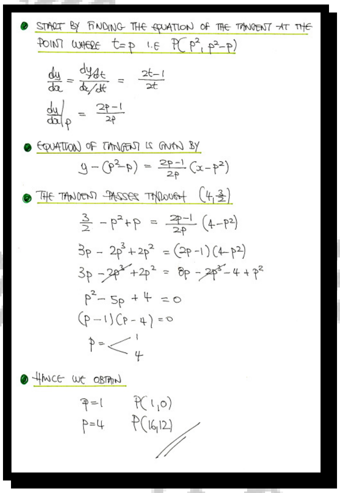
The point P lies on the curve given parametrically as

$$x = t^2, \quad y = t^2 - t, \quad t \in \mathbb{R}.$$

The tangent to the curve at P passes through the point with coordinates $\left(4, \frac{3}{2}\right)$.

Determine the possible coordinates of P.

MP2-C , $P(1,0) \cup P(16,12)$


START BY FINDING THE EQUATION OF THE TANGENT AT THE POINT WHERE $t = p$ I.E $P(p^2, p^2 - p)$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t-1}{2t}$$

$$\left.\frac{dy}{dx}\right|_p = \frac{2p-1}{2p}$$

EQUATION OF TANGENT IS GIVEN BY

$$y - (p^2 - p) = \frac{2p-1}{2p}(x - p^2)$$

THE TANGENT PASSES THROUGH $\left(4, \frac{3}{2}\right)$

$$\frac{3}{2} - p^2 + p = \frac{2p-1}{2p}(4 - p^2)$$

$$3p - 2p^3 + 2p^2 = (2p-1)(4-p^2)$$

$$3p - 2p^3 + 2p^2 = 8p - 2p^3 - 4 + p^2$$

$$p^2 - 5p + 4 = 0$$

$$(p-1)(p-4) = 0$$

$$p = 1, 4$$

HENCE WE OBTAIN

$p = 1$ $P(1,0)$

$p = 4$ $P(16,12)$

Question 133 (****+)

A curve C is given parametrically by

$$x = a + \tan t, \quad y = b + \cot^2 t, \quad 0 < t < \frac{\pi}{2},$$

where a and b are non zero constants.

a) Show that …

i. … $\dfrac{dy}{dx} = -2\cot^3 t$.

ii. … a Cartesian equation of C is

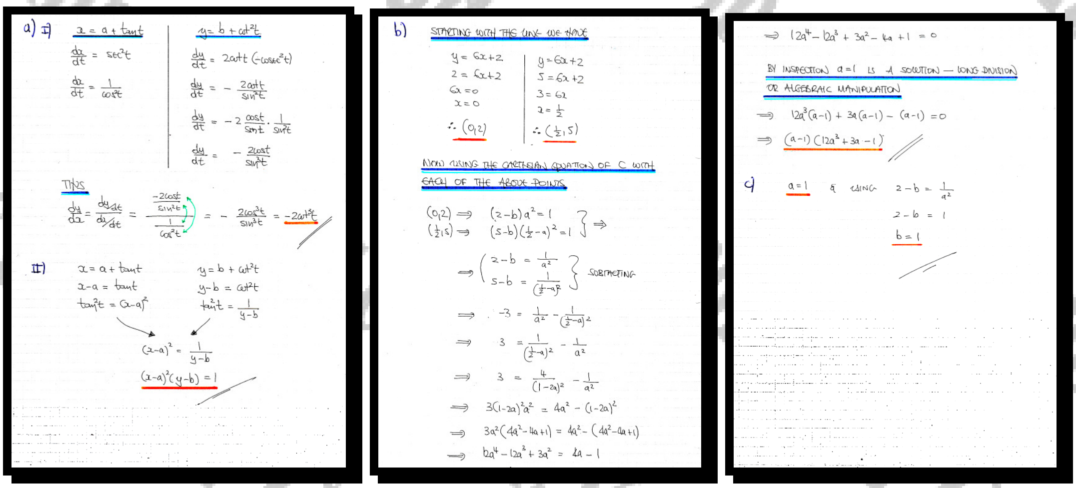
$$(y-b)(x-a)^2 = 1.$$

b) Given that C meets the straight line with equation $y = 6x + 2$ at the points where $y = 2$ and $y = 5$, show further that a is a solution of the equation

$$(a-1)\left(12a^3 + 3a - 1\right) = 0.$$

c) Hence, state a possible value for a and a possible value for b.

$\boxed{}$, $\boxed{a = -1}$, $\boxed{b = 1}$



Question 134 (*****)

A curve C is given parametrically by the equations

$$x = 2 + 2\sin\theta, \quad y = 2\cos\theta + \sin 2\theta, \quad 0 \le \theta < 2\pi .$$

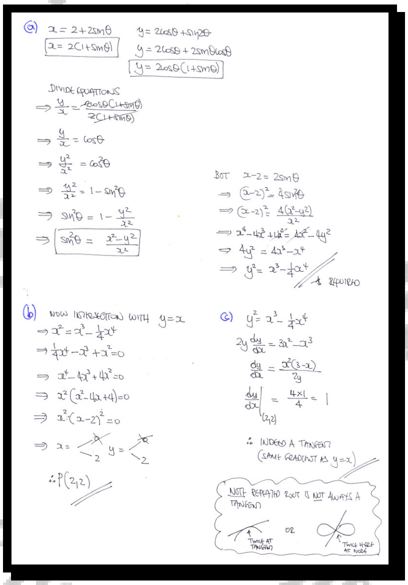
a) By considering a simplified expression for $\frac{y}{x}$, show that a Cartesian equation of C is given by

$$y^2 = x^3 - \frac{1}{4}x^4 .$$

b) Given that C meets the straight line with equation $y = x$ at the origin and at the point P, determine the coordinates of P.

c) Use differentiation to show that the straight line with equation $y = x$ is in fact a tangent to C at the point P.

$P(2,2)$



(a) $x = 2 + 2\sin\theta$ $y = 2\cos\theta + \sin 2\theta$

$x = 2(1+\sin\theta)$ $y = 2\cos\theta + 2\sin\theta\cos\theta$

$y = 2\cos\theta(1+\sin\theta)$

DIVIDE EQUATIONS

$\Rightarrow \frac{y}{x} = \frac{2\cos\theta(1+\sin\theta)}{2(1+\sin\theta)}$

$\Rightarrow \frac{y}{x} = \cos\theta$

$\Rightarrow \frac{y^2}{x^2} = \cos^2\theta$

$\Rightarrow \frac{y^2}{x^2} = 1 - \sin^2\theta$

$\Rightarrow \sin^2\theta = 1 - \frac{y^2}{x^2}$

$\Rightarrow \sin^2\theta = \frac{x^2 - y^2}{x^2}$

BUT $x - 2 = 2\sin\theta$

$\Rightarrow (x-2)^2 = 4\sin^2\theta$

$\Rightarrow (x-2)^2 = \frac{4(x^2-y^2)}{x^2}$

$\Rightarrow x^4 - 4x^3 + 4x^2 = 4x^2 - 4y^2$

$\Rightarrow 4y^2 = 4x^3 - x^4$

$\Rightarrow y^2 = x^3 - \frac{1}{4}x^4$ — AS REQUIRED

(b) NOW INTERSECTION WITH $y = x$

$\Rightarrow x^2 = x^3 - \frac{1}{4}x^4$

$\Rightarrow \frac{1}{4}x^4 - x^3 + x^2 = 0$

$\Rightarrow x^4 - 4x^3 + 4x^2 = 0$

$\Rightarrow x^2(x^2 - 4x + 4) = 0$

$\Rightarrow x^2(x-2)^2 = 0$

$\Rightarrow x = 0, 2 \quad y = 0, 2$

$\therefore P(2,2)$

(c) $y^2 = x^3 - \frac{1}{4}x^4$

$2y\frac{dy}{dx} = 3x^2 - x^3$

$\frac{dy}{dx} = \frac{x^2(3-x)}{2y}$

$\left.\frac{dy}{dx}\right|_{(2,2)} = \frac{4 \times 1}{4} = 1$

$\therefore$ INDEED A TANGENT (SAME GRADIENT AS $y = x$)

NOTE REPEATED ROOT IS NOT ALWAYS A TANGENT

TWICE AT TANGENT OR TWICE HERE AT NODE

Question 135 (*****)

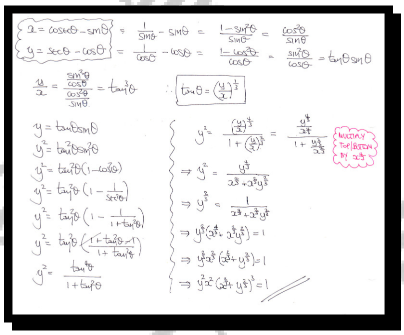
A parametric relationship is given by

$$x = \operatorname{cosec}\theta - \sin\theta \,, \quad y = \sec\theta - \cos\theta \,, \quad 0 < \theta < \frac{\pi}{2} .$$

Show that a Cartesian equation for this relationship is

$$y^2 x^2 \left(x^{\frac{2}{3}} + y^{\frac{2}{3}} \right)^3 = 1 .$$

proof



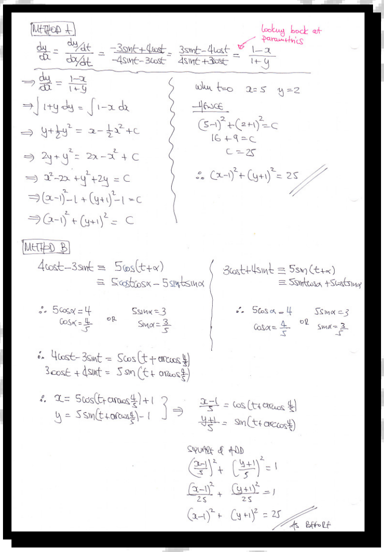
Question 136 (*****)

The curve C has parametric equations

$$x = 4\cos t - 3\sin t + 1, \quad y = 3\cos t + 4\sin t - 1, \quad 0 \le t < 2\pi .$$

Find a Cartesian equation of the curve.

$$\boxed{(x-1)^2 + (y+1)^2 = 25}$$



Question 137 (*****)

A curve C is given parametrically by

$$x = t^2 - p^2, \qquad y = 2tp\,,$$

where t and p are real parameters.

