INEQUALITIES

FURTHER PRACTICE

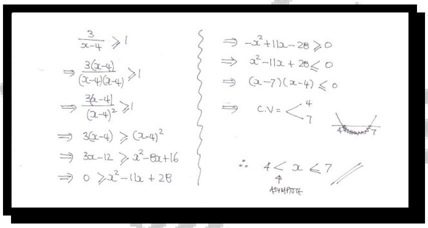
RATIONAL INEQUALITIES

Question 1 (**)

Solve the following inequality.

$$\frac{3}{x-4} \geq 1.$$

$\boxed{4 < x \leq 7}$



$\frac{3}{x-4} \geqslant 1$

$\Rightarrow \frac{3(x-4)}{(x-4)(x-4)} \geqslant 1$

$\Rightarrow \frac{3(x-4)}{(x-4)^2} \geqslant 1$

$\Rightarrow 3(x-4) \geqslant (x-4)^2$

$\Rightarrow 3x-12 \geqslant x^2-8x+16$

$\Rightarrow 0 \geqslant x^2-11x+28$

$\Rightarrow -x^2+11x-28 \geqslant 0$

$\Rightarrow x^2-11x+28 \leqslant 0$

$\Rightarrow (x-7)(x-4) \leqslant 0$

$\Rightarrow$ C.V $= \begin{cases} 4 \\ 7 \end{cases}$

$\therefore 4 < x \leqslant 7$

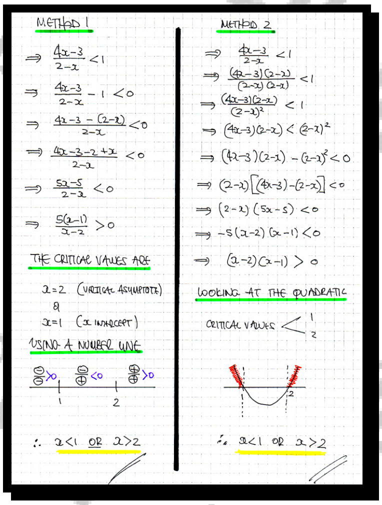
ASYMPTOTE

Question 2 (**)

Solve the following rational inequality.

$$\frac{4x-3}{2-x} < 1.$$

$\boxed{}$, $\boxed{x < 1 \;\cup\; x > 2}$



METHOD 1

$\Rightarrow \frac{4x-3}{2-x} < 1$

$\Rightarrow \frac{4x-3}{2-x} - 1 < 0$

$\Rightarrow \frac{4x-3-(2-x)}{2-x} < 0$

$\Rightarrow \frac{4x-3-2+x}{2-x} < 0$

$\Rightarrow \frac{5x-5}{2-x} < 0$

$\Rightarrow \frac{5(x-1)}{x-2} > 0$

THE CRITICAL VALUES ARE

$x=2$ (VERTICAL ASYMPTOTE)

&

$x=1$ (x INTERCEPT)

USING A NUMBER LINE

$\frac{\ominus}{\ominus} > 0$, $\frac{\ominus}{\oplus} < 0$, $\frac{\oplus}{\oplus} > 0$

1 2

$\therefore x<1$ OR $x>2$

METHOD 2

$\Rightarrow \frac{4x-3}{2-x} < 1$

$\Rightarrow \frac{(4x-3)(2-x)}{(2-x)(2-x)} < 1$

$\Rightarrow \frac{(4x-3)(2-x)}{(2-x)^2} < 1$

$\Rightarrow (4x-3)(2-x) < (2-x)^2$

$\Rightarrow (4x-3)(2-x) - (2-x)^2 < 0$

$\Rightarrow (2-x)\left[(4x-3)-(2-x)\right] < 0$

$\Rightarrow (2-x)(5x-5) < 0$

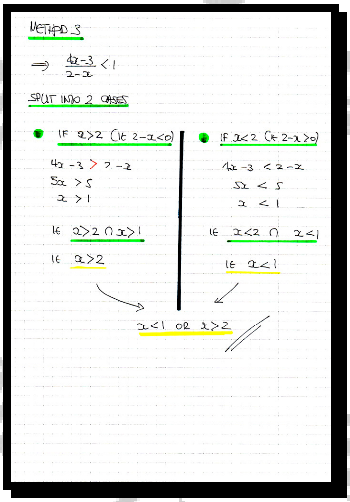
$\Rightarrow -5(x-2)(x-1) < 0$

$\Rightarrow (x-2)(x-1) > 0$

LOOKING AT THE QUADRATIC

CRITICAL VALUES $\begin{cases} 1 \\ 2 \end{cases}$

$\therefore x<1$ OR $x>2$



METHOD 3

$\Rightarrow \frac{4x-3}{2-x} < 1$

SPLIT INTO 2 CASES

- IF $x>2$ (IE $2-x<0$)

$4x-3 > 2-x$

$5x > 5$

$x > 1$

IE $x>2 \cap x>1$

IE $x>2$

- IF $x<2$ (IE $2-x>0$)

$4x-3 < 2-x$

$5x < 5$

$x < 1$

IE $x<2 \cap x<1$

IE $x<1$

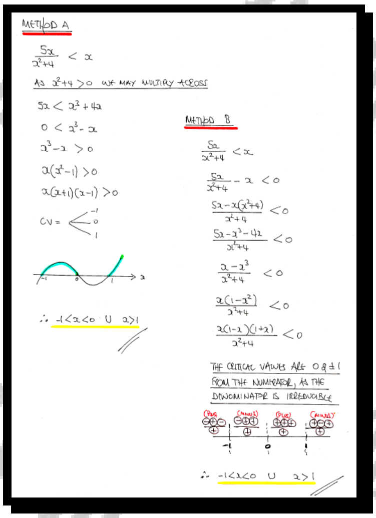
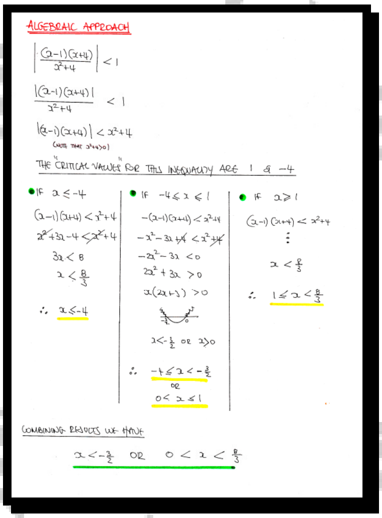
$x<1$ OR $x>2$

Question 3 (**)

Find the set of values of x, that satisfy the following inequality.

$$\frac{5x}{x^2+4} < x .$$

$\boxed{}$, $\boxed{-1 < x < 0}$, $\boxed{x > 1}$

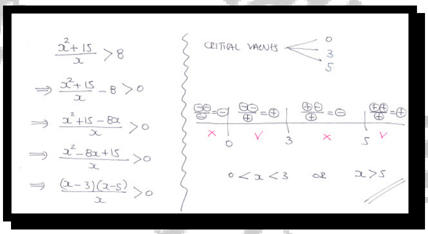



Question 4 (**)

