NUMERICAL SOLUTIONS OF EQUATIONS

ITERATIVE METHODS

Question 1 (**)

$$x^3 + 10x - 4 = 0.$$

a) Show that the above equation has a root α, which lies between 0 and 1.

The recurrence relation

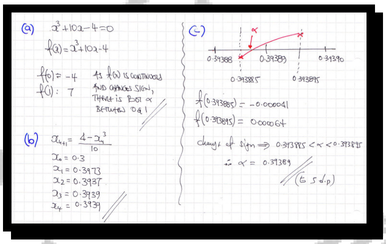
$$x_{n+1} = \frac{4 - x_n^3}{10}$$

starting with $x_0 = 0.3$ is to be used to find α.

b) Find, to 4 decimal places, the value of x_1, x_2, x_3 and x_4.

c) By considering the sign of an appropriate function $f(x)$ in a suitable interval, show clearly that $\alpha = 0.39389$, correct to 5 decimal places.

$\boxed{}$, $\boxed{x_1 = 0.3973, \quad x_2 = 0.3937, \quad x_3 = 0.3939, \quad x_4 = 0.3939}$



Question 2 **(**)**

$$e^{-x} + \sqrt{x} = 2$$

a) Show that the above equation has a root α, which lies between 3 and 4.

The recurrence relation

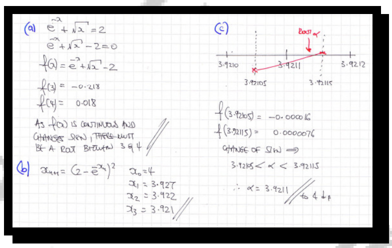
$$x_{n+1} = \left(2 - e^{-x_n}\right)^2$$

starting with $x_0 = 4$ is to be used to find α.

b) Find, to 3 decimal places, the value of x_1, x_2 and x_3.

c) By considering the sign of an appropriate function $f(x)$ in a suitable interval, show clearly that $\alpha = 3.9211$, correct to 4 decimal places.

$\boxed{}$, $\boxed{x_1 = 3.927, \ x_2 = 3.922, \ x_3 = 3.921}$


(a) $e^{-x} + \sqrt{x} = 2$

$e^{-x} + \sqrt{x} - 2 = 0$

$f(x) = e^{-x} + \sqrt{x} - 2$

$f(3) = -0.218$

$f(4) = 0.018$

AS $f(x)$ IS CONTINUOUS AND CHANGES SIGN, THERE MUST BE A ROOT BETWEEN 3 & 4

(b) $x_{n+1} = (2 - e^{-x_n})^2$ $x_0 = 4$

$x_1 = 3.927$

$x_2 = 3.922$

$x_3 = 3.921$

(c)

$f(3.92105) = -0.000016$

$f(3.92115) = 0.0000076$

CHANGE OF SIGN $\Rightarrow$

$3.92105 < \alpha < 3.92115$

$\therefore \alpha = 3.9211$ to 4 d.p.

Question 3 (**)

$$e^{3x} = x + 20$$

a) Show that the above equation has a root α between 1 and 2.

The recurrence relation

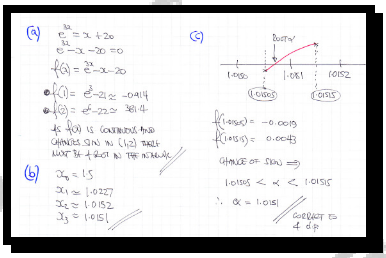
$$x_{n+1} = \frac{1}{3}\ln\left(x_n + 20\right)$$

starting with $x_0 = 1.5$ is to be used to find α.

b) Find to 4 decimal places, the value of x_1, x_2 and x_3.

c) By considering the sign of an appropriate function $f(x)$ in a suitable interval, show clearly that $\alpha = 1.0151$, correct to 4 decimal places.

$\boxed{}$, $\boxed{x_1 = 1.0227, \quad x_2 = 1.0152, \quad x_3 = 1.0151}$



(a) $e^{3x} = x + 20$

$e^{3x} - x - 20 = 0$

$f(x) = e^{3x} - x - 20$

• $f(1) = e^3 - 21 \simeq -0.914$

• $f(2) = e^6 - 22 \simeq 381.4$

AS $f(x)$ IS CONTINUOUS AND CHANGES SIGN IN $(1,2)$ THERE MUST BE A ROOT IN THE INTERVAL

(b) $x_0 = 1.5$

$x_1 \simeq 1.0227$

$x_2 \simeq 1.0152$

$x_3 \simeq 1.0151$

(c)

$f(1.01505) = -0.0019$

$f(1.01515) = 0.0043$

CHANGE OF SIGN $\Rightarrow$

$1.01505 < \alpha < 1.01515$

$\therefore \alpha = 1.0151$ CORRECT TO 4 d.p.

Question 4 (**+)

$$f(x) = 4x - 3\sin x - 1, \quad 0 \le x \le 2\pi.$$

a) Show that the equation $f(x) = 0$ has a solution α in the interval $(0.7, 0.8)$.

An iterative formula, of the form given below, is used to find α.

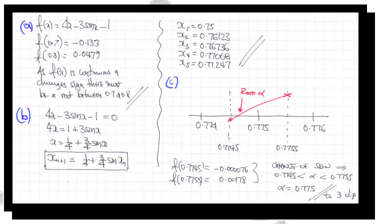
$$x_{n+1} = A + B\sin x_n, \quad x_1 = 0.75,$$

where A and B are constants.

b) Find, to 5 decimal places, the value of x_2, x_3, x_4 and x_5.

c) By considering the sign of $f(x)$ in a suitable interval show clearly that $\alpha = 0.775$, correct to 3 decimal places.

$\boxed{}$, $\boxed{x_2 = 0.76123, \ x_3 = 0.76736, \ x_4 = 0.77068, \ x_5 = 0.77247}$



Question 5 (**+)

$$x^3 - x^2 = 6x + 6, \quad x \in \mathbb{R}.$$

a) Show that the above equation has a root α in the interval $(3, 4)$.

b) Show that the above equation can be written as

$$x = \sqrt{\frac{6x+6}{x-1}}.$$

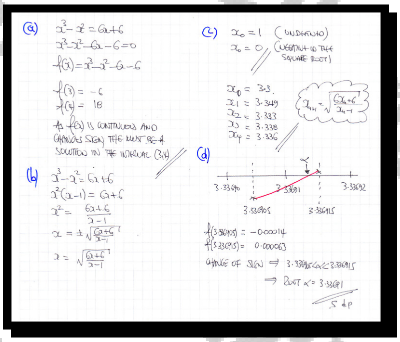
An iterative formula of the form given in part **(b)**, starting with x_0 is used to find α.

c) Give two different values for x_0 that would not produce an answer for x_1.

d) Starting with $x_0 = 3.3$ find the value of x_1, x_2, x_3 and x_4, giving each of the answers correct to 3 decimal places.

e) By considering the sign of an appropriate function in a suitable interval, show clearly that $\alpha = 3.33691$, correct to 5 decimal places.

$\boxed{}$, $\boxed{x_0 \neq 1,\ 0,\ 0.5 \text{ etc}}$, $\boxed{x_1 = 3.349,\ x_2 = 3.333,\ x_3 = 3.338,\ x_4 = 3.336}$



(a) $x^3 - x^2 = 6x+6$

$x^3 - x^2 - 6x - 6 = 0$

$f(x) = x^3 - x^2 - 6x - 6$

$f(3) = -6$

$f(4) = 18$

AS $f(x)$ IS CONTINUOUS AND CHANGES SIGN, THERE MUST BE A SOLUTION IN THE INTERVAL $(3,4)$

(b) $x^3 - x^2 = 6x+6$

$x^2(x-1) = 6x+6$

$x^2 = \frac{6x+6}{x-1}$

$x = \pm\sqrt{\frac{6x+6}{x-1}}$

$x = \sqrt{\frac{6x+6}{x-1}}$

(c) $x_0 = 1$ (UNDEFINED)

$x_0 = 0$ (NEGATIVE IN THE SQUARE ROOT)

$x_0 = 3.3$

$x_1 = 3.349$

$x_2 = 3.333$

$x_3 = 3.338$

$x_4 = 3.336$

$x_{n+1} = \sqrt{\frac{6x_n+6}{x_n-1}}$

(d)

α

3.33690 3.33691 3.33692

3.336905 3.336915

$f(3.336905) = -0.00014$

$f(3.336915) = 0.000063$

CHANGE OF SIGN $\Rightarrow 3.336905 < \alpha < 3.336915$

$\Rightarrow$ ROOT $\alpha = 3.33691$

5 d.p

Question 6 (***)

The curves C_1 and C_2 have respective equations

$$y = 9 - x^2, \ x \in \mathbb{R} \quad \text{and} \quad y = e^x, \ x \in \mathbb{R}.$$

a) Sketch in the same diagram the graph of C_1 and the graph of C_2.

The sketch must include the coordinates of the points where each of the curves meet the coordinate axes.

b) By considering the graphs sketched in part (a), show clearly that the equation

$$\left(9 - x^2\right)e^{-x} = 1$$

has exactly one positive root and one negative root.

To find the negative root of the equation the following iterative formula is used

$$x_{n+1} = -\sqrt{9 - e^{x_n}}, \ x_1 = -3.$$

c) Find, to 5 decimal places, the value of x_2, x_3 and x_4.

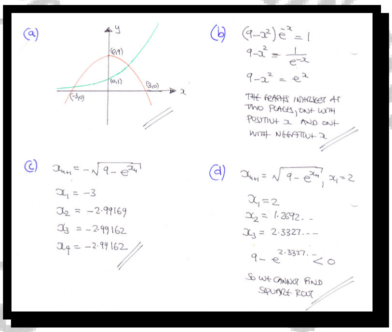
To find the positive root of the equation the following iterative formula is used

$$x_{n+1} = \sqrt{9 - e^{x_n}}, \ x_1 = 2.$$

d) Explain, clearly but briefly, why do these iterations fail.

$\boxed{}$, $\boxed{C_1 : (-3,0),(3,0),(0,9)}$, $\boxed{C_2 : (0,1)}$,

$\boxed{x_2 = -2.99169, \quad x_3 = -2.99162, \quad x_4 = -2.99162}$



Question 7 (***)

$$x^3 + 3x = 5, \quad x \in \mathbb{R}.$$

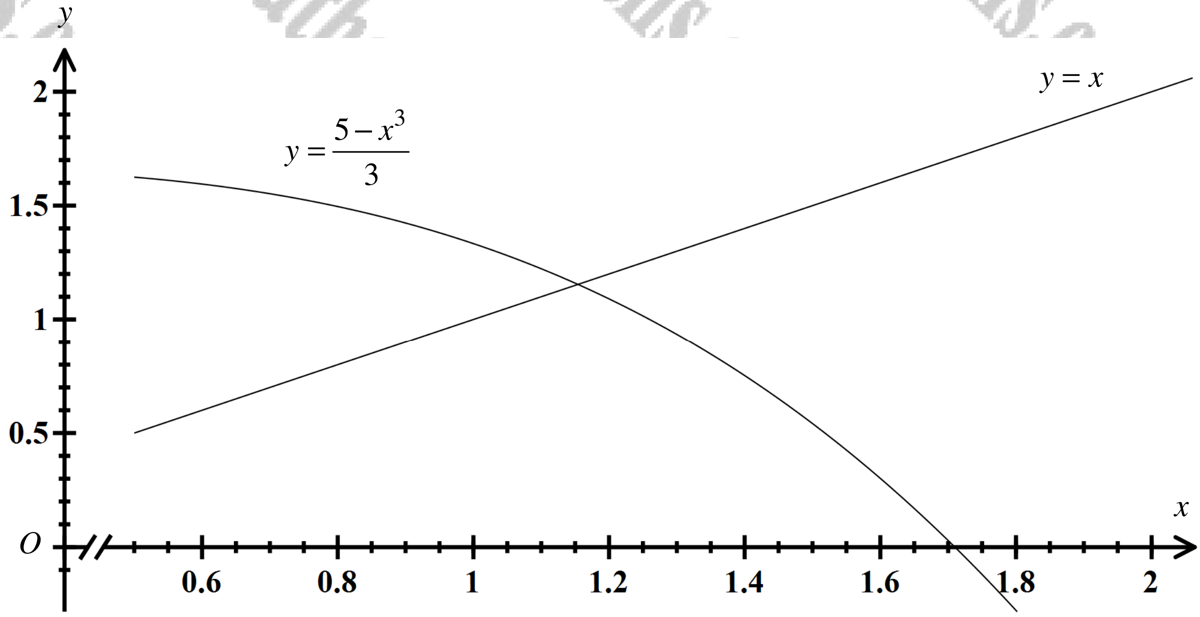
a) Show that the above equation has a root α, between 1 and 2.

An attempt is made to find α using the iterative formula

$$x_{n+1} = \frac{5 - x_n^{\ 3}}{3}, \quad x_1 = 1.$$

b) Find, to 2 decimal places, the value of x_2, x_3, x_4, x_5 and x_6.

The diagram below is used to investigate the results of these iterations.



[continues overleaf]

[continued from overleaf]

c) On a copy of this diagram draw a "staircase" or "cobweb" pattern marking the position of x_1, x_2, x_3 and x_4, further stating the results of these iterations.

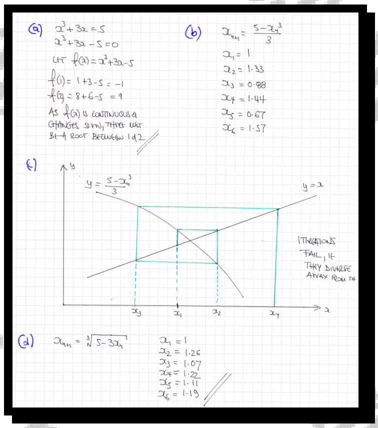
d) Use the iterative formula

$$x_{n+1} = \sqrt[3]{5-3x_n}\ ,\ x_1 = 1,$$

to find, to 2 decimal places, the value of x_2, x_3, x_4, x_5 and x_6.

$\boxed{}$, $\boxed{x_2 = 1.33,\quad x_3 = 0.88,\quad x_4 = 1.44,\quad x_5 = 0.67,\quad x_6 = 1.57}$,

$\boxed{x_2 = 1.26,\quad x_3 = 1.07,\quad x_4 = 1.22,\quad x_5 = 1.11,\quad x_6 = 1.19}$



Question 8 (***)

$$x^3 = 5x+1, \; x \in \mathbb{R}.$$

a) Show that the above equation has a root α between 2 and 3.

The iterative formula

$$x_{n+1} = \sqrt[3]{5x_n + 1}, \; x_1 = 2,$$

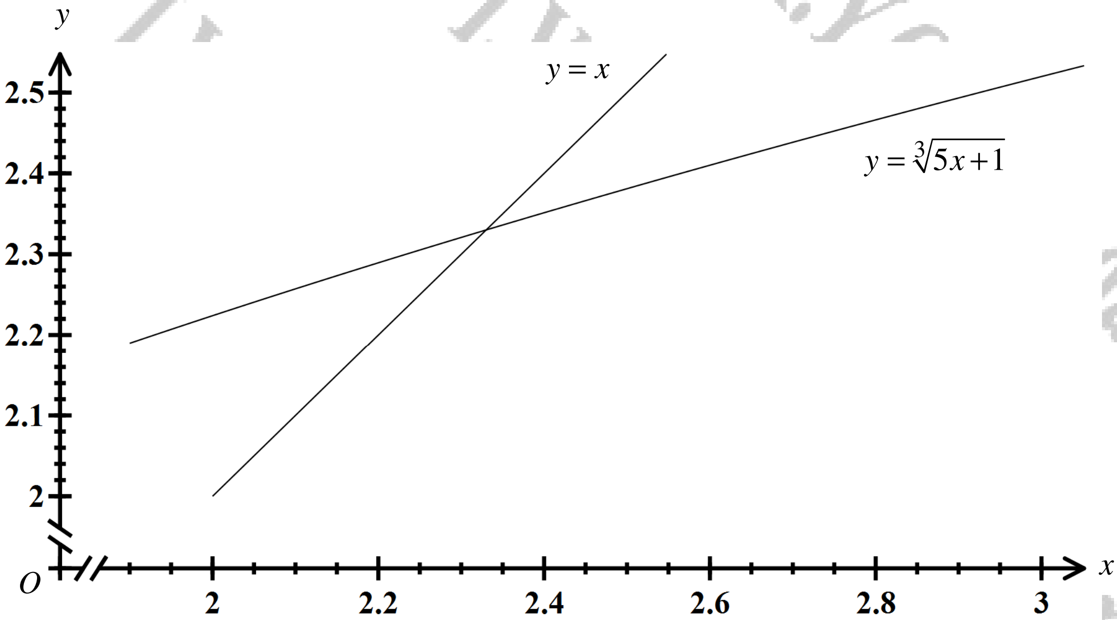
is to be used to find α

b) Find, to 2 decimal places, the value of x_2, x_3 and x_4.

[continues overleaf]

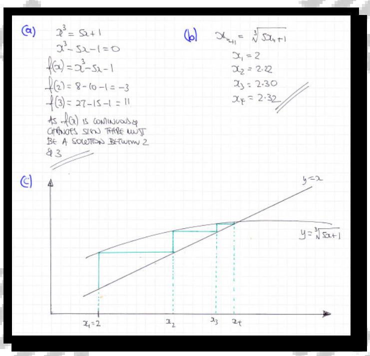
[continued from overleaf]

The diagram below is used to investigate the results of these iterations.



c) On a copy of this diagram draw a "staircase" or "cobweb" pattern showing how these iterations converge to α, marking the position of x_1, x_2, x_3 and x_4.