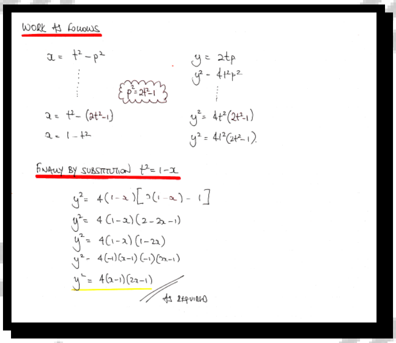
The parameters t and p are related by the equation

$$p^2 = 2t^2 - 1.$$

Show that a Cartesian equation for C is

$$y^2 = 4(x-1)(2x-1).$$

| | , | proof |


WORK AS FOLLOWS

$x = t^2 - p^2$ $\quad$ $y = 2tp$

$y^2 = 4t^2p^2$

$p^2 = 2t^2 - 1$

$x = t^2 - (2t^2 - 1)$ $\quad$ $y^2 = 4t^2(2t^2-1)$

$x = 1 - t^2$ $\quad$ $y^2 = 4t^2(2t^2-1)$

FINALLY BY SUBSTITUTION $t^2 = 1 - x$

$y^2 = 4(1-x)[2(1-x) - 1]$

$y^2 = 4(1-x)(2-2x-1)$

$y^2 = 4(1-x)(1-2x)$

$y^2 = 4(-1)(x-1)(-1)(2x-1)$

$y^2 = 4(x-1)(2x-1)$

AS REQUIRED

Question 138 (*****)

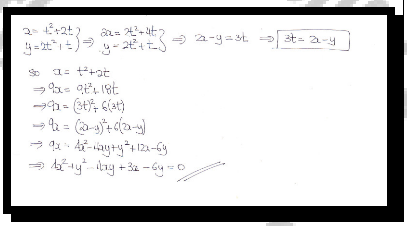
The curve C has parametric equations

$$x = t^2 + 2t, \quad y = 2t^2 + t, \quad t \in \mathbb{R}.$$

Show that a Cartesian equation of the curve is given by

$$4x^2 + y^2 - 4xy + 3x - 6y = 0.$$

[ ], [proof]



$x = t^2 + 2t$, $y = 2t^2 + t$ $\Rightarrow$ $2x = 2t^2 + 4t$, $y = 2t^2 + t$ $\Rightarrow$ $2x - y = 3t$ $\Rightarrow$ $\boxed{3t = 2x - y}$

so $x = t^2 + 2t$

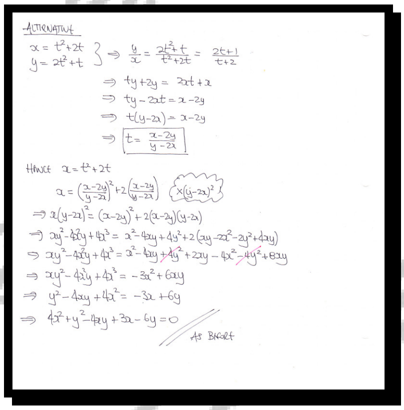
$\Rightarrow 9x = 9t^2 + 18t$

$\Rightarrow 9x = (3t)^2 + 6(3t)$

$\Rightarrow 9x = (2x-y)^2 + 6(2x-y)$

$\Rightarrow 9x = 4x^2 - 4xy + y^2 + 12x - 6y$

$\Rightarrow 4x^2 + y^2 - 4xy + 3x - 6y = 0$



ALTERNATIVE

$x = t^2 + 2t$, $y = 2t^2 + t$ $\Rightarrow \frac{y}{x} = \frac{2t^2 + t}{t^2 + 2t} = \frac{2t+1}{t+2}$

$\Rightarrow ty + 2y = 2xt + x$

$\Rightarrow ty - 2xt = x - 2y$

$\Rightarrow t(y - 2x) = x - 2y$

$\Rightarrow \boxed{t = \frac{x-2y}{y-2x}}$

Hence $x = t^2 + 2t$

$x = \left(\frac{x-2y}{y-2x}\right)^2 + 2\left(\frac{x-2y}{y-2x}\right)$ $\quad \times (y-2x)^2$

$\Rightarrow x(y-2x)^2 = (x-2y)^2 + 2(x-2y)(y-2x)$

$\Rightarrow xy^2 - 4x^2y + 4x^3 = x^2 - 4xy + 4y^2 + 2(xy - 2x^2 - 2y^2 + 4xy)$

$\Rightarrow xy^2 - 4x^2y + 4x^3 = x^2 - 4xy + 4y^2 + 2xy - 4x^2 - 4y^2 + 8xy$

$\Rightarrow xy^2 - 4x^2y + 4x^3 = -3x^2 + 6xy$

$\Rightarrow y^2 - 4xy + 4x^2 = -3x + 6y$

$\Rightarrow 4x^2 + y^2 - 4xy + 3x - 6y = 0$

AS BEFORE

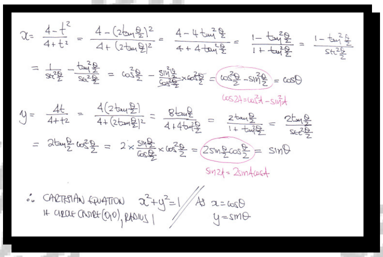
Question 139 **(*****)**

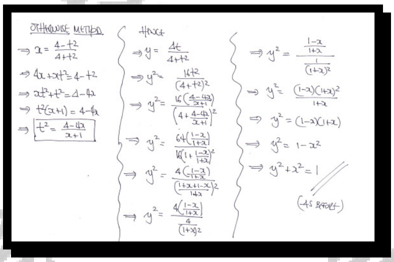
A curve C is given parametrically by the equations

$$x=\frac{4-t^2}{4+t^2}, \quad y=\frac{4t}{4+t^2}, \quad t\in\mathbb{R}.$$

By using the substitution $t=\tan\frac{\theta}{2}$, or otherwise, show that the Cartesian equation of C represents a circle.

$\boxed{y^2+x^2=1}$





Question 140 **(*****)**

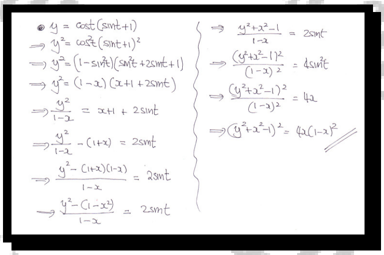
A curve is defined by the parametric equations

$$x=\sin^2 t, \quad y=\sin t\cos t+\cos t, \; 0\le t<2\pi.$$

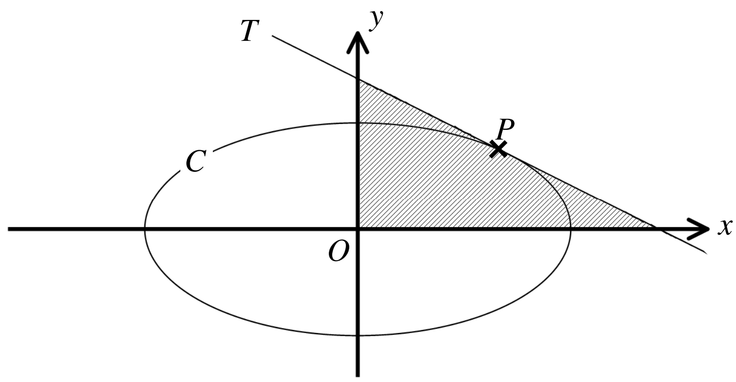
Show that the Cartesian equation of the curve is

$$\left(x^2+y^2-1\right)^2=4x(1-x)^2.$$

$\boxed{\text{proof}}$



Question 141 (*****)



The figure above shows the curve C with parametric equations

$$x = 4\cos\theta, \quad y = 3\sin\theta, \quad 0 \le \theta < 2\pi .$$

The point P lies on C where $\theta = \alpha$, where $0 < \alpha < \frac{\pi}{2}$.

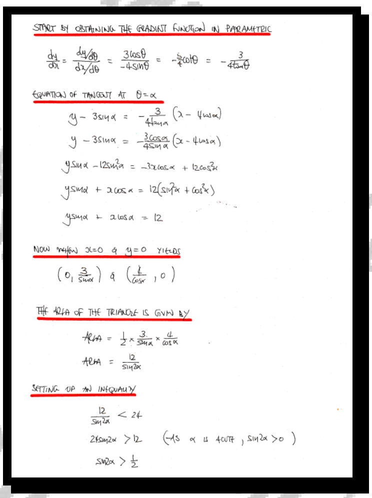
The line T is a tangent to C at P.

The tangent T meets the coordinate axes at the points A and B.

The area of the triangle OAB, where O is the origin, is less than 24 square units.

Find the range of the possible values of α.

$\boxed{}$, $\boxed{\frac{\pi}{12} \le \alpha \le \frac{5\pi}{12}}$



START BY OBTAINING THE GRADIENT FUNCTION IN PARAMETRIC

$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3\cos\theta}{-4\sin\theta} = -\frac{3}{4}\cot\theta = -\frac{3}{4\tan\theta}$

EQUATION OF TANGENT AT $\theta = \alpha$

$y - 3\sin\alpha = -\frac{3}{4\tan\alpha}(x - 4\cos\alpha)$

$y - 3\sin\alpha = -\frac{3\cos\alpha}{4\sin\alpha}(x - 4\cos\alpha)$

$y\sin\alpha - 12\sin^2\alpha = -3x\cos\alpha + 12\cos^2\alpha$

$y\sin\alpha + x\cos\alpha = 12(\sin^2\alpha + \cos^2\alpha)$

$y\sin\alpha + x\cos\alpha = 12$

NOW WHEN $x=0$ & $y=0$ YIELDS

$\left(0, \frac{3}{\sin\alpha}\right)$ & $\left(\frac{4}{\cos\alpha}, 0\right)$

THE AREA OF THE TRIANGLE IS GIVEN BY

$\text{AREA} = \frac{1}{2} \times \frac{3}{\sin\alpha} \times \frac{4}{\cos\alpha}$

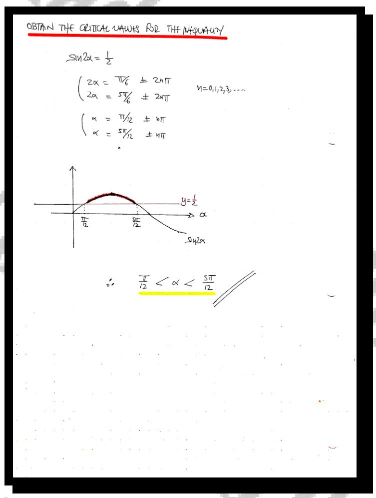
$\text{AREA} = \frac{12}{\sin 2\alpha}$

SETTING UP AN INEQUALITY

$\frac{12}{\sin 2\alpha} < 24$

$24\sin 2\alpha > 12$ (AS α IS ACUTE, $\sin 2\alpha > 0$)

$\sin 2\alpha > \frac{1}{2}$



OBTAIN THE CRITICAL VALUES FOR THE INEQUALITY

$\sin 2\alpha = \frac{1}{2}$

$2\alpha = \pi/6 \pm 2n\pi$
$2\alpha = 5\pi/6 \pm 2n\pi$ $\quad n = 0,1,2,3,\ldots$

$\alpha = \pi/12 \pm n\pi$
$\alpha = 5\pi/12 \pm n\pi$

$\therefore \frac{\pi}{12} < \alpha < \frac{5\pi}{12}$

Question 142 (*****)