Find the set of values of x, that satisfy the following inequality.

$$\frac{x^2+15}{x} > 8 .$$

$\boxed{0 < x < 3 \quad \cup \quad x > 5}$



Question 5 (**)

Solve the following rational inequality.

$$\frac{5}{(x+2)(4-x)} < 1.$$

$$\boxed{x < -2 \quad \cup \quad -1 < x < 3 \quad \cup \quad x > 4}$$

$$\frac{5}{(x+2)(4-x)} < 1$$

$$\Rightarrow \frac{5}{(x+2)(x-4)} > -1$$

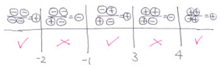
$$\Rightarrow \frac{5}{(x+2)(x-4)} + 1 > 0$$

$$\Rightarrow \frac{5+(x+2)(x-4)}{(x+2)(x-4)} > 0$$

$$\Rightarrow \frac{x^2-2x-3}{(x+2)(x-4)} > 0$$

$$\Rightarrow \frac{(x-3)(x+1)}{(x+2)(x-4)} > 0$$

CRITICAL VALUES: 3, −1, −2, 4



$x<-2$ OR $-1<x<3$ OR $x>4$

Question 6 (**)

Solve the following inequality.

$$\frac{2}{x+1} < x$$

$$\boxed{-2 < x < -1 \quad \cup \quad x > 1}$$

WORK AS FOLLOWS

$$\Rightarrow \frac{2}{x+1} < x$$

$$\Rightarrow \frac{2(x+1)}{(x+1)(x+1)} < x$$

$$\Rightarrow \frac{2(x+1)}{(x+1)^2} < x$$

MULTIPLYING ACROSS NOW IS NOT AN ISSUE AS $(x+1)^2$ CANNOT BE NEGATIVE

$$\Rightarrow 2(x+1) < x(x+1)^2$$

$$\Rightarrow 2(x+1) - x(x+1)^2 < 0$$

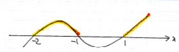
$$\Rightarrow (x+1)\left[2-x(x+1)\right] < 0$$

$$\Rightarrow (x+1)(2-x^2-x) < 0$$

$$\Rightarrow (x+1)(x^2+x-2) > 0$$

$$\Rightarrow (x+1)(x-1)(x+2) > 0$$

THIS IS A "POSITIVE" CUBIC, SKETCHED BELOW



$\therefore$ $-2<x<-1$ OR $x>1$

Question 7 (**+)

Find the set of values of x, that satisfy the following inequality.

$$x \le \frac{2x+5}{x-2}.$$

$$\boxed{x \le -1 \quad \cup \quad 2 < x \le 5}$$

$$x \leqslant \frac{2x+5}{x-2}$$

$$\Rightarrow x - \frac{2x+5}{x-2} \leqslant 0$$

$$\Rightarrow \frac{x(x-2)-(2x+5)}{x-2} \leqslant 0$$

$$\Rightarrow \frac{x^2-2x-2x-5}{x-2} \leqslant 0$$

$$\Rightarrow \frac{x^2-4x-5}{x-2} \leqslant 0$$

$$\Rightarrow \frac{(x-5)(x+1)}{x-2} \leqslant 0$$

CRITICAL VALUES ARE $5, -1$ & 2

$$\frac{\ominus\ominus}{\ominus} = \ominus \quad \Big| \quad \frac{\ominus\oplus}{\ominus} = \oplus \quad \Big| \quad \frac{\ominus\oplus}{\oplus} = \ominus \quad \Big| \quad \frac{\oplus\oplus}{\oplus} = \oplus$$

✓ $\quad -1 \quad$ ✗ $\quad 2 \quad$ ✓ $\quad 5 \quad$ ✗

$x \leqslant -1$ <u>OR</u> $2 < x \leqslant 5$

Question 8 (**+)

Find the set of values of x, that satisfy the following inequality.

$$\frac{3}{x+3} > \frac{x-4}{x}.$$

$$\boxed{-3 < x < -2 \quad \cup \quad 0 < x < 6}$$

$$\frac{3}{x+3} > \frac{x-4}{x}$$

$$\Rightarrow \frac{3}{x+3} - \frac{x-4}{x} > 0$$

$$\Rightarrow \frac{3x-(x-4)(x+3)}{x(x+3)} > 0$$

$$\Rightarrow \frac{3x-(x^2-x-12)}{x(x+3)} > 0$$

$$\Rightarrow \frac{-x^2+4x+12}{x(x+3)} > 0$$

$$\Rightarrow \frac{x^2-4x-12}{x(x+3)} < 0$$

$$\Rightarrow \frac{(x-6)(x+2)}{x(x+3)} < 0$$

C.V = $6, -2, 0, -3$

$\frac{++}{++}$

✗ $\quad -3 \quad$ ✓ $\quad -2 \quad$ ✗ $\quad 0 \quad$ ✓ $\quad 6 \quad$ ✗

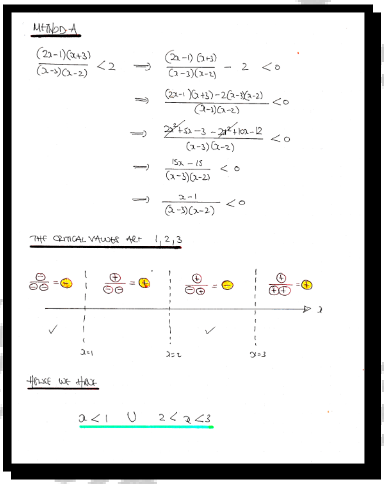
$-3 < x < -2$ <u>OR</u> $0 < x < 6$

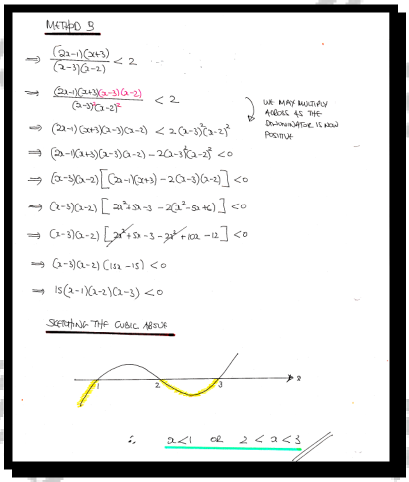
Question 9 (**+)

Solve the following inequality.

$$\frac{(2x-1)(x+3)}{(x-3)(x-2)} < 2 .$$

$\boxed{}$, $\boxed{x<1 \cup 2<x<3}$



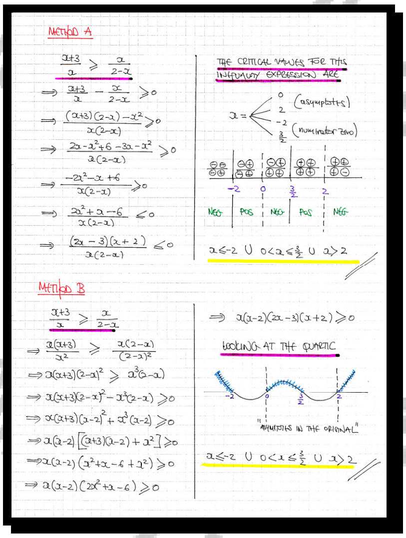


Question 10 (***)