, $x_2 = 2.22$, $x_3 = 2.30$, $x_4 = 2.32$



Question 9 (***)

A cubic equation has the following equation.

$$x^3 + 1 = 4x, \ x \in \mathbb{R}.$$

a) Show that the above equation has a root α, which lies between 0 and 1.

b) Show further that the above equation can be written as

$$x = \frac{1}{4 - x^2}.$$

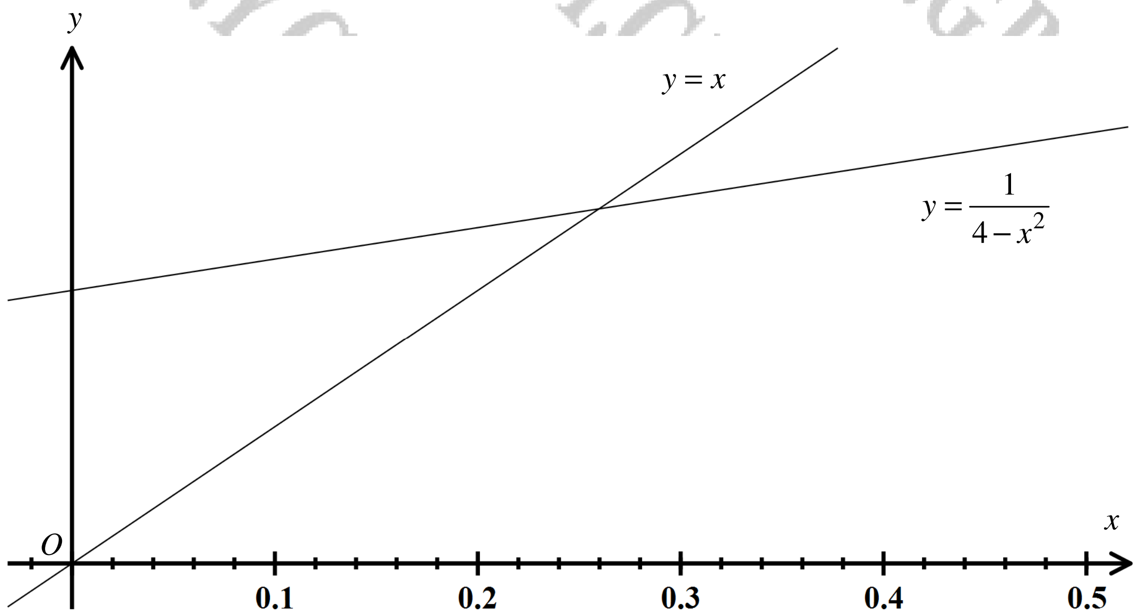
An iterative formula, based on the rearrangement of part **(b)**, is to be used to find α.

c) Starting with $x_1 = 0.1$, find to 4 decimal places, the value of x_2, x_3 and x_4.

[continues overleaf]

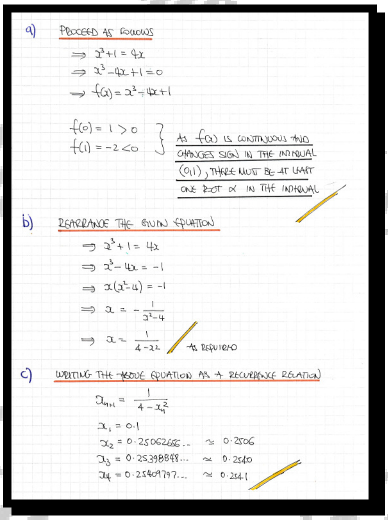
[continued from overleaf]

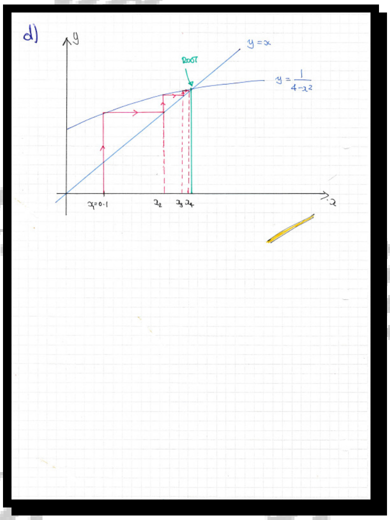
The diagram below is used to show the convergence of these iterations.



d) Draw on a copy of this diagram a "staircase" or "cobweb" pattern showing how these iterations converge to α, marking the position of x_1, x_2, x_3 and x_4.

$\boxed{}$, $\boxed{}$, $\boxed{x_2 = 0.2506, \quad x_3 = 0.2540, \quad x_4 = 0.2541}$





Question 10 (***)

$$f(x) = x^3 - 6x^2 + 12x - 11, \quad x \in \mathbb{R}.$$

a) Show that the equation $f(x) = 0$ has a root α between 3 and 4.

An approximation to the value of α is to be found by using the iterative formula

$$x_{n+1} = \sqrt[3]{6x_n^{\ 2} - 12x_n + 11}, \quad x_1 = 3.$$

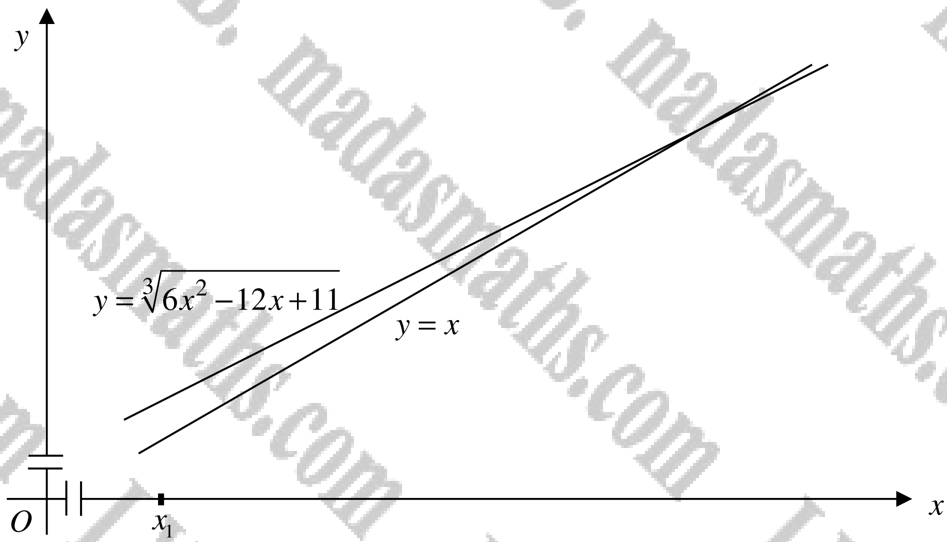
b) Find, to 3 decimal places, the value of x_2, x_3 and x_4.

[continues overleaf]

[continued from overleaf]

The diagram below shows the graphs of

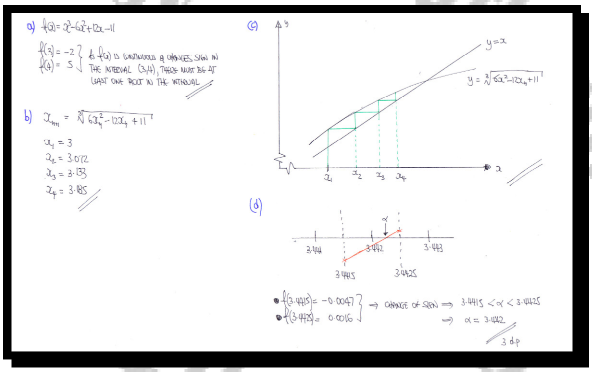
$$y = x \qquad \text{and} \qquad y = \sqrt[3]{6x^2 - 12x + 11}.$$



c) Draw on a copy of the diagram a "staircase" or "cobweb" pattern showing how these iterations converge to α, marking the position of x_2, x_3 and x_4.

d) By considering the sign of $f(x)$ in a suitable interval show that $\alpha = 3.442$, correct to 3 decimal places.

$\boxed{}$, $\boxed{x_2 = 3.072,\ x_3 = 3.133,\ x_4 = 3.185}$



Question 11 (***)

$$4\arccos x = x+1, \; x \in \mathbb{R}, \; -1 \le x \le 1.$$

The above equation has a single root α.

a) Show that $0.5 < \alpha < 1$.

An iterative formula of the form $x_{n+1} = \cos\left(f\left(x_n\right)\right)$ is used to find α.

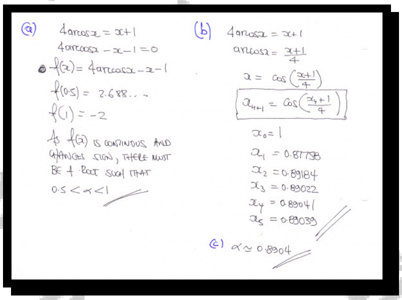
b) Use this iterative formula, starting with $x_0 = 1$, to find the value of x_1, x_2, x_3, x_4 and x_5.

Give the answers correct to 5 decimal places.

c) Write down the value of α to an appropriate accuracy.

$\boxed{}$, $\boxed{x_1 = 0.87758, \quad x_2 = 0.89184, \quad x_3 = 0.89022, \quad x_4 = 0.89041, \quad x_5 = 0.89039}$,

$\boxed{\alpha = 0.8904}$



(a) $4\arccos x = x+1$

$4\arccos x - x - 1 = 0$

$f(x) = 4\arccos x - x - 1$

$f(0.5) = 2.688\ldots$

$f(1) = -2$

As $f(x)$ IS CONTINOUS AND CHANGES SIGN, THERE MUST BE A ROOT SUCH THAT $0.5 < \alpha < 1$

(b) $4\arccos x = x+1$

$\arccos x = \frac{x+1}{4}$

$x = \cos\left(\frac{x+1}{4}\right)$

$\boxed{x_{n+1} = \cos\left(\frac{x_n+1}{4}\right)}$

$x_0 = 1$

$x_1 = 0.87758$

$x_2 = 0.89184$

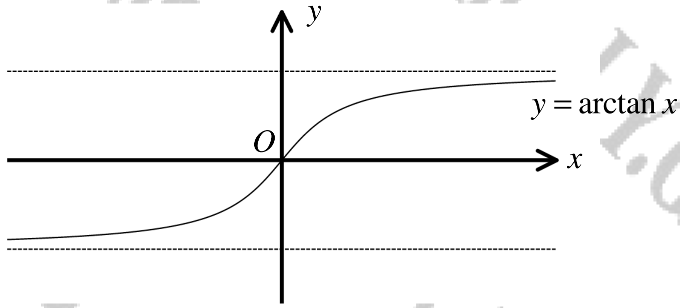
$x_3 = 0.89022$

$x_4 = 0.89041$

$x_5 = 0.89039$

(c) $\alpha \simeq 0.8904$

Question 12 (***)



The diagram above shows the graph of

$$y = \arctan x\,.$$

a) Write down the equations of the two horizontal asymptotes.

b) Copy the diagram above and use it to show that the equation

$$3x - \arctan x = 1$$

has only one positive real root.

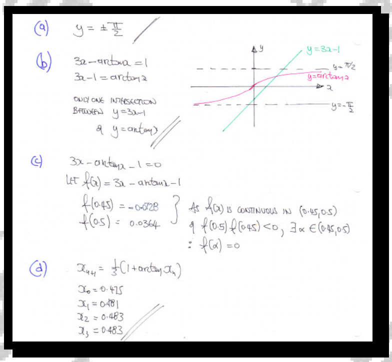
c) Show that this root lies between 0.45 and 0.5.

d) Use the iterative formula

$$x_{n+1} = \tfrac{1}{3}\left(1 + \arctan x_n\right), \; x_0 = 0.475$$

to find, to 3 decimal places, the value of x_1, x_2 and x_3.

$\boxed{}$, $\boxed{y = \pm\dfrac{\pi}{2}}$, $\boxed{x_1 = 0.481, \quad x_2 = 0.483, \quad x_3 = 0.483}$



Question 13 (***+)

The curve C has equation

$$y = x^3 - 3x^2 - 3,$$

and crosses the x axis at the point $A(\alpha, 0)$.

a) Show that α lies between 3 and 4.

b) Show further that the equation $x^3 - 3x^2 - 3 = 0$ can be rearranged to

$$x = 3 + \frac{3}{x^2}, \quad x \neq 0.$$

The equation rearrangement of part **(b)** is written as the following recurrence relation

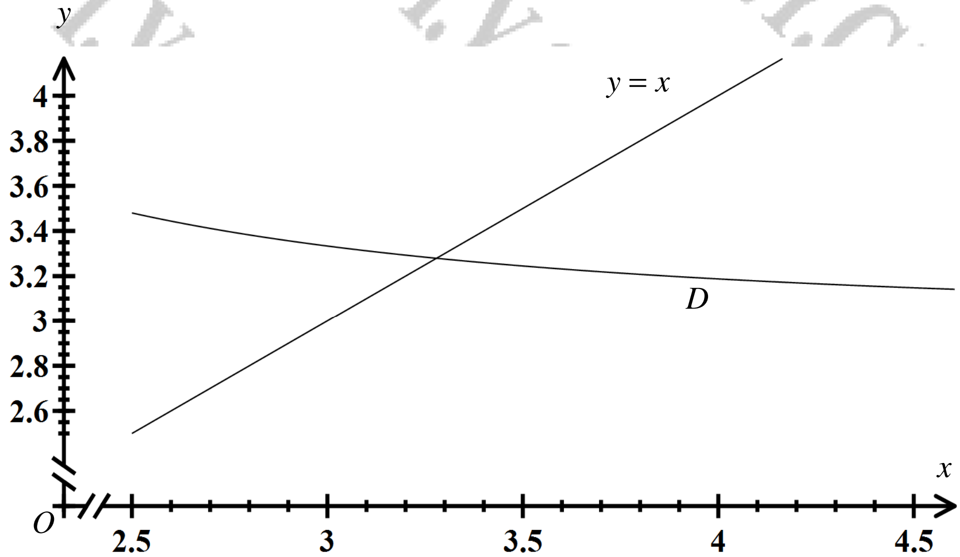
$$x_{n+1} = 3 + \frac{3}{{x_n}^2}, \quad x_1 = 4.$$

c) Use the above iterative formula to find, to 4 decimal places, the value of x_2, x_3, x_4 and x_5.

[continues overleaf]

[continued from overleaf]

The diagram below is used to describe how the iteration formula converges to α, and shows the graph of $y = x$ and another curve D.

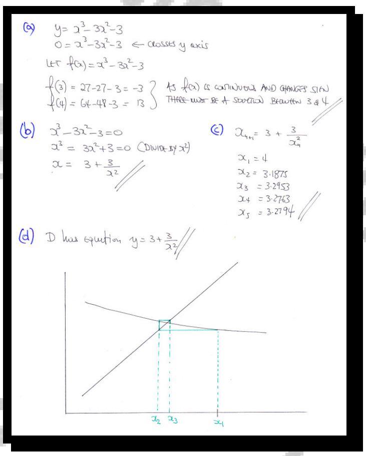


d) Write down the equation of D.

e) On a copy of the diagram draw a "staircase" or a "cob-web" pattern to show how the convergence to the root α is taking place, marking clearly the position of x_1, x_2 and x_3.

$\boxed{}$, $\boxed{x_1 = 3.1875, \quad x_2 = 3.2953, \quad x_3 = 3.2763, \quad x_4 = 3.2794}$,

$\boxed{D: y = 3 + \frac{3}{x^2}}$



Question 14 (***+)

$$x^3 - 1 - \frac{1}{x} = 0, \; x \neq 0.$$

a) By sketching two suitable graphs in the same diagram, show that the above equation has one positive root α and one negative root β.

The sketch must include the coordinates of the points where the curves meet the coordinate axes.

b) Explain why $\alpha > 1$.

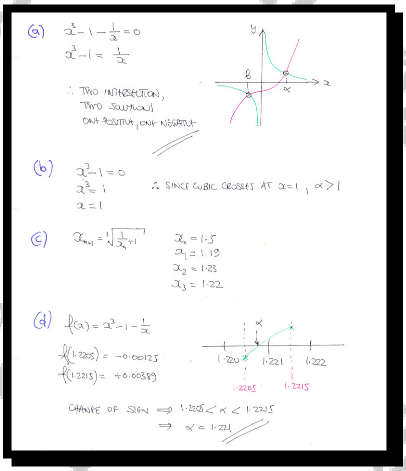
To find α the following iterative formula is used

$$x_{n+1} = \sqrt[3]{\frac{1}{x_n} + 1}, \; x_0 = 1.5.$$

c) Find, to 2 decimal places, the value of x_1, x_2 and x_3.

d) By considering the sign of an appropriate function $f(x)$ in a suitable interval, show clearly that $\alpha = 1.221$, correct to 3 decimal places.

$\boxed{}$, $\boxed{x_1 = 1.19, \; x_2 = 1.23, \; x_3 = 1.22}$



Question 15 (***+)

$$f(x) = x^4 + 3x - 1, \; x \in \mathbb{R}.$$

a) By sketching two suitable graphs in the same set of axes, determine the number of real roots of the equation $f(x) = 0$.

b) Show that the equation $f(x) = 0$ has a root α between 0 and 1.

The recurrence relation

$$x_{n+1} = \frac{1 - x_n^4}{3}$$

starting with $x_0 = 0.3$ is to be used to find α.

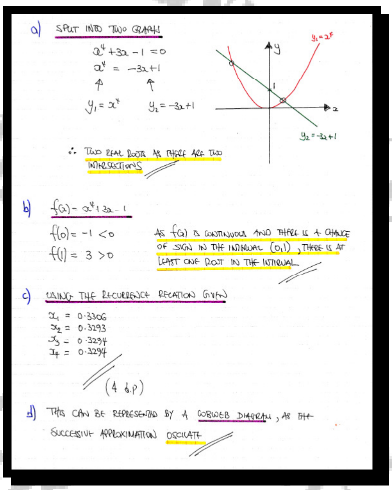
c) Find to 4 decimal places, the value of x_1, x_2, x_3 and x_4.

d) Explain whether the convergence to the root α, can be represented by a cobweb or a staircase diagram.

e) By considering the sign of $f(x)$ in a suitable interval, show that $\alpha = 0.32941$, correct to 5 decimal places.

