MODULUS FUNCTION

EXAM QUESTIONS

Question 1 (**)

$$f(x) = |3x+2|, \ x \in \mathbb{R}.$$

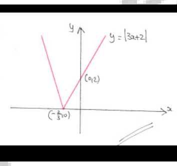
a) Sketch the graph of $f(x)$, clearly indicating the coordinates of any points where the graph of $f(x)$ meets the coordinate axes.

b) Solve the equation

$$f(x) = 1.$$

$\boxed{}$, $\boxed{(0,2),\left(-\frac{2}{3},0\right)}$, $\boxed{x = -\frac{1}{3}, -1}$

(a)



(b) $|3x+2| = 1$

$3x+2 = 1$
$3x+2 = -1$

$3x = -1$
$3x = -3$

$x = -\frac{1}{3}, -1$

Question 2 (**)

Solve the equation

$$\left|x^2 - 2x - 4\right| = 4.$$

$\boxed{}$, $\boxed{x = -2, 0, 2, 4}$

$|x^2-2x-4| = 4$

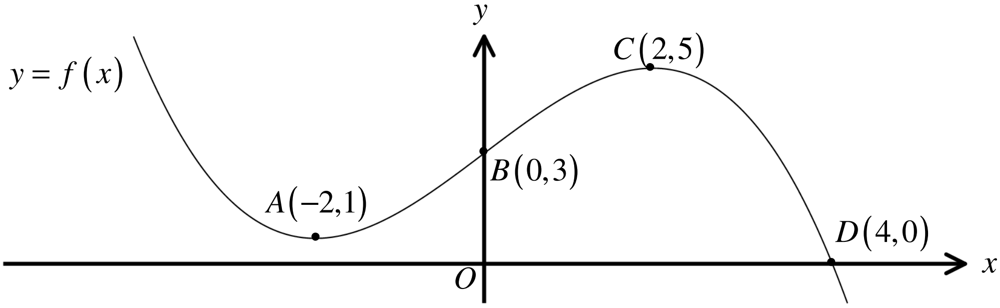
$x^2-2x-4 = 4$ or $x^2-2x-4 = -4$

$x^2-2x-8 = 0$ $x^2-2x = 0$

$(x-4)(x+2) = 0$ $x(x-2) = 0$

$x = -2, 0, 2, 4$

Question 3 (**)



The figure above shows part of the graph of the curve with equation $y = f(x)$.

The graph meets the coordinate axes at $B(0,3)$ and $D(4,0)$ and has stationary points at $A(-2,1)$ and $C(2,5)$.

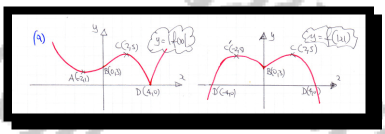
Sketch on separate diagrams the graph of …

a) … $y = |f(x)|$

b) … $y = f(|x|)$

Indicate the new coordinates of the points A, B, C and D in these graphs.

, graph



Question 4 (**)

A curve C has equation

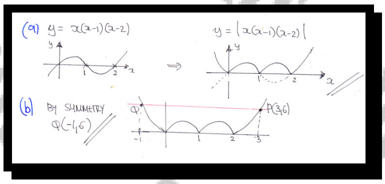
$$y = |x(x-1)(x-2)|, \quad x \in \mathbb{R}.$$

a) Sketch the graph of C, indicating the coordinates of any intercepts with the coordinate axes.

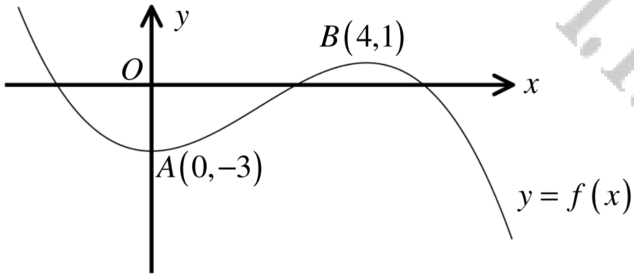
The straight line with equation $y = 6$ intersects the graph of C at the point $P(3,6)$ and at the point Q.

b) State the coordinates of Q

$\boxed{(0,0),(1,0),(2,0)}$, $\boxed{Q(-1,6)}$



Question 5 **(**)**



The figure above shows part of the graph of the curve with equation $y = f(x)$.

The graph has stationary points at $A(0,-3)$ and $B(4,1)$.

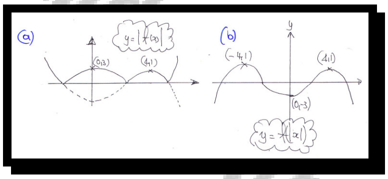
Sketch on separate diagrams the graph of …

a) … $y = |f(x)|$.

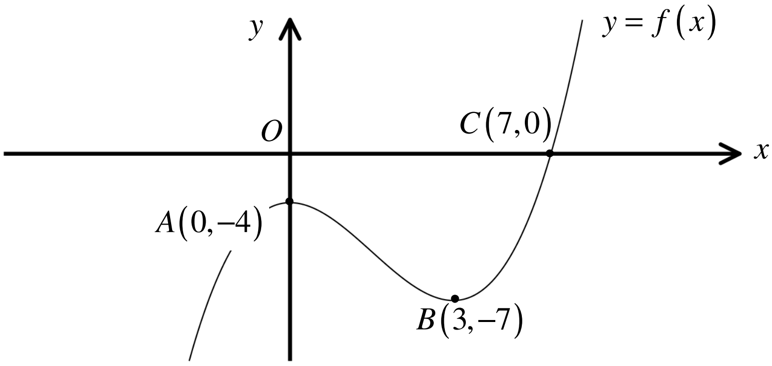
b) … $y = f(|x|)$.

Indicate the new coordinates of the points A and B in these graphs.

$\boxed{}$, $\boxed{\text{graph}}$



Question 6 (**+)



The figure above shows part of the graph of the curve with equation $y = f(x)$.

The graph meets the coordinate axes at $A(0,-4)$ and $C(7,0)$ and has a stationary point at $B(3,-7)$.

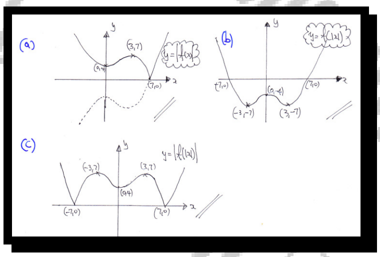
Sketch on separate diagrams, indicating the new coordinates of the points A, B and C, the graph of …

a) … $y = |f(x)|$.

b) … $y = f(|x|)$.

c) … $y = |f(|x|)|$.

, graph



Question 7 (+)**

The functions f and g are defined by

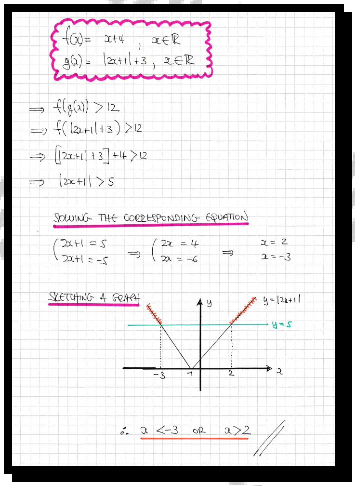
$$f(x) = x + 4, \quad x \in \mathbb{R}$$

$$g(x) = |2x + 1| + 3, \quad x \in \mathbb{R}.$$

Solve the inequality

$$fg(x) > 12.$$

, $\boxed{x < -3 \text{ or } x > 2}$



Question 8 **(**+)**

The functions f and g are defined by

$$f(x) = e^{2x} - 1, \quad x \in \mathbb{R}$$

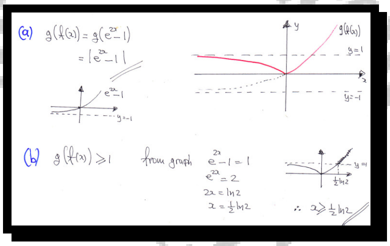
$$g(x) = |x|, \quad x \in \mathbb{R}.$$

a) Find the composite function $gf(x)$, and sketch its graph.

b) Solve the inequality

$$gf(x) \geq 1.$$

$\boxed{}$, $\boxed{gf(x) = \left|e^{2x} - 1\right|}$, $\boxed{x \geq \frac{1}{2}\ln 2}$

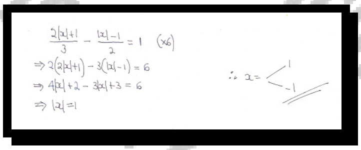


Question 9 **(**+)**

Solve the equation

$$\frac{2|x|+1}{3} - \frac{|x|-1}{2} = 1.$$

$\boxed{}$, $\boxed{x = \pm 1}$



Question 10 (***)

The functions f and g are defined as

$$f(x) = |2x-4|, \quad x \in \mathbb{R}$$

$$g(x) = |x|, \quad x \in \mathbb{R}.$$

a) Sketch in the same diagram the graph of f and the graph of g.

Mark clearly in the sketch the coordinates of any x or y intercepts.

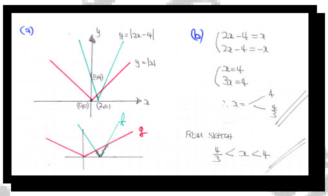
b) Solve the equation

$$f(x) = g(x).$$

c) Hence, or otherwise, solve the inequality

$$f(x) < g(x).$$

$\boxed{(0,0),(2,0),(0,4)}$, $\boxed{x = \frac{4}{3}, 4}$, $\boxed{\frac{4}{3} < x < 4}$



Question 11 (***)

Solve the following modulus inequality.

$$12 - 2|2x-3| \geq 7 .$$

$\boxed{}$, $\boxed{\frac{1}{4} \leq x \leq \frac{11}{4}}$

$$\boxed{12-2|2x-3| \geqslant 7}$$

$\Rightarrow -2|2x-3| \geqslant -5$

$\Rightarrow |2x-3| \leqslant \frac{5}{2}$

$\Rightarrow 2\left|x-\frac{3}{2}\right| \leqslant \frac{5}{2}$

$\Rightarrow \left|x-\frac{3}{2}\right| \leqslant \frac{5}{4}$

THE STATEMENT ABOVE READS
"THE DIFFERENCE OF x FROM $\frac{3}{2}$ IS LESS THAN $\frac{5}{4}$"

$-5/4 \quad +5/4$

$\frac{1}{4} \quad \frac{3}{2} \quad \frac{11}{4}$

$\therefore \frac{1}{4} \leqslant x \leqslant \frac{11}{4}$

ALTERNATIVE – SOLVE AN EQUATION INSTEAD

$\Rightarrow 12 - 2|2x-3| = 7$

$\Rightarrow 5 = 2|2x-3|$

$\Rightarrow \frac{5}{2} = |2x-3|$

$\Rightarrow \begin{cases} 2x-3 = \frac{5}{2} \\ 2x-3 = -\frac{5}{2} \end{cases}$

$\Rightarrow \begin{cases} 2x = \frac{11}{2} \\ 2x = \frac{1}{2} \end{cases}$

$\Rightarrow x = \begin{cases} 11/4 \\ 1/4 \end{cases}$

THE SOLUTION INTERVAL IS EITHER BETWEEN $\frac{1}{4}$ & $\frac{11}{4}$ OR OUTSIDE

TRY A NUMBER, SAY $x=1$, IN THE INITIAL INEQUALITY

$12-2|2\times1-3| \geqslant 7$

$12 - 2\times1 \geqslant 7$

$10 \geqslant 7$

IF $x=1$ WORKS

$\frac{1}{4} \quad x=1 \quad \frac{11}{4}$

$\therefore \frac{1}{4} \leqslant x \leqslant \frac{11}{4}$

Question 12 (***)

Solve the following modulus equation.

$$4x + |3x+2| = 1 .$$

$\boxed{}$, $\boxed{x = -\frac{1}{7}}$

$4x + |3x+2| = 1$

$|3x+2| = 1-4x$

$\begin{cases} 3x+2 = 1-4x \\ 3x+2 = -1+4x \end{cases}$

$7x = -1$

$-x = -3$

$\therefore x = \begin{cases} -\frac{1}{7} \\ \cancel{3} \text{ DOESN'T WORK} \end{cases}$

$4\times3 + |3\times3+2| = 12+11 \neq 1$

$4\left(-\frac{1}{7}\right) + \left|3\left(-\frac{1}{7}\right)+2\right| = -\frac{4}{7} + \frac{10}{7} = 1$

Question 13 (***)

Find the solutions of the following equation.

$$\left|2x^2-5\right|=13.$$

$\boxed{x=\pm 3}$

$$\left|2x^2-5\right|=13 \implies \begin{cases} 2x^2-5=13 \\ 2x^2-5=-13 \end{cases} \implies \begin{cases} 2x^2=18 \\ \cancel{2x^2=-8} \end{cases} \implies x^2=9 \implies x=\pm 3$$

Question 14 (***)

A curve has equation

$$y=\left|3x-2\right|,\ x\in\mathbb{R}.$$

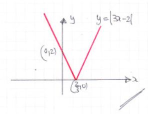
a) Sketch the graph of the above curve, indicating the coordinates of any intercepts with the coordinate axes.

b) Hence solve the equation

$$\left|3x-2\right|=x.$$

$\boxed{(0,2),\left(\tfrac{2}{3},0\right)}$, $\boxed{x=\tfrac{1}{2},1}$

(a)



(b) $|3x-2|=x$

$$\begin{cases} 3x-2=x \\ 3x-2=-x \end{cases}$$

$$\begin{cases} 2x=2 \\ 4x=2 \end{cases}$$

$$x = 1,\ \tfrac{1}{2}$$ both fine

Question 15 (***)

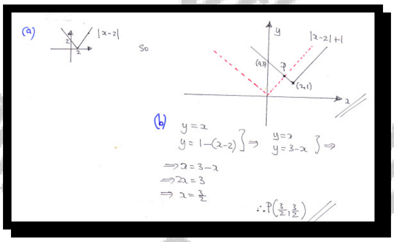
The curve C_1 and the curve C_2 have respective equations

$$y = |x| \quad \text{and} \quad y = |x-2| + 1.$$

a) Sketch the graph of C_2, indicating the coordinates of any intercepts with the coordinate axes.

b) Determine the coordinates of the point of intersection between the graph of C_1 and the graph of C_2.

$$\boxed{\left(\frac{3}{2}, \frac{3}{2}\right)}$$



Question 16 (***)

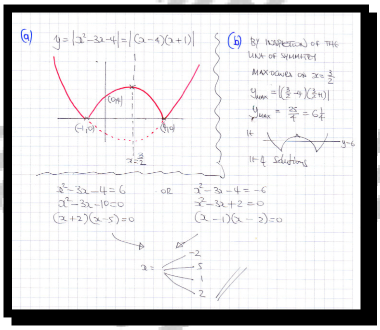
$$f(x) = \left| x^2 - 3x - 4 \right|, \quad x \in \mathbb{R}.$$

a) Sketch the graph of $f(x)$, clearly indicating the coordinates of any points where the graph of $f(x)$ meets the coordinate axes.

b) Solve the equation

$$f(x) = 6.$$

$\boxed{(0,4),(-1,0),(4,0)}$, $\boxed{x = -2, 1, 2, 5}$



Question 17 (***)

A curve has equation

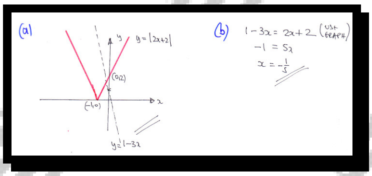
$$y = |2x+2|, \; x \in \mathbb{R}.$$

a) Sketch the graph of the above curve, indicating the coordinates of any intercepts with the coordinate axes.

b) Hence solve the equation

$$1-3x = |2x+2|.$$

$\boxed{(0,2), \; (-1,0)}, \; \boxed{x = -\frac{1}{5}}$



Question 18 (***)

$$f(x) = \frac{2}{x-3}, \quad x \in \mathbb{R},\ x \neq 2$$

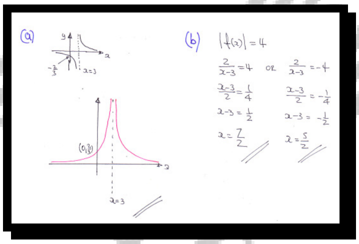
a) Sketch the graph of $y = |f(x)|$, clearly indicating the coordinates of any points where the graph of $y = |f(x)|$ meets the coordinate axes.

(The sketch should include the equation of the vertical asymptote of the curve.)

b) Solve the equation

$$|f(x)| = 4.$$

$\boxed{\left(0, \frac{2}{3}\right),\ \ x = 3}, \boxed{x = \frac{5}{2}, \frac{7}{2}}$



Question 19 (***)

A curve has equation

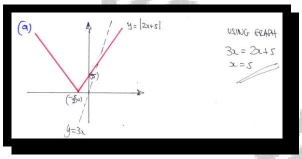
$$y = |2x + 5|, \ x \in \mathbb{R}.$$

a) Sketch the graph of the above curve, indicating the coordinates of any intercepts with the coordinate axes.

b) Hence solve the equation

$$|2x + 5| = 3x.$$

$\boxed{(0,5), \left(-\frac{5}{2},0\right)}$, $\boxed{x = 5}$



Question 20 (***)

A curve has equation

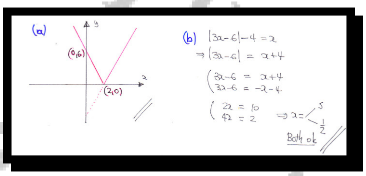
$$y = |3x - 6|, \ x \in \mathbb{R}.$$

a) Sketch the graph of the above curve, indicating the coordinates of any intercepts with the coordinate axes.

b) Solve the equation

$$|3x - 6| - 4 = x.$$

$\boxed{(0,6), \ (2,0)}$, $\boxed{x = \frac{1}{2}, \ 5}$



Question 21 (***)

The functions f and g have equations

$$f(x) = |x|, \ x \in \mathbb{R},$$

$$g(x) = |5x+1|, \ x \in \mathbb{R}.$$

a) Sketch in the same diagram the graph of $f(x)$ and the graph of $g(x)$.

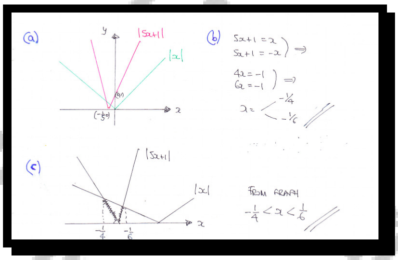
The sketch must include the coordinates of any points where the graphs meet the coordinate axes.

b) Find the x coordinates of the points of intersection between the two graphs.

