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PARAMETRIC EQUATIONS

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ALGEBRAIC ELIMINATIONS

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Question 1

Find a Cartesian equation for each of the following parametric relationships.

a) $x = t + 1, \quad y = 4 - 3t, \quad t \in \mathbb{R}$

b) $x = 2t + 1, \quad y = 3t - 2, \quad t \in \mathbb{R}$

c) $x = \frac{2}{t}, \quad y = 2t - 1, \quad t \in \mathbb{R}, \quad t \neq 0$

d) $x = 2t + 1, \quad y = t^2 - 1, \quad t \in \mathbb{R}$

$$\boxed{y = 7 - 3x}, \quad \boxed{3x - 2y = 7}, \quad \boxed{y = \frac{4}{x} - 1}, \quad \boxed{(x-1)^2 = 4(y+1)}$$

Handwritten solutions for Question 1:

a) $\begin{cases} x = t + 1 \\ y = 4 - 3t \end{cases} \Rightarrow t = x - 1$ Substitute for t in $y = 4 - 3(x - 1) = 4 - 3x + 3 = 7 - 3x$

b) $\begin{cases} x = 2t + 1 \\ y = 3t - 2 \end{cases} \xrightarrow{\begin{matrix} \times 3 \\ \times 2 \end{matrix}} \begin{cases} 3x = 6t + 3 \\ 2y = 6t - 4 \end{cases} \xrightarrow{\text{subtract}} 3x - 2y = 7$

c) $\begin{cases} x = \frac{2}{t} \\ y = 2t - 1 \end{cases} \Rightarrow t = \frac{2}{x}$ Then $y = 2\left(\frac{2}{x}\right) - 1 \Rightarrow y = \frac{4}{x} - 1$

d) $\begin{cases} x = 2t + 1 \\ y = t^2 - 1 \end{cases} \Rightarrow \begin{cases} (x-1)^2 = (2t)^2 \\ y+1 = t^2 \end{cases} \Rightarrow \frac{(x-1)^2}{4t^2} = \frac{(x-1)^2}{4(y+1)} \Rightarrow 4(y+1) = (x-1)^2$
 or $y = \frac{1}{4}(x-1)^2 - 1$
 or $y = \frac{1}{4}x^2 - \frac{1}{2}x - \frac{3}{4}$

Question 2

Find a Cartesian equation for each of the following parametric relationships.

a) $x = 2t - 1, \quad y = 4 - 3t, \quad t \in \mathbb{R}$

b) $x = 2t - 1, \quad y = \frac{1}{t+1}, \quad t \in \mathbb{R}, \quad t \neq -1$

c) $x = t^2, \quad y = 2t^3, \quad t \in \mathbb{R}$

d) $x = \frac{1}{4t-1}, \quad y = \frac{t}{4t-1}, \quad t \in \mathbb{R}, \quad t \neq \frac{1}{4}$

$$\boxed{3x+2y=5}, \quad \boxed{y=\frac{2}{x+3}}, \quad \boxed{y^2=4x^3}, \quad \boxed{y=\frac{1}{4}(x+1)}$$

[illegible]

Question 3

Find a Cartesian equation for each of the following parametric relationships.

a) $x = 1 - 4t^2, \quad y = 1 + 2t, \quad t \in \mathbb{R}$

b) $x = 3 - 4t, \quad y = 1 + \frac{2}{t}, \quad t \in \mathbb{R} \quad t \neq 0$

c) $x = t + 2, \quad y = \ln(t - 1), \quad t \in \mathbb{R} \quad t > 1$

d) $x = e^{t-1}, \quad y = t + 7, \quad t \in \mathbb{R}$

$$\boxed{x = 2y - y^2}, \quad \boxed{y = 1 - \frac{8}{x-3}}, \quad \boxed{y = \ln(x-3)}, \quad \boxed{y = 8 + \ln x}$$

Handwritten solutions for Question 3:

a) $\begin{cases} x = 1 - 4t^2 \\ y = 1 + 2t \end{cases} \Rightarrow \begin{cases} 4t^2 = 1 - x \\ y - 1 = 2t \end{cases} \Rightarrow \begin{cases} 4t^2 = 1 - x \\ (y-1)^2 = 4t^2 \end{cases} \Rightarrow 1 - x = (y-1)^2$

b) $\begin{cases} x = 3 - 4t \\ y = 1 + \frac{2}{t} \end{cases} \Rightarrow \begin{cases} 4t = 3 - x \\ y = 1 + \frac{2}{t} \end{cases} \Rightarrow y = 1 + \frac{2}{\frac{3-x}{4}} \Rightarrow y = 1 + \frac{8}{3-x} \Rightarrow y - 1 = \frac{8}{3-x} \Rightarrow (y-1)(3-x) = 8 \Rightarrow 3y - x - 3 = \frac{8}{y-1} \Rightarrow (y-1)^2(3y-x-3) = 8$

c) $\begin{cases} x = t + 2 \\ y = \ln(t-1) \end{cases} \Rightarrow \begin{cases} t = x - 2 \\ y = \ln(t-1) \end{cases} \Rightarrow y = \ln(x-3)$

d) $\begin{cases} x = e^{t-1} \\ y = t + 7 \end{cases} \Rightarrow \begin{cases} \ln x = t - 1 \\ y = t + 7 \end{cases} \Rightarrow \begin{cases} \ln x = t - 1 \\ y - 7 = t \end{cases} \Rightarrow y - 7 = \ln x + 1 \Rightarrow y = 8 + \ln x$

Question 4

Find a Cartesian equation for each of the following parametric expressions.

a) $x = t^2 - 1, \quad y = \frac{t^3}{5}, \quad t \in \mathbb{R}$

b) $x = \sqrt{t} + 1, \quad y = t - 1, \quad t \in \mathbb{R}, t \geq 0$

c) $x = 3t - 1, \quad y = (t - 2)(t + 1), \quad t \in \mathbb{R}$

d) $x = \frac{1}{t - 2}, \quad y = t^2, \quad t \in \mathbb{R}, t \neq 2$

$$\boxed{25y^2 = (x + 1)^3}, \quad \boxed{y = x^2 - 2x}, \quad \boxed{9y = (x - 5)(x + 4)}, \quad \boxed{y = \left(\frac{2x + 1}{x}\right)^2}$$

Handwritten solutions for Question 4:

a) $x = t^2 - 1 \Rightarrow x + 1 = t^2 \Rightarrow (x + 1)^3 = t^6$
 $y = \frac{t^3}{5} \Rightarrow 5y = t^3 \Rightarrow (5y)^2 = t^6$
 $\therefore (x + 1)^3 = 25y^2$

b) $x = \sqrt{t} + 1 \Rightarrow x - 1 = \sqrt{t} \Rightarrow (x - 1)^2 = t$
 $y = t - 1 \Rightarrow y + 1 = t$
 $\therefore y = (x - 1)^2 - 1$
 $y = x^2 - 2x + 1 - 1$
 $y = x^2 - 2x$

c) $x = 3t - 1 \Rightarrow x + 1 = 3t \Rightarrow t = \frac{x + 1}{3}$
 $y = (t - 2)(t + 1) = \left(\frac{x + 1}{3} - 2\right)\left(\frac{x + 1}{3} + 1\right)$
 $y = \left(\frac{x + 1 - 6}{3}\right)\left(\frac{x + 1 + 3}{3}\right) = \left(\frac{x - 5}{3}\right)\left(\frac{x + 4}{3}\right)$
 $9y = (x - 5)(x + 4)$

d) $x = \frac{1}{t - 2} \Rightarrow \frac{1}{x} = t - 2 \Rightarrow \frac{1}{x} + 2 = t$
 $y = t^2 = \left(\frac{1}{x} + 2\right)^2$
 $y = \left(\frac{1 + 2x}{x}\right)^2$