$\boxed{}$, $\boxed{x_1 = 0.3306, \; x_2 = 0.3293, \; x_3 = 0.3294, \; x_4 = 0.3294}$,

$\boxed{\text{cobweb, because of oscillation}}$



a) SPLIT INTO TWO GRAPHS

$x^4 + 3x - 1 = 0$

$x^4 = -3x + 1$

$y_1 = x^4 \qquad y_2 = -3x + 1$

∴ TWO REAL ROOTS AS THERE ARE TWO INTERSECTIONS

b) $f(x) = x^4 + 3x - 1$

$f(0) = -1 < 0$

$f(1) = 3 > 0$

AS $f(x)$ IS CONTINUOUS AND THERE IS A CHANGE OF SIGN IN THE INTERVAL $(0,1)$, THERE IS AT LEAST ONE ROOT IN THE INTERVAL

c) USING THE RECURRENCE RELATION GIVEN

$x_1 = 0.3306$

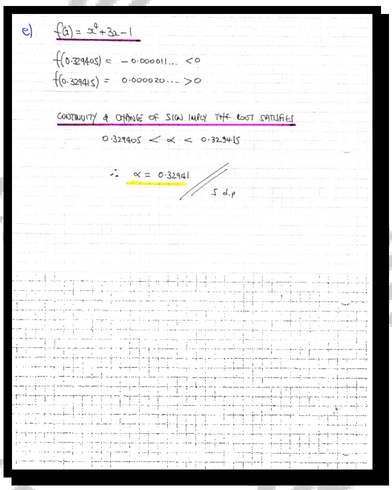
$x_2 = 0.3293$

$x_3 = 0.3294$

$x_4 = 0.3294$

(4 d.p)

d) THIS CAN BE REPRESENTED BY A COBWEB DIAGRAM, AS THE SUCCESSIVE APPROXIMATION OSCILLATE



e) $f(x) = x^4 + 3x - 1$

$f(0.329405) = -0.000011\ldots < 0$

$f(0.329415) = 0.000020\ldots > 0$

CONTINUITY & CHANGE OF SIGN IMPLY THE ROOT SATISFIES

$0.329405 < \alpha < 0.329415$

∴ $\alpha = 0.32941$ (5 d.p)

Question 16 (***+)

The curve with equation $y = 2^x$ intersects the straight line with equation $y = 3 - 2x$ at the point P, whose x coordinate is α.

a) Show clearly that …

i. … $0.5 < \alpha < 1$.

ii. … α is the solution of the equation

$$x = \frac{\ln(3-2x)}{\ln 2}.$$

An iterative formula based on the equation of part (**a**ii) is used to find α.

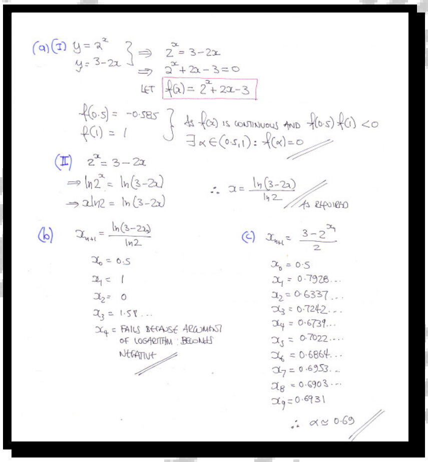
b) Starting with $x_0 = 0.5$, find the value of x_1, x_2 and x_3, explaining why a valid value of x_4 cannot be produced.

c) Use the iterative formula

$$x_{n+1} = \frac{3 - 2^{x_n}}{2}, \quad x_0 = 0.5,$$

with as many iterations as necessary, to determine the value of α correct to 2 decimal places.

$\boxed{}$, $\boxed{x_1 = 1, \quad x_2 = 0, \quad x_3 = 1.584}$, $\boxed{\alpha = 0.69}$



(a)(I) $\left.\begin{array}{l} y = 2^x \\ y = 3-2x \end{array}\right\} \Rightarrow 2^x = 3-2x$

$\Rightarrow 2^x + 2x - 3 = 0$

LET $\boxed{f(x) = 2^x + 2x - 3}$

$\left.\begin{array}{l} f(0.5) = -0.585 \\ f(1) = 1 \end{array}\right\}$ AS $f(x)$ IS CONTINUOUS AND $f(0.5)f(1) < 0$

$\exists \alpha \in (0.5, 1) : f(\alpha) = 0$

(II) $2^x = 3 - 2x$

$\Rightarrow \ln 2^x = \ln(3-2x)$

$\Rightarrow x\ln 2 = \ln(3-2x)$

$\therefore x = \frac{\ln(3-2x)}{\ln 2}$ AS REQUIRED

(b) $x_{n+1} = \frac{\ln(3-2x_n)}{\ln 2}$

$x_0 = 0.5$

$x_1 = 1$

$x_2 = 0$

$x_3 = 1.58\ldots$

x_4 = FAILS BECAUSE ARGUMENT OF LOGARITHM BECOMES NEGATIVE

(c) $x_{n+1} = \frac{3 - 2^{x_n}}{2}$

$x_0 = 0.5$

$x_1 = 0.7928\ldots$

$x_2 = 0.6337\ldots$

$x_3 = 0.7242\ldots$

$x_4 = 0.6739\ldots$

$x_5 = 0.7022\ldots$

$x_6 = 0.6864\ldots$

$x_7 = 0.6953\ldots$

$x_8 = 0.6903\ldots$

$x_9 = 0.6931$

$\therefore \alpha \simeq 0.69$

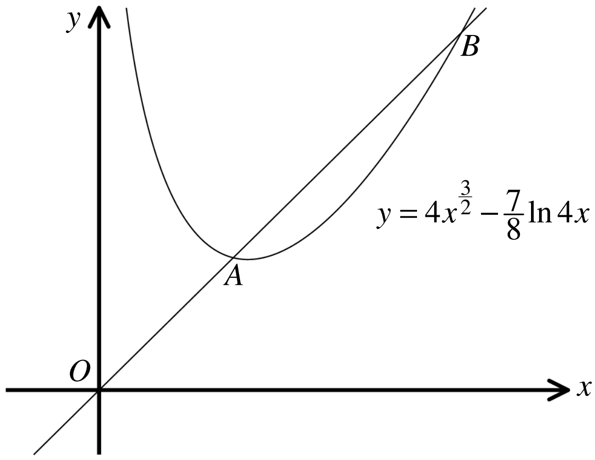
Question 17 **(***+)**

A curve C has equation

$$y = 4x^{\frac{3}{2}} - \frac{7}{8}\ln 4x, \quad x \in \mathbb{R}, \quad x > 0.$$

The point A is on C, where $x = \frac{1}{4}$.

a) Find an equation of the normal to the curve at A.



This normal meets the curve again at the point B, as shown in the figure above.

b) Show that the x coordinate of B satisfies the equation

$$x = \left(\frac{16x + 7\ln 4x}{32}\right)^{\frac{2}{3}}.$$

[continues overleaf]

[continued from overleaf]

The recurrence relation

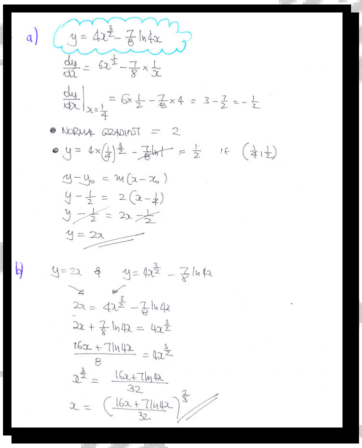
$$x_{n+1} = \left(\frac{16x_n + 7\ln 4x_n}{32}\right)^{\frac{2}{3}}, \quad x_0 = 0.7$$

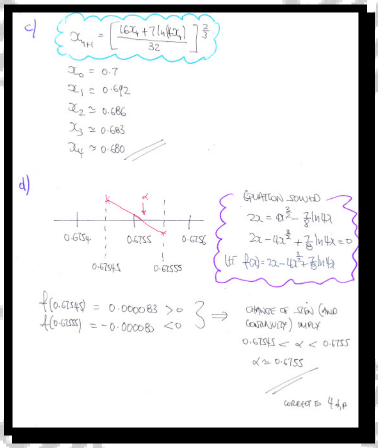
is to be used to find the x coordinate of B.

c) Find, to 3 decimal places, the value of x_1, x_2, x_3 and x_4.

d) Show that the x coordinate of B is 0.6755, correct to 4 decimal places.

$\boxed{y = 2x}$, $\boxed{x_1 = 0.692, \quad x_2 = 0.686, \quad x_3 = 0.683, \quad x_4 = 0.680}$





Question 18 (****)

The curve C has equation

$$y = x\cos x, \ 0 \le x \le \frac{\pi}{2}.$$

The curve has a single turning point at M.

a) Show that x coordinate of M is a solution of the equation

$$x = \arctan\left(\frac{1}{x}\right).$$

b) Show further that the equation

$$x = \arctan\left(\frac{1}{x}\right)$$

has root α between 0.8 and 1.

The iterative formula

$$x_{n+1} = \arctan\left(\frac{1}{x_n}\right) \text{ with } x_1 = 0.9$$

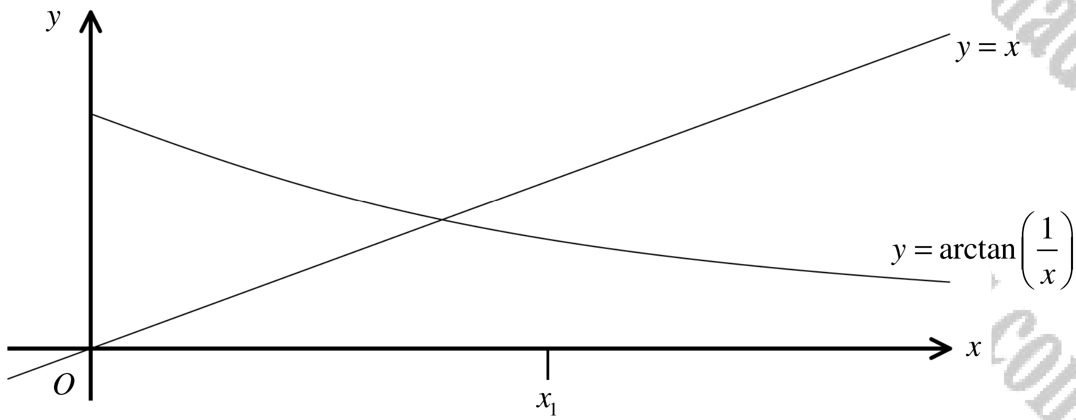
is to be used to find α.

c) Find, to 3 decimal places, the value of x_2, x_3 and x_4.

[continues overleaf]

[continued from overleaf]

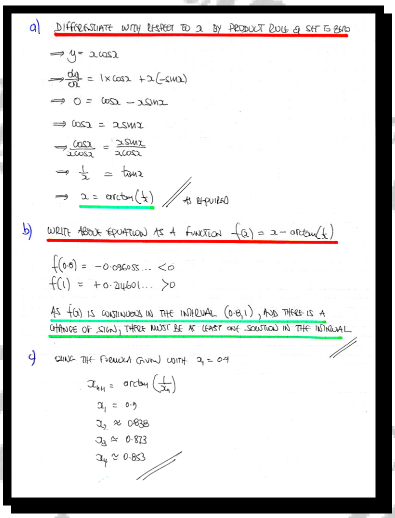
The diagram below shows the graphs of $y = x$ and $y = \arctan\left(\frac{1}{x}\right)$.



d) Use a copy of the above diagram to show how the convergence to the root α takes place, by constructing a staircase or cobweb pattern.

Indicate clearly the positions of x_2, x_3 and x_4.

$\boxed{\quad}$, $\boxed{x_2 = 0.838, \quad x_3 = 0.873, \quad x_4 = 0.853}$



a) DIFFERENTIATE WITH RESPECT TO x BY PRODUCT RULE & SET TO ZERO

$\Rightarrow y = x\cos x$

$\Rightarrow \frac{dy}{dx} = 1\times\cos x + x(-\sin x)$

$\Rightarrow 0 = \cos x - x\sin x$

$\Rightarrow \cos x = x\sin x$

$\Rightarrow \frac{\cos x}{x\cos x} = \frac{x\sin x}{x\cos x}$

$\Rightarrow \frac{1}{x} = \tan x$

$\Rightarrow x = \arctan\left(\frac{1}{x}\right)$ // AS REQUIRED

b) WRITE ABOVE EQUATION AS A FUNCTION $f(x) = x - \arctan\left(\frac{1}{x}\right)$

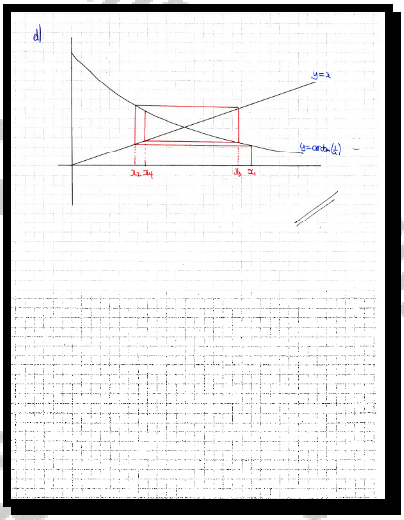
$f(0.8) = -0.096055\ldots < 0$
$f(1) = +0.214601\ldots > 0$

AS $f(x)$ IS CONTINUOUS IN THE INTERVAL $(0.8, 1)$, AND THERE IS A CHANGE OF SIGN, THERE MUST BE AT LEAST ONE SOLUTION IN THE INTERVAL //

c) USING THE FORMULA GIVEN WITH $x_1 = 0.9$

$x_{n+1} = \arctan\left(\frac{1}{x_n}\right)$

$x_1 = 0.9$
$x_2 \approx 0.838$
$x_3 \approx 0.873$
$x_4 \approx 0.853$ //



Question 19 (****)

$$f(x) \equiv \left|4e^{2x} - 28\right|, \; x \in \mathbb{R}.$$

a) Sketch the graph of $f(x)$.

The sketch must include the coordinates of any intersections with the coordinate axes and the equations of any asymptotes.

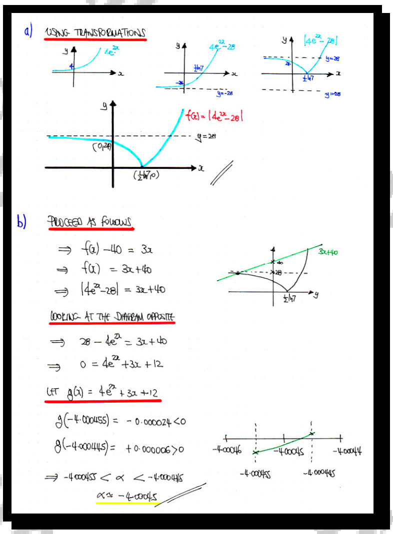
The equation $f(x) - 40 = 3x$ has a negative root α.

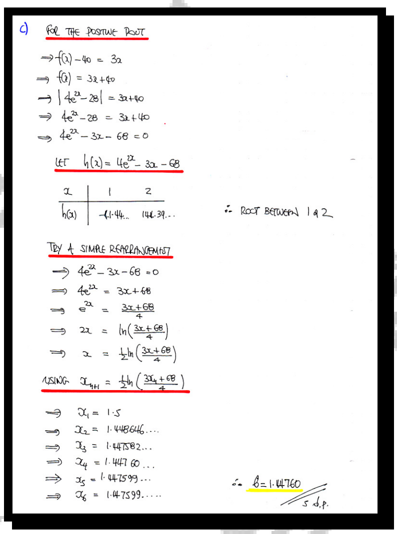
b) Show that $\alpha = -4.00045$, correct to 5 decimal places.

The equation $f(x) - 40 = 3x$ also has a positive root β.

c) Use a numerical method, based on an iterative method, to determine the value of β correct to 5 decimal places.

$\boxed{}$, $\boxed{\beta \approx 1.44760}$





Question 20 (****)

The curve C has equation

$$y = \frac{3x+1}{x^3 - x^2 + 5}.$$

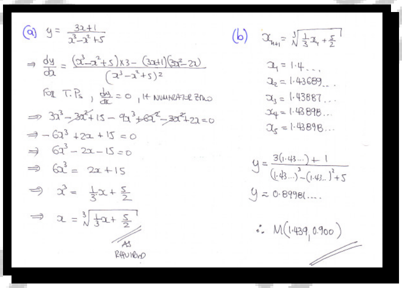
The curve has a single stationary point at M, with approximate coordinates $(1.4, 0.9)$.

a) Show that the x coordinate of M is a solution of the equation

$$x = \sqrt[3]{\frac{1}{3}x + \frac{5}{2}}.$$

b) By using an iterative formula based on the equation of part **(a)**, determine the coordinates of M correct to three decimal places.

$\boxed{}$, $\boxed{M(1.439, 0.900)}$


(a) $y = \frac{3x+1}{x^3-x^2+5}$

$\Rightarrow \frac{dy}{dx} = \frac{(x^3-x^2+5)\times 3 - (3x+1)(3x^2-2x)}{(x^3-x^2+5)^2}$

For T.Ps, $\frac{dy}{dx} = 0$, i.e. NUMERATOR ZERO

$\Rightarrow 3x^3 - \cancel{3x^2} + 15 - 9x^3 \cancel{+6x^2} - \cancel{3x^2} + 2x = 0$

$\Rightarrow -6x^3 + 2x + 15 = 0$

$\Rightarrow 6x^3 - 2x - 15 = 0$

$\Rightarrow 6x^3 = 2x + 15$

$\Rightarrow x^3 = \frac{1}{3}x + \frac{5}{2}$

$\Rightarrow x = \sqrt[3]{\frac{1}{3}x + \frac{5}{2}}$

AS REQUIRED

(b) $x_{n+1} = \sqrt[3]{\frac{1}{3}x_n + \frac{5}{2}}$

$x_1 = 1.4\ldots$

$x_2 = 1.43689\ldots$

$x_3 = 1.43887\ldots$

$x_4 = 1.43898\ldots$

$x_5 = 1.43898\ldots$

$y = \frac{3(1.43\ldots)+1}{(1.43\ldots)^3 - (1.43\ldots)^2 + 5}$

$y \approx 0.89981\ldots$

$\therefore M(1.439, 0.900)$

Question 21 (****)

The curves C_1 and C_2 have respective equations

$$y_1 = 3\arcsin(x-1) \quad \text{and} \quad y_1 = 2\arccos(x-1).$$

a) Sketch in the same set of axes the graph of C_1 and the graph of C_2.