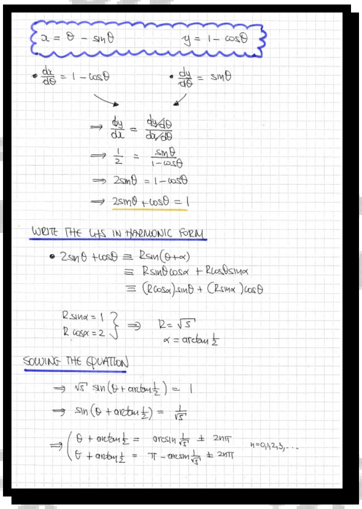
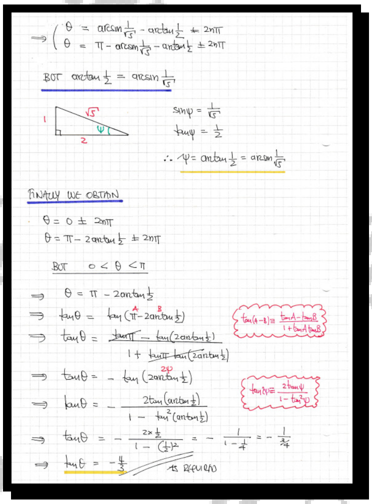
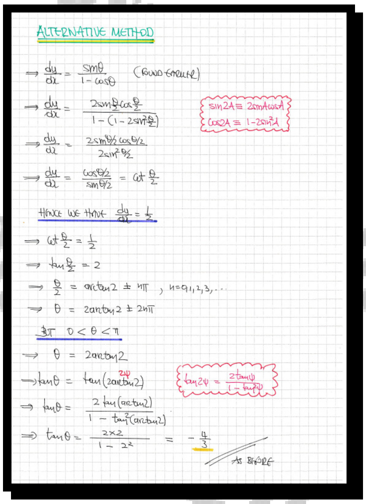
A cycloid is given by the parametric equations

$$x = \theta - \sin\theta, \quad y = 1 - \cos\theta, \quad 0 < \theta < \pi .$$

The gradient at the point P on this cycloid is $\frac{1}{2}$.

Show that at the point P, $\tan\theta = -\frac{4}{3}$.

[ ], [proof]

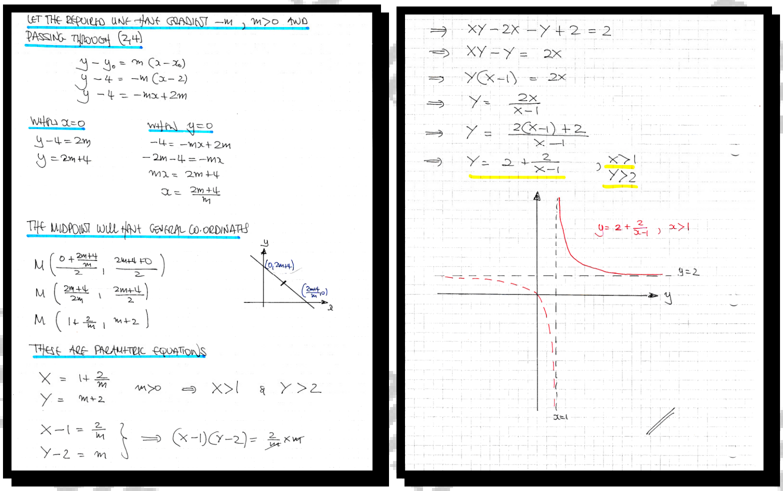




Question 143 (*****)

A straight line with negative gradient passes though the point with coordinates $(2,4)$. The point M the midpoint of the two intercepts of this line with the coordinate axes.

Sketch a detailed graph of the locus of M.

SP-X , graph



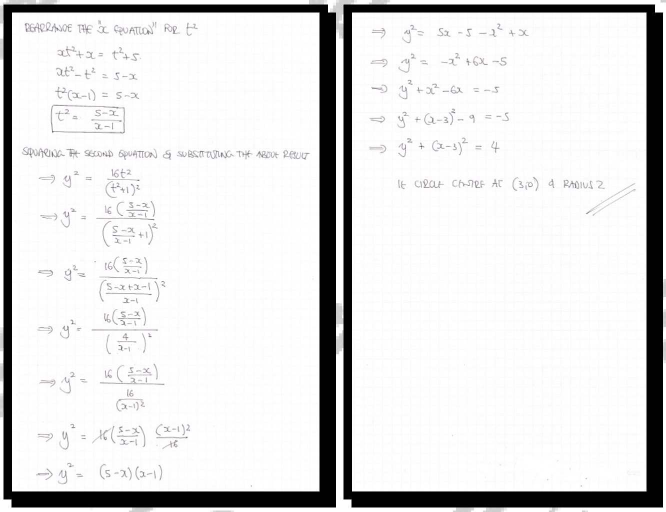
Question 144 **(*****)**

The curve has parametric equations

$$x = \frac{t^2+5}{t^2+1}, \quad y = \frac{4t}{t^2+1}, \quad t \in \mathbb{R}.$$

Show, by eliminating the parameter t, that the curve is a circle, stating the coordinates of its centre, and the size of its radius.

$\boxed{}$, $\boxed{(3,0)}$, $\boxed{R=2}$



REARRANGE THE "x EQUATION" FOR t^2

$xt^2 + x = t^2 + 5$

$xt^2 - t^2 = 5 - x$

$t^2(x-1) = 5 - x$

$\boxed{t^2 = \frac{5-x}{x-1}}$

SQUARING THE SECOND EQUATION & SUBSTITUTING THE ABOVE RESULT

$\Rightarrow y^2 = \frac{16t^2}{(t^2+1)^2}$

$\Rightarrow y^2 = \frac{16\left(\frac{5-x}{x-1}\right)}{\left(\frac{5-x}{x-1}+1\right)^2}$

$\Rightarrow y^2 = \frac{16\left(\frac{5-x}{x-1}\right)}{\left(\frac{5-x+x-1}{x-1}\right)^2}$

$\Rightarrow y^2 = \frac{16\left(\frac{5-x}{x-1}\right)}{\left(\frac{4}{x-1}\right)^2}$

$\Rightarrow y^2 = \frac{16\left(\frac{5-x}{x-1}\right)}{\frac{16}{(x-1)^2}}$

$\Rightarrow y^2 = 16\left(\frac{5-x}{x-1}\right)\frac{(x-1)^2}{16}$

$\Rightarrow y^2 = (5-x)(x-1)$

$\Rightarrow y^2 = 5x - 5 - x^2 + x$

$\Rightarrow y^2 = -x^2 + 6x - 5$

$\Rightarrow y^2 + x^2 - 6x = -5$

$\Rightarrow y^2 + (x-3)^2 - 9 = -5$

$\Rightarrow y^2 + (x-3)^2 = 4$

IE CIRCLE CENTRE AT $(3,0)$ & RADIUS 2

Question 145 **(*****)**

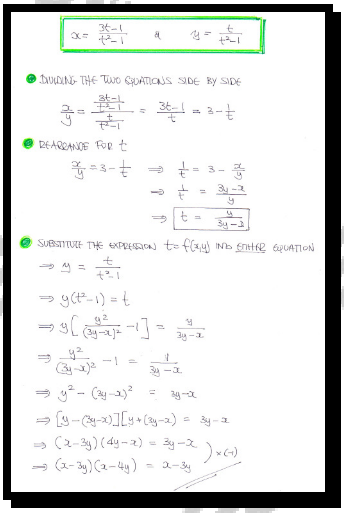
The curve C has parametric equations

$$x = \frac{3t-1}{t^2-1}, \quad y = \frac{t}{t^2-1}, \quad t \in \mathbb{R}.$$

Show by eliminating the parameter t, that a Cartesian equation of C is

$$(x-2y)(x-4y) = x-3y$$

[ ], [proof]



$x = \frac{3t-1}{t^2-1}$ & $y = \frac{t}{t^2-1}$

● DIVIDING THE TWO EQUATIONS SIDE BY SIDE

$\frac{x}{y} = \frac{\frac{3t-1}{t^2-1}}{\frac{t}{t^2-1}} = \frac{3t-1}{t} = 3 - \frac{1}{t}$

● REARRANGE FOR t

$\frac{x}{y} = 3 - \frac{1}{t} \implies \frac{1}{t} = 3 - \frac{x}{y}$

$\implies \frac{1}{t} = \frac{3y-x}{y}$

$\implies t = \frac{y}{3y-x}$

● SUBSTITUTE THE EXPRESSION $t = f(x,y)$ INTO EITHER EQUATION

$\implies y = \frac{t}{t^2-1}$

$\implies y(t^2-1) = t$

$\implies y\left[\frac{y^2}{(3y-x)^2} - 1\right] = \frac{y}{3y-x}$

$\implies \frac{y^2}{(3y-x)^2} - 1 = \frac{1}{3y-x}$

$\implies y^2 - (3y-x)^2 = 3y-x$

$\implies [y-(3y-x)][y+(3y-x)] = 3y-x$

$\implies (x-3y)(4y-x) = 3y-x$) $\times(-1)$

$\implies (x-3y)(x-4y) = x-3y$

Question 146 **(*****)**

A curve is given parametrically by the equations

$$x = \sin t, \;\; y = \cos^3 t, \;\; 0 \le t < 2\pi .$$

a) Find a simplified expression for $\frac{dy}{dx}$, in terms of t.