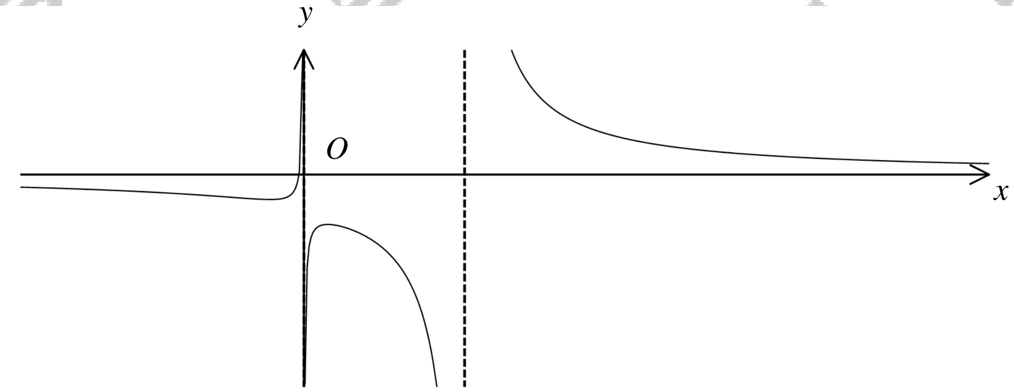
c) Hence solve the inequality

$$|5x+1| < |x|.$$

$\boxed{x = -\frac{1}{4}, -\frac{1}{6}}$, $\boxed{-\frac{1}{4} < x < -\frac{1}{6}}$



Question 22 (***)



The figure above shows the graph of the curve with equation $y = f(x)$.

The graph of $y = f(x)$ …

… has as asymptotes the lines with equations $y = 0$, $x = 0$ and $x = 10$.

… crosses the x axis at the point $A\left(-\frac{1}{2}, 0\right)$.

… has local minimum and local maximum at $B(-3,-1)$ and $C(2,-4)$, respectively.

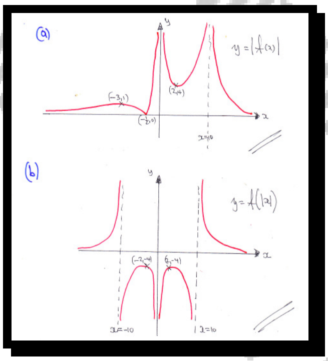
Sketch on separate diagrams the graph of …

a) … $y = |f(x)|$

b) … $y = f(|x|)$

Each of the two sketches must clearly indicate the coordinates of the new position of A, B and C, and the equations of any asymptotes

$\boxed{\text{C3F}}$, $\boxed{\text{graph}}$



Question 23 (***)

A curve has equation

$$f(x) = \left|2x^2 - 4x - 11\right|, \ x \in \mathbb{R}.$$

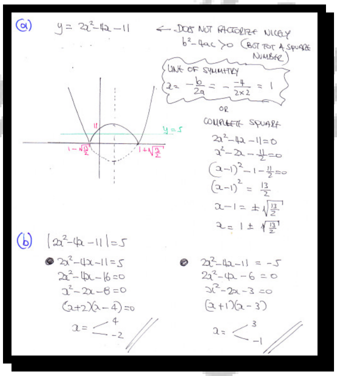
a) Sketch the graph of $f(x)$.

The sketch must include the coordinates of any points where the graphs meet the coordinate axes.

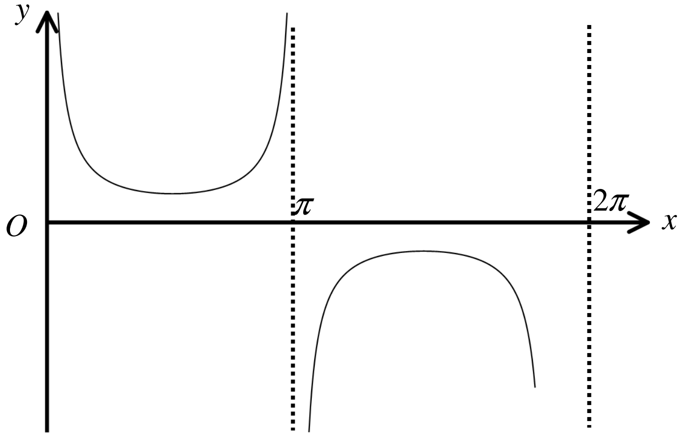
b) Solve the equation

$$f(x) = 5.$$

$\boxed{x = -2, -1, 3, 4}$



Question 24 (***)



The figure above shows the graph of

$$y = \operatorname{cosec} x \text{, for } 0 \le x < 360° .$$

a) Sketch the graph of

$$y = |\operatorname{cosec} x| \text{, for } 0 \le x < 360° .$$

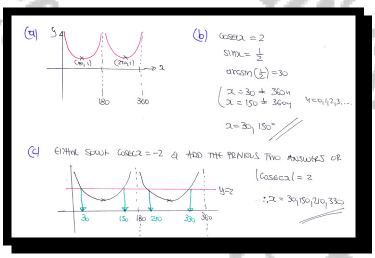
b) Solve the equation

$$\operatorname{cosec} x = 2 \text{, for } 0 \le x < 360° .$$

c) Hence solve the equation

$$|\operatorname{cosec} x| = 2 \text{ for } 0 \le x < 360° .$$

$\boxed{}$, $\boxed{x = 30°, 150°}$, $\boxed{x = 30°, 150°, 210°, 330°}$



Question 25 (***)

$$f(x) = |2x - k|, \ k > 0, \ x \in \mathbb{R}$$

a) Sketch the graph of $f(x)$.

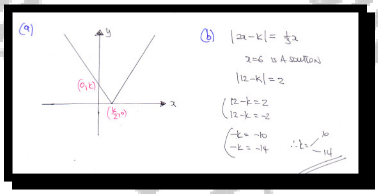
Indicate clearly in the sketch the coordinates of any point where the graph of $f(x)$ meets the coordinate axes.

b) Given that $x = 6$ is a solution of the equation

$$f(x) = \frac{1}{3}x,$$

find the possible values of k.

$\boxed{(0,k), \left(\frac{k}{2}, 0\right)}$, $\boxed{x = 10, 14}$



Question 26 (***)

Two curves y_1 and y_2 have equations

$$y_1 = |3x+3| \quad \text{and} \quad y_2 = |x^2 - 1|.$$

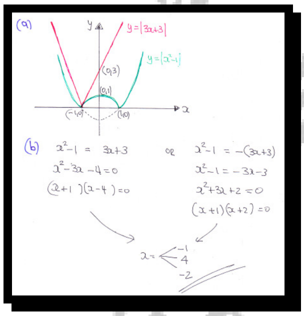
a) Sketch the graph of y_1 and the graph of y_2 on the same diagram.

Mark clearly in the sketch the coordinates of any x or y intercepts.

b) Solve the equation

$$|x^2 - 1| = |3x+3|.$$

$$\boxed{x = -1, -2, 4}$$



Question 27 (***)

A curve C has equation

$$y = \left|5 - x^2\right|, \; x \in \mathbb{R}.$$

a) Sketch the graph of C.

Mark clearly in the sketch the coordinates of any x or y intercepts.

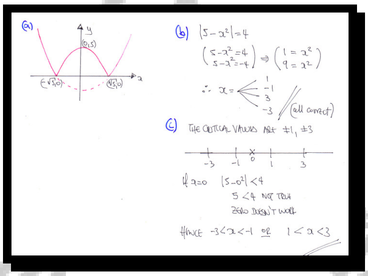
b) Solve the equation

$$\left|5 - x^2\right| = 4.$$

c) Hence, or otherwise, solve the inequality

$$\left|5 - x^2\right| < 4.$$

$\boxed{\left(\sqrt{5},0\right), \left(-\sqrt{5},0\right), (0,5)}$, $\boxed{x = \pm 1, \; x = \pm 3}$, $\boxed{-3 < x < -1 \text{ or } 1 < x < 3}$



Question 28 (***)

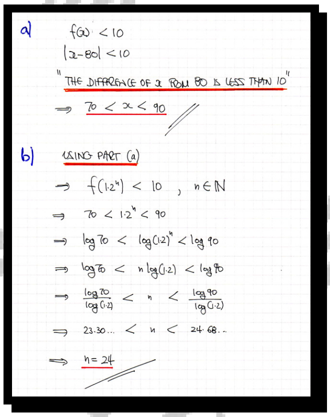
$$f(x) = |x - 80|, \ x \in \mathbb{R}.$$

a) Solve the inequality

$$f(x) < 10.$$

b) Find the value of the integer n, such that $f\left(1.2^n\right) < 10$.

$\boxed{}$, $\boxed{70 < x < 90}$, $\boxed{n = 24}$


a) $f(x) < 10$

$|x-80| < 10$

"THE DIFFERENCE OF x FROM 80 IS LESS THAN 10"

$\Rightarrow 70 < x < 90$

b) USING PART (a)

$\Rightarrow f(1.2^n) < 10, \quad n \in \mathbb{N}$

$\Rightarrow 70 < 1.2^n < 90$

$\Rightarrow \log 70 < \log(1.2)^n < \log 90$

$\Rightarrow \log 70 < n\log(1.2) < \log 90$

$\Rightarrow \frac{\log 70}{\log(1.2)} < n < \frac{\log 90}{\log(1.2)}$

$\Rightarrow 23.30\ldots < n < 24.68\ldots$

$\Rightarrow n = 24$

Question 29 (***)

The function f is defined as

$$f : x \mapsto |2x - 5|, \quad x \in \mathbb{R}.$$

a) Sketch the graph of $f(x)$.

Mark clearly in the sketch the coordinates of any x or y intercepts.

b) Solve the equation

$$f(x) = x.$$

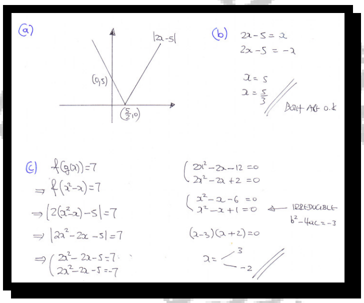
The function g is defined as

$$g : x \mapsto x^2 - x, \quad x \in \mathbb{R}.$$

c) Solve the equation

$$fg(x) = 7.$$

$\boxed{\text{C3}}$, $\boxed{\left(\frac{5}{2},0\right),(0,5)}$, $\boxed{x = \frac{5}{3}, 5}$, $\boxed{x = -2, 3}$



Question 30 (***)

$$f(x) \equiv 1 + \frac{4x}{2x-5} - \frac{15}{2x^2 - 7x + 5}, \; x \in \mathbb{R}, \; x \neq \frac{5}{2}, \; x \neq k.$$

a) Show that

$$f(x) \equiv \frac{3x+2}{x-k}, \; x \in \mathbb{R}, \; x \neq k,$$

stating the value of k.

b) Express $f(x)$ in the form

$$A + \frac{B}{x-k}, \; x \in \mathbb{R}, \; x \neq k,$$

where A and B are integers to be found.

c) Sketch on separate set of axes the graph of …

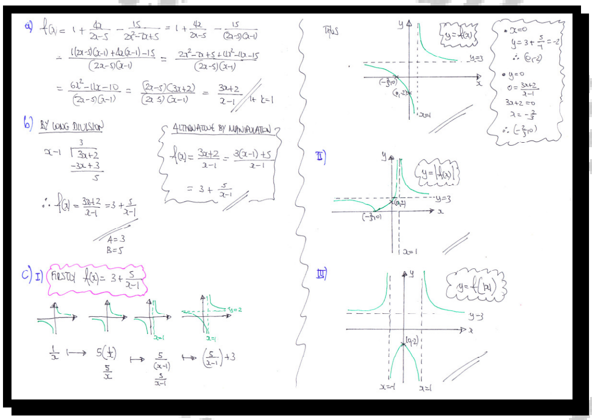
i. … $y = f(x)$.

ii. … $y = |f(x)|$.

iii. … $y = f(|x|)$.

Each sketch must include the coordinates of any points where the graph meets the coordinate axes and the equations of any asymptotes.

$\boxed{}$, $\boxed{k=1}$, $\boxed{A=3}$, $\boxed{B=5}$

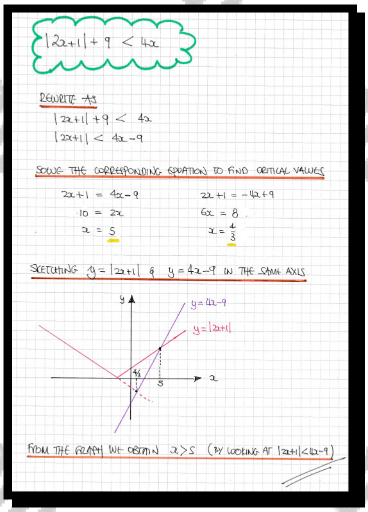


Question 31 (***)

Find the solution interval of the following modulus inequality.

$$|2x+1|+9<4x\,.$$

$\boxed{}$, $\boxed{x>5}$



$$|2x+1|+9 < 4x$$

REWRITE AS

$|2x+1|+9 < 4x$

$|2x+1| < 4x-9$

SOLVE THE CORRESPONDING EQUATION TO FIND CRITICAL VALUES

$2x+1 = 4x-9$ $\quad$ $2x+1 = -4x+9$

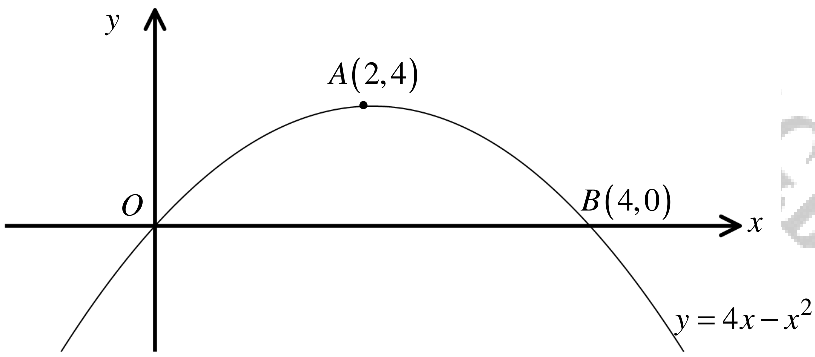
$10 = 2x$ $\quad$ $6x = 8$

$x = 5$ $\quad$ $x = \frac{4}{3}$

SKETCHING $y=|2x+1|$ & $y=4x-9$ IN THE SAME AXIS

FROM THE GRAPH WE OBTAIN $x>5$ (BY LOOKING AT $|2x+1|<4x-9$)

Question 32 (***+)



The figure above shows part of the graph of the curve with equation

$$y = 4x - x^2, \; x \in \mathbb{R}.$$

The graph meets the coordinate axes at the origin O and $B(4,0)$, and has a stationary point at $A(2,4)$.

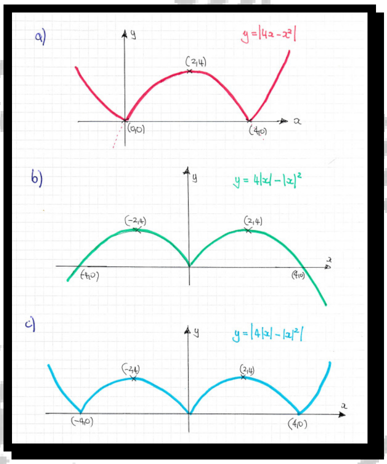
Sketch on separate diagrams, indicating the new coordinates of the points A and B, the graph of …

a) … $y = \left|4x - x^2\right|$.

b) … $y = 4|x| - |x|^2$.

c) … $y = \left|4|x| - |x|^2\right|$.

graph



Question 33 (***+)

The curve C_1 has equation

$$y = |x-1|, \; x \in \mathbb{R}.$$

The curve C_2 has equation

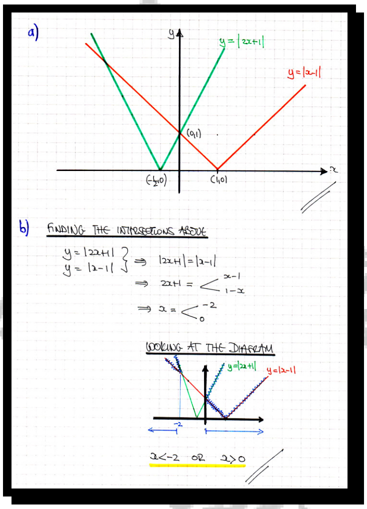
$$y = |2x+1|, \; x \in \mathbb{R}.$$

a) Sketch the graph of C_1 and the graph of C_2 on the same set of axes, indicating the coordinates of any intercepts of the graphs with the coordinate axes.

b) Hence, solve the inequality

$$|2x+1| \ge |x-1|.$$

$\boxed{}$, $\boxed{(0,1),\ (1,0)}$, $\boxed{(0,1),\ \left(-\tfrac{1}{2},0\right)}$, $\boxed{x \le -2 \text{ or } x \ge 0}$



Question 34 (***+)

The curves C_1 and C_2 have equations

$$C_1: \ y = 4x + 3, \ x \in \mathbb{R},$$

$$C_2: \ y = |3x + 2|, \ x \in \mathbb{R}.$$

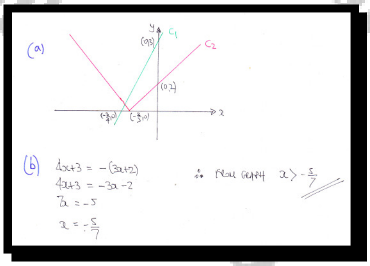
a) Sketch in the same diagram the graph of C_1 and the graph of C_2.

The sketch must include the coordinates of any points where the graphs meet the coordinate axes.

b) Solve the inequality

$$4x + 3 > |3x + 2|.$$

$$\boxed{x > -\frac{5}{7}}$$



Question 35 (***+)

The function f is given by

$$f : x \to 4 - |2x|, \quad x \in \mathbb{R}$$

a) Sketch the graph of $f(x)$.

Mark clearly in the sketch the coordinates of any x or y intercepts.