The sketch must include the coordinates.

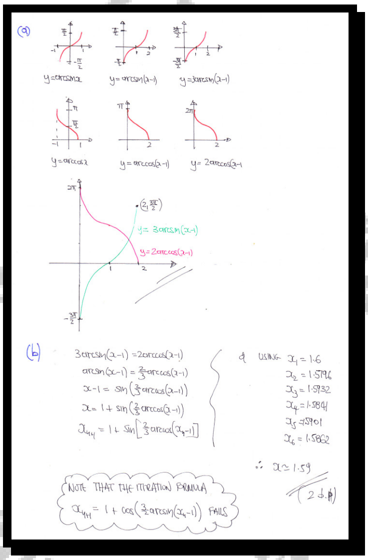
- … of any points where each of the graphs meet the coordinate axes.
- … of the endpoints of each of the graphs.

b) Use a suitable iteration formula of the form

$$x_{n+1} = f(x_n) \text{ with } x_1 = 1.6,$$

to find an approximate value for the x coordinate of the point of intersection between the graph of C_1 and the graph of C_2.

$\boxed{}$, $\boxed{x \approx 1.59}$



Question 22 (****)

At the point P which lies on the curve with equation

$$x = \ln\left(y^3 - 4y\right),$$

the gradient is 2.

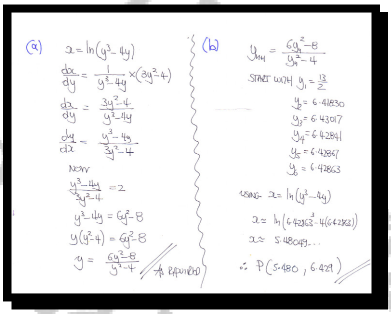
The point P is close to the point with coordinates $\left(\frac{11}{2}, \frac{13}{2}\right)$.

a) Show that the y coordinate of P is a solution of the equation

$$y = \frac{6y^2 - 8}{y^2 - 4}.$$

b) By using an iterative formula based on the equation of part **(a)**, determine the coordinates of P correct to three decimal places.

$\boxed{}$, $\boxed{P(5.480, 6.429)}$



(a) $x = \ln(y^3 - 4y)$

$\frac{dx}{dy} = \frac{1}{y^3 - 4y} \times (3y^2 - 4)$

$\frac{dx}{dy} = \frac{3y^2 - 4}{y^3 - 4y}$

$\frac{dy}{dx} = \frac{y^3 - 4y}{3y^2 - 4}$

NOW

$\frac{y^3 - 4y}{3y^2 - 4} = 2$

$y^3 - 4y = 6y^2 - 8$

$y(y^2 - 4) = 6y^2 - 8$

$y = \frac{6y^2 - 8}{y^2 - 4}$ // AS REQUIRED

(b) $y_{n+1} = \frac{6y_n^2 - 8}{y_n^2 - 4}$

START WITH $y_1 = \frac{13}{2}$

$y_2 = 6.41830$

$y_3 = 6.43017$

$y_4 = 6.42841$

$y_5 = 6.42867$

$y_6 = 6.42863$

USING $x = \ln(y^3 - 4y)$

$x \simeq \ln(6.42863^3 - 4(6.42863))$

$x \simeq 5.48049\ldots$

$\therefore P(5.480, 6.429)$ //

Question 23 (****)

At the point P which lies on the curve with equation

$$y = \frac{x}{y + \ln y},$$

the gradient is 2.

The point P is close to the point with coordinates $(-0.3, 0.3)$.

a) Show that the y coordinate of P is a solution of the equation

$$y = e^{-\frac{1}{2}(4y+1)}.$$

b) By using an iterative formula based on the equation of part **(a)**, determine the coordinates of P correct to three decimal places.

$\boxed{}$, $\boxed{P(-0.262, 0.320)}$

a) START BY REARRANGING THE EQUATION FOR x – THEN DIFFERENTIATE

$$\Rightarrow y = \frac{x}{y + \ln y}$$
$$\Rightarrow y^2 + y\ln y = x$$
$$\Rightarrow x = y^2 + y\ln y$$
$$\Rightarrow \frac{dx}{dy} = 2y + 1 \times \ln y + y \times \frac{1}{y}$$
$$\Rightarrow \frac{dx}{dy} = 2y + \ln y + 1$$
$$\Rightarrow \frac{dy}{dx} = \frac{1}{2y + \ln y + 1}$$

SOLVING $\frac{dy}{dx} = 2$ YIELDS

$$\Rightarrow 2 = \frac{1}{2y + \ln y + 1}$$
$$\Rightarrow 4y + 2\ln y + 2 = 1$$
$$\Rightarrow 4y + 2\ln y + 1 = 0$$
$$\Rightarrow 2\ln y = -1 - 4y$$
$$\Rightarrow \ln y = -\frac{1}{2} - 2y$$
$$\Rightarrow \ln y = -\frac{1}{2}(4y+1)$$

$$\therefore y = e^{-\frac{1}{2}(4y+1)}$$ AS REQUIRED

b) USING THE ITERATIVE FORMULA $y = e^{-\frac{1}{2}(4y+1)}$

STARTING WITH $y_1 = 0.3$

$y_2 = 0.33287\ldots$
$y_3 = 0.311691\ldots$
$y_4 = 0.325178\ldots$

THE CONVERGENCE IS BY OSCILLATION BUT VERY SLOW

$y_5 = 0.316524\ldots$
$y_6 = 0.32205\ldots$
$y_7 = 0.31851\ldots$
$y_8 = 0.32077\ldots$
$y_9 = 0.31932\ldots$
$y_{10} = 0.32025\ldots$
$y_{11} = 0.31965\ldots$
$y_{12} = 0.32003\ldots$
$y_{13} = 0.31979\ldots$
$y_{14} = 0.31995\ldots$
$y_{15} = 0.31985\ldots$

$\therefore y = 0.320$ (CORRECT TO 3 d.p)

USING $y = 0.3199$ IN $x = y^2 + y\ln y$ WE OBTAIN $x = -0.262$

$\therefore P(-0.262, 0.320)$

Question 24 (****)

A curve has equation

$$y = \frac{x+1}{x^3 + 2x + 1}.$$

It is given that the curve has a local maximum at the point M, whose approximate coordinates are $(-1.7, 0.1)$.

a) Show that the x coordinate of M is a solution of the equation

$$x = -\frac{3x^2+1}{2x^2}.$$

b) By using an iterative formula based on the equation of part **(a)**, determine the coordinates of M correct to three decimal places.

c) State, with a reason, whether the convergence taking place using the formula of part **(b)** is of the "cobweb" type or the "staircase" type.

$\boxed{}$, $\boxed{M(-1.678, 0.096)}$

a) START BY DIFFERENTIATION

$$y = \frac{x+1}{x^3+2x+1} \implies \frac{dy}{dx} = \frac{(x^3+2x+1)\times 1 - (x+1)(3x^2+2)}{(x^3+2x+1)^2}$$

$$\implies \frac{dy}{dx} = \frac{x^3+\cancel{2x}+1-(3x^3+3x^2+\cancel{2x}+2)}{(x^3+2x+1)^2}$$

$$\implies \frac{dy}{dx} = \frac{-2x^3-3x^2-1}{(x^3+2x+1)^2}$$

SOLVING FOR ZERO TO SEARCH FOR STATIONARY POINTS

$$\implies -2x^3-3x^2-1=0$$

$$\implies -3x^2-1 = 2x^3$$

$$\implies -\frac{3x^2+1}{2x^2} = x$$

AS REQUIRED

b) USING THE ABOVE FORMULA AS A RECURRENCE RELATION

$$x_{n+1} = -\frac{3x_n^2+1}{2x_n^2}, \quad x_1 = -1.7$$

$x_2 = -1.67301\ldots$
$x_3 = -1.67864\ldots$
$x_4 = -1.67744\ldots$
$x_5 = -1.67769$
$x_6 = -1.67764$
$x_7 = -1.67765$

NOW USING $x_7 = -1.67765\ldots$ WE OBTAIN

$$y = \frac{x_7+1}{(x_7)^3+2x_7+1} = 0.095753\ldots$$

$\therefore M(-1.678, 0.096)$

3 d.p

c) AS CONVERGENCE TAKES PLACE BY OSCILLATIONS, WE HAVE A "COBWEB" TYPE DIAGRAM

Question 25 (****)

It is required to find the approximate coordinates of the points of intersection between the graphs of

$$y_1 = 9 - x^2, \ x \in \mathbb{R} \qquad \text{and} \qquad y_2 = \ln(x-1), \ x \in \mathbb{R}, \ x > 1.$$

a) Show that the two graphs intersect at a single point P.

b) Explain why the x coordinate of P lies between 2 and 3.

The recurrence formula

$$x_{n+1} = \sqrt{9 - \ln(x_n - 1)},$$

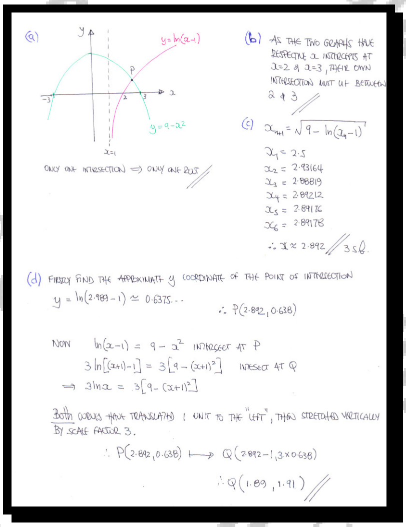
starting with a suitable value for x_1, is to be used to find the x coordinate of P.

c) Calculate the x coordinate of P, correct to three decimal places.

d) By considering two suitable transformations, determine correct to two decimal places the coordinates of the points of intersection between the graph of

$$y_3 = 3\left[9 - (x+1)^2\right], \ x \in \mathbb{R} \qquad \text{and} \qquad y_4 = 3\ln x, \ x \in \mathbb{R}, \ x > 0.$$

$\boxed{\text{C.S.}}$, $\boxed{x \approx 2.892}$, $\boxed{(1.89, 1.91)}$



Question 26 (****)

$$y = \arctan x, \ x \in \mathbb{R}.$$

a) By writing $y = \arctan x$ as $x = \tan y$ show that

$$\frac{dy}{dx} = \frac{1}{1+x^2}.$$

The curve C has equation

$$y = \arctan x - 4\ln\left(1+x^2\right) - 3x^2, \ x \in \mathbb{R}.$$

b) Show that the x coordinate of the stationary point of C is a root of the equation

$$6x^3 + 14x - 1 = 0.$$

c) Show that the above equation has a root α in the interval $(0,1)$.

The iterative formula

$$x_{n+1} = \frac{1-6x_n^3}{14} \quad \text{with } x_0 = 0,$$

is used to find this root.

d) Find, correct to 6 decimal places, the value of x_1, x_2 and x_3.

e) Hence write down the value of α, correct to 5 decimal places.

$\boxed{}$, $\boxed{x_1 = 0.071429, \quad x_2 = 0.071272, \quad x_3 = 0.071273}$, $\boxed{\alpha = 0.07127}$

(a) $y = \arctan x$

$\Rightarrow x = \tan y$

$\Rightarrow \frac{dx}{dy} = \sec^2 y$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{1+\tan^2 y}$

BUT $\boxed{x = \tan y}$

$\therefore \frac{dy}{dx} = \frac{1}{1+x^2}$ // AS REQUIRED

(b) $y = \arctan x - 4\ln(1+x^2) - 3x^2$

$\frac{dy}{dx} = \frac{1}{1+x^2} - 4\left(\frac{1}{1+x^2}\right)\times 2x - 6x$

$\Rightarrow \frac{dy}{dx} = \frac{1}{1+x^2} - \frac{8x}{1+x^2} - 6x$

$\Rightarrow \frac{dy}{dx} = \frac{1-8x}{1+x^2} - 6x$

FOR TURNING POINTS $\frac{dy}{dx} = 0$

$\Rightarrow 0 = \frac{1-8x}{1+x^2} - 6x$

$\Rightarrow 6x = \frac{1-8x}{1+x^2}$

$\Rightarrow 6x + 6x^3 = 1 - 8x$

$\therefore 6x^3 + 14x - 1 = 0$ // AS REQUIRED

(c) $f(x) = 6x^3 + 14x - 1$

$\left.\begin{array}{l} f(0) = -1 \\ f(1) = 19 \end{array}\right\}$ AS $f(x)$ IS CONTINUOUS AND CHANGES SIGN IN THE INTERVAL $(0,1)$, THERE MUST BE A ROOT IN THE INTERVAL //

(d) $x_{n+1} = \frac{1-6x_n^3}{14}$

$x_0 = 0$

$x_1 = 0.071429$

$x_2 = 0.071272$

$x_3 = 0.071273$ //

(e) $\alpha = 0.07127$ // (5 d.p)

Question 27 (****)

$$y = \arctan x, \quad x \in \mathbb{R}.$$

a) By writing $y = \arctan x$ as $x = \tan y$ show that

$$\frac{dy}{dx} = \frac{1}{1+x^2}.$$

The curve C has equation

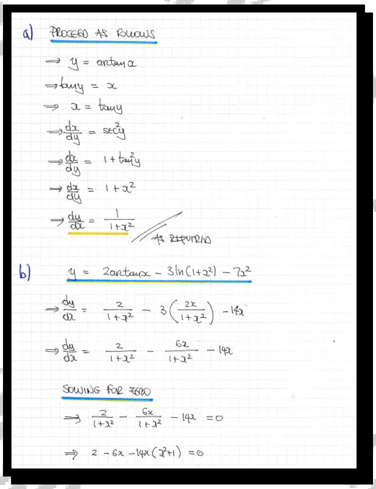
$$y = 2\arctan x - 3\ln\left(1+x^2\right) - 7x^2, \quad x \in \mathbb{R}.$$

b) Show that the x coordinate of the stationary point of C is a solution of the cubic equation

$$7x^3 + 10x - 1 = 0.$$

c) Hence show further that the x coordinate of the stationary point of C is 0.099314, correct to 6 decimal places.

$\boxed{}$, $\boxed{\text{proof}}$



a) PROCEED AS FOLLOWS

$\Rightarrow y = \arctan x$

$\Rightarrow \tan y = x$

$\Rightarrow x = \tan y$

$\Rightarrow \frac{dx}{dy} = \sec^2 y$

$\Rightarrow \frac{dx}{dy} = 1 + \tan^2 y$

$\Rightarrow \frac{dx}{dy} = 1 + x^2$

$\Rightarrow \frac{dy}{dx} = \frac{1}{1+x^2}$ // AS REQUIRED

b) $y = 2\arctan x - 3\ln(1+x^2) - 7x^2$

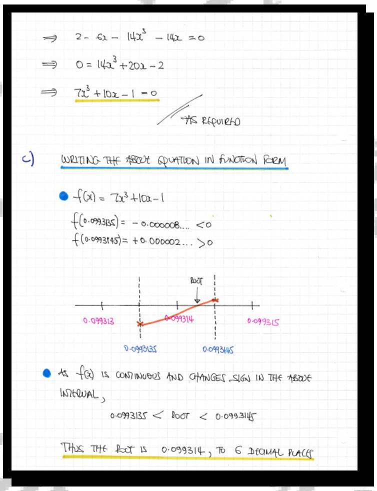
$\Rightarrow \frac{dy}{dx} = \frac{2}{1+x^2} - 3\left(\frac{2x}{1+x^2}\right) - 14x$

$\Rightarrow \frac{dy}{dx} = \frac{2}{1+x^2} - \frac{6x}{1+x^2} - 14x$

SOLVING FOR ZERO

$\Rightarrow \frac{2}{1+x^2} - \frac{6x}{1+x^2} - 14x = 0$

$\Rightarrow 2 - 6x - 14x(x^2+1) = 0$



$\Rightarrow 2 - 6x - 14x^3 - 14x = 0$

$\Rightarrow 0 = 14x^3 + 20x - 2$

$\Rightarrow 7x^3 + 10x - 1 = 0$ // AS REQUIRED

c) WRITING THE ABOVE EQUATION IN FUNCTION FORM

- $f(x) = 7x^3 + 10x - 1$

  $f(0.0993135) = -0.000008\ldots < 0$

  $f(0.0993145) = +0.000002\ldots > 0$

- AS $f(x)$ IS CONTINUOUS AND CHANGES SIGN IN THE ABOVE INTERVAL,

  $0.0993135 < \text{ROOT} < 0.0993145$

THUS THE ROOT IS 0.099314, TO 6 DECIMAL PLACES

Question 28 (****)

$$y = \arcsin x, \quad -1 \le x \le 1.$$

a) By writing $y = \arcsin x$ as $x = \sin y$ show that

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}.$$

The curve C has equation

$$y = 3\arcsin x - 4x^{\frac{3}{2}} + 5, \quad 0 \le x \le 1.$$

b) Show that the x coordinates of the stationary points of C are the solutions of the equation

$$4x^3 - 4x + 1 = 0.$$

c) Show that the above equation has a root α in the interval $(0, 0.5)$.

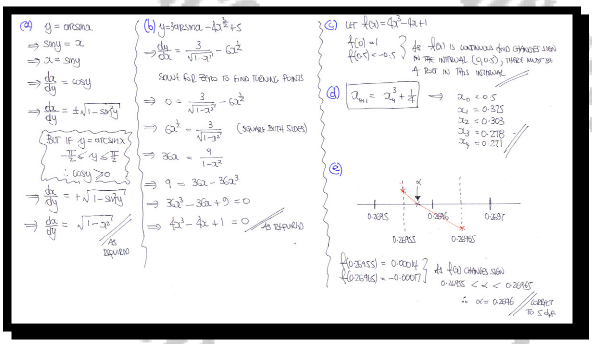
The root α can be found by using the iterative formula

$$x_{n+1} = x_n^3 + \frac{1}{4}, \quad \text{with } x_0 = 0.5.$$

d) Find, correct to 3 decimal places, the value of x_1, x_2, x_3 and x_4.

e) By considering the sign of an appropriate function $f(x)$ in a suitable interval, show that $\alpha = 0.2696$, correct to 4 decimal places.

