IMPLICIT EQUATIONS

EXAM QUESTIONS

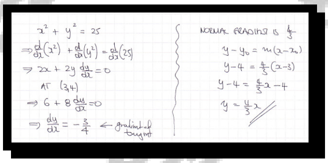
Question 1 **(**)**

A circle has equation

$$x^2 + y^2 = 25.$$

Use implicit differentiation to find an equation of the normal to the circle at the point with coordinates $(3,4)$.

$$\boxed{y = \frac{4}{3}x}$$


$x^2 + y^2 = 25$

$\Rightarrow \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(25)$

$\Rightarrow 2x + 2y\frac{dy}{dx} = 0$

AT $(3,4)$

$\Rightarrow 6 + 8\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = -\frac{3}{4}$ ← gradient of tangent

NORMAL GRADIENT IS $\frac{4}{3}$

$y - y_0 = m(x - x_0)$

$y - 4 = \frac{4}{3}(x-3)$

$y - 4 = \frac{4}{3}x - 4$

$y = \frac{4}{3}x$

Question 2 **(**)**

A circle has equation

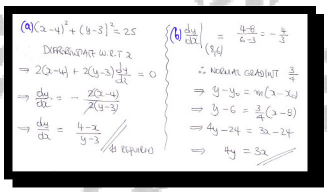
$$(x-4)^2 + (y-3)^2 = 25.$$

a) Show clearly that

$$\frac{dy}{dx} = \frac{4-x}{y-3}.$$

b) Find an equation of the normal to the circle at the point $(8,6)$.

$$\boxed{4y = 3x}$$


(a) $(x-4)^2 + (y-3)^2 = 25$

DIFFERENTIATE W.R.T x

$\Rightarrow 2(x-4) + 2(y-3)\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = -\frac{\cancel{2}(x-4)}{\cancel{2}(y-3)}$

$\Rightarrow \frac{dy}{dx} = \frac{4-x}{y-3}$ // AS REQUIRED

(b) $\left.\frac{dy}{dx}\right|_{(8,6)} = \frac{4-8}{6-3} = -\frac{4}{3}$

∴ NORMAL GRADIENT $\frac{3}{4}$

$\Rightarrow y - y_0 = m(x - x_0)$

$\Rightarrow y - 6 = \frac{3}{4}(x-8)$

$\Rightarrow 4y - 24 = 3x - 24$

$\Rightarrow 4y = 3x$

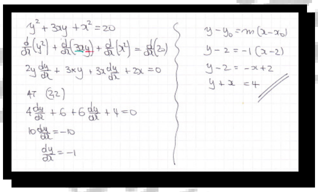
Question 3 **(**)**

A curve is given implicitly by the equation

$$y^2 + 3xy + x^2 = 20 .$$

Find an equation for the tangent to the curve at the point $P(2,2)$.

$\boxed{}$, $\boxed{x + y = 4}$



$y^2 + 3xy + x^2 = 20$

$\frac{d}{dx}(y^2) + \frac{d}{dx}(3xy) + \frac{d}{dx}(x^2) = \frac{d}{dx}(20)$

$2y\frac{dy}{dx} + 3 \times y + 3x\frac{dy}{dx} + 2x = 0$

AT $(2,2)$

$4\frac{dy}{dx} + 6 + 6\frac{dy}{dx} + 4 = 0$

$10\frac{dy}{dx} = -10$

$\frac{dy}{dx} = -1$

$y - y_0 = m(x - x_0)$

$y - 2 = -1(x - 2)$

$y - 2 = -x + 2$

$y + x = 4$

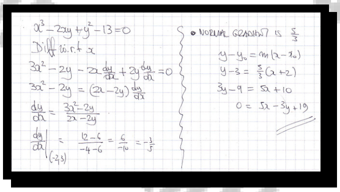
Question 4 **(**)**

A curve C has equation

$$x^3 - 2xy + y^2 - 13 = 0 .$$

Find an equation for the normal to C at the point $P(-2,3)$.

$\boxed{5x - 3y + 19 = 0}$



$x^3 - 2xy + y^2 - 13 = 0$

Diff w.r.t x

$3x^2 - 2y - 2x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$

$3x^2 - 2y = (2x - 2y)\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{3x^2 - 2y}{2x - 2y}$

$\left.\frac{dy}{dx}\right|_{(-2,3)} = \frac{12 - 6}{-4 - 6} = \frac{6}{-10} = -\frac{3}{5}$

• NORMAL GRADIENT IS $\frac{5}{3}$

$y - y_0 = m(x - x_0)$

$y - 3 = \frac{5}{3}(x + 2)$

$3y - 9 = 5x + 10$

$0 = 5x - 3y + 19$

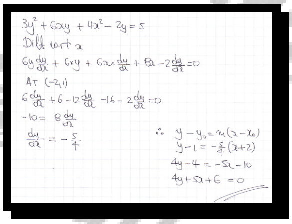
Question 5 (**)

A curve is given implicitly by the equation

$$3y^2 + 6xy + 4x^2 - 2y = 5.$$

Find an equation for the tangent to the curve at the point $P(-2,1)$.

$$\boxed{4y + 5x + 6 = 0}$$



$3y^2 + 6xy + 4x^2 - 2y = 5$

Diff w.r.t x

$6y\frac{dy}{dx} + 6y + 6x\frac{dy}{dx} + 8x - 2\frac{dy}{dx} = 0$

At $(-2,1)$

$6\frac{dy}{dx} + 6 - 12\frac{dy}{dx} - 16 - 2\frac{dy}{dx} = 0$

$-10 = 8\frac{dy}{dx}$

$\frac{dy}{dx} = -\frac{5}{4}$

$\therefore\ y - y_0 = m(x - x_0)$

$y - 1 = -\frac{5}{4}(x+2)$

$4y - 4 = -5x - 10$

$4y + 5x + 6 = 0$

Question 6 (**+)

A curve has implicit equation

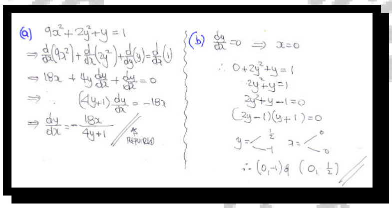
$$9x^2 + 2y^2 + y = 1.$$

a) Show clearly that

$$\frac{dy}{dx} = -\frac{18x}{4y+1}.$$

b) Hence find the coordinates of the points on the curve where $\frac{dy}{dx} = 0$.

$$\boxed{(0,-1), \left(0, \tfrac{1}{2}\right)}$$



(a) $9x^2 + 2y^2 + y = 1$

$\Rightarrow \frac{d}{dx}(9x^2) + \frac{d}{dx}(2y^2) + \frac{d}{dx}(y) = \frac{d}{dx}(1)$

$\Rightarrow 18x + 4y\frac{dy}{dx} + \frac{dy}{dx} = 0$

$\Rightarrow (4y+1)\frac{dy}{dx} = -18x$

$\Rightarrow \frac{dy}{dx} = -\frac{18x}{4y+1}$ AS REQUIRED

(b) $\frac{dy}{dx} = 0 \Rightarrow x = 0$

$\therefore 0 + 2y^2 + y = 1$

$2y^2 + y = 1$

$2y^2 + y - 1 = 0$

$(2y-1)(y+1) = 0$

$y = \begin{cases} \frac{1}{2} \\ -1 \end{cases}$ $x = \begin{cases} 0 \\ 0 \end{cases}$

$\therefore (0,-1)$ & $(0, \frac{1}{2})$

Question 7 (**+)

A curve is given by

$$2\cos x + \tan y = 2\sqrt{3}.$$

a) Show clearly that

$$\frac{dy}{dx} = 2\sin x \cos^2 y.$$

b) Find an equation of the normal to the curve at the point $\left(\frac{\pi}{6}, \frac{\pi}{3}\right)$, giving the answer in the form $ax + by = \pi$, where a and b are integers.

C4J , $\boxed{4x + y = \pi}$

(a) $2\cos x + \tan y = 2\sqrt{3}$

Diff w.r.t x

$\Rightarrow -2\sin x + \sec^2 y \frac{dy}{dx} = 0$

$\Rightarrow \sec^2 y \frac{dy}{dx} = 2\sin x$

$\Rightarrow \frac{dy}{dx} = \frac{2\sin x}{\sec^2 y}$

$\Rightarrow \frac{dy}{dx} = 2\sin x \cos^2 y$ // AS REQUIRED

(b) $\left.\frac{dy}{dx}\right|_{\left(\frac{\pi}{6},\frac{\pi}{3}\right)} = 2\sin\frac{\pi}{6}\cos^2\frac{\pi}{3}$

$= \frac{1}{4}$

$\therefore$ NORMAL GRADIENT IS -4

$y - y_0 = m(x - x_0)$

$y - \frac{\pi}{3} = -4\left(x - \frac{\pi}{6}\right)$

$y - \frac{\pi}{3} = -4x + \frac{2\pi}{3}$

$y + 4x = \pi$ //

Question 8 (**+)

A curve is described by the implicit relationship

$$y^3 + xy = 2y + 4x - 10.$$

Find an equation of the normal to the curve at the point where $y = 1$.

$\boxed{3y + 4x = 15}$

$y^3 + xy = 2y + 4x - 10$

$\bullet\ y = 1$

$1 + x = 2 + 4x - 10$

$9 = 3x$

$x = 3$

$\therefore (3,1)$

Diff w.r.t x

$3y^2\frac{dy}{dx} + y + x\frac{dy}{dx} = 2\frac{dy}{dx} + 4$

AT $(3,1)$

$3\frac{dy}{dx} + 1 + 3\frac{dy}{dx} = 2\frac{dy}{dx} + 4$

$4\frac{dy}{dx} = 3$

$\frac{dy}{dx} = \frac{3}{4}$, NORMAL GRADIENT $-\frac{4}{3}$

NORMAL: $y - y_0 = m(x - x_0)$

$y - 1 = -\frac{4}{3}(x - 3)$

$3y - 3 = -4x + 12$

$3y + 4x = 15$ //

Question 9 **(**+)**

A curve C has implicit equation

$$x^3 + 2xy = \mathrm{e}^y .$$

Show **clearly** that

$$\frac{dy}{dx} = \frac{x^3 + 2y}{x^3 + 2xy - 2x} .$$

$\boxed{}$, $\boxed{\text{proof}}$

$x^3 + 2xy = e^y$

Diff w.r.t x

$\Rightarrow x^3 + 2y + 2x\frac{dy}{dx} = e^y\frac{dy}{dx}$

$\Rightarrow x^3 + 2y = (e^y - 2x)\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{x^3+2y}{e^y - 2x}$

$\Rightarrow \frac{dy}{dx} = \frac{x^3+2y}{x^3+2xy-2x}$

// AS REQUIRED

Question 10 **(**+)**

A curve has equation

$$4\cos y = 3 - 2\sin x,\ x \in \mathbb{R},\ y \in \mathbb{R} .$$

Show that the straight line with equation

$$4y - 2x = \pi$$

is the tangent to the curve at the point with coordinates $\left(\frac{\pi}{6}, \frac{\pi}{3}\right)$.

$\boxed{\text{proof}}$

$4\cos y = 3 - 2\sin x$

Diff w.r.t x

$-4\sin y\frac{dy}{dx} = -2\cos x$

$\frac{dy}{dx} = \frac{-2\cos x}{-4\sin y} = \frac{\cos x}{2\sin y}$

$\left.\frac{dy}{dx}\right|_{\substack{x=\frac{\pi}{6}\\ y=\frac{\pi}{3}}} = \frac{\cos\pi/6}{2\sin\pi/3} = \frac{\cancel{\sqrt{3}/2}}{2\times\cancel{\sqrt{3}/2}} = \frac{1}{2}$

EQUATION OF TANGENT

$y - y_0 = m(x - x_0)$

$y - \frac{\pi}{3} = \frac{1}{2}\left(x - \frac{\pi}{6}\right)$

$y - \frac{\pi}{3} = \frac{1}{2}x - \frac{\pi}{12}$

$12y - 4\pi = 6x - \pi$

$12y - 6x = 3\pi$

$4y - 2x = \pi$

// AS REQUIRED

Question 11 (***)

A curve has implicit equation

$$y^2 + 3xy - 2x^2 + 17 = 0.$$

Find an equation of the tangent to the curve at the point $(-2,3)$.

$\boxed{x = -2}$

$y^2 + 3xy - 2x^2 + 17 = 0$

Diff w.r.t x

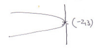
$2y\frac{dy}{dx} + 3y + 3x\frac{dy}{dx} - 4x = 0$

$(2y + 3x)\frac{dy}{dx} = 4x - 3y$

$\frac{dy}{dx} = \frac{4x - 3y}{2y + 3x}$

$\left.\frac{dy}{dx}\right|_{(-2,3)} = \frac{-8-9}{6-6} = \frac{-17}{0} = \infty$

IE INFINITE GRADIENT $\Rightarrow$ VERTICAL LINE



$\therefore x = -2$

Question 12 (***)

The curve C has equation

$$yx(2x-y)+1=0.$$

a) Show clearly that

$$\frac{dy}{dx}=\frac{y^2-4xy}{2x^2-2xy}.$$

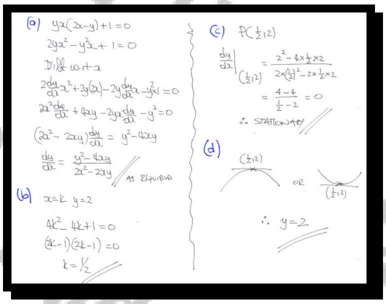
The point $P(k,2)$ lies on C.

b) Find the value of k.

c) Show that P is a stationary point of C.

d) Hence, state an equation of the tangent to C at P

$\boxed{}$, $\boxed{k=\frac{1}{2}}$, $\boxed{y=2}$



(a) $yx(2x-y)+1=0$

$2yx^2-y^2x+1=0$

Diff w.r.t x

$2\frac{dy}{dx}x^2+2y(2x)-2y\frac{dy}{dx}x-y^2\times1=0$

$2x^2\frac{dy}{dx}+4xy-2yx\frac{dy}{dx}-y^2=0$

$(2x^2-2xy)\frac{dy}{dx}=y^2-4xy$

$\frac{dy}{dx}=\frac{y^2-4xy}{2x^2-2xy}$ AS REQUIRED

(b) $x=k$ $y=2$

$4k^2-4k+1=0$

$(2k-1)(2k-1)=0$

$k=\frac{1}{2}$

(c) $P(\frac{1}{2},2)$

$\left.\frac{dy}{dx}\right|_{(\frac{1}{2},2)}=\frac{2^2-4\times\frac{1}{2}\times2}{2\times(\frac{1}{2})^2-2\times\frac{1}{2}\times2}=\frac{4-4}{\frac{1}{2}-2}=0$

$\therefore$ STATIONARY

(d)

$(\frac{1}{2},2)$ OR $(\frac{1}{2},2)$

$\therefore y=2$

Question 13 (***)

The curve C has equation

$$2\cos 3x \sin y = 1, \quad 0 \le x, y \le \pi .$$

The point $P\left(\frac{\pi}{12}, \frac{\pi}{4}\right)$ lies on C.

Show that an equation of the tangent to C at P is

$$y = 3x .$$

$\boxed{}$, $\boxed{\text{proof}}$

DIFFERENTIATING IMPLICITLY W.R.T x

$$\Rightarrow \frac{d}{dx}\left(2\cos 3x \sin y\right) = \frac{d}{dx}(1)$$

$$\Rightarrow -6\sin 3x \sin y + 2\cos 3x \cos y \frac{dy}{dx} = 0$$

AT $\left(\frac{\pi}{12}, \frac{\pi}{4}\right)$ WE OBTAIN

$$\Rightarrow -6\sin\frac{\pi}{4}\sin\frac{\pi}{4} + 2\cos\frac{\pi}{4}\cos\frac{\pi}{4}\left.\frac{dy}{dx}\right|_P = 0$$

$$\Rightarrow -6\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) + 2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)\left.\frac{dy}{dx}\right|_P = 0$$

$$\Rightarrow -3 + \left.\frac{dy}{dx}\right|_P = 0$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_P = 3$$

FINALLY WE HAVE

$$y - y_0 = m(x - x_0)$$

$$y - \frac{\pi}{4} = 3\left(x - \frac{\pi}{12}\right)$$

$$y - \frac{\pi}{4} = 3x - \frac{\pi}{4}$$

$$y = 3x$$

AS REQUIRED

Question 14 (***)

A curve has equation

$$3x^2 - xy + y^2 + 2x - 4y = 1 .$$

a) Show clearly that

$$\frac{dy}{dx} = \frac{2 + 6x - y}{4 - 2y + x} .$$

b) Hence show further that the value of x at the stationary points of the curve satisfies the equation

$$x^2 = \frac{5}{33} .$$

proof

(a) $3x^2 - xy + y^2 + 2x - 4y = 1$

Diff w.r.t x

$\Rightarrow 6x - 1xy - x \times \frac{dy}{dx} + 2y\frac{dy}{dx} + 2 - 4\frac{dy}{dx} = 0$

$\Rightarrow 6x - y + 2 = x\frac{dy}{dx} - 2y\frac{dy}{dx} + 4\frac{dy}{dx} = 0$

$\Rightarrow 6x + 2 - y = (x - 2y + 4)\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{6x + 2 - y}{x + 4 - 2y}$ // AS REQUIRED

(b) T.P. $\Rightarrow \frac{dy}{dx} = 0$

$\Rightarrow 6x + 2 - y = 0$

$\Rightarrow \boxed{y = 6x + 2}$

SOLVING SIMULTANEOUSLY

$\Rightarrow 3x^2 - x(6x+2) + (6x+2)^2 + 2x - 4(6x+2) = 1$

$\Rightarrow 3x^2 - 6x^2 \cancel{- 2x} + 36x^2 \cancel{+ 24x} + 4 \cancel{+ 2x} \cancel{- 24x} - 8 = 1$

$\Rightarrow 33x^2 = 5$

$\Rightarrow x^2 = \frac{5}{33}$ //

Question 15 (***)

A curve has equation

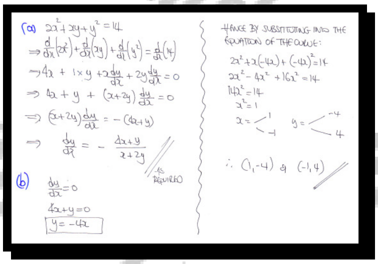
$$2x^2 + xy + y^2 = 14.$$

a) Show clearly that

$$\frac{dy}{dx} = -\frac{4x+y}{x+2y}.$$

b) Hence, find the coordinates of the stationary points of the curve.

$$\boxed{(1,-4),\ (-1,4)}$$



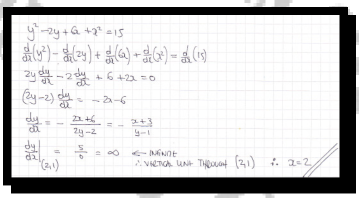
Question 16 (***)

A curve is described by the implicit relationship

$$y^2 - 2y + 6x + x^2 = 15.$$

Find an equation for the tangent to the curve at the point $P(2,1)$.

$$\boxed{x = 2}$$



Question 17 (***)

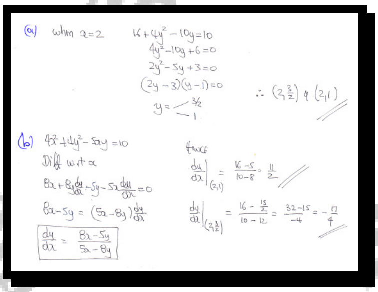
The equation of a curve is given by

$$4x^2 + 4y^2 - 5xy = 10.$$

a) Find the y coordinates of the points on the curve where $x = 2$.

b) Find the gradient at these points.

$\boxed{(2,1) \,\&\, \left(2,\frac{3}{2}\right)}$, $\boxed{\left.\frac{dy}{dx}\right|_{(2,1)} = \frac{11}{2}}$, $\boxed{\left.\frac{dy}{dx}\right|_{\left(2,\frac{3}{2}\right)} = -\frac{17}{4}}$


(a) when $x=2$

$16 + 4y^2 - 10y = 10$

$4y^2 - 10y + 6 = 0$

$2y^2 - 5y + 3 = 0$

$(2y-3)(y-1) = 0$

$y = \frac{3}{2}$ or 1

$\therefore \left(2,\frac{3}{2}\right)$ & $(2,1)$

(b) $4x^2 + 4y^2 - 5xy = 10$

Diff w.r.t x

$8x + 8y\frac{dy}{dx} - 5y - 5x\frac{dy}{dx} = 0$

$8x - 5y = (5x - 8y)\frac{dy}{dx}$

$\boxed{\frac{dy}{dx} = \frac{8x-5y}{5x-8y}}$

thus

$\left.\frac{dy}{dx}\right|_{(2,1)} = \frac{16-5}{10-8} = \frac{11}{2}$

$\left.\frac{dy}{dx}\right|_{\left(2,\frac{3}{2}\right)} = \frac{16-\frac{15}{2}}{10-12} = \frac{32-15}{-4} = -\frac{17}{4}$

Question 18 (***)

A curve C has implicit equation

$$x^2 - 4xy + y^2 = 13.$$

a) Show clearly that

$$\frac{dy}{dx} = \frac{x-2y}{2x-y}.$$

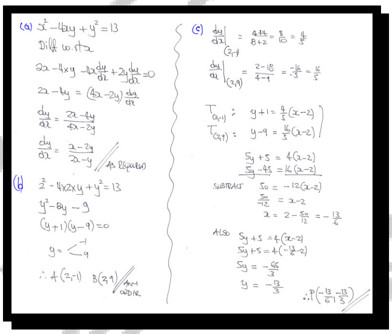
The points A and B are the two points on C whose x coordinate is 2.

b) Find the y coordinates of A and B.

The tangents to C at A and B, meet at the point P.

c) Find the exact coordinates of P.

$\boxed{}$, $\boxed{A(2,9),\ B(2,-1)}$, $\boxed{P\left(-\frac{13}{6}, -\frac{13}{3}\right)}$



Question 19 **(***)**

A curve C is defined implicitly by the equation

$$(2x-y)^3 = 37+3x^2 .$$

Find the value of the gradient at the point on C with coordinates $(3,2)$.

$\boxed{}$, $\boxed{\frac{13}{8}}$

$(2x-y)^3 = 37+3x^2$

Diff w.r.t x

$3(2x-y)^2 \times \left(2-\frac{dy}{dx}\right) = 0+6x$

AT $(3,2)$

$\Rightarrow 3(6-2)^2\left(2-\frac{dy}{dx}\right) = 18$

$\Rightarrow 48\left(2-\frac{dy}{dx}\right) = 18$

$\Rightarrow 2-\frac{dy}{dx} = \frac{3}{8}$

$\Rightarrow \frac{13}{8} = \frac{dy}{dx}$

$\Rightarrow \left.\frac{dy}{dx}\right|_{(3,2)} = \frac{13}{8}$ //

Question 20 **(***+)**

The point $P(2,3)$ lies on the curve with equation

$$4x^3-6xy+3^y = 23 .$$

Show that the gradient of the curve at P is

$$\frac{k}{4-9\ln 3},$$

where k is a positive integer to be found.