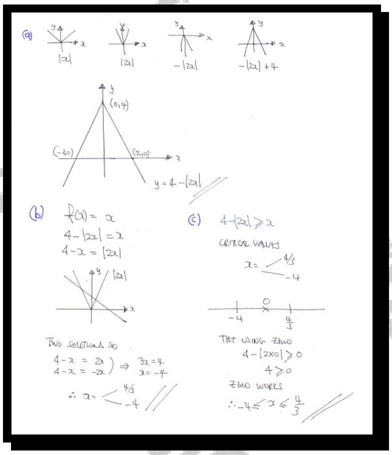
b) Solve the equation

$$f(x) = x.$$

c) Hence, or otherwise, solve the inequality

$$f(x) \geq x.$$

$\boxed{(2,0),(-2,0),(0,4)}$, $\boxed{x=-4,\ x=\frac{4}{3}}$, $\boxed{-4 \leq x \leq \frac{4}{3}}$



Question 36 (***+)

The curve C_1 and the curve C_2 have respective equations

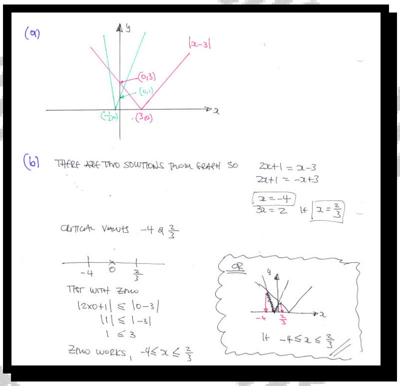
$$y = |x-3| \quad \text{and} \quad y = |2x+1|.$$

a) Sketch the graphs of C_1 and C_2 on the same diagram, indicating the coordinates of any intercepts of the graphs with the coordinate axes.

b) Solve the inequality

$$|2x+1| \le |x-3|.$$

$\boxed{(0,3),(3,0)}$, $\boxed{(0,1),\left(-\frac{1}{2},0\right)}$, $\boxed{-4 \le x \le \frac{2}{3}}$



Question 37 (***+)

$$f(x) \equiv \ln x, \ x \in \mathbb{R}, \ x > 0$$

$$g(x) \equiv 2\ln(x + \mathrm{e}), \ x \in \mathbb{R}, \ x > -\mathrm{e}.$$

a) Describe mathematically the transformations which map the graph of $f(x)$ onto the graph of $g(x)$.

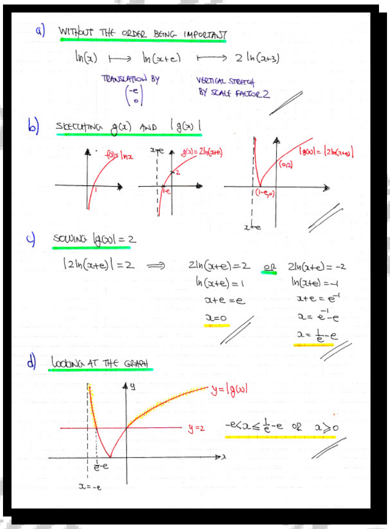
b) Sketch the graph of $y = |g(x)|$, indicating the coordinates of any intercepts of the graph with the coordinate axes.

c) Solve the equation

$$|g(x)| = 2.$$

d) Hence solve the inequality $|g(x)| \geq 2$.

$\boxed{\text{SYN}}$, $\boxed{x = 0, \ \mathrm{e}^{-1} - \mathrm{e}}$, $\boxed{-\mathrm{e} < x \leq \mathrm{e}^{-1} - \mathrm{e} \ \text{ or } \ x \geq 0}$

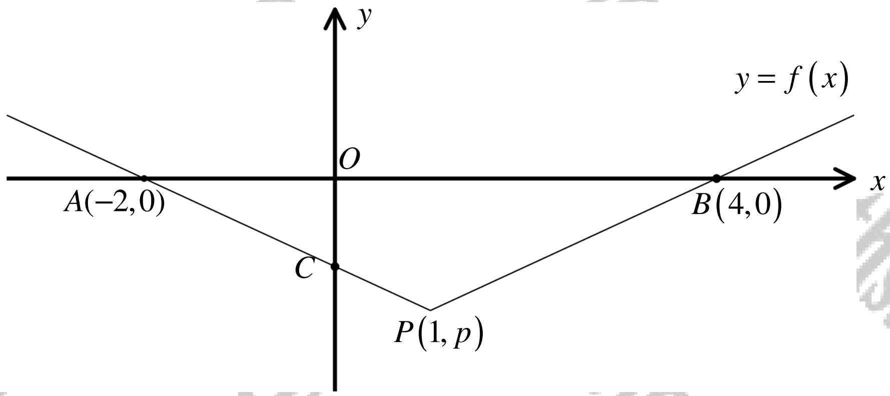


Question 38 (***+)

The figure below shows the graph of

$$y = f(x), \ x \in \mathbb{R},$$

which consists of two straight line segments that meet at the point $P(1, p)$.



The points A, B and C are the points where $f(x)$ crosses the coordinate axes.

Sketch, in separate diagrams, the graph of …

a) … $y = f(x+1)$.

b) … $y = f(|x|)$.

Each of these sketches must show the coordinates of any intersections with the x axis and the new position of the point P.

[continues overleaf]

[continued from overleaf]

It is now given that

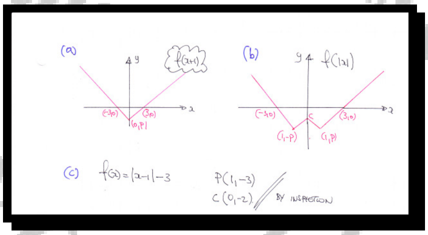
$$f(x) = |x-1| - 3, \quad x \in \mathbb{R}.$$

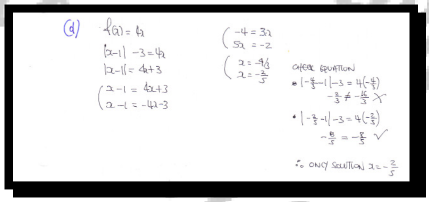
c) Find the full coordinates of the points P and C.

d) Solve the equation

$$f(x) = 4x.$$

$\boxed{}$, $\boxed{P(1,-3)}$, $\boxed{C(0,-2)}$, $\boxed{x = -\frac{2}{5}}$





Question 39 (***+)

$$f(x) = 2|x+1|, \quad x \in \mathbb{R}$$

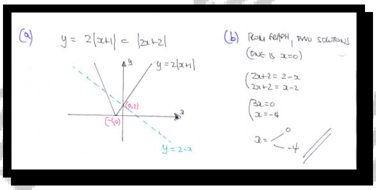
a) Sketch the graph of $f(x)$.

Indicate clearly in the sketch the coordinates of any point where the graph of $f(x)$ meets the coordinate axes.

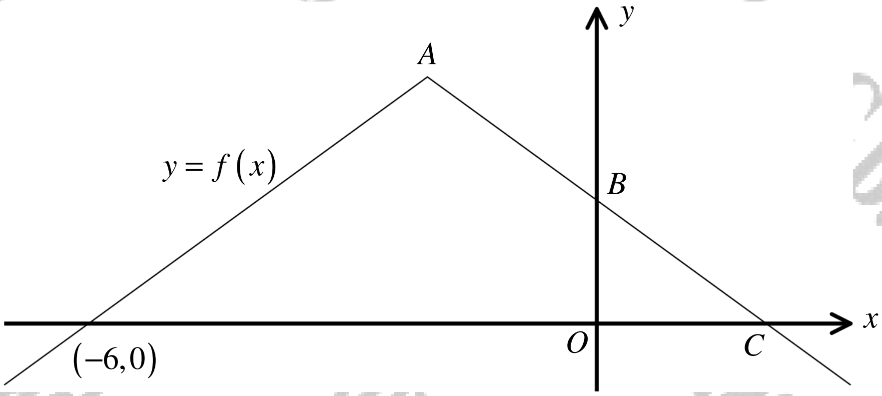
b) Hence solve the equation

$$f(x) = 2 - x .$$

$\boxed{(-1,0),(0,2)}$, $\boxed{x = 0, -4}$



Question 40 (***+)



The figure below shows the graph of

$$y = f(x),\ x \in \mathbb{R},\ x \in \mathbb{R},$$

which consists of two straight line segments which meet at the point A.

The graph of $f(x)$ crosses the coordinate axes at the points $(-6,0)$, B and C.

Sketch, in separate diagrams, the graph of …

a) … $y = |f(x)|$.

b) … $y = f(|x|)$.

[continues overleaf]

[continued from overleaf]

It is now given that

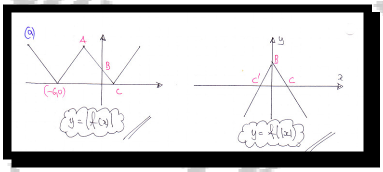
$$f(x) = 4 - |x+2|, \ x \in \mathbb{R}.$$

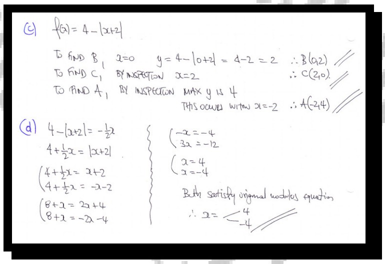
c) Find the coordinates of the points A, B and C.

d) Solve the equation

$$f(x) = -\frac{1}{2}x.$$

$\boxed{}$, $\boxed{A(-2,4)}$, $\boxed{B(0,2)}$, $\boxed{C(2,0)}$, $\boxed{x = \pm 4}$




(c) $f(x) = 4 - |x+2|$

TO FIND B, $x=0$ $\quad y = 4 - |0+2| = 4 - 2 = 2$ $\quad \therefore B(0,2)$

TO FIND C, BY INSPECTION $x=2$ $\quad \therefore C(2,0)$

TO FIND A, BY INSPECTION MAX y IS 4

THIS OCCURS WITH $x=-2$ $\quad \therefore A(-2,4)$

(d) $4 - |x+2| = -\frac{1}{2}x$

$4 + \frac{1}{2}x = |x+2|$

$4 + \frac{1}{2}x = x+2$
$4 + \frac{1}{2}x = -x-2$

$8 + x = 2x + 4$
$8 + x = -2x - 4$

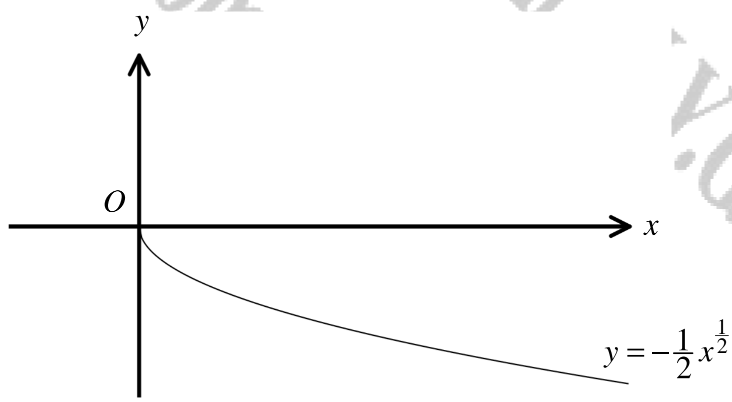
$-x = -4$
$3x = -12$

$x = 4$
$x = -4$

Both satisfy original modulus equation

$\therefore x = 4, -4$

Question 41 (***+)



The figure above shows the graph of the curve with equation

$$y=-\frac{1}{2}x^{\frac{1}{2}},\ x\geq 0.$$

Sketch on separate diagrams, the graphs of …

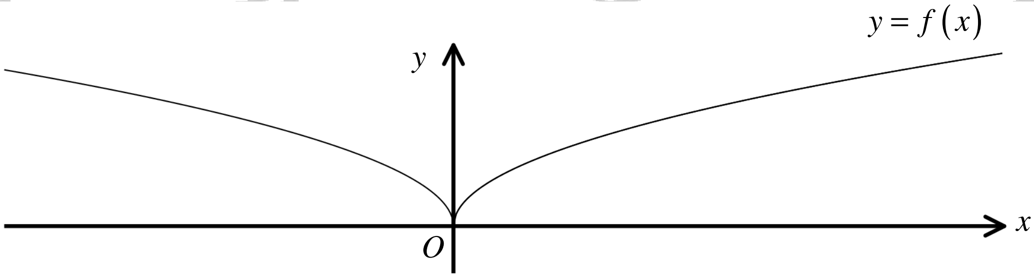
a) … $y=\left|-\frac{1}{2}x^{\frac{1}{2}}\right|$.

b) … $y=-\frac{1}{2}|x|^{\frac{1}{2}}$.

[continues overleaf]

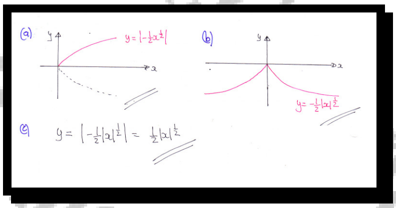
[continued from overleaf]

The figure below shows the graph of $y = f(x)$, which is related to the graph of $y = -\frac{1}{2}x^{\frac{1}{2}}$.



c) Write an equation for the graph of $y = f(x)$.

$$\boxed{y = \left|-\frac{1}{2}|x|^{\frac{1}{2}}\right| = \frac{1}{2}|x|^{\frac{1}{2}}}$$



Question 42 (***+)

The function f is defined by

$$f(x) = |2x - 3| - 1, \ x \in \mathbb{R}.$$

a) Sketch the graph of $f(x)$.

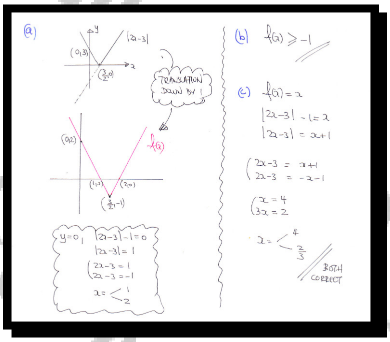
Mark clearly in the sketch the coordinates of any x or y intercepts as well as the coordinates of any minima or maxima.

b) State the range of $f(x)$.

c) Solve the equation

$$f(x) = x.$$

$$\boxed{(1,0), \ (2,0), \ (0,2), \ \min\left(\frac{3}{2}, -1\right)}, \ \boxed{f(x) \geq -1}, \ \boxed{x = \frac{2}{3}, 4}$$



Question 43 (***+)

The function $f(x)$ is given by

$$f(x) = |2x-4|, \quad x \in \mathbb{R}.$$

a) Sketch the graph of $y = f(x)$.
Mark clearly in the sketch the coordinates of any x or y intercepts.

b) Find the coordinates of the intersections between the graphs of

$$y = |2x-4| \quad \text{and} \quad y = x.$$

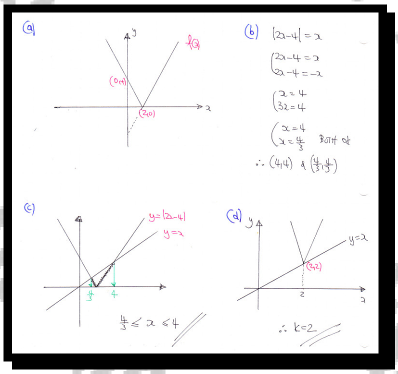
c) Solve the inequality

$$|2x-4| \le x.$$

The graph of $y = |2x-4| + k$, where k is a constant, **touches** the straight line with equation $y = x$.

d) State the value of k.

$\boxed{(2,0),(0,4)}$, $\boxed{\left(\frac{4}{3},\frac{4}{3}\right),(4,4)}$, $\boxed{\frac{4}{3} \le x \le 4}$, $\boxed{k=2}$

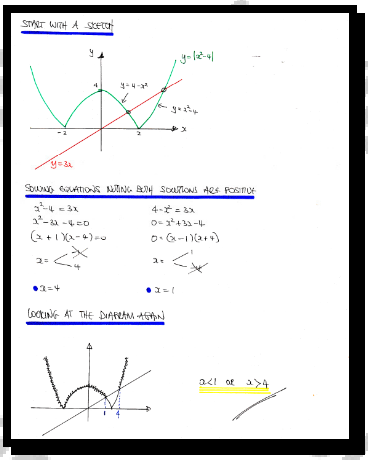


Question 44 (***+)

Find the set of values of x for which

$$\left|x^2-4\right| > 3x .$$

$\boxed{}$, $\boxed{x<1 \text{ or } x>4}$



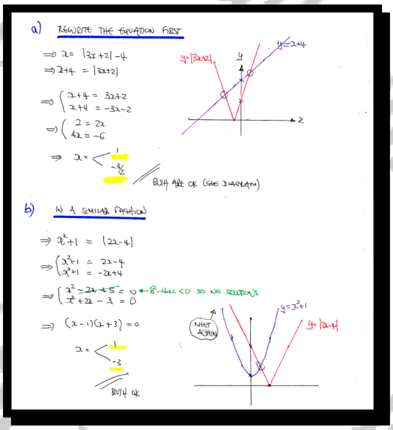
Question 45 (***+)

Solve each of the following equations.

a) $x = |3x+2| - 4$.

b) $x^2+1 = |2x-4|$.

$\boxed{}$, $\boxed{x = 1 \;\cup\; -\frac{3}{2}}$, $\boxed{x = 1 \;\cup\; -3}$



Question 46 (***+)

The function f is defined by

$$f(x) = \left|8 - e^{3x}\right|, \; x \in \mathbb{R}.$$

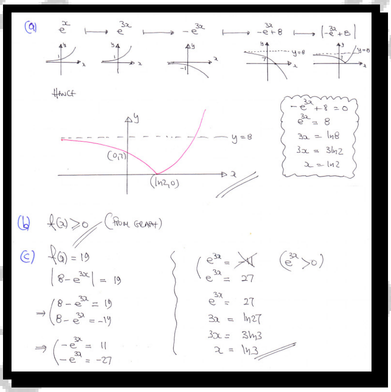
a) Sketch the graph of $f(x)$.

Mark clearly in the sketch the coordinates of any x or y intercepts as well as the equations of any asymptotes.

b) State the range of $f(x)$.

c) Solve the equation $f(x) = 19$.

$\boxed{(\ln 2, 0), (0, 7), \; x = 8}$, $\boxed{f(x) \geq 0}$, $\boxed{x = \ln 3}$



Question 47 (***+)

$$\frac{4x-1}{2(x-1)}-\frac{3}{2(x-1)(2x-1)}-2,\ x\neq\frac{1}{2},\ x\neq 1.$$

a) Show that the above expression can be simplified to $\dfrac{3}{2x-1}$.

b) Sketch the graph of the curve with equation

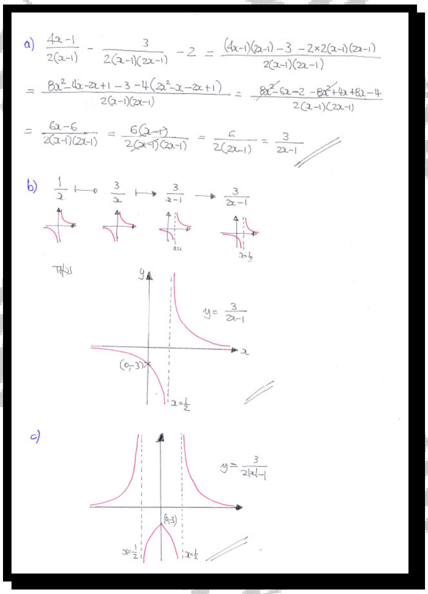
$$y=\frac{3}{2x-1},\ x\neq\frac{1}{2},$$

c) Hence sketch the graph of the curve with equation

$$y=\frac{3}{2|x|-1},\ x\neq\pm\frac{1}{2},$$

Each of sketches in parts **(b)** and **(c)**, must include the equation of the vertical asymptote of the curve, and the coordinates of any points where the curve meets the coordinate axes.