$\boxed{}$, $\boxed{x_1 = 0.375, \quad x_2 = 0.303, \quad x_3 = 0.278, \quad x_4 = 0.271}$, $\boxed{\alpha = 0.07127}$



(a) $y = \arcsin x$
$\Rightarrow \sin y = x$
$\Rightarrow x = \sin y$
$\Rightarrow \frac{dx}{dy} = \cos y$
$\Rightarrow \frac{dx}{dy} = \pm\sqrt{1-\sin^2 y}$
BUT IF $y = \arcsin x$
$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$\therefore \cos y \ge 0$
$\Rightarrow \frac{dx}{dy} = +\sqrt{1-\sin^2 y}$
$\Rightarrow \frac{dx}{dy} = \sqrt{1-x^2}$ // AS REQUIRED

(b) $y = 3\arcsin x - 4x^{\frac{3}{2}} + 5$
$\Rightarrow \frac{dy}{dx} = \frac{3}{\sqrt{1-x^2}} - 6x^{\frac{1}{2}}$
SOLVE FOR ZERO TO FIND TURNING POINTS
$\Rightarrow 0 = \frac{3}{\sqrt{1-x^2}} - 6x^{\frac{1}{2}}$
$\Rightarrow 6x^{\frac{1}{2}} = \frac{3}{\sqrt{1-x^2}}$ (SQUARE BOTH SIDES)
$\Rightarrow 36x = \frac{9}{1-x^2}$
$\Rightarrow 9 = 36x - 36x^3$
$\Rightarrow 36x^3 - 36x + 9 = 0$
$\Rightarrow 4x^3 - 4x + 1 = 0$ // AS REQUIRED

(c) LET $f(x) = 4x^3 - 4x + 1$
$f(0) = 1$
$f(0.5) = -0.5$
AS $f(x)$ IS CONTINUOUS AND CHANGES SIGN IN THE INTERVAL $(0, 0.5)$, THERE MUST BE A ROOT IN THIS INTERVAL //

(d) $\boxed{x_{n+1} = x_n^3 + \frac{1}{4}}$ $\Rightarrow$
$x_0 = 0.5$
$x_1 = 0.375$
$x_2 = 0.303$
$x_3 = 0.278$
$x_4 = 0.271$ //

(e)
α
0.2695
0.2696
0.2697
0.26955
0.26965

$f(0.26955) = 0.00014$
$f(0.26965) = -0.00017$
AS $f(x)$ CHANGES SIGN
$0.26955 < \alpha < 0.26965$
$\therefore \alpha = 0.2696$ // CORRECT TO 5 d.p.

Question 29 (****)

It is known that the cubic equation

$$x^3 - 2x = 5, \quad x \in \mathbb{R},$$

has a single real solution α, which is close to 2.1.

Four iterative formulas based on rearrangements of this equation are being considered for their suitability to approximate the value of α to greater accuracy.

- $x_{n+1} = \frac{1}{2}\left(x_n^3 - 5\right)$
- $x_{n+1} = \sqrt[3]{2x_n + 5}$
- $x_{n+1} = \dfrac{5}{x_n^2 - 2}$
- $x_{n+1} = \sqrt{2 + \dfrac{5}{x_n}}$

a) Use a differentiation method, and **without** carrying any direct iterations, briefly describe the suitability of these four formulas.

In these descriptions you must make a reference to rates of convergence or divergence, and cobweb or staircase diagrams.

b) Use one of these four formulas to approximate the value of α, correct to 6 decimal places

$\boxed{}$, $\boxed{\alpha \approx 2.094551}$

WRITE EACH FORMULA IN y NOTATION & DIFFERENTIATE

- $y = \frac{1}{2}(x^3 - 5)$

  $\frac{dy}{dx} = \frac{3}{2}x^2$

- $y = \left(2 + \frac{5}{x}\right)^{\frac{1}{2}}$

  $\frac{dy}{dx} = \frac{1}{2}\left(2 + \frac{5}{x}\right)^{-\frac{1}{2}} \times \left(-\frac{5}{x^2}\right)$

- $y = (2x+5)^{\frac{1}{3}}$

  $\frac{dy}{dx} = \frac{2}{3}(2x+5)^{-\frac{2}{3}}$

- $y = 5(x^2-2)^{-1}$

  $\frac{dy}{dx} = -5(x^2-2)^{-2}(2x)$

  $\frac{dy}{dx} = \frac{-10x}{(x^2-2)^2}$

NEXT EVALUATE THESE DERIVATIVES IN THE NEIGHBOURHOOD OF $x = 2.1$

$\frac{dy}{dx} = 3x^2 \quad \left.\frac{dy}{dx}\right|_{x=2.1} = +6.615 > 1$ RAPIDLY DIVERGES WITHOUT OSCILLATION (STAIRCASE AWAY FROM α)

$\frac{dy}{dx} = \frac{2}{3}(2x+5)^{-\frac{2}{3}} \quad \left.\frac{dy}{dx}\right|_{x=2.1} = +0.1518\ldots$ RAPIDLY CONVERGES WITHOUT OSCILLATION AS THIS FIGURE IS BETWEEN 0 & 1 (STAIRCASE TOWARDS α)

$\frac{dy}{dx} = \frac{-10x}{(x^2-2)^2} \quad \left.\frac{dy}{dx}\right|_{x=2.1} = -3.6156 < -1$ DIVERGES WITH OSCILLATION AS THIS FIGURE IS BELOW -1 (COBWEB AWAY FROM α)

$\frac{dy}{dx} = \frac{1}{2}\left(2+\frac{5}{x}\right)^{-\frac{1}{2}} \times \left(-\frac{5}{x^2}\right) \quad \left.\frac{dy}{dx}\right|_{x=2.1} = -0.2708$ (RAPIDLY) CONVERGES WITH OSCILLATION AS THIS FIGURE IS VERY CLOSE TO 0 (COBWEB TOWARDS α)

QUICK SUMMARY FOR THE VALUE OF THE DERIVATIVE CLOSE TO THE ROOT

$-1 \qquad 0 \qquad 1$

NEGATIVE $\Rightarrow$ COBWEB | POSITIVE $\Rightarrow$ STAIRCASE

DIVERGES | CONVERGES | CONVERGES | DIVERGES

PICKING THE FORMULA WHICH PRODUCED THE CLOSEST TO ZERO

$x_{n+1} = (2x_n + 5)^{\frac{1}{3}}$

$x_1 = 2.1$

$x_2 = 2.095379106$

$x_3 = 2.094677239$

$x_4 = 2.094570591$

$x_5 = 2.094554385$

$x_6 = 2.094551923$

$x_7 = 2.094551549$

$x_8 = 2.094551492$

$\therefore \alpha \approx 2.094551$ // 6 d.p

Question 30 (****+)

A function f is defined in the largest real domain by the equation

$$f(x) \equiv \frac{50x^2 - 142x + 95}{2x - 5}.$$

a) State the domain of f.

b) Evaluate $f(1)$, $f(2)$ and $f(3)$, and hence briefly discuss the number of possible intersections of f with the coordinate axes, ...

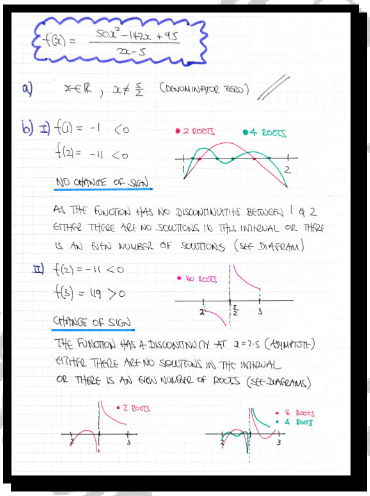
i. ... in the interval $[1,2]$.

ii. ... in the interval $[2,3]$.

c) Express $f(x)$ in the form $Ax + B + \frac{C}{2x-5}$, where A, B and C are constants.

d) Calculate, correct to 3 decimal places, the x coordinates of the stationary points of f.

$\boxed{x \in \mathbb{R},\ x \neq \frac{5}{2}}$, $\boxed{f(x) \equiv 25x - 8.5 + \frac{52.5}{2x-5}}$, $\boxed{x \approx 3.525,\quad x \approx 1.475}$



$f(x) = \frac{50x^2 - 142x + 95}{2x-5}$

a) $x \in \mathbb{R}$, $x \neq \frac{5}{2}$ (DENOMINATOR ZERO)

b) I) $f(1) = -1 < 0$

$f(2) = -11 < 0$

NO CHANGE OF SIGN

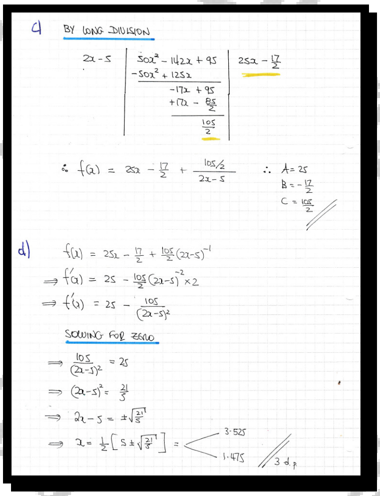
AS THE FUNCTION HAS NO DISCONTINUITIES BETWEEN 1 & 2 EITHER THERE ARE NO SOLUTIONS IN THIS INTERVAL OR THERE IS AN EVEN NUMBER OF SOLUTIONS (SEE DIAGRAM)

II) $f(2) = -11 < 0$

$f(3) = 119 > 0$

CHANGE OF SIGN

THE FUNCTION HAS A DISCONTINUITY AT $x = 2.5$ (ASYMPTOTE) EITHER THERE ARE NO SOLUTIONS IN THE INTERVAL OR THERE IS AN EVEN NUMBER OF ROOTS (SEE DIAGRAMS)



c) BY LONG DIVISION

$$\begin{array}{r|l|l} 2x-5 & 50x^2 - 142x + 95 & 25x - \frac{17}{2} \\ & -50x^2 + 125x & \\ \hline & -17x + 95 & \\ & +17x - \frac{85}{2} & \\ \hline & \frac{105}{2} & \end{array}$$

$\therefore f(x) = 25x - \frac{17}{2} + \frac{105/2}{2x-5}$ $\therefore A = 25$, $B = -\frac{17}{2}$, $C = \frac{105}{2}$

d) $f(x) = 25x - \frac{17}{2} + \frac{105}{2}(2x-5)^{-1}$

$\Rightarrow f'(x) = 25 - \frac{105}{2}(2x-5)^{-2} \times 2$

$\Rightarrow f'(x) = 25 - \frac{105}{(2x-5)^2}$

SOLVING FOR ZERO

$\Rightarrow \frac{105}{(2x-5)^2} = 25$

$\Rightarrow (2x-5)^2 = \frac{21}{5}$

$\Rightarrow 2x - 5 = \pm\sqrt{\frac{21}{5}}$

$\Rightarrow x = \frac{1}{2}\left[5 \pm \sqrt{\frac{21}{5}}\right] = 3.525,\ 1.475$ 3 d.p.

Question 31 (****+)

$$y = \arcsin x, \ -1 \le x \le 1.$$

a) By writing $y = \arcsin x$ as $x = \sin y$ show that

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}.$$

The curve C has equation

$$y = 2\arcsin x - 4x^{\frac{3}{2}}, \ 0 \le x \le 1.$$

b) Show that the x coordinates of the stationary points of C are the solutions of the equation

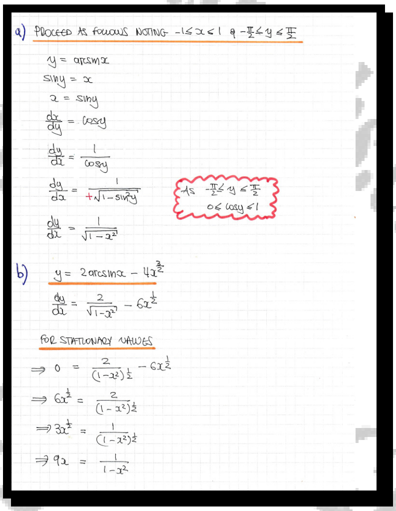
$$9x^3 - 9x + 1 = 0.$$

c) Show further that one of the roots, α, of the equation of part **(b)** is 0.9390, correct to 4 decimal places.

It is further given that the equation of part **(b)** has 2 more real roots, $\beta \approx -1.0515$, and γ.

d) Determine the value of γ, correct to 3 places.

$\boxed{}$, $\boxed{\gamma \approx 0.112}$



a) PROCEED AS FOLLOWS NOTING $-1 \le x \le 1$ & $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$y = \arcsin x$

$\sin y = x$

$x = \sin y$

$\frac{dx}{dy} = \cos y$

$\frac{dy}{dx} = \frac{1}{\cos y}$

$\frac{dy}{dx} = \frac{1}{+\sqrt{1-\sin^2 y}}$

AS $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, $0 \le \cos y \le 1$

$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

b) $y = 2\arcsin x - 4x^{\frac{3}{2}}$

$\frac{dy}{dx} = \frac{2}{\sqrt{1-x^2}} - 6x^{\frac{1}{2}}$

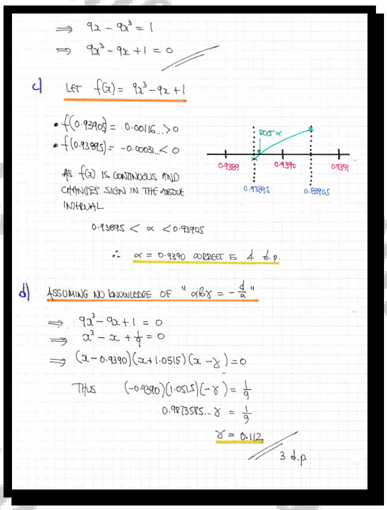
FOR STATIONARY VALUES

$\Rightarrow 0 = \frac{2}{(1-x^2)^{\frac{1}{2}}} - 6x^{\frac{1}{2}}$

$\Rightarrow 6x^{\frac{1}{2}} = \frac{2}{(1-x^2)^{\frac{1}{2}}}$

$\Rightarrow 3x^{\frac{1}{2}} = \frac{1}{(1-x^2)^{\frac{1}{2}}}$

$\Rightarrow 9x = \frac{1}{1-x^2}$



$\Rightarrow 9x - 9x^3 = 1$

$\Rightarrow 9x^3 - 9x + 1 = 0$

c) LET $f(x) = 9x^3 - 9x + 1$

- $f(0.93905) = 0.00116\ldots > 0$
- $f(0.93895) = -0.00031\ldots < 0$

AS $f(x)$ IS CONTINUOUS AND CHANGES SIGN IN THE ABOVE INTERVAL

$0.93895 < \alpha < 0.93905$

$\therefore \alpha = 0.9390$ CORRECT TO 4 d.p.

d) ASSUMING NO KNOWLEDGE OF "$\alpha\beta\gamma = -\frac{d}{a}$"

$\Rightarrow 9x^3 - 9x + 1 = 0$

$\Rightarrow x^3 - x + \frac{1}{9} = 0$

$\Rightarrow (x - 0.9390)(x + 1.0515)(x - \gamma) = 0$

THUS $(-0.9390)(1.0515)(-\gamma) = \frac{1}{9}$

$0.9873585\ldots\gamma = \frac{1}{9}$

$\gamma = 0.112$ 3 d.p.

Question 32 (****+)

The curve C has equation

$$f(x) \equiv 3x^4 + 8x^3 + 3x^2 - 12x - 6, \quad x \in \mathbb{R}.$$

The curve has a single stationary point whose x coordinate lies in the interval $[n, n+1]$, where $n \in \mathbb{Z}$.

a) Determine with full justification the value of n.

A suitable equation is rearranged to produced three recurrence relations, each of which may be used to find the x coordinate of the stationary point of C.

These recurrence relations, all starting with $x_0 = \frac{1}{2}n$ are shown below.

$$\textbf{(i)}\ x_{n+1} = \frac{2}{2x_n^{\ 2} + 4x_n + 1} \qquad \textbf{(ii)}\ x_{n+1} = \frac{1}{x_n^{\ 2}} - \frac{1}{2x_n} - 2 \qquad \textbf{(iii)}\ x_{n+1} = \sqrt{\frac{2 - x_n}{4 + 2x_n}}$$

b) Use a differentiation method, to investigate the result in attempting to find an approximate value for the x coordinate of the stationary point of $f(x)$, with each of these three recurrence relations.

The method must include ...

- ... whether the attempt is successful
- ... whether the convergence or divergence is a "cobweb" case or a "staircase" case.
- ... which recurrence relation converges at the fastest rate.

You may not answer part (**b**), by simply generating sequences.