Question 5

Find a Cartesian equation for each of the following parametric equations.

a) $x = 4t + 3, \quad y = \frac{1}{2t} - 1, \quad t \in \mathbb{R}, t \neq 0$

b) $x = 3 - 4t, \quad y = \frac{2}{t} + 1, \quad t \in \mathbb{R}, t \neq 0$

c) $x = \frac{1}{t-1}, \quad y = \frac{1}{t+2}, \quad t \in \mathbb{R}, t \neq -1, -2$

d) $x = \frac{t}{2t-1}, \quad y = \frac{t}{t+1}, \quad t \in \mathbb{R}, t \neq -1, \frac{1}{2}$

$$\boxed{y = \frac{2}{x-3} - 1}, \quad \boxed{y = \frac{x-11}{x-3}}, \quad \boxed{y = \frac{x}{1+3x}}, \quad \boxed{y = \frac{x}{3x-1}}$$

Handwritten solutions for Question 5:

a) $x = 4t + 3 \Rightarrow t = \frac{x-3}{4}$
 $y = \frac{1}{2t} - 1 = \frac{1}{2(\frac{x-3}{4})} - 1 = \frac{2}{x-3} - 1$

b) $x = 3 - 4t \Rightarrow t = \frac{3-x}{4}$
 $y = \frac{2}{t} + 1 = \frac{2}{\frac{3-x}{4}} + 1 = \frac{8}{3-x} + 1 = \frac{8 + 3 - x}{3-x} = \frac{11-x}{3-x}$

c) $x = \frac{1}{t-1} \Rightarrow t-1 = \frac{1}{x} \Rightarrow t = \frac{1}{x} + 1$
 $y = \frac{1}{t+2} = \frac{1}{\frac{1}{x} + 1 + 2} = \frac{1}{\frac{1}{x} + 3} = \frac{x}{1+3x}$

d) $x = \frac{t}{2t-1} \Rightarrow 2xt - x = t \Rightarrow 2xt - t = x \Rightarrow t(2x-1) = x \Rightarrow t = \frac{x}{2x-1}$
 $y = \frac{t}{t+1} = \frac{\frac{x}{2x-1}}{\frac{x}{2x-1} + 1} = \frac{x}{x + 2x - 1} = \frac{x}{3x-1}$

Question 6

Find a Cartesian equation for each of the following parametric expressions.

a) $x = 1 - 3t$, $y = 1 + 2t^3$, $t \in \mathbb{R}$

b) $x = \frac{1}{2t-3}$, $y = \frac{t}{2t-3}$, $t \in \mathbb{R}$, $t \neq \frac{3}{2}$

c) $x = \frac{2}{2t-5}$, $y = \frac{t}{4-t}$, $t \in \mathbb{R}$, $t \neq 4$, $t \neq \frac{5}{2}$

d) $x = t + e^t$, $y = t - e^t$, $t \in \mathbb{R}$

$$\boxed{y = 1 + \frac{2}{27}(1-x)^3}, \quad \boxed{2y - 3x = 1}, \quad \boxed{y = \frac{5x+2}{3x-2}}, \quad \boxed{x - y = 2e^{\frac{1}{2}(x+y)}}$$

Handwritten solutions for Question 6:

(a) $x = 1 - 3t$, $y = 1 + 2t^3$
 $3t = 1 - x$, $y - 1 = 2t^3$
 $27t^3 = (1-x)^3$, $27(y-1) = (1-x)^3$
 $27y - 27 = (1-x)^3$
 $\therefore 27y = (1-x)^3 + 27$
 $y = \frac{(1-x)^3}{27} + 1$

(b) $x = \frac{1}{2t-3}$, $y = \frac{t}{2t-3}$
 $\frac{1}{x} = 2t - 3$
 $\frac{1}{x} + 3 = 2t$
 $\frac{1}{x} + 3 + \frac{3}{2} = \frac{3}{2} + 2t$
 $\frac{1}{x} + \frac{9}{2} = \frac{3}{2} + 2t$
 $\frac{1}{x} + 3 = 2t$
 $y = \frac{t}{2t-3} = \frac{t}{\frac{1}{x} + 3}$
 $y = \frac{1}{1 + 3x}$

(c) $x = \frac{2}{2t-5}$, $y = \frac{t}{4-t}$
 $\frac{2}{x} = 2t - 5$
 $\frac{2}{x} + 5 = 2t$
 $\frac{2}{x} + 5 + \frac{5}{2} = \frac{5}{2} + 2t$
 $\frac{2}{x} + \frac{15}{2} = \frac{5}{2} + 2t$
 $\frac{2}{x} + 5 = 2t$
 $y = \frac{t}{4-t} = \frac{t}{\frac{2}{x} + 5}$
 $y = \frac{2}{2 + 5x}$

(d) $x = t + e^t$, $y = t - e^t$
 $x - y = t + e^t - (t - e^t) = 2e^t$
 $x - y = 2e^{\frac{1}{2}(x+y)}$

Question 7

Find a Cartesian equation for each of the following parametric expressions.

a) $x = t + \frac{1}{t}, \quad y = t - \frac{1}{t}, \quad t \in \mathbb{R}, \quad t \neq 0$

b) $x = t^2 + \frac{1}{t}, \quad y = t^2 - \frac{1}{t}, \quad t \in \mathbb{R}, \quad t \neq 0$

c) $x = 3t + \frac{1}{t^2}, \quad y = 3t - \frac{1}{t^2}, \quad t \in \mathbb{R}, \quad t \neq 0$

d) $x = \frac{1-t^2}{1+t^2}, \quad y = \frac{2t}{1+t^2}, \quad t \in \mathbb{R}$

$x^2 - y^2 = 4, \quad (x+y)(x-y)^2 = 8, \quad (x-y)(x+y)^2 = 72, \quad x^2 + y^2 = 1$

Handwritten solutions for Question 7:

a) $x = t + \frac{1}{t}, \quad y = t - \frac{1}{t}$
 $x+y = t + \frac{1}{t} + t - \frac{1}{t} = 2t$
 $x-y = t + \frac{1}{t} - t + \frac{1}{t} = \frac{2}{t}$
 $\therefore (x+y)(x-y) = 2t \cdot \frac{2}{t} = 4$
 $(x+y)(x-y)^2 = 4 \cdot \left(\frac{2}{t}\right)^2 = \frac{16}{t^2}$
 $(x-y)(x+y)^2 = \frac{2}{t} \cdot (2t)^2 = 8t$
 $\therefore (x+y)(x-y)^2 = 8$
 $(x-y)(x+y)^2 = 72$

b) $x = t^2 + \frac{1}{t}, \quad y = t^2 - \frac{1}{t}$
 $x+y = t^2 + \frac{1}{t} + t^2 - \frac{1}{t} = 2t^2$
 $x-y = t^2 + \frac{1}{t} - t^2 + \frac{1}{t} = \frac{2}{t}$
 $\therefore (x+y)(x-y) = 2t^2 \cdot \frac{2}{t} = 4t$
 $(x+y)(x-y)^2 = 4t \cdot \left(\frac{2}{t}\right)^2 = \frac{16}{t}$
 $(x-y)(x+y)^2 = \frac{2}{t} \cdot (2t^2)^2 = 8t^3$
 $\therefore (x+y)(x-y)^2 = 8$
 $(x-y)(x+y)^2 = 72$