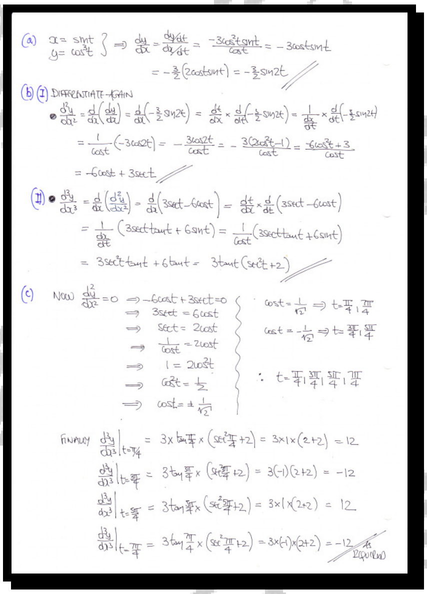
b) Show that …

i. … $\frac{d^2y}{dx^2} = -6\cos t + 3\sec t$.

ii. … $\frac{d^3y}{dx^3} = 3\tan t\left(2+\sec^2 t\right)$.

c) Show further that the value of $\frac{d^3y}{dx^3}$ at the points where $\frac{d^2y}{dx^2} = 0$ is ± 12.

$$\boxed{\frac{dy}{dx} = -\frac{3}{2}\sin 2t}$$



Question 147 (*****)

A curve is given by the parametric equations

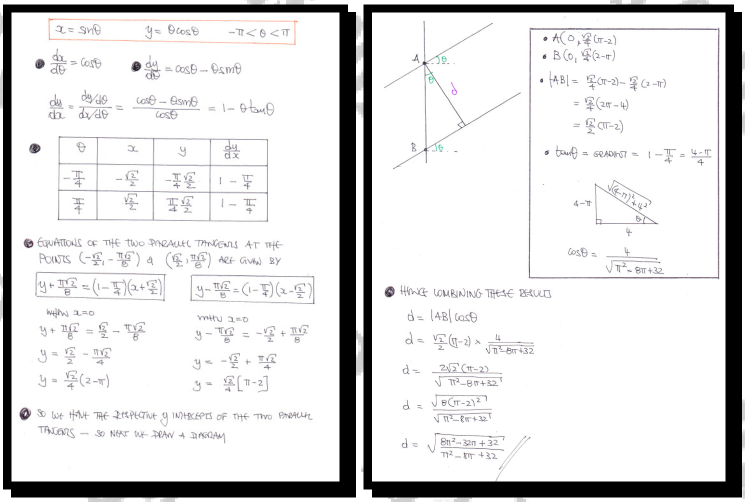
$$x = \sin\theta, \quad y = \theta\cos\theta, \quad -\pi < \theta < \pi.$$

The tangents to the curve, at the points where $\theta = -\frac{\pi}{4}$ and $\theta = \frac{\pi}{4}$, are parallel to one another , at a distance d apart.

Show that

$$d = \sqrt{\frac{8\pi^2 - 32\pi + 32}{\pi^2 - 8\pi + 32}}.$$

SP X, proof



$x = \sin\theta \qquad y = \theta\cos\theta \qquad -\pi < \theta < \pi$

- $\frac{dx}{d\theta} = \cos\theta$
- $\frac{dy}{d\theta} = \cos\theta - \theta\sin\theta$

$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cos\theta - \theta\sin\theta}{\cos\theta} = 1 - \theta\tan\theta$

θ	x	y	$\frac{dy}{dx}$
$-\frac{\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\pi}{4}\frac{\sqrt{2}}{2}$	$1-\frac{\pi}{4}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\pi}{4}\frac{\sqrt{2}}{2}$	$1-\frac{\pi}{4}$

- EQUATIONS OF THE TWO PARALLEL TANGENTS AT THE POINTS $\left(-\frac{\sqrt{2}}{2}, -\frac{\pi\sqrt{2}}{8}\right)$ & $\left(\frac{\sqrt{2}}{2}, \frac{\pi\sqrt{2}}{8}\right)$ ARE GIVEN BY

$y + \frac{\pi\sqrt{2}}{8} = \left(1-\frac{\pi}{4}\right)\left(x + \frac{\sqrt{2}}{2}\right)$

WHEN $x=0$

$y + \frac{\pi\sqrt{2}}{8} = \frac{\sqrt{2}}{2} - \frac{\pi\sqrt{2}}{8}$

$y = \frac{\sqrt{2}}{2} - \frac{\pi\sqrt{2}}{4}$

$y = \frac{\sqrt{2}}{4}(2-\pi)$

$y - \frac{\pi\sqrt{2}}{8} = \left(1-\frac{\pi}{4}\right)\left(x - \frac{\sqrt{2}}{2}\right)$

WHEN $x=0$

$y - \frac{\pi\sqrt{2}}{8} = -\frac{\sqrt{2}}{2} + \frac{\pi\sqrt{2}}{8}$

$y = -\frac{\sqrt{2}}{2} + \frac{\pi\sqrt{2}}{4}$

$y = \frac{\sqrt{2}}{4}[\pi - 2]$

- SO WE HAVE THE RESPECTIVE y INTERCEPTS OF THE TWO PARALLEL TANGENTS — SO NEXT WE DRAW A DIAGRAM

- $A\left(0, \frac{\sqrt{2}}{4}(\pi-2)\right)$
- $B\left(0, \frac{\sqrt{2}}{4}(2-\pi)\right)$
- $|AB| = \frac{\sqrt{2}}{4}(\pi-2) - \frac{\sqrt{2}}{4}(2-\pi)$
  $= \frac{\sqrt{2}}{4}(2\pi-4)$
  $= \frac{\sqrt{2}}{2}(\pi-2)$
- $\tan\theta$ = GRADIENT $= 1 - \frac{\pi}{4} = \frac{4-\pi}{4}$

$\cos\theta = \frac{4}{\sqrt{\pi^2 - 8\pi + 32}}$

- HENCE COMBINING THESE RESULTS

$d = |AB|\cos\theta$

$d = \frac{\sqrt{2}}{2}(\pi-2) \times \frac{4}{\sqrt{\pi^2-8\pi+32}}$

$d = \frac{2\sqrt{2}(\pi-2)}{\sqrt{\pi^2-8\pi+32}}$

$d = \frac{\sqrt{8(\pi-2)^2}}{\sqrt{\pi^2-8\pi+32}}$

$d = \sqrt{\frac{8\pi^2-32\pi+32}{\pi^2-8\pi+32}}$

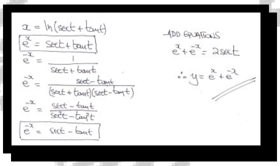
Question 148 (*****)

A curve is given parametrically by

$$x = \ln(\sec t + \tan t), \quad y = 2\sec t, \quad t \in \mathbb{R}, \quad t \neq \frac{(2n-1)\pi}{2}.$$

Find a Cartesian equation for the curve in the form $y = f(x)$.

$$\boxed{y = e^{x} + e^{-x}}$$


$x = \ln(\sec t + \tan t)$
$e^{x} = \sec t + \tan t$
$e^{-x} = \frac{1}{\sec t + \tan t}$
$e^{-x} = \frac{\sec t - \tan t}{(\sec t + \tan t)(\sec t - \tan t)}$
$e^{-x} = \frac{\sec t - \tan t}{\sec^2 t - \tan^2 t}$
$e^{-x} = \sec t - \tan t$
- ADD EQUATIONS
$e^{x} + e^{-x} = 2\sec t$
$\therefore y = e^{x} + e^{-x}$

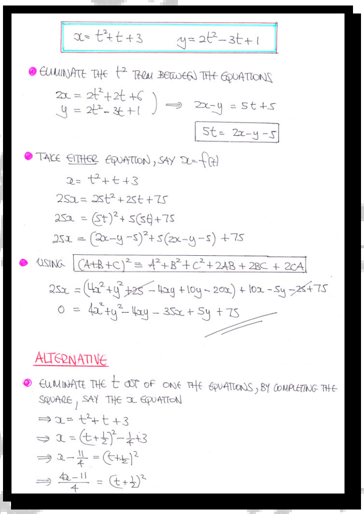
Question 149 (*****)

A curve is given parametrically by

$$x = t^2 + t + 3, \quad y = 2t^2 - 3t + 1, \quad t \in \mathbb{R}$$

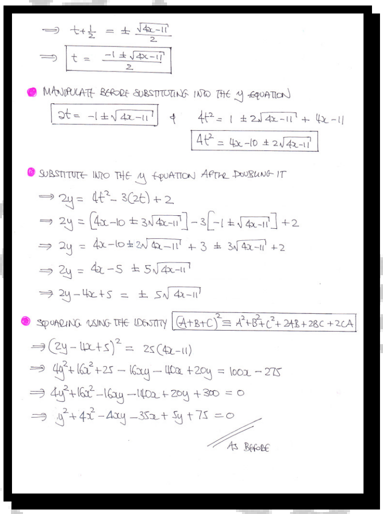
Find a Cartesian equation for the curve in the form $f(x, y) = 0$.

$$\boxed{\quad}, \ \boxed{4x^2 + y^2 - 4xy + 5y - 35x + 75 = 0}$$


$x = t^2 + t + 3 \qquad y = 2t^2 - 3t + 1$

- ELIMINATE THE t^2 TERM BETWEEN THE EQUATIONS
$\left.\begin{array}{l} 2x = 2t^2 + 2t + 6 \\ y = 2t^2 - 3t + 1 \end{array}\right\} \Rightarrow 2x - y = 5t + 5$
$5t = 2x - y - 5$
- TAKE EITHER EQUATION, SAY $x = f(t)$
$x = t^2 + t + 3$
$25x = 25t^2 + 25t + 75$
$25x = (5t)^2 + 5(5t) + 75$
$25x = (2x - y - 5)^2 + 5(2x - y - 5) + 75$
- USING $(A+B+C)^2 \equiv A^2 + B^2 + C^2 + 2AB + 2BC + 2CA$
$25x = (4x^2 + y^2 + 25 - 4xy + 10y - 20x) + 10x - 5y - 25 + 75$
$0 = 4x^2 + y^2 - 4xy - 35x + 5y + 75$

ALTERNATIVE
- ELIMINATE THE t OUT OF ONE THE EQUATIONS, BY COMPLETING THE SQUARE, SAY THE x EQUATION
$\Rightarrow x = t^2 + t + 3$
$\Rightarrow x = \left(t + \frac{1}{2}\right)^2 - \frac{1}{4} + 3$
$\Rightarrow x - \frac{11}{4} = \left(t + \frac{1}{2}\right)^2$
$\Rightarrow \frac{4x - 11}{4} = \left(t + \frac{1}{2}\right)^2$


$\Rightarrow t + \frac{1}{2} = \pm \frac{\sqrt{4x-11}}{2}$
$\Rightarrow t = \frac{-1 \pm \sqrt{4x-11}}{2}$
- MANIPULATE BEFORE SUBSTITUTING INTO THE y EQUATION
$2t = -1 \pm \sqrt{4x-11}$ & $4t^2 = 1 \pm 2\sqrt{4x-11} + 4x - 11$
$4t^2 = 4x - 10 \pm 2\sqrt{4x-11}$
- SUBSTITUTE INTO THE y EQUATION AFTER DOUBLING IT
$\Rightarrow 2y = 4t^2 - 3(2t) + 2$
$\Rightarrow 2y = \left[4x - 10 \pm 3\sqrt{4x-11}\right] - 3\left[-1 \pm \sqrt{4x-11}\right] + 2$
$\Rightarrow 2y = 4x - 10 \pm 2\sqrt{4x-11} + 3 \pm 3\sqrt{4x-11} + 2$
$\Rightarrow 2y = 4x - 5 \pm 5\sqrt{4x-11}$
$\Rightarrow 2y - 4x + 5 = \pm 5\sqrt{4x-11}$
- SQUARING USING THE IDENTITY $(A+B+C)^2 \equiv A^2 + B^2 + C^2 + 2AB + 2BC + 2CA$
$\Rightarrow (2y - 4x + 5)^2 = 25(4x - 11)$
$\Rightarrow 4y^2 + 16x^2 + 25 - 16xy - 40x + 20y = 100x - 275$
$\Rightarrow 4y^2 + 16x^2 - 16xy - 140x + 20y + 300 = 0$
$\Rightarrow y^2 + 4x^2 - 4xy - 35x + 5y + 75 = 0$
AS BEFORE

Question 150 (*****)

Eliminate θ from the following pair of equations.