$\boxed{}$, $\boxed{k=10}$

$4x^3-6xy+3^y = 23$

$\Rightarrow \frac{d}{dx}(4x^3) - \frac{d}{dx}(6xy) + \frac{d}{dx}(3^y) = \frac{d}{dx}(23)$

$\Rightarrow 12x^2 - 6y - 6x\frac{dy}{dx} + 3^y\ln 3\,\frac{dy}{dx} = 0$

AT $(2,3)$

$\Rightarrow 48 - 18 - 12\left.\frac{dy}{dx}\right|_{(2,3)} + 27\ln 3\left.\frac{dy}{dx}\right|_{(2,3)} = 0$

$\Rightarrow (-12+27\ln 3)\left.\frac{dy}{dx}\right|_{(2,3)} = -30$

$\Rightarrow \left.\frac{dy}{dx}\right|_{(2,3)} = \frac{-30}{-12+27\ln 3}$

$\Rightarrow \left.\frac{dy}{dx}\right|_{(2,3)} = \frac{10}{4-9\ln 3}$ // ie $k=10$

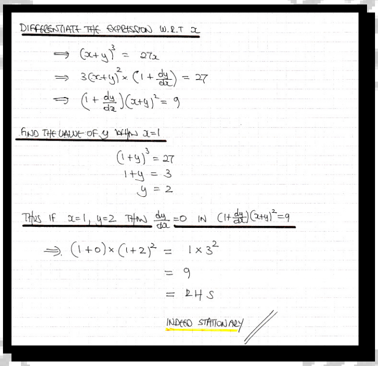
Question 21 (***+)

A curve C is defined implicitly by

$$\left(x+y\right)^3 = 27x\,, \quad x, y \in \mathbb{R}\,.$$

Verify that the point on C where $x = 1$ is a stationary point.

$\boxed{}$, $\boxed{\text{proof}}$



DIFFERENTIATE THE EXPRESSION W.R.T x

$\Rightarrow (x+y)^3 = 27x$

$\Rightarrow 3(x+y)^2 \times \left(1+\frac{dy}{dx}\right) = 27$

$\Rightarrow \left(1+\frac{dy}{dx}\right)(x+y)^2 = 9$

FIND THE VALUE OF y WHEN $x=1$

$(1+y)^3 = 27$

$1+y = 3$

$y = 2$

THUS IF $x=1$, $y=2$ THEN $\frac{dy}{dx}=0$ IN $\left(1+\frac{dy}{dx}\right)(x+y)^2=9$

$\Rightarrow (1+0)\times(1+2)^2 = 1\times 3^2$

$= 9$

$= \text{RHS}$

INDEED STATIONARY

Question 22 (***+)

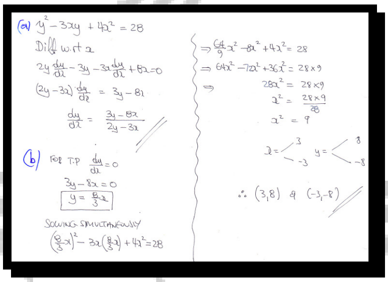
A curve C is defined implicitly by

$$y^2 - 3xy + 4x^2 = 28, \quad x \in \mathbb{R},\ y \in \mathbb{R}.$$

a) Find, in terms of x and y, a simplified expression for $\frac{dy}{dx}$.

b) Determine the coordinates of the stationary points of C.

$\boxed{}$, $\boxed{\frac{dy}{dx} = \frac{3y-8x}{2y-3x}}$, $\boxed{(-3,-8),\ (3,8)}$



(a) $y^2 - 3xy + 4x^2 = 28$

Diff w.r.t x

$2y\frac{dy}{dx} - 3y - 3x\frac{dy}{dx} + 8x = 0$

$(2y-3x)\frac{dy}{dx} = 3y - 8x$

$\frac{dy}{dx} = \frac{3y-8x}{2y-3x}$

(b) FOR T.P $\frac{dy}{dx} = 0$

$3y - 8x = 0$

$\boxed{y = \frac{8}{3}x}$

SOLVING SIMULTANEOUSLY

$\left(\frac{8}{3}x\right)^2 - 3x\left(\frac{8}{3}x\right) + 4x^2 = 28$

$\Rightarrow \frac{64}{9}x^2 - 8x^2 + 4x^2 = 28$

$\Rightarrow 64x^2 - 72x^2 + 36x^2 = 28 \times 9$

$\Rightarrow 28x^2 = 28 \times 9$

$x^2 = \frac{28 \times 9}{28}$

$x^2 = 9$

$x = 3, -3 \quad y = 8, -8$

$\therefore (3,8)$ & $(-3,-8)$

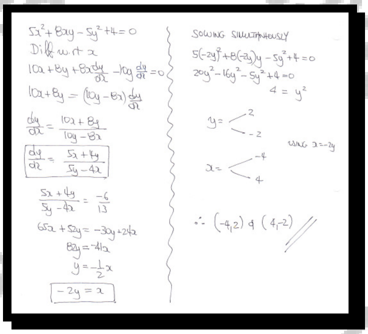
Question 23 (***+)

A curve has equation

$$5x^2 + 8xy - 5y^2 + 4 = 0.$$

Find the coordinates of the two points on the curve at which $\frac{dy}{dx} = -\frac{6}{13}$.

$\boxed{(-4,2),\ (4,-2)}$



$5x^2 + 8xy - 5y^2 + 4 = 0$

Diff w.r.t x

$10x + 8y + 8x\frac{dy}{dx} - 10y\frac{dy}{dx} = 0$

$10x + 8y = (10y - 8x)\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{10x+8y}{10y-8x}$

$\boxed{\frac{dy}{dx} = \frac{5x+4y}{5y-4x}}$

$\frac{5x+4y}{5y-4x} = \frac{-6}{13}$

$65x + 52y = -30y + 24x$

$82y = -41x$

$y = -\frac{1}{2}x$

$\boxed{-2y = x}$

SOLVING SIMULTANEOUSLY

$5(-2y)^2 + 8(-2y)y - 5y^2 + 4 = 0$

$20y^2 - 16y^2 - 5y^2 + 4 = 0$

$4 = y^2$

$y = 2, -2$

USING $x = -2y$

$x = -4, 4$

$\therefore (-4,2)$ & $(4,-2)$

Question 24 (***+)

A curve C is given implicitly by

$$x^2 - xy + y^2 = x.$$

Find the coordinates of the points on C at which the gradient is zero.

$$\boxed{(1,1) \ \& \ \left(\frac{1}{3}, -\frac{1}{3}\right)}$$

$x^2 - xy + y^2 = x$

Diff w.r.t x

$\Rightarrow 2x - 1 \times y - x\frac{dy}{dx} + 2y\frac{dy}{dx} = 1$

$\Rightarrow 2x - y - x\frac{dy}{dx} + 2y\frac{dy}{dx} = 1$

$\Rightarrow (2y - x)\frac{dy}{dx} = 1 - 2x + y$

$\Rightarrow \frac{dy}{dx} = \frac{1 - 2x + y}{2y - x}$

NOW $\frac{dy}{dx} = 0$

$1 - 2x + y = 0$

$\boxed{y = 2x - 1}$

SOLVING SIMULTANEOUSLY

$\left.\begin{array}{l} y = 2x - 1 \\ x^2 - xy + y^2 = x \end{array}\right) \Rightarrow$

$\Rightarrow x^2 - x(2x-1) + (2x-1)^2 = x$

$\Rightarrow x^2 - 2x^2 + \cancel{x} + 4x^2 - 4x + 1 = \cancel{x}$

$\Rightarrow 3x^2 - 4x + 1 = 0$

$\Rightarrow (x-1)(3x-1) = 0$

$\Rightarrow x = \begin{cases} 1 \\ \frac{1}{3} \end{cases} \quad y = \begin{cases} 2 \times 1 - 1 = 1 \\ 2 \times \frac{1}{3} - 1 = -\frac{1}{3} \end{cases}$

$\therefore \ (1,1) \ \& \ \left(\frac{1}{3}, -\frac{1}{3}\right)$ //

Question 25 (***+)

The equation of a curve is given implicitly by

$$2\ln y = x \ln x, \quad x, y \in \mathbb{R} \quad x, y > 0.$$

Find the exact value of the gradient at the point on the curve where $x = 4$.

$$\boxed{}, \ \boxed{8(1 + \ln 4)}$$

DIFFERENTIATING IMPLICITLY W.R.T x

$\Rightarrow 2\ln y = x\ln x$

$\Rightarrow \frac{d}{dx}(2\ln y) = \frac{d}{dx}(x\ln x)$

$\Rightarrow 2 \times \frac{1}{y}\frac{dy}{dx} = 1 \times \ln x + x \times \frac{1}{x}$

$\Rightarrow \frac{2}{y}\frac{dy}{dx} = 1 + \ln x$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2}y(1 + \ln x)$

NOW WHEN $x = 4$

$2\ln y = 4\ln 4$

$\ln y = 2\ln 4$

$\ln y = \ln 16$

$y = 16$

FINALLY WE OBTAIN

$\left.\frac{dy}{dx}\right|_{(4,16)} = \frac{1}{2} \times 16 \times (1 + \ln 4) = 8(1 + \ln 4) = 8(1 + 2\ln 2)$ //

ALTERNATIVE BY REARRANGING FIRST

$\Rightarrow 2\ln y = x\ln x$

$\Rightarrow \ln y = \frac{1}{2}x\ln x$

$\Rightarrow e^{\ln y} = e^{\frac{1}{2}x\ln x}$

$\Rightarrow y = e^{\frac{1}{2}x\ln x}$

NOW DIFFERENTIATING W.R.T x

$\Rightarrow \frac{dy}{dx} = e^{\frac{1}{2}x\ln x} \times \left(\frac{1}{2}\ln x + \frac{1}{2}x \times \frac{1}{x}\right)$

$\Rightarrow \frac{dy}{dx} = e^{\frac{1}{2}x\ln x} \times \left(\frac{1}{2}\right.$

$\Rightarrow \left.\frac{dy}{dx}\right|_{x=4} = e^{2\ln 4} \times \left(\frac{1}{2}\ln 4 + \frac{1}{2}\right)$

$= e^{\ln 16} \times \frac{1}{2}(1 + \ln 4)$

$= 16 \times \frac{1}{2}(1 + 2\ln 2)$

$= 8(1 + 2\ln 2)$ // AS BEFORE

Question 26 (***+)

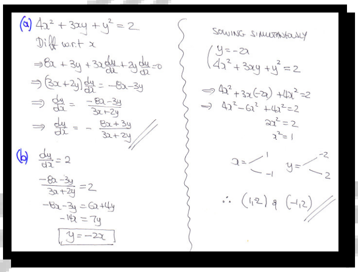
A curve C is given implicitly by the equation

$$4x^2 + 3xy + y^2 = 2, \quad x \in \mathbb{R}, \quad y \in \mathbb{R}.$$

a) Find, in terms of x and y, an expression for $\frac{dy}{dx}$.

b) Find the coordinates of the points on C, where the gradient is 2.

$\boxed{\frac{dy}{dx} = -\frac{8x+3y}{3x+2y}}$, $\boxed{(-1,2),\ (1,-2)}$



(a) $4x^2 + 3xy + y^2 = 2$

Diff w.r.t x

$\Rightarrow 8x + 3y + 3x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$

$\Rightarrow (3x+2y)\frac{dy}{dx} = -8x-3y$

$\Rightarrow \frac{dy}{dx} = \frac{-8x-3y}{3x+2y}$

$\Rightarrow \frac{dy}{dx} = -\frac{8x+3y}{3x+2y}$

(b) $\frac{dy}{dx} = 2$

$\frac{-8x-3y}{3x+2y} = 2$

$-8x-3y = 6x+4y$

$-14x = 7y$

$\boxed{y=-2x}$

SOLVING SIMULTANEOUSLY

$\begin{cases} y = -2x \\ 4x^2 + 3xy + y^2 = 2 \end{cases}$

$\Rightarrow 4x^2 + 3x(-2x) + 4x^2 = 2$

$\Rightarrow 4x^2 - 6x^2 + 4x^2 = 2$

$2x^2 = 2$

$x^2 = 1$

$x = 1, -1 \qquad y = -2, 2$

$\therefore (1,-2)$ & $(-1,2)$

Question 27 (***+)

A curve C has implicit equation

$$x^2 + 4xy + 2y^2 = 7 .$$

a) Show clearly that …

i. … $\dfrac{dy}{dx} = -\dfrac{x+2y}{2x+2y}$.

ii. … the equation of the tangent to the curve at $P(1,1)$ is

$$3x + 4y = 7 .$$

The tangent to the curve at the point Q is parallel to the tangent to the curve at P .

b) Find the coordinates of Q .

$\boxed{}$, $\boxed{Q(-1,-1)}$

(a)(I) $x^2 + 4xy + 2y^2 = 7$

Diff w.r.t x

$2x + \left(4xy + 4x \times \frac{dy}{dx}\right) + 4y\frac{dy}{dx} = 0$

$2x + 4y + 4x\frac{dy}{dx} + 4y\frac{dy}{dx} = 0$

$(4x + 4y)\frac{dy}{dx} = -2x - 4y$

$\frac{dy}{dx} = \frac{-2x-4y}{4x+4y}$

$\frac{dy}{dx} = -\frac{x+2y}{2x+2y}$ AS REQUIRED

(II) $\left.\frac{dy}{dx}\right|_{P(1,1)} = -\frac{1+2}{2+2} = \frac{-3}{4}$

TANGENT AT P

$y - y_0 = m(x - x_0)$

$y - 1 = -\frac{3}{4}(x-1)$

$4y - 4 = -3x + 3$

$4y + 3x = 7$

(b) $\frac{dy}{dx} = -\frac{3}{4}$

$-\frac{x+2y}{2x+2y} = -\frac{3}{4}$

$4x + 8y = 6x + 6y$

$2y = 2x$

$\boxed{y = x}$

$\begin{cases} x^2 + 4xy + 2y^2 = 7 \\ y = x \end{cases}$

$x^2 + 4x^2 + 2x^2 = 7$

$7x^2 = 7$

$x^2 = 1$

$x = \begin{cases} 1 \\ -1 \end{cases}$ $y = \begin{cases} 1 \\ -1 \end{cases}$ ← P(1,1)

Q(−1,−1)

Question 28 (***+)

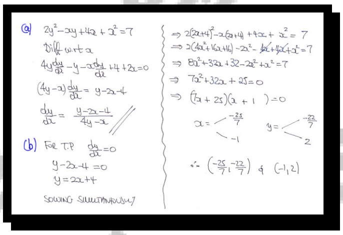
A curve C is defined implicitly by

$$2y^2 - xy + 4x + x^2 = 7, \quad x, y \in \mathbb{R}.$$

a) Find an expression for $\frac{dy}{dx}$, in terms of x and y.

b) Show that $(-1,2)$ is one of the stationary points of C and determine the exact coordinates of the other stationary point.

$\boxed{}$, $\boxed{\frac{dy}{dx} = \frac{y-2x-4}{4y-x}}$, $\boxed{\left(-\frac{25}{7}, -\frac{22}{7}\right)}$



(a) $2y^2 - xy + 4x + x^2 = 7$

Diff w.r.t x

$4y\frac{dy}{dx} - y - x\frac{dy}{dx} + 4 + 2x = 0$

$(4y - x)\frac{dy}{dx} = y - 2x - 4$

$\frac{dy}{dx} = \frac{y-2x-4}{4y-x}$

(b) FOR T.P $\frac{dy}{dx} = 0$

$y - 2x - 4 = 0$

$y = 2x + 4$

SOLVING SIMULTANEOUSLY

$\Rightarrow 2(2x+4)^2 - x(2x+4) + 4x + x^2 = 7$

$\Rightarrow 2(4x^2+16x+16) - 2x^2 - \cancel{4x} + \cancel{4x} + x^2 = 7$

$\Rightarrow 8x^2 + 32x + 32 - 2x^2 + x^2 = 7$

$\Rightarrow 7x^2 + 32x + 25 = 0$

$\Rightarrow (7x+25)(x+1) = 0$

$x = -\frac{25}{7}, \ -1 \qquad y = -\frac{22}{7}, \ 2$

$\therefore \left(-\frac{25}{7}, -\frac{22}{7}\right)$ & $(-1,2)$

Question 29 (***+)

A curve C is defined implicitly by

$$4xy-(x+2)^2 = y^2-5.$$

a) Find a simplified expression for $\frac{dy}{dx}$, in terms of x and y.

b) Hence determine the coordinates of the two stationary points of C.

$\boxed{}$, $\boxed{\frac{dy}{dx}=\frac{x-2y+2}{2x-y}}$, $\boxed{(0,1),\ \left(\frac{4}{3},\frac{5}{3}\right)}$

a) DIFFERENTIATE WITH RESPECT TO x

$\Rightarrow \frac{d}{dx}(4xy) - \frac{d}{dx}\left((x+2)^2\right) = \frac{d}{dx}(y^2) - \frac{d}{dx}(5)$

PRODUCT RULE CHAIN RULE

$\Rightarrow 4y + 4x\frac{dy}{dx} - 2(x+2) = 2y\frac{dy}{dx} - 0$

$\Rightarrow (4x-2y)\frac{dy}{dx} = 2(x+2) - 4y$

$\Rightarrow \frac{dy}{dx} = \frac{2x-4y+4}{4x-2y}$

$\Rightarrow \frac{dy}{dx} = \frac{x-2y+2}{2x-y}$ //

b) "STATIONARY" $\Rightarrow \frac{dy}{dx}=0$

$\therefore\ x-2y+2=0$

$x = 2y-2$

SUBSTITUTE INTO THE EQUATION OF THE CURVE

$\Rightarrow 4y(2y-2) - (2y-\cancel{2}+\cancel{2})^2 = y^2-5$

$\Rightarrow 8y^2-8y-4y^2 = y^2-5$

$\Rightarrow 3y^2-8y+5=0$

$\Rightarrow (3y-5)(y-1)=0$

$y = \begin{cases} 5/3 \\ 1 \end{cases}$ $\quad x = \begin{cases} 2\times\frac{5}{3}-2 = \frac{4}{3} \\ 2(1)-2=0 \end{cases}$

$\therefore \left(\frac{4}{3},\frac{5}{3}\right)$ & $(0,1)$ //

Question 30 (***+)

A curve C is defined implicitly by

$$6^x + 6xy + y^2 = 9 .$$

a) Show clearly that

$$\frac{dy}{dx} = -\frac{6y + 6^x \ln 6}{6x + 2y} .$$

b) Find the gradient at each of the two points on C, where $x = 2$.

Give the answers in the form $a + b\ln 6$, where a and b are integers.

$\boxed{}$, $\boxed{\left.\frac{dy}{dx}\right|_{(2,-3)} = 3 - 6\ln 6}$, $\boxed{\left.\frac{dy}{dx}\right|_{(2,-9)} = -9 + 6\ln 6}$

a) $6^x + 6xy + y^2 = 9$

diff w.r.t x

$\Rightarrow 6^x \ln 6 + \left(6y + 6x\frac{dy}{dx}\right) + 2y\frac{dy}{dx} = 0$

$\Rightarrow 6^x \ln 6 + 6y + 6x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$

$\Rightarrow 6x\frac{dy}{dx} + 2y\frac{dy}{dx} = -6y - 6^x \ln 6$

$\Rightarrow (6x + 2y)\frac{dy}{dx} = -6y - 6^x \ln 6$

$\frac{dy}{dx} = -\frac{6y + 6^x \ln 6}{6x + 2y}$ // AS REQUIRED

b) WHEN $x = 2$

$6^2 + 12y + y^2 = 9$

$y^2 + 12y + 27 = 0$

$(y+9)(y+3) = 0$

$y = \begin{cases} -3 \\ -9 \end{cases}$

$\therefore (2,-3)$ & $(2,-9)$

THUS

- $\left.\frac{dy}{dx}\right|_{(2,-3)} = -\frac{-18 + 36\ln 6}{12 - 6} = \frac{18 - 36\ln 6}{6} = 3 - 6\ln 6$ //

- $\left.\frac{dy}{dx}\right|_{(2,-9)} = -\frac{-54 + 36\ln 6}{12 - 18} = \frac{54 - 36\ln 6}{-6} = -9 + 6\ln 6$ //

Question 31 (***+)

A curve C has implicit equation

$$2xy = 2^x + y^2.$$

a) Show clearly that

$$\frac{dy}{dx} = \frac{y - 2^{x-1}\ln 2}{y - x}.$$

The point P lies on C, where $x = 2$.

b) Find an equation of the tangent to C at P.

$\boxed{\text{MP2-H}}$, $\boxed{x = 2}$

a) DIFFERENTIATE W.R.T x

$$\Rightarrow \frac{d}{dx}(2xy) = \frac{d}{dx}(2^x + y^2)$$

$$\Rightarrow 2y + 2x\frac{dy}{dx} = 2^x \ln 2 + 2y\frac{dy}{dx}$$

$$\frac{d}{dx}(a^x) = a^x \ln a$$

$$\Rightarrow 2x\frac{dy}{dx} - 2y\frac{dy}{dx} = 2^x \ln 2 - 2y$$

$$\Rightarrow (2x - 2y)\frac{dy}{dx} = -2y + 2^x \ln 2$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2y + 2^x \ln 2}{2x - 2y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2(y - 2^{x-1}\ln 2)}{-2(y - x)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y - 2^{x-1}\ln 2}{y - x}$$

AS REQUIRED

b) FIRST FIND THE FULL COORDINATES OF P

WHEN $x = 2$ $\Rightarrow 2xy = 2^x + y^2$

$$\Rightarrow 4y = 4 + y^2$$

$$\Rightarrow 0 = y^2 - 4y + 4$$

$$\Rightarrow 0 = (y - 2)^2$$

$$\Rightarrow y = 2$$

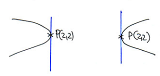
$\therefore P(2,2)$

NEXT FIND THE GRADIENT AT P

$$\left.\frac{dy}{dx}\right|_{(2,2)} = \frac{2 - 2\ln 2}{2 - 2} = \frac{2 - 2\ln 2}{0} = \infty$$ ← INFINITE GRADIENT

HENCE THE TANGENT AT $P(2,2)$ IS "VERTICAL"

I.E EITHER OR



$\therefore$ EQUATION OF TANGENT IS $x = 2$

Question 32 (***+)

A curve C has implicit equation

$$\sin 3x + \sin 2y = \sqrt{2}, \quad 0 \le x, y \le \frac{\pi}{3}.$$

The point P lies on C and its x coordinate is $\frac{\pi}{12}$.

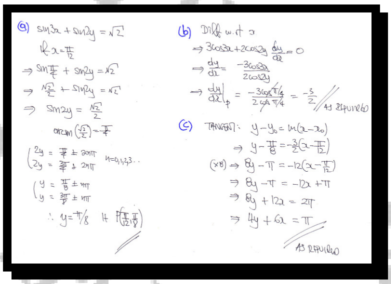
a) Find the y coordinate of P.

b) Show that the gradient at P is $-\frac{3}{2}$.

c) Show further that the equation of the tangent to C at P is

$$4y + 6x = \pi.$$

$$\boxed{P\left(\frac{\pi}{12}, \frac{\pi}{8}\right)}$$



(a) $\sin 3x + \sin 2y = \sqrt{2}$

If $x = \frac{\pi}{12}$

$\Rightarrow \sin\frac{\pi}{4} + \sin 2y = \sqrt{2}$

$\Rightarrow \frac{\sqrt{2}}{2} + \sin 2y = \sqrt{2}$

$\Rightarrow \sin 2y = \frac{\sqrt{2}}{2}$

$\arcsin\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$

$2y = \frac{\pi}{4} \pm 2n\pi$, $2y = \frac{3\pi}{4} \pm 2n\pi$ $\quad n = 0, 1, 2, 3 \ldots$

$y = \frac{\pi}{8} \pm n\pi$, $y = \frac{3\pi}{8} \pm n\pi$

$\therefore y = \pi/8$ $\quad$ If $P\left(\frac{\pi}{12}, \frac{\pi}{8}\right)$

(b) Diff w.r.t x

$\Rightarrow 3\cos 3x + 2\cos 2y \frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = \frac{-3\cos 3x}{2\cos 2y}$

$\Rightarrow \left.\frac{dy}{dx}\right|_P = \frac{-3\cos \pi/4}{2\cos \pi/4} = -\frac{3}{2}$ AS REQUIRED

(c) TANGENT: $y - y_0 = m(x - x_0)$

$\Rightarrow y - \frac{\pi}{8} = -\frac{3}{2}\left(x - \frac{\pi}{12}\right)$

$(\times 8) \Rightarrow 8y - \pi = -12\left(x - \frac{\pi}{12}\right)$

$\Rightarrow 8y - \pi = -12x + \pi$

$\Rightarrow 8y + 12x = 2\pi$

$\Rightarrow 4y + 6x = \pi$

AS REQUIRED

Question 33 **(***+)**

A curve has implicit equation

$$\frac{3x^2}{y} - 5y = 2(x+8), \qquad x \in \mathbb{R},\ y \in \mathbb{R},\ y \neq 0.$$

Find the coordinates of the stationary point of the curve.