c) $x = 3t + \frac{1}{t^2}, \quad y = 3t - \frac{1}{t^2}$
 $x+y = 3t + \frac{1}{t^2} + 3t - \frac{1}{t^2} = 6t$
 $x-y = 3t + \frac{1}{t^2} - 3t + \frac{1}{t^2} = \frac{2}{t^2}$
 $\therefore (x+y)(x-y) = 6t \cdot \frac{2}{t^2} = \frac{12}{t}$
 $(x+y)(x-y)^2 = \frac{12}{t} \cdot \left(\frac{2}{t^2}\right)^2 = \frac{48}{t^3}$
 $(x-y)(x+y)^2 = \frac{2}{t^2} \cdot (6t)^2 = 72$
 $\therefore (x+y)(x-y)^2 = 8$
 $(x-y)(x+y)^2 = 72$

d) $x = \frac{1-t^2}{1+t^2}, \quad y = \frac{2t}{1+t^2}$
 $x^2 + y^2 = \frac{(1-t^2)^2 + (2t)^2}{(1+t^2)^2} = \frac{1 - 2t^2 + t^4 + 4t^2}{(1+t^2)^2} = \frac{1 + 2t^2 + t^4}{(1+t^2)^2} = \frac{(1+t^2)^2}{(1+t^2)^2} = 1$

Question 8

Find a Cartesian equation for each of the following parametric equations.

a) $x = t - \frac{1}{t^3}, \quad y = \frac{1}{t} - t^3, \quad t \in \mathbb{R}, \quad t \neq 0$

b) $x = \frac{4t}{1+t^2}, \quad y = \frac{1-3t^2}{1+t^2}, \quad t \in \mathbb{R}$

c) $x = \frac{1}{t} + \frac{1}{t^2}$, $y = \frac{1}{t} - \frac{1}{t^2}$, $t \in \mathbb{R}, t \neq 0$

d) $x = \frac{t^2}{1+t^3}, \quad y = \frac{2t}{1+t^3}, \quad t \in \mathbb{R}, t \neq -1$

$$\boxed{(y^2 - x^2)^2 + x^3 y^3 = 0}, \quad \boxed{x^2 + (y+1)^2 = 4}, \quad \boxed{(x+y)^2 = 2(x-y)}, \quad \boxed{y^3 + 8x^3 = 4xy}$$

[illegible]

(c)
$$z = \frac{1}{t} + \frac{1}{1-t} \quad \rightarrow \quad xy = \frac{z}{t} \Rightarrow (xy)^2 = \frac{z^2}{t^2}$$
$$y = \frac{1}{t} - \frac{1}{1-t} \quad \rightarrow \quad x-y = \frac{z}{t^2} \Rightarrow \frac{1}{x-y} = \frac{t^2}{z}$$
$$4 \text{ hoc } (xy)^2 \cdot \frac{1}{x-y} = \frac{z}{t^2} \times \frac{t^2}{z}$$
$$\frac{(xy)^2}{x-y} = 1 \quad \text{or} \quad (xy)^2 = z(x-y)$$

(d)
$$z = \frac{t^2}{1+t^2} \quad \rightarrow \quad \frac{z}{y} = \frac{\frac{t^2}{1+t^2}}{\frac{2t}{1+t^2}} = \frac{t^2}{2t} = \frac{t}{2} \quad \therefore \boxed{t = \frac{2z}{y}}$$
$$\text{Hence } y = \frac{2t}{1+t^2} = \frac{4z}{y} \Rightarrow y = \frac{2(4z)}{1+(16z^2/y^2)}$$
$$\Rightarrow y = \frac{8z}{1+16z^2/y^2}$$
$$\Rightarrow y(1 + \frac{16z^2}{y^2}) = \frac{8z}{y}$$
$$\Rightarrow y(\frac{y^2 + 16z^2}{y^2}) = \frac{8z}{y}$$
$$\Rightarrow \frac{1}{y^2} (y^2 + 16z^2) = \frac{8z}{y^2}$$
$$\Rightarrow y^2 + 16z^2 = 8zy$$

TRIGONOMETRIC ELIMINATIONS

Question 1

Find a Cartesian equation for each of the following parametric relationships.

- a) $x = 5 \cos t$, $y = 5 \sin t$, $0 \leq t < 2\pi$
- b) $x = 1 + 2 \cos \theta$, $y = 2 + \sin \theta$, $0 \leq \theta < 2\pi$
- c) $x = 2 \tan \theta$, $y = \cos \theta$, $0 \leq \theta < 2\pi$
- d) $x = \cos t$, $y = \operatorname{cosec} t$, $0 \leq t < 2\pi$
- e) $x = \tan \theta$, $y = \sin \theta$, $0 \leq \theta < 2\pi$
- f) $x = \cos \theta$, $y = \cos 2\theta$, $0 \leq \theta < 2\pi$
- g) $x = \frac{1}{2} \cos 2\theta$, $y = 2 \sin \theta$, $0 \leq \theta < 2\pi$
- h) $x = \cos t$, $y = \sin 2t$, $0 \leq t < 2\pi$

$$\boxed{x^2 + y^2 = 25}, \quad \boxed{(x-1)^2 + 4(y-2)^2 = 4}, \quad \boxed{y^2 = \frac{4}{4+x^2}}, \quad \boxed{y^2 = \frac{1}{1-x^2}}, \quad \boxed{y^2 = \frac{x^2}{1+x^2}},$$

$$\boxed{y = 2x^2 - 1}, \quad \boxed{y^2 = 2 - 4x}, \quad \boxed{y^2 = 4x^2(1-x^2)}$$

Handwritten solutions for the parametric equations in Question 1:

- a) $x = 5 \cos t$, $y = 5 \sin t$
 $\frac{x}{5} = \cos t$, $\frac{y}{5} = \sin t$
 $\left(\frac{x}{5}\right)^2 + \left(\frac{y}{5}\right)^2 = 1$
 $\frac{x^2}{25} + \frac{y^2}{25} = 1$
 $x^2 + y^2 = 25$
- b) $x = 1 + 2 \cos \theta$, $y = 2 + \sin \theta$
 $\frac{x-1}{2} = \cos \theta$, $y-2 = \sin \theta$
 $\left(\frac{x-1}{2}\right)^2 + (y-2)^2 = 1$
 $\frac{(x-1)^2}{4} + (y-2)^2 = 1$
 $(x-1)^2 + 4(y-2)^2 = 4$
- c) $x = 2 \tan \theta$, $y = \cos \theta$
 $\frac{x}{2} = \tan \theta$, $y = \cos \theta$
 $\frac{1}{1+y^2} = \frac{1}{1+\cos^2 \theta}$
 $\frac{1}{1+y^2} = \frac{1}{1+\frac{x^2}{4}}$
 $y^2 = \frac{4}{4+x^2}$
- d) $x = \cos t$, $y = \operatorname{cosec} t$
 $x = \cos t$, $\frac{1}{y} = \sin t$
 $\frac{1}{y^2} = 1 - x^2$
 $y^2 = \frac{1}{1-x^2}$
- e) $x = \tan \theta$, $y = \sin \theta$
 $x = \tan \theta$, $y = \frac{1}{\sec \theta}$
 $y^2 = \frac{1}{1+x^2}$
- f) $x = \cos \theta$, $y = \cos 2\theta$
 $x = \cos \theta$, $y = 2 \cos^2 \theta - 1$
 $y = 2x^2 - 1$
- g) $x = \frac{1}{2} \cos 2\theta$, $y = 2 \sin \theta$
 $x = \frac{1}{2} (1 - 2 \sin^2 \theta)$
 $x = \frac{1}{2} (1 - \frac{y^2}{4})$
 $4x = 2 - \frac{y^2}{4}$
 $y^2 = 2 - 4x$
- h) $x = \cos t$, $y = \sin 2t$
 $x = \cos t$, $y = 2 \sin t \cos t$
 $y = 2x \sqrt{1-x^2}$
 $y^2 = 4x^2(1-x^2)$