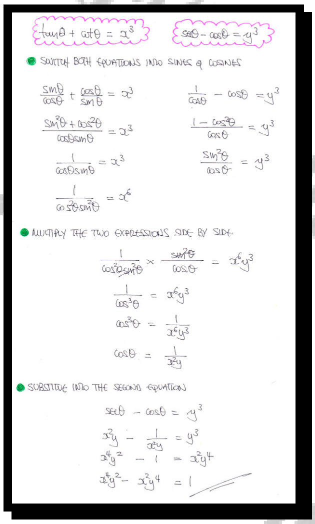
$$\tan\theta + \cot\theta = x^3$$

$$\sec\theta - \cos\theta = y^3$$

Write the answer in the form

$$f(x, y) = 1 .$$

SPM , $x^4y^2 - y^4x^2 = 1$



$\tan\theta + \cot\theta = x^3$ $\quad$ $\sec\theta - \cos\theta = y^3$

- SWITCH BOTH EQUATIONS INTO SINES & COSINES

$\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} = x^3$ $\quad$ $\frac{1}{\cos\theta} - \cos\theta = y^3$

$\frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta} = x^3$ $\quad$ $\frac{1-\cos^2\theta}{\cos\theta} = y^3$

$\frac{1}{\cos\theta\sin\theta} = x^3$ $\quad$ $\frac{\sin^2\theta}{\cos\theta} = y^3$

$\frac{1}{\cos^2\theta\sin^2\theta} = x^6$

- MULTIPLY THE TWO EXPRESSIONS SIDE BY SIDE

$\frac{1}{\cos^2\theta\sin^2\theta} \times \frac{\sin^2\theta}{\cos\theta} = x^6y^3$

$\frac{1}{\cos^3\theta} = x^6y^3$

$\cos^3\theta = \frac{1}{x^6y^3}$

$\cos\theta = \frac{1}{x^2y}$

- SUBSTITUTE INTO THE SECOND EQUATION

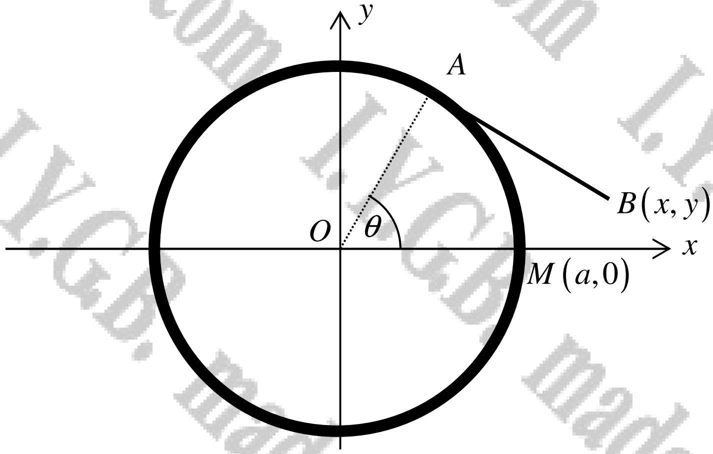
$\sec\theta - \cos\theta = y^3$

$x^2y - \frac{1}{x^2y} = y^3$

$x^4y^2 - 1 = x^2y^4$

$x^4y^2 - x^2y^4 = 1$

Question 151 (*****)



The figure above shows a set of coordinate axes superimposed with a cotton reel.

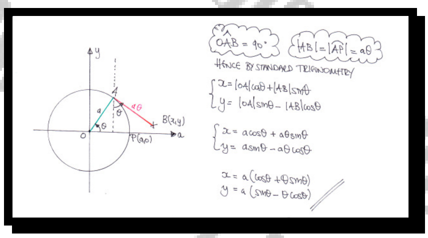
Cotton thread is being unwound from around the circumference of the fixed circular reel of radius a and centre at O.

The free end of the cotton thread is marked as the point $B(x, y)$ which was originally at $P(a,0)$.

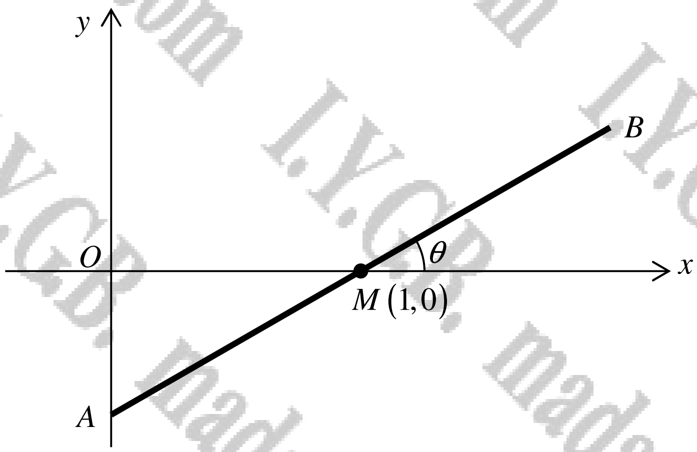
The unwound part of the cotton thread AB is kept straight and θ is the angle OA subtends at the positive x axis, as shown in the figure above.

Find the parametric equations that satisfy the locus of $B(x, y)$, as the cotton thread is unwound in the fashion described.

$\boxed{}$, $\boxed{x = a(\cos\theta + \theta\sin\theta), \quad y = a(\sin\theta - \theta\cos\theta)}$



Question 152 (*****)



The figure above shows a rigid rod AB of length 4 units which can slide through a hinge located at the point $M(1,0)$. The hinge allows the rod to turn in any direction in the x-y plane. The end of the rod marked as A can slide on the y axis so that $|OA| \le 4$. Let θ be the angle of inclination of the rod to the positive x axis.

a) Show that as A slides on the on the y axis, the locus of B satisfies the parametric equations

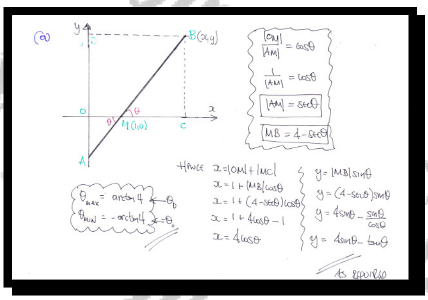
$$x = 4\cos\theta, \quad y = 4\sin\theta - \tan\theta, \quad -\theta_0 \le \theta \le \theta_0,$$

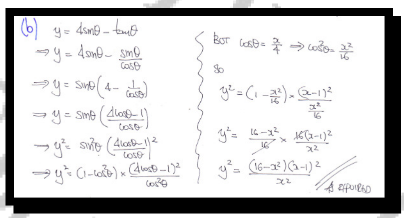
stating the exact value of θ_0.

b) Show further that a Cartesian equation of this locus is given by

$$y^2 = \frac{(16 - x^2)(x-1)^2}{x^2}.$$

[ ], proof





Question 153 **(*****)**

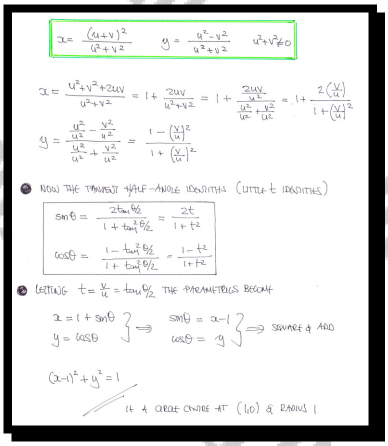
The curve C has parametric equations

$$x = \frac{(u+v)^2}{u^2+v^2}, \quad y = \frac{u^2-v^2}{u^2+v^2},$$

where u and v are real parameters with $u^2+v^2 \neq 0$.

By considering the tangent half angle trigonometric identities, or otherwise, show that C is a circle, stating the coordinates of its centre and the size of its radius.

$\boxed{\quad}$, $\boxed{(1,0)}$, $\boxed{R=1}$



$x = \frac{(u+v)^2}{u^2+v^2} \qquad y = \frac{u^2-v^2}{u^2+v^2} \qquad u^2+v^2 \neq 0$

$x = \frac{u^2+v^2+2uv}{u^2+v^2} = 1 + \frac{2uv}{u^2+v^2} = 1 + \frac{\frac{2uv}{u^2}}{\frac{u^2}{u^2}+\frac{v^2}{u^2}} = 1 + \frac{2\left(\frac{v}{u}\right)}{1+\left(\frac{v}{u}\right)^2}$

$y = \frac{\frac{u^2}{u^2}-\frac{v^2}{u^2}}{\frac{u^2}{u^2}+\frac{v^2}{u^2}} = \frac{1-\left(\frac{v}{u}\right)^2}{1+\left(\frac{v}{u}\right)^2}$

- NOW THE TANGENT HALF-ANGLE IDENTITIES (LITTLE t IDENTITIES)

$\sin\theta = \frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} = \frac{2t}{1+t^2}$

$\cos\theta = \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} = \frac{1-t^2}{1+t^2}$

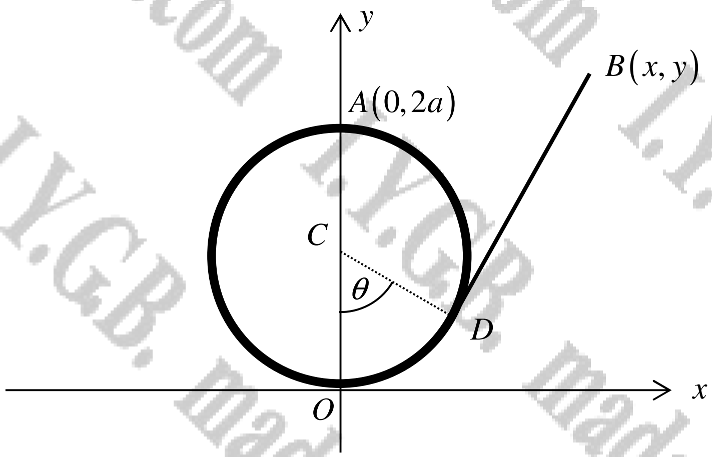
- LETTING $t = \frac{v}{u} = \tan\frac{\theta}{2}$ THE PARAMETRICS BECOME

$x = 1+\sin\theta, \; y = \cos\theta \implies \sin\theta = x-1, \; \cos\theta = y \implies$ SQUARE & ADD

$(x-1)^2 + y^2 = 1$

IT A CIRCLE CENTRE AT $(1,0)$ & RADIUS 1

Question 154 (*****)



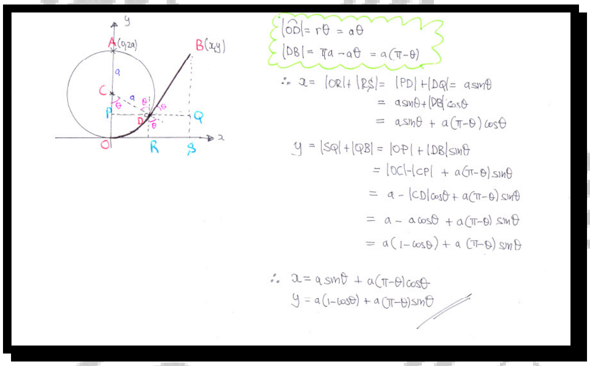
The figure above shows a set of coordinate axes superimposed with a circular cotton reel of radius a and centre at $C(0,a)$.