Determine the range of values of x that satisfy the following inequality.

$$\frac{x+3}{x} \geq \frac{x}{2-x}.$$

$\boxed{}$, $\boxed{x \leq -2 \ \cup \ 0 < x \leq \frac{3}{2} \ \cup \ x > 2}$



METHOD A

$$\frac{x+3}{x} \geqslant \frac{x}{2-x}$$

$$\Rightarrow \frac{x+3}{x} - \frac{x}{2-x} \geqslant 0$$

$$\Rightarrow \frac{(x+3)(2-x)-x^2}{x(2-x)} \geqslant 0$$

$$\Rightarrow \frac{2x-x^2+6-3x-x^2}{x(2-x)} \geqslant 0$$

$$\Rightarrow \frac{-2x^2-x+6}{x(2-x)} \geqslant 0$$

$$\Rightarrow \frac{2x^2+x-6}{x(2-x)} \leqslant 0$$

$$\Rightarrow \frac{(2x-3)(x+2)}{x(2-x)} \leqslant 0$$

THE CRITICAL VALUES FOR THIS INEQUALITY EXPRESSION ARE

$x = 0, 2$ (asymptotes)

$x = -2, \frac{3}{2}$ (numerator zero)

$\frac{\ominus\ominus}{\ominus\oplus}$	$\frac{\ominus\oplus}{\ominus\oplus}$	$\frac{\ominus\oplus}{\oplus\oplus}$	$\frac{\oplus\oplus}{\oplus\oplus}$	$\frac{\oplus\oplus}{\oplus\ominus}$

-2, 0, $\frac{3}{2}$, 2

NEG | POS | NEG | POS | NEG

$x \leqslant -2 \cup 0 < x \leqslant \frac{3}{2} \cup x > 2$

METHOD B

$$\frac{x+3}{x} \geqslant \frac{x}{2-x}$$

$$\Rightarrow \frac{x(x+3)}{x^2} \geqslant \frac{x(2-x)}{(2-x)^2}$$

$$\Rightarrow x(x+3)(2-x)^2 \geqslant x^3(2-x)$$

$$\Rightarrow x(x+3)(2-x)^2 - x^3(2-x) \geqslant 0$$

$$\Rightarrow x(x+3)(x-2)^2 + x^3(x-2) \geqslant 0$$

$$\Rightarrow x(x-2)\left[(x+3)(x-2)+x^2\right] \geqslant 0$$

$$\Rightarrow x(x-2)(x^2+x-6+x^2) \geqslant 0$$

$$\Rightarrow x(x-2)(2x^2+x-6) \geqslant 0$$

$$\Rightarrow x(x-2)(2x-3)(x+2) \geqslant 0$$

LOOKING AT THE QUARTIC

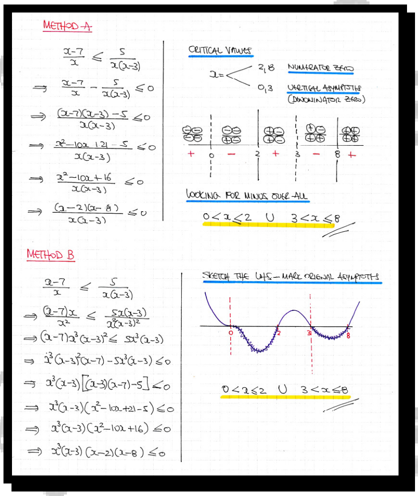
$x \leqslant -2 \cup 0 < x \leqslant \frac{3}{2} \cup x > 2$

Question 11 (***)

Determine the solution interval of the following inequality.

$$\frac{x-7}{x} \le \frac{5}{x(x-3)}.$$

$\boxed{}$, $\boxed{0 < x \le 2 \ \cup \ 3 < x \le 8}$


METHOD A

$$\frac{x-7}{x} \leqslant \frac{5}{x(x-3)}$$
$$\Rightarrow \frac{x-7}{x} - \frac{5}{x(x-3)} \leqslant 0$$
$$\Rightarrow \frac{(x-7)(x-3)-5}{x(x-3)} \leqslant 0$$
$$\Rightarrow \frac{x^2-10x+21-5}{x(x-3)} \leqslant 0$$
$$\Rightarrow \frac{x^2-10x+16}{x(x-3)} \leqslant 0$$
$$\Rightarrow \frac{(x-2)(x-8)}{x(x-3)} \leqslant 0$$

CRITICAL VALUES

$x = 2, 8$ NUMERATOR ZERO

$x = 0, 3$ VERTICAL ASYMPTOTES (DENOMINATOR ZERO)

$+ \quad 0 \quad - \quad 2 \quad + \quad 3 \quad - \quad 8 \quad +$

LOOKING FOR MINUS OVERALL

$0 < x \leqslant 2 \ \cup \ 3 < x \leqslant 8$

METHOD B

$$\frac{x-7}{x} \leqslant \frac{5}{x(x-3)}$$
$$\Rightarrow \frac{(x-7)x}{x^2} \leqslant \frac{5x(x-3)}{x^2(x-3)^2}$$
$$\Rightarrow (x-7)x^3(x-3)^2 \leqslant 5x^3(x-3)$$
$$\Rightarrow x^3(x-3)^2(x-7) - 5x^3(x-3) \leqslant 0$$
$$\Rightarrow x^3(x-3)\left[(x-3)(x-7)-5\right] \leqslant 0$$
$$\Rightarrow x^3(x-3)(x^2-10x+21-5) \leqslant 0$$
$$\Rightarrow x^3(x-3)(x^2-10x+16) \leqslant 0$$
$$\Rightarrow x^3(x-3)(x-2)(x-8) \leqslant 0$$

SKETCH THE LHS – MARK ORIGINAL ASYMPTOTES

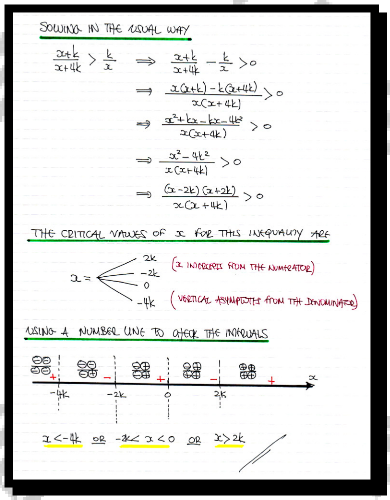
$0 < x \leqslant 2 \ \cup \ 3 < x \leqslant 8$

Question 12 (***)

Find, in terms of the positive constant k, the solution set of the following inequality.