$\boxed{}$, $\boxed{(-1,-3)}$

REWRITE THE EQUATION BEFORE DIFFERENTIATION

$\Rightarrow \frac{3x^2}{y} - 5y = 2(x+8)$

$\Rightarrow 3x^2 - 5y^2 = 2y(x+8)$

$\Rightarrow 3x^2 - 5y^2 = 2xy + 16y$

DIFFERENTIATE WITH RESPECT TO x

$\Rightarrow \frac{d}{dx}(3x^2) - \frac{d}{dx}(5y^2) = \frac{d}{dx}(2xy) + \frac{d}{dx}(16y)$

$\Rightarrow 6x - 10y\frac{dy}{dx} = \left[2y + 2x\frac{dy}{dx}\right] + 16\frac{dy}{dx}$

FOR STATIONARY POINTS $\frac{dy}{dx} = 0$

$\Rightarrow 6x = 2y$

$\Rightarrow y = 3x$

ANY STATIONARY POINTS MUST LIE ON THE LINE $y = 3x$ – SOLVING SIMULTANEOUSLY WITH THE EQUATION OF THE CURVE

$$\left.\begin{array}{l} 3x^2 - 5y^2 = 2xy + 16y \\ y = 3x \end{array}\right\} \Rightarrow 3x^2 - 5(3x)^2 = 2x(3x) + 16(3x)$$

$\Rightarrow 3x^2 - 45x^2 = 6x^2 + 48x$

$\Rightarrow 0 = 48x^2 + 48x$

$\Rightarrow 48x(x+1) = 0$

$\Rightarrow x = \begin{cases} 0 \\ -1 \end{cases} \quad y = \begin{cases} 0 \\ -3 \end{cases}$

$\therefore$ ONLY POINT IS $(-1,-3)$ AS $y \neq 0$

Question 34 **(***+)**

A curve C has implicit equation

$$xy(x-y)+16=0, \quad x \neq y, \quad x, y \neq 0.$$

Find the coordinates of the stationary point of C.

$\boxed{\quad}$, $\boxed{(2,4)}$

EXPANDING & DIFFERENTIATING IMPLICITLY

$\Rightarrow xy(x-y)+16=0$

$\Rightarrow x^2y - xy^2 + 16 = 0$

$\Rightarrow \frac{d}{dx}(x^2y) - \frac{d}{dx}(xy^2) = \frac{d}{dx}(-16)$

$\Rightarrow 2xy + x^2\frac{dy}{dx} - y^2 - x\left(2y\frac{dy}{dx}\right) = 0$

$\Rightarrow 2xy + x^2\frac{dy}{dx} - y^2 - 2xy\frac{dy}{dx} = 0$

FOR STATIONARY POINTS $\frac{dy}{dx}=0$

$\Rightarrow 2xy - y^2 = 0$

$\Rightarrow 2x - y = 0 \quad (y \neq 0)$

$\Rightarrow y = 2x$

SUBSTITUTE INTO THE EQUATION

$\Rightarrow x(2x)[x-2x)+16=0$

$\Rightarrow -2x^3+16=0$

$\Rightarrow 16 = 2x^3$

$\Rightarrow 8 = x^3$

$\Rightarrow x = 2$

$\therefore (2,4)$

Question 35 **(***+)**

A curve C has implicit equation

$$ax^2 + xy - 2y^2 + b = 0,$$

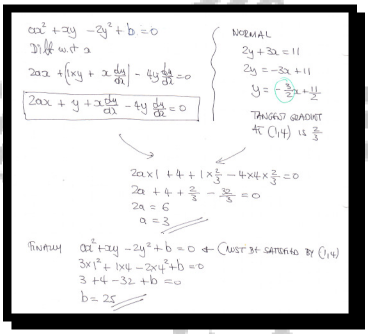
where a and b are constants.

The normal to the curve at the point $P(1,4)$ has equation

$$2y + 3x = 11.$$

Determine the value of a and the value of b.

$\boxed{}$, $\boxed{a = 3}$, $\boxed{b = 25}$



$ax^2 + xy - 2y^2 + b = 0$

Diff w.r.t x

$2ax + \left(1 \times y + x\frac{dy}{dx}\right) - 4y\frac{dy}{dx} = 0$

$\boxed{2ax + y + x\frac{dy}{dx} - 4y\frac{dy}{dx} = 0}$

NORMAL

$2y + 3x = 11$

$2y = -3x + 11$

$y = -\frac{3}{2}x + \frac{11}{2}$

TANGENT GRADIENT AT $(1,4)$ IS $\frac{2}{3}$

$2a \times 1 + 4 + 1 \times \frac{2}{3} - 4 \times 4 \times \frac{2}{3} = 0$

$2a + 4 + \frac{2}{3} - \frac{32}{3} = 0$

$2a = 6$

$a = 3$

FINALLY $ax^2 + xy - 2y^2 + b = 0$ ← (MUST BE SATISFIED BY $(1,4)$)

$3 \times 1^2 + 1 \times 4 - 2 \times 4^2 + b = 0$

$3 + 4 - 32 + b = 0$

$b = 25$

Question 36 (***+)

The equation of a curve is given by

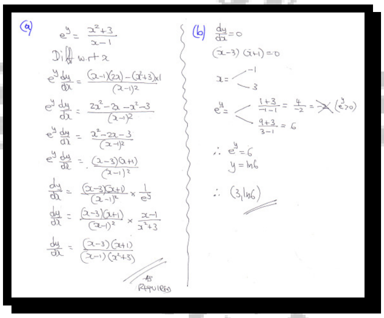
$$e^{y} = \frac{x^2+3}{x-1}.$$

a) Show clearly that

$$\frac{dy}{dx} = \frac{(x-3)(x+1)}{\left(x^2+3\right)(x-1)}.$$

b) Find the exact coordinates of the turning point of the curve.

$\boxed{}$, $\boxed{(3, \ln 6)}$



Question 37 (*+)**

The curve C is given implicitly by

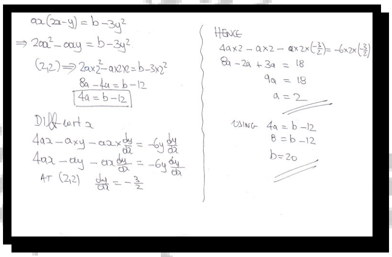
$$ax(2x-y)=b-3y^2,$$

where a and b are non zero constants.

The point $(2,2)$ lies on C and the gradient at that point is $-\frac{3}{2}$.

Find the value of a and the value of b.

$\boxed{\text{C4R}}$, $\boxed{a=2,\ b=20}$



$ax(2x-y)=b-3y^2$

$\Rightarrow 2ax^2-axy=b-3y^2$

$(2,2) \Rightarrow 2a\times 2^2-a\times 2\times 2=b-3\times 2^2$

$8a-4a=b-12$

$\boxed{4a=b-12}$

Diff w.r.t x

$4ax-a\times y-ax\times\frac{dy}{dx}=-6y\frac{dy}{dx}$

$4ax-ay-ax\frac{dy}{dx}=-6y\frac{dy}{dx}$

AT $(2,2)$ $\frac{dy}{dx}=-\frac{3}{2}$

HENCE

$4a\times 2-a\times 2-a\times 2\times\left(-\frac{3}{2}\right)=-6\times 2\times\left(-\frac{3}{2}\right)$

$8a-2a+3a=18$

$9a=18$

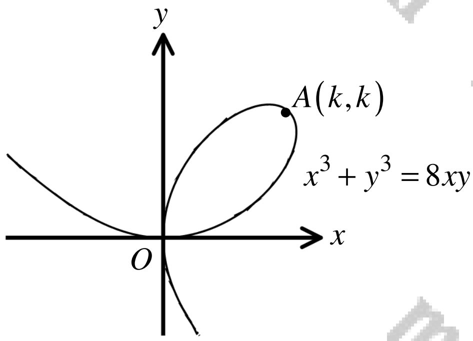
$a=2$

USING $4a=b-12$

$8=b-12$

$b=20$

Question 38 (****)



The figure above shows a curve known as "the folium of Descartes", with equation

$$x^3 + y^3 = 8xy \, .$$

The point $A(k,k)$, where k is a non zero constant, lies on the curve.

a) Find the value of k .

b) Show that the gradient at A is -1.

$\boxed{}$, $\boxed{k = 4}$

(a) $x^3+y^3 = 8xy$

$A(k,k) \Rightarrow k^3+k^3 = 8k^2$

$\Rightarrow 2k^3 = 8k^2$

$\Rightarrow k^3 = 4k^2$

$\Rightarrow k^3 - 4k^2 = 0$

$\Rightarrow k^2(k-4)=0$

$\Rightarrow k = \cancel{0}, 4$

(b) Diff w.r.t x

$3x^2 + 3y^2\frac{dy}{dx} = 8y + 8x\frac{dy}{dx}$

AT $A(4,4)$

$48 + 48\frac{dy}{dx} = 32 + 32\frac{dy}{dx}$

$16\frac{dy}{dx} = -16$

$\frac{dy}{dx} = -1$

AS REQUIRED

Question 39 **(****)**

A curve C has implicit equation

$$ax^3 - 3xy + by^2 = 224,$$

where a and b are non zero constants.

The normal to the curve at the point $P(-2,6)$ has equation

$$15x - 13y + 108 = 0.$$

Determine the value of a and the value of b.

$\boxed{a=8}$, $\boxed{b=7}$

START BY OBTAINING THE GRADIENT FUNCTION

$$\Rightarrow ax^3 - 3xy + by^2 = 224$$
$$\Rightarrow \frac{d}{dx}\left(ax^3 - 3xy + by^2\right) = \frac{d}{dx}(224)$$
$$\Rightarrow 3ax^2 - 3y - 3x\frac{dy}{dx} + 2by\frac{dy}{dx} = 0$$

OBTAIN THE GRADIENT AT $(-2,6)$

$$\Rightarrow 3a(-2)^2 - 3\times6 - 3(-2)\frac{dy}{dx} + 2b(6)\frac{dy}{dx} = 0$$
$$\Rightarrow 12a - 18 + 6\frac{dy}{dx} + 12b\frac{dy}{dx} = 0$$
$$\Rightarrow (6+12b)\frac{dy}{dx} = 18 - 12a$$
$$\Rightarrow \frac{dy}{dx} = \frac{18-12a}{6+12b}$$
$$\Rightarrow \frac{dy}{dx} = \frac{3-2a}{1+2b} \leftarrow \text{GRADIENT OF THE TANGENT}$$

$\therefore$ GRADIENT OF THE NORMAL IS $\dfrac{2b+1}{2a-3}$

NEXT WE REARRANGE THE NORMAL TO "READ" ITS GRADIENT

$$\Rightarrow 15x - 13y + 108 = 0$$
$$\Rightarrow 15x + 108 = 13y$$
$$\Rightarrow y = \underbrace{\frac{15}{13}}_{\frac{2b+1}{2a-3}}x + \frac{108}{13}$$
$$\Rightarrow \frac{2b+1}{2a-3} = \frac{15}{13}$$
$$\Rightarrow 13(2b+1) = 15(2a-3)$$
$$\Rightarrow 26b + 13 = 30a - 45$$
$$\Rightarrow 26b - 30a = -58$$
$$\Rightarrow 30a - 26b = 58$$
$$\Rightarrow 15a - 13b = 29$$

THE POINT $(-2,6)$ MUST ALSO SATISFY THE EQUATION OF THE CURVE

$$\Rightarrow ax^3 - 3xy + by^2 = 224$$
$$\Rightarrow -8a - 3(-2)\times6 + b\times36 = 224$$
$$\Rightarrow -8a + 36 + 36b = 224$$
$$\Rightarrow -8a + 36b = 188$$
$$\Rightarrow -2a + 9b = 47$$

FINALLY SOLVING THE EQUATIONS SIMULTANEOUSLY

$$\left.\begin{array}{r} 15a - 13b = 29 \\ -2a + 9b = 47 \end{array}\right) \underset{\times15}{\overset{\times2}{\Rightarrow}} \left.\begin{array}{r} 30a - 26b = 58 \\ -30a + 135b = 705 \end{array}\right\} \Rightarrow$$

$$\Rightarrow 109b = 763$$
$$\Rightarrow b = 7$$

& $-2a + 9b = 47$

$$-2a + 63 = 47$$
$$-2a = -16$$
$$a = 8$$

Question 40 (****)

The equation of a curve is given implicitly by

$$4y + y^2 e^{3x} = x^3 + C,$$

where C is a non zero constant.

a) Find a simplified expression for $\dfrac{dy}{dx}$.

The point $P(1,k)$, where $k > 0$, is a stationary point of the curve.

b) Find an exact value for C.

$$\boxed{\frac{dy}{dx} = \frac{3\left(x^2 - y^2 e^{3x}\right)}{2\left(2 + y e^{3x}\right)}}, \quad \boxed{C = 4e^{-\frac{3}{2}}}$$

(a) $4y + e^{3x}y^2 = x^3 + C$

Diff w.r.t x

$4\frac{dy}{dx} + 3e^{3x}y^2 + 2y\frac{dy}{dx}e^{3x} = 3x^2$

$\left(4 + 2ye^{3x}\right)\frac{dy}{dx} = 3x^2 - 3e^{3x}y^2$

$\frac{dy}{dx} = \frac{3\left(x^2 - y^2e^{3x}\right)}{2\left(2 + ye^{3x}\right)}$

(b) $x = 1$ IS STATIONARY

$\left.\frac{dy}{dx}\right|_{(1,y)} = 0$

$\Rightarrow 1 - y^2e^3 = 0$

$\Rightarrow 1 = y^2e^3$

$\Rightarrow y^2 = \frac{1}{e^3}$

$\Rightarrow y = \frac{1}{e^{3/2}} = e^{-3/2}$

THUS $P(1, e^{-3/2})$

THUS $4(e^{-3/2}) + e^3 \times \frac{1}{e^3} = 1 + C$

$4e^{-3/2} + \cancel{1} = \cancel{1} + C$

$C = 4e^{-3/2}$

Question 41 (****)

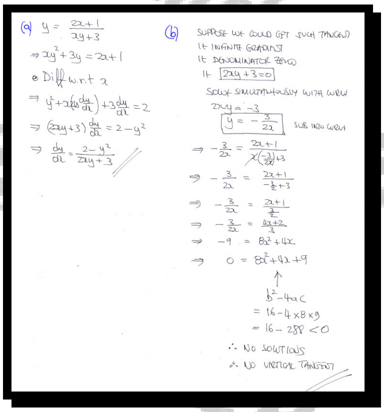
A curve C has implicit equation

$$y = \frac{2x+1}{xy+3}.$$

a) Find an expression for $\frac{dy}{dx}$, in terms of x and y.

b) Show that there is **no** point on C, where the tangent is parallel to the y axis.

$\boxed{C4}$, $\boxed{\frac{dy}{dx} = \frac{2-y^2}{2xy+3}}$



(a) $y = \frac{2x+1}{xy+3}$

$\Rightarrow xy^2 + 3y = 2x+1$

• Diff w.r.t x

$\Rightarrow y^2 + x\left(2y\frac{dy}{dx}\right) + 3\frac{dy}{dx} = 2$

$\Rightarrow (2xy+3)\frac{dy}{dx} = 2 - y^2$

$\Rightarrow \frac{dy}{dx} = \frac{2-y^2}{2xy+3}$

(b) SUPPOSE WE COULD GET SUCH TANGENT

IE INFINITE GRADIENT

IE DENOMINATOR ZERO

IE $\boxed{2xy+3=0}$

SOLVE SIMULTANEOUSLY WITH CURVE

$2xy = -3$

$\boxed{y = -\frac{3}{2x}}$ SUB INTO CURVE

$\Rightarrow -\frac{3}{2x} = \frac{2x+1}{x\left(\frac{-3}{2x}\right)+3}$

$\Rightarrow -\frac{3}{2x} = \frac{2x+1}{-\frac{3}{2}+3}$

$\Rightarrow -\frac{3}{2x} = \frac{2x+1}{\frac{3}{2}}$

$\Rightarrow -\frac{3}{2x} = \frac{4x+2}{3}$

$\Rightarrow -9 = 8x^2 + 4x$

$\Rightarrow 0 = 8x^2 + 4x + 9$

$b^2 - 4ac$

$= 16 - 4 \times 8 \times 9$

$= 16 - 288 < 0$

$\therefore$ NO SOLUTIONS

$\therefore$ NO VERTICAL TANGENT

Question 42 (****)

A curve has implicit equation

$$8x^4 + 32xy^3 + 16y^4 = 1 .$$

Find the coordinates of any points on the curve whose gradient is $\frac{1}{2}$.

$\boxed{}$, $\boxed{\left(0,\frac{1}{2}\right) \cup \left(0,-\frac{1}{2}\right)}$

DIFFERENTIATE IMPLICITLY WITH RESPECT TO x

$\Rightarrow 8x^4 + 32xy^3 + 16y^4 = 1$

$\Rightarrow \frac{d}{dx}(8x^4) + \frac{d}{dx}(32xy^3) + \frac{d}{dx}(16y^4) = \cancel{\frac{d}{dx}(1)}$

$\Rightarrow 32x^3 + 32y^3 + 96xy^2\frac{dy}{dx} + 64y^3\frac{dy}{dx} = 0$

$\Rightarrow x^3 + y^3 + 3xy^2\frac{dy}{dx} + 2y^3\frac{dy}{dx} = 0$

NOW GRADIENT OF $\frac{1}{2}$

$\Rightarrow x^3 + \cancel{y^3} + \frac{1}{2}(3xy^2) + \cancel{y^3} = 0$

$\Rightarrow 2x^3 + 3xy^2 = 0$

$\Rightarrow x(2x^2 + 3y^2) = 0$

THE TRIVIAL SOLUTION $x=y=0$, DOES NOT SATISFY THE CURVE,

$\Rightarrow x=0 \quad y \neq 0$

RETURNING TO THE EQUATION WITH $x=0$

$\Rightarrow 16y^4 - 1 = 0$

$\Rightarrow y^4 = \frac{1}{16}$

$\Rightarrow y = \pm\frac{1}{2}$

$\therefore \left(0,-\frac{1}{2}\right) \text{ \& } \left(0,\frac{1}{2}\right)$

Question 43 (****)

A curve C has implicit equation

$$(xy-2)(y+5)=10\,.$$

The curve crosses the y axis at the point A.

The straight line L is the tangent to C at A.

a) State the coordinates of A.

b) Find an equation for L.

c) Determine the coordinates of the point where L meets C again.

$\boxed{A(0,-10)}$, $\boxed{y=25x-10}$, $\boxed{\left(\frac{3}{5},5\right)}$

(a) $(xy-2)(y+5)=10$

$x=0 \Rightarrow -2(y+5)=10$

$y+5=-5$

$y=-10$

$\therefore A(0,-10)$

(b) $xy^2+5xy-2y-10=10$

Diff w.r.t x

$y^2+x\left(2y\frac{dy}{dx}\right)+5y+5x\frac{dy}{dx}-2\frac{dy}{dx}=0$

$y^2+2xy\frac{dy}{dx}+5y+5x\frac{dy}{dx}-2\frac{dy}{dx}=0$

$(2xy+5x-2)\frac{dy}{dx}=-5y-y^2$

$\frac{dy}{dx}=-\frac{5y+y^2}{2xy+5x-2}$

$\left.\frac{dy}{dx}\right|_{(0,-10)}=-\frac{-50+100}{-2}=25$

$\therefore L:\quad y=25x-10$

(c) SOLVING SIMULTANEOUSLY

$\left.\begin{array}{l}(xy-2)(y+5)=10\\ y=25x-10\end{array}\right\} \Rightarrow$

$\Rightarrow [x(25x-10)-2][25x-10+5]=10$

$\Rightarrow [25x^2-10x-2][25x-5]=10$

$\Rightarrow 5(25x^2-10x-2)(5x-1)=10$

$\Rightarrow (5x-1)(25x^2-10x-2)=2$

$\Rightarrow 125x^3-50x^2-\cancel{10x}$
$\quad -25x^2+\cancel{10x}+2=2$

$\Rightarrow 125x^3-75x^2+2=2$

$\Rightarrow 125x^3-75x^2=0$

$\Rightarrow 25x^2(5x-3)=0$

$x=\begin{cases}0 & \leftarrow \text{POINT OF TANGENCY}\\ \frac{3}{5}\end{cases}$

USING $y=25x-10$

$y=25\left(\frac{3}{5}\right)-10$

$y=5$

$\therefore \left(\frac{3}{5},5\right)$

Question 44 **(****)**

A curve C has implicit equation

$$4^x + 2xy + y^2 = 13 .$$

a) Show clearly that

$$\frac{dy}{dx} = -\frac{y + 4^x \ln 2}{x + y} .$$

There are two points on C whose x coordinate is 2 .

b) Find the gradient at each of these two points.

Give the answers in the form $a + b\ln 2$, where a and b are integers.

$$\boxed{\left.\frac{dy}{dx}\right|_{(2,-1)} = 1 - 16\ln 2} , \quad \boxed{\left.\frac{dy}{dx}\right|_{(2,-3)} = -3 + 16\ln 2}$$

a) $4^x + 2xy + y^2 = 13$

$\Rightarrow \frac{d}{dx}(4^x) + \frac{d}{dx}(2xy) + \frac{d}{dx}(y^2) = \frac{d}{dx}(13)$

$\Rightarrow 4^x \times \ln 4 + 2y + 2x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$

$\Rightarrow (2x + 2y)\frac{dy}{dx} = -2y - 4^x \ln 4$

$\Rightarrow \frac{dy}{dx} = \frac{-2y - 4^x \ln 4}{2x + 2y}$

$\Rightarrow \frac{dy}{dx} = -\frac{2y + 4^x \ln 4}{2x + 2y}$

$\Rightarrow \frac{dy}{dx} = -\frac{2y + 4^x(2\ln 2)}{2x + 2y}$

$\Rightarrow \frac{dy}{dx} = -\frac{y + 4^x \ln 2}{x + y}$

b) WITH $x = 2$

$4^2 + 2 \times 2 \times y + y^2 = 13$

$16 + 4y + y^2 = 13$

$y^2 + 4y + 3 = 0$

$(y+1)(y+3) = 0$

$\therefore y = \begin{cases} -1 \\ -3 \end{cases}$

THUS

$\left.\frac{dy}{dx}\right|_{(2,-1)} = -\frac{-1 + 16\ln 2}{1} = 1 - 16\ln 2$

$\left.\frac{dy}{dx}\right|_{(2,-3)} = -\frac{-3 + 16\ln 2}{-1} = -3 + 16\ln 2$

Question 45 (****)

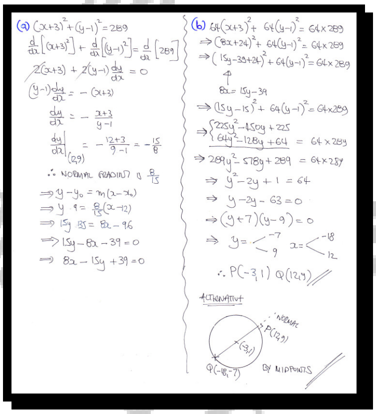
A circle has equation

$$(x+3)^2+(y-1)^2=289.$$

a) Find an equation for the normal to the curve at the point $P(12,9)$.

b) Find the coordinates of the point Q, where the normal to the circle at P intersects the circle again.