$\boxed{\text{MI}}$, $\boxed{[0,1]}$

a) START BY DIFFERENTIATION TO LOCATE THE STATIONARY POINT

$f(x) = 3x^4 + 8x^3 + 3x^2 - 12x - 6$

$f'(x) = 12x^3 + 24x^2 + 6x - 12$

$f'(x) = 6(2x^3 + 4x^2 + x - 2)$

BY INSPECTION

$f'(0) = -12 < 0$

$f'(1) = 30 > 0$

AS $f'(x)$ IS CONTINUOUS AND CHANGES SIGN IN THE INTERVAL $[0,1]$ THERE IS AT LEAST A ROOT OF $f'(x) = 0$ IN THIS INTERVAL

b) INVESTIGATE EACH OF THE THREE RELATIONS USING THE MIDPOINT OF THE INTERVAL $x = 0.5$

- $x_{n+1} = \frac{2}{2x_n^2 + 4x_n + 1}$, $\frac{d}{dx}\left[2(2x^2+4x+1)^{-1}\right] = \frac{-2(4x+4)}{(2x^2+4x+1)^2}$

  EVALUATE AT $x = 0.5$ GIVES $-\frac{48}{49}$

- $x_{n+1} = \frac{1}{x_n^2} - \frac{1}{2x_n} - 2$, $\frac{d}{dx}\left[\frac{1}{x^2} - \frac{1}{2x} - 2\right] = -\frac{2}{x^3} + \frac{1}{2x^2}$

  EVALUATE AT $x = 0.5$ GIVES -14

- $x_{n+1} = \sqrt{\frac{2 - x_n}{4 + 2x_n}}$, $\frac{d}{dx}\left[(2-x)^{\frac{1}{2}}(4+2x)^{-\frac{1}{2}}\right] = \dots$

$\dots = -\frac{1}{2}(2-x)^{-\frac{1}{2}}(4+2x)^{-\frac{1}{2}} - (2-x)^{\frac{1}{2}}(4+2x)^{-\frac{3}{2}}$

EVALUATE AT $x = 0.5$

$= -\frac{1}{2}\sqrt{\frac{2}{3}}\frac{1}{\sqrt{5}} - \sqrt{\frac{3}{2}} \times \frac{1}{5\sqrt{5}}$

$= -\sqrt{\frac{1}{30}} - \sqrt{\frac{3}{250}}$

$= -0.29211$

HENCE WE DEDUCE

- $x_{n+1} = \frac{1}{x_n^2} - \frac{1}{2x_n} - 2$ DIVERGES BY OSCILLATION (COBWEB) AS $|-14| > 1$ & NEGATIVE. RATE OF DIVERGENCE WILL BE RAPID ($14 >> 1$)
- $x_{n+1} = \frac{2}{2x_n^2 + 4x_n + 1}$ CONVERGES BY OSCILLATION (COBWEB) AS $\left|-\frac{48}{49}\right| < 1$ & NEGATIVE. SLOW CONVERGING AS IT IS ALMOST 1
- $x_{n+1} = \sqrt{\frac{2 - x_n}{4 + 2x_n}}$ CONVERGES BY OSCILLATION (COBWEB) AS $|-0.292..| < 1$ & NEGATIVE. CONVERGES FAST AS 0.292 IS CLOSER TO ZERO THAN TO 1

Question 33 (****+)

The curve C has equation

$$y = \sqrt{e^{2x} - 2x}\,, \quad e^{2x} > 2x\,.$$

The tangent to C at the point P, where $x = p$, passes through the origin.

a) Show that $x = p$ is a solution of the equation

$$(1-x)e^{2x} = x\,.$$

b) Show further that the equation of part **(a)** has a root between 0.8 and 1.

The iterative formula

$$x_{n+1} = 1 - x_n e^{-2x_n}$$

with $x_0 = 0.8$ is used to find this root.

c) Find, correct to 3 decimal places, the value of x_1, x_2, x_3 and x_4.

d) Hence show that the value of p is 0.8439, correct to 4 decimal places.

$\boxed{}$, $\boxed{x_1 = 0.838, \quad x_2 = 0.843, \quad x_3 = 0.844, \quad x_4 = 0.844}$

(a) $y = (e^{2x} - 2x)^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2}(e^{2x} - 2x)^{-\frac{1}{2}}(2e^{2x} - 2) = \frac{e^{2x} - 1}{\sqrt{e^{2x} - 2x}}$

$\left.\frac{dy}{dx}\right|_{x=p} = \frac{e^{2p} - 1}{\sqrt{e^{2p} - 2p}}$

when $x = p$ $\quad y = \sqrt{e^{2p} - 2p}$ $\quad$ i.e. $\left(p, \sqrt{e^{2p} - 2p}\right)$

• EQUATION OF TANGENT

$y - \sqrt{e^{2p} - 2p} = \frac{e^{2p} - 1}{\sqrt{e^{2p} - 2p}}(x - p)$

• TANGENT PASSES THROUGH $(0,0)$

$\Rightarrow -\sqrt{e^{2p} - 2p} = \frac{e^{2p} - 1}{\sqrt{e^{2p} - 2p}}(-p)$

$\Rightarrow -(e^{2p} - 2p) = -pe^{2p} + p$

$\Rightarrow -e^{2p} + 2p = -pe^{2p} + p$

$\Rightarrow p = e^{2p} - pe^{2p}$

$\Rightarrow p = e^{2p}(1 - p)$ // AS REQUIRED

(b) $(1-x)e^{2x} = x$

$(1-x)e^{2x} - x = 0$

$\boxed{\text{LET } f(x) = (1-x)e^{2x} - x}$

$f(0.8) = 0.191$

$f(1) = -1$

AS $f(x)$ IS CONTINUOUS & CHANGES SIGN, THERE MUST BE A ROOT BETWEEN 0.8 & 1

(c) $x_{n+1} = 1 - x_n e^{-2x_n}$

$x_0 = 0.8$

$x_1 = 0.838$

$x_2 = 0.843$

$x_3 = 0.844$

$x_4 = 0.844$ //

(d) $0.84385 < 0.8439 < 0.84395$

ROOT

0.8438 $\quad$ 0.8439 $\quad$ 0.8440

$f(0.84385) = 0.00046$

$f(0.84395) = -0.000014$

$\therefore p = 0.8439$ to 4 d.p //

Question 34 (*****)

The point P lies on the curve C with equation

$$y = \sqrt{1 + 2e^{2x^2}} \ , \ x \in \mathbb{R} \ .$$

Given that the tangent to C at P passes through the origin, determine the coordinates of P, correct to 3 significant figures.

$\boxed{}$, $\boxed{P(\pm 0.761, 2.71)}$

- FIRSTLY LET US NOTE THAT $y = \sqrt{1+2e^{2x^2}}$ IS EVEN, SO THERE ARE TWO SUCH POINTS WITH IDENTICAL y CO-ORDS & "OPPOSITE" x CO-ORDINATES
- LET THE POINT P HAVE CO-ORDINATES $\left(p, \sqrt{1+2e^{2p^2}}\right)$
- DIFFERENTIATE AND FIND THE EQUATION OF THE TANGENT AT P

$\Rightarrow y = \left(1+2e^{2x^2}\right)^{\frac{1}{2}}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(1+2e^{2x^2}\right)^{-\frac{1}{2}} \times 2e^{2x^2} \times 4x$

$\Rightarrow \frac{dy}{dx} = \frac{4xe^{2x^2}}{\sqrt{1+2e^{2x^2}}}$

$\Rightarrow \left.\frac{dy}{dx}\right|_P = \frac{4pe^{2p^2}}{\sqrt{1+2e^{2p^2}}}$

EQUATION OF TANGENT IS

$\Rightarrow y - \sqrt{1+2e^{2p^2}} = \frac{4pe^{2p^2}}{\sqrt{1+2e^{2p^2}}}(x-p)$

BUT TANGENT PASSES THROUGH THE ORIGIN $(0,0)$

$\Rightarrow -\sqrt{1+2e^{2p^2}} = \frac{4pe^{2p^2}}{\sqrt{1+2e^{2p^2}}}(-p)$

$\Rightarrow 1+2e^{2p^2} = 4p^2e^{2p^2}$

$\Rightarrow \boxed{e^{-2p^2} + 2 = 4p^2}$

- REARRANGE THE EQUATION AS

$f(p) = 4p^2 - e^{-2p^2} - 2$ & $p = +\frac{1}{2}\sqrt{2+e^{-2p^2}}$

$f(0) = -3 < 0$

$f(1) = 2 - e^{-1} > 0$

- AS CURVE IS CONTINUOUS, START SAY WITH $x = p = 0.5$

$\boxed{x_{n+1} = \frac{1}{2}\left(2+e^{-2x^2}\right)^{\frac{1}{2}}}$

$x_1 = 0.5$

$x_2 = 0.80724\ldots$

$x_3 = 0.75360\ldots$

$x_4 = 0.76177\ldots$

$x_5 = 0.76048\ldots$

$x_6 = 0.76068$

$x_7 = 0.76065$

$\therefore x \approx 0.761$ TO 3 S.F

AND USING $y = \sqrt{1+2e^{2x^2}}$ WE OBTAIN $y \approx 2.71$ TO 3 S.F

$\therefore P(\pm 0.761, 2.71)$

Question 35 (*****)

A curve C is defined in the largest real domain by the equation

$$y = \log_x 2 .$$

a) Sketch a detailed graph of C.

The point P, where $x = 2$ lies on C.

The normal to C at P meets C again at the point Q.

b) Show that the x coordinate of Q is a solution of the equation

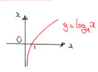
$$[1 + x\ln 4 - \ln 16]\ln x = \ln 2 .$$

c) Use an iterative formula of the form $x_{n+1} = e^{f(x_n)}$, with a suitable starting value, to find the coordinates of Q, correct to 3 decimal places.

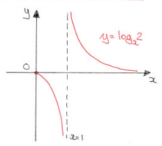
$\boxed{\text{MP}}$, $\boxed{Q(0.518, -1.054)}$

a) TO SKETCH, WE EMPLOY THE RULES OF LOGARITHMS

$$y = \log_x 2 = \frac{1}{\log_2 x}$$



"RECIPROCATING" THE GRAPH WE OBTAIN



b) DIFFERENTIATING USING MORE LOGARITHM RULES

$$y = \log_x 2 = \frac{\log_e 2}{\log_e x} = \frac{\ln 2}{\ln x} = (\ln 2)(\ln x)^{-1}$$

$$\frac{dy}{dx} = -(\ln 2)(\ln x)^{-2} \times \frac{1}{x} = -\frac{\ln 2}{x(\ln x)^2}$$

$$\left.\frac{dy}{dx}\right|_{x=2} = -\frac{\ln 2}{2(\ln 2)^2} = -\frac{1}{2\ln 2} = -\frac{1}{\ln 4}$$

NORMAL GRADIENT IS $\ln 4$ & THE POINT HAS FULL CO-ORDINATES $(2,1)$

EQUATION OF THE NORMAL IS GIVEN BY

$$y - 1 = (\ln 4)(x - 2)$$

SOLVING SIMULTANEOUSLY WITH THE EQUATION OF THE CURVE USING THE FORM

$$y = \frac{\ln 2}{\ln x}$$

$$\Rightarrow \frac{\ln 2}{\ln x} - 1 = x\ln 4 - 2\ln 4$$

$$\Rightarrow \ln 2 - \ln x = x\ln x\ln 4 - 2\ln x\ln 4$$

$$\Rightarrow \ln 2 = \ln x + x\ln x\ln 4 - 2\ln x\ln 4$$

$$\Rightarrow \ln 2 = \ln x[1 + x\ln 4 - 2\ln 4]$$

$$\Rightarrow [1 + x\ln 4 - \ln 16]\ln x = \ln 2$$

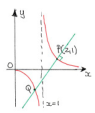
c) "EXPONENTIATING" GIVES

$$\Rightarrow \ln x = \frac{\ln 2}{1 + x\ln 4 - \ln 16}$$

$$\Rightarrow x = e^{\frac{\ln 2}{1 + x\ln 4 - \ln 16}}$$

$$\Rightarrow x_{n+1} = e^{\frac{\ln 2}{1 + x_n\ln 4 - \ln 16}}$$

STARTING SAY WITH $x_1 = 0.5$



$x_1 = 0.5$
$x_2 = 0.526168$
$x_3 = 0.514549$
$x_4 = 0.519774$
$x_5 = 0.517437$
$x_6 = 0.518485$
$x_7 = 0.518016$
$x_8 = 0.518226$
$\vdots$

$\therefore x \simeq 0.518$

AND USING

$$y = \frac{\ln 2}{\ln(0.518)}$$

$y \simeq -1.054$

$\therefore Q(0.518, -1.054)$

LINEAR INTERPOLATION

Question 1 (**+)

The following cubic equation is to be solved numerically.

$$x^3 = 8x - 1, \ x \in \mathbb{R}.$$

a) Show that the equation has a root α, in the interval $[2,3]$.

b) Use linear interpolation three successive times, to find α correct to an appropriate degree of accuracy.

$\boxed{}$, $\boxed{\alpha \approx 2.76}$

a) REWRITE THE EQUATION AS A FUNCTION

$\Rightarrow x^3 = 8x - 1$

$\Rightarrow x^3 - 8x + 1 = 0$

$f(x) = x^3 - 8x + 1$

- $f(2) = 8 - 16 + 1 = -7 < 0$
- $f(3) = 27 - 24 + 1 = 4 > 0$

AS $f(x)$ IS CONTINUOUS AND CHANGES SIGN IN $[2,3]$, THERE IS AT LEAST ONE ROOT α IN $(2,3)$

b) PROCEED TO INTERPOLATE (SIMILAR TRIANGLES)

$\Rightarrow \frac{3 - \alpha_1}{\alpha_1 - 2} = \frac{4}{7}$

$\Rightarrow 21 - 7\alpha_1 = 4\alpha_1 - 8$

$\Rightarrow 29 = 11\alpha_1$

$\Rightarrow \alpha_1 = 2.636...$

NEXT THE SIGN OF $f(\alpha_1)$

- $f(2.636...) = -1.767... < 0 \Rightarrow 2.636 < \alpha < 3$

INTERPOLATE AGAIN

$\Rightarrow \frac{3 - \alpha_2}{\alpha_2 - 2.636} = \frac{4}{1.767...}$

$\Rightarrow 5.301... - 1.767\alpha_2 = 4\alpha_2 - 10.544...$

$\Rightarrow 15.845... = 5.767\alpha_2$

$\Rightarrow \alpha_2 = 2.748...$

CHECK THE SIGN OF $f(\alpha_2)$

- $f(2.748...) = -0.2394... < 0 \Rightarrow 2.748 < \alpha < 3$

FINAL INTERPOLATION

$\Rightarrow \frac{3 - \alpha_3}{\alpha_3 - 2.748...} = \frac{4}{0.2394}$

$\Rightarrow 0.7181... 0.2394\alpha_3 = 4\alpha_3 - 10.992...$

$\Rightarrow 11.7101... = 4.2394... \alpha$

$\Rightarrow \alpha_3 = 2.7622...$

HENCE $\alpha = 2.76$ (2 d.p) AFTER 3 INTERPOLATIONS

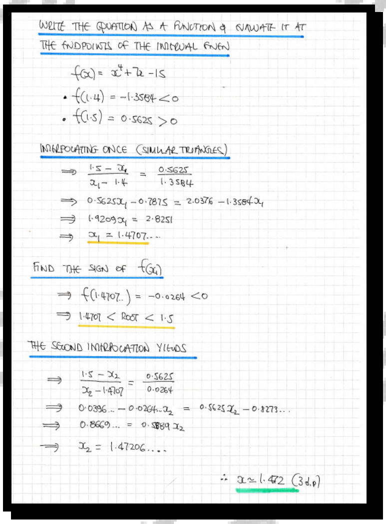
Question 2 (**+)

The following quartic equation is to be solved numerically.

$$x^4 + 7x - 15 = 0, \; x \in \mathbb{R}.$$

Given that the above quartic has a real root α in the interval $[1.4, 1.5]$, use linear interpolation twice, to find α correct to an appropriate degree of accuracy.

$\boxed{}$, $\boxed{\alpha \approx 1.472}$



WRITE THE EQUATION AS A FUNCTION & EVALUATE IT AT THE ENDPOINTS OF THE INTERVAL GIVEN

$f(x) = x^4 + 7x - 15$

- $f(1.4) = -1.3584 < 0$
- $f(1.5) = 0.5625 > 0$

INTERPOLATING ONCE (SIMILAR TRIANGLES)

$\Rightarrow \dfrac{1.5 - x_1}{x_1 - 1.4} = \dfrac{0.5625}{1.3584}$

$\Rightarrow 0.5625x_1 - 0.7875 = 2.0376 - 1.3584x_1$

$\Rightarrow 1.9209x_1 = 2.8251$

$\Rightarrow x_1 = 1.4707\ldots$

FIND THE SIGN OF $f(x_1)$

$\Rightarrow f(1.4707\ldots) = -0.0264 < 0$

$\Rightarrow 1.4707 < \text{ROOT} < 1.5$

THE SECOND INTERPOLATION YIELDS

$\Rightarrow \dfrac{1.5 - x_2}{x_2 - 1.4707} = \dfrac{0.5625}{0.0264}$

$\Rightarrow 0.0396\ldots - 0.0264\ldots x_2 = 0.5625x_2 - 0.8273\ldots$

$\Rightarrow 0.8669\ldots = 0.5889x_2$

$\Rightarrow x_2 = 1.47206\ldots$

$\therefore x \simeq 1.472$ (3 d.p.)

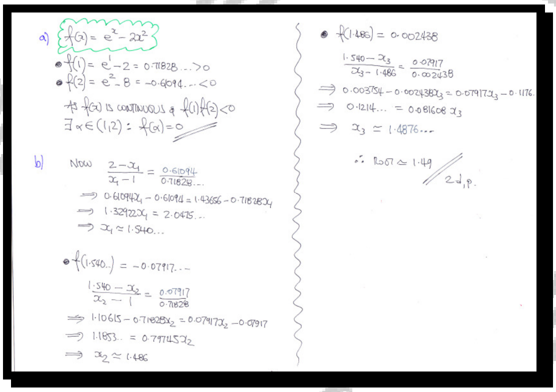
Question 3 (***)

$$e^x - 2x^2 = 0 .$$

a) Show that the above equation has a root α, which lies between 1 and 2.

b) Use linear interpolation three times, starting in the interval $[1,2]$ to find, correct to 2 decimal places the value of α.

$$\boxed{x_1 \approx 1.540, \quad x_2 \approx 1.486, \quad x_3 \approx 1.488, \quad \alpha \approx 1.49}$$



a) $f(x) = e^x - 2x^2$

- $f(1) = e^1 - 2 = 0.71828\ldots > 0$
- $f(2) = e^2 - 8 = -0.61094\ldots < 0$

As $f(x)$ is continuous & $f(1)f(2) < 0$

$\exists\, \alpha \in (1,2) : f(\alpha) = 0$

b) Now $\dfrac{2 - x_1}{x_1 - 1} = \dfrac{0.61094}{0.71828\ldots}$

$\Rightarrow 0.61094x_1 - 0.61094 = 1.43656 - 0.718280x_1$

$\Rightarrow 1.329220x_1 = 2.0475\ldots$

$\Rightarrow x_1 \simeq 1.540\ldots$

- $f(1.540\ldots) = -0.07917\ldots$

$\dfrac{1.540 - x_2}{x_2 - 1} = \dfrac{0.07917}{0.71828}$

$\Rightarrow 1.10615 - 0.718280x_2 = 0.07917x_2 - 0.07917$

$\Rightarrow 1.1853\ldots = 0.797450x_2$

$\Rightarrow x_2 \simeq 1.486$

- $f(1.486) = 0.002438$

$\dfrac{1.540 - x_3}{x_3 - 1.486} = \dfrac{0.07917}{0.002438}$

$\Rightarrow 0.003754 - 0.002438x_3 = 0.07917x_3 - 0.1176.$

$\Rightarrow 0.1214\ldots = 0.081608\,x_3$

$\Rightarrow x_3 \simeq 1.4876\ldots$

$\therefore$ Root $\simeq 1.49$ 2 d.p.