$$\frac{x+k}{x+4k} > \frac{k}{x}.$$

$\boxed{}$, $\boxed{x < -4k \ \cup \ -2k < x < 0 \ \cup \ x > 2k}$

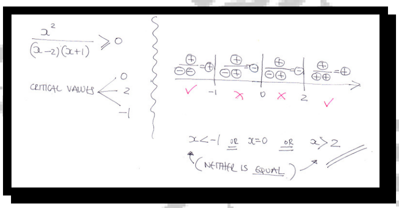


Question 13 (***+)

Solve the following rational inequality.

$$\frac{x^2}{(x-2)(x+1)} \geq 0.$$

$\boxed{x < -1 \ \cup \ x = 0 \ \cup \ x > 2}$

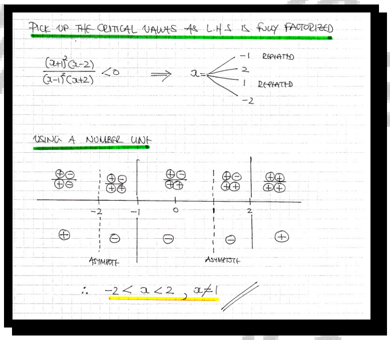


Question 14 (***+)

Solve the following rational inequality.

$$\frac{(x+1)^2(x-2)}{(x-1)^2(x+2)} < 0.$$

$\boxed{}$, $\boxed{-2 < x < 2, \quad x \neq 1}$

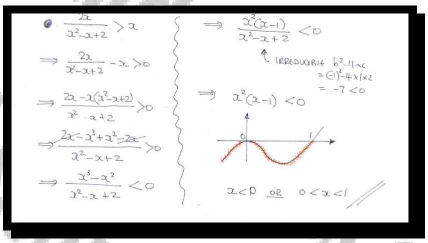


Question 15 (***+)

Solve the following rational inequality.

$$\frac{2x}{x^2 - x + 2} > x.$$

$\boxed{x < 0 \quad \cup \quad 0 < x < 1}$

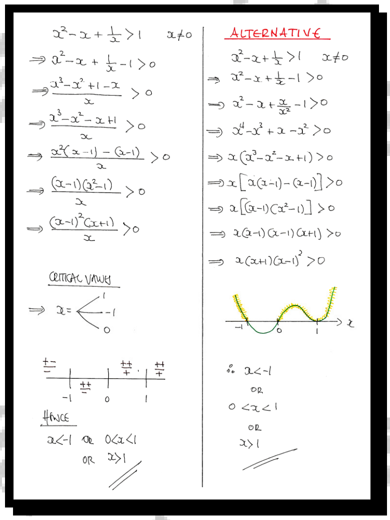


Question 16 (****+)

Solve the following inequality.

$$x^2 - x + \frac{1}{x} > 1.$$

$\boxed{}$, $\boxed{x < -1 \;\cup\; 0 < x < 1 \;\cup\; x > 1}$



$x^2 - x + \frac{1}{x} > 1 \quad x \neq 0$

$\Rightarrow x^2 - x + \frac{1}{x} - 1 > 0$

$\Rightarrow \frac{x^3 - x^2 + 1 - x}{x} > 0$

$\Rightarrow \frac{x^3 - x^2 - x + 1}{x} > 0$

$\Rightarrow \frac{x^2(x-1) - (x-1)}{x} > 0$

$\Rightarrow \frac{(x-1)(x^2-1)}{x} > 0$

$\Rightarrow \frac{(x-1)^2(x+1)}{x} > 0$

CRITICAL VALUES

$\Rightarrow x = 1, -1, 0$

HENCE

$x < -1$ OR $0 < x < 1$ OR $x > 1$

ALTERNATIVE

$x^2 - x + \frac{1}{x} > 1 \quad x \neq 0$

$\Rightarrow x^2 - x + \frac{1}{x} - 1 > 0$

$\Rightarrow x^2 - x + \frac{x}{x^2} - 1 > 0$

$\Rightarrow x^4 - x^3 + x - x^2 > 0$

$\Rightarrow x(x^3 - x^2 - x + 1) > 0$

$\Rightarrow x[x(x-1) - (x-1)] > 0$

$\Rightarrow x[(x-1)(x^2-1)] > 0$

$\Rightarrow x(x-1)(x-1)(x+1) > 0$

$\Rightarrow x(x+1)(x-1)^2 > 0$

$\therefore x < -1$ OR $0 < x < 1$ OR $x > 1$

Created by T. Madas

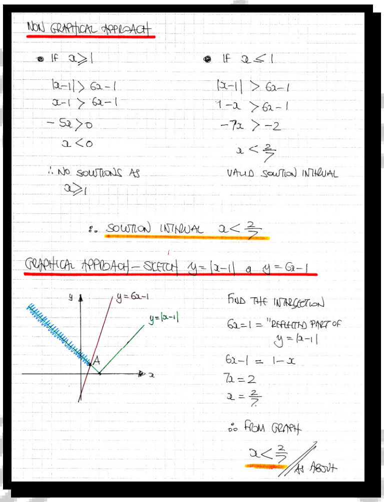
MODULUS INEQUALTIES

Question 1 (**)

Find the set of values of x that satisfy the inequality

$$|x-1| > 6x-1$$

$\boxed{}$, $\boxed{x < \frac{2}{7}}$

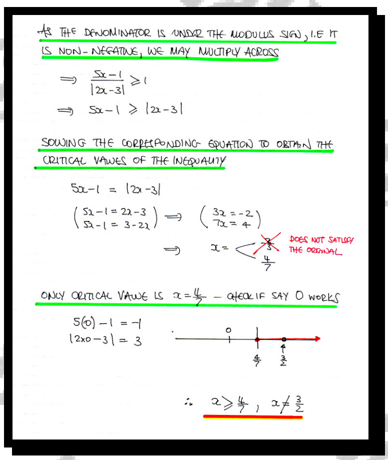


Question 2 (***)

Find the set of values of x that satisfy the inequality

$$\frac{5x-1}{|2x-3|} \geq 1.$$

$\boxed{}$, $\boxed{x \geq \frac{4}{7}, \quad x \neq \frac{3}{2}}$

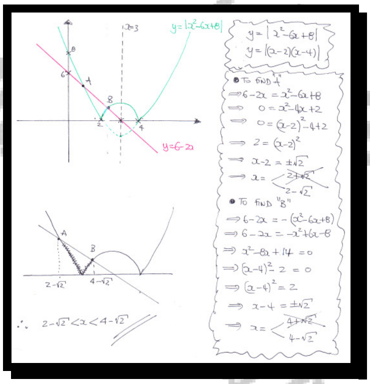


Question 3 **(***)**

Find the set of values of x that satisfy the inequality

$$\left|x^2 - 6x + 8\right| < 6 - 2x .$$

$$\boxed{2-\sqrt{2} < x < 4-\sqrt{2}}$$

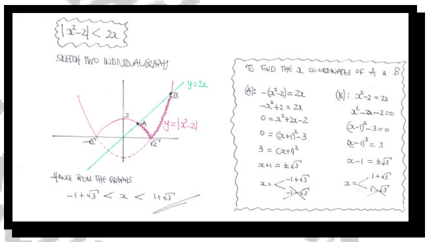


Question 4 **(**+)**

Find the set of values of x that satisfy the inequality

$$\left|x^2 - 2\right| < 2x .$$

$$\boxed{-1+\sqrt{3} < x < 1+\sqrt{3}}$$



Question 5 (****)