Question 2

Find a Cartesian equation for each of the following parametric relationships.

a) $x = 2\cos t$, $y = 2\sin t$, $0 \leq t < 2\pi$

b) $x = 4 + 3\cos t$, $y = -2 + 3\sin t$, $0 \leq t < 2\pi$

c) $x = 4 + \cos t$, $y = 2\sin t$, $0 \leq t < 2\pi$

d) $x = \sin t$, $y = \sec t$, $0 \leq t < 2\pi$

$$x^2 + y^2 = 4, \quad (x-4)^2 + (y+2)^2 = 9, \quad 4(x-4)^2 + y^2 = 4, \quad y^2 = \frac{1}{1-x^2}$$

Handwritten solutions for the four parametric equations:

- a) $x = 2\cos t$, $y = 2\sin t$ $\Rightarrow \frac{x}{2} = \cos t$, $\frac{y}{2} = \sin t$ $\Rightarrow \cos^2 t + \sin^2 t = 1$ $\Rightarrow \frac{x^2}{4} + \frac{y^2}{4} = 1$ $\Rightarrow x^2 + y^2 = 4$
- b) $x = 4 + 3\cos t$, $y = -2 + 3\sin t$ $\Rightarrow \frac{x-4}{3} = \cos t$, $\frac{y+2}{3} = \sin t$ $\Rightarrow \cos^2 t + \sin^2 t = 1$ $\Rightarrow \frac{(x-4)^2}{9} + \frac{(y+2)^2}{9} = 1$ $\Rightarrow (x-4)^2 + (y+2)^2 = 9$
- c) $x = 4 + \cos t$, $y = 2\sin t$ $\Rightarrow \frac{x-4}{1} = \cos t$, $\frac{y}{2} = \sin t$ $\Rightarrow \cos^2 t + \sin^2 t = 1$ $\Rightarrow (x-4)^2 + \frac{y^2}{4} = 1$ $\Rightarrow 4(x-4)^2 + y^2 = 4$
- d) $x = \sin t$, $y = \sec t$ $\Rightarrow \frac{x}{1} = \sin t$, $\frac{y}{1} = \sec t$ $\Rightarrow \sin^2 t + \sec^2 t = 1$ $\Rightarrow \frac{x^2}{1} + \frac{y^2}{1-x^2} = 1$ $\Rightarrow y^2 = \frac{1}{1-x^2}$

Question 3

Eliminate θ to obtain a Cartesian equation for the following parametric equations.

a) $x = \tan \theta$, $y = \sec \theta$, $0 \leq \theta < 2\pi$

b) $x = 2\sin \theta$, $y = 3\operatorname{cosec} \theta$, $0 \leq \theta < 2\pi$

c) $x = \sin \theta$, $y = \sec^2 \theta$, $0 \leq \theta < 2\pi$

d) $x = \cos \theta$, $y = \tan^2 \theta$, $0 \leq \theta < 2\pi$

$$\boxed{y^2 = x^2 + 1}, \quad \boxed{y = \frac{6}{x}}, \quad \boxed{y = \frac{1}{1-x^2}}, \quad \boxed{y = \frac{1}{x^2} - 1}$$

Handwritten solutions for Question 3:

a) $x = \tan \theta$, $y = \sec \theta$
 $1 + \tan^2 \theta = \sec^2 \theta \Rightarrow 1 + x^2 = y^2$
 $y^2 = x^2 + 1$

b) $x = 2\sin \theta$, $y = 3\operatorname{cosec} \theta$
 $\frac{x}{2} = \sin \theta$, $\frac{y}{3} = \operatorname{cosec} \theta$
 $\frac{x}{2} = \frac{1}{\frac{y}{3}} \Rightarrow \frac{x}{2} = \frac{3}{y} \Rightarrow xy = 6$
 $y = \frac{6}{x}$

c) $x = \sin \theta$, $y = \sec^2 \theta$
 $\frac{x}{1} = \sin \theta$, $\frac{y}{1} = \sec^2 \theta$
 $\frac{x^2}{1} + \frac{y}{1} = 1$
 $\frac{y}{1} = 1 - x^2$
 $y = \frac{1}{1-x^2}$

d) $x = \cos \theta$, $y = \tan^2 \theta$
 $\frac{x}{1} = \cos \theta$, $\frac{y}{1} = \tan^2 \theta$
 $1 + \frac{y}{1} = \frac{1}{\frac{x}{1}^2}$
 $1 + y = \frac{1}{x^2}$
 $y = \frac{1}{x^2} - 1$

Question 4

Eliminate the parameter θ to obtain a Cartesian equation for each of the following parametric equations.

a) $x = \sin \theta$, $y = \tan^2 \theta$, $0 \leq \theta < 2\pi$

b) $x = 2 \sec \theta$, $y = \sin^2 \theta$, $0 \leq \theta < 2\pi$

c) $x = 3 \cos \theta$, $y = 2 \cot \theta$, $0 \leq \theta < 2\pi$

d) $x = \frac{1}{2} \cos \theta$, $y = 2 \cos 2\theta$, $0 \leq \theta < 2\pi$

$$\boxed{y = \frac{x^2}{1-x^2}}, \quad \boxed{y = 1 - \frac{4}{x^2}}, \quad \boxed{y^2 = \frac{4x^2}{9-x^2}}, \quad \boxed{y = 2(8x^2 - 1)}$$

Handwritten solutions for Question 4:

a) $x = \sin \theta \Rightarrow \frac{x}{y} = \frac{\sin \theta}{\tan^2 \theta} \Rightarrow \frac{1}{1 + \cot^2 \theta} = \frac{\sin \theta}{\tan^2 \theta}$
 $\frac{1}{1 + \frac{1}{x^2}} = \frac{x}{\frac{1}{x^2}} \Rightarrow \frac{x}{\frac{1+x^2}{x^2}} = \frac{x^3}{1+x^2}$
 $y = \frac{x^3}{1+x^2}$

b) $x = 2 \sec \theta \Rightarrow \frac{x}{y} = \frac{2 \sec \theta}{\sin^2 \theta} \Rightarrow \frac{x}{y} = \frac{2}{\sin^2 \theta \cos \theta}$
 $\frac{x}{y} = \frac{2}{\frac{1}{\cos^2 \theta} \cos \theta} \Rightarrow \frac{x}{y} = 2 \cos \theta$
 $\frac{x}{2 \cos \theta} = y \Rightarrow \frac{x}{2} = y \cos \theta$
 $\frac{x}{2} = y \sqrt{1 - \sin^2 \theta} = y \sqrt{1 - \frac{y^2}{x^2}}$
 $\frac{x^2}{4} = y^2 \sqrt{1 - \frac{y^2}{x^2}}$
 $\frac{x^2}{4} = y^2 \frac{\sqrt{x^2 - y^2}}{x}$
 $\frac{x^3}{4} = y^2 \sqrt{x^2 - y^2}$