$\boxed{}$, $\boxed{\text{proof}}$, $\boxed{\text{graph}}$



Question 48 (***+)

The functions f and g are defined as

$$f(x) = \frac{1}{3}(x+2a), \ x \in \mathbb{R},$$

$$g(x) = |2x-a|, \ x \in \mathbb{R},$$

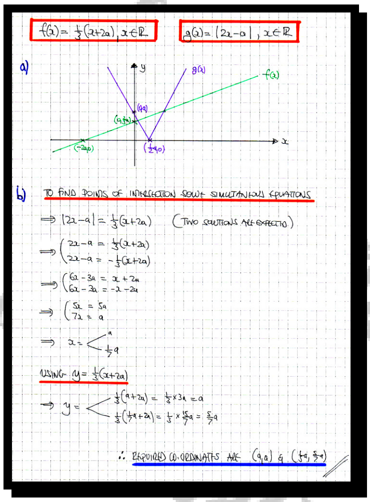
where a is a positive constant.

a) Sketch in the same set of axes the graph of $f(x)$ and the graph of $g(x)$.

The sketch must include the coordinates of any points where these graphs meet the coordinate axes.

b) Find, in terms of a, the coordinates of the points of intersection between the graphs of $f(x)$ and $g(x)$.

$\boxed{}$, $\boxed{(a,a), \ \left(\frac{1}{7}a, \frac{5}{7}a\right)}$



$f(x) = \frac{1}{3}(x+2a), x \in \mathbb{R}$ $\quad$ $g(x) = |2x-a|, x \in \mathbb{R}$

a)

b) TO FIND POINTS OF INTERSECTION SOLVE SIMULTANEOUS EQUATIONS

$\Rightarrow |2x-a| = \frac{1}{3}(x+2a)$ (TWO SOLUTIONS ARE EXPECTED)

$\Rightarrow \begin{cases} 2x-a = \frac{1}{3}(x+2a) \\ 2x-a = -\frac{1}{3}(x+2a) \end{cases}$

$\Rightarrow \begin{cases} 6x-3a = x+2a \\ 6x-3a = -x-2a \end{cases}$

$\Rightarrow \begin{cases} 5x = 5a \\ 7x = a \end{cases}$

$\Rightarrow x = \begin{cases} a \\ \frac{1}{7}a \end{cases}$

USING $y = \frac{1}{3}(x+2a)$

$\Rightarrow y = \begin{cases} \frac{1}{3}(a+2a) = \frac{1}{3}\times 3a = a \\ \frac{1}{3}(\frac{1}{7}a+2a) = \frac{1}{3}\times\frac{15}{7}a = \frac{5}{7}a \end{cases}$

$\therefore$ REQUIRED CO-ORDINATES ARE (a,a) & $(\frac{1}{7}a, \frac{5}{7}a)$

Question 49 **(***+)**

The functions f and g are defined as

$$f(x) = |2x-1|, \; x \in \mathbb{R},$$

$$g(x) = \ln(x+2), \; x \in \mathbb{R}, \; x > -2.$$

a) State the range of $f(x)$.

b) Find, in exact form, the solutions of the equation

$$gf(x) = 2.$$

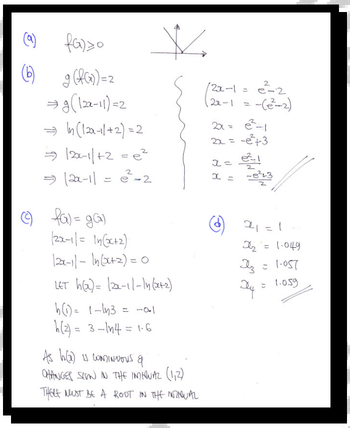
c) Show that the equation $f(x) = g(x)$ has a solution between 1 and 2.

d) Use the iteration formula

$$x_{n+1} = \tfrac{1}{2}\left[1 + \ln(x_n + 2)\right], \; x_1 = 1,$$

to find the values of x_2, x_3 and x_4, correct to three decimal places.

$\boxed{}$, $\boxed{f(x) \geq 0}$, $\boxed{x = \tfrac{1}{2}\left(e^2 - 1\right), \; x = \tfrac{1}{2}\left(3 - e^2\right)}$, $\boxed{x_2 = 1.049, \; x_3 = 1.057, \; x_4 = 1.059}$


(a) $f(x) \geq 0$

(b) $g(f(x)) = 2$

$\Rightarrow g(|2x-1|) = 2$

$\Rightarrow \ln(|2x-1|+2) = 2$

$\Rightarrow |2x-1| + 2 = e^2$

$\Rightarrow |2x-1| = e^2 - 2$

$2x-1 = e^2 - 2$, $2x - 1 = -(e^2-2)$

$2x = e^2 - 1$, $2x = -e^2 + 3$

$x = \frac{e^2-1}{2}$, $x = \frac{-e^2+3}{2}$

(c) $f(x) = g(x)$

$|2x-1| = \ln(x+2)$

$|2x-1| - \ln(x+2) = 0$

LET $h(x) = |2x-1| - \ln(x+2)$

$h(1) = 1 - \ln 3 = -0.1$

$h(2) = 3 - \ln 4 = 1.6$

AS $h(x)$ IS CONTINUOUS & CHANGES SIGN IN THE INTERVAL $(1,2)$ THERE MUST BE A ROOT IN THE INTERVAL

(d) $x_1 = 1$

$x_2 = 1.049$

$x_3 = 1.057$

$x_4 = 1.059$

Question 50 (***+)

It is given that $|y| = 2$, $y \in \mathbb{R}$.

a) Find the possible values of $|3y-1|$.

It is next given that $5 \le t \le 13$, $t \in \mathbb{R}$.

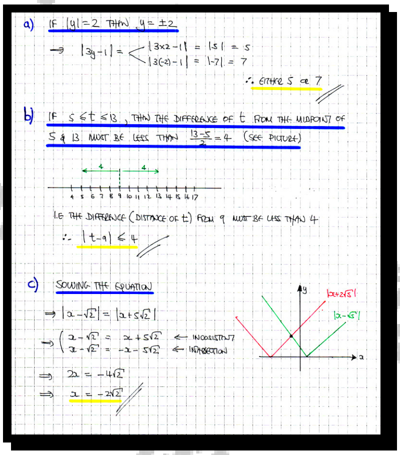
b) Express the above inequality in the form $|t-a| \le b$, where a and b are positive integers to be stated.

It is finally given that

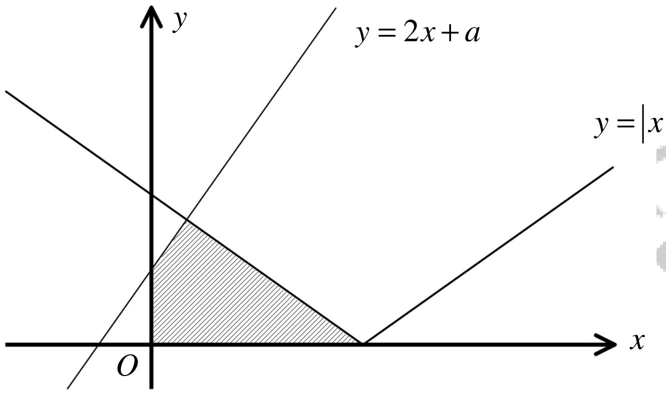
$$\left|x-\sqrt{2}\right| = \left|x+5\sqrt{2}\right|, \quad x \in \mathbb{R}.$$

c) Determine the value of x.

$\boxed{}$, $\boxed{|3y-1|=5, \quad |3y-1|=7}$, $\boxed{|t-9| \le 4}$, $\boxed{x=-2\sqrt{2}}$



Question 51 (***+)



The figure above shows the graphs of

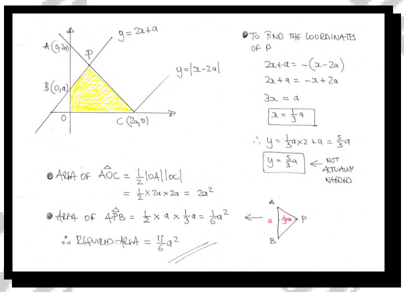
$$C_1 : \ y = |x - 2a|, \ x \in \mathbb{R},$$

$$C_2 : \ y = 2x + a, \ x \in \mathbb{R}.$$

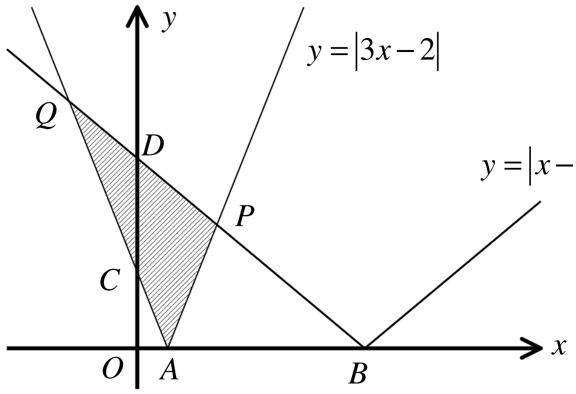
The finite region bounded by C_1, C_2 and the coordinate axes is shown shaded in the above diagram.

Find, in terms of a, the exact area of the shaded region.

$\boxed{}$, $\boxed{\text{area} = \frac{11}{6}a}$



Question 52 (***+)



The figure above shows the graphs of

$$C_1: \ y=|3x-2|, \ x \in \mathbb{R},$$

$$C_2: \ y=|x-5|, \ x \in \mathbb{R}.$$

a) State the coordinates of the points where each of the graphs meet …

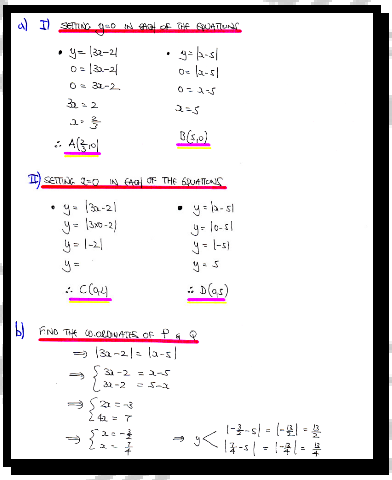
i. … the x axis, indicted by A and B.

ii. … the y axis, indicted by C and D.

The two graphs intersect at the points P and Q.

b) Find the exact area of the triangle APQ.

$\boxed{}$, $\boxed{A\left(\frac{2}{3},0\right), \ B(5,0)}$, $\boxed{C(0,2), \ D(0,5)}$, $\boxed{\text{area} = \frac{169}{24}}$



a) I) SETTING $y=0$ IN EACH OF THE EQUATIONS

- $y=|3x-2|$, $0=|3x-2|$, $0=3x-2$, $3x=2$, $x=\frac{2}{3}$, $\therefore A\left(\frac{2}{3},0\right)$
- $y=|x-5|$, $0=|x-5|$, $0=x-5$, $x=5$, $B(5,0)$

II) SETTING $x=0$ IN EACH OF THE EQUATIONS

- $y=|3x-2|$, $y=|3\times0-2|$, $y=|-2|$, $y=$, $\therefore C(0,2)$
- $y=|x-5|$, $y=|0-5|$, $y=|-5|$, $y=5$, $\therefore D(0,5)$

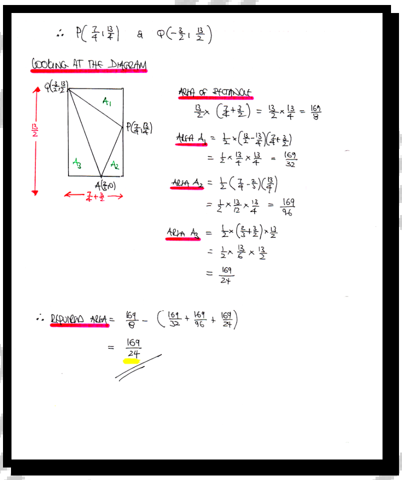
b) FIND THE CO-ORDINATES OF P & Q

$\Rightarrow |3x-2|=|x-5|$

$\Rightarrow \begin{cases} 3x-2=x-5 \\ 3x-2=5-x \end{cases}$

$\Rightarrow \begin{cases} 2x=-3 \\ 4x=7 \end{cases}$

$\Rightarrow \begin{cases} x=-\frac{3}{2} \\ x=\frac{7}{4} \end{cases} \Rightarrow y < \begin{matrix} \left|-\frac{3}{2}-5\right| = \left|-\frac{13}{2}\right| = \frac{13}{2} \\ \left|\frac{7}{4}-5\right| = \left|-\frac{13}{4}\right| = \frac{13}{4} \end{matrix}$



$\therefore P\left(\frac{7}{4},\frac{13}{4}\right)$ & $Q\left(-\frac{3}{2},\frac{13}{2}\right)$

LOOKING AT THE DIAGRAM

AREA OF RECTANGLE

$\frac{13}{2}\times\left(\frac{7}{4}+\frac{3}{2}\right) = \frac{13}{2}\times\frac{13}{4} = \frac{169}{8}$

AREA $A_1 = \frac{1}{2}\times\left(\frac{13}{2}-\frac{13}{4}\right)\left(\frac{7}{4}+\frac{3}{2}\right) = \frac{1}{2}\times\frac{13}{4}\times\frac{13}{4} = \frac{169}{32}$

AREA $A_2 = \frac{1}{2}\left(\frac{7}{4}-\frac{2}{3}\right)\left(\frac{13}{4}\right) = \frac{1}{2}\times\frac{13}{12}\times\frac{13}{4} = \frac{169}{96}$

AREA $A_3 = \frac{1}{2}\times\left(\frac{2}{3}+\frac{3}{2}\right)\times\frac{13}{2} = \frac{1}{2}\times\frac{13}{6}\times\frac{13}{2} = \frac{169}{24}$

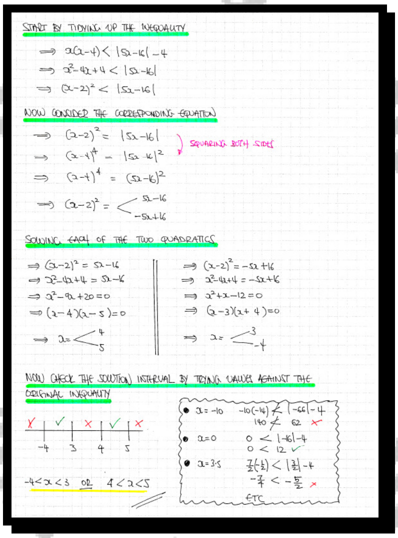
$\therefore$ REQUIRED AREA $= \frac{169}{8} - \left(\frac{169}{32}+\frac{169}{96}+\frac{169}{24}\right) = \frac{169}{24}$

Question 53 (****)

Find the solution interval for the following inequality.

$$x(x-4) < |5x-16| - 4 .$$

$\boxed{}$, $\boxed{-4 < x < 3 \cup 4 < x < 5}$



START BY TIDYING UP THE INEQUALITY

$\Rightarrow x(x-4) < |5x-16| - 4$

$\Rightarrow x^2-4x+4 < |5x-16|$

$\Rightarrow (x-2)^2 < |5x-16|$

NOW CONSIDER THE CORRESPONDING EQUATION

$\Rightarrow (x-2)^2 = |5x-16|$

$\Rightarrow (x-2)^4 = |5x-16|^2$ SQUARING BOTH SIDES

$\Rightarrow (x-2)^4 = (5x-16)^2$

$\Rightarrow (x-2)^2 = \begin{cases} 5x-16 \\ -5x+16 \end{cases}$

SOLVING EACH OF THE TWO QUADRATICS

$\Rightarrow (x-2)^2 = 5x-16$

$\Rightarrow x^2-4x+4 = 5x-16$

$\Rightarrow x^2-9x+20=0$

$\Rightarrow (x-4)(x-5)=0$

$\Rightarrow x = \begin{cases} 4 \\ 5 \end{cases}$

$\Rightarrow (x-2)^2 = -5x+16$

$\Rightarrow x^2-4x+4 = -5x+16$

$\Rightarrow x^2+x-12=0$

$\Rightarrow (x-3)(x+4)=0$

$\Rightarrow x = \begin{cases} 3 \\ -4 \end{cases}$

NOW CHECK THE SOLUTION INTERVAL BY TRYING VALUES AGAINST THE ORIGINAL INEQUALITY

✗ ✓ ✗ ✓ ✗

$-4 \quad 3 \quad 4 \quad 5$

- $x=-10$: $-10(-14) \not< |-66|-4$; $140 \not< 62$ ✗
- $x=0$: $0 < |-16|-4$; $0 < 12$ ✓
- $x=3.5$: $\frac{7}{2}\left(-\frac{1}{2}\right) < \left|\frac{3}{2}\right|-4$; $-\frac{7}{4} < -\frac{5}{2}$ ✗

ETC

$-4 < x < 3$ OR $4 < x < 5$

Question 54 (****)

The functions f and g are defined as

$$f(x) = 4a^2 - x^2, \ x \in \mathbb{R},$$

$$g(x) = |4x - a|, \ x \in \mathbb{R},$$

where a is a constant, such that $a \geq 1$.

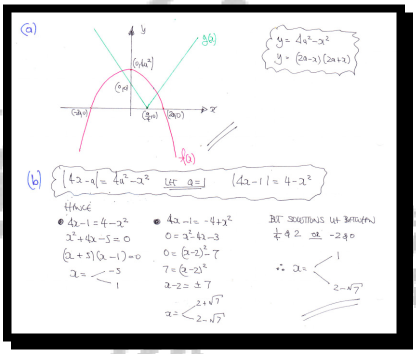
a) Sketch in the same diagram the graph of $f(x)$ and the graph of $g(x)$.

The sketch must include the coordinates of any points where each of the graphs meets the coordinate axes.

b) Find, in exact form where appropriate, the solutions of the equation

$$4 - x^2 = |4x - 1|.$$

$\boxed{}$, $\boxed{x = 2 - \sqrt{7}, \ x = 1}$



Question 55 (****)

The straight line L with equation

$$y = x + 3, \; x \in \mathbb{R},$$

intersects the curve C with equation

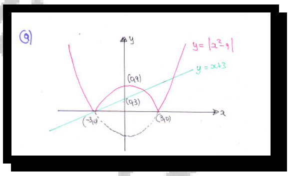
$$y = \left| x^2 - 9 \right|, \; x \in \mathbb{R},$$

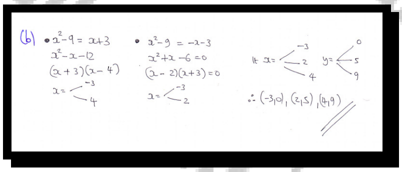
at three distinct points.

a) Sketch on the same set of axes the graph of L and the graph of C. The sketch must include the coordinates of any x or y intercepts.

b) Find the coordinates of the points of intersections between L and C.