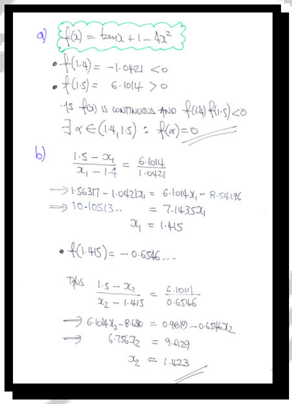
Question 4 (***)

$$\tan x = 4x^2 - 1.$$

a) Show that the above equation has a root α, which lies between 1.4 and 1.5.

b) Use linear interpolation twice, starting in the interval $[1.4, 1.5]$ to find, correct to 3 decimal places two approximations for α.

$$\boxed{x_1 \approx 1.415, \quad x_2 \approx 1.423}$$



a) $f(x) = \tan x + 1 - 4x^2$

- $f(1.4) = -1.0421 < 0$
- $f(1.5) = 6.1014 > 0$

As $f(x)$ is continuous and $f(1.4)f(1.5) < 0$

$\exists\, \alpha \in (1.4, 1.5) : f(\alpha) = 0$

b) $\dfrac{1.5 - x_1}{x_1 - 1.4} = \dfrac{6.1014}{1.0421}$

$\Rightarrow 1.56317 - 1.04214x_1 = 6.1014x_1 - 8.54196$

$\Rightarrow 10.10513\ldots = 7.14354x_1$

$x_1 = 1.415$

- $f(1.415) = -0.6546\ldots$

Thus $\dfrac{1.5 - x_2}{x_2 - 1.415} = \dfrac{6.1014}{0.6546}$

$\Rightarrow 6.1014x_2 - 8.6335 = 0.9819 - 0.6546x_2$

$\Rightarrow 6.756x_2 = 9.6129$

$x_2 \simeq 1.423$

NEWTON RAPHSON METHOD

Question 1 (**)

$$x^3 + 10x - 4 = 0.$$

a) Show that the above equation has a root α, which lies between 0 and 1.

b) Use the Newton-Raphson method **twice**, starting with $x_1 = 0.5$ to find, correct to 4 decimal places, an approximation for α.

$\boxed{}$, $\boxed{x = 0.3939}$

a) WRITE IN FUNCTION NOTATION

$f(x) = x^3 + 10x - 4$

$f(0) = -4 < 0$

$f(1) = +7 > 0$

AS $f(x)$ IS CONTINUOUS IN $(0,1)$ & CHANGES SIGN, THERE IS AT LEAST ONE ROOT IN $(0,1)$

b) "PREPARE THE NEWTON – RAPHSON ITEMS"

$f(x) = x^3 + 10x - 4$

$f'(x) = 3x^2 + 10$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{x_n^3 + 10x_n - 4}{3x_n^2 + 10}$$

STARTING WITH $x_1 = 0.5$

$$x_2 = 0.5 - \frac{0.5^3 + 10(0.5) - 4}{3(0.5)^2 + 10} \simeq 0.39534\ldots \left(\frac{17}{43}\right)$$

REAPPLY ONCE MORE

$x_3 = 0.3938891165\ldots$

$\therefore \alpha \simeq 0.3939$ 4. d.p.

Question 2 **(***+)**

It is known that the cubic equation below has a root α, which is close to 1.25.

$$x^3 + x = 3.$$

Use an iterative formula based on the Newton Raphson method to find the value of α, correct to 6 decimal places.

$\boxed{}$, $\boxed{\alpha \approx 1.213411}$

REARRANGE THE EQUATION, AND WRITE IT AS A FUNCTION

$\Rightarrow x^3 + x = 3$

$\Rightarrow x^3 + x - 3 = 0$

$\Rightarrow f(x) = x^3 + x - 3$

SET UP A RECURRENCE RELATION BASED ON NEWTON RAPHSON

- $f'(x) = 3x^2 + 1$
- $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

$\Rightarrow x_{n+1} = x_n - \dfrac{x_n^3 + x_n - 3}{3x_n^2 + 1}$

$\Rightarrow x_{n+1} = \dfrac{3x_n^3 + \cancel{x_n} - (x_n^3 + \cancel{x_n} - 3)}{3x_n^2 + 1}$

$\Rightarrow \boxed{x_{n+1} = \dfrac{2x_n^3 + 3}{3x_n^2 + 1}}$

PRODUCING ITERATIONS, STARTING WITH $x_1 = 1.25$, USING THIS FORMULA

$\Rightarrow x_1 = 1.25$

$\Rightarrow x_2 = 1.214285714$

$\Rightarrow x_3 = 1.213412176$

$\Rightarrow x_4 = 1.213411663$

$\Rightarrow x_5 = 1.213411663$

$\therefore \alpha = 1.213411$ 6 d.p

Question 3 (***+)

The curve with equation $y = 2^x$ intersects the straight line with equation $y = 3 - 2x$ at the point P, whose x coordinate is α.

a) Show that $0 < \alpha < 1$.

b) Starting with $x = 0.5$, use the Newton Raphson method to find the value of α, correct to 3 decimal places.

$\boxed{}$, $\boxed{\alpha \approx 0.692}$

a) SOLVING SIMULTANEOUSLY

$$\Rightarrow 3 - 2x = 2^x$$
$$\Rightarrow 2^x + 2x - 3 = 0$$
$$\Rightarrow f(x) = 2^x + 2x - 3$$

$f(0) = 1 + 0 - 3 = -2 < 0$

$f(1) = 2 + 2 - 3 = 1 > 0$

AS $f(x)$ IS CONTINUOUS AND CHANGES SIGN IN THE INTERVAL $(0,1)$, THERE MUST BE AT LEAST ONE ROOT IN $(0,1)$

b) PREPARING FOR NEWTON-RAPHSON METHOD

$$f(x) = 2^x + 2x - 3$$
$$f'(x) = 2^x \ln 2 + 2$$
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{2^{x_n} + 2x_n - 3}{2^{x_n}\ln 2 + 2}$$
$$= \frac{x_n 2^{x_n}\ln 2 + \cancel{2x_n} - 2^{x_n} - \cancel{2x_n} + 3}{2^{x_n}\ln 2 + 2}$$
$$\therefore x_{n+1} = \frac{3 + 2^{x_n}(x_n \ln 2 - 1)}{2^{x_n}\ln 2 + 2}$$

USING ABOVE FORMULA WITH $x_1 = 0.5$

$x_1 = 0.5$

$x_2 = 0.6965556034\ldots$

$x_3 = 0.6921557681\ldots$

$x_4 = 0.6921533549\ldots$

$\therefore \alpha \simeq 0.692$ (3 d.p.)

Question 4 (***+)

A curve has equation

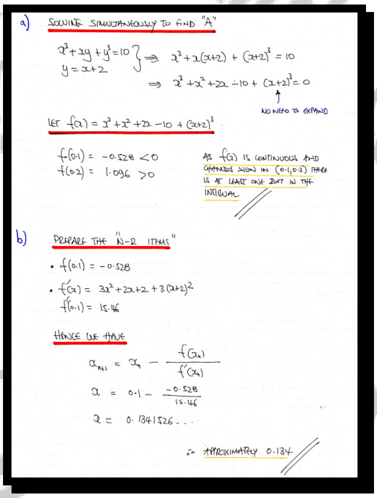
$$x^3 + xy + y^3 = 10.$$

The straight line with equation $y = x + 2$ meets this curve at the point A.

a) Show that the x coordinate of A lies in the interval $(0.1, 0.2)$.

b) Use the Newton Raphson method once, starting with $x = 0.1$, to find a better approximation for the x coordinate of A.

[illegible] , $\boxed{x \approx 0.134}$



a) SOLVING SIMULTANEOUSLY TO FIND "A"

$$\left.\begin{aligned} x^3 + xy + y^3 &= 10 \\ y &= x+2 \end{aligned}\right\} \Rightarrow x^3 + x(x+2) + (x+2)^3 = 10$$

$$\Rightarrow x^3 + x^2 + 2x - 10 + (x+2)^3 = 0$$

NO NEED TO EXPAND

LET $f(x) = x^3 + x^2 + 2x - 10 + (x+2)^3$

$f(0.1) = -0.528 < 0$
$f(0.2) = 1.096 > 0$

AS $f(x)$ IS CONTINUOUS AND CHANGES SIGN IN $(0.1, 0.2)$ THERE IS AT LEAST ONE ROOT IN THE INTERVAL

b) PREPARE THE "N-R ITEMS"

- $f(0.1) = -0.528$
- $f'(x) = 3x^2 + 2x + 2 + 3(x+2)^2$
  $f'(0.1) = 15.46$

HENCE WE HAVE

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x = 0.1 - \frac{-0.528}{15.46}$$

$$x = 0.1341526\ldots$$

$\therefore$ APPROXIMATELY 0.134

Question 5 (****)

A curve has equation

$$x^3 + y = xy.$$

The straight line with equation $y + 3x + 1 = 0$ meets this curve at the point A.

a) Show that the x coordinate of A lies in the interval $(-0.4, -0.3)$.

b) If P and Q are integers, use an iterative procedure based on the formula

$$x_{n+1} = \frac{1}{2}\left[Px^3 + Qx^2 - 1\right],\ x_1 = -0.35,$$

to find the x coordinate of A, correct to 2 decimal places.

The straight line with equation $y + 3x + 1 = 0$ meets the above mentioned curve at another point B, whose the x coordinate lies in the interval $(0.8, 0.9)$.

c) Use the Newton Raphson method twice, starting with $x = 0.8$, to find a better approximation for the x coordinate of B.

$\boxed{}$, $\boxed{x_A \approx 0.34}$, $\boxed{x_B \approx 0.834}$

a) SOLVING SIMULTANEOUSLY TO FIND INTERSECTIONS

$$\left.\begin{array}{l} y = -3x - 1 \\ x^3 + y - xy = 0 \end{array}\right\} \Rightarrow x^3 + (-3x-1) - x(-3x-1) = 0$$
$$\Rightarrow x^3 - 3x - 1 + 3x^2 + x = 0$$
$$\Rightarrow x^3 + 3x^2 - 2x - 1 = 0$$

LET $f(x) = x^3 + 3x^2 - 2x - 1$

$$\left.\begin{array}{l} f(-0.3) = -0.157 < 0 \\ f(-0.4) = +0.216 > 0 \end{array}\right\}$$ AS $f(x)$ IS CONTINUOUS AND CHANGES SIGN, THERE IS AT LEAST ONE ROOT IN THE INTERVAL

b) REARRANGING $f(x) = 0$ FOR $2x$

$$\Rightarrow x^3 + 3x^2 - 1 = 2x$$
$$\Rightarrow x = \frac{1}{2}(x^3 + 3x^2 - 1)$$

I.E. $x_{n+1} = \frac{1}{2}\left[x_n^3 + 3x_n^2 - 1\right]$ $(P = 1 \quad Q = 3)$

$x_1 = -0.35$
$x_2 = -0.3376875\ldots$
$x_3 = -0.3482044621\ldots$
$x_4 = -0.3392397384\ldots$
$x_5 = -0.346895065\ldots$
$x_6 = -0.3403677354\ldots$
$x_7 = -0.3459409414\ldots$
$x_8 = -0.3411879053\ldots$
$x_9 = -0.3452449232\ldots$
$x_{10} = -0.3417844861\ldots$
$x_{11} = -0.3447381044\ldots$
$x_{12} = -0.3422185$

$\therefore x_A \simeq -0.34$

c) PREPARING THE "N-R" METHOD

$$f(x) = x^3 + 3x^2 - 2x - 1$$
$$f'(x) = 3x^2 + 6x - 2$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{x_n^3 + 3x_n^2 - 2x_n - 1}{3x_n^2 + 6x_n - 1}, \quad x_1 = 0.8$$

$$x_2 \approx 0.8 - \frac{0.8^3 + 3\times0.8^2 - 2\times0.8 - 1}{3\times0.8^2 + 6\times0.8 - 2} \approx 0.83557\ldots$$

$$x_3 \approx 0.834245\ldots$$

$\therefore x_B \approx 0.834$

Question 6 (****)

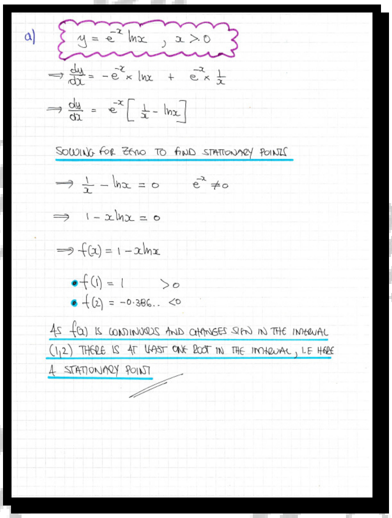
A curve C has equation

$$y = e^{-x} \ln x, \ x > 0.$$

a) Show that the x coordinate of the stationary point of C lies between 1 and 2.

b) Use an iterative formula based on the Newton Raphson method to find the x coordinate of the stationary point of C, correct to 8 decimal places.

$\boxed{x \approx 1.76322283}$


a) $y = e^{-x} \ln x, \ x > 0$

$\Rightarrow \frac{dy}{dx} = -e^{-x} \times \ln x + e^{-x} \times \frac{1}{x}$

$\Rightarrow \frac{dy}{dx} = e^{-x}\left[\frac{1}{x} - \ln x\right]$

SOLVING FOR ZERO TO FIND STATIONARY POINTS

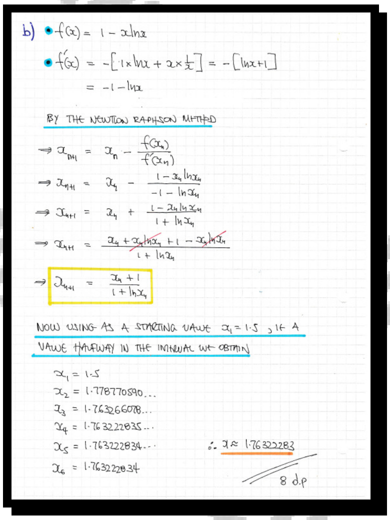
$\Rightarrow \frac{1}{x} - \ln x = 0 \qquad e^{-x} \neq 0$

$\Rightarrow 1 - x\ln x = 0$

$\Rightarrow f(x) = 1 - x\ln x$

- $f(1) = 1 \quad > 0$
- $f(2) = -0.386... \quad < 0$

AS $f(x)$ IS CONTINUOUS AND CHANGES SIGN IN THE INTERVAL (1,2) THERE IS AT LEAST ONE ROOT IN THE INTERVAL, I.E HERE A STATIONARY POINT


b)
- $f(x) = 1 - x\ln x$
- $f'(x) = -\left[1 \times \ln x + x \times \frac{1}{x}\right] = -\left[\ln x + 1\right] = -1 - \ln x$

BY THE NEWTON RAPHSON METHOD

$\Rightarrow x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$\Rightarrow x_{n+1} = x_n - \frac{1 - x_n \ln x_n}{-1 - \ln x_n}$

$\Rightarrow x_{n+1} = x_n + \frac{1 - x_n \ln x_n}{1 + \ln x_n}$

$\Rightarrow x_{n+1} = \frac{x_n + \cancel{x_n \ln x_n} + 1 - \cancel{x_n \ln x_n}}{1 + \ln x_n}$

$\Rightarrow \boxed{x_{n+1} = \frac{x_n + 1}{1 + \ln x_n}}$

NOW USING AS A STARTING VALUE $x_1 = 1.5$, IE A VALUE HALFWAY IN THE INTERVAL WE OBTAIN

$x_1 = 1.5$
$x_2 = 1.778770590...$
$x_3 = 1.763266078...$
$x_4 = 1.763222835...$
$x_5 = 1.763222834...$
$x_6 = 1.763222834$

$\therefore x \approx 1.76322283$

8 d.p

Question 7 (****)

At the point P, which lies on the curve with equation

$$x = \ln\left(y^3 - y\right),$$

the gradient is 4.

The point P is close to the point with coordinates $(7.5, 12)$.

a) Show that the y coordinate of P is a solution of the equation

$$y^3 - 12y^2 - y + 4 = 0.$$

b) Use the Newton Raphson method once on the equation of part **(a)**, in order to determine the coordinates of P, correct to two decimal places.

$\boxed{}$, $\boxed{P(7.46, 12.06)}$

a) START BY DIFFERENTIATION

$$\Rightarrow x = \ln(y^3 - y)$$
$$\Rightarrow \frac{dx}{dy} = \frac{3y^2-1}{y^3-y}$$
$$\Rightarrow \frac{dy}{dx} = \frac{y^3-y}{3y^2-1}$$

SETTING $\frac{dy}{dx} = 4$

$$\Rightarrow \frac{y^3-y}{3y^2-1} = 4$$
$$\Rightarrow y^3 - y = 12y^2 - 4$$
$$\Rightarrow y^3 - 12y^2 - y + 4 = 0$$

AS REQUIRED

b) LET $f(y) = y^3 - 12y^2 - y + 4$

- $f'(y) = 3y^2 - 24y - 1$
- $f(12) = -8$
- $f'(12) = 143$

BY THE NEWTON RAPHSON METHOD

$$\Rightarrow y_{r+1} = y_r - \frac{f(y_r)}{f'(y_r)}$$
$$\Rightarrow y = 12 - \frac{f(12)}{f'(12)}$$
$$\Rightarrow y = 12 - \frac{-8}{143}$$
$$\Rightarrow y = \frac{1724}{143}$$
$$\Rightarrow y \simeq 12.05594\ldots$$

& HENCE $x = 7.46176\ldots$

$\therefore$ $P(7.46, 12.06)$

AS REQUIRED

Question 8 (****)

It is required to find the single real root α of the following equation

$$x^2 = \frac{2}{\sqrt{x}} + \frac{3}{x^2}, \quad x > 0.$$

a) Show that the α lies between 1 and 2.

b) Use the Newton Raphson method to show that α can be found by the iterative formula

$$x_{n+1} = \frac{x_n^5 + 3x_n^{\frac{5}{2}} + 9x_n}{2x_n^4 + x_n^{\frac{3}{2}} + 6},$$

starting with a suitable value for x_1.

c) Hence find the value of α, correct to 8 decimal places.