$$f(x) \equiv \frac{3x-1}{x+2}, \quad x \in \mathbb{R}, \quad x \neq -2.$$

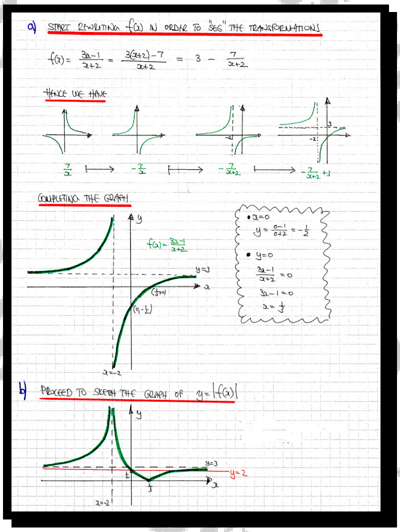
a) Sketch the graph of $f(x)$.

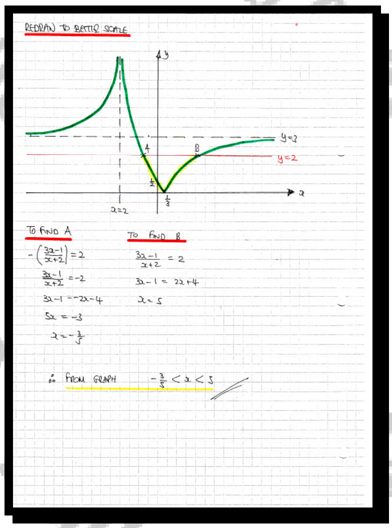
The sketch must include the coordinates of any points where the graph meets the coordinate axes and the equations of any asymptotes.

b) Hence, or otherwise, solve the inequality

$$\left|\frac{3x-1}{x+2}\right| < 2.$$

$\boxed{\phantom{\text{graph}}}$, $\boxed{-\frac{3}{5} < x < 5}$



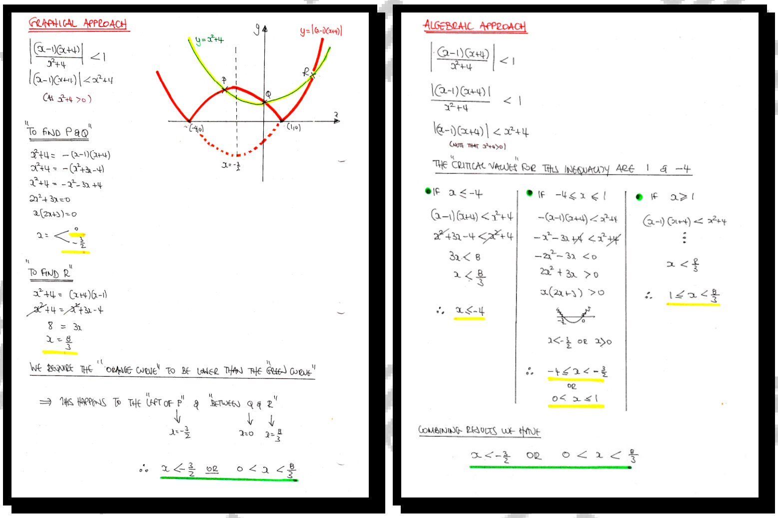


Question 6 (****+)

Find the set of values of x, that satisfy the following inequality.

$$\left|\frac{(x-1)(x+4)}{x^2+4}\right| < 1 .$$

$\boxed{}$, $\boxed{x < -\frac{3}{2} \text{ or } 0 < x < \frac{8}{3}}$



Question 7 (*****)

Determine the range of values of x that satisfy the inequality

$$\left|\frac{x+3}{x}\right| \geq \left|\frac{x}{2-x}\right|.$$

$\boxed{}$, $\boxed{-2 \leq x < 0 \quad \cup \quad 0 < x \leq \frac{3}{2} \quad \cup \quad x \geq 6}$

$$\left|\frac{x+3}{x}\right| \geqslant \left|\frac{x}{2-x}\right|$$

$$\Rightarrow \frac{|x+3|}{|x|} \geqslant \frac{|x|}{|2-x|}$$

EVERYTHING IS NON NEGATIVE, SO WE MAY MULTIPLY ACROSS, BUT LET US NOTE THAT $x \neq 0, x \neq 2$

$$\Rightarrow |x+3||2-x| \geqslant |x|^2$$

$$\Rightarrow |(x+3)(2-x)| \geqslant x^2$$

$$\Rightarrow |(x+3)(x-2)| \geqslant x^2$$

$$\Rightarrow |x^2+x-6| \geqslant x^2$$

THERE ARE 3 CRITICAL VALUES FOR THIS INEQUALITY, $x=-3, x=0, x=2$

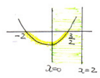
- IF $x>2$

$$x^2+x-6 \geqslant x^2$$
$$x \geqslant 6$$

$\therefore$ $x \geqslant 6$ ONLY

- IF $0<x<2$

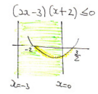
$$-x^2-x+6 \geqslant x^2$$
$$-2x^2-x+6 \geqslant 0$$
$$2x^2+x-6 \leqslant 0$$
$$(2x-3)(x+2) \leqslant 0$$



$\therefore$ $0<x \leqslant \frac{3}{2}$

- IF $-3 \leqslant x<0$

$$-x^2-x+6 \geqslant x^2$$
$$-2x^2-x+6 \geqslant 0$$
$$2x^2+x-6 \leqslant 0$$
$$(2x-3)(x+2) \leqslant 0$$



$\therefore$ $-2 \leqslant x<0$

- IF $x \leqslant -3$

$$x^2+x-6 \geqslant x^2$$
$$x \geqslant 6$$

NO COMMON SOLUTIONS

COLLECTING ALL THE SOLUTION INTERVALS

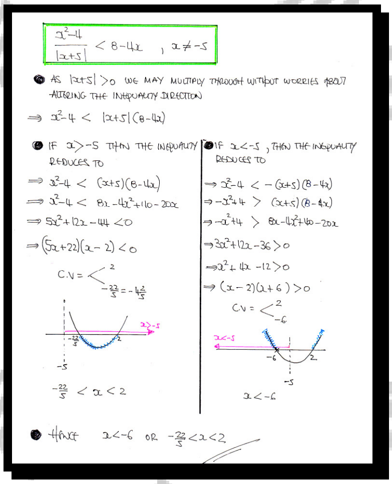
$$-2 \leqslant x<0 \text{ OR } 0<x \leqslant \tfrac{3}{2} \text{ OR } x \geqslant 6$$

Question 8 (*****)