$\boxed{8x-15y+39=0}$, $\boxed{Q(-18,-7)}$



(a) $(x+3)^2+(y-1)^2=289$

$\frac{d}{dx}\left[(x+3)^2\right]+\frac{d}{dx}\left[(y-1)^2\right]=\frac{d}{dx}\left[289\right]$

$\cancel{2}(x+3)+\cancel{2}(y-1)\frac{dy}{dx}=0$

$(y-1)\frac{dy}{dx}=-(x+3)$

$\frac{dy}{dx}=-\frac{x+3}{y-1}$

$\left.\frac{dy}{dx}\right|_{(12,9)}=-\frac{12+3}{9-1}=-\frac{15}{8}$

$\therefore$ NORMAL GRADIENT IS $\frac{8}{15}$

$\Rightarrow y-y_0=m(x-x_0)$

$\Rightarrow y-9=\frac{8}{15}(x-12)$

$\Rightarrow 15y-135=8x-96$

$\Rightarrow 15y-8x-39=0$

$\Rightarrow 8x-15y+39=0$

(b) $64(x+3)^2+64(y-1)^2=64\times289$

$\Rightarrow (8x+24)^2+64(y-1)^2=64\times289$

$\Rightarrow (15y-39+24)^2+64(y-1)^2=64\times289$

$8x=15y-39$

$\Rightarrow (15y-15)^2+64(y-1)^2=64\times289$

$\Rightarrow \begin{cases}225y^2-450y+225\\64y^2-128y+64\end{cases}=64\times289$

$\Rightarrow 289y^2-578y+289=64\times284$

$\Rightarrow y^2-2y+1=64$

$\Rightarrow y-2y-63=0$

$\Rightarrow (y+7)(y-9)=0$

$\Rightarrow y=\begin{cases}-7\\9\end{cases}\quad x=\begin{cases}-18\\12\end{cases}$

$\therefore P(-3,1)\ Q(12,9)$

ALTERNATIVE

Question 46 (****)

The point $P(4,2)$ lies on the curve with equation

$$2^x y + 2^y x = 6xy .$$

Show that the gradient of the curve at P is

$$\frac{1 - a\ln 2}{b\ln 2 - 1},$$

where a and b are positive integers to be found.

$\boxed{\text{C4}}$, $\boxed{a = 4, b = 2}$

$$2^x y + 2^y x = 6xy$$

$$\Rightarrow \frac{d}{dx}\left(2^x y\right) + \frac{d}{dx}\left(2^y x\right) = \frac{d}{dx}\left(6xy\right)$$

$$\Rightarrow 2^x \ln 2 \times y + 2^x \times \frac{dy}{dx} + 2^y \ln 2 \frac{dy}{dx} \times x + 2^y \times 1 = 6 \times y + 6x \times \frac{dy}{dx}$$

• AT $(4,2)$

$$\Rightarrow 16\ln 2 \times 2 + 16\left.\frac{dy}{dx}\right|_{(4,2)} + 4\ln 2 \left.\frac{dy}{dx}\right|_{(4,2)} \times 4 + 4 = 12 + 24\left.\frac{dy}{dx}\right|_{(4,2)}$$

$$\Rightarrow 32\ln 2 + 16\left.\frac{dy}{dx}\right|_{(4,2)} + 16\ln 2 \left.\frac{dy}{dx}\right|_{(4,2)} + 4 = 12 + 24\left.\frac{dy}{dx}\right|_{(4,2)}$$

$$\Rightarrow \left(16\ln 2 - 8\right)\left.\frac{dy}{dx}\right|_{(4,2)} = 8 - 32\ln 2$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{(4,2)} = \frac{8 - 32\ln 2}{16\ln 2 - 8}$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{(4,2)} = \frac{1 - 4\ln 2}{2\ln 2 - 1}$$

IE $a = 4$
$b = 2$

Question 47 **(****)**

A curve C is defined implicitly by

$$\sin 2x \cot y = 1, \quad x \in \mathbb{R}, \quad y \in \mathbb{R}, \; 0 < x < \frac{\pi}{2}, \; 0 < y < \frac{\pi}{2}.$$

a) Show clearly that

$$\frac{dy}{dx} = \cot 2x \sin 2y.$$

The point $A\left(\frac{\pi}{4}, \frac{\pi}{12}\right)$ is a turning point of C.

b) Use $\frac{d^2y}{dx^2}$ to show that A is a local maximum.

$\boxed{}$, $\boxed{\text{proof}}$

a) DIFFERENTIATE WITH RESPECT TO x — BY PRODUCT RULE ON L.H.S

$\Rightarrow \frac{d}{dx}(\sin 2x \cot y) = \frac{d}{dx}(1)$

$\Rightarrow 2\cos 2x \cot y + \sin 2x(-\operatorname{cosec}^2 y)\frac{dy}{dx} = 0$

$\Rightarrow 2\cos 2x \cot y = \sin 2x \operatorname{cosec}^2 y \frac{dy}{dx}$

$\Rightarrow \frac{2\cos 2x}{\sin 2x} = \frac{\operatorname{cosec}^2 y}{\cot y} \frac{dy}{dx}$

$\Rightarrow 2\cot 2x = \operatorname{cosec}^2 y \tan y \frac{dy}{dx}$

$\Rightarrow 2\cot 2x = \frac{1}{\sin^2 y} \frac{\sin y}{\cos y} \frac{dy}{dx}$

$\Rightarrow 2\cot 2x = \frac{1}{\sin y \cos y} \frac{dy}{dx}$

$\Rightarrow (2\sin y \cos y)\cot 2x = \frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \cot 2x \sin 2y$

AS REQUIRED

b) DIFFERENTIATE AGAIN W.R.T x BY THE PRODUCT RULE

$\Rightarrow \frac{d^2y}{dx^2} = -2\operatorname{cosec}^2 2x \sin 2y + \cot 2x\left(2\cos 2y \frac{dy}{dx}\right)$

BUT $\left.\frac{dy}{dx}\right|_{\left(\frac{\pi}{4}, \frac{\pi}{12}\right)} = 0$

$\Rightarrow \left.\frac{d^2y}{dx^2}\right|_{\left(\frac{\pi}{4}, \frac{\pi}{12}\right)} = -2\operatorname{cosec}^2\left(\frac{\pi}{2}\right)\sin\frac{\pi}{6} = -2 \times 1 \times \frac{1}{2} = -1 < 0$

INDEED A LOCAL MAX

Question 48 (****)

A curve has equation

$$\sin x + \cos y = \frac{1}{2}, \quad 0 \le x < 2\pi, \; 0 \le y < 2\pi .$$

Find the coordinates of the points on the curve, where the tangent to the curve is parallel to the y axis.

$$\boxed{\left(\frac{7\pi}{6}, 0\right), \left(\frac{11\pi}{6}, 0\right)}$$

$\sin x + \cos y = \frac{1}{2}$

$\Rightarrow \frac{d}{dx}(\sin x) + \frac{d}{dx}(\cos y) = \frac{d}{dx}\left(\frac{1}{2}\right)$

$\Rightarrow \cos x - \sin y \frac{dy}{dx} = 0$

$\Rightarrow \cos x = \sin y \frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{\cos x}{\sin y}$

INFINITE GRADIENT $\Rightarrow \sin y = 0$

$\therefore \arcsin 0 = 0$

$\begin{cases} y = 0 \pm 2n\pi \\ y = \pi \pm 2n\pi \end{cases}$ $\quad \therefore y = 0, \pi$

$n = 0, 1, 2, 3, \ldots$

• IF $y = 0$

$\sin x + \cos 0 = \frac{1}{2}$

$\sin x + 1 = \frac{1}{2}$

$\sin x = -\frac{1}{2}$

$\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$

$\begin{cases} x = -\frac{\pi}{6} \pm 2n\pi \\ x = \frac{7\pi}{6} \pm 2n\pi \end{cases}$ $\quad n = 0, 1, 2, 3, \ldots$

$x = \frac{7\pi}{6}, \frac{11\pi}{6}$

• IF $y = \pi$

$\sin x + \cos \pi = \frac{1}{2}$

$\sin x - 1 = \frac{1}{2}$

$\cancel{\sin x = \frac{3}{2}}$

$\therefore \left(\frac{7\pi}{6}, 0\right)$ OR $\left(\frac{11\pi}{6}, 0\right)$

Question 49 (****)

The equation of a curve is given by

$$x^2 - 2y^2 - xy - x + 5y + 34 = 0.$$

a) Show clearly that

$$\frac{dy}{dx} = \frac{2x - y - 1}{x + 4y - 5}.$$

b) Find the exact value of gradient at the point on the curve with coordinates

$$\left(1 + 4\sqrt{2}, -5 - \sqrt{2}\right).$$

c) Determine the coordinates of the turning points of the curve.

$\boxed{-\frac{1}{8}\left(2 + 3\sqrt{2}\right)}$, $\boxed{(3,5),\ (-1,-3)}$

(a) $x^2 - 2y^2 - xy - x + 5y + 34 = 0$

Diff w.r.t. x

$\Rightarrow 2x - 4y\frac{dy}{dx} - 1 \times y - x\frac{dy}{dx} - 1 + 5\frac{dy}{dx} = 0$

$\Rightarrow 2x - y - 1 = (4y + x - 5)\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{2x - y - 1}{4y + x - 5}$ // AS REQUIRED

(b) $\left.\frac{dy}{dx}\right|_{\substack{x = 1 + 4\sqrt{2} \\ y = -5 - \sqrt{2}}} = \frac{2(1 + 4\sqrt{2}) - (-5 - \sqrt{2}) - 1}{4(-5 - \sqrt{2}) + (1 + 4\sqrt{2}) - 5} = \frac{2 + 8\sqrt{2} + 5 + \sqrt{2} - 1}{-20 \cancel{- 4\sqrt{2}} + 1 \cancel{+ 4\sqrt{2}} - 5} = \frac{6 + 9\sqrt{2}}{-24}$

$= \frac{2 + 3\sqrt{2}}{-8} = -\frac{1}{8}\left(2 + 3\sqrt{2}\right)$ //

(c) T.P. $\Rightarrow \frac{dy}{dx} = 0$

$\Rightarrow 2x - y - 1 = 0$

$\Rightarrow \boxed{y = 2x - 1}$

SOLVING SIMULTANEOUSLY

$\Rightarrow x^2 - 2(2x-1)^2 - x(2x-1) - x + 5(2x-1) + 34 = 0$

$\Rightarrow x^2 - 2(4x^2 - 4x + 1) - 2x^2 \cancel{+ x} \cancel{- x} + 10x - 5 + 34 = 0$

$\Rightarrow x^2 - 8x^2 + 8x - 2 - 2x^2 + 10x - 5 + 34 = 0$

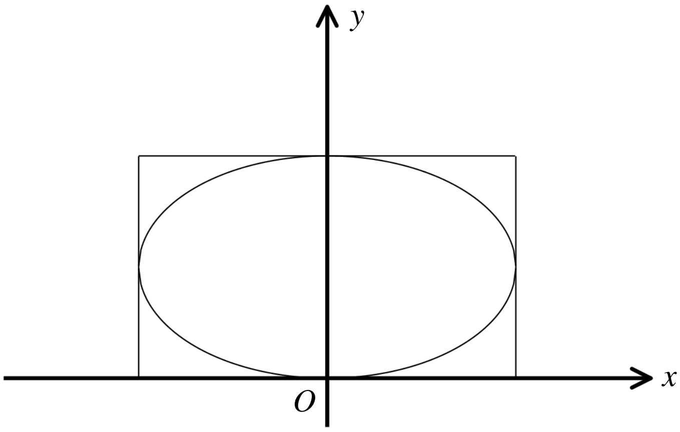
$\Rightarrow -9x^2 + 18x + 27 = 0$

$\Rightarrow x^2 - 2x - 3 = 0$

$\Rightarrow (x + 1)(x - 3) = 0$

$\Rightarrow x = \begin{cases} -1 \\ 3 \end{cases}$ $\quad y = \begin{cases} -3 \\ 5 \end{cases}$ $\quad \therefore (-1, -3)$ & $(3, 5)$ //

Question 50 (****)



The figure above shows the curve with equation

$$x^2 - 8y + 4y^2 = 0.$$

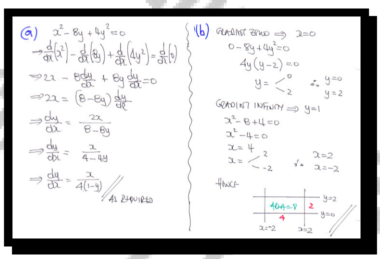
a) Show that

$$\frac{dy}{dx} = \frac{x}{4(1-y)}.$$

The curve fits perfectly inside a rectangle whose sides are parallel to the coordinate axes, so they are tangents to the curve.

b) Show further that the area of the rectangle is 8 square units.

$\boxed{}$, $\boxed{\text{proof}}$



Question 51 (****)

The equation of a curve is given implicitly by

$$y^2 - x^2 = 1, \quad |y| \geq 1 \ .$$

Show clearly that

$$\frac{d^2y}{dx^2} = \frac{1}{y^3}.$$

$\boxed{}$, $\boxed{\text{proof}}$

DIFFERENTIATING IMPLICITLY WITH RESPECT TO x

$$\Rightarrow y^2 - x^2 = 1$$

$$\Rightarrow 2y\frac{dy}{dx} - 2x = 0$$

$$\Rightarrow y\frac{dy}{dx} = x$$

DIFFERENTIATE AGAIN WITH RESPECT TO x

$$\Rightarrow \frac{d}{dx}\left(y\frac{dy}{dx}\right) = \frac{d}{dx}(x)$$

$$\Rightarrow \frac{dy}{dx}\frac{dy}{dx} + y\frac{d^2y}{dx^2} = 1$$

$$\Rightarrow y\frac{d^2y}{dx^2} = 1 - \left(\frac{dy}{dx}\right)^2$$

BUT WE FOUND THAT $\frac{dy}{dx} = \frac{x}{y}$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{x^2}{y^2}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{y^2-1}{y^2}$$

$y^2 - x^2 = 1$
$y^2 - 1 = x^2$

FINALLY WE HAVE

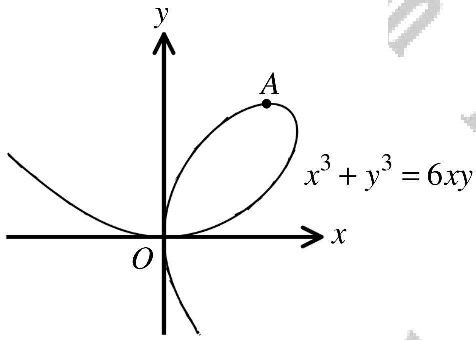
$$y\frac{d^2y}{dx^2} = 1 - \frac{y^2-1}{y^2}$$

$$y\frac{d^2y}{dx^2} = \frac{y^2-(y^2-1)}{y^2}$$

$$y\frac{d^2y}{dx^2} = \frac{1}{y^2}$$

$$\frac{d^2y}{dx^2} = \frac{1}{y^3} \quad //$$

Question 52 (****)



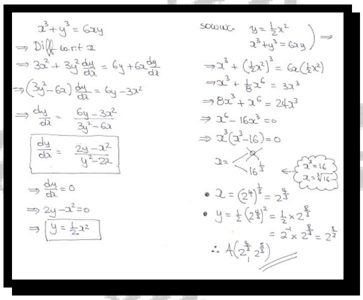
The diagram above shows a curve known as "the folium of Descartes", with equation

$$x^3 + y^3 = 6xy .$$

The curve is stationary at the origin O and at the point A.

Find the exact coordinates of A in the form $\left(2^n, 2^m\right)$, where n and m are fractions to be found.

$\boxed{A\left(2^{\frac{4}{3}}, 2^{\frac{5}{3}}\right)}$



$x^3+y^3=6xy$

$\Rightarrow$ Diff w.r.t x

$\Rightarrow 3x^2+3y^2\frac{dy}{dx}=6y+6x\frac{dy}{dx}$

$\Rightarrow (3y^2-6x)\frac{dy}{dx}=6y-3x^2$

$\Rightarrow \frac{dy}{dx}=\frac{6y-3x^2}{3y^2-6x}$

$\boxed{\frac{dy}{dx}=\frac{2y-x^2}{y^2-2x}}$

$\Rightarrow \frac{dy}{dx}=0$

$\Rightarrow 2y-x^2=0$

$\Rightarrow \boxed{y=\frac{1}{2}x^2}$

SOLVING $\left.\begin{array}{l} y=\frac{1}{2}x^2 \\ x^3+y^3=6xy \end{array}\right) \Rightarrow$

$\Rightarrow x^3+\left(\frac{1}{2}x^2\right)^3=6x\left(\frac{1}{2}x^2\right)$

$\Rightarrow x^3+\frac{1}{8}x^6=3x^3$

$\Rightarrow 8x^3+x^6=24x^3$

$\Rightarrow x^6-16x^3=0$

$\Rightarrow x^3(x^3-16)=0$

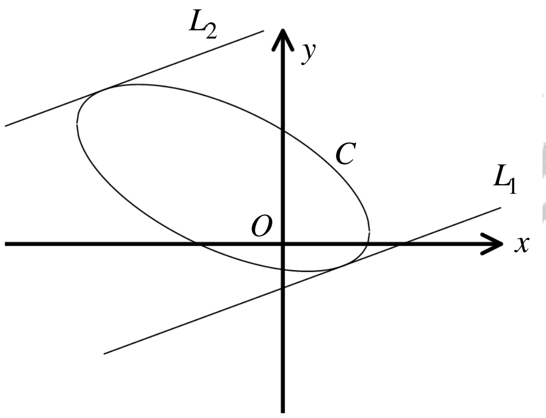
$x = \cancel{0}$, $16^{\frac{1}{3}}$ $\leftarrow$ $x^3=16$, $x=\sqrt[3]{16}$

• $x=(2^4)^{\frac{1}{3}}=2^{\frac{4}{3}}$

• $y=\frac{1}{2}\left(2^{\frac{4}{3}}\right)^2=\frac{1}{2}\times2^{\frac{8}{3}}$
$=2^{-1}\times2^{\frac{8}{3}}=2^{\frac{5}{3}}$

$\therefore A\left(2^{\frac{4}{3}},2^{\frac{5}{3}}\right)$

Question 53 (****)



The figure above shows the curve C with the equation

$$4y - 2xy + 6 = y^2 + 3x^2 .$$

a) Show clearly that

$$\frac{dy}{dx} = \frac{y+3x}{2-x-y} .$$

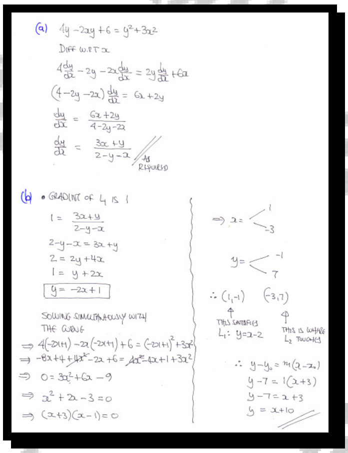
The straight lines L_1 and L_2 are parallel to each other and are both tangents to C.

The equation of L_1 is

$$y = x - 2 .$$

b) Find an equation of L_2

$\boxed{\text{O.K.}}$, $\boxed{y = x + 10}$



(a) $4y-2xy+6=y^2+3x^2$

DIFF W.R.T x

$4\frac{dy}{dx}-2y-2x\frac{dy}{dx}=2y\frac{dy}{dx}+6x$

$(4-2y-2x)\frac{dy}{dx}=6x+2y$

$\frac{dy}{dx}=\frac{6x+2y}{4-2y-2x}$

$\frac{dy}{dx}=\frac{3x+y}{2-y-x}$ AS REQUIRED

(b) • GRADIENT OF L_1 IS 1

$1=\frac{3x+y}{2-y-x}$

$2-y-x=3x+y$

$2=2y+4x$

$1=y+2x$

$\boxed{y=-2x+1}$

SOLVING SIMULTANEOUSLY WITH THE CURVE

$\Rightarrow 4(-2x+1)-2x(-2x+1)+6=(-2x+1)^2+3x^2$

$\Rightarrow -8x+4+\cancel{4x^2}-2x+6=\cancel{4x^2}-4x+1+3x^2$

$\Rightarrow 0=3x^2+6x-9$

$\Rightarrow x^2+2x-3=0$

$\Rightarrow (x+3)(x-1)=0$

$\Rightarrow x = 1, -3$

$y = -1, 7$

$\therefore (1,-1)$ $(-3,7)$

THIS SATISFIES $L_1: y=x-2$

THIS IS WHERE L_2 TOUCHES

$\therefore y-y_0=m(x-x_0)$

$y-7=1(x+3)$

$y-7=x+3$

$y=x+10$

Question 54 (****)

A curve C is given implicitly by

$$x^2 + 4y^2 - 8x - 16y + 28 = 0.$$

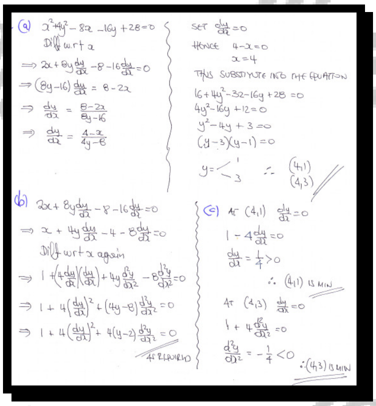
a) Find the coordinates of the turning points of C.

b) Show clearly that

$$1 + 4\left(\frac{dy}{dx}\right)^2 + 4(y-2)\frac{d^2y}{dx^2} = 0.$$

c) Hence determine the nature of these turning points.

$\boxed{(4,3) \,\&\, (4,1)}$, $\boxed{\text{max at } (4,3) \text{ \& min at } (4,1)}$



(a) $x^2+4y^2-8x-16y+28=0$

Diff w.r.t x

$\Rightarrow 2x+8y\frac{dy}{dx}-8-16\frac{dy}{dx}=0$

$\Rightarrow (8y-16)\frac{dy}{dx}=8-2x$

$\Rightarrow \frac{dy}{dx}=\frac{8-2x}{8y-16}$

$\Rightarrow \frac{dy}{dx}=\frac{4-x}{4y-8}$

SET $\frac{dy}{dx}=0$

HENCE $4-x=0$

$x=4$

THUS SUBSTITUTE INTO THE EQUATION

$16+4y^2-32-16y+28=0$

$4y^2-16y+12=0$

$y^2-4y+3=0$

$(y-3)(y-1)=0$

$y = \begin{cases} 1 \\ 3 \end{cases}$ $\therefore$ $(4,1)$ $(4,3)$

(b) $2x+8y\frac{dy}{dx}-8-16\frac{dy}{dx}=0$

$\Rightarrow x+4y\frac{dy}{dx}-4-8\frac{dy}{dx}=0$

Diff wrt x again

$\Rightarrow 1+\left(4\frac{dy}{dx}\right)\left(\frac{dy}{dx}\right)+4y\frac{d^2y}{dx^2}-8\frac{d^2y}{dx^2}=0$

$\Rightarrow 1+4\left(\frac{dy}{dx}\right)^2+(4y-8)\frac{d^2y}{dx^2}=0$

$\Rightarrow 1+4\left(\frac{dy}{dx}\right)^2+4(y-2)\frac{d^2y}{dx^2}=0$

AS REQUIRED

(c) AT $(4,1)$ $\frac{dy}{dx}=0$

$1-4\frac{dy}{dx}=0$

$\frac{dy}{dx}=\frac{1}{4}>0$

$\therefore$ $(4,1)$ IS MIN

AT $(4,3)$ $\frac{dy}{dx}=0$

$1+4\frac{d^2y}{dx^2}=0$

$\frac{d^2y}{dx^2}=-\frac{1}{4}<0$

$\therefore (4,3)$ IS MIN

Question 55 (****)

A curve C has equation

$$y = 2^{\sin 2x}, \ x \in \mathbb{R}.$$

a) By taking logarithms on both sides of this equation, or otherwise, find an expression for $\frac{dy}{dx}$ in terms of x.

b) Find an equation of the tangent to the curve at the point where $x = \frac{\pi}{4}$.