$\boxed{\alpha \approx 1.63756623...}$

a) REWRITE THE EQUATION IN FUNCTION FORM

$$\boxed{x^2 = \frac{2}{\sqrt{x}} + \frac{3}{x^2}}$$

$$x^2 - \frac{2}{\sqrt{x}} + \frac{3}{x^2} = 0$$

$$f(x) = x^2 - \frac{2}{\sqrt{x}} - \frac{3}{x^2}$$

- $f(1) = 1 - 2 - 3 = -4 < 0$
- $f(2) = 4 - \sqrt{2} - \frac{3}{4} = 1.8357.... > 0$

AS $f(x)$ IS CONTINOUS ON $(1,2)$, AND CHANGES SIGN ON $(1,2)$ THERE EXISTS AT LEAST A VALUE α IN $(1,2)$ SO THAT $f(\alpha)=0$

b) $$\boxed{\begin{aligned} f(x) &= x^2 - 2x^{-\frac{1}{2}} - 3x^{-2} \\ f'(x) &= 2x + x^{-\frac{3}{2}} + 6x^{-3} \end{aligned}}$$

NEWTON RAPHSON STATES

$$\Rightarrow x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\Rightarrow x_{n+1} = x_n - \frac{x_n^2 - 2x_n^{-\frac{1}{2}} - 3x_n^{-2}}{2x_n + x_n^{-\frac{3}{2}} + 6x_n^{-3}}$$

$$\Rightarrow x_{n+1} = x_n - \frac{x_n^5 - 2x_n^{\frac{5}{2}} - 3x_n}{2x_n^4 + x_n^{\frac{3}{2}} + 6}$$

$$\Rightarrow x_{n+1} = \frac{2x_n^5 + x_n^{\frac{5}{2}} + 6x_n - x_n^5 + 2x_n^{\frac{5}{2}} + 3x_n}{2x_n^4 + x_n^{\frac{3}{2}} + 6}$$

$$\Rightarrow x_{n+1} = \frac{x_n^5 + 3x_n^{\frac{5}{2}} + 9x_n}{2x_n^4 + x_n^{\frac{3}{2}} + 6}$$

AS REQUIRED

c) USING ANY VALUE IN THE INTERVAL AND THE RECURRANCE OF PART (b)

$$\boxed{x_{n+1} = \frac{x_n^5 + 3x^{\frac{5}{2}} + 9x_n}{2x_n^4 + x_n^{\frac{3}{2}} + 6}}$$

- $x_1 = 1.5$
- $x_2 = 1.634594485....$
- $x_3 = 1.6375655406....$
- $x_4 = 1.637566228...$
- $x_5 = 1.637566228...$

$\therefore$ REQUIRED ROOT IS 1.63756623

CORRECT TO 8 d.p

Question 9 (****)

It is required to find the approximate coordinates of the points of intersection between the graphs of

$$y_1 = 1 - x^2,\ x \in \mathbb{R} \qquad \text{and} \qquad y_2 = \ln(x+1),\ x \in \mathbb{R},\ x > -1.$$

a) Show that the two graphs intersect at a single point P, explaining further why the x coordinate of P lies between 0 and 1.

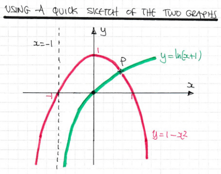
b) Use the Newton Raphson method once, starting $x = 0.7$, to calculate the x coordinate of P, giving the answer correct to 3 decimal places.

c) By considering two suitable transformations, determine correct to 2 decimal places the coordinates of the points of intersection between the graph of

$$y_3 = 2\left[1 - (2x+1)^2\right],\ x \in \mathbb{R} \qquad \text{and} \qquad y_4 = 2\ln(2x+2),\ x \in \mathbb{R},\ x > -\tfrac{1}{2}.$$

$\boxed{}$, $\boxed{x \approx 0.690}$, $\boxed{(-0.16, 1.05)}$

a) USING A QUICK SKETCH OF THE TWO GRAPHS



ONE INTERSECTION AS SEEN ABOVE – AS THE x INTERCEPTS OF $y = 1 - x^2$ & $\ln(x+1)$ ARE 1 & 0, THE x CO-ORDINATE OF P MUST LIE BETWEEN THEM

b) FORM AN EQUATION & WRITE IT AS A FUNCTION

$$\Rightarrow \ln(1+x) = 1 - x^2$$
$$\Rightarrow x^2 - 1 + \ln(1+x) = 0$$
$$\Rightarrow f(x) = x^2 - 1 + \ln(1+x)$$
$$\Rightarrow f'(x) = 2x + \frac{1}{x+1}$$

$$\therefore\ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x \simeq 0.7 - \frac{(0.7)^2 - 1 + \ln(1+0.7)}{2 \times 0.7 + \frac{1}{0.7+1}} \simeq 0.690$$

c) LOOKING AT THE TWO GRAPHS, THEY HAVE BOTH UNDERGONE IDENTICAL TRANSFORMATIONS

- TRANSLATION, 1 UNIT, "LEFT"
- HORIZONTAL STRETCH, SCALE FACTOR $\frac{1}{2}$
- VERTICAL STRETCH, SCALE FACTOR 2

IF $x \simeq 0.690 \Rightarrow y \simeq 0.524$

$$\therefore\ (0.690, 0.524)$$
$$\downarrow$$
$$(-0.310, 0.524)$$
$$\downarrow$$
$$(-0.155, 0.524)$$
$$\downarrow$$
$$(-0.155, 1.048) \qquad \therefore\ (-0.16, 1.05)$$

Question 10 (****)

An arithmetic series has first term 2 and common difference X.

A geometric series has first term 2 and common ratio X.

The sum of the 11^{th} term of the arithmetic series and the 11^{th} term of the geometric series is 900.

a) Show that X is a solution of the equation

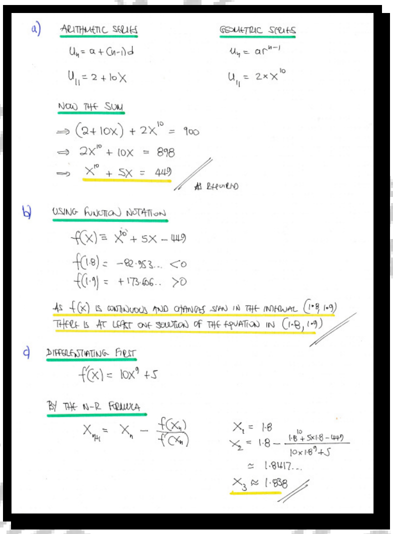
$$X^{10} + 5X = 449 .$$

b) Show further that

$$1.8 < X < 1.9 .$$

c) Use the Newton Raphson approximation method **twice**, with a starting value of 1.8, to find an approximate value for X, giving the answer correct to 3 decimal places.

$\boxed{}$, $\boxed{X \approx 1.838}$



a) ARITHMETIC SERIES

$u_n = a + (n-1)d$

$u_{11} = 2 + 10X$

GEOMETRIC SERIES

$u_n = ar^{n-1}$

$u_{11} = 2 \times X^{10}$

NOW THE SUM

$\Rightarrow (2 + 10X) + 2X^{10} = 900$

$\Rightarrow 2X^{10} + 10X = 898$

$\Rightarrow X^{10} + 5X = 449$ // AS REQUIRED

b) USING FUNCTION NOTATION

$f(X) \equiv X^{10} + 5X - 449$

$f(1.8) = -82.953\ldots < 0$

$f(1.9) = +173.606\ldots > 0$

AS $f(x)$ IS CONTINUOUS AND CHANGES SIGN IN THE INTERVAL $(1.8, 1.9)$ THERE IS AT LEAST ONE SOLUTION OF THE EQUATION IN $(1.8, 1.9)$ //

c) DIFFERENTIATING FIRST

$f'(X) = 10X^9 + 5$

BY THE N-R FORMULA

$X_{n+1} = X_n - \frac{f(X_n)}{f'(X_n)}$

$X_1 = 1.8$

$X_2 = 1.8 - \frac{1.8^{10} + 5\times 1.8 - 449}{10 \times 1.8^9 + 5}$

$\simeq 1.8417\ldots$

$X_3 \approx 1.838$ //

Question 11 (****+)

The curve C has equation

$$y = \sqrt{e^{2x}+1}, \quad x \in \mathbb{R}.$$

The tangent to the curve at the point P, where $x = p$, passes through the origin.

a) Show that $x = p$ is a solution of the equation

$$(x-1)e^{2x} = 1.$$

b) Show further that the equation of part **(a)** has a root between 1 and 2.

c) By using the Newton Raphson method once, starting with $x = 1$, find an approximation for this root, correct to 1 decimal place.

It is further given that the Newton Raphson method fails on this occasion.

d) Use an appropriate method to verify that the root of the equation of part **(a)** is 1.10886 correct to 5 decimal places.

$\boxed{\alpha \approx 1.1}$

a) START BY OBTAINING THE EQUATION OF THE TANGENT AT $\left(p, \sqrt{e^{2p}+1}\right)$

$\Rightarrow y = \left(e^{2x}+1\right)^{\frac{1}{2}}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(e^{2x}+1\right)^{-\frac{1}{2}}\left(2e^{2x}\right)$

$\Rightarrow \frac{dy}{dx} = \frac{e^{2x}}{\left(e^{2x}+1\right)^{\frac{1}{2}}}$

$\Rightarrow \left.\frac{dy}{dx}\right|_p = \frac{e^{2p}}{\left(e^{2p}+1\right)^{\frac{1}{2}}}$

$\Rightarrow y - \left(e^{2p}+1\right)^{\frac{1}{2}} = \frac{e^{2p}}{\left(e^{2p}+1\right)^{\frac{1}{2}}}(x-p)$

NOW IT IS GIVEN THAT THE ABOVE TANGENT PASSES THROUGH $(0,0)$

$\Rightarrow -\left(e^{2p}+1\right)^{\frac{1}{2}} = \frac{e^{2p}}{\left(e^{2p}+1\right)^{\frac{1}{2}}}(-p)$

$\Rightarrow -\left(e^{2p}+1\right)^{\frac{1}{2}} = \frac{-pe^{2p}}{\left(e^{2p}+1\right)^{\frac{1}{2}}}$

$\Rightarrow e^{2p}+1 = pe^{2p}$

$\Rightarrow 1 = pe^{2p} - e^{2p}$

$\Rightarrow 1 = e^{2p}(p-1)$

OR WRITTEN IN x, AS $x=p$

$\Rightarrow (x-1)e^{2x} = 1$

AS REQUIRED

b) REWRITE THE EQUATION AS A FUNCTION

$f(x) = (x-1)e^{2x} - 1$

$f(1) = -1 < 0$

$f(2) = e^{4} - 1 > 0$

AS $f(x)$ IS CONTINUOUS AND CHANGES SIGN IN $[1,2]$

THERE IS AT LEAST ONE SOLUTION IN THE INTERVAL

c) $f'(x) = 1 \times e^{2x} + 2(x-1)e^{2x} = e^{2x}[1+2x-2] = (2x-1)e^{2x}$

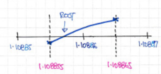
$x_{r+1} = x_r - \frac{f(x_r)}{f'(x_r)}$

$\alpha = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{-1}{e^2} = 1.1$

d) USING THE GRAPH OF THE ABOVE FUNCTION

$f(1.108855) = -0.000029.. < 0$

$f(1.108865) = 0.000083 > 0$



AS THERE IS A CHANGE OF SIGN IN THE ABOVE INTERVAL

$1.108855 < \text{ROOT} < 1.108865$

$\therefore$ ROOT $\simeq$ 1.10886 CORRECT TO 5 d.p.

Question 12 (****+)

The point P has x coordinate 2 and lies on the curve with equation

$$xy = e^x, \qquad xy > 0.$$

a) Determine an equation of the tangent to the curve at P.

The tangent to the curve found in part **(a)** meets the curve again at the point Q.

b) Show that the x coordinate of Q is -0.6, correct to one significant figure.

c) Use the Newton Raphson method **twice** to find a better approximation for the x coordinate of Q, giving the answer correct to 4 significant figures.

$\boxed{}$, $\boxed{4y = e^2 x}$, $\boxed{\approx -0.5569}$

a) BY IMPLICIT DIFFERENTIATION OR REARRANGING AND USING THE QUOTIENT RULE

$$xy = e^x \implies y = \frac{e^x}{x}$$
$$\implies \frac{dy}{dx} = \frac{xe^x - e^x \times 1}{x^2} = \frac{e^x(x-1)}{x^2}$$

WHEN $x=2$, $y = \frac{e^2}{2} = \frac{1}{2}e^2$ & $\frac{dy}{dx} = \frac{1}{4}e^2$

$$\implies y - y_0 = m(x - x_0)$$
$$\implies y - \tfrac{1}{2}e^2 = \tfrac{1}{4}e^2(x-2)$$
$$\implies \cancel{y - \tfrac{1}{2}e^2} = \tfrac{1}{4}e^2x \cancel{- \tfrac{1}{2}e^2}$$
$$\implies y = \tfrac{1}{4}e^2x$$

b) SOLVING SIMULTANEOUSLY "TANGENT" & CURVE

$$\left.\begin{array}{l} xy = e^x \\ y = \frac{1}{4}e^2x \end{array}\right\} \implies x\left(\tfrac{1}{4}e^2x\right) = e^x$$
$$\implies \tfrac{1}{4}x^2e^2 = e^x$$
$$\implies x^2e^2 = 4e^x$$
$$\implies x^2e^2 - 4e^x = 0$$

LET $f(x) = x^2e^2 - 4e^x$

$f(-0.65) = 1.03369\ldots > 0$

$f(-0.55) = -0.0726\ldots < 0$

AS $f(x)$ IS CONTINUOUS, AND CHANGES SIGN IN THE INTERVAL $[-0.65, -0.55]$, THERE EXISTS AT LEAST A SOLUTION IN THIS INTERVAL

$$\implies -0.65 < x < -0.55$$
$$\implies x = -0.6$$

CORRECT TO 1 s.f.

c) USING THE "FUNCTION" OF PART (b)

$$f(x) = x^2e^2 - 4e^x$$
$$f'(x) = 2e^2x - 4e^x$$

BY NEWTON RAPHSON

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$x_{n+1} = x_n - \frac{x_n^2e^2 - 4e^{x_n}}{2e^2x_n - 4e^{x_n}}$$

TIDY IS OPTIONAL AS MODERN CALCULATORS CAN HANDLE THIS

$$x_{n+1} = \frac{2e^2x_n^2 - 4x_ne^{x_n} - x_n^2e^2 + 4e^{x_n}}{2e^2x_n - 4e^{x_n}}$$
$$x_{n+1} = \frac{e^2x_n^2 + 4e^{x_n}(1-x_n)}{2e^2x_n - 4e^{x_n}}$$

- USING $x_1 = -0.6$ WE OBTAIN

  $x_2 = -0.55798\ldots$

  $x_3 = -0.556929\ldots$

- USING $x_1 = -0.55$ (MORE SENSIBLE)

  $x_2 = -0.556957\ldots$

  $x_3 = -0.556929\ldots$

$\therefore$ REQUIRED x CO-ORDINATE -0.5569

4 sf

Question 13 (****+)

It is required to find the real solutions of the equation

$$x^2 = 2^x .$$

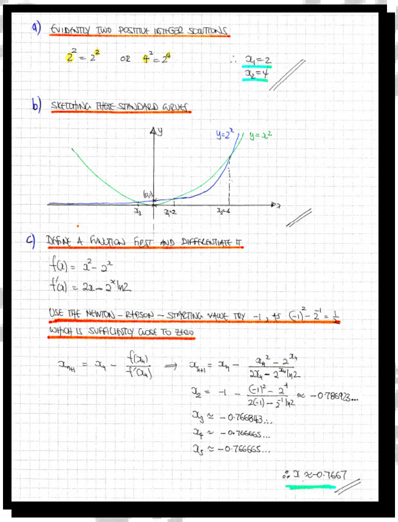
a) State the 2 integer solutions of the equation.

b) Sketch in the same set of axes the graph of $y = x^2$ and the graph of $y = 2^x$.

c) Use the Newton Raphson method, with a suitable function and an appropriate starting value, to find the third real root of this equation correct to 4 decimal places.

You may use as many steps as necessary in part **(c)**, to obtain the required accuracy.

 , $\boxed{x_1 = 2 \cup x_2 = 4}$, $\boxed{x_3 \approx -0.7667}$



a) EVIDENTLY TWO POSITIVE INTEGER SOLUTIONS

$2^2 = 2^2$ OR $4^2 = 2^4$ $\therefore x_1 = 2$, $x_2 = 4$

b) SKETCHING THESE STANDARD CURVES

c) DEFINE A FUNCTION FIRST AND DIFFERENTIATE IT

$f(x) = x^2 - 2^x$

$f'(x) = 2x - 2^x \ln 2$

USE THE NEWTON – RAPHSON – STARTING VALUE TRY -1, AS $(-1)^2 - 2^{-1} = \frac{1}{2}$ WHICH IS SUFFICIENTLY CLOSE TO ZERO

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \Rightarrow x_{n+1} = x_n - \frac{x_n^2 - 2^{x_n}}{2x_n - 2^{x_n}\ln 2}$

$x_2 = -1 - \frac{(-1)^2 - 2^{-1}}{2(-1) - 2^{-1}\ln 2} \approx -0.786923...$

$x_3 \approx -0.766843...$

$x_4 \approx -0.766665...$

$x_5 \approx -0.766665...$

$\therefore x \approx -0.7667$