A piece of cotton thread, of length πa, is fixed at one end at O and is being unwound from around the circumference of the fixed circular reel. The free end of the cotton thread is marked as the point $B(x,y)$ which was originally at $A(0,2a)$.

The unwound part of the cotton thread BD is kept straight and θ is the angle OCD as shown in the figure above.

Find the parametric equations that satisfy the locus of $B(x,y)$, as the cotton thread is unwound in the fashion described, for which $x>0$, $y>0$.

$$x = a\left[\sin\theta + (\pi-\theta)\cos\theta\right], \quad y = a\left[1-\cos\theta+(\pi-\theta)\sin\theta\right]$$



Question 155 (*****)

The straight line L has equation

$$\frac{x}{p}+\frac{y}{q}=1,$$

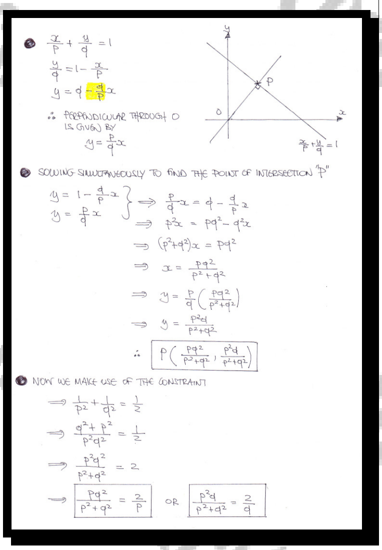
where p and q are non zero parameters, constrained by the equation

$$\frac{1}{p^2}+\frac{1}{q^2}=\frac{1}{2}.$$

The point P is the foot of the perpendicular from the origin O to L.

Show that for all values of p and q, P lies on a circle C, stating its radius.

$\boxed{}$, $\boxed{R=\sqrt{2}}$



- $\frac{x}{p}+\frac{y}{q}=1$

$\frac{y}{q}=1-\frac{x}{p}$

$y=q-\frac{q}{p}x$

∴ PERPENDICULAR THROUGH O IS GIVEN BY

$y=\frac{p}{q}x$

- SOLVING SIMULTANEOUSLY TO FIND THE POINT OF INTERSECTION "P"

$\left.\begin{array}{l} y=1-\frac{q}{p}x \\ y=\frac{p}{q}x \end{array}\right\} \Rightarrow \frac{p}{q}x=q-\frac{q}{p}x$

$\Rightarrow p^2x=pq^2-q^2x$

$\Rightarrow (p^2+q^2)x=pq^2$

$\Rightarrow x=\frac{pq^2}{p^2+q^2}$

$\Rightarrow y=\frac{p}{q}\left(\frac{pq^2}{p^2+q^2}\right)$

$\Rightarrow y=\frac{p^2q}{p^2+q^2}$

$\therefore \boxed{P\left(\frac{pq^2}{p^2+q^2},\frac{p^2q}{p^2+q^2}\right)}$

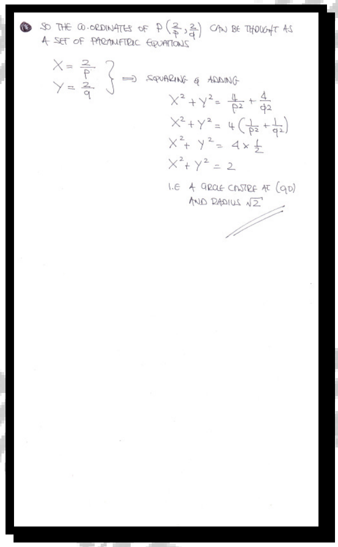
- NOW WE MAKE USE OF THE CONSTRAINT

$\Rightarrow \frac{1}{p^2}+\frac{1}{q^2}=\frac{1}{2}$

$\Rightarrow \frac{q^2+p^2}{p^2q^2}=\frac{1}{2}$

$\Rightarrow \frac{p^2q^2}{p^2+q^2}=2$

$\Rightarrow \boxed{\frac{pq^2}{p^2+q^2}=\frac{2}{p}}$ OR $\boxed{\frac{p^2q}{p^2+q^2}=\frac{2}{q}}$



- SO THE CO-ORDINATES OF $P\left(\frac{2}{p},\frac{2}{q}\right)$ CAN BE THOUGHT AS A SET OF PARAMETRIC EQUATIONS

$\left.\begin{array}{l} X=\frac{2}{p} \\ Y=\frac{2}{q} \end{array}\right\} \Rightarrow$ SQUARING & ADDING

$X^2+Y^2=\frac{4}{p^2}+\frac{4}{q^2}$

$X^2+Y^2=4\left(\frac{1}{p^2}+\frac{1}{q^2}\right)$

$X^2+Y^2=4\times\frac{1}{2}$

$X^2+Y^2=2$

I.E A CIRCLE CENTRE AT (0,0) AND RADIUS $\sqrt{2}$

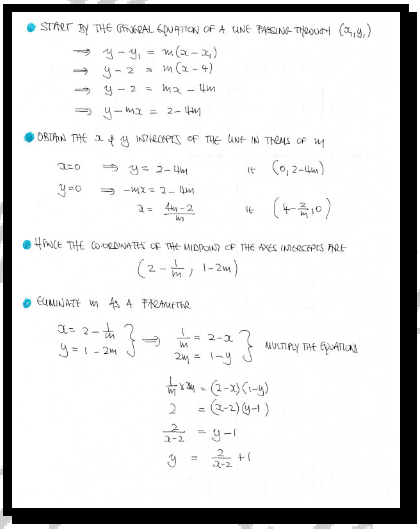
Question 156 (*****)

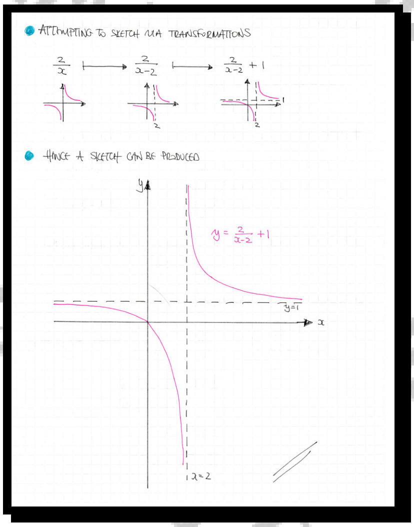
A family of straight lines passes through the point with coordinates $(4,2)$.

The variable point M denotes the midpoint of the x and y intercepts of this family of straight lines.

Sketch a detailed graph of the curve that M traces, for this family of straight lines.

[ ], [graph]





Question 157 **(*****)**

The point P lies on the curve given parametrically as

$$x = t^2, \quad y = t^2 - t, \quad t \in \mathbb{R}.$$

The tangent to the curve at P meets the y axis at the point A and the straight line with equation $y = x$ at the point B.

P is moving along the curve so that its x coordinate is increasing at the constant rate of 15 units of distance per unit time.

Determine the rate at which the area of the triangle OAB is increasing at the instant when the coordinates of P are $(36, 30)$.

SP-C, 45

START BY FINDING THE EQUATION OF THE TANGENT AT A GENERAL POINT P ON THE CURVE, I.E. WHERE $t = p$, SO $P(p^2, p^2 - p)$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t-1}{2t}$$

$$\left.\frac{dy}{dx}\right|_{t=p} = \frac{2p-1}{2p}$$

EQUATION OF THE TANGENT AT P IS GIVEN BY

$$y - (p^2 - p) = \frac{2p-1}{2p}(x - p^2)$$

WHEN $x = 0$

$$\Longrightarrow y - p^2 + p = \frac{2p-1}{2p}(-p^2)$$

$$\Longrightarrow y - p^2 + p = \frac{2p-1}{2}(-p)$$

$$\Longrightarrow y = p^2 - p - p^2 + \tfrac{1}{2}p$$

$$\Longrightarrow y = -\tfrac{1}{2}p \qquad \therefore A\left(0, -\tfrac{1}{2}p\right)$$

WHEN $y = x$

$$\Longrightarrow x - p^2 + p = \frac{2p-1}{2p}(x - p^2)$$

$$\Longrightarrow 2px - 2p^3 + 2p^2 = (2p-1)(x - p^2)$$

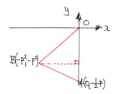
$$\Longrightarrow 2px - 2p^3 + 2p^2 = (2p-1)x - p^2(2p-1)$$

$$\Longrightarrow p^2(2p-1) - 2p^3 + 2p^2 = (2p-1)x - 2px$$

$$\Longrightarrow 2p^3 - p^2 - 2p^3 + 2p^2 = 2px - x - 2px$$

$$\Longrightarrow x = -p^2 \qquad \therefore B(-p^2, -p^2)$$

QUICK SKETCH, TAKING $p > 0$ WITHOUT LOSS OF GENERALITY



AREA OF THE TRIANGLE IS

$$A(p) = \tfrac{1}{2}|-p^2|\left|-\tfrac{1}{2}p\right|$$

$$A(p) = \tfrac{1}{4}p^3$$

NOW $P(p^2, p^2 - 1)$ I.E. $x = p^2$ & $\frac{dx}{dT} = 20$ (T = TIME)

SO WE HAVE

$$\frac{dA}{dT} = \frac{dA}{dp} \times \frac{dp}{dx} \times \frac{dx}{dT}$$

$$\frac{dA}{dT} = \left(\tfrac{3}{4}p^2\right)\left(\frac{1}{2p}\right) \times 20$$

$$\frac{dA}{dT} = \frac{15}{2}p$$

$$\left.\frac{dA}{dT}\right|_{(36,30)} = \left.\frac{dA}{dT}\right|_{p=6} = \frac{15}{2} \times 6 = 45 \text{ UNITS}^2/\text{UNIT TIME}$$

Question 158 (*****)

A curve has Cartesian equation

$$y = \frac{1}{2}x^2, \quad x \in \mathbb{R}.$$

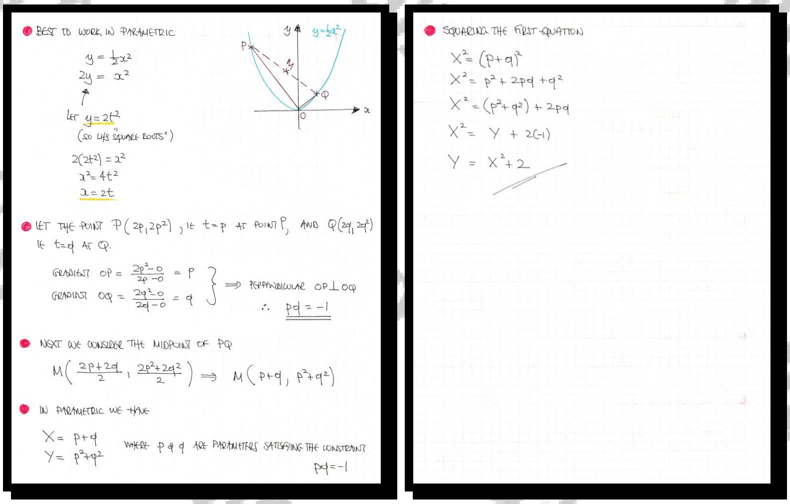
The points P and Q both lie on the curve so that POQ is a right angle, where O is the origin.