$\boxed{(-3,0), \; (3,0), \; (0,3), \; (0,9)}$, $\boxed{(-3,0), \; (2,5), \; (4,7)}$





Question 56 (****)

The function f is defined by

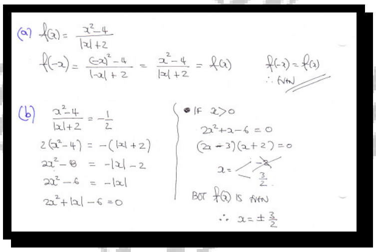
$$f(x) = \frac{x^2 - 4}{|x| + 2}, \quad x \in \mathbb{R}.$$

a) Show that $f(x)$ is even.

b) Solve the equation

$$f(x) = -\tfrac{1}{2}.$$

$\boxed{\quad}$, $\boxed{x = \pm\frac{3}{2}}$



(a) $f(x) = \frac{x^2-4}{|x|+2}$

$f(-x) = \frac{(-x)^2-4}{|-x|+2} = \frac{x^2-4}{|x|+2} = f(x)$ $\quad f(-x) = f(x)$ $\therefore$ EVEN

(b) $\frac{x^2-4}{|x|+2} = -\frac{1}{2}$

$2(x^2-4) = -(|x|+2)$

$2x^2 - 8 = -|x| - 2$

$2x^2 - 6 = -|x|$

$2x^2 + |x| - 6 = 0$

• IF $x > 0$

$2x^2 + x - 6 = 0$

$(2x-3)(x+2) = 0$

$x = \cancel{-2}$, $\frac{3}{2}$

BUT $f(x)$ IS EVEN

$\therefore x = \pm\frac{3}{2}$

Question 57 (****)

The functions f and g are defined as

$$f(x) = x^2 - 4x + 3, \ x \in \mathbb{R}, \ x \geq 2$$

$$g(x) = |x - 15|, \ x \in \mathbb{R}.$$

a) Find an expression for $gf(x)$ and state its domain.

b) Sketch the graph of $gf(x)$ and hence state its range.

The sketch must include the coordinates of any points where the graph meets the coordinate axes, and the starting point of the graph.

c) Solve the equation

$$gf(x) = 12.$$

$\boxed{gf(x) = |x^2 - 4x - 12|, \ x \geq 2}$, $\boxed{(2,16), \ (6,0)}$, $\boxed{x = 4, \ \ x = 2 + 2\sqrt{7}}$

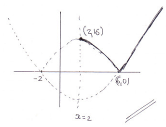
(a) $g(f(x)) = g(x^2-4x+3) = |(x^2-4x+3)-15|$
$= |x^2-4x+3-15| = |x^2-4x-12|$
$= |(x-6)(x+2)|$

$x \geq 2$ IN $\xrightarrow{f(x)}$ OUT

IN $x \in \mathbb{R}$ $\xrightarrow{g(x)}$ OUT

DOMAIN $x \geq 2$

(b)



$g(f(2)) = |(-4)(4)| = 16$

$g(f(x)) \geq 0$

(c) $g(f(x)) = 12$
$|x^2-4x-12| = 12$

$\begin{cases} x^2-4x-12 = 12 \\ x^2-4x-12 = -12 \end{cases}$

$\begin{cases} x^2-4x-24 = 0 \\ x^2-4x = 0 \end{cases}$

HENCE $(x-2)^2-4-24=0$
$(x-2)^2 = 28$
$x-2 = \pm\sqrt{28}$
$x = 2 \pm \sqrt{28}$

OR $x^2-4x=0$
$x(x-4)=0$
$x = \begin{smallmatrix} 0 \\ 4 \end{smallmatrix}$

$\therefore x = 4$ OR $2+\sqrt{28}$

Question 58 (****)

$$f(x) = a - |x - 2a|, \ x \in \mathbb{R},$$

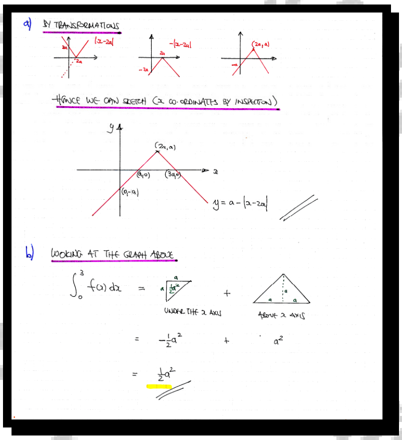
where a is a positive constant.

a) Sketch the graph of $f(x)$.

The sketch must include the coordinates of any points where the graph meets the coordinates axes, and the coordinates of the cusp of the curve.

b) Find the value of $\displaystyle\int_0^{3a} f(x) \, dx$.

$\boxed{\text{SYN-K}}$, $\boxed{\frac{1}{2}a^2}$



Question 59 (****)

The functions f and g are defined as

$$f(x) = |x| - a, \; x \in \mathbb{R},$$

$$g(x) = |2x + 4a|, \; x \in \mathbb{R},$$

where a is a positive constant.

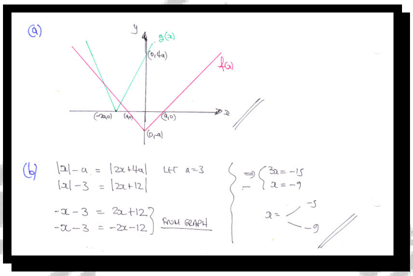
a) Sketch in the same diagram the graph of $f(x)$ and the graph of $g(x)$.

The sketch must include the coordinates of any points where the graphs meet the coordinate axes.

b) Find the solutions of the equation

$$|x| - 3 = |2x + 12|.$$

$\boxed{}$, $\boxed{x = -5, \; x = -9}$



Question 60 **(****)**

The function f is defined as

$$f(x) = a\ln(bx), \ x \in \mathbb{R}, \ x > 0$$

where a and b are positive constants.

a) Given that the graph of $f(x)$ passes through the points $\left(\frac{1}{3}, 0\right)$ and $(3,4)$, find the exact value of a and the value of b.

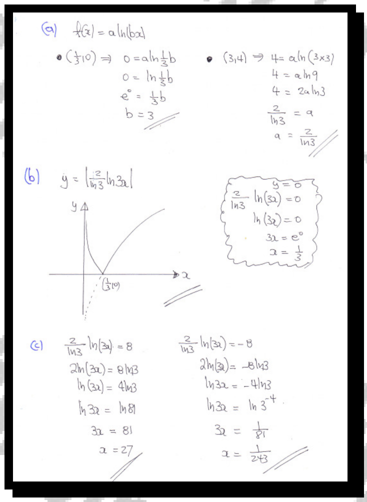
b) Sketch the graph of

$$y = |f(x)|.$$

c) Find, in exact form where appropriate, the solutions of the equation

$$|f(x)| = 8.$$

$\boxed{}$, $\boxed{a = \dfrac{2}{\ln 3}}$, $\boxed{b = 3}$, $\boxed{x = \dfrac{1}{243}, 27}$



(a) $f(x) = a\ln(bx)$

• $\left(\frac{1}{3},0\right) \Rightarrow 0 = a\ln\frac{1}{3}b$, $0 = \ln\frac{1}{3}b$, $e^0 = \frac{1}{3}b$, $b = 3$

• $(3,4) \Rightarrow 4 = a\ln(3\times 3)$, $4 = a\ln 9$, $4 = 2a\ln 3$, $\frac{2}{\ln 3} = a$, $a = \frac{2}{\ln 3}$

(b) $y = \left|\frac{2}{\ln 3}\ln 3x\right|$

$y = 0$, $\frac{2}{\ln 3}\ln(3x) = 0$, $\ln(3x) = 0$, $3x = e^0$, $x = \frac{1}{3}$

(c) $\frac{2}{\ln 3}\ln(3x) = 8$, $2\ln(3x) = 8\ln 3$, $\ln(3x) = 4\ln 3$, $\ln 3x = \ln 81$, $3x = 81$, $x = 27$

$\frac{2}{\ln 3}\ln(3x) = -8$, $2\ln(3x) = -8\ln 3$, $\ln 3x = -4\ln 3$, $\ln 3x = \ln 3^{-4}$, $3x = \frac{1}{81}$, $x = \frac{1}{243}$

Question 61 (****)

The function f is defined by

$$f : x \mapsto |2x-3| - 1, \quad x \in \mathbb{R}.$$

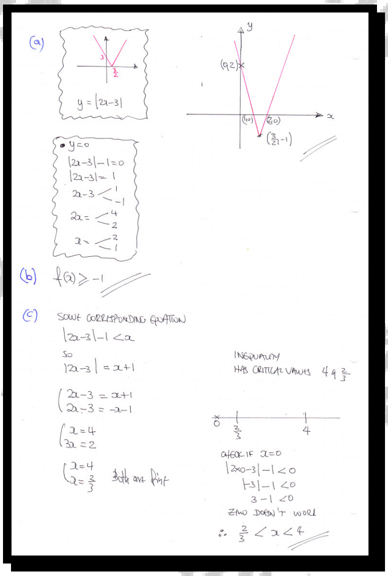
a) Sketch the graph of f.
Mark clearly in the sketch the coordinates of any x or y intercepts as well as the coordinates of any minima or maxima.

b) State the range of f.

c) Solve the inequality

$$|2x-3| - 1 < x.$$

$$\boxed{(1,0),(2,0)(0,2),\min\left(\tfrac{3}{2},-1\right)}, \ \boxed{f(x) \geq -1}, \ \boxed{\tfrac{2}{3} < x < 4}$$



Question 62 (****)

The curve C has equation

$$y = 2\left|x^2 - 6x + 8\right| , \; x \in \mathbb{R} .$$

The straight line L has equation

$$y = 3x - 9 , \; x \in \mathbb{R} .$$

a) Sketch in the same diagram the graph of C and L.

Mark clearly in the sketch the coordinates of any x or y intercepts.

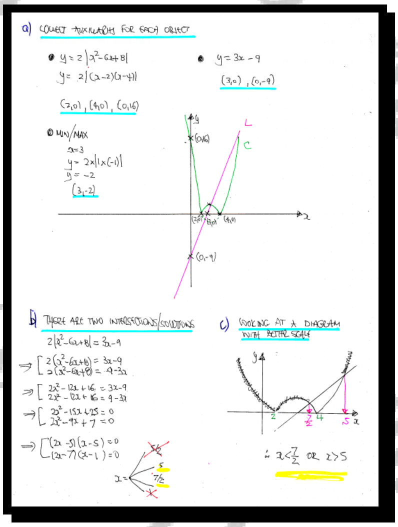
b) Solve the equation

$$2\left|x^2 - 6x + 8\right| = 3x - 9 .$$

c) Hence or otherwise, solve the inequality

$$2\left|x^2 - 6x + 8\right| > 3x - 9 .$$

$\boxed{}$, $\boxed{C:(2,0),(4,0),(0,16)}$, $\boxed{L:(3,0),(0,-9)}$, $\boxed{x = 5, \frac{7}{2}}$, $\boxed{x < \frac{7}{2} \text{ or } x > 5}$



Question 63 (****)

$$f(x) = 9x^2 + 6x + 2, \ x \in \mathbb{R}.$$

$$g(x) = (x+1)(x+3), \ x \in \mathbb{R}$$

Show clearly that …

a) … $f(x) = |f(x)|$.

b) … the equation $g(|x|) = 2$ has no solutions.

$\boxed{}$, $\boxed{\text{proof}}$

(a) $f(x) = 9x^2 + 6x + 2 = \ldots$ COMPLETE THE SQUARE
$= (9x^2 + 6x + 1) + 1$
$= (3x+1)^2 + 1$
$\geqslant 1$

$f(x)$ LIES ENTIRELY ABOVE THE x AXIS
$\therefore f(x) = |f(x)|$

(b) • $g(x) = (x+1)(x+3) = x^2 + 4x + 3$
$g(|x|) = |x|^2 + 4|x| + 3$
BUT $|x|$ & $|x|^2$ ARE AT LEAST ZERO
$\therefore g(|x|) \geqslant 3$
$\therefore g(|x|) = 2$ HAS NO SOLUTION

Question 64 (****)

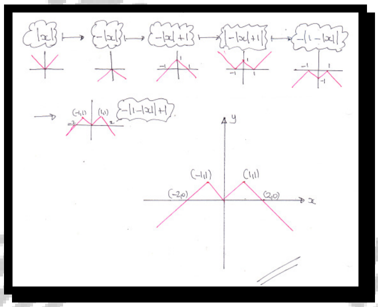
Sketch the graph of

$$y = 1 - \left|1 - |x|\right|, \; x \in \mathbb{R}.$$

The sketch must include the coordinates …

- … of any points where the graph meets the coordinate axes
- … of any cusps of the graph.

graph



Question 65 (****)

The functions f and g are defined as

$$f(x) \equiv |3x+a|+b, \quad x \in \mathbb{R}$$

$$g(x) \equiv 2x+5, \quad x \in \mathbb{R},$$

where a and b are positive constants.

The graph of f meets the graph of g at the points P and Q.

Given that the coordinates of P are $(0,5)$, find the coordinates of Q in terms of a.

$\boxed{}$, $\boxed{Q\left(-\frac{1}{2}a, 5-\frac{4}{5}a\right)}$

USING THE FACT THAT $P(0,5)$ LIES ON BOTH OBJECTS

$y = |3x+a|+b$

$5 = |0+a|+b$

$5 = |a|+b$ $\quad a>0$

$a+b=5$

$b = 5-a$

NOW SOLVING SIMULTANEOUSLY

$|3x+a|+b = 2x+5$

$|3x+a|+\cancel{5}-a = 2x+\cancel{5}$

$|3x+a| = 2x+a$

SOLVING WE OBTAIN

$\left(\begin{array}{l} 3x+a = 2x+a \\ 3x+a = -(2x+a) \end{array} \right.$

$\left(\begin{array}{l} \cancel{x=0} \text{ (ALREADY KNOWN)} \\ 5x = -2a \end{array} \right.$

$x = -\frac{2}{5}a$

FINALLY USING $y = 2x+5$

$y = 2\left(-\frac{2}{5}a\right)+5$

$y = -\frac{4}{5}a+5$

$\therefore Q\left(-\frac{2}{5}a, 5-\frac{4}{5}a\right)$

Question 66 (****+)

The functions f and g are defined as

$$f(x) = |a - 2x| + a, \ x \in \mathbb{R}$$

$$g(x) = |3x + a|, \ x \in \mathbb{R},$$

where a is a positive constant.

a) Sketch in the same set of axes the graph of $f(x)$ and the graph of $g(x)$.

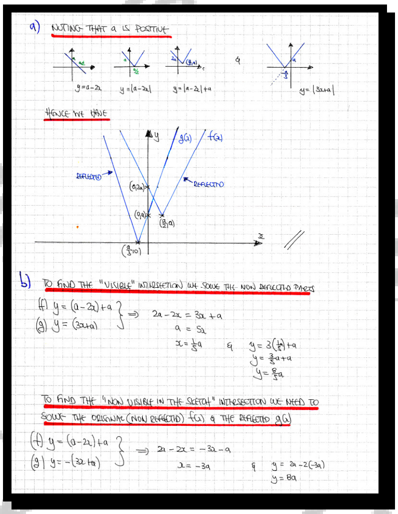
The sketch must include the coordinates of any points where the graphs meet the coordinate axes.

b) Determine, in terms of a, the coordinates of any points of intersection between the two graphs.

c) Find a simplified expression for $gf(x)$.

d) Solve, in terms of a, the equation $gf(x) = 10a$.

$\boxed{}$, $\boxed{(0,a),\left(-\tfrac{1}{3}a,0\right) \ \& \ (0,2a)}$, $\boxed{(a,4a)}$ $\boxed{fg(x) = 3|a-2x| + 4a}$, $\boxed{x = -\tfrac{1}{2}a, \ \tfrac{3}{2}a}$



a) NOTING THAT a IS POSITIVE

$y = a - 2x$ $\quad$ $y = |a - 2x|$ $\quad$ $y = |a-2x| + a$ $\quad$ & $\quad$ $y = |3x + a|$

HENCE WE HAVE

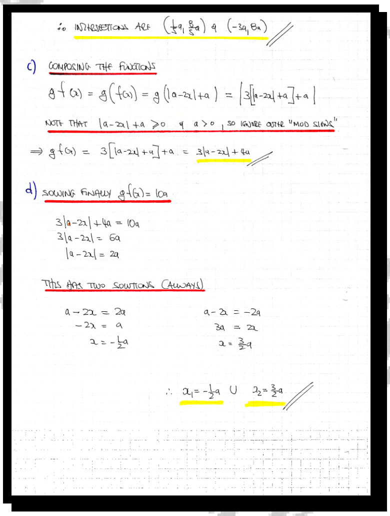
$g(x)$ $\quad$ $f(x)$ $\quad$ REFLECTED $\quad$ REFLECTED $\quad$ $(0,2a)$ $\quad$ $(0,a)$ $\quad$ $\left(\frac{a}{2},a\right)$ $\quad$ $\left(\frac{a}{3},0\right)$

b) TO FIND THE "VISIBLE" INTERSECTION WE SOLVE THE NON REFLECTED PARTS

$$\left.\begin{array}{l}(f)\ y = (a-2x)+a \\ (g)\ y = (3x+a)\end{array}\right\} \Rightarrow \begin{aligned} 2a - 2x &= 3x + a \\ a &= 5x \\ x &= \tfrac{1}{5}a \end{aligned} \quad \& \quad \begin{aligned} y &= 3\left(\tfrac{1}{5}a\right) + a \\ y &= \tfrac{3}{5}a + a \\ y &= \tfrac{8}{5}a \end{aligned}$$

TO FIND THE "NON VISIBLE IN THE SKETCH" INTERSECTION WE NEED TO SOLVE THE ORIGINAL (NON REFLECTED) $f(x)$ & THE REFLECTED $g(x)$

$$\left.\begin{array}{l}(f)\ y = (a-2x)+a \\ (g)\ y = -(3x+a)\end{array}\right\} \Rightarrow \begin{aligned} 2a - 2x &= -3x - a \\ x &= -3a \end{aligned} \quad \& \quad \begin{aligned} y &= 2a - 2(-3a) \\ y &= 8a \end{aligned}$$



$\therefore$ INTERSECTIONS ARE $\left(\frac{1}{5}a, \frac{8}{5}a\right)$ & $(-3a, 8a)$

c) COMPOSING THE FUNCTIONS

$$gf(x) = g(f(x)) = g(|a-2x|+a) = \Big|3\big[|a-2x|+a\big]+a\Big|$$

NOTE THAT $|a-2x| + a \geqslant 0$ & $a > 0$, SO IGNORE OUTER "MOD SIGNS"

$$\Rightarrow gf(x) = 3\big[|a-2x|+a\big]+a = 3|a-2x| + 4a$$

d) SOLVING FINALLY $gf(x) = 10a$

$$\begin{aligned} 3|a-2x| + 4a &= 10a \\ 3|a-2x| &= 6a \\ |a-2x| &= 2a \end{aligned}$$

THIS HAS TWO SOLUTIONS (ALWAYS)

$$\begin{aligned} a - 2x &= 2a \\ -2x &= a \\ x &= -\tfrac{1}{2}a \end{aligned} \qquad \begin{aligned} a - 2x &= -2a \\ 3a &= 2x \\ x &= \tfrac{3}{2}a \end{aligned}$$

$$\therefore x_1 = -\tfrac{1}{2}a \ \cup \ x_2 = \tfrac{3}{2}a$$

Question 67 (****+)

The function f is defined as

$$f : x \mapsto \ln|4x-12|, \quad x \in \mathbb{R}, \; x \neq 3 .$$

Consider the following sequence of transformations T_1, T_2 and T_3

$$\ln x \xrightarrow{T_1} \ln|x| \xrightarrow{T_2} \ln|x-12| \xrightarrow{T_3} \ln|4x-12| .$$

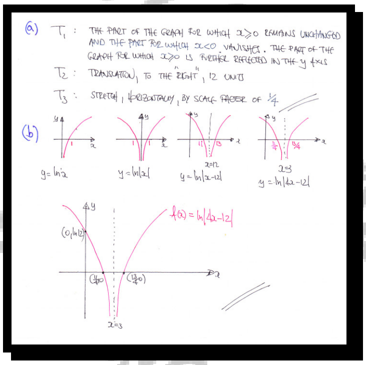
a) Describe geometrically the transformations T_1, T_2 and T_3

b) Hence sketch the graph of f.
Indicate clearly any intersections with the axes and the equation of its asymptote.