c) $x = 3 \cos \theta \Rightarrow \frac{x}{y} = \frac{3 \cos \theta}{2 \cot \theta} \Rightarrow \frac{x}{y} = \frac{3 \cos^2 \theta}{2}$
 $\frac{x}{y} = \frac{3}{2} \cos^2 \theta$
 $\frac{x}{y} = \frac{3}{2} (1 - \sin^2 \theta)$
 $\frac{x}{y} = \frac{3}{2} (1 - \frac{y^2}{x^2})$
 $\frac{x}{y} = \frac{3}{2} - \frac{3y^2}{2x^2}$
 $\frac{x}{y} = \frac{3}{2} - \frac{3y^2}{2x^2}$
 $\frac{x}{y} = \frac{3}{2} - \frac{3y^2}{2x^2}$

d) $x = \frac{1}{2} \cos \theta \Rightarrow 2x = \cos \theta$
 $y = 2 \cos 2\theta \Rightarrow y = 2(2 \cos^2 \theta - 1)$
 $y = 4 \cos^2 \theta - 2$
 $y + 2 = 4 \cos^2 \theta$
 $y + 2 = 4(2x^2)$
 $y + 2 = 8x^2$
 $y = 8x^2 - 2$

Question 5

Find a Cartesian equation for each of the following parametric relationships.

a) $x = 2\sin^2 \theta$, $y = \cot \theta$, $0 \leq \theta < 2\pi$

b) $x = 2\sin \theta$, $y = \cos 2\theta$, $0 \leq \theta < 2\pi$

c) $x = 2\cos \theta$, $y = 6\cos 2\theta$, $0 \leq \theta < 2\pi$

d) $x = 2\cos \theta$, $y = 6\sin 2\theta$, $0 \leq \theta < 2\pi$

$$\boxed{y^2 = \frac{2-x}{x}}, \quad \boxed{y = 1 - \frac{1}{2}x^2}, \quad \boxed{y = 3x^2 - 6}, \quad \boxed{y^2 = 9x^2(4 - x^2)}$$

Handwritten solutions for Question 5:

(a) $x = 2\sin^2 \theta \Rightarrow \frac{x}{2} = \sin^2 \theta \Rightarrow \frac{x}{2} = 1 - \cos^2 \theta \Rightarrow \cos^2 \theta = 1 - \frac{x}{2} \Rightarrow \cos \theta = \pm \sqrt{1 - \frac{x}{2}} \Rightarrow y = \cot \theta = \frac{\cos \theta}{\sin \theta} = \frac{\cos \theta}{\sqrt{1 - \cos^2 \theta}} = \frac{\cos \theta}{\sqrt{1 - (1 - \frac{x}{2})}} = \frac{\cos \theta}{\sqrt{\frac{x}{2}}} = \frac{\cos \theta}{\sqrt{x/2}} \Rightarrow y^2 = \frac{\cos^2 \theta}{x/2} = \frac{1 - \frac{x}{2}}{x/2} = \frac{2-x}{x}$

(b) $x = 2\sin \theta \Rightarrow \sin \theta = \frac{x}{2} \Rightarrow \cos 2\theta = 1 - 2\sin^2 \theta = 1 - 2(\frac{x}{2})^2 = 1 - \frac{1}{2}x^2 \Rightarrow y = \cos 2\theta = 1 - \frac{1}{2}x^2$

(c) $x = 2\cos \theta \Rightarrow \cos \theta = \frac{x}{2} \Rightarrow \cos 2\theta = 2\cos^2 \theta - 1 = 2(\frac{x}{2})^2 - 1 = \frac{1}{2}x^2 - 1 \Rightarrow y = 6\cos 2\theta = 6(\frac{1}{2}x^2 - 1) = 3x^2 - 6$

(d) $x = 2\cos \theta \Rightarrow \cos \theta = \frac{x}{2} \Rightarrow \sin 2\theta = 2\sin \theta \cos \theta \Rightarrow y = 6\sin 2\theta = 6(2\sin \theta \cos \theta) = 12\sin \theta \cos \theta \Rightarrow y^2 = 144\sin^2 \theta \cos^2 \theta \Rightarrow y^2 = 144(1 - \cos^2 \theta)\cos^2 \theta \Rightarrow y^2 = 144(1 - \frac{x^2}{4})\frac{x^2}{4} = 9x^2(4 - x^2)$

Question 6

Eliminate the parameter θ to obtain a Cartesian equation for each of the following parametric equations.

a) $x = \sin^2 \theta$, $y = \sin 2\theta$, $0 \leq \theta < 2\pi$

b) $x = \sin \theta + \cos \theta$, $y = \sin \theta - \cos \theta$, $0 \leq \theta < 2\pi$

c) $x = \cos 2\theta$, $y = \tan \theta$, $0 \leq \theta < 2\pi$

d) $x = \tan \theta$, $y = 2 \sin 2\theta$, $0 \leq \theta < 2\pi$

$$\boxed{y^2 = 4x(1-x)}, \quad \boxed{x^2 + y^2 = 2}, \quad \boxed{y^2 = \frac{1-x}{1+x}}, \quad \boxed{y = \frac{4x}{1+x^2}}$$

(a) $x = \sin^2 \theta$
 $y = \sin 2\theta$
 $\Rightarrow x = \sin^2 \theta$
 $y = 2 \sin \theta \cos \theta$
 $\Rightarrow y^2 = 4 \sin^2 \theta \cos^2 \theta$
 $\Rightarrow y^2 = 4x(1-x)$

(b) $x = \sin \theta + \cos \theta$
 $y = \sin \theta - \cos \theta$
 $\Rightarrow x^2 = (\sin \theta + \cos \theta)^2 = \sin^2 \theta + \cos^2 \theta + 2 \sin \theta \cos \theta = 1 + \sin 2\theta$
 $y^2 = (\sin \theta - \cos \theta)^2 = \sin^2 \theta + \cos^2 \theta - 2 \sin \theta \cos \theta = 1 - \sin 2\theta$
 $\Rightarrow x^2 + y^2 = 2$

(c) $x = \cos 2\theta$
 $y = \tan \theta$
 $\Rightarrow x = 2 \cos^2 \theta - 1$
 $\Rightarrow x+1 = 2 \cos^2 \theta$
 $\Rightarrow \frac{x+1}{2} = \cos^2 \theta$
 $\Rightarrow \frac{x+1}{2} = \frac{1}{1+y^2}$
 $\Rightarrow y^2 = \frac{1-x}{1+x}$

(d) $x = \tan \theta$
 $y = 2 \sin 2\theta$
 $\Rightarrow x = \tan \theta$
 $\Rightarrow x^2 = \tan^2 \theta$
 $\Rightarrow x^2 + 1 = \sec^2 \theta$
 $\Rightarrow \frac{1}{1+x^2} = \cos^2 \theta$
 $\Rightarrow \frac{y}{2} = \sin \theta \cos \theta$
 $\Rightarrow y = \frac{2 \sin \theta \cos \theta}{\frac{1}{1+x^2}} = \frac{2 \sin \theta \cos \theta (1+x^2)}{1}$
 $\Rightarrow y = \frac{4x}{1+x^2}$