$\boxed{}$, $\boxed{\frac{dy}{dx} = 2^{\sin 2x} \times 2\ln 2 \times \cos 2x = 2^{1+\sin 2x} \times \ln 2 \times \cos 2x}$, $\boxed{y = 2}$

(a) $y = 2^{\sin 2x}$

$\Rightarrow \ln y = \ln 2^{\sin 2x}$

$\Rightarrow \ln y = \sin 2x \times \ln 2$

• Diff w.r.t x

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = 2\cos 2x \times \ln 2$

$\Rightarrow \frac{dy}{dx} = 2\ln 2 \times \cos 2x \times y$

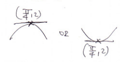
$\Rightarrow \frac{dy}{dx} = 2\ln 2 \times \cos 2x \times 2^{\sin 2x}$

(b) when $x = \frac{\pi}{4}$

$y = 2^{\sin\frac{\pi}{2}} = 2$

$\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{4}} = 2\ln 2 \times \cos\frac{\pi}{2} \times 2^{\sin\pi/2} = 0$

• If $\left(\frac{\pi}{4}, 2\right)$ IS STATIONARY



$\therefore y = 2$

Question 56 (****)

The equation of a curve is given by the implicit relationship

$$\frac{x}{x+1}+\frac{y}{y+1}=x^2 .$$

Show that at the point on the curve with coordinates $(1,1)$, the gradient is 7.

$\boxed{}$, $\boxed{\text{proof}}$

MULTIPLY THROUGH AND TIDY

$$\Rightarrow \frac{x}{x+1}+\frac{y}{y+1}=x^2$$
$$\Rightarrow x(y+1)+y(x+1)=x^2(x+1)(y+1)$$
$$\Rightarrow xy+x+xy+y=x^2(xy+x+y+1)$$
$$\Rightarrow 2xy+x+y=x^3y+x^3+x^2y+x^2$$

DIFFERENTIATE WITH RESPECT TO x

$$\Rightarrow \frac{d}{dx}(2xy)+\frac{d}{dx}(x)+\frac{d}{dx}(y)=\frac{d}{dx}(x^3y)+\frac{d}{dx}(x^3)+\frac{d}{dx}(x^2y)+\frac{d}{dx}(x^2)$$
$$\Rightarrow 2y+2x\frac{dy}{dx}+1+\frac{dy}{dx}=3x^2y+x^3\frac{dy}{dx}+3x^2+2xy+x^2\frac{dy}{dx}+2x$$

AT $x=1$ & $y=1$

$$\Rightarrow 2+2\frac{dy}{dx}+1+\frac{dy}{dx}=3+\frac{dy}{dx}+3+2+\frac{dy}{dx}+2$$
$$\Rightarrow 3+3\frac{dy}{dx}=2\frac{dy}{dx}+10$$
$$\Rightarrow \frac{dy}{dx}=7$$

AS REQUIRED

ALTERNATIVE WITHOUT ANY INITIAL TIDY UP

$$\Rightarrow \frac{d}{dx}\left[\frac{x}{x+1}\right]+\frac{d}{dx}\left[\frac{y}{y+1}\right]=\frac{d}{dx}\left[x^2\right]$$
$$\Rightarrow \frac{d}{dx}\left[\frac{(x+1)-1}{(x+1)}\right]+\frac{d}{dx}\left[\frac{(y+1)-1}{(y+1)}\right]=2x$$
$$\Rightarrow \frac{d}{dx}\left[1-\frac{1}{x+1}\right]+\frac{d}{dx}\left[1-\frac{1}{y+1}\right]=2x$$
$$\Rightarrow \frac{d}{dx}\left[1-(x+1)^{-1}\right]+\frac{d}{dx}\left[1-(y+1)^{-1}\right]=2x$$
$$\Rightarrow 0+(x+1)^{-2}+0+(y+1)^{-2}\times\frac{dy}{dx}=2x$$
$$\Rightarrow \frac{1}{(x+1)^2}+\frac{1}{(y+1)^2}\frac{dy}{dx}=2x$$

AT $x=1$, $y=1$

$$\Rightarrow \frac{1}{4}+\frac{1}{4}\frac{dy}{dx}=2$$
$$\Rightarrow 1+\frac{dy}{dx}=8$$
$$\Rightarrow \frac{dy}{dx}=7$$

AS REQUIRED

TIDY THE EQUATION FIRST

$$\Rightarrow \frac{x}{x+1}+\frac{y}{y+1}=x^2$$

$\times(x+1)(y+1)$

$$\Rightarrow x(y+1)+y(x+1)=x^2(x+1)(y+1)$$
$$\Rightarrow xy+x+xy+y=x^2(xy+x+y+1$$
$$\Rightarrow 2xy+x+y=x^3y+x^3+x^2y+x^2$$
$$\Rightarrow 2xy+y-x^3y-x^2y=x^3+x^2-x$$
$$\Rightarrow y(2x+1-x^3-x^2)=x^3+x^2-x$$

DIFFERENTIATE WITH RESPECT TO x

$$\frac{dy}{dx}(2x+1-x^3-x^2)+y(2+0-3x^2-2x)=3x^2+2x-1$$

AT $(1,1)$ WE OBTAIN

$$\left.\frac{dy}{dx}\right|_{(1,1)}(\cancel{2+1-1-1})+1(2-3-2)=3+2-1$$
$$\left.\frac{dy}{dx}\right|_{(1,1)}+(-3)=4$$
$$\left.\frac{dy}{dx}\right|_{(1,1)}=7$$

AS REQUIRED

ALTERNATIVE IS TO DIFFERENTIATE W.R.T x STRAIGHT AWAY

$$\frac{(x+1)\times1-x(1)}{(x+1)^2}+\frac{(y+1)\frac{dy}{dx}-y(1)\frac{dy}{dx}}{(y+1)^2}=2x$$
$$\frac{1}{(x+1)^2}+\frac{dy}{dx}\left(\frac{1}{(y+1)^2}\right)=2x \quad \text{ETC}$$

Question 57 (****)

A curve C has implicit equation

$$\frac{(x+2y)^2}{4x-y} + y = 3x+2 .$$

a) Show clearly that

$$\frac{dy}{dx} = \frac{2kx-ky+8}{6y+kx+2},$$

where k is a constant to be found.

b) Find the gradient at each of the points on C, where $x=2$.

$\boxed{\quad\quad\quad}$, $\boxed{\text{gradient} = -\frac{9}{2}, \frac{5}{6}}$

a) REARRANGE THE EQUATION FIRST

$$\Rightarrow \frac{(x+2y)^2}{4x-y} + y = 3x+2$$

$$\Rightarrow (x+2y)^2 + y(4x-y) = (3x+2)(4x-y)$$

$$\Rightarrow x^2+4xy+4y^2+4xy-y^2 = 12x^2-3xy+8x-2y$$

$$\Rightarrow -11x^2+3y^2+11xy-8x+2y = 0$$

$$\Rightarrow 11x^2-3y^2-11xy+8x-2y = 0$$

DIFFERENTIATE WITH RESPECT TO x

$$\Rightarrow 22x - 6y\frac{dy}{dx} - 11y - 11x\frac{dy}{dx} + 8 - 2\frac{dy}{dx} = 0$$

$$\Rightarrow 22x - 11y + 8 = 11x\frac{dy}{dx} + 6y\frac{dy}{dx} + 2\frac{dy}{dx}$$

$$\Rightarrow (11x+6y+2)\frac{dy}{dx} = 22x-11y+8$$

$$\Rightarrow \frac{dy}{dx} = \frac{22x-11y+8}{6y+11x+2} \quad\text{IE } k=11$$

b) FIRSTLY WHEN $x=2$

$$\Rightarrow 11\times2^2 - 3y^2 - 11\times2y + 8\times2 - 2y = 0$$

$$\Rightarrow 44 - 3y^2 - 22y + 16 - 2y = 0$$

$$\Rightarrow 0 = 3y^2+24y-60$$

$$\Rightarrow y^2+8y-20 = 0$$

$$\Rightarrow (y+10)(y-2) = 0$$

$$\Rightarrow y = \begin{cases} 2 \\ -10 \end{cases} \quad\text{IE } (2,2) \text{ \& } (2,-10)$$

FINALLY WE HAVE

$$\left.\frac{dy}{dx}\right|_{(2,2)} = \frac{22\times2 - 11\times2 + 8}{6\times2 + 11\times2 + 2} = \frac{44-22+8}{12+22+2} = \frac{30}{36} = \frac{5}{6}$$

$$\left.\frac{dy}{dx}\right|_{(2,-10)} = \frac{22\times2 - 11(-10) + 8}{6(-10) + 11\times2 + 2} = \frac{44+110+8}{-60+22+2} = \frac{162}{-36} = -\frac{9}{2}$$

Question 58 (****)

A curve C is given by the implicit equation

$$xy + y^2 = x^2 + 5.$$

a) Show clearly that

$$\frac{dy}{dx} = \frac{2x-y}{x+2y}.$$

b) Find the coordinates of the turning points of C.

c) Show further that

$$2\frac{dy}{dx} + 2\left(\frac{dy}{dx}\right)^2 + (x+2y)\frac{d^2y}{dx^2} = 2.$$

d) Hence determine the nature of these turning points.

$\boxed{(1,2) \,\&\, (-1,-2)}$, $\boxed{\text{max at } (-1,-2) \text{ & min at } (1,2)}$

(a) $xy + y^2 = x^2 + 5$

Diff w.r.t x

$1xy + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 2x$

$(x+2y)\frac{dy}{dx} = 2x - y$

$\frac{dy}{dx} = \frac{2x-y}{x+2y}$ AS REQUIRED

(b) $\frac{dy}{dx} = 0$

$2x - y = 0$

$\boxed{y = 2x}$

HENCE, SUBSTITUTE INTO THE EQUATION OF THE CURVE:

$x(2x) + (2x)^2 = x^2 + 5$

$2x^2 + 4x^2 = x^2 + 5$

$5x^2 = 5$

$x^2 = 1$

$x = \begin{cases} 1 \\ -1 \end{cases}$ $y = \begin{cases} 2 \\ -2 \end{cases}$

$\therefore (1,2)$ & $(-1,-2)$

(c) $y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 2x$

DIFF AGAIN WRT x

$\Rightarrow \frac{dy}{dx} + 1\frac{dy}{dx} + x\frac{d^2y}{dx^2} + 2\frac{dy}{dx}\left(\frac{dy}{dx}\right) + 2y\frac{d^2y}{dx^2} = 2$

$\Rightarrow 2\frac{dy}{dx} + x\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 + 2y\frac{d^2y}{dx^2} = 2$

$\Rightarrow 2\frac{dy}{dx} + 2\left(\frac{dy}{dx}\right)^2 + (x+2y)\frac{d^2y}{dx^2} = 2$

AS REQUIRED

(d) • AT $(1,2)$, $\frac{dy}{dx} = 0$

$5\frac{d^2y}{dx^2} = 2$

$\frac{d^2y}{dx^2} = \frac{2}{5} > 0$

$\therefore (1,2)$ IS A MIN

• AT $(-1,-2)$, $\frac{dy}{dx} = 0$

$-5\frac{d^2y}{dx^2} = 2$

$\frac{d^2y}{dx^2} = -\frac{2}{5} < 0$

$\therefore (-1,-2)$ IS A MAX

Question 59 (****)

$$y = \arcsin x, \quad -1 \le y \le 1.$$

a) Show clearly that

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}.$$

The point $P\left(\frac{1}{6}, k\right)$, where k is a constant, lies on the curve with equation

$$\arcsin 3x + 2\arcsin y = \frac{\pi}{2}, \ |x| \le \frac{1}{3}, \ |y| \le 1.$$

b) Find the value of the gradient at P.

$\boxed{}$, $\boxed{-\frac{3}{2}}$

a) BY "INVERSE" RULE

$\Rightarrow y = \arcsin x$

$\Rightarrow \sin y = x$

$\Rightarrow x = \sin y$

$\Rightarrow \frac{dx}{dy} = \cos y$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos y}$

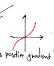
$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{1}{\cos^2 y}$

$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{1}{1-\sin^2 y}$

$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{1}{1-x^2}$

$\Rightarrow \frac{dy}{dx} = \pm\frac{1}{\sqrt{1-x^2}}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$



(arcsin x has positive gradient)

b) FIRSTLY FIND THE VALUE OF k

$\Rightarrow \arcsin(3x) + 2\arcsin y = \frac{\pi}{2}$

$\Rightarrow \arcsin\frac{1}{2} + 2\arcsin y = \frac{\pi}{2}$

$\Rightarrow \frac{\pi}{6} + 2\arcsin y = \frac{\pi}{2}$

$\Rightarrow 2\arcsin y = \frac{\pi}{3}$

$\Rightarrow \arcsin y = \frac{\pi}{6}$

$\Rightarrow y = \frac{1}{2}$

i.e. $k = \frac{1}{2}$

NOW DIFFERENTIATING THE EQUATION IMPLICITLY

$\Rightarrow \arcsin 3x + 2\arcsin y = \frac{\pi}{2}$

$\Rightarrow \frac{d}{dx}(\arcsin 3x) + \frac{d}{dx}(2\arcsin y) = \frac{d}{dx}\left(\frac{\pi}{2}\right)$

$\Rightarrow \frac{1}{\sqrt{1-(3x)^2}} \times 3 + \frac{2}{\sqrt{1-y^2}}\frac{dy}{dx} = 0$

$\Rightarrow \frac{3}{\sqrt{1-9x^2}} + \frac{2}{\sqrt{1-y^2}}\frac{dy}{dx} = 0$

At $\left(\frac{1}{6}, \frac{1}{2}\right)$

$\Rightarrow \frac{3}{\sqrt{1-9\left(\frac{1}{6}\right)^2}} + \frac{2}{\sqrt{1-\left(\frac{1}{2}\right)^2}}\frac{dy}{dx} = 0$

$\Rightarrow \frac{3}{\sqrt{\frac{3}{4}}} + \frac{2}{\sqrt{\frac{3}{4}}}\frac{dy}{dx} = 0$

$\Rightarrow 3 + 2\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = -\frac{3}{2}$

Question 60 (****)

A curve C is given implicitly by

$$x^2 + 3xy - 2y^2 + 17 = 0.$$

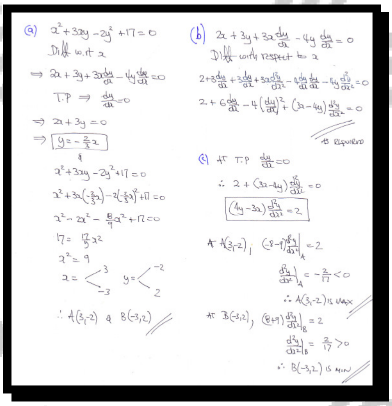
a) Find the coordinates of the turning points of C.

b) Show further that

$$2 + 6\frac{dy}{dx} - 4\left(\frac{dy}{dx}\right)^2 + (3x - 4y)\frac{d^2y}{dx^2} = 0.$$

c) Hence determine the nature of these turning points.

$\boxed{(3,-2) \,\&\, (-3,2)}$, $\boxed{\text{max at } (3,-2) \text{ \& min at } (-3,2)}$



(a) $x^2 + 3xy - 2y^2 + 17 = 0$

Diff w.r.t x

$\Rightarrow 2x + 3y + 3x\frac{dy}{dx} - 4y\frac{dy}{dx} = 0$

T.P $\Rightarrow \frac{dy}{dx} = 0$

$\Rightarrow 2x + 3y = 0$

$\Rightarrow \boxed{y = -\frac{2}{3}x}$

&

$x^2 + 3xy - 2y^2 + 17 = 0$

$x^2 + 3x\left(-\frac{2}{3}x\right) - 2\left(-\frac{2}{3}x\right)^2 + 17 = 0$

$x^2 - 2x^2 - \frac{8}{9}x^2 + 17 = 0$

$17 = \frac{17}{9}x^2$

$x^2 = 9$

$x = 3, -3 \quad y = -2, 2$

$\therefore A(3,-2)$ & $B(-3,2)$

(b) $2x + 3y + 3x\frac{dy}{dx} - 4y\frac{dy}{dx} = 0$

Diff with respect to x

$2 + 3\frac{dy}{dx} + 3\frac{dy}{dx} + 3x\frac{d^2y}{dx^2} - 4\frac{dy}{dx}\frac{dy}{dx} - 4y\frac{d^2y}{dx^2} = 0$

$2 + 6\frac{dy}{dx} - 4\left(\frac{dy}{dx}\right)^2 + (3x - 4y)\frac{d^2y}{dx^2} = 0$

AS REQUIRED

(c) AT T.P $\frac{dy}{dx} = 0$

$\therefore 2 + (3x - 4y)\frac{d^2y}{dx^2} = 0$

$\boxed{(4y - 3x)\frac{d^2y}{dx^2} = 2}$

AT $A(3,-2)$; $(-8-9)\left.\frac{d^2y}{dx^2}\right|_A = 2$

$\left.\frac{d^2y}{dx^2}\right|_A = -\frac{2}{17} < 0$

$\therefore A(3,-2)$ IS MAX

AT $B(-3,2)$, $(8+9)\left.\frac{d^2y}{dx^2}\right|_B = 2$

$\left.\frac{d^2y}{dx^2}\right|_B = \frac{2}{17} > 0$

$\therefore B(-3,2)$ IS MIN

Question 61 (****)

The curve C has equation

$$y = \frac{\ln y}{x-y}, \quad y > 0.$$

Show that the equation of the tangent to C at the point where $y = \mathrm{e}$ can be written as

$$\mathrm{e}(x-y) = 1.$$

$\boxed{}$, $\boxed{\text{proof}}$

METHOD A – WITHOUT IMPLICIT DIFFERENTIATION

- START BY REARRANGING THE EQUATION OF THE CURVE FOR x

$$\Rightarrow y = \frac{\ln y}{x-y}$$
$$\Rightarrow xy - y^2 = \ln y$$
$$\Rightarrow xy = y^2 + \ln y$$
$$\Rightarrow x = y + \frac{\ln y}{y}$$

- WHEN $y=e$

$$\Rightarrow x = e + \frac{\ln e}{e} = e + \frac{1}{e} \qquad \therefore P\left(e+\tfrac{1}{e}, e\right)$$

- DIFFERENTIATE WITH RESPECT TO y

$$\Rightarrow \frac{dx}{dy} = 1 + \frac{y \times \frac{1}{y} - \ln y \times 1}{y^2}$$
$$\Rightarrow \frac{dx}{dy} = 1 + \frac{1-\ln y}{y^2}$$
$$\Rightarrow \left.\frac{dx}{dy}\right|_{y=e} = 1 + \frac{1-\ln e}{e^2} = 1 + \cancel{\frac{1-1}{e^2}} = 1$$
$$\Rightarrow \frac{dy}{dx} = 1$$

- EQUATION OF TANGENT AT $P\left(e+\frac{1}{e}, e\right)$

$$\Rightarrow y - e = 1\left(x - e - \tfrac{1}{e}\right)$$
$$\Rightarrow y \cancel{-e} = x \cancel{-e} - \tfrac{1}{e}$$
$$\Rightarrow \frac{1}{e} = x - y \qquad \therefore e(x-y) = 1$$

METHOD B – BY IMPLICIT DIFFERENTIATION

- FIRSTLY WHEN $y=e$

$$\Rightarrow y = \frac{\ln y}{x-y}$$
$$\Rightarrow e = \frac{\ln e}{x-e}$$
$$\Rightarrow e = \frac{1}{x-e}$$
$$\Rightarrow x - e = \frac{1}{e}$$
$$\Rightarrow x = e + \frac{1}{e} \qquad \therefore P\left(e+\tfrac{1}{e}, e\right)$$

- MULTIPLY THE DENOMINATOR ACROSS AND DIFFERENTIATE W.R.T x

$$\Rightarrow yx - y^2 = \ln y$$
$$\Rightarrow \frac{d}{dx}(yx - y^2) = \frac{d}{dx}(\ln y)$$
$$\Rightarrow x\frac{dy}{dx} + y - 2y\frac{dy}{dx} = \frac{1}{y}\frac{dy}{dx}$$

- EVALUATE THE ABOVE EXPRESSION AT $P\left(e+\frac{1}{e}, e\right)$

$$\Rightarrow \left(e+\tfrac{1}{e}\right)\left.\frac{dy}{dx}\right|_P + e - 2e\left.\frac{dy}{dx}\right|_P = \frac{1}{e}\left.\frac{dy}{dx}\right|_P$$
$$\Rightarrow e = \left(\tfrac{1}{e} + 2e - e - \tfrac{1}{e}\right)\left.\frac{dy}{dx}\right|_P$$
$$\Rightarrow e = e\left.\frac{dy}{dx}\right|_P$$
$$\Rightarrow \left.\frac{dy}{dx}\right|_P = 1$$

- AND THE EQUATION OF THE TANGENT CAN BE FOUND AS BEFORE

Question 62 (****)

A curve C is given by the implicit equation

$$x^2 + 2xy - 3y^2 = 4x + 4y - 20.$$

a) Show clearly that

$$\frac{dy}{dx} = \frac{x+y-2}{3y-x+2}.$$

b) Find the coordinates of the turning points of C.

c) Show further that

$$(x-3y-2)\frac{d^2y}{dx^2} - 3\left(\frac{dy}{dx}\right)^2 + 2\frac{dy}{dx} + 1 = 0.$$

d) Hence determine the nature of these turning points.