The point M represents the midpoint of PQ.

Show that as the position of P varies along the curve, M traces the curve with equation

$$y = x^2 - 2.$$

, proof



Question 159 (*****)

A curve is given parametrically by the equations

$$x = 2t^2 - 3t + 1, \ x = t^2 + t + 1, \ t \in \mathbb{R}.$$

The tangents to the curve, at two distinct points P and Q, intersect each other at the point with coordinates $(2,9)$.

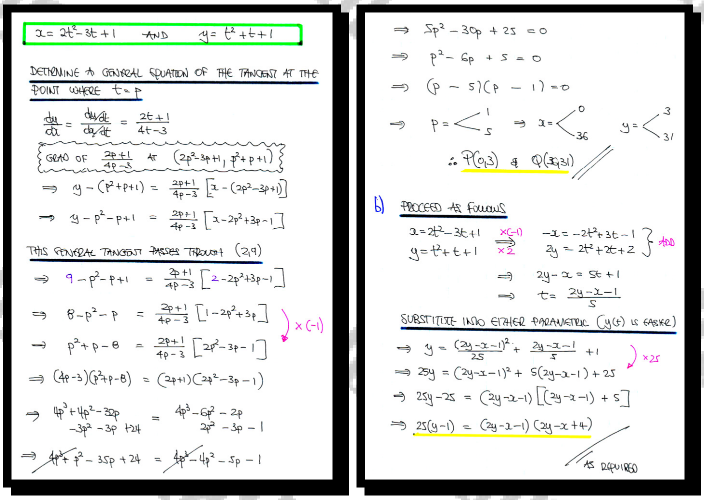
a) Determine the coordinates of P and Q.

b) Show that the Cartesian equation of the curve is

$$25(y-1) = (2y-x-1)(2y-x+4).$$

You may not use a verification method in this part.

[ ], $P(0,3)$, $Q(36,31)$



$x = 2t^2 - 3t + 1$ AND $y = t^2 + t + 1$

DETERMINE A GENERAL EQUATION OF THE TANGENT AT THE POINT WHERE $t = p$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t+1}{4t-3}$$

GRAD OF $\frac{2p+1}{4p-3}$ AT $(2p^2-3p+1, p^2+p+1)$

$$\Rightarrow y - (p^2+p+1) = \frac{2p+1}{4p-3}\left[x - (2p^2-3p+1)\right]$$

$$\Rightarrow y - p^2 - p + 1 = \frac{2p+1}{4p-3}\left[x - 2p^2 + 3p - 1\right]$$

THIS GENERAL TANGENT PASSES THROUGH $(2,9)$

$$\Rightarrow 9 - p^2 - p + 1 = \frac{2p+1}{4p-3}\left[2 - 2p^2 + 3p - 1\right]$$

$$\Rightarrow 8 - p^2 - p = \frac{2p+1}{4p-3}\left[1 - 2p^2 + 3p\right]$$

$\times(-1)$

$$\Rightarrow p^2 + p - 8 = \frac{2p+1}{4p-3}\left[2p^2 - 3p - 1\right]$$

$$\Rightarrow (4p-3)(p^2+p-8) = (2p+1)(2p^2-3p-1)$$

$$\Rightarrow \begin{matrix} 4p^3 + 4p^2 - 32p \\ -3p^2 - 3p + 24 \end{matrix} = \begin{matrix} 4p^3 - 6p^2 - 2p \\ 2p^2 - 3p - 1 \end{matrix}$$

$$\Rightarrow 4p^3 + p^2 - 35p + 24 = 4p^3 - 4p^2 - 5p - 1$$

$$\Rightarrow 5p^2 - 30p + 25 = 0$$

$$\Rightarrow p^2 - 6p + 5 = 0$$

$$\Rightarrow (p-5)(p-1) = 0$$

$$\Rightarrow p = \begin{cases} 1 \\ 5 \end{cases} \Rightarrow x = \begin{cases} 0 \\ 36 \end{cases} \quad y = \begin{cases} 3 \\ 31 \end{cases}$$

$\therefore P(0,3)$ & $Q(36,31)$

b) PROCEED AS FOLLOWS

$x = 2t^2 - 3t + 1$ $\xrightarrow{\times(-1)}$ $-x = -2t^2 + 3t - 1$

$y = t^2 + t + 1$ $\xrightarrow{\times 2}$ $2y = 2t^2 + 2t + 2$ ADD

$$\Rightarrow 2y - x = 5t + 1$$

$$\Rightarrow t = \frac{2y-x-1}{5}$$

SUBSTITUTE INTO EITHER PARAMETRIC ($y(t)$ IS EASIER)

$$\Rightarrow y = \frac{(2y-x-1)^2}{25} + \frac{2y-x-1}{5} + 1$$

$\times 25$

$$\Rightarrow 25y = (2y-x-1)^2 + 5(2y-x-1) + 25$$

$$\Rightarrow 25y - 25 = (2y-x-1)\left[(2y-x-1) + 5\right]$$

$$\Rightarrow 25(y-1) = (2y-x-1)(2y-x+4)$$

AS REQUIRED

Question 160 (*****)

The points P and Q are two distinct points which lie on the curve with equation

$$y=\frac{1}{x},\ x\in\mathbb{R},\ x\neq 0.$$

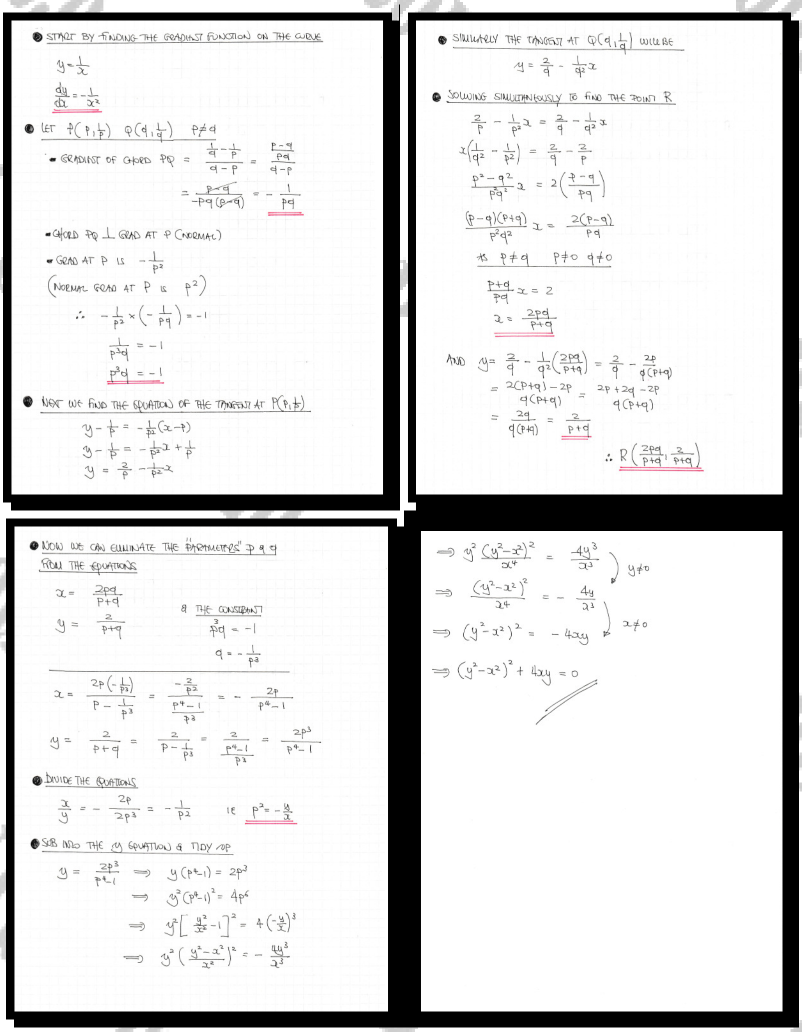
P and Q are free to move on the curve so that the straight line segment PQ is a normal to the curve at P.

The tangents to the curve at P and Q meet at the point R.

Show that R is moving on the curve with Cartesian equation

$$\left(y^2-x^2\right)^2+4xy=0.$$

proof



Question 161 (*****)

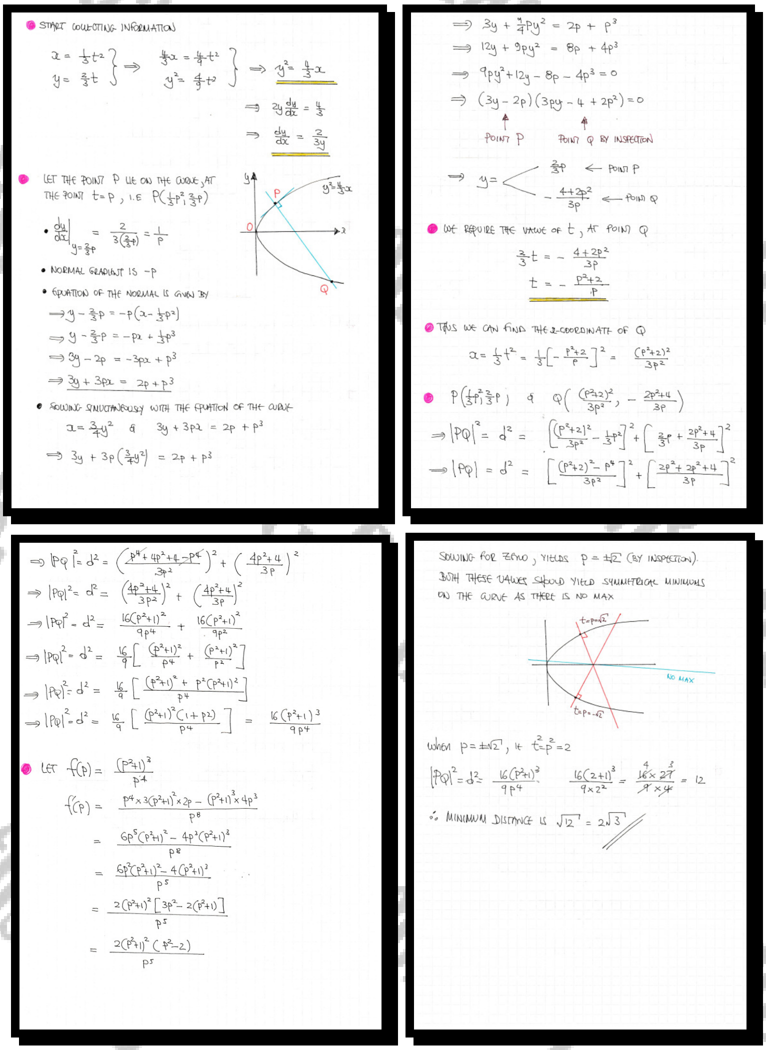
A curve is given parametrically by

$$x = \frac{1}{3}t^2, \quad y = \frac{2}{3}t, \quad t \in \mathbb{R}.$$

The normal to the curve at the point P meets the curve again at the point Q.