$\boxed{T_1 = \text{maintains the graph for } x \geq 0\text{, and further reflects this section in the } y \text{ axis}}$,

$\boxed{T_2 = \text{translation, "right", 12 units}}$, $\boxed{T_3 = \text{enlargement in } x\text{, scale factor } \frac{1}{4}}$, $\boxed{(0, \ln 12)}$,

$\boxed{}$, $\boxed{\left(\frac{11}{4}, 0\right)}$, $\boxed{\left(\frac{13}{4}, 0\right)}$, $\boxed{x = 3}$



Question 68 (****+)

The function f is defined as

$$f : x \mapsto \ln\left(\tfrac{1}{2}x - 4\right),\ x \in \mathbb{R},\ x > 8.$$

Consider the following sequence of transformations T_1 and T_2.

$$\ln x \xrightarrow{T_1} \ln(x-4) \xrightarrow{T_2} \ln\left(\tfrac{1}{2}x - 4\right).$$

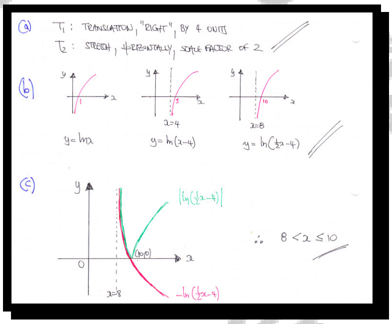
a) Describe geometrically the transformations T_1 and T_2.

b) Hence sketch the graph of f.
Indicate clearly any intersections with the axes and the equation of its asymptote.

c) Find the set of values that satisfy the equation

$$-\ln\left(\tfrac{1}{2}x - 4\right) = \left|\ln\left(\tfrac{1}{2}x - 4\right)\right|.$$

$\boxed{T_1 = \text{translation, "right", 4 units}}$, $\boxed{T_2 = \text{enlargement in } x\text{, scale factor 2}}$, $\boxed{(10,0)}$,
$\boxed{8 < x \le 10}$



Question 69 (****+)

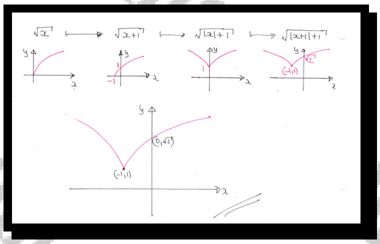
By considering the following sequence of transformations T_1, T_2 and T_3

$$\sqrt{x} \xrightarrow{T_1} \sqrt{x+1} \xrightarrow{T_2} \sqrt{|x|+1} \xrightarrow{T_3} \sqrt{|x+1|+1}$$

sketch the graph of $y=\sqrt{|x+1|+1}$.

Indicate the coordinates of any intersections with the axes, and the coordinates of the cusp of the curve.

$\boxed{\quad}$, $\boxed{(0,\sqrt{2})}$, $\boxed{(1,1)}$

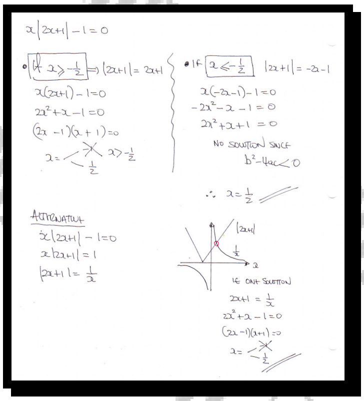


Question 70 (****+)

Solve the equation

$$x|2x+1|-1=0.$$

$\boxed{x=\frac{1}{2}}$



Question 71 (****+)

A curve is defined by

$$f(x) = k\left(x^2 - 4x\right), \ x \in \mathbb{R},$$

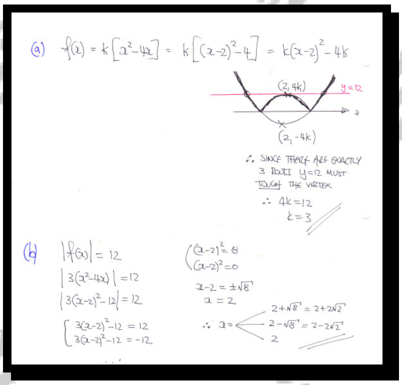
where k is a positive constant

The equation $|f(x)| = 12$ has three distinct roots.

a) Determine the value of k.

b) Find the three roots of the equation, in exact surd form where appropriate.

$\boxed{}$, $\boxed{k = 3}$, $\boxed{x = 2, \ 2 \pm 2\sqrt{2}}$



Question 72 (****+)

Evaluate the following integral

$$\int_{-1}^{1} 2\left(x+|x|\right)-7x|x| \ dx \, .$$

$\boxed{\text{SP}}$, $\boxed{2}$

$$\int_{-1}^{1} 2(x+|x|)-7x|x| \, dx$$

$$= \int_{-1}^{1} \underbrace{\cancel{2x}}_{\text{ODD}} + \underbrace{2|x|}_{\text{EVEN}} - \underbrace{\cancel{7x|x|}}_{\substack{\text{ODD}\times\text{EVEN} \\ \text{ODD}}} \, dx = \int_{-1}^{1} 2|x| \, dx$$

$$= 2\int_{0}^{1} 2|x| \, dx = \int_{0}^{1} 4x \, dx = \left[2x^2\right]_{0}^{1}$$

$$= 2-0$$

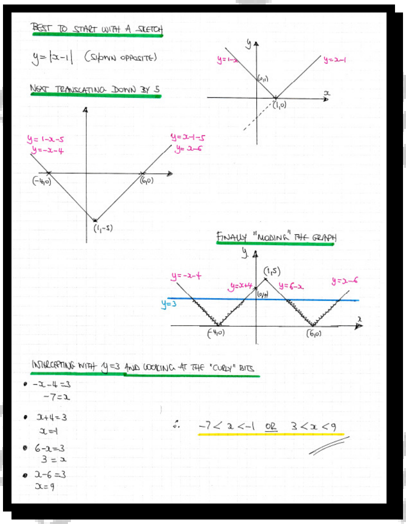
$$= 2$$

Question 73 (****+)

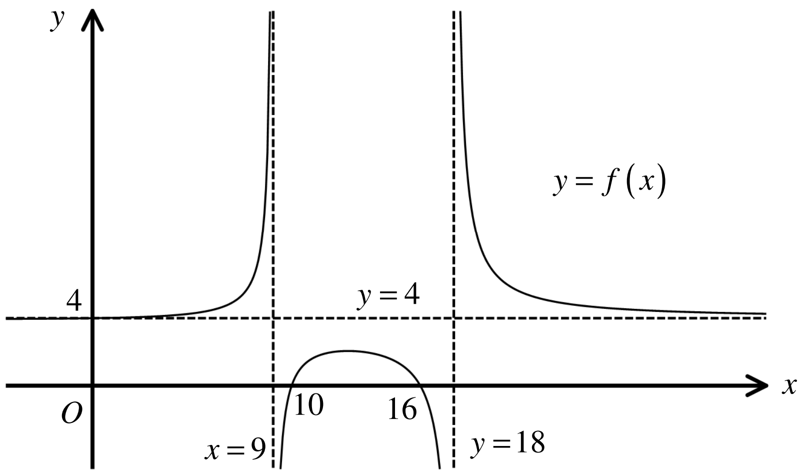
Find the solution interval of the following modulus inequality.

$$||x-1|-5|<3.$$

V , , $\boxed{-7<x<-1 \;\cup\; 3<x<9}$



Question 74 (****+)

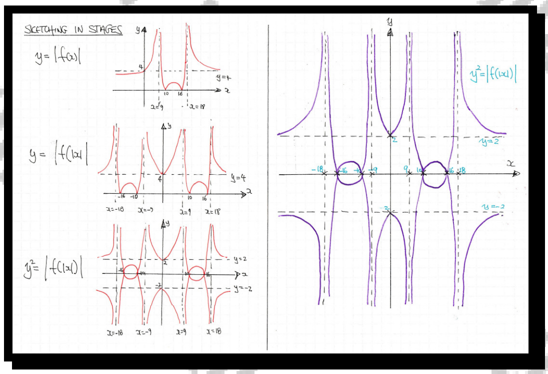


The figure above shows the curve with equation $y = f(x)$.

The equations of the three asymptotes to the curve, and the three intercepts of the curve with the coordinate axes are marked in the figure.

Sketch a detailed graph of $y^2 = \left|f(|x|)\right|$.

, graph



Question 75 (*****)

Determine the range of values of x that satisfy the inequality

$$\left|\frac{x+3}{x}\right| \geq \left|\frac{x}{2-x}\right|.$$

$\boxed{}$, $\boxed{-2 \leq x < 0 \quad \cup \quad 0 < x \leq \frac{3}{2} \quad \cup \quad x \geq 6}$

$\left|\frac{x+3}{x}\right| \geqslant \left|\frac{x}{2-x}\right|$

$\Rightarrow \frac{|x+3|}{|x|} \geqslant \frac{|x|}{|2-x|}$

EVERYTHING IS NON NEGATIVE, SO WE MAY MULTIPLY ACROSS, BUT LET US NOTE THAT $x \neq 0, x \neq 2$

$\Rightarrow |x+3||2-x| \geqslant |x|^2$

$\Rightarrow |(x+3)(2-x)| \geqslant x^2$

$\Rightarrow |(x+3)(x-2)| \geqslant x^2$

$\Rightarrow |x^2+x-6| \geqslant x^2$

THERE ARE 3 CRITICAL VALUES FOR THIS INEQUALITY, $x=-3, x=0, x=2$

- IF $x>2$

$x^2+x-6 \geqslant x^2$

$x \geqslant 6$

$\therefore x \geqslant 6$ ONLY

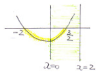
- IF $0<x<2$

$-x^2-x+6 \geqslant x^2$

$-2x^2-x+6 \geqslant 0$

$2x^2+x-6 \leqslant 0$

$(2x-3)(x+2) \leqslant 0$



$\therefore 0<x \leqslant \frac{3}{2}$

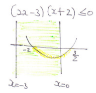
- IF $-3 \leqslant x < 0$

$-x^2-x+6 \geqslant x^2$

$-2x^2-x+6 \geqslant 0$

$2x^2+x-6 \leqslant 0$

$(2x-3)(x+2) \leqslant 0$



$\therefore -2 \leqslant x < 0$

- IF $x \leqslant -3$

$x^2+x-6 \geqslant x^2$

$x \geqslant 6$

NO COMMON SOLUTIONS

COLLECTING ALL THE SOLUTION INTERVALS

$-2 \leqslant x < 0$ OR $0 < x \leqslant \frac{3}{2}$ OR $x \geqslant 6$

Question 76 (*****)

$$f(x) = \big|2 - |x+2|\big| - 4, \; x \in \mathbb{R}.$$

a) Sketch the graph of $f(x)$

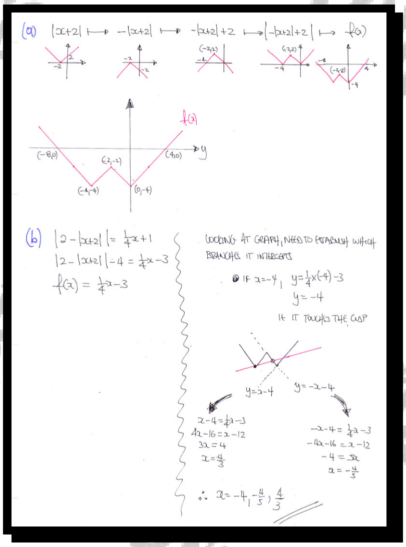
The sketch must include the coordinates …

- … of any points where the graph meets the coordinate axes
- … of any cusps of the graph.

b) Hence, or otherwise, solve the equation

$$\big|2 - |x+2|\big| = \frac{1}{4}x + 1.$$

$$\boxed{x = -4, -\frac{4}{5}, \frac{4}{3}}$$



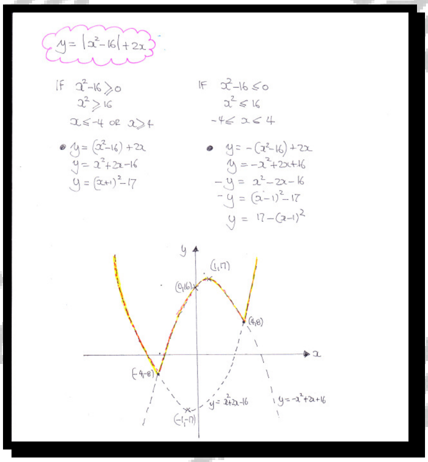
Question 77 (*****)

Sketch the graph of

$$y = \left|x^2 - 16\right| + 2x, \quad x \in \mathbb{R}.$$

The sketch must include the coordinates of any cusps or any stationary points

[illegible], graph

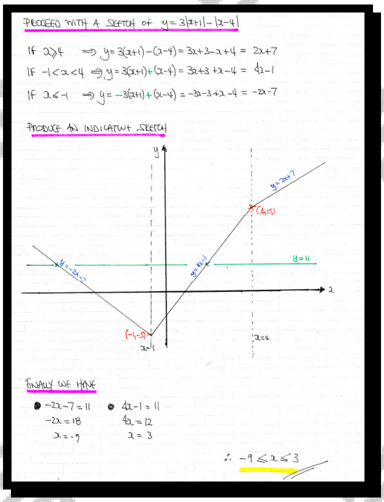


Question 78 (*****)

Solve the following modulus inequality

$$3|x+1|-|x-4| \le 11, \; x \in \mathbb{R}.$$

$-9 \le x \le 3$



Question 79 (*****)

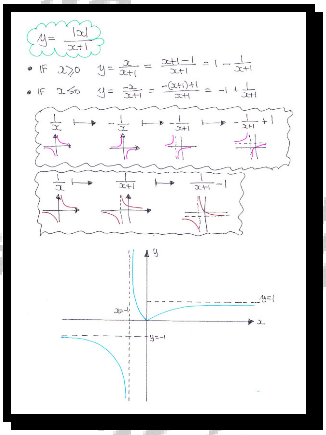
Sketch the graph of

$$y = \frac{|x|}{x+1}, \ x \in \mathbb{R}.$$

The sketch must include the equations of any asymptotes of the curve, and the coordinates of any points where the curve meets the coordinate axes.

[*No credit will be given to non analytical sketches based on plotting coordinates*]

$\boxed{}$, $\boxed{\text{graph}}$



Question 80 (*****)

Find, in exact surd form, the solutions of the following equation

$$x^2 + |2x-3| - |4-x| = 0.$$

$$\boxed{x_1 = \frac{-3+\sqrt{37}}{2}, \quad x_2 = \frac{1-\sqrt{5}}{2}}$$

$x^2 + |2x-3| - |4-x| = 0$

- THE EQUATION HAS 2 CRITICAL VALUES (WITH RESPECT TO THE MOD)

$x = \frac{3}{2}$ & $x = 4$

- IF $x \geqslant 4 \implies x^2 + (2x-3) - (x-4) = 0$

$x^2 + x + 1 = 0$

NO SOLUTIONS

- IF $\frac{3}{2} \leqslant x \leqslant 4 \implies x^2 + (2x-3) - (4-x) = 0$

$\implies x^2 + 3x - 7 = 0$

$\implies \left(x+\frac{3}{2}\right)^2 - \frac{9}{4} - \frac{28}{4} = 0$

$\implies \left(x+\frac{3}{2}\right)^2 - \frac{37}{4} = 0$

$\implies x = -\frac{3}{2} + \frac{\sqrt{37}}{2}$ ← IN RANGE, or $\cancel{-\frac{3}{2} - \frac{\sqrt{37}}{2}} < \frac{3}{2}$

- IF $x \leqslant \frac{3}{2} \implies x^2 + (3-2x) - (4-x) = 0$

$\implies x^2 - x - 1 = 0$

$\implies \left(x-\frac{1}{2}\right)^2 - \frac{1}{4} - 1 = 0$

$\implies \left(x-\frac{1}{2}\right)^2 = \frac{5}{4}$

$\implies x = \cancel{\frac{1}{2} + \frac{\sqrt{5}}{2}} > \frac{3}{2}$, or $\frac{1}{2} - \frac{\sqrt{5}}{2}$ ← IN RANGE

$\therefore x = -\frac{3}{2} + \frac{\sqrt{37}}{2}$, $\frac{1}{2} - \frac{\sqrt{5}}{2}$

Question 81 (*****)

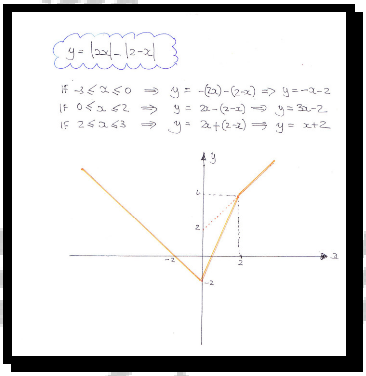
Sketch the graph of

$$y = |2x| - |2 - x|, \ x \in \mathbb{R}.$$

The sketch must include the coordinates of any points where the curve meets the coordinate axes.