$\boxed{\text{N/K}}$, $\boxed{(0,2)\ \&\ (4,-2)}$, $\boxed{\text{max at } (4,-2)\ \&\ \text{min at } (0,2)}$

a) DIFFERENTIATE WITH RESPECT TO x

$$\Rightarrow \frac{d}{dx}(x^2) + \frac{d}{dx}(2xy) - \frac{d}{dx}(3y^2) = \frac{d}{dx}(4x) + \frac{d}{dx}(4y) - \frac{d}{dx}(20)$$
$$\Rightarrow 2x + 2y + 2x\frac{dy}{dx} - 6y\frac{dy}{dx} = 4 + 4\frac{dy}{dx} - 0$$
$$\Rightarrow 2x + 2y - 4 = 6y\frac{dy}{dx} - 2x\frac{dy}{dx} + 4\frac{dy}{dx}$$
$$\Rightarrow 2x + 2y - 4 = (6y - 2x + 4)\frac{dy}{dx}$$
$$\Rightarrow \frac{dy}{dx} = \frac{2x+2y-4}{6y-2x+4}$$
$$\Rightarrow \frac{dy}{dx} = \frac{x+y-2}{3y-x+2} \quad \text{AS REQUIRED}$$

b) SOLVING $\frac{dy}{dx} = 0$

$$\frac{x+y-2}{3y-x+2} = 0$$
$$x + y - 2 = 0$$
$$y = 2 - x$$

SUBSTITUTE INTO THE EQUATION OF THE CURVE

$$\Rightarrow x^2 + 2x(2-x) - 3(2-x)^2 = 4x + 4(2-x) - 20$$
$$\Rightarrow x^2 + \cancel{4x} - 2x^2 - \cancel{12} + 12x - 3x^2 = \cancel{4x} + \cancel{8} - 4x - \cancel{20}$$
$$\Rightarrow 0 = 4x^2 - 16x$$
$$\Rightarrow 0 = x^2 - 4x$$
$$\Rightarrow 0 = x(x-4)$$

$$x = \begin{cases} 0 \\ 4 \end{cases} \qquad y = \begin{cases} 2 \\ -2 \end{cases} \qquad \therefore (0,2) \ \& \ (4,-2)$$

c) STARTING WITH

$$2x + 2y - 4 = (6y - 2x + 4)\frac{dy}{dx} \quad \div 2$$
$$x + y - 2 = (3y - x + 2)\frac{dy}{dx}$$

PRODUCT RULE

DIFFERENTIATE AGAIN W.R.T x

$$1 + \frac{dy}{dx} - 0 = \left(3\frac{dy}{dx} - 1 + 0\right)\frac{dy}{dx} + (3y - x + 2)\frac{d^2y}{dx^2}$$
$$1 + \frac{dy}{dx} = 3\left(\frac{dy}{dx}\right)^2 - \frac{dy}{dx} + (3y - x + 2)\frac{d^2y}{dx^2}$$
$$(x - 3y - 2)\frac{d^2y}{dx^2} - 3\left(\frac{dy}{dx}\right)^2 + 2\frac{dy}{dx} + 1 = 0 \quad \text{AS REQUIRED}$$

d) CHECKING $(0,2)$ & NOTE $\frac{dy}{dx} = 0$ AT $(0,2)$

$$(0 - 6 - 2)\frac{d^2y}{dx^2} + 1 = 0$$
$$\frac{d^2y}{dx^2} = \frac{1}{8} > 0 \qquad \therefore (0,2) \text{ IS A LOCAL MIN}$$

CHECKING $(4,-2)$ & $\frac{dy}{dx} = 0$ AT $(4,-2)$

$$(4 + 6 - 2)\frac{d^2y}{dx^2} + 1 = 0$$
$$\frac{d^2y}{dx^2} = -\frac{1}{8} < 0 \qquad \therefore (4,-2) \text{ IS A LOCAL MAX}$$

Question 63 (****)

A curve C is given by the implicit equation

$$x^2 + 4xy + 2y^2 + 18 = 0 .$$

a) Show clearly that

$$\frac{dy}{dx} = -\frac{x+2y}{2x+2y} .$$

b) Find the coordinates of the turning points of C.

c) Show further that

$$1 + 4\frac{dy}{dx} + 2\left(\frac{dy}{dx}\right)^2 + 2(x+y)\frac{d^2y}{dx^2} = 0 .$$

d) Hence determine the nature of these turning points.

$\boxed{}$, $\boxed{(-6,3)\ \&\ (6,-3)}$, $\boxed{\text{max at } (6,-3)\ \&\ \text{min at } (-6,3)}$

a) DIFFERENTIATE THE ENTIRE EQUATION WITH RESPECT TO x

$\Rightarrow x^2 + 4xy + 2y^2 + 18 = 0$

$\Rightarrow \frac{d}{dx}(x^2) + \frac{d}{dx}(4xy) + \frac{d}{dx}(2y^2) + \frac{d}{dx}(18) = \frac{d}{dx}(0)$

$\Rightarrow 2x + \left[4y + 4x\frac{dy}{dx}\right] + 4y\frac{dy}{dx} + 0 = 0$

$\Rightarrow 4x\frac{dy}{dx} + 4y\frac{dy}{dx} = -2x - 4y$

$\Rightarrow 2x\frac{dy}{dx} + 2y\frac{dy}{dx} = -x - 2y$

$\Rightarrow (2x+2y)\frac{dy}{dx} = -(x+2y)$

$\Rightarrow \frac{dy}{dx} = -\frac{x+2y}{2x+2y}$ // AS REQUIRED

b) SOLVING $\frac{dy}{dx} = 0$

$x + 2y = 0$

$x = -2y$

SUBSTITUTE INTO THE EQUATION OF THE CURVE

$\Rightarrow x^2 + 4xy + 2y^2 + 18 = 0$

$\Rightarrow (-2y)^2 + 4(-2y)y + 2y^2 + 18 = 0$

$\Rightarrow 4y^2 - 8y^2 + 2y^2 + 18 = 0$

$\Rightarrow 18 = 2y^2$

$\Rightarrow y^2 = 9$

$\Rightarrow y = \begin{cases} 3 \\ -3 \end{cases} \quad x = \begin{cases} -6 \\ 6 \end{cases}$

$\therefore (-6,3)$ & $(6,-3)$ //

c) DIFFERENTIATE AGAIN W.R.T x

$\Rightarrow 2x\frac{dy}{dx} + 2y\frac{dy}{dx} = -x - 2y$

$\Rightarrow \frac{d}{dx}\left(2x\frac{dy}{dx}\right) + \frac{d}{dx}\left(2y\frac{dy}{dx}\right) = \frac{d}{dx}(-x) + \frac{d}{dx}(-2y)$

$\Rightarrow \left[2\frac{dy}{dx} + 2x\frac{d^2y}{dx^2}\right] + \left[2\frac{dy}{dx}\frac{dy}{dx} + 2y\frac{d^2y}{dx^2}\right] = -1 - 2\frac{dy}{dx}$

$\Rightarrow 2\frac{dy}{dx} + 2x\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 + 2y\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 1 = 0$

$\Rightarrow 1 + 4\frac{dy}{dx} + 2\left(\frac{dy}{dx}\right)^2 + 2(x+y)\frac{d^2y}{dx^2} = 0$ // AS REQUIRED

d) AT THESE POINTS $\frac{dy}{dx} = 0$

$(-6,3) \Rightarrow 1 + 0 + 0 + 2(-6+3)\frac{d^2y}{dx^2} = 0$

$\Rightarrow 1 = 6\frac{d^2y}{dx^2}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{1}{6} > 0$

$(-6,3)$ IS A LOCAL MIN //

$(6,-3) \Rightarrow 1 + 0 + 0 + 2(6-3)\frac{d^2y}{dx^2}$

$\Rightarrow 1 = -6\frac{d^2y}{dx^2}$

$\Rightarrow \frac{d^2y}{dx^2} = -\frac{1}{6} < 0$

$(6,-3)$ IS A LOCAL MAX //

Question 64 **(****)**

It is given that

$$\frac{d}{du}(\arcsin u) = \frac{1}{\sqrt{1-u^2}}, \quad |u| \le 1.$$

Hence show that if $y = \sin\left(\frac{1}{2}\arcsin 2x\right)$, then …

a) … $\left(1-4x^2\right)\left(\frac{dy}{dx}\right)^2 = 1 - y^2$.

b) … $\left(1-4x^2\right)\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + y = 0$.

proof

(a) $y = \sin\left(\frac{1}{2}\arcsin 2x\right)$

$\frac{dy}{dx} = \cos\left(\frac{1}{2}\arcsin 2x\right) \times \cancel{\frac{1}{2}} \frac{1}{\sqrt{1-(2x)^2}} \times \cancel{2}$

$\frac{dy}{dx} = \cos\left(\frac{1}{2}\arcsin 2x\right) \times \frac{1}{\sqrt{1-4x^2}}$

$\left(\frac{dy}{dx}\right)^2 = \frac{\cos^2\left(\frac{1}{2}\arcsin 2x\right)}{(1-4x^2)}$

$\left(\frac{dy}{dx}\right)^2 = \frac{1-\sin^2\left(\frac{1}{2}\arcsin 2x\right)}{(1-4x^2)}$

$(1-4x^2)\left(\frac{dy}{dx}\right)^2 = 1-\sin^2\left(\frac{1}{2}\arcsin 2x\right)$

$(1-4x^2)\left(\frac{dy}{dx}\right)^2 = 1-y^2$

(b) Differentiate again w.r.t x

$-8x\left(\frac{dy}{dx}\right)^2 + 2(1-4x^2)\left(\frac{dy}{dx}\right)\frac{d^2y}{dx^2} = -2y\left(\frac{dy}{dx}\right)$

$-8x\frac{dy}{dx} + 2(1-4x^2)\frac{d^2y}{dx^2} = -2y$

$2(1-4x^2)\frac{d^2y}{dx^2} - 8x\frac{dy}{dx} + 2y = 0$

$(1-4x^2)\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + y = 0$

AS REQUIRED

Question 65 (****)

A curve C has implicit equation

$$y e^{y} = x^{x}, \ x > 0$$

Show clearly that

$$\frac{dy}{dx} = \frac{y(1 + \ln x)}{1 + y}.$$

$\boxed{}$, $\boxed{\text{proof}}$

$y e^{y} = x^{x}$

$\ln(y e^{y}) = \ln x^{x}$

$\ln y + \ln e^{y} = x \ln x$

$\boxed{\ln y + y = x \ln x}$

HENCE, DIFFERENTIATE W.R.T x

$\frac{1}{y}\frac{dy}{dx} + \frac{dy}{dx} = 1 \times \ln x + x \times \frac{1}{x}$

$\left(\frac{1}{y} + 1\right)\frac{dy}{dx} = \ln x + 1$

$\frac{dy}{dx} = \frac{1 + \ln x}{\frac{1}{y} + 1}$

MULTIPLY TOP & BOTTOM BY y

$\frac{dy}{dx} = \frac{y(1 + \ln x)}{1 + y}$ // AS REQUIRED

Question 66 (****+)

A curve C is given implicitly by

$$4y^2 + 3xy - 2x^2 = 2x - 2y - 12.$$

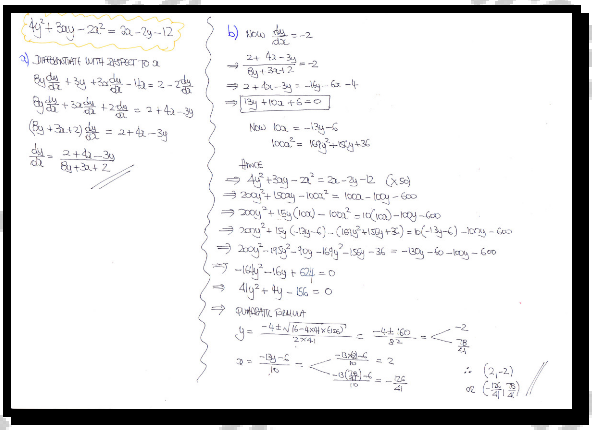
a) Show clearly that

$$\frac{dy}{dx} = \frac{2+4x-3y}{8y+3y+2}.$$

The tangent to C at two distinct points has gradient -2.

b) Find the coordinates of these two points.

$$\boxed{(2,-2) \ \& \ \left(-\frac{126}{41}, \frac{78}{41}\right)}$$



$4y^2 + 3xy - 2x^2 = 2x - 2y - 12$

a) DIFFERENTIATE WITH RESPECT TO x

$8y\frac{dy}{dx} + 3y + 3x\frac{dy}{dx} - 4x = 2 - 2\frac{dy}{dx}$

$8y\frac{dy}{dx} + 3x\frac{dy}{dx} + 2\frac{dy}{dx} = 2 + 4x - 3y$

$(8y + 3x + 2)\frac{dy}{dx} = 2 + 4x - 3y$

$\frac{dy}{dx} = \frac{2+4x-3y}{8y+3x+2}$

b) NOW $\frac{dy}{dx} = -2$

$\Rightarrow \frac{2+4x-3y}{8y+3x+2} = -2$

$\Rightarrow 2 + 4x - 3y = -16y - 6x - 4$

$\Rightarrow \boxed{13y + 10x + 6 = 0}$

NOW $10x = -13y - 6$

$100x^2 = 169y^2 + 156y + 36$

HENCE

$\Rightarrow 4y^2 + 3xy - 2x^2 = 2x - 2y - 12 \quad (\times 50)$

$\Rightarrow 200y^2 + 150xy - 100x^2 = 100x - 100y - 600$

$\Rightarrow 200y^2 + 15y(10x) - 100x^2 = 10(10x) - 100y - 600$

$\Rightarrow 200y^2 + 15y(-13y-6) - (169y^2 + 156y + 36) = 10(-13y-6) - 100y - 600$

$\Rightarrow 200y^2 - 195y^2 - 90y - 169y^2 - 156y - 36 = -130y - 60 - 100y - 600$

$\Rightarrow -164y^2 - 16y + 624 = 0$

$\Rightarrow 41y^2 + 4y - 156 = 0$

$\Rightarrow$ QUADRATIC FORMULA

$y = \frac{-4 \pm \sqrt{16 - 4\times 41 \times (-156)}}{2\times 41} = \frac{-4 \pm 160}{82} = \begin{cases} -2 \\ \frac{78}{41} \end{cases}$

$x = \frac{-13y-6}{10} = \begin{cases} \frac{-13\times(-2)-6}{10} = 2 \\ \frac{-13\left(\frac{78}{41}\right)-6}{10} = -\frac{126}{41} \end{cases}$

$\therefore (2,-2)$ or $\left(-\frac{126}{41}, \frac{78}{41}\right)$

Question 67 (****+)

A curve C is given implicitly by

$$2x^2 + xy - y^2 - 4x - y + 20 = 0 .$$

a) Show clearly that

$$\frac{dy}{dx} = \frac{4x + y - 4}{2y - x + 1} .$$

b) Find the coordinates of the stationary points of C.

c) Show further that

$$4 + 2\frac{dy}{dx} - 2\left(\frac{dy}{dx}\right)^2 + (x - 2y - 1)\frac{d^2y}{dx^2} = 0 .$$

d) Hence determine the nature of the stationary points of part **(b)**.

$\boxed{\text{N.C}}$, $\boxed{(2,-4) \,\&\, (0,4)}$, $\boxed{\text{max at } (2,-4) \text{ & min at } (0,4)}$

a) DIFFERENTIATE THE EQUATION WITH RESPECT TO x

$$\Rightarrow 2x^2 + xy - y^2 - 4x - y + 20 = 0$$
$$\Rightarrow \frac{d}{dx}(2x^2) + \frac{d}{dx}(xy) - \frac{d}{dx}(y^2) - \frac{d}{dx}(4x) - \frac{d}{dx}(y) + \frac{d}{dx}(20) = \frac{d}{dx}(0)$$
$$\Rightarrow 4x + y + x\frac{dy}{dx} - 2y\frac{dy}{dx} - 4 - \frac{dy}{dx} + 0 = 0$$
$$\Rightarrow (x - 2y - 1)\frac{dy}{dx} = -4x - y + 4$$
$$\Rightarrow \frac{dy}{dx} = \frac{-4x - y + 4}{x - 2y - 1} \quad \text{MULTIPLY "TOP \& BOTTOM" BY } -1$$
$$\Rightarrow \frac{dy}{dx} = \frac{4x + y - 4}{2y - x + 1}$$

AS REQUIRED

b) SOLVING $\frac{dy}{dx} = 0$ $\Rightarrow 4x + y - 4 = 0$

$$\Rightarrow y = 4 - 4x$$

SUBSTITUTE INTO THE EQUATION OF THE CURVE

$$\Rightarrow 2x^2 + x(4 - 4x) - (4 - 4x)^2 - 4x - (4 - 4x) + 20 = 0$$
$$\Rightarrow 2x^2 + 4x - 4x^2 - (16 - 32x + 16x^2) - \cancel{4x} - 4 + \cancel{4x} + 20 = 0$$
$$\Rightarrow 2x^2 + 4x - 4x^2 - 16 + 32x - 16x^2 - 4 + 20 = 0$$
$$\Rightarrow -18x^2 + 36x = 0$$
$$\Rightarrow -18x(x - 2) = 0$$

$$x = \begin{cases} 0 \\ 2 \end{cases} \qquad y = \begin{cases} 4 - 4 \times 0 = 4 \\ 4 - 4 \times 2 = -4 \end{cases}$$

$\therefore (0,4) \;\&\; (2,-4)$

c) STARTING FROM

$$\Rightarrow 4x + y + x\frac{dy}{dx} - 2y\frac{dy}{dx} - 4 - \frac{dy}{dx} = 0$$
$$\Rightarrow \frac{d}{dx}(4x) + \frac{d}{dx}(y) + \frac{d}{dx}\left(x\frac{dy}{dx}\right) - 2\frac{d}{dx}\left(y\frac{dy}{dx}\right) - \frac{d}{dx}(4) - \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(0)$$
$$\Rightarrow 4 + \frac{dy}{dx} + \left[1\frac{dy}{dx} + x\frac{d^2y}{dx^2}\right] - 2\left[\frac{dy}{dx}\frac{dy}{dx} + y\frac{d^2y}{dx^2}\right] - 0 - \frac{d^2y}{dx^2} = 0$$
$$\Rightarrow 4 + \frac{dy}{dx} + \frac{dy}{dx} + x\frac{d^2y}{dx^2} - 2\left(\frac{dy}{dx}\right)^2 - 2y\frac{d^2y}{dx^2} - \frac{d^2y}{dx^2} = 0$$
$$\Rightarrow 4 + 2\frac{dy}{dx} - 2\left(\frac{dy}{dx}\right)^2 + (x - 2y - 1)\frac{d^2y}{dx^2} = 0$$

AS REQUIRED

d) CHECKING $(0,4)$, $\frac{dy}{dx} = 0$

$$4 + 0 - 0 + (0 - 8 - 1)\frac{d^2y}{dx^2} = 0$$
$$4 = 9\frac{d^2y}{dx^2}$$
$$\frac{d^2y}{dx^2} = \frac{4}{9} > 0$$

$\therefore (0,4)$ IS A LOCAL MIN

CHECKING $(2,-4)$ $\frac{dy}{dx} = 0$

$$4 + 0 - 0 + (2 + 8 - 1)\frac{d^2y}{dx^2} = 0$$
$$9\frac{d^2y}{dx^2} = -4$$
$$\frac{d^2y}{dx^2} = -\frac{4}{9} < 0$$

$\therefore (2,-4)$ IS A LOCAL MAX

Question 68 (****+)

A curve C has implicit equation

$$x^2 - 2y^2 + 4xy - 4x - 6y + 4 = 0.$$

a) Show clearly that

$$\frac{dy}{dx} = \frac{2y + x + a}{2y - 2x + b},$$

where a and b are integers to be found.

The straight line l_1 with equation $y = 2x - 3$ is a tangent to C at the point P.

The straight line l_2 is parallel to l_1 and is also a tangent to C at a different point Q.

b) Find an equation of l_2.

$\boxed{}$, $\boxed{a = -2,\ b = 3}$, $\boxed{y = 2x - \frac{10}{3}}$

a) $x^2 - 2y^2 + 4xy - 4x - 6y + 4 = 0$

DIFFERENTIATE W.R.T x

$\Rightarrow 2x - 4y\frac{dy}{dx} + \left[4 \times y + 4x \times \frac{dy}{dx}\right] - 4 - 6\frac{dy}{dx} = 0$

$\Rightarrow 2x - 4y\frac{dy}{dx} + 4y + 4x\frac{dy}{dx} - 4 - 6\frac{dy}{dx} = 0$

$\Rightarrow (4x - 4y - 6)\frac{dy}{dx} = 4 - 2x - 4y$

$\Rightarrow \frac{dy}{dx} = \frac{4 - 2x - 4y}{4x - 4y - 6} = \frac{4y + 2x - 4}{4y - 4x + 6}$

$\Rightarrow \frac{dy}{dx} = \frac{2y + x - 2}{2y - 2x + 3}$ // ie $a = -2$, $b = 3$

b) GRADIENT OF l_1 IS 2

$\Rightarrow \frac{dy}{dx} = 2$

$\Rightarrow \frac{2y + x - 2}{2y - 2x + 3} = 2$

$\Rightarrow 2y + x - 2 = 4y - 4x + 6$

$\Rightarrow 5x - 2y = 8$

$\boxed{2y = 5x - 8}$

DOUBLE THE EQUATION OF THE CURVE BEFORE SUBSTITUTING.

$\Rightarrow 2x^2 - 4y^2 + 8xy - 8x - 12y + 8 = 0$

$\Rightarrow 2x^2 - (5x-8)^2 + 4x(5x-8) - 8x - 6(5x-8) + 8 = 0$

$\Rightarrow 2x^2 - (25x^2 - 80x + 64) + 20x^2 - 32x - 8x - 30x + 48 + 8 = 0$

$\Rightarrow 2x^2 - 25x^2 + 80x - 64 + 20x^2 - 32x - 8x - 30x + 48 + 8 = 0$

$\Rightarrow -3x^2 - 10x - 8 = 0$

$\Rightarrow 3x^2 + 10x + 8 = 0$

$\Rightarrow (3x - 4)(x - 2) = 0$

$\Rightarrow x = \begin{cases} 2 \\ \frac{4}{3} \end{cases}$ NOW $y = \frac{5x - 8}{2}$

$\Rightarrow y = \begin{cases} 1 \\ -\frac{2}{3} \end{cases}$

$\frac{5 \times \frac{4}{3} - 8}{2} = \frac{20 - 24}{6} = -\frac{4}{6} = -\frac{2}{3}$

$\therefore \boxed{(2, 1) \ \& \ \left(\frac{4}{3}, -\frac{2}{3}\right)}$

LIES ON l_1 BY INSPECTION

THUS USING $\left(\frac{4}{3}, -\frac{2}{3}\right)$ & GRADIENT 2

$y + \frac{2}{3} = 2\left(x - \frac{4}{3}\right)$

$y + \frac{2}{3} = 2x - \frac{8}{3}$

$y = 2x - \frac{10}{3}$ //

Question 69 (****+)

A curve has implicit equation

$$2x\sin y + 2\cos 2y = 1, \quad 0 \le y \le 2\pi .$$

Determine the equations of the two straight lines, which are parallel to the y axis, and are tangents to the above curve.

$\boxed{}$, $\boxed{x = \pm\frac{3}{2}}$

DIFFERENTIATING IMPLICITLY WITH RESPECT TO x

$\Rightarrow \frac{d}{dx}\left[2x\sin y\right] + \frac{d}{dx}\left[2\cos 2y\right] = \frac{d}{dx}(1)$

$\Rightarrow 2\sin y + 2x\cos y\frac{dy}{dx} - 4\sin 2y\frac{dy}{dx} = 0$

$\Rightarrow 2\sin y = 4\sin 2y\frac{dy}{dx} - 2x\cos y\frac{dy}{dx}$

$\Rightarrow \sin y = 2\sin 2y\frac{dy}{dx} - x\cos y\frac{dy}{dx}$

$\Rightarrow \sin y = (2\sin 2y - x\cos y)\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{\sin y}{2\sin 2y - x\cos y}$

NOW FOR A "VERTICAL" TANGENT WE NEED INFINITE GRADIENT SO THE DENOMINATOR ABOVE MUST BE ZERO

$\Rightarrow 2\sin 2y - x\cos y = 0$

$\Rightarrow 4\sin y\cos y - x\cos y = 0$

$\Rightarrow \cos y\left[4\sin y - x\right] = 0$

EITHER $\cos y = 0$

$\Rightarrow y = \frac{\pi}{2}, \frac{3\pi}{2}$

$\Rightarrow \begin{cases} 2x\sin\frac{\pi}{2} + 2\cos\pi = 1 \\ 2x\sin\frac{3\pi}{2} + 2\cos 3\pi = 1 \end{cases}$

$\Rightarrow \begin{cases} 2x - 2 = 1 \\ -2x - 2 = 1 \end{cases}$

$\Rightarrow x = \frac{3}{2}, -\frac{3}{2}$

OR $x = 4\sin y$

$\Rightarrow 2(4\sin y)\sin y + 2\cos 2y = 1$

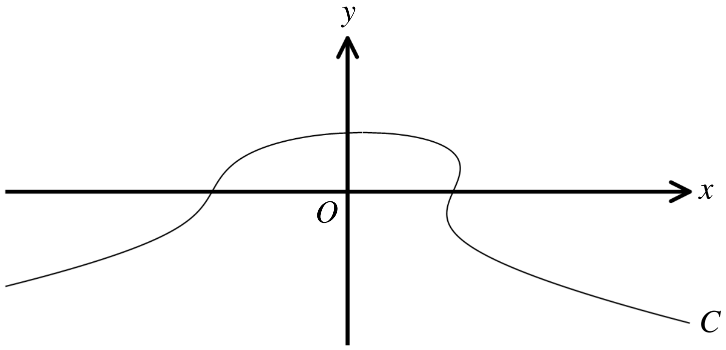
$\Rightarrow 8\sin^2 y + 2(1 - 2\sin^2 y) = 1$

$\Rightarrow 4\sin^2 y = -1$

$\Rightarrow \sin^2 y = -\frac{1}{4}$

NO SOLUTIONS HERE

Question 70 (****+)



The figure above shows part of the curve C with equation

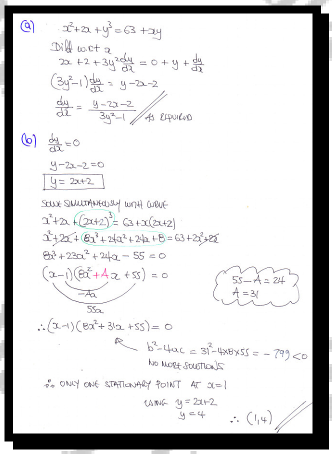
$$x^2 + 2x + y^3 = 63 + xy.$$

a) Show clearly that

$$\frac{dy}{dx} = \frac{y - 2x - 2}{3y^2 - x}$$

b) Show further that C has only one stationary point at $(1,4)$.