Show that the minimum value of $|PQ|$ is $\sqrt{12}$.

[ ], [proof]



START COLLECTING INFORMATION

$x = \frac{1}{3}t^2,\ y = \frac{2}{3}t \implies \frac{4}{3}x = \frac{4}{9}t^2,\ y^2 = \frac{4}{9}t^2 \implies y^2 = \frac{4}{3}x$

$\implies 2y\frac{dy}{dx} = \frac{4}{3}$

$\implies \frac{dy}{dx} = \frac{2}{3y}$

LET THE POINT P LIE ON THE CURVE, AT THE POINT $t = p$, I.E. $P\left(\frac{1}{3}p^2, \frac{2}{3}p\right)$

- $\left.\frac{dy}{dx}\right|_{y=\frac{2}{3}p} = \frac{2}{3\left(\frac{2}{3}p\right)} = \frac{1}{p}$
- NORMAL GRADIENT IS $-p$
- EQUATION OF THE NORMAL IS GIVEN BY

$\implies y - \frac{2}{3}p = -p\left(x - \frac{1}{3}p^2\right)$

$\implies y - \frac{2}{3}p = -px + \frac{1}{3}p^3$

$\implies 3y - 2p = -3px + p^3$

$\implies 3y + 3px = 2p + p^3$

- SOLVING SIMULTANEOUSLY WITH THE EQUATION OF THE CURVE

$x = \frac{3}{4}y^2$ & $3y + 3px = 2p + p^3$

$\implies 3y + 3p\left(\frac{3}{4}y^2\right) = 2p + p^3$

$\implies 3y + \frac{9}{4}py^2 = 2p + p^3$

$\implies 12y + 9py^2 = 8p + 4p^3$

$\implies 9py^2 + 12y - 8p - 4p^3 = 0$

$\implies (3y - 2p)(3py - 4 + 2p^2) = 0$

POINT P — POINT Q BY INSPECTION

$\implies y = \frac{2}{3}p$ ← POINT P, $\quad y = -\frac{4+2p^2}{3p}$ ← POINT Q

WE REQUIRE THE VALUE OF t, AT POINT Q

$\frac{2}{3}t = -\frac{4+2p^2}{3p}$

$t = -\frac{p^2+2}{p}$

THUS WE CAN FIND THE x-COORDINATE OF Q

$x = \frac{1}{3}t^2 = \frac{1}{3}\left[-\frac{p^2+2}{p}\right]^2 = \frac{(p^2+2)^2}{3p^2}$

$P\left(\frac{1}{3}p^2, \frac{2}{3}p\right)$ & $Q\left(\frac{(p^2+2)^2}{3p^2}, -\frac{2p^2+4}{3p}\right)$

$\implies |PQ|^2 = d^2 = \left[\frac{(p^2+2)^2}{3p^2} - \frac{1}{3}p^2\right]^2 + \left[\frac{2}{3}p + \frac{2p^2+4}{3p}\right]^2$

$\implies |PQ| = d^2 = \left[\frac{(p^2+2)^2 - p^4}{3p^2}\right]^2 + \left[\frac{2p^2 + 2p^2 + 4}{3p}\right]^2$

$\implies |PQ|^2 = d^2 = \left(\frac{p^4 + 4p^2 + 4 - p^4}{3p^2}\right)^2 + \left(\frac{4p^2+4}{3p}\right)^2$

$\implies |PQ|^2 = d^2 = \left(\frac{4p^2+4}{3p^2}\right)^2 + \left(\frac{4p^2+4}{3p}\right)^2$

$\implies |PQ|^2 = d^2 = \frac{16(p^2+1)^2}{9p^4} + \frac{16(p^2+1)^2}{9p^2}$

$\implies |PQ|^2 = d^2 = \frac{16}{9}\left[\frac{(p^2+1)^2}{p^4} + \frac{(p^2+1)^2}{p^2}\right]$

$\implies |PQ|^2 = d^2 = \frac{16}{9}\left[\frac{(p^2+1)^2 + p^2(p^2+1)^2}{p^4}\right]$

$\implies |PQ|^2 = d^2 = \frac{16}{9}\left[\frac{(p^2+1)^2(1+p^2)}{p^4}\right] = \frac{16(p^2+1)^3}{9p^4}$

LET $f(p) = \frac{(p^2+1)^3}{p^4}$

$f'(p) = \frac{p^4 \times 3(p^2+1)^2 \times 2p - (p^2+1)^3 \times 4p^3}{p^8}$

$= \frac{6p^5(p^2+1)^2 - 4p^3(p^2+1)^3}{p^8}$

$= \frac{6p^2(p^2+1)^2 - 4(p^2+1)^3}{p^5}$

$= \frac{2(p^2+1)^2\left[3p^2 - 2(p^2+1)\right]}{p^5}$

$= \frac{2(p^2+1)^2(p^2-2)}{p^5}$

SOLVING FOR ZERO, YIELDS $p = \pm\sqrt{2}$ (BY INSPECTION).

BOTH THESE VALUES SHOULD YIELD SYMMETRICAL MINIMUMS ON THE CURVE AS THERE IS NO MAX

when $p = \pm\sqrt{2}$, ie $t^2 = p^2 = 2$

$|PQ|^2 = d^2 = \frac{16(p^2+1)^3}{9p^4}$ $\frac{16(2+1)^3}{9 \times 2^2} = \frac{16 \times 27}{9 \times 4} = 12$

∴ MINIMUM DISTANCE IS $\sqrt{12} = 2\sqrt{3}$

Question 162 (*****)

The function f maps points from a Cartesian x-y plane onto the same Cartesian x-y plane by

$$f:(x,y) \mapsto \left(\frac{1-x^2-y^2}{x^2+(1-y)^2}, \frac{-2x}{x^2+(1-y)^2} \right), \quad x \in \mathbb{R}, \quad y \in \mathbb{R}, \quad (x,y) \neq (0,1).$$

The set of points, S, which lie on the x axis are mapped by f onto a new set of points S', which in turn are mapped by f onto a new set of points S''.

Use algebra to determine the equation of S''.

$\boxed{}$, $\boxed{x=0}$

$$f:(x,y) \longmapsto \left[\frac{1-x^2-y^2}{x^2+(1-y)^2}, \frac{-2x}{x^2+(1-y)^2} \right] \quad (x,y) \neq (0,1)$$

IF A POINT LIES ON THE x AXIS THEN $y=0$

$$\Rightarrow f:(x,y) \longmapsto \left[\frac{1-x^2}{x^2+1}, \frac{-2x}{x^2+1} \right]$$

$$\Rightarrow (X,Y) \longmapsto \left[\frac{1-x^2}{1+x^2}, \frac{-2x}{1+x^2} \right]$$

ELIMINATE THE x WHICH ACTS AS A PARAMETER AS FOLLOWS

$\Rightarrow X = \frac{1-x^2}{1+x^2}$

$\Rightarrow X + x^2X = 1 - x^2$

$\Rightarrow x^2 + x^2X = 1 - X$

$\Rightarrow x^2(1+X) = 1 - X$

$\Rightarrow x^2 = \frac{1-X}{1+X}$

$\Rightarrow Y = \frac{-2x}{1+x^2}$

$\Rightarrow Y^2 = \frac{4x^2}{(1+x^2)^2}$

$\Rightarrow Y^2 = \frac{4\left(\frac{1-X}{1+X}\right)}{\left[1+\frac{1-X}{1+X}\right]^2}$

$\Rightarrow Y^2 = \frac{4\left(\frac{1-X}{1+X}\right)}{\left[\frac{1+X+1-X}{1+X}\right]^2}$

$\Rightarrow Y^2 = \frac{\frac{4(1-X)}{(1+X)}}{\frac{4}{(1+X)^2}}$

$\Rightarrow Y^2 = (1-X)(1+X)$

$\Rightarrow Y^2 = 1 - X^2$

$\Rightarrow X^2 + Y^2 = 1$

OR USING LOWER CASE NOTATION $x^2+y^2=1$

NOW TRANSFORM THE POINTS WHICH LIE ON THIS CIRCLE AGAIN

NOTICE THAT $x^2+y^2=1 \Rightarrow 1-x^2-y^2=0$

$\therefore X = 0$

$Y = \frac{-2x}{x^2+y^2-2y+1} = -\frac{2x}{2-2y} = \frac{x}{y-1}$ $\quad x \neq 0, \; y \neq 1$

$\therefore$ THE LOCUS IS $X=0$, IE THE y AXIS

ALTERNATIVE ELIMINATION

$X = \frac{1-x^2}{1+x^2}$ $\qquad$ $Y = \frac{-2x}{1+x^2}$

LET $x = \tan\theta$

$X = \frac{1-\tan^2\theta}{1+\tan^2\theta}$

$X = \frac{1-\tan^2\theta}{\sec^2\theta}$

$X = \cos^2\theta - \tan^2\theta\cos^2\theta$

$X = \cos^2\theta - \frac{\sin^2\theta}{\cos^2\theta} \times \cos^2\theta$

$X = \cos 2\theta$

$Y = \frac{-2\tan\theta}{1+\tan^2\theta}$

$Y = \frac{-2\tan\theta}{\sec^2\theta}$

$Y = -2\tan\theta\cos^2\theta$

$Y = -2\frac{\sin\theta}{\cos\theta} \times \cos^2\theta$

$Y = -2\sin\theta\cos\theta$

$Y = -\sin 2\theta$

$\therefore X^2 + Y^2 = (\cos 2\theta)^2 + (-\sin 2\theta)^2 = 1$ (AS BEFORE)