Find the set of values of x that satisfy the inequality

$$\frac{x^2-4}{|x+5|} < 8-4x .$$

$$\boxed{x < -6 \quad \cup \quad -\frac{22}{5} < x < 2}$$



$\boxed{\dfrac{x^2-4}{|x+5|} < 8-4x \quad , \quad x \neq -5}$

- AS $|x+5| > 0$ WE MAY MULTIPLY THROUGH WITHOUT WORRIES ABOUT ALTERING THE INEQUALITY DIRECTION

$\Rightarrow x^2-4 < |x+5|(8-4x)$

- IF $x > -5$ THEN THE INEQUALITY REDUCES TO

$\Rightarrow x^2-4 < (x+5)(8-4x)$

$\Rightarrow x^2-4 < 8x-4x^2+40-20x$

$\Rightarrow 5x^2+12x-44 < 0$

$\Rightarrow (5x+22)(x-2) < 0$

C.V $= \begin{cases} 2 \\ -\frac{22}{5} = -4\frac{2}{5} \end{cases}$

$-\frac{22}{5} < x < 2$

- IF $x < -5$, THEN THE INEQUALITY REDUCES TO

$\Rightarrow x^2-4 < -(x+5)(8-4x)$

$\Rightarrow -x^2+4 > (x+5)(8-4x)$

$\Rightarrow -x^2+4 > 8x-4x^2+40-20x$

$\Rightarrow 3x^2+12x-36 > 0$

$\Rightarrow x^2+4x-12 > 0$

$\Rightarrow (x-2)(x+6) > 0$

C.V $= \begin{cases} 2 \\ -6 \end{cases}$

$x < -6$

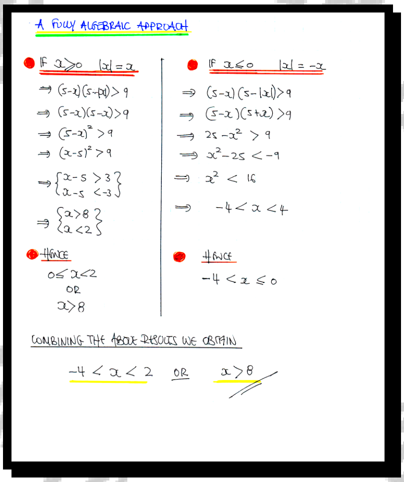
- HENCE $x < -6$ OR $-\frac{22}{5} < x < 2$

Question 9 (*****)

Solve the following inequality.

$$(5-x)(5-|x|)>9\,,\ x\in\mathbb{R}$$

$\boxed{}$, $\boxed{-4<x<2,\ \cup\ x>8}$



A FULLY ALGEBRAIC APPROACH

- IF $x\geqslant 0$ $|x|=x$

$\Rightarrow (5-x)(5-|x|)>9$
$\Rightarrow (5-x)(5-x)>9$
$\Rightarrow (5-x)^2>9$
$\Rightarrow (x-5)^2>9$
$\Rightarrow \begin{Bmatrix} x-5>3 \\ x-5<-3 \end{Bmatrix}$
$\Rightarrow \begin{Bmatrix} x>8 \\ x<2 \end{Bmatrix}$

- HENCE

$0\leqslant x<2$
OR
$x>8$

- IF $x\leqslant 0$ $|x|=-x$

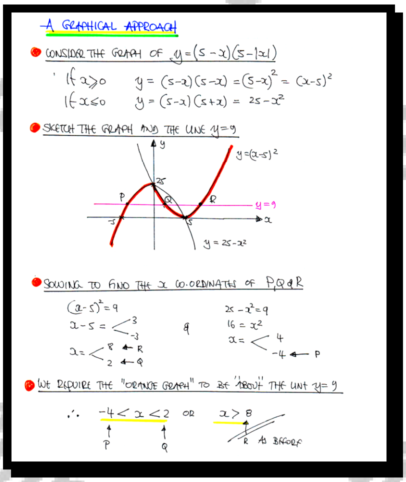
$\Rightarrow (5-x)(5-|x|)>9$
$\Rightarrow (5-x)(5+x)>9$
$\Rightarrow 25-x^2>9$
$\Rightarrow x^2-25<-9$
$\Rightarrow x^2<16$
$\Rightarrow -4<x<4$

- HENCE

$-4<x\leqslant 0$

COMBINING THE ABOVE RESULTS WE OBTAIN

$-4<x<2$ OR $x>8$



A GRAPHICAL APPROACH

- CONSIDER THE GRAPH OF $y=(5-x)(5-|x|)$

If $x\geqslant 0$ $y=(5-x)(5-x)=(5-x)^2=(x-5)^2$
If $x\leqslant 0$ $y=(5-x)(5+x)=25-x^2$

- SKETCH THE GRAPH AND THE LINE $y=9$

- SOLVING TO FIND THE x CO-ORDINATES OF P, Q & R

$(x-5)^2=9$
$x-5=\begin{matrix}3\\-3\end{matrix}$
$x=\begin{matrix}8 \leftarrow R\\2 \leftarrow Q\end{matrix}$

$25-x^2=9$
$16=x^2$
$x=\begin{matrix}4\\-4 \leftarrow P\end{matrix}$

- WE REQUIRE THE "ORANGE GRAPH" TO BE "ABOVE" THE LINE $y=9$

$\therefore$ $-4<x<2$ OR $x>8$
P Q R AS BEFORE

Question 10 (*****)

Solve the following inequality in the largest real domain.

$$\frac{x^2-2|x|-8}{6|x|^3-5x^2+12|x|}\le 0.$$

$\boxed{}$, $\boxed{-4 \le x \le 4, \quad x \ne 0}$

- FIRSTLY LET US NOT THAT THE L.H.S IS EVEN

$$\Rightarrow \frac{x^2-2|x|-8}{6|x|^3-5x^2+12|x|} \leqslant 0$$

$$\Rightarrow \frac{x^2-2x-8}{6x^3-5x^2+12x} \leqslant 0 \qquad (\text{FOR } x>0)$$

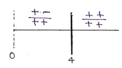
$$\Rightarrow \frac{(x+2)(x-4)}{x(6x^2-5x+12)} \leqslant 0$$

IRREDUCIBLE AS $(-5)^2-4\times6\times12<0$

- HENCE WE HAVE CRITICAL VALUES

$$x = \begin{cases} 0 & (\text{ASYMPTOTE}) \\ \cancel{-2} & x>0 \\ 4 & (x \text{ INTERCEPT} \end{cases}$$

- CHECK BETWEEN THESE VALUES



$\therefore \; 0 < x \leqslant 4$

- AS THE FUNCTION ON THE L.H.S IS EVEN WE HAVE

$$-4 \leqslant x \leqslant 4, \quad x \neq 0$$

Question 11 (*****)

Solve the following modulus inequality.

$$3|x+1|-|x-4| \le 11, \ x \in \mathbb{R}.$$

$\boxed{}$, $\boxed{}$, $\boxed{-9 \le x \le 3}$

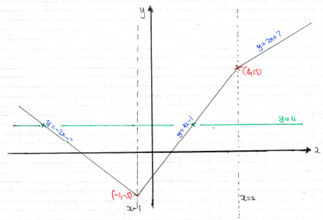
PROCEED WITH A SKETCH OF $y = 3|x+1| - |x-4|$