Eliminate the parameter θ to obtain a Cartesian equation for each of the following parametric expressions.

b) $x = \sin 2\theta, \quad y = \cot \theta, \quad 0 \leq \theta < 2\pi$

c) $x = \sin^2 \theta$, $y = \tan 2\theta$, $0 \leq \theta < 2\pi$

d) $x = \operatorname{cosec} \theta - \sin \theta$, $y = \sec \theta - \cos \theta$, $0 \leq \theta < 2\pi$

$$\boxed{16x^2 = y(4-y)}, \quad \boxed{y(2-xy) = x}, \quad \boxed{y^2 = \frac{4x(1-x)}{(1-2x)^2}}, \quad \boxed{y^2 x^2 \left(x^{\frac{2}{3}} + y^{\frac{2}{3}}\right)^3 = 1}$$

(A)

$$\begin{aligned} z &= \sin \theta \cos \phi \\ y &= 4 \cos \theta \end{aligned}$$

$$\Rightarrow \begin{aligned} z^2 &= \sin^2 \theta \cos^2 \phi \\ &= z^2 (-\cos \theta) \cos^2 \phi \\ &= (4z) (4(-\cos \theta)) \times 4 \cos^2 \phi \\ &= 4z^2 = 4 \cos \theta (-4 \cos \theta) \\ &= 4z^2 = y(4-y) \end{aligned}$$

(B)

$$\begin{aligned} z &= \sin \theta \\ y &= \cos \theta \end{aligned}$$

$$\Rightarrow \begin{aligned} z &= 2 \sin \theta \cos \theta \\ z^2 &= 4 \sin \theta \cos \theta \\ z^2 &= 4 \sin \theta (1 - \sin \theta) \end{aligned}$$

$$\text{Now } y^2 = \cos^2 \theta = \cos \theta (-1)$$

$$\frac{y^2 + 1}{z^2 + 1} = \frac{\cos^2 \theta + 1}{\sin^2 \theta + 1}$$

$$\frac{y^2 + 1}{z^2 + 1} = \frac{1}{y^2 + 1}$$

Thus $z^2 = 4 = \frac{y}{y+1} \times (1 - \frac{y}{y+1})$

$$\Rightarrow z^2 = \frac{4}{y^2 + 1} \times \frac{y^2 + 1}{y^2 + 1}$$

$$\Rightarrow z^2 = \frac{4y^2}{(y^2 + 1)^2}$$

$$\Rightarrow z = \frac{2y}{y^2 + 1}$$

(22) $2y^2 + xz = 2y$

$$\Rightarrow z = \frac{2y - 2y^2}{y}$$

$$\Rightarrow z = y(2 - y)$$

If $y(2 - y) = 2$

[illegible]

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PARAMETRIC ALGEBRA

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Question 1

Find the x and y intercepts for each pair of parametric equations.

a) $x = 2t + 1, \quad y = 2t + 6, \quad t \in \mathbb{R}$

b) $x = t^2, \quad y = (t+1)(t+2), \quad t \in \mathbb{R}$

c) $x = \frac{t-1}{t+1}, \quad y = \frac{2t}{t^2+1}, \quad t \in \mathbb{R}, \quad t \neq -1$

$(0,5) \& (-5,0), \quad (0,2) \& (4,0), (1,0), \quad (0,1) \& (-1,0)$

Question 2

A curve is defined by the following parametric equations

$$x = 4at^2, \quad y = a(2t+1), \quad t \in \mathbb{R}.$$

where a is non zero constant.

Given the curves passes through the point $A(4,0)$, find the value of a .

$a = 4$

Question 3

A curve is given by the parametric equations

$$x = 2t^2 - 1, \quad y = 3(t+1), \quad t \in \mathbb{R}.$$

Find the coordinates of the points of intersection of this curve and the line with equation

$$3x - 4y = 3.$$

$$(17, 12) \text{ \& } (1, 0)$$

Handwritten solution for Question 3:

$$\begin{aligned} x &= 2t^2 - 1 \\ y &= 3(t+1) \\ 3x - 4y &= 3 \end{aligned}$$

Solving simultaneously:

$$\begin{aligned} \Rightarrow 3(2t^2 - 1) - 4(3(t+1)) &= 3 \\ \Rightarrow 6t^2 - 3 - 12(t+1) &= 3 \\ \Rightarrow 6t^2 - 3 - 12t - 12 &= 3 \\ \Rightarrow 6t^2 - 12t - 15 &= 0 \\ \Rightarrow 2t^2 - 4t - 5 &= 0 \\ \Rightarrow (t+1)(t-3) &= 0 \\ \Rightarrow t &= -1 \text{ or } 3 \end{aligned}$$

For $t = -1$: $x = 2(-1)^2 - 1 = 1$, $y = 3(-1+1) = 0$
 For $t = 3$: $x = 2(3)^2 - 1 = 17$, $y = 3(3+1) = 12$
 $\therefore (1, 0) \text{ \& } (17, 12)$

Question 4

The curve C_1 has Cartesian equation

$$x^2 + y^2 = 9x - 4.$$

The curve C_2 has parametric equations

$$x = t^2, \quad y = 2t, \quad t \in \mathbb{R}.$$

Find the coordinates of the points of intersection of C_1 and C_2 .

$$(4, 4), (4, -4), (1, 2), (1, -2)$$

Handwritten solution for Question 4:

$$\begin{aligned} x^2 + y^2 &= 9x - 4 \\ x &= t^2 \\ y &= 2t \end{aligned}$$

Solving simultaneously:

$$\begin{aligned} (t^2)^2 + (2t)^2 &= 9(t^2) - 4 \\ t^4 + 4t^2 &= 9t^2 - 4 \\ t^4 - 5t^2 + 4 &= 0 \\ (t^2 - 1)(t^2 - 4) &= 0 \\ t^2 &= 1 \text{ or } 4 \\ t &= \pm 1 \text{ or } \pm 2 \end{aligned}$$

For $t = 1$: $x = 1$, $y = 2$
 For $t = -1$: $x = 1$, $y = -2$
 For $t = 2$: $x = 4$, $y = 4$
 For $t = -2$: $x = 4$, $y = -4$
 $\therefore (1, 2), (1, -2), (4, 4), (4, -4)$

Question 5

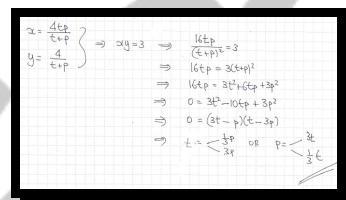
The curve with Cartesian equation $xy = 3$ is also traced by the following parametric equations

$$x = \frac{4tp}{t+p}, \quad y = \frac{4}{t+p}, \quad t, p \in \mathbb{R}, \quad t \neq p$$

where t and p are parameters.