[*No credit will be given to non analytical sketches based on plotting coordinates*]

graph

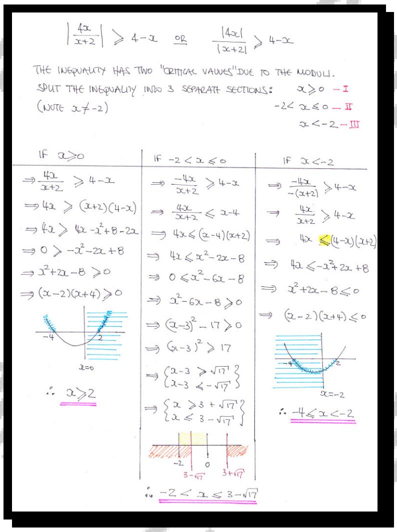


Question 82 (*****)

Find the set of values of x that satisfy the inequality

$$\left|\frac{4x}{x+2}\right| \ge 4-x\,.$$

$\boxed{}$, $\boxed{-4 \le x < -2 \quad \cup \quad -2 < x \le 3-\sqrt{17} \quad \cup \quad x \ge 2}$



$$\left|\frac{4x}{x+2}\right| \geqslant 4-x \quad \underline{\text{OR}} \quad \frac{|4x|}{|x+2|} \geqslant 4-x$$

THE INEQUALITY HAS TWO "CRITICAL VALUES" DUE TO THE MODULI.
SPLIT THE INEQUALITY INTO 3 SEPARATE SECTIONS: $x \geqslant 0$ — I
(NOTE $x \neq -2$) $-2 < x \leqslant 0$ — II
$x < -2$ — III

IF $x \geqslant 0$	IF $-2 < x \leqslant 0$	IF $x < -2$
$\Rightarrow \frac{4x}{x+2} \geqslant 4-x$	$\Rightarrow \frac{-4x}{x+2} \geqslant 4-x$	$\Rightarrow \frac{-4x}{-(x+2)} \geqslant 4-x$
$\Rightarrow 4x \geqslant (x+2)(4-x)$	$\Rightarrow \frac{4x}{x+2} \leqslant x-4$	$\Rightarrow \frac{4x}{x+2} \geqslant 4-x$
$\Rightarrow 4x \geqslant 4x - x^2 + 8 - 2x$	$\Rightarrow 4x \leqslant (x-4)(x+2)$	$\Rightarrow 4x \leqslant (4-x)(x+2)$
$\Rightarrow 0 \geqslant -x^2 - 2x + 8$	$\Rightarrow 4x \leqslant x^2 - 2x - 8$	$\Rightarrow 4x \leqslant -x^2 + 2x + 8$
$\Rightarrow x^2 + 2x - 8 \geqslant 0$	$\Rightarrow 0 \leqslant x^2 - 6x - 8$	$\Rightarrow x^2 + 2x - 8 \leqslant 0$
$\Rightarrow (x-2)(x+4) \geqslant 0$	$\Rightarrow x^2 - 6x - 8 \geqslant 0$	$\Rightarrow (x-2)(x+4) \leqslant 0$
-4, 2, $x=0$	$\Rightarrow (x-3)^2 - 17 \geqslant 0$	-4, 2, $x=-2$
$\therefore \ x \geqslant 2$	$\Rightarrow (x-3)^2 \geqslant 17$	$\therefore \ -4 \leqslant x < -2$
	$\Rightarrow \begin{cases} x-3 \geqslant \sqrt{17} \\ x-3 \leqslant -\sqrt{17} \end{cases}$	
	$\Rightarrow \begin{cases} x \geqslant 3+\sqrt{17} \\ x \leqslant 3-\sqrt{17} \end{cases}$	
	-2, 0, $3-\sqrt{17}$, $3+\sqrt{17}$	
	$\therefore \ -2 < x \leqslant 3-\sqrt{17}$	

Question 83 (*****)

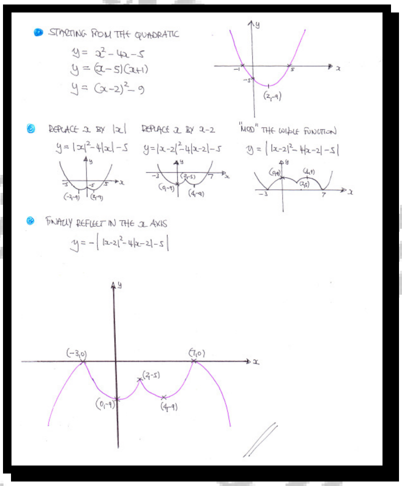
By considering a sequence of four transformations, or otherwise, sketch the graph of

$$y = -\left| \left| x-2 \right|^2 - 4\left| x-2 \right| - 5 \right|.$$

Indicate the coordinates of any intersections with the axes, and the coordinates of the cusp of the curve.

[*No credit will be given to non analytical sketches based on plotting coordinates*]

$\boxed{}$, $\boxed{(-3,0),\ (7,0),\ (0,-9),\ (2,-5)}$



Question 84 (*****)

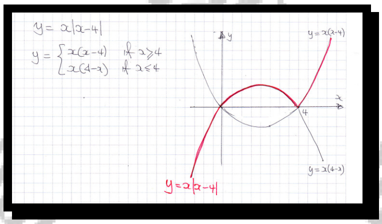
By considering the graphs of two separate curves, or otherwise, sketch the graph of

$$y = x|x-4|.$$

Indicate the coordinates of any intersections with the axes, and the coordinates of the cusp of the curve.

[*No credit will be given to non analytical sketches based on plotting coordinates*]

$$\boxed{(0,0),(4,0)}$$



Question 85 (*****)

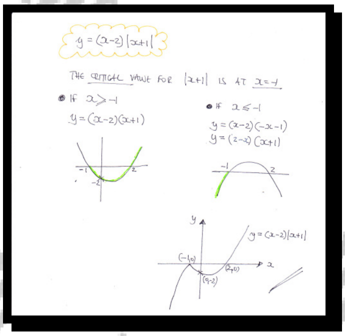
By considering the graphs of two separate curves, or otherwise, sketch the graph of

$$y = (x-2)|x+1|.$$

Indicate the coordinates of any intersections with the axes, and the coordinates of the cusp of the curve.

[*No credit will be given to non analytical sketches based on plotting coordinates*]

graph



Question 86 (*****)

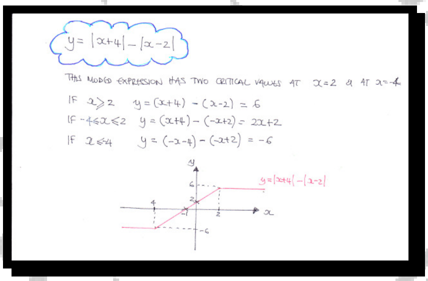
By considering the graphs of three separate lines, or otherwise, sketch the graph of

$$y = |x+4| - |x-2|$$

Indicate the coordinates of any intersections with the axes, and the coordinates of the cusp of the curve.

[*No credit will be given to non analytical sketches based on plotting coordinates*]

graph

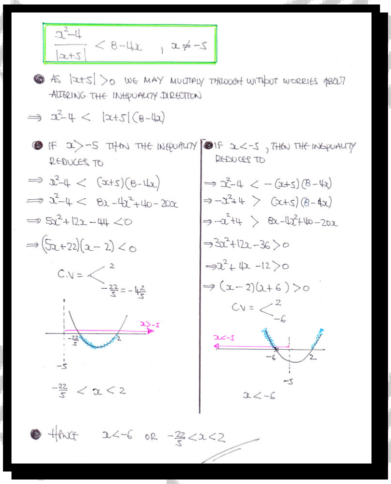


Question 87 (*****)

Find the set of values of x that satisfy the inequality

$$\frac{x^2-4}{|x+5|} < 8-4x \, .$$

$\boxed{\quad}$, $\boxed{x<-6 \quad \cup \quad -\frac{22}{5}<x<2}$



Question 88 (*****)

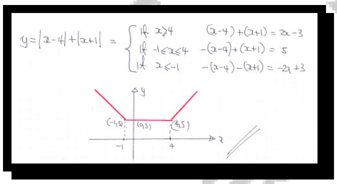
By considering the graphs of three separate lines, or otherwise, sketch the graph of

$$y=|x-4|+|x+1|$$

Indicate the coordinates of any intersections with the axes, and the coordinates of the cusp of the curve.

[*No credit will be given to non analytical sketches based on plotting coordinates*]

$\boxed{(-1,5),(0,5),(4,5)}$



Question 89 (*****)

The point $A(1,-1)$ lies on the curve with equation

$$y = \left| x^2 - |2x| \right| - 2x, \ x \in \mathbb{R}.$$

The tangent to the curve at A meets the curve at three more points B, C and D.

Sketch the curve and its tangent at A in a single set of axes.

Give the coordinates of B, C and D in exact form where appropriate.

$\boxed{}$, $\boxed{\text{graph}}$

$$\boxed{y = \left| x^2 - |2x| \right| - 2x, \ x \in \mathbb{R}}$$

- THE CRITICAL VALUES OF THE MODULUS ARE $x=0$ (FROM $|2x|=0$), $x=2$ (FROM $x^2-2x=0$), $x=-2$ (FROM $x^2+2x=0$)
- CONSIDER THE GRAPH IN SEPARATE REGIONS

• IF $x \geqslant 2$
$y = (x^2-(2x))-2x$
$y = x^2-4x$

• IF $0 \leqslant x \leqslant 2$
$y = -(x^2-(2x))-2x$
$y = -x^2$

• IF $-2 \leqslant x \leqslant 0$
$y = -(x^2+(2x))-2x$
$y = -x^2-4x$

• IF $x \leqslant -2$
$y = (x^2+(2x))-2x$
$y = x^2$

- FINDING THE EQUATION OF THE TANGENT AT $x=1$

$y = -x^2$
$\frac{dy}{dx} = -2x$
$\left.\frac{dy}{dx}\right|_{x=1} = -2$
$A(1,-1)$

$y - y_0 = m(x-x_0)$
$y+1 = -2(x-1)$
$y+1 = -2x+2$
$y = 1-2x$

• IF $x \geqslant 2$
$\Rightarrow x^2-4x = 1-2x$
$\Rightarrow x^2-2x-1=0$
$\Rightarrow (x-1)^2-1=1$
$\Rightarrow (x-1)^2=2$
$\Rightarrow x-1 = \pm\sqrt{2}$
$\Rightarrow x = \begin{cases} 1+\sqrt{2} \\ \cancel{1-\sqrt{2}} \end{cases}$

$y = 1-2x$
$y = 1-2(1+\sqrt{2})$
$y = -1-2\sqrt{2}$

$B(1+\sqrt{2}, -1-2\sqrt{2})$

• IF $-2 \leqslant x \leqslant 2$
$\Rightarrow -x^2-4x = 1-2x$
$\Rightarrow -x^2-2x-1=0$
$\Rightarrow x^2+2x+1=0$
$\Rightarrow (x+1)^2=0$
$\Rightarrow x=-1$

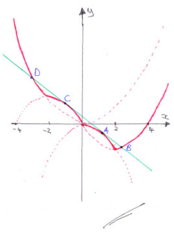
$y = 1-2x$
$y = 3$

$C(-1,3)$
↑
TANGENT POINT

• IF $x < -2$
$\Rightarrow x^2 = 1-2x$
$\Rightarrow x^2+2x-1=0$
$\Rightarrow (x+1)^2-1=1$
$\Rightarrow (x+1)^2=2$
$\Rightarrow x+1=\pm\sqrt{2}$
$\Rightarrow x = \begin{cases} \cancel{-1+\sqrt{2}} \\ -1-\sqrt{2} \end{cases}$

$y = 1-2x$
$y = 1-2(-1-\sqrt{2})$
$y = 3+2\sqrt{2}$

$D(-1-\sqrt{2}, 3+2\sqrt{2})$



Question 90 (*****)

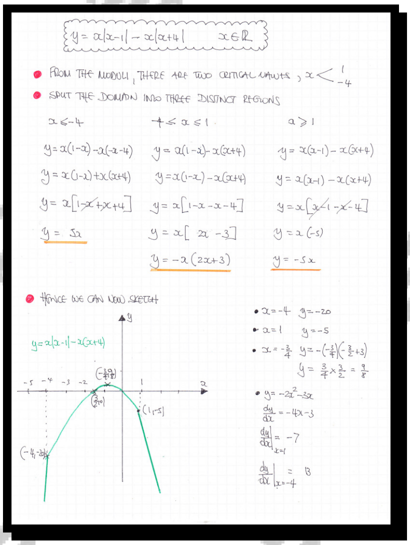
Sketch the graph of

$$y = x|x-1| - x|x+4|, \quad x \in \mathbb{R}.$$

Indicate the coordinates of any intersections with the axes, and the coordinates of any cusps of the curve.

[*No credit will be given to non analytical sketches based on plotting coordinates*]

$\boxed{}$, $\boxed{\text{graph}}$



$y = x|x-1| - x|x+4| \qquad x \in \mathbb{R}$

- FROM THE MODULI, THERE ARE TWO CRITICAL VALUES, $x <^{1}_{-4}$
- SPLIT THE DOMAIN INTO THREE DISTINCT REGIONS

$x \leqslant -4$

$y = x(1-x) - x(-x-4)$

$y = x(1-x) + x(x+4)$

$y = x[1 - \cancel{x} + \cancel{x} + 4]$

$y = 5x$

$-4 \leqslant x \leqslant 1$

$y = x(1-x) - x(x+4)$

$y = x(1-x) - x(x+4)$

$y = x[1 - x - x - 4]$

$y = x[\ 2x - 3]$

$y = -x(2x+3)$

$x \geqslant 1$

$y = x(x-1) - x(x+4)$

$y = x(x-1) - x(x+4)$

$y = x[\cancel{x} - 1 - \cancel{x} - 4]$

$y = x(-5)$

$y = -5x$

- HENCE WE CAN NOW SKETCH

- $x = -4 \quad y = -20$
- $x = 1 \quad y = -5$
- $x = -\frac{3}{4} \quad y = -\left(-\frac{3}{4}\right)\left(-\frac{3}{2}+3\right)$

  $y = \frac{3}{4} \times \frac{3}{2} = \frac{9}{8}$

- $y = -2x^2 - 3x$

  $\frac{dy}{dx} = -4x - 3$

  $\left.\frac{dy}{dx}\right|_{x=1} = -7$

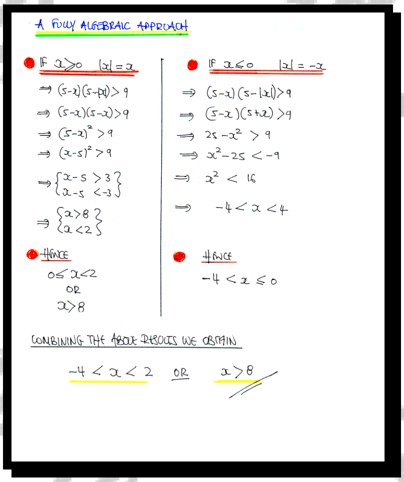
  $\left.\frac{dy}{dx}\right|_{x=-4} = 13$

Question 91 (*****)

Solve the following inequality.

$$(5-x)(5-|x|)>9, \ x \in \mathbb{R}$$

$\boxed{}$, $\boxed{-4<x<2, \ \cup \ \ x>8}$



A FULLY ALGEBRAIC APPROACH

- IF $x \geqslant 0 \quad |x|=x$

$\Rightarrow (5-x)(5-|x|)>9$

$\Rightarrow (5-x)(5-x)>9$

$\Rightarrow (5-x)^2>9$

$\Rightarrow (x-5)^2>9$

$\Rightarrow \begin{Bmatrix} x-5>3 \\ x-5<-3 \end{Bmatrix}$

$\Rightarrow \begin{Bmatrix} x>8 \\ x<2 \end{Bmatrix}$

- HENCE

$0 \leqslant x < 2$

OR

$x>8$

- IF $x \leqslant 0 \quad |x|=-x$

$\Rightarrow (5-x)(5-|x|)>9$

$\Rightarrow (5-x)(5+x)>9$

$\Rightarrow 25-x^2>9$

$\Rightarrow x^2-25<-9$

$\Rightarrow x^2<16$

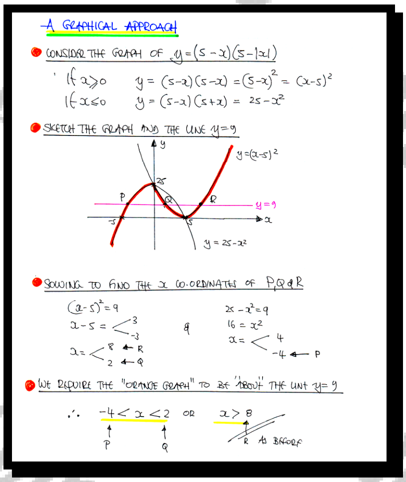
$\Rightarrow -4<x<4$

- HENCE

$-4<x \leqslant 0$

COMBINING THE ABOVE RESULTS WE OBTAIN

$-4<x<2 \quad$ OR $\quad x>8$



A GRAPHICAL APPROACH

- CONSIDER THE GRAPH OF $y=(5-x)(5-|x|)$

IF $x \geqslant 0 \quad y=(5-x)(5-x)=(5-x)^2=(x-5)^2$

IF $x \leqslant 0 \quad y=(5-x)(5+x)=25-x^2$

- SKETCH THE GRAPH AND THE LINE $y=9$

- SOLVING TO FIND THE x CO-ORDINATES OF P, Q & R

$(x-5)^2=9$

$x-5=\begin{cases} 3 \\ -3 \end{cases}$

$x=\begin{cases} 8 \leftarrow R \\ 2 \leftarrow Q \end{cases}$

&

$25-x^2=9$

$16=x^2$

$x=\begin{cases} 4 \\ -4 \leftarrow P \end{cases}$

- WE REQUIRE THE "ORANGE GRAPH" TO BE "ABOVE" THE LINE $y=9$

$\therefore \ \underset{P}{-4}<x<\underset{Q}{2} \quad$ OR $\quad x>\underset{R}{8}$ AS BEFORE

Question 92 (*****)

Solve the following inequality in the largest real domain.

$$\frac{x^2-2|x|-8}{6|x|^3-5x^2+12|x|} \le 0 .$$

$\boxed{}$, $\boxed{-4 \le x \le 4, \quad x \ne 0}$

FIRSTLY LET US NOTE THAT THE L.H.S IS EVEN

$$\Rightarrow \frac{x^2-2|x|-8}{6|x|^3-5x^2+12|x|} \le 0$$

$$\Rightarrow \frac{x^2-2x-8}{6x^3-5x^2+12x} \le 0 \qquad (\text{FOR } x>0)$$

$$\Rightarrow \frac{(x+2)(x-4)}{x(6x^2-5x+12)} \le 0$$

IRREDUCIBLE AS $(-5)^2-4\times6\times12<0$

HENCE WE HAVE CRITICAL VALUES

$x = 0$ (ASYMPTOTE), $\cancel{-2}$ $x>0$, 4 (x INTERCEPT)

CHECK BETWEEN THESE VALUES

0 $\frac{+\,-}{+\,+}$ 4 $\frac{+\,+}{+\,+}$

$\therefore \ 0 < x \le 4$

AS THE FUNCTION ON THE L.H.S IS EVEN WE HAVE

$-4 \le x \le 4, \quad x \ne 0$

Question 93 (*****)

Sketch the curve with equation

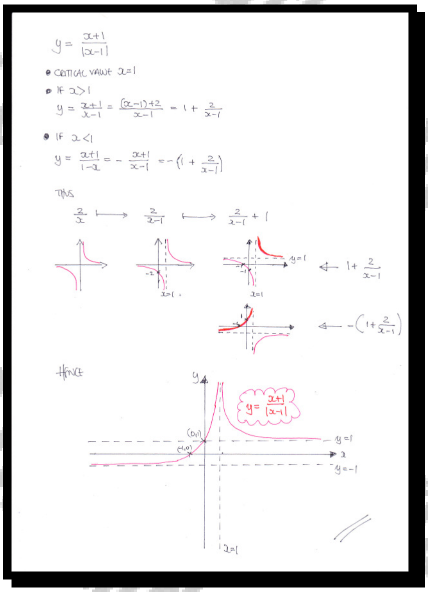
$$y = \frac{x+1}{|x-1|}, \ x \in \mathbb{R}, \ x \neq 1.$$

The sketch must include …

- … the coordinates of all the points where the curve meets the coordinate axes.
- … the equations of the asymptotes of the curve.