IF $x \geq 4 \Rightarrow y = 3(x+1) - (x-4) = 3x+3-x+4 = 2x+7$

IF $-1 < x < 4 \Rightarrow y = 3(x+1) + (x-4) = 3x+3+x-4 = 4x-1$

IF $x \leq -1 \Rightarrow y = -3(x+1) + (x-4) = -3x-3+x-4 = -2x-7$

PRODUCE AN INDICATIVE SKETCH



FINALLY WE HAVE

- $-2x-7 = 11$
  $-2x = 18$
  $x = -9$
- $4x-1 = 11$
  $4x = 12$
  $x = 3$

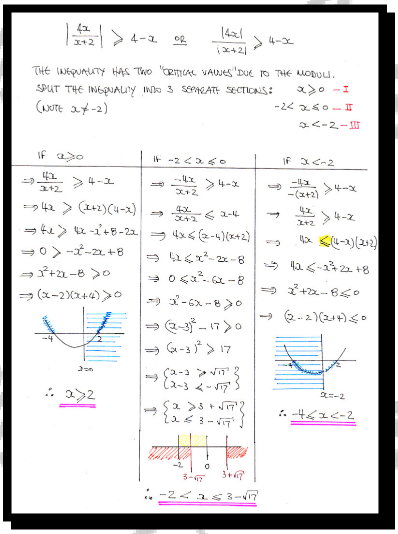
$\therefore -9 \leq x \leq 3$

Question 12 (*****)

Find the set of values of x that satisfy the inequality

$$\left|\frac{4x}{x+2}\right| \geq 4 - x .$$

$\boxed{}$, $\boxed{-4 \leq x < -2 \quad \cup \quad -2 < x \leq 3 - \sqrt{17} \quad \cup \quad x \geq 2}$



$\left|\frac{4x}{x+2}\right| \geqslant 4-x$ OR $\frac{|4x|}{|x+2|} \geqslant 4-x$

THE INEQUALITY HAS TWO "CRITICAL VALUES" DUE TO THE MODULI. SPLIT THE INEQUALITY INTO 3 SEPARATE SECTIONS: (NOTE $x \neq -2$)

$x \geqslant 0$ — I
$-2 < x \leqslant 0$ — II
$x < -2$ — III

IF $x \geqslant 0$

$\Rightarrow \frac{4x}{x+2} \geqslant 4-x$
$\Rightarrow 4x \geqslant (x+2)(4-x)$
$\Rightarrow 4x \geqslant 4x - x^2 + 8 - 2x$
$\Rightarrow 0 \geqslant -x^2 - 2x + 8$
$\Rightarrow x^2 + 2x - 8 \geqslant 0$
$\Rightarrow (x-2)(x+4) \geqslant 0$

$\therefore \ x \geqslant 2$

IF $-2 < x \leqslant 0$

$\Rightarrow \frac{-4x}{x+2} \geqslant 4-x$
$\Rightarrow \frac{4x}{x+2} \leqslant x-4$
$\Rightarrow 4x \leqslant (x-4)(x+2)$
$\Rightarrow 4x \leqslant x^2 - 2x - 8$
$\Rightarrow 0 \leqslant x^2 - 6x - 8$
$\Rightarrow x^2 - 6x - 8 \geqslant 0$
$\Rightarrow (x-3)^2 - 17 \geqslant 0$
$\Rightarrow (x-3)^2 \geqslant 17$
$\Rightarrow \left\{\begin{array}{l} x-3 \geqslant \sqrt{17} \\ x-3 \leqslant -\sqrt{17} \end{array}\right\}$
$\Rightarrow \left\{\begin{array}{l} x \geqslant 3+\sqrt{17} \\ x \leqslant 3-\sqrt{17} \end{array}\right\}$

$\therefore \ -2 < x \leqslant 3-\sqrt{17}$

IF $x < -2$

$\Rightarrow \frac{-4x}{-(x+2)} \geqslant 4-x$
$\Rightarrow \frac{4x}{x+2} \geqslant 4-x$
$\Rightarrow 4x \leqslant (4-x)(x+2)$
$\Rightarrow 4x \leqslant -x^2 + 2x + 8$
$\Rightarrow x^2 + 2x - 8 \leqslant 0$
$\Rightarrow (x-2)(x+4) \leqslant 0$

$\therefore \ -4 \leqslant x < -2$

VARIOUS INEQUALITIES

Question 1 **(**)**

Solve the inequality

$$\frac{2x-1}{7} < \frac{3x+2}{3}.$$

$$\boxed{x > -\frac{17}{15}}$$

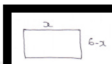
$$\frac{2x-1}{7} < \frac{3x+2}{3}$$
$$\Rightarrow 6x-3 < 21x+14$$
$$\Rightarrow -15x < 17$$
$$\Rightarrow x > -\frac{17}{15}$$

Question 2 **(**+)**

An advertising sign has a rectangular design so the its length is x metres and its width is $(6-x)$ metres.

Given that the area of the advertising sign must be at least 5 square metres, determine the range of possible values of x.

$$\boxed{1 \le x \le 5}$$



$$\Rightarrow x(6-x) \geqslant 5$$
$$\Rightarrow 6x - x^2 \geqslant 5$$
$$\Rightarrow -x^2+6x-5 \geqslant 0$$
$$\Rightarrow x^2-6x+5 \leqslant 0$$
$$\Rightarrow (x-1)(x-5) \leqslant 0$$
$$\text{C.V.} = \begin{cases} 1 \\ 5 \end{cases}$$

$$1 \leqslant x \leqslant 5$$

Question 3 **(**+)**

$$T = 8x - 12y + 7.$$

It is further given that $-\frac{1}{2} < x < \frac{7}{8}$ and $-\frac{1}{6} < y < \frac{2}{3}$.

Determine the range of possible values of T.

$\boxed{}$, $\boxed{-5 < T < 16}$

PROCEED AS FOLLOWS

$-\frac{1}{2} < x < \frac{7}{8}$ $\qquad$ $-\frac{1}{6} < y < \frac{2}{3}$

$-4 < 8x < 7$ $\qquad$ $-2 < 12y < 8$

$-12y > -8$ OR $-12y < +2$

$-8 < -12y < 2$

NOW WE HAVE

$T = 8x - 12y + 7 \Longrightarrow$

$-4 < 8x < 7$

$-8 < -12y < 2$

$-12 < 8x - 12y < 9$

$-5 < 8x - 12y + 7 < 16$

$-5 < T < 16$

Question 4 **(***)**

Show clearly, without approximating and without using any calculating aid that

$$\sqrt{2} + \sqrt{5} > \sqrt{7}.$$

$\boxed{\text{proof}}$

IF $a > b > 0 \Rightarrow a^2 > b^2$

$\Rightarrow (\sqrt{2} + \sqrt{5})^2 = 2 + 2\sqrt{2}\sqrt{5} + 5$

$= 7 + 2\sqrt{10}$

> 7

$= (\sqrt{7})^2$

$\therefore \sqrt{2} + \sqrt{5} > \sqrt{7}$

Question 5 (***+)