$\boxed{}$, $\boxed{\text{proof}}$


(a) $x^2+2x+y^3=63+xy$

Diff w.r.t x

$2x+2+3y^2\frac{dy}{dx} = 0 + y + \frac{dy}{dx}$

$(3y^2-1)\frac{dy}{dx} = y-2x-2$

$\frac{dy}{dx} = \frac{y-2x-2}{3y^2-1}$ // AS REQUIRED

(b) $\frac{dy}{dx}=0$

$y-2x-2=0$

$\boxed{y=2x+2}$

SOLVE SIMULTANEOUSLY WITH CURVE

$x^2+2x+(2x+2)^3 = 63+x(2x+2)$

$x^2+\cancel{2x}+8x^3+24x^2+24x+8 = 63+2x^2+\cancel{2x}$

$8x^3+23x^2+24x-55=0$

$(x-1)(8x^2+Ax+55)=0$

$\underbrace{-Ax}$ $\underbrace{55x}$

$55-A=24$

$A=31$

$\therefore (x-1)(8x^2+31x+55)=0$

$b^2-4ac = 31^2-4\times8\times55 = -799<0$

NO MORE SOLUTIONS

$\therefore$ ONLY ONE STATIONARY POINT AT $x=1$

USING $y=2x+2$

$y=4$

$\therefore (1,4)$ //

Question 71 (****+)

If $\tan 3y = 3\tan x$ show clearly that

$$\frac{dy}{dx} = \frac{1}{1+8\sin^2 x}.$$

proof

DIFFERENTIATE BOTH SIDES W.R.T x

$$\Rightarrow \tan 3y = 3\tan x$$

$$\Rightarrow \frac{d}{dx}(\tan 3y) = \frac{d}{dx}(3\tan x)$$

$$\Rightarrow \cancel{3}\sec^2 3y \frac{dy}{dx} = \cancel{3}\sec^2 x$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sec^2 x}{\sec^2 3y}$$

ELIMINATE y BY IDENTITIES

$$\Rightarrow \frac{dy}{dx} = \frac{\sec^2 x}{1+\tan^2 3y}$$

$\boxed{1+\tan^2 A \equiv \sec^2 A}$

$$\Rightarrow \frac{dy}{dx} = \frac{\sec^2 x}{1+(3\tan x)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sec^2 x}{1+9\tan^2 x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{1}{\cos^2 x}}{1+\frac{9\sin^2 x}{\cos^2 x}}$$

MULTIPLY "TOP/BOTTOM" BY $\cos^2 x$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos^2 x + 9\sin^2 x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos^2 x + \sin^2 x + 8\sin^2 x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1+8\sin^2 x}$$

AS REQUIRED

ALTERNATIVE APPROACH

$$\tan 3y = 3\tan x$$

$$3y = \arctan(3\tan x) \pm n\pi, \quad n = 0,1,2,3,\dots$$

$$y = \tfrac{1}{3}\arctan(3\tan x) \pm n\pi$$

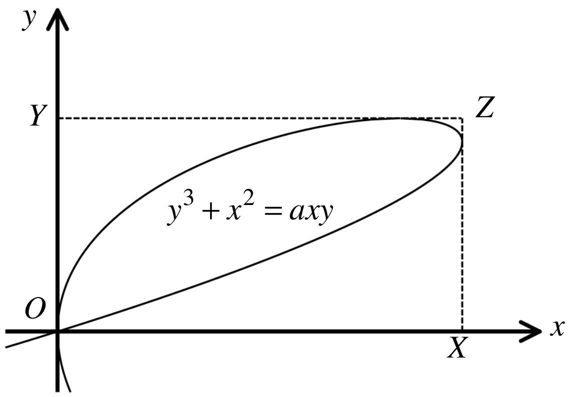
NOW DIFFERENTIATING W.R.T x

$$\frac{dy}{dx} = \frac{1}{\cancel{3}} \times \frac{1}{1+(3\tan x)^2} \times \cancel{3}\sec^2 x$$

$$\frac{dy}{dx} = \frac{\sec^2 x}{1+9\tan^2 x}$$

AND THE SOLUTION NOW MERGES WITH THE PREVIOUS METHOD

Question 72 (****+)



The figure above shows the curve with equation

$$y^3 + x^2 = axy,$$

where a is a positive constant.

The point Y lies on the y axis so that the straight line segment YZ is a tangent to the curve parallel to the x axis. Similarly the point X lies on the x axis so that the straight line segment XZ is a tangent to the curve parallel to the y axis.

The area of the rectangle $OYZX$, where O is the origin, is 288 square units.

Determine the value of a.

$\boxed{}$, $\boxed{a = 6}$

START BY OBTAINING THE GRADIENT FUNCTION

$\Rightarrow y^3 + x^2 = axy$

$\Rightarrow \frac{d}{dx}\left[y^3 + x^2\right] = \frac{d}{dx}\left[axy\right]$

$\Rightarrow 3y^2\frac{dy}{dx} + 2x = ay + ax\frac{dy}{dx}$

$\Rightarrow (3y^2 - ax)\frac{dy}{dx} = ay - 2x$

$\Rightarrow \frac{dy}{dx} = \frac{ay - 2x}{3y^2 - ax}$

LOOK FOR "HORIZONTAL" TANGENTS

$\frac{dy}{dx} = 0 \Rightarrow ay - 2x = 0$

$\Rightarrow ay = 2x$

$\Rightarrow x = \frac{ay}{2}$

SUBSTITUTE INTO THE EQUATION OF THE CURVE

$\Rightarrow y^3 + \left(\frac{a}{2}y\right)^2 = a\left(\frac{a}{2}y\right)y$

$\Rightarrow y^3 + \frac{1}{4}a^2y^2 = \frac{1}{2}a^2y^2$

$\Rightarrow y^3 - \frac{1}{4}a^2y^2 = 0$

$\Rightarrow \frac{1}{4}y^2\left[4y - a^2\right] = 0$

$\therefore y = \cancel{0}$ or $\frac{1}{4}a^2$ $\quad x = \cancel{0}$ or $\frac{1}{8}a^3$

(NOT ACTUALLY NEEDED)

NEXT LOOK FOR "VERTICAL" TANGENTS

$\frac{dy}{dx} = \infty \Rightarrow 3y^2 - ax = 0$

$\Rightarrow 3y^2 = ax$

$\Rightarrow x = \frac{3y^2}{a}$

SUBSTITUTE INTO THE EQUATION OF THE CURVE

$\Rightarrow y^3 + \left(\frac{3y^2}{a}\right)^2 = a\left(\frac{3y^2}{a}\right)y$

$\Rightarrow y^3 + \frac{9}{a^2}y^4 = 3y^3$

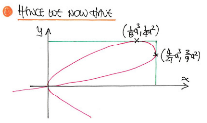
$\Rightarrow 2y^3 - \frac{9}{a^2}y^4 = 0$

$\Rightarrow 2a^2y^3 - 9y^4 = 0$

$\Rightarrow y^3(2a^2 - 9y) = 0$

$\Rightarrow y = \cancel{0}$ or $\frac{2a^2}{9}$ $\quad \Rightarrow x = \cancel{0}$ or $\frac{3}{a}\left(\frac{2a^2}{9}\right)^2 = \frac{12a^4}{81a} = \frac{4}{27}a^3$

HENCE WE NOW HAVE



AREA = 288

$\Rightarrow \frac{1}{4}a^2 \times \frac{4}{27}a^3 = 288$

$\Rightarrow \frac{1}{27}a^5 = 288$

$\Rightarrow a^5 = 27 \times 288 = 27 \times 2 \times 144 = 3^3 \times 2 \times (3 \times 2 \times 2)^2$

$\Rightarrow a^5 = 3^5 \times 2^5 = 6^5$

$\Rightarrow a = 6$

Question 73 (*****)

The curve C has implicit equation

$$xy + x^3y + ay = 1,$$

where a is a positive constant.

Use implicit differentiation to show that the gradient at every point on C is negative.

$\boxed{}$, $\boxed{\text{proof}}$

$xy + x^3y + ay = 1$

diff w.r.t x

$$\Rightarrow \left[1 \times y + x\frac{dy}{dx}\right] + \left[3x^2y + x^3\frac{dy}{dx}\right] + a\frac{dy}{dx} = 0$$

$$\Rightarrow y + x\frac{dy}{dx} + 3x^2y + x^3\frac{dy}{dx} + a\frac{dy}{dx} = 0$$

$$\Rightarrow (x + x^3 + a)\frac{dy}{dx} = -y - 3x^2y$$

$$\Rightarrow \frac{dy}{dx} = -\frac{y + 3x^2y}{x + x^3 + a}$$

EVIDENTLY $y \neq 0$ FROM ORIGINAL EQUATION SO MULTIPLY TOP & BOTTOM BY y

$$\Rightarrow \frac{dy}{dx} = -\frac{y(y + 3x^2y)}{y(x + x^3 + a)}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{y^2 + 3x^2y^2}{xy + yx^3 + ay} \leftarrow 1$$

$$\Rightarrow \frac{dy}{dx} = -(y^2 + 3x^2y^2) < 0$$

SINCE THE BRACKETED EXPRESSION IS POSITIVE

ALTERNATIVE WITHOUT IMPLICIT

- $xy + x^3y + ay = 1$

$$\Rightarrow y\left[x + x^3 + a\right] = 1$$

$$\Rightarrow y = \frac{1}{x + x^3 + a}$$

$$\Rightarrow y = (x + x^3 + a)^{-1}$$

$$\Rightarrow \frac{dy}{dx} = -(x + x^3 + a)^{-2} \times (1 + 3x^2)$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\overbrace{1 + 3x^2}^{\geq 1}}{\underbrace{(x + x^3 + a)^2}_{> 0}} < 0 \qquad a \neq 0$$

AS THE FRACTION IS POSITIVE HAVING POSITIVE TERMS OR SQUARES ONLY

Question 74 (****+)

A curve has equation

$$y = 2^{3e^{2x}}, \quad x \in \mathbb{R}.$$

Express $\frac{dy}{dx}$ in terms of y.

$\boxed{}$, $\boxed{\frac{dy}{dx} = 2y\ln y}$

DIFFERENTIATE THE EXPRESSION W.R.T x, USING THE FACT $\frac{d}{dx}\left[a^{f(x)}\right] = a^{f(x)} \times \ln a \times f'(x)$

$$y = 2^{3e^{2x}} \implies \frac{dy}{dx} = 2^{3e^{2x}} \times \ln 2 \times 6e^{2x}$$

$$\implies \frac{dy}{dx} = y \ln 2 \times 2 \times (3e^{2x})$$

NOW WE NOTE THAT

$$\ln y = \ln 2^{3e^{2x}}$$

$$\ln y = (3e^{2x})(\ln 2)$$

$$\implies \frac{dy}{dx} = 2y\ln y$$

ALTERNATIVE BY "TAKING LOGS" FIRST FOLLOWED BY IMPLICIT DIFFERENTIATION

$$\implies y = 2^{3e^{2x}}$$

$$\implies \ln y = \ln 2^{3e^{2x}}$$

$$\implies \ln y = (\ln 2)(3e^{2x})$$

DIFFERENTIATE WITH RESPECT TO x

$$\implies \frac{1}{y}\frac{dy}{dx} = (\ln 2) \times 6e^{2x}$$

$$\implies \frac{dy}{dx} = y \times (\ln 2) \times 6e^{2x}$$

$$\implies \frac{dy}{dx} = 2y \times (\ln 2)(3e^{2x})$$

$$\implies \frac{dy}{dx} = 2y\ln y$$

Question 75 (*****)

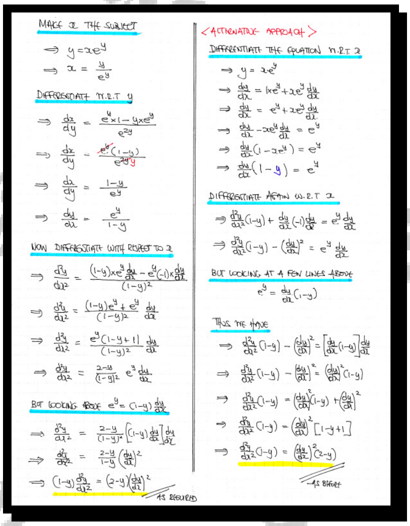
The curve C has implicit equation

$$y = x\mathrm{e}^{y}, \quad x \neq 0, \quad y \neq 1, \quad y \neq 2.$$

Show clearly that

$$(1-y)\frac{d^2y}{dx^2} = (2-y)\left(\frac{dy}{dx}\right)^2.$$

$\boxed{}$, $\boxed{\text{proof}}$



Question 76 (*****)

It is given that

$$x = t^{\frac{1}{2}}, \ t > 0.$$

Given further that y is a function of x, show clearly that

$$\frac{d^2y}{dx^2} = 2\frac{dy}{dt} + 4t\frac{d^2y}{dt^2}.$$

$\boxed{}$, $\boxed{\text{proof}}$

BY IMPLICIT DIFFERENTIATION NOTING THAT $y = y(x)$ & $x = x(t)$

$$\Rightarrow x = t^{\frac{1}{2}}$$

$$\frac{dx}{dy} = \frac{d}{dy}\left(t^{\frac{1}{2}}\right) = \frac{1}{2}t^{-\frac{1}{2}} \times \frac{dt}{dy} = \frac{1}{2t^{\frac{1}{2}}}\frac{dt}{dy}$$

$$\therefore \boxed{\frac{dy}{dx} = 2t^{\frac{1}{2}}\frac{dy}{dt}}$$

DIFFERENTIATE THE ABOVE RESULT WITH RESPECT TO x NOW

$$\Rightarrow \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(2t^{\frac{1}{2}}\frac{dy}{dt}\right)$$

$$\Rightarrow \frac{d^2y}{dx^2} = \underbrace{t^{-\frac{1}{2}}\frac{dt}{dx} \times \frac{dy}{dt} + 2t^{\frac{1}{2}}\frac{d}{dx}\left(\frac{dy}{dt}\right)}_{\text{PRODUCT RULE}}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{1}{t^{\frac{1}{2}}}\frac{dy}{dx} + 2t^{\frac{1}{2}}\frac{d^2y}{dt^2} \times \frac{dt}{dx}$$

NOW IF $x = t^{\frac{1}{2}}$

$$\frac{dx}{dt} = \frac{1}{2}t^{-\frac{1}{2}}$$

$$\frac{dt}{dx} = 2t^{\frac{1}{2}}$$

$\frac{dy}{dx}$ $\qquad$ $\frac{dt}{dx}$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{1}{t^{1/2}}\left(2t^{\frac{1}{2}}\frac{dy}{dt}\right) + 2t^{\frac{1}{2}}\frac{d^2y}{dt^2}\left(2t^{\frac{1}{2}}\right)$$

$$\Rightarrow \frac{d^2y}{dx^2} = 2\frac{dy}{dt} + 4t\frac{d^2y}{dt^2}$$

AS REQUIRED

Question 77 (*****)

$$y^2 - x^2 = 4, \ |y| \geq 2.$$

Use differentiation to show that

$$\frac{d^2y}{dx^2} = \frac{4}{y^3}.$$

$\boxed{}$, $\boxed{\text{proof}}$

REARRANGE THE EQUATION AND DIFFERENTIATE W.R.T x

$$\Rightarrow y^2 - x^2 = 4$$
$$\Rightarrow y^2 = x^2 + 4$$
$$\Rightarrow \frac{d}{dx}\left(y^2\right) = \frac{d}{dx}\left(x^2+4\right)$$
$$\Rightarrow 2y\frac{d}{dx}(y) = 2x$$
$$\Rightarrow \boxed{y\frac{dy}{dx} = x}$$

DIFFERENTIATE AGAIN W.R.T x

$$\Rightarrow \frac{d}{dx}\left(y\frac{dy}{dx}\right) = \frac{d}{dx}(x)$$
$$\Rightarrow \frac{d}{dx}(y) \times \frac{dy}{dx} + y\frac{d}{dx}\left(\frac{dy}{dx}\right) = 1$$
$$\Rightarrow \boxed{\frac{dy}{dx} \times \frac{dy}{dx} + y\frac{d^2y}{dx^2} = 1}$$

BUT $\frac{dy}{dx} = \frac{x}{y}$ FROM ABOVE

$$\Rightarrow \frac{x}{y} \times \frac{x}{y} + y\frac{d^2y}{dx^2} = 1$$
$$\Rightarrow \frac{x^2}{y^2} + y\frac{d^2y}{dx^2} = 1$$
$$\Rightarrow x^2 + y^3\frac{dy}{dx} = y^2$$
$$\Rightarrow y^3\frac{dy}{dx} = y^2 - x^2$$

$y^2 - x^2 = 4$

$$\Rightarrow y^3\frac{dy}{dx} = 4$$
$$\Rightarrow \frac{dy}{dx} = \frac{4}{y^3}$$

Question 78 (*****)

A curve is defined implicitly as

$$y^3 - x^2 + x(3y+2) - 3y = 2 .$$

The y axis is a tangent to the curve at the point A and the point B is another intercept of the curve with the y axis.

The tangent to the curve at the point B meets the curve again at the point C.

Determine the exact coordinate of C.

$$\boxed{C\left(\frac{41}{64},\frac{783}{512}\right)}$$

LOCATE THE y INTERCEPTS, IE $x=0$

$$y^3 - x^2 + x(3y+2) - 3y = 2$$
$$y^3 - 3y = 2$$

BY INSPECTION $y=-1$ IS A ROOT – DIVIDE OR MANIPULATE AS $(y+1)$ IS A KNOWN FACTOR

$$y^3 - 3y - 2 = 0$$
$$y^2(y+1) - y(y+1) - 2(y+1) = 0$$
$$(y^2 - y - 2)(y+1) = 0$$
$$(y+1)(y-2)(y+1) = 0$$
$$(y-2)(y+1)^2 = 0$$
$$y = \begin{cases} 2 \\ -1 \end{cases}$$

$\therefore A(0,-1)$ REPEATED ROOT

$B(0,2)$

DIFFERENTIATE IMPLICITLY BY ANY METHOD

$$f(x,y) = y^3 - x^2 + x(3y+2) - 3y - 2$$

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{-2x+3y+2}{3y^2+3x-3} = \frac{2x-3y-2}{3y^2+3x-3}$$

$$\left.\frac{dy}{dx}\right|_{B(0,2)} = \frac{-6-2}{12-3} = \frac{-8}{9} = -\frac{8}{9}$$

$\therefore$ EQUATION OF TANGENT AT $(0,2)$

$$y = -\frac{8}{9}x + 2$$

SOLVING SIMULTANEOUSLY

$$y^3 - x^2 + x(3y+2) - 3y = 2 \quad \& \quad y = -\frac{8}{9}x + 2$$

$\times 64$

$$64y^3 - 64x^2 + 64x(3y+2) - 192y = 128$$
$$64y^3 - 81(2-y)^2 + 72(2-y)(3y+2) - 192y = 128$$
$$64y^3 - 81(2-y)^2 - 72(y-2)(3y+2) - 192y = 128$$

$$9y = -8x + 18$$
$$8x = 18 - 9y$$
$$8x = 9(2-y)$$
$$64x^2 = 81(2-y)^2$$
$$64x = 72(2-y)$$

TIDY FURTHER INTO A CUBIC WITH REPEATED ROOT AT $y=2$

$$64y^3 - 81(4-4y+y^2) - 72(3y^2-4y-4) - 192y = 128$$
$$64y^3 - 324 + 324y - 81y^2 - 216y^2 + 288y + 288 - 192y - 128 = 0$$
$$64y^3 - 297y^2 + 420y - 164 = 0$$

$$\begin{array}{r} 216 \\ +81 \\ \hline 297 \end{array} \quad \begin{array}{r} 324 \\ +288 \\ \hline 612 \end{array} \quad \begin{array}{r} 612 \\ -192 \\ \hline 420 \end{array} \quad \begin{array}{r} 324 \\ +128 \\ \hline 452 \end{array} \quad \begin{array}{r} 452 \\ -288 \\ \hline 164 \end{array}$$

NO OBVIOUS CANCELLATIONS BUT $(y-2)^2$ MUST BE A FACTOR (POINT OF TANGENCY)

$$\therefore (y-2)^2(64y-41) = 0$$

$$\therefore y_c = \frac{41}{64}$$

AND USING $y = -\frac{8}{9}x + 2$

$$\frac{41}{64} = -\frac{8}{9}x + 2$$
$$41 \times 9 = -8 \times 64x + 2 \times 64 \times 9$$
$$369 = -512x + 1152$$
$$512x = 783$$
$$x = \frac{783}{512}$$

$$\begin{array}{r} 128 \\ \times 9 \\ \hline 1152 \end{array} \quad \begin{array}{r} 1152 \\ 369 \\ \hline 783 \end{array}$$

$$\therefore C\left(\frac{41}{64}, \frac{783}{512}\right)$$

Question 79 (*****)

The curve C has equation

$$y = \ln(1+\cos x), \quad x \in \mathbb{R}, \quad -\pi < x < \pi .$$

Show clearly that

$$\frac{d^4y}{dx^4} + e^{-y}\left(\frac{dy}{dx}\right)^2 + 2e^{-2y} = 0 .$$

$\boxed{}$, $\boxed{\text{proof}}$

START BY DIRECT DIFFERENTIATION

$$y = \ln(1+\cos x)$$

$$\frac{dy}{dx} = \frac{-\sin x}{1+\cos x} = -\frac{\sin x}{1+\cos x}$$

$$\frac{d^2y}{dx^2} = -\frac{\cos x(1+\cos x) - \sin x(-\sin x)}{(1+\cos x)^2}$$

$$= -\frac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2} = -\frac{1+\cos x}{(1+\cos x)} = -\frac{1}{1+\cos x}$$