[*No credit will be given to non analytical sketches based on plotting coordinates*]

graph



Question 94 (*****)

A curve C has equation

$$y = \frac{3|x|-1}{2x^2+2-|x+2|}, \quad x \in \mathbb{R}, \quad x \neq 0, \quad x \neq \frac{1}{2}.$$

Find, in exact simplified surd form, the y coordinate of the stationary point of C.

$\boxed{}$, $\boxed{y = 7 - 2\sqrt{10}}$

- FOR THE SAKE OF SIMPLICITY WHEN IT COMES TO DIFFERENTIATION, LET US NOTE THAT

$$\frac{d}{dx}\left[|x|\right] = \text{sign}(x) = \begin{cases} 1 & \text{if } x>0 \\ -1 & \text{if } x<0 \end{cases}$$

- SO BY THE QUOTIENT RULE & CHAIN RULE WHERE NEEDED, WE OBTAIN

$$y = \frac{3|x|-1}{2x^2+2-|x+2|} \quad \text{WITH CRITICAL VALUES } x = \begin{cases} 0 \\ -2 \end{cases}$$

$$\frac{dy}{dx} = \frac{\left[2x^2+2-|x+2|\right]\left[3\,\text{sign}(x)\right] - \left[3|x|-1\right]\left[4x - \text{sign}(x+2)\right]}{\left[2x^2+2-|x+2|\right]^2}$$

- LOOKING FOR STATIONARY POINTS BY CONSIDERING THE NUMERATOR ONLY

IF $x>0$

$$\boxed{\begin{aligned} |x| &= x \\ |x+2| &= x+2 \\ \text{sign}(x) &= 1 \\ \text{sign}(x+2) &= 1 \end{aligned}}$$

$$\left[2x^2+2-(x+2)\right]\left[3\times1\right] - \left[3x-1\right]\left[4x-1\right] = 0$$
$$3(2x^2-x) - (12x^2-7x+1) = 0$$
$$6x^2-3x-12x^2+7x-1 = 0$$
$$0 = 6x^2-4x+1$$

$$b^2-4ac = (-4)^2-4\times6\times1 = -8<0$$

NO STATIONARY POINTS IN THIS RANGE

IF $-2<x<0$

$$\boxed{\begin{aligned} |x| &= -x \\ |x+2| &= x+2 \\ \text{sign}(x) &= -1 \\ \text{sign}(x+2) &= 1 \end{aligned}}$$

$$\left[2x^2+2-(x+2)\right](-3) - (-3x-1)(4x-1) = 0$$
$$(2x^2-x)(-3) + (3x+1)(4x-1) = 0$$
$$-6x^2+3x+12x^2+x-1 = 0$$
$$6x^2+4x-1 = 0$$
$$x^2+\frac{2}{3}x-\frac{1}{6} = 0$$
$$\left(x+\frac{1}{3}\right)^2 - \frac{1}{9} - \frac{1}{6} = 0$$
$$\left(x+\frac{1}{3}\right)^2 = \frac{1}{9}+\frac{1}{6}$$
$$\left(x+\frac{1}{3}\right)^2 = \frac{4+6}{36}$$
$$\left(x+\frac{1}{3}\right)^2 = \frac{10}{36}$$
$$x+\frac{1}{3} = \pm\frac{\sqrt{10}}{6}$$
$$x = -\frac{1}{3} \pm \frac{\sqrt{10}}{6} = \begin{cases} \cancel{-\frac{1}{3}+\frac{\sqrt{10}}{6}} > 0 \\ -\frac{1}{3}-\frac{\sqrt{10}}{6} \longleftarrow \text{IS IN RANGE} \end{cases}$$

- AS THERE IS A SINGLE STATIONARY POINT WE NEED NOT LOOK IN THE RANGE $x<-2$

- TO FIND THE y CO-ORDINATE FINALLY FOR $-2<x<0$

$$\Longrightarrow y = \frac{3|x|-1}{2x^2+2-|x+2|} = \frac{-3x-1}{2x^2+2-(x+2)} = \frac{-3x-1}{2x^2-x} = \frac{3x+1}{x-2x^2}$$

$$\Longrightarrow y = \frac{3\left(-\frac{1}{3}-\frac{1}{6}\sqrt{10}\right)+1}{-\frac{1}{3}-\frac{1}{6}\sqrt{10}-2\left(-\frac{1}{3}-\frac{1}{6}\sqrt{10}\right)^2} = \frac{\cancel{-1}-\frac{1}{2}\sqrt{10}\cancel{+1}}{-\frac{1}{3}-\frac{1}{6}\sqrt{10}-2\left(\frac{1}{9}+\frac{1}{9}\sqrt{10}+\frac{5}{18}\right)}$$

$$\Longrightarrow y = \frac{-\frac{1}{2}\sqrt{10}}{-\frac{1}{3}-\frac{1}{6}\sqrt{10}-\frac{2}{9}-\frac{2}{9}\sqrt{10}-\frac{5}{9}} \begin{matrix}\times 18 \\ = \\ \times 18\end{matrix} \frac{-9\sqrt{10}}{-6-3\sqrt{10}-4-4\sqrt{10}-10}$$

$$\Longrightarrow y = \frac{-9\sqrt{10}}{-20-7\sqrt{10}} = \frac{9\sqrt{10}}{20+7\sqrt{10}} = \frac{9\sqrt{10}(20-7\sqrt{10})}{(20+7\sqrt{10})(20-7\sqrt{10})}$$

$$\Longrightarrow y = \frac{9\sqrt{10}(20-7\sqrt{10})}{400-490} = \frac{9\sqrt{10}(20-7\sqrt{10})}{-90} = \frac{\sqrt{10}(-20+7\sqrt{10})}{10}$$

$$\Longrightarrow y = \frac{-20\sqrt{10}+70}{10} = -2\sqrt{10}+7$$

$$\therefore \; y = 7-2\sqrt{10}$$

AS REQUIRED

Question 95 (*****)

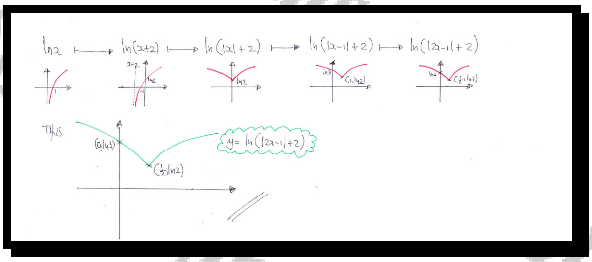
By considering a sequence of transformations, or otherwise, sketch the graph of

$$y = \ln\left(|2x-1|+2\right), \quad x \in \mathbb{R}.$$

Indicate the coordinates of any intersections with the axes, and the coordinates of the cusp of the curve.

[*No credit will be given to non analytical sketches based on plotting coordinates*]

graph

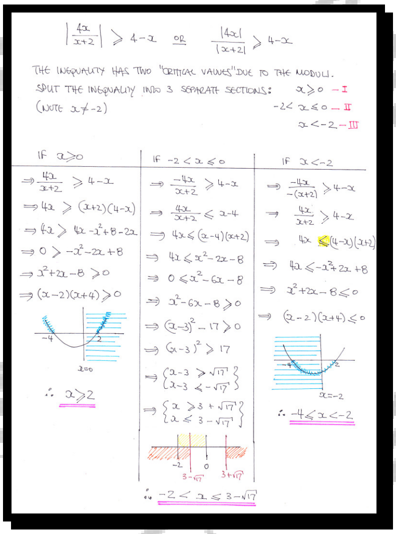


Question 96 (*****)

Find the set of values of x that satisfy the inequality

$$\left|\frac{4x}{x+2}\right| \geq 4 - x .$$

$\boxed{}$, $\boxed{-4 \leq x < -2 \quad \cup \quad -2 < x \leq 3-\sqrt{17} \quad \cup \quad x \geq 2}$



$\left|\frac{4x}{x+2}\right| \geqslant 4-x$ OR $\frac{|4x|}{|x+2|} \geqslant 4-x$

THE INEQUALITY HAS TWO "CRITICAL VALUES" DUE TO THE MODULI.
SPLIT THE INEQUALITY INTO 3 SEPARATE SECTIONS: $x \geqslant 0$ — I, $-2 < x \leqslant 0$ — II, $x < -2$ — III
(NOTE $x \neq -2$)

IF $x \geqslant 0$

$\Rightarrow \frac{4x}{x+2} \geqslant 4-x$
$\Rightarrow 4x \geqslant (x+2)(4-x)$
$\Rightarrow 4x \geqslant 4x - x^2 + 8 - 2x$
$\Rightarrow 0 \geqslant -x^2 - 2x + 8$
$\Rightarrow x^2 + 2x - 8 \geqslant 0$
$\Rightarrow (x-2)(x+4) \geqslant 0$

$x=0$

$\therefore \underline{x \geqslant 2}$

IF $-2 < x \leqslant 0$

$\Rightarrow \frac{-4x}{x+2} \geqslant 4-x$
$\Rightarrow \frac{4x}{x+2} \leqslant x-4$
$\Rightarrow 4x \leqslant (x-4)(x+2)$
$\Rightarrow 4x \leqslant x^2 - 2x - 8$
$\Rightarrow 0 \leqslant x^2 - 6x - 8$
$\Rightarrow x^2 - 6x - 8 \geqslant 0$
$\Rightarrow (x-3)^2 - 17 \geqslant 0$
$\Rightarrow (x-3)^2 \geqslant 17$
$\Rightarrow \begin{cases} x-3 \geqslant \sqrt{17} \\ x-3 \leqslant -\sqrt{17} \end{cases}$
$\Rightarrow \begin{cases} x \geqslant 3+\sqrt{17} \\ x \leqslant 3-\sqrt{17} \end{cases}$

$\therefore \underline{-2 < x \leqslant 3-\sqrt{17}}$

IF $x < -2$

$\Rightarrow \frac{-4x}{-(x+2)} \geqslant 4-x$
$\Rightarrow \frac{4x}{x+2} \geqslant 4-x$
$\Rightarrow 4x \leqslant (4-x)(x+2)$
$\Rightarrow 4x \leqslant -x^2 + 2x + 8$
$\Rightarrow x^2 + 2x - 8 \leqslant 0$
$\Rightarrow (x-2)(x+4) \leqslant 0$

$x=-2$

$\therefore \underline{-4 \leqslant x < -2}$

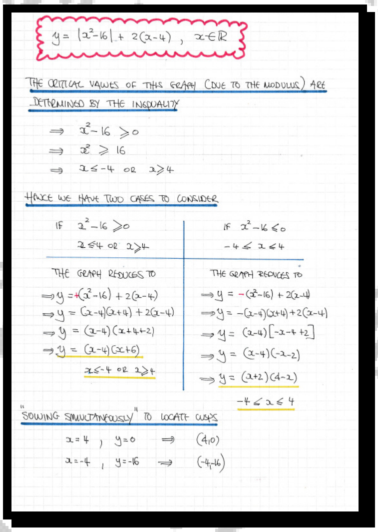
Question 97 (*****)

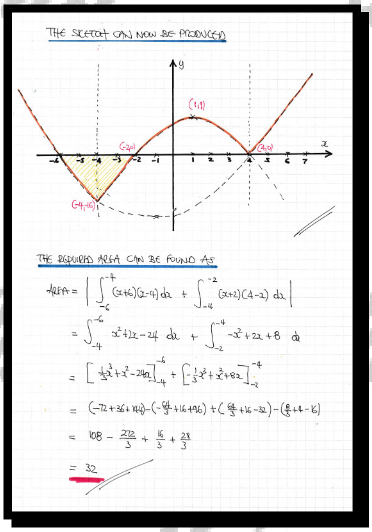
The curve C has equation

$$y = \left|x^2 - 16\right| + 2(x-4), \quad x \in \mathbb{R}.$$

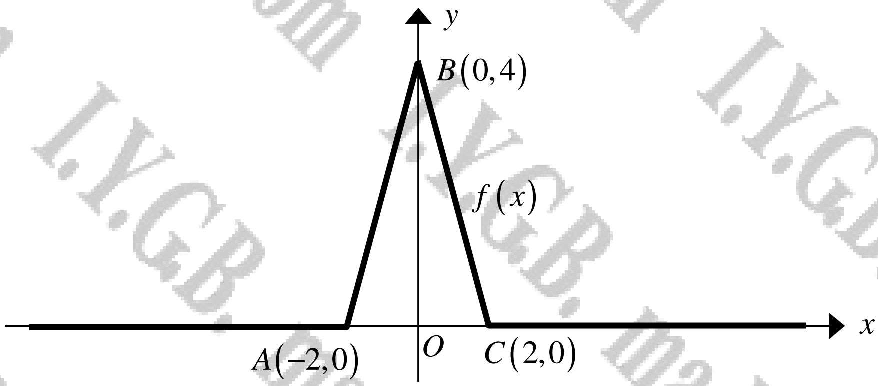
Sketch a detailed graph of C and hence show that the area of the finite region bounded by C and the x axis, for which $y < 0$, is 32 square units.

$\boxed{}$, $\boxed{\text{proof}}$





Question 98 (*****)



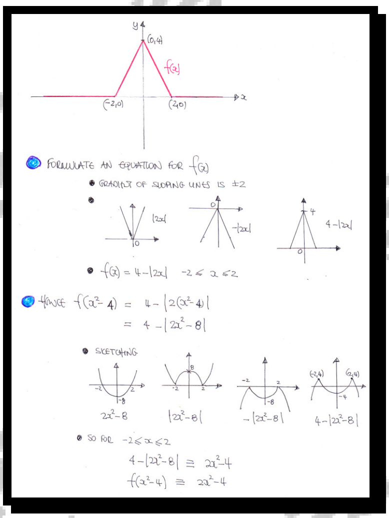
The figure above shows the graph of the function $f(x)$, consisting entirely of straight line sections. The coordinates of the joints of these straight line sections which make up the graph of $f(x)$ are also marked in the figure.

Given further that

$$\int_{-2}^{2} k + f\left(x^2 - 4\right)\, dx = 0,$$

determine as an exact fraction the value of the constant k.

$\boxed{}$, $\boxed{k = \frac{4}{3}}$



FORMULATE AN EQUATION FOR $f(x)$

- GRADIENT OF SLOPING LINES IS ± 2
- $|2x|$, $-|2x|$, $4-|2x|$
- $f(x) = 4 - |2x| \quad -2 \leqslant x \leqslant 2$

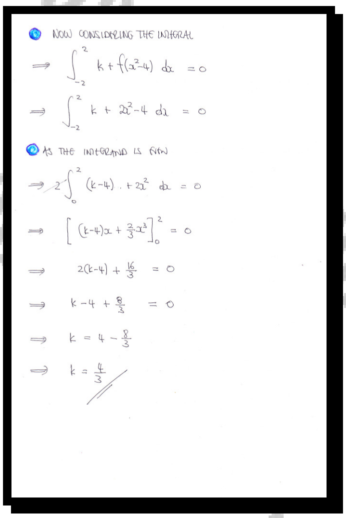
HENCE $f(x^2-4) = 4 - |2(x^2-4)|$
$= 4 - |2x^2 - 8|$

- SKETCHING

$2x^2-8$, $|2x^2-8|$, $-|2x^2-8|$, $4-|2x^2-8|$

- SO FOR $-2 \leqslant x \leqslant 2$

$4 - |2x^2-8| \equiv 2x^2-4$
$f(x^2-4) \equiv 2x^2-4$



NOW CONSIDERING THE INTEGRAL

$\Rightarrow \int_{-2}^{2} k + f(x^2-4)\, dx = 0$

$\Rightarrow \int_{-2}^{2} k + 2x^2 - 4\, dx = 0$

AS THE INTEGRAND IS EVEN

$\Rightarrow 2\int_{0}^{2} (k-4) + 2x^2\, dx = 0$

$\Rightarrow \left[(k-4)x + \frac{2}{3}x^3\right]_0^2 = 0$

$\Rightarrow 2(k-4) + \frac{16}{3} = 0$

$\Rightarrow k - 4 + \frac{8}{3} = 0$

$\Rightarrow k = 4 - \frac{8}{3}$

$\Rightarrow k = \frac{4}{3}$

Question 99 (*****)

Sketch the curve with equation

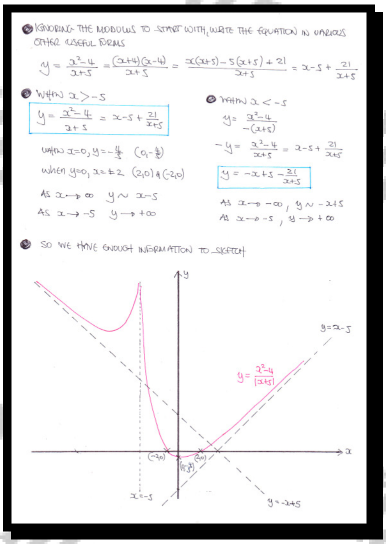
$$y = \frac{x^2 - 4}{|x+5|}, \ x \in \mathbb{R}, \ x \neq -5.$$

The sketch must include …

- … the coordinates of all the points where the curve meets the coordinate axes.
- … the equations of the asymptotes of the curve.

[*No credit will be given to non analytical sketches based on plotting coordinates*]

graph



Question 100 (*****)

Sketch, in the largest real domain, the graph of

$$y = \ln\left|\left|x+4\right|-6\right|.$$

Indicate the coordinates of any intersections with the axes, the equations of any asymptotes and the coordinates of any cusps of the curve.

$\boxed{}$, $\boxed{\text{graph}}$