Given that $k > 0$ show clearly that

$$\frac{k+1}{\sqrt{k}} \geq 2 .$$

$\boxed{\quad}$, $\boxed{\text{proof}}$

CONSIDER THE EXPANSION OF $(\sqrt{k}-1)^2$

$$\Rightarrow (\sqrt{k}-1)^2 \geq 0$$
$$\Rightarrow (\sqrt{k})^2 - 2\times1\times\sqrt{k} + 1^2 \geq 0$$
$$\Rightarrow k - 2\sqrt{k} + 1 \geq 0$$
$$\Rightarrow k + 1 \geq 2\sqrt{k}$$

AS $\sqrt{k} > 0$ WE MAY DIVIDE IT

$$\Rightarrow \frac{k+1}{\sqrt{k}} \geq 2$$

// AS REQUIRED

ALTERNATIVE BY DIFFERENTIATION

FIRSTLY LET US NOTE THAT AS k GETS LARGER, THE WHOLE EXPRESSION GETS LARGER WITHOUT BOUND, SO ANY STATIONARY POINT WILL BE AN ABSOLUTE MINIMUM

$$\text{e.g.} \quad \lim_{k\to\infty}\left(\frac{k+1}{\sqrt{k}}\right) = \lim_{k\to\infty}\left(\sqrt{k} + \frac{1}{\sqrt{k}}\right)$$

$$y = \frac{k+1}{\sqrt{k}} = \frac{k}{k^{\frac{1}{2}}} + \frac{1}{k^{\frac{1}{2}}} = k^{\frac{1}{2}} + k^{-\frac{1}{2}}$$
$$\frac{dy}{dk} = \frac{1}{2}k^{-\frac{1}{2}} - \frac{1}{2}k^{-\frac{3}{2}}$$

SOLVING FOR ZERO, TO LOOK FOR MINIMUM

$$0 = \frac{1}{2}k^{-\frac{1}{2}} - \frac{1}{2}k^{-\frac{3}{2}}$$
$$\Rightarrow \frac{1}{2}k^{-\frac{1}{2}} = \frac{1}{2}k^{-\frac{3}{2}}$$
$$\Rightarrow k^{-\frac{1}{2}} = k^{-\frac{3}{2}}$$
$$\Rightarrow \frac{1}{k^{\frac{1}{2}}} = \frac{1}{k^{3/2}}$$
$$\Rightarrow \frac{k^{3/2}}{k^{1/2}} = 1$$

AS $k>0$, WE MAY DIVIDE

$$\Rightarrow k = 1$$

$$\therefore \left(\frac{k+1}{\sqrt{k}}\right)_{\text{MIN}} = \frac{1+1}{\sqrt{1}} = \frac{2}{1} = 2$$

// AS REQUIRED

Question 6 (****+)

Show clearly, without approximating and without using any calculating aid that

a) $\sqrt{6+2\sqrt{6}} > \sqrt{3}+\sqrt{2}$.

b) $\sqrt[3]{3} > \sqrt{2}$.

c) $\sqrt{2}-1 > \sqrt{3}-\sqrt{2}$.

proof

IF $a>b>0 \Rightarrow a^n > b^n$ IF $n=2,3,4,5,\dots$

(a) $\sqrt{6+2\sqrt{6}}$ SQUARES TO $6+2\sqrt{6}$

$(\sqrt{3}+\sqrt{2})^2 = 3+2\sqrt{3}\sqrt{2}+2 = 5+2\sqrt{6}$

SINCE $6+2\sqrt{6} > 5+2\sqrt{6} \Longrightarrow \sqrt{6+2\sqrt{6}} > \sqrt{3}+\sqrt{2}$

(b) $\sqrt[3]{3} = 3^{\frac{1}{3}}$ & $(3^{\frac{1}{3}})^6 = 3^2 = 9$

$\sqrt{2} = 2^{\frac{1}{2}}$ & $(2^{\frac{1}{2}})^6 = 2^3 = 8$

AS $9>8 \Rightarrow \sqrt[3]{3} > \sqrt{2}$

(c) SUPPOSE THE INEQUALITY HOLDS

$\Rightarrow \sqrt{2}-1 > \sqrt{3}-\sqrt{2}$

$\Rightarrow 2\sqrt{2} > \sqrt{3}+1$

SQUARE BOTH SIDES, NOTE BOTH SIDES ARE POSITIVE

$\Rightarrow 8 > (\sqrt{3}+1)^2$

$\Rightarrow 8 > 3+2\sqrt{3}+1$

$\Rightarrow 8 > 4+2\sqrt{3}$

$\Rightarrow 8 = 4+2\sqrt{4} > 4+2\sqrt{3}$

$\therefore$ ORIGINAL INEQUALITY HOLDS INDEED

Question 7 **(****+)**

Show clearly that for all real numbers α, β and γ

$$\alpha^2 + \beta^2 + \gamma^2 \geq \alpha\beta + \beta\gamma + \gamma\alpha .$$

$\boxed{}$, $\boxed{\text{proof}}$

- STARTING FROM $\boxed{(\alpha-\beta)^2 \geq 0}$

$$\alpha^2 - 2\alpha\beta + \beta^2 \geq 0$$

$$\left.\begin{array}{l} \alpha^2 + \beta^2 \geq 2\alpha\beta \\ \alpha^2 + \gamma^2 \geq 2\alpha\gamma \\ \beta^2 + \gamma^2 \geq 2\beta\gamma \end{array}\right\} \text{ADDING THESE 3 INEQUALITIES}$$

- SIMILARLY :

$$\Longrightarrow 2\alpha^2 + 2\beta^2 + 2\gamma^2 \geq 2\alpha\beta + 2\beta\gamma + 2\gamma\alpha$$

$$\Longrightarrow \alpha^2 + \beta^2 + \gamma^2 \geq \alpha\beta + \beta\gamma + \gamma\alpha$$

ALTERNATIVE BY THE AM-GM INEQUALITY

"AM $\geq$ GM"

$$\frac{A+B}{2} \geq \sqrt[2]{AB}$$

$$\frac{A^2+2AB+B^2}{4} \geq AB$$

$$A^2+2AB+B^2 \geq 4AB$$

$$\boxed{A^2+B^2 \geq 2AB}$$

HENCE

$$\left.\begin{array}{l} \alpha^2+\beta^2 \geq 2\alpha\beta \\ \beta^2+\gamma^2 \geq 2\beta\gamma \\ \gamma^2+\alpha^2 \geq 2\alpha\gamma \end{array}\right\} \Longrightarrow \text{ADDING } 2(\alpha^2+\beta^2+\gamma^2) \geq 2(\alpha\beta+\beta\gamma+\gamma\alpha)$$

$$\alpha^2+\beta^2+\gamma^2 \geq \alpha\beta+\beta\gamma+\gamma\alpha$$

(NOTE THE TECHNICALITY THAT IN THE AM-GM INEQUALITY $AB \geq 0$)