NOW PROCEED AS FOLLOWS

$$\Rightarrow y = \ln(1+\cos x) \quad \text{AND} \quad \frac{d^2y}{dx^2} = -\frac{1}{1+\cos x}$$

$$\Rightarrow e^{y} = 1+\cos x$$

$$\Rightarrow e^{-y} = \frac{1}{1+\cos x}$$

$$\Rightarrow -e^{-y} = -\frac{1}{1+\cos x}$$

$$\Rightarrow -e^{-y} = \frac{d^2y}{dx^2}$$

$$\Rightarrow \boxed{\frac{d^2y}{dx^2} + e^{-y} = 0}$$

DIFFERENTIATE THE ABOVE EXPRESSION WITH RESPECT TO x GIVES

$$\Rightarrow \frac{d^3y}{dx^3} - e^{-y}\frac{dy}{dx} = 0$$

FINALLY, ONE MORE DIFFERENTIATION W.R.T x AND TIDY

$$\Rightarrow \frac{d^4y}{dx^4} - \frac{d}{dx}\left[e^{-y}\frac{dy}{dx}\right] = 0$$

$$\Rightarrow \frac{d^4y}{dx^4} - \left[-e^{-y}\frac{dy}{dx} \times \frac{dy}{dx} + e^{-y}\frac{d^2y}{dx^2}\right] = 0$$

$$\Rightarrow \frac{d^4y}{dx^4} + e^{-y}\left(\frac{dy}{dx}\right)^2 - e^{-y}\frac{d^2y}{dx^2} = 0$$

$$\Rightarrow \frac{d^4y}{dx^4} + e^{-y}\left(\frac{dy}{dx}\right)^2 - e^{-y}\left(-e^{-y}\right) = 0$$

$$\Rightarrow \frac{d^4y}{dx^4} + e^{-y}\left(\frac{dy}{dx}\right)^2 + e^{-2y} = 0$$

AS REQUIRED

Question 80 (*****)

A curve is defined implicitly by the equation.

$$\sqrt{x+y}-\sqrt{x-y}=\sqrt{k},$$

where k is a positive constant.

a) Use implicit differentiation, directly onto the above equation, to show that

$$\frac{dy}{dx}=\frac{k}{2y}.$$

b) Verify the result of part **(a)** by differentiating the equation of the curve, having first made y the subject of the equation.

proof

a) DIFFERENTIATE THE EXPRESSION WITH RESPECT TO x

$$\Rightarrow \frac{d}{dx}\left[(x+y)^{\frac{1}{2}}\right]-\frac{d}{dx}\left[(x-y)^{\frac{1}{2}}\right]=\frac{d}{dx}\left(k^{\frac{1}{2}}\right)$$

$$\Rightarrow \tfrac{1}{2}(x+y)^{-\frac{1}{2}}\left(1+\frac{dy}{dx}\right)-\tfrac{1}{2}(x-y)^{-\frac{1}{2}}\left(1-\frac{dy}{dx}\right)=0$$

$$\Rightarrow (x+y)^{-\frac{1}{2}}\left(1+\frac{dy}{dx}\right)-(x-y)^{-\frac{1}{2}}\left(1-\frac{dy}{dx}\right)=0$$

$$\Rightarrow (x+y)^{-\frac{1}{2}}+(x+y)^{-\frac{1}{2}}\frac{dy}{dx}-(x-y)^{-\frac{1}{2}}+(x-y)^{\frac{1}{2}}\frac{dy}{dx}=0$$

$$\Rightarrow \left[(x+y)^{-\frac{1}{2}}+(x-y)^{-\frac{1}{2}}\right]\frac{dy}{dx}=(x-y)^{-\frac{1}{2}}-(x+y)^{-\frac{1}{2}}$$

$$\Rightarrow \frac{dy}{dx}=\frac{(x-y)^{-\frac{1}{2}}-(x+y)^{-\frac{1}{2}}}{(x+y)^{-\frac{1}{2}}+(x-y)^{-\frac{1}{2}}}$$

$$\Rightarrow \frac{dy}{dx}=\frac{\frac{1}{(x-y)^{\frac{1}{2}}}-\frac{1}{(x+y)^{\frac{1}{2}}}}{\frac{1}{(x+y)^{\frac{1}{2}}}+\frac{1}{(x-y)^{\frac{1}{2}}}}$$

MULTIPLY "TOP AND BOTTOM" OF THE R.H.S BY $(x+y)^{\frac{1}{2}}(x-y)^{\frac{1}{2}}$

$$\Rightarrow \frac{dy}{dx}=\frac{(x+y)^{\frac{1}{2}}-(x-y)^{\frac{1}{2}}}{(x-y)^{\frac{1}{2}}-(x+y)^{\frac{1}{2}}}$$

$$\Rightarrow \frac{dy}{dx}=\frac{\sqrt{x+y}-\sqrt{x-y}}{\sqrt{x-y}+\sqrt{x+y}}$$

$$\Rightarrow \frac{dy}{dx}=\frac{\sqrt{k}}{\sqrt{x-y}+\sqrt{x+y}}$$

$$\Rightarrow \frac{dy}{dx}=\frac{\sqrt{k}}{\sqrt{x+y}+\sqrt{x-y}}$$

$$\Rightarrow \frac{dy}{dx}=\frac{\sqrt{k}\left[\sqrt{x+y}-\sqrt{x-y}\right]}{\left[\sqrt{x+y}+\sqrt{x-y}\right]\left[\sqrt{x+y}-\sqrt{x-y}\right]}$$ "RATIONALIZING"

$$\Rightarrow \frac{dy}{dx}=\frac{\sqrt{k}\sqrt{k}}{(\sqrt{x+y})^2-(\sqrt{x-y})^2}$$

$$\Rightarrow \frac{dy}{dx}=\frac{k}{(x+y)-(x-y)}$$

$$\Rightarrow \frac{dy}{dx}=\frac{k}{2y}$$

b) SQUARE THE EQUATION

$$\Rightarrow \left[\sqrt{x+y}-\sqrt{x-y}\right]^2=k$$

$$\Rightarrow (x+y)-2\sqrt{x+y}\sqrt{x-y}+(x-y)=k$$

$$\Rightarrow x+\cancel{y}-2\sqrt{(x+y)(x-y)}+x-\cancel{y}=k$$

$$\Rightarrow 2x-2\sqrt{x^2-y^2}=k$$

$$\Rightarrow 2x-k=2\sqrt{x^2-y^2}$$

SQUARING AGAIN

$$\Rightarrow 4x^2-4kx+k^2=4(x^2-y^2)$$

$$\Rightarrow \cancel{4x^2}-4kx+k^2=\cancel{4x^2}-4y^2$$

$$\Rightarrow 4y^2=4kx-k^2$$

$$\Rightarrow y^2=kx-\tfrac{1}{4}k^2$$

$$\Rightarrow y=\pm\left(kx-\tfrac{1}{4}k^2\right)^{\frac{1}{2}}$$

DIFFERENTIATE W.R.T x

$$\frac{dy}{dx}=\pm\tfrac{1}{2}k\left(kx-\tfrac{1}{4}k^2\right)^{-\frac{1}{2}}=\tfrac{1}{2}k\left[\pm\left(kx-\tfrac{1}{4}x^2\right)^{\frac{1}{2}}\right]^{-1}=\tfrac{1}{2}ky^{-1}$$

$$\therefore \frac{dy}{dx}=\frac{k}{2y}$$

Question 81 (*****)

The point T lies on the curve with equation

$$x^2 + y^2 - 5xy = 15 .$$

The tangent to the curve at T passes through the point with coordinates $(2,6)$.

Determine the two possible sets of coordinates for T.

$\boxed{T(-1,2) \ \cup \ T(-2,11)}$

START BY OBTAINING THE GRADIENT FUNCTION OF THE CURVE

$$\Rightarrow x^2 + y^2 - 5xy = 15$$
$$\Rightarrow \frac{d}{dx}\left(x^2 + y^2 - 5xy\right) = \frac{d}{dx}(15)$$
$$\Rightarrow 2x + 2y\frac{dy}{dx} - 5y - 5x\frac{dy}{dx} = 0$$
$$\Rightarrow (2y - 5x)\frac{dy}{dx} = 5y - 2x$$
$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{5y - 2x}{2y - 5x}}$$

IF THE CO-ORDINATES OF THE REQUIRED POINT ARE $T(a,b)$, THEN THE EQUATION OF THE TANGENT IS

$$\boxed{y - b = \frac{5b - 2a}{2b - 5a}(x - a)}$$

THIS TANGENT PASSES THROUGH THE POINT $P(2,6)$

$$\Rightarrow 6 - b = \frac{5b - 2a}{2b - 5a}(2 - a)$$
$$\Rightarrow b - 6 = \frac{5b - 2a}{2b - 5a}(a - 2)$$
$$\Rightarrow (b - 6)(2b - 5a) = (a - 2)(5b - 2a)$$
$$\Rightarrow 2b^2 - 5ab - 12b + 30a = 5ab - 2a^2 - 10b + 4a$$
$$\Rightarrow 2a^2 + 2b^2 - 10ab + 26a - 2b = 0$$
$$\Rightarrow \boxed{a^2 + b^2 - 5ab + 13a - b = 0}$$

BUT THE POINT $T(a,b)$ MUST SATISFY THE EQUATION OF THE CURVE, SO

$$a^2 + b^2 - 5ab = 15$$

HENCE WE HAVE

$$\left.\begin{array}{l} a^2 + b^2 - 5ab = 15 \\ a^2 + b^2 - 5ab + 13a - b = 0 \end{array}\right\} \Rightarrow 15 + 13a - b = 0$$
$$\Rightarrow b = 15 + 13a.$$

$$\Rightarrow a^2 + (15 + 13a)^2 - 5a(15 + 13a) = 15$$
$$\Rightarrow a^2 + 225 + 390a + 169a^2 - 75a - 65a^2 = 15$$
$$\Rightarrow 105a^2 + 315a + 210 = 0$$
$$\Rightarrow a^2 + 3a + 2 = 0$$
$$\Rightarrow (a + 1)(a + 2) = 0$$
$$\Rightarrow a = \begin{cases} -1 \\ -2 \end{cases} \qquad b = \begin{cases} 2 \\ -11 \end{cases}$$

$\therefore T(-1,2)$ OR $(-2,-11)$

Question 82 (*****)

A curve has the following implicit equation

$$x^2+3xy-y^2+4x=1.$$

Two tangents to the curve, at some points on the curve, both pass through the point with coordinates $(6,-4)$.

Determine the equations of these two tangents.

$\boxed{}$, $\boxed{(3x+5y+2)(11x+28y+46)=0}$

START BY FINDING THE GRADIENT FUNCTION

$\Rightarrow x^2+3xy-y^2+4x=1$

$\Rightarrow \frac{d}{dx}\left[x^2+3xy-y^2+4x\right]=\frac{d}{dx}(1)$

$\Rightarrow 2x+3y+3x\frac{dy}{dx}-2y\frac{dy}{dx}+4=0$

$\Rightarrow (3x-2y)\frac{dy}{dx}=-2x-3y-4$

$\Rightarrow \frac{dy}{dx}=-\frac{2x+3y+4}{3x-2y}$

$\Rightarrow \boxed{\frac{dy}{dx}=\frac{2x+3y+4}{2y-3x}}$

LET THE CO-ORDINATES OF T BE (a,b)

$\Rightarrow \boxed{\left.\frac{dy}{dx}\right|_{(a,b)}=\frac{2a+3b+4}{2b-3a}}$

THE EQUATION OF THE TANGENT AT $T(a,b)$ IS

$\Rightarrow \boxed{y-b=\frac{2a+3b+4}{2b-3a}(x-a)}$

THIS TANGENT PASSES THROUGH $P(6,-4)$

$\Rightarrow -4-b=\frac{2a+3b+4}{2b-3a}(6-a)$

$\Rightarrow b+4=\frac{2a+3b+4}{2b-3a}(a-6)$

$\Rightarrow (b+4)(2b-3a)=(a-6)(2a+3b+4)$

$\Rightarrow 2b^2-3ab+8b-\cancel{12a}=2a^2+3ab+4a-\cancel{12a}-18b-24$

$\Rightarrow 2b^2-2a^2-6ab+26b-4a+24=0$

$\Rightarrow b^2-a^2-3ab+13b-2a+12=0$

$\Rightarrow a^2-b^2+3ab-13b+2a-12=0$

BUT $T(a,b)$ LIES ON THE CURVE SO IT MUST SATISFY

$a^2+3ab-b^2+4a=1$

$a^2-b^2+3ab=1-4a$

SUBSTITUTE INTO THE PREVIOUS EQUATION

$\Rightarrow (1-4a)-13b+2a-12=0$

$\Rightarrow -2a-13b-11=0$

$\Rightarrow 2a=-13b-11$

SOLVING SIMULTANEOUSLY WITH THE EQUATION OF THE CURVE

$\Rightarrow a^2-b^2+3ab=1-4a.$

$\Rightarrow 4a^2-4b^2+12ab=4-16a$

$\Rightarrow (2a)^2-4b^2+6b(2a)=4-8(2a)$

$\Rightarrow (-13b-11)^2-4b^2+6b(-13b-11)=4-8(-13b-11)$

$\Rightarrow \begin{pmatrix}169b^2+286b+121\\-4b^2\\-78b^2-66b\end{pmatrix}=4+104b+88$

$\Rightarrow 87b^2+220b+121=104b+92$

$\Rightarrow 87b^2+116b+29=0$

BY INSPECTION $x=-1$ IS A SOLUTION $(87+29=116)$

(OR NOTE THAT THE EQUATION IS DIVISIBLE BY 29)

$\Rightarrow 3b^2+4b+1=0$

$\Rightarrow (3b+1)(b+1)=0$

$\Rightarrow b=\left\langle\begin{matrix}-1\\-\frac{1}{3}\end{matrix}\right.$

$\Rightarrow a=\left\langle\begin{matrix}\frac{-13(-1)-11}{2}=1\\ \frac{-13(-\frac{1}{3})-11}{2}=\frac{\frac{13}{3}-11}{2}=\frac{13-33}{6}=-\frac{10}{3}\end{matrix}\right.$

IE $T(1,-1)$ OR $T\left(-\frac{10}{3},-\frac{1}{3}\right)$

HENCE THE POSSIBLE EQUATIONS OF THE TANGENT ARE

$\boxed{y-b=\frac{2a+3b+4}{2b-3a}(x-a)}$

$\Rightarrow y+1=\frac{2-3+4}{-2-3}(x-1)$

$\Rightarrow y+1=\frac{3}{-5}(x-1)$

$\Rightarrow -5y-5=3x-3$

$\Rightarrow 0=3x+5y+2$

$\Rightarrow 3x+5y+2=0$

$\Rightarrow y+\frac{1}{3}=\frac{-\frac{20}{3}-1+4}{-\frac{2}{3}+10}\left(x+\frac{10}{3}\right)$

$\Rightarrow y+\frac{1}{3}=\frac{3-\frac{20}{3}}{10-\frac{2}{3}}\left(x+\frac{10}{3}\right)$

$\Rightarrow y+\frac{1}{3}=\frac{9-20}{30-2}\left(x+\frac{10}{3}\right)$

$\Rightarrow y+\frac{1}{3}=\frac{-11}{28}\left(x+\frac{10}{3}\right)$

$\Rightarrow 28y+\frac{28}{3}=-11x-\frac{110}{3}$

$\Rightarrow 84y+28=-33x-110$

$\Rightarrow 11x+28y+46=0$

Question 83 **(*****)**

A curve is defined implicitly by the equation

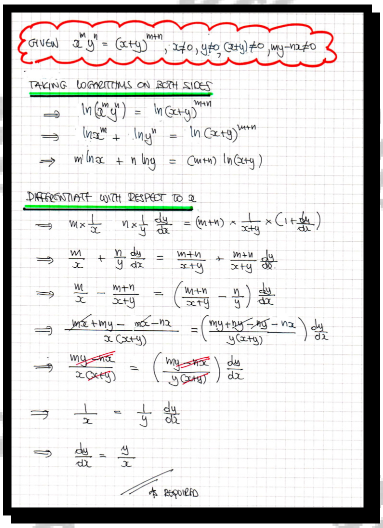
$$x^m y^n = (x+y)^{m+n},$$

where m and n are rational constants, and $x \neq 0$, $y \neq 0$, $x+y \neq 0$, $my - nx \neq 0$.

Show that

$$\frac{dy}{dx} = \frac{y}{x}.$$

$\boxed{}$, $\boxed{\text{proof}}$



GIVEN $x^m y^n = (x+y)^{m+n}$, $x \neq 0$, $y \neq 0$, $(x+y) \neq 0$, $my - nx \neq 0$

TAKING LOGARITHMS ON BOTH SIDES

$\Rightarrow \ln(x^m y^n) = \ln(x+y)^{m+n}$

$\Rightarrow \ln x^m + \ln y^n = \ln(x+y)^{m+n}$

$\Rightarrow m \ln x + n \ln y = (m+n) \ln(x+y)$

DIFFERENTIATE WITH RESPECT TO x

$\Rightarrow m \times \frac{1}{x} \quad n \times \frac{1}{y} \frac{dy}{dx} = (m+n) \times \frac{1}{x+y} \times \left(1 + \frac{dy}{dx}\right)$

$\Rightarrow \frac{m}{x} + \frac{n}{y}\frac{dy}{dx} = \frac{m+n}{x+y} + \frac{m+n}{x+y}\frac{dy}{dx}$

$\Rightarrow \frac{m}{x} - \frac{m+n}{x+y} = \left(\frac{m+n}{x+y} - \frac{n}{y}\right)\frac{dy}{dx}$

$\Rightarrow \frac{\cancel{mx} + my - \cancel{mx} - nx}{x(x+y)} = \left(\frac{my + \cancel{ny} - \cancel{ny} - nx}{y(x+y)}\right)\frac{dy}{dx}$

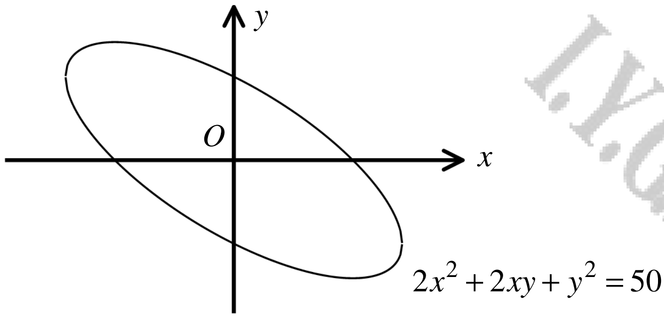
$\Rightarrow \frac{my \cancel{- nx}}{x\cancel{(x+y)}} = \left(\frac{my \cancel{- nx}}{y\cancel{(x+y)}}\right)\frac{dy}{dx}$

$\Rightarrow \frac{1}{x} = \frac{1}{y}\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{y}{x}$

AS REQUIRED

Question 84 (*****)



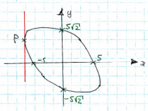
The figure above shows the curve with equation

$$2x^2 + 2xy + y^2 = 50.$$

Determine the area of the finite region bounded by the x axis and the part of the curve for which $y \geq 0$.

$\boxed{}$, $\boxed{25\pi}$

- FIRSTLY PRODUCE A SKETCH WITH x & y INTERCEPTS



- NEXT FIND THE x CO-ORDINATE OF THE POINT P (VERTICAL TANGENT)

$$2x^2 + 2xy + y^2 = 50$$
$$\frac{d}{dx}\left(2x^2 + 2xy + y^2\right) = 0$$
$$4x + 2y + 2x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$
$$2(y+2x) = -2(x+y)\frac{dy}{dx}$$
$$\frac{dy}{dx} = -\frac{y+2x}{x+y} \leftarrow \text{INFINITE GRADIENT} \Longrightarrow \text{DENOMINATOR ZERO}$$

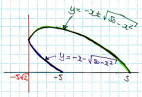
- SOLVING SIMULTANEOUSLY WITH THE EQUATION OF THE CURVE WE OBTAIN

$$y + x = 0 \qquad 2x^2 + 2x(-x) + (-x)^2 = 50$$
$$y = -x \qquad x^2 = 50$$
$$x = \pm 5\sqrt{2}$$
$$\therefore P\left(-5\sqrt{2}, 5\sqrt{2}\right)$$

- NEXT REARRANGE THE EQUATION OF THE CURVE, IN THE FORM $y = f(x)$

$$\Longrightarrow y^2 + 2xy + 2x^2 = 50$$
$$\Longrightarrow (y+x)^2 - x^2 + 2x^2 = 50$$
$$\Longrightarrow (y+x)^2 = 50 - x^2$$
$$\Longrightarrow y + x = \pm\sqrt{50 - x^2}$$
$$\Longrightarrow y = -x \pm \sqrt{50 - x^2}$$

- THUS WE NOW HAVE THE "PICTURE" OPPOSITE – IN BOTH CASES WE HAVE TO INTEGRATE $\sqrt{50 - x^2}$, SO PREPARE THIS PART FIRST



$$\int_{x_1}^{x_2} \sqrt{50 - x^2}\, dx = \dots \text{ BY SUBSTITUTION} \qquad \boxed{\begin{array}{l} x = \sqrt{50}\sin\theta \\ dx = \sqrt{50}\cos\theta \end{array}}$$
$$= \int_{\theta_1}^{\theta_2} \sqrt{50 - 50\sin^2\theta}\left(\sqrt{50}\cos\theta\, d\theta\right)$$
$$= \int_{\theta_1}^{\theta_2} \sqrt{50\left(1 - \sin^2\theta\right)} \cdot \sqrt{50}\cos\theta\, d\theta$$
$$= \int_{\theta_1}^{\theta_2} 50\cos^2\theta\, d\theta = \int_{\theta_1}^{\theta_2} 25 + 25\cos 2\theta\, d\theta$$
$$= \left[25\theta + \frac{25}{2}\sin 2\theta\right]_{\theta_1}^{\theta_2}$$

- HENCE THE REQUIRED AREA CAN BE FOUND

$$= \int_{-5\sqrt{2}}^{5} -x + \sqrt{50 - x^2}\, dx - \int_{-5\sqrt{2}}^{-5} -x - \sqrt{50 - x^2}\, dx$$
$$= \int_{-5\sqrt{2}}^{5} -x\, dx + \int_{-5\sqrt{2}}^{-5} x\, dx + \int_{-5\sqrt{2}}^{5} \sqrt{50 - x^2}\, dx + \int_{-5\sqrt{2}}^{-5} \sqrt{50 - x^2}\, dx$$

- CHANGING THE LIMITS IN THE SUBSTITUTION INTEGRALS

$$\boxed{\begin{array}{l} x = 5\sqrt{2}\sin\theta \\ x = 5 \longmapsto \theta = \frac{\pi}{4} \\ x = -5 \longmapsto \theta = -\frac{\pi}{4} \\ x = -5\sqrt{2} \longmapsto \theta = -\frac{\pi}{2} \end{array}}$$

$$= \left[-\frac{1}{2}x^2\right]_{-5\sqrt{2}}^{5} + \left[\frac{1}{2}x^2\right]_{-5\sqrt{2}}^{-5} + \left[25\theta + \frac{25}{2}\sin 2\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{4}} + \left[25\theta + \frac{25}{2}\sin 2\theta\right]_{-\frac{\pi}{2}}^{-\frac{\pi}{4}}$$
$$= \cancel{\left[-\frac{25}{2} + \frac{50}{2}\right]} + \cancel{\left[\frac{25}{2} - \frac{50}{2}\right]} + \left[\left(\cancel{\frac{25\pi}{4}} + \cancel{\frac{25}{2}}\right) - \left(-\frac{25\pi}{2} + 0\right)\right]$$
$$+ \left[\left(\cancel{-\frac{25\pi}{4}} - \cancel{\frac{25}{2}}\right) - \left(-\frac{25\pi}{2} + 0\right)\right]$$

$$= 25\pi$$