Find the relationship between the two parameters t and p in the form $p = f(t)$.

$$p = 3t \quad \text{or} \quad p = \frac{1}{3}t$$



Handwritten solution for Question 5:

$$\begin{aligned} x &= \frac{4tp}{t+p} \\ y &= \frac{4}{t+p} \end{aligned} \quad \Rightarrow \quad xy = 3 \quad \Rightarrow \quad \frac{16tp}{(t+p)^2} = 3$$

$$\Rightarrow 16tp = 3(t+p)^2$$

$$\Rightarrow 16tp = 3t^2 + 6tp + 3p^2$$

$$\Rightarrow 0 = 3t^2 - 10tp + 3p^2$$

$$\Rightarrow 0 = (3t - p)(t - 3p)$$

$$\Rightarrow t = \frac{1}{3}p \quad \text{or} \quad p = 3t$$

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PARAMETRIC DIFFERENTIATION

Created by T. Madas

Question 1

A curve is given parametrically by the equations

$$x = 1 - \cos 2\theta, \quad y = \sin 2\theta, \quad 0 \leq \theta < 2\pi.$$

The point P lies on this curve, and the value of θ at that point is $\frac{\pi}{6}$.

Show that an equation of the normal at the point P is given by

$$y + \sqrt{3}x = \sqrt{3}.$$

proof

Handwritten mathematical proof for the normal line equation:

$$\begin{aligned} x &= 1 - \cos 2\theta \\ y &= \sin 2\theta \end{aligned} \Rightarrow \frac{dx}{d\theta} = 2\sin 2\theta, \quad \frac{dy}{d\theta} = 2\cos 2\theta$$

$$\therefore \frac{dy}{dx} = \frac{2\cos 2\theta}{2\sin 2\theta} = \frac{\cos 2\theta}{\sin 2\theta} \quad \therefore \frac{dx}{dy} = \frac{\sin 2\theta}{\cos 2\theta} = \frac{1}{\cot 2\theta} = \frac{1}{\frac{1}{\tan 2\theta}} = \tan 2\theta$$

At $\theta = \frac{\pi}{6}$, $2\theta = \frac{\pi}{3}$

$$x = 1 - \cos \frac{\pi}{3} = 1 - \frac{1}{2} = \frac{1}{2}$$

$$y = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

Normal line: $m = -\frac{1}{\tan 2\theta} = -\frac{1}{\tan \frac{\pi}{3}} = -\frac{1}{\sqrt{3}}$

$$y - \frac{\sqrt{3}}{2} = -\frac{1}{\sqrt{3}} \left(x - \frac{1}{2} \right)$$

$$y - \frac{\sqrt{3}}{2} = -\frac{1}{\sqrt{3}}x + \frac{1}{2\sqrt{3}}$$

$$y + \frac{1}{\sqrt{3}}x = \frac{\sqrt{3}}{2} + \frac{1}{2\sqrt{3}} = \frac{3 + 1}{2\sqrt{3}} = \frac{4}{2\sqrt{3}} = \frac{2}{\sqrt{3}}$$

$$y + \frac{1}{\sqrt{3}}x = \frac{2}{\sqrt{3}} \Rightarrow y + \sqrt{3}x = 2$$

Question 2

A curve is given parametrically by the equations

$$x = \frac{2}{t}, \quad y = t^2 - 1, \quad t \in \mathbb{R}, \quad t \neq 0.$$

The point $P(4, y)$ lies on this curve.

Show that an equation of the tangent at the point P is given by

$$x + 8y + 2 = 0.$$

proof

Handwritten proof on grid paper:

$$\begin{aligned} x &= \frac{2}{t} = 2t^{-1} \\ y &= t^2 - 1 \end{aligned}$$

When $x=4$, $t=\frac{1}{2}$

$$\therefore y = \left(\frac{1}{2}\right)^2 - 1 = -\frac{3}{4}$$

$$\therefore P\left(4, -\frac{3}{4}\right) \text{ at } t=\frac{1}{2}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{-2t^{-2}} = -t^3$$

$$\left. \frac{dy}{dx} \right|_{t=\frac{1}{2}} = -\left(\frac{1}{2}\right)^3 = -\frac{1}{8}$$

$$y - y_1 = m(x - x_1)$$

$$y + \frac{3}{4} = -\frac{1}{8}(x - 4)$$

$$8y + 6 = -x + 4$$

$$x + 8y + 2 = 0$$

Question 3

A curve is given parametrically by the equations

$$x = 3t - 2\sin t, \quad y = t^2 + t \cos t, \quad 0 \leq t < 2\pi.$$

Show that an equation of the tangent at the point on the curve where $t = \frac{\pi}{2}$ is given by

$$y = \frac{\pi}{6}(x + 2).$$

proof

Handwritten proof for Question 3:

$$\begin{aligned} x &= 3t - 2\sin t \Rightarrow \frac{dx}{dt} = 3 - 2\cos t \\ y &= t^2 + t \cos t \Rightarrow \frac{dy}{dt} = 2t + \cos t - t \sin t \\ \therefore \frac{dy}{dx} &= \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t + \cos t - t \sin t}{3 - 2\cos t} \\ \text{At } t = \frac{\pi}{2}: \quad \frac{dy}{dx} &= \frac{2(\frac{\pi}{2}) + \cos(\frac{\pi}{2}) - (\frac{\pi}{2})\sin(\frac{\pi}{2})}{3 - 2\cos(\frac{\pi}{2})} = \frac{\pi - \frac{\pi}{2}}{3} = \frac{\pi}{6} \\ \text{When } t = \frac{\pi}{2}: \quad x &= 3(\frac{\pi}{2}) - 2\sin(\frac{\pi}{2}) = \frac{3\pi}{2} - 2 \quad \left(\frac{3\pi}{2} - 1, \frac{\pi^2}{4}\right) \\ y &= (\frac{\pi}{2})^2 + (\frac{\pi}{2})\cos(\frac{\pi}{2}) = \frac{\pi^2}{4} \\ \text{Equation of tangent: } y - \frac{\pi^2}{4} &= \frac{\pi}{6}\left(x - \frac{3\pi}{2} + 2\right) \\ y - \frac{\pi^2}{4} &= \frac{\pi}{6}x - \frac{\pi^2}{4} + \frac{\pi}{3} \\ y &= \frac{\pi}{6}x + \frac{\pi}{3} \\ y &= \frac{\pi}{6}(x + 2) \end{aligned}$$

Question 4

Find the turning points of the curve given parametrically by the equations

$$x = 1 - \cos 2t, \quad y = \sin 2t, \quad 0 \leq t < 2\pi.$$

Determine the nature of these turning points.

max(1,1), min(1,-1)

Handwritten proof for Question 4:

$$\begin{aligned} x &= 1 - \cos 2t \\ y &= \sin 2t \\ \text{For MAX/MIN: } \frac{dx}{dt} &= 0 \Rightarrow \frac{dx}{dt} = 2\sin 2t = 0 \\ 2t &= 0, \pi, 2\pi, 3\pi, 4\pi \\ t &= 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi \\ \text{Points: } A(1,1), B(1,-1), C(1,1), D(1,-1) \\ \text{Nature: } A(1,1) \text{ is MAX, } B(1,-1) \text{ is MIN} \\ \frac{d^2y}{dx^2} &= \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{\cos 2t}{2\sin 2t}\right)}{2\sin 2t} = \frac{\frac{-2\sin 2t \cdot 2\cos 2t - \cos 2t \cdot 2\sin 2t}{4\sin^2 2t}}{2\sin 2t} \\ &= \frac{-4\sin 2t \cos 2t}{8\sin^3 2t} = -\frac{1}{2\sin^2 2t} \\ \therefore \text{At } (1,1): t = \frac{\pi}{2}, \frac{d^2y}{dx^2} &= -1 < 0 \Rightarrow (1,1) \text{ is MAX} \\ \text{At } (1,-1): t = \frac{3\pi}{2}, \frac{d^2y}{dx^2} &= -1 < 0 \Rightarrow (1,-1) \text{ is MIN} \end{aligned}$$

Question 5

A curve is given parametrically by the equations

$$x = 3 \sin 2\theta, \quad y = 4 \cos 2\theta, \quad 0 \leq \theta \leq 2\pi.$$

The point P is such so that $\cos \theta = \frac{3}{5}$ with $0 \leq \theta \leq \frac{\pi}{2}$.

Show that an equation of the tangent at the point P is

$$32x - 7y = 100.$$

proof

Handwritten mathematical proof for the tangent equation at point P :

$$\begin{aligned} x &= 3 \sin 2\theta = 6 \sin \theta \cos \theta \\ y &= 4 \cos 2\theta = 4(2 \cos^2 \theta - 1) = 8 \cos^2 \theta - 4 \end{aligned}$$

Given $\cos \theta = \frac{3}{5}$, then $\sin \theta = \frac{4}{5}$.
 $\therefore P$ has coordinates $(6 \times \frac{3}{5} \times \frac{4}{5}, 8 \times (\frac{3}{5})^2 - 4) = (\frac{72}{25}, \frac{32}{25})$.

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-8 \sin 2\theta}{6 \cos 2\theta} = \frac{-16 \sin \theta \cos \theta}{6(2 \cos^2 \theta - 1)}$$

$$\frac{dy}{dx} \bigg|_P = \frac{-16 \times \frac{3}{5} \times \frac{4}{5}}{6 \left(2 \left(\frac{3}{5} \right)^2 - 1 \right)} = \frac{-\frac{192}{25}}{-\frac{42}{25}} = \frac{32}{7}$$

EQUATION OF TANGENT $\Rightarrow y - y_1 = m(x - x_1)$
 $\Rightarrow y - \frac{32}{25} = \frac{32}{7} \left(x - \frac{72}{25} \right)$
 $\Rightarrow 175y - 106 = 80x - 256$
 $\Rightarrow 175y = 80x - 150$
 $\Rightarrow 7y = 32x - 100$
 $\Rightarrow 32x - 7y = 100$

Question 6

For the curve given parametrically by

$$x = \frac{t}{1-t}, \quad y = \frac{t^2}{1-t}, \quad t \in \mathbb{R}, t \neq 1,$$

find the coordinates of the turning points and determine their nature.

$$\max(-2, -4), \min(0, 0)$$

Handwritten solution for Question 6:

- $x = \frac{t}{1-t}$
- $y = \frac{t^2}{1-t}$
- $\frac{dx}{dt} = \frac{(1-t)(1) - t(-1)}{(1-t)^2} = \frac{1-t+t}{(1-t)^2} = \frac{1}{(1-t)^2}$
- $\frac{dy}{dt} = \frac{(1-t)(2t) - t^2(-1)}{(1-t)^2} = \frac{2t-t^2+t^2}{(1-t)^2} = \frac{2t}{(1-t)^2}$
- $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{2t}{(1-t)^2}}{\frac{1}{(1-t)^2}} = 2t$
- $\frac{dy}{dx} = 0 \Rightarrow 2t = 0 \Rightarrow t = 0 \Rightarrow x = 0, y = 0$
- $\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dt} \left(\frac{dy}{dx} \right) \cdot \frac{dt}{dx} = \frac{d}{dt} (2t) \cdot \frac{1}{\frac{dx}{dt}} = 2 \cdot (1-t)^2 = 2(1-t)^2$
- $\frac{d^2y}{dx^2} \Big|_{t=0} = 2(1-0)^2 = 2 > 0 \Rightarrow (0,0) \text{ is a min}$
- $\frac{d^2y}{dx^2} \Big|_{t=2} = 2(1-2)^2 = 2 > 0 \Rightarrow (-2,-4) \text{ is a min}$

Question 7

A curve is given parametrically by the equations

$$x = \frac{2t}{1+t^2}, \quad y = \frac{1-t^2}{1+t^2}, \quad t \in \mathbb{R}.$$

The point $P\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ lies on this curve.

Show that an equation of the tangent at the point P is given by

$$x + y = \sqrt{2}.$$

proof

$\bullet z = \frac{2t}{1+t^2}$
 $\frac{dz}{dt} = \frac{2(1+t^2) - 2t(2t)}{(1+t^2)^2}$
 $\frac{dz}{dt} = \frac{2-4t^2}{(1+t^2)^2}$
 $\frac{dz}{dt} = \frac{2(1-2t^2)}{(1+t^2)^2}$
 $\bullet \frac{dy}{dt} = \frac{1-t^2}{1+t^2}$
 $\frac{dy}{dt} = \frac{(1+t^2)(2t) - (1-t^2)2t}{(1+t^2)^2}$
 $\frac{dy}{dt} = \frac{2t - 2t^3 - 2t + 2t^3}{(1+t^2)^2}$
 $\frac{dy}{dt} = \frac{0}{(1+t^2)^2}$
 $\bullet \frac{dy}{dz} = \frac{\frac{dy}{dt}}{\frac{dz}{dt}} = \frac{\frac{0}{(1+t^2)^2}}{\frac{2(1-2t^2)}{(1+t^2)^2}} = \frac{0}{2(1-2t^2)} = \frac{0}{2-2t^2} = \frac{0}{2(1-t^2)}$
 $\bullet \frac{dz}{dz} = \frac{2t}{1+t^2} \Leftrightarrow 2 \cdot \frac{dz}{dz} t = \frac{1-t^2}{1+t^2} = \frac{1-t^2}{1+t^2}$
 $\Rightarrow (t^2+1) \frac{dz}{dz} = \frac{1-t^2}{1+t^2}$
 $\Rightarrow t^2+1 = 2t^2 \quad \Rightarrow 2-2t^2 = (2+t^2)t^2$
 $\Rightarrow t^2 - 2t^2 = t + 0 \quad \Rightarrow \frac{2-t^2}{2+t^2} = t^2$
 $\Rightarrow (t - \sqrt{2})^2 = 0 \quad \Rightarrow t^4 = 3 - 2\sqrt{2}$
 $\Rightarrow t - \sqrt{2} = -1 \quad \Rightarrow t^6 = (1 - 2\sqrt{2} + \sqrt{2})^2$
 $\Rightarrow t = \frac{1+\sqrt{2}}{-1+\sqrt{2}} \quad \Rightarrow t^2 = (\sqrt{2}-1)^2$
 $\Rightarrow t = \frac{1+\sqrt{2}}{-1+\sqrt{2}} \quad \Rightarrow t = \sqrt{2}-1$
 $\swarrow$
 $\text{AT } P, t = -1+\sqrt{2}$
 $\frac{dz}{dz} = \frac{2(-1+\sqrt{2})}{(-1+\sqrt{2})^2-1} = \frac{2(-1+\sqrt{2})}{3-2\sqrt{2}-1} = \frac{2(-1+\sqrt{2})}{2-2\sqrt{2}} = \frac{2(-1+\sqrt{2})}{-2(1-\sqrt{2})}$
 $= -1$
 $\therefore y-y_1 = M(x-x_1)$
 $y - \frac{2}{3} = - (x - \frac{\sqrt{2}}{2})$
 $y - \frac{2}{3} = -x + \frac{\sqrt{2}}{2}$
 $y - \frac{2}{3} = -x + \frac{\sqrt{2}}{2}$
 $\therefore 2+y = \sqrt{2}$