IMPLICIT DIFFERENTIATION

BASIC DIFFERENTIATION

Y.G.B. FREE SAMPLE T. MADAS

Question 1

For each of the following implicit relationships, find an expression for $\frac{dy}{dx}$, in terms of x and y.

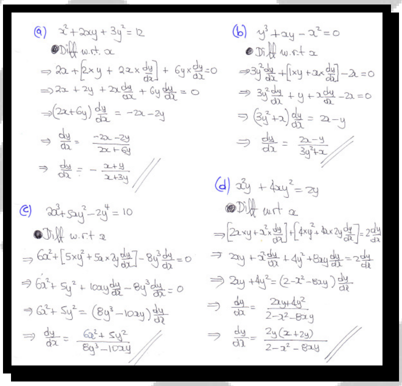
a) $x^2 + 2xy + 3y^2 = 12$

b) $y^3 + xy - x^2 = 0$

c) $2x^3 + 5xy^2 - 2y^4 = 10$

d) $x^2 y + 4xy^2 = 2y$

$$\boxed{\frac{dy}{dx} = -\frac{x+y}{x+3y}}, \quad \boxed{\frac{dy}{dx} = \frac{2x-y}{3y^2+x}}, \quad \boxed{\frac{dy}{dx} = \frac{6x^2+5y^2}{8y^3-10xy}}, \quad \boxed{\frac{dy}{dx} = \frac{2y(x+2y)}{2-x^2-8xy}}$$



(a) $x^2+2xy+3y^2=12$

• Diff w.r.t. x

$\Rightarrow 2x + \left[2\times y + 2x\times\frac{dy}{dx}\right] + 6y\times\frac{dy}{dx} = 0$

$\Rightarrow 2x + 2y + 2x\frac{dy}{dx} + 6y\frac{dy}{dx} = 0$

$\Rightarrow (2x+6y)\frac{dy}{dx} = -2x-2y$

$\Rightarrow \frac{dy}{dx} = \frac{-2x-2y}{2x+6y}$

$\Rightarrow \frac{dy}{dx} = -\frac{x+y}{x+3y}$

(b) $y^3+xy-x^2=0$

• Diff w.r.t x

$\Rightarrow 3y^2\frac{dy}{dx} + \left[1\times y + x\times\frac{dy}{dx}\right] - 2x = 0$

$\Rightarrow 3y^2\frac{dy}{dx} + y + x\frac{dy}{dx} - 2x = 0$

$\Rightarrow (3y^2+x)\frac{dy}{dx} = 2x-y$

$\Rightarrow \frac{dy}{dx} = \frac{2x-y}{3y^2+x}$

(c) $2x^3+5xy^2-2y^4=10$

• Diff w.r.t x

$\Rightarrow 6x^2 + \left[5\times y^2 + 5x\times 2y\frac{dy}{dx}\right] - 8y^3\frac{dy}{dx} = 0$

$\Rightarrow 6x^2 + 5y^2 + 10xy\frac{dy}{dx} - 8y^3\frac{dy}{dx} = 0$

$\Rightarrow 6x^2+5y^2 = (8y^3-10xy)\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{6x^2+5y^2}{8y^3-10xy}$

(d) $x^2y + 4xy^2 = 2y$

• Diff w.r.t x

$\Rightarrow \left[2x\times y + x^2\times\frac{dy}{dx}\right] + \left[4\times y^2 + 4x\times 2y\frac{dy}{dx}\right] = 2\frac{dy}{dx}$

$\Rightarrow 2xy + x^2\frac{dy}{dx} + 4y^2 + 8xy\frac{dy}{dx} = 2\frac{dy}{dx}$

$\Rightarrow 2xy + 4y^2 = (2-x^2-8xy)\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{2xy+4y^2}{2-x^2-8xy}$

$\Rightarrow \frac{dy}{dx} = \frac{2y(x+2y)}{2-x^2-8xy}$

Question 2

For each of the following implicit relationships, find an expression for $\frac{dy}{dx}$, in terms of x and y.

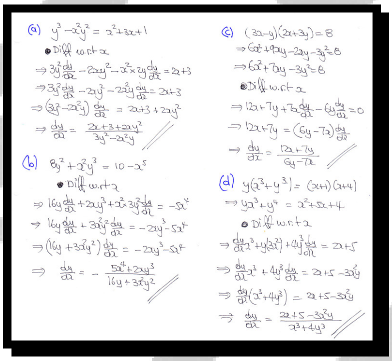
a) $y^3 - x^2y^2 = x^2 + 3x + 1$

b) $8y^2 + x^2y^3 = 10 - x^5$

c) $(3x - y)(2x + 3y) = 8$

d) $y\left(x^3 + y^3\right) = (x+1)(x+4)$

$$\boxed{\frac{dy}{dx} = \frac{2xy^2 + 2x + 3}{3y^2 - 2yx^2}}, \boxed{\frac{dy}{dx} = -\frac{2xy^3 + 5x^4}{16y + 3x^2y^2}}, \boxed{\frac{dy}{dx} = \frac{12x + 7y}{6y - 7x}}, \boxed{\frac{dy}{dx} = \frac{2x + 5 - 3yx^2}{4y^3 + x^3}}$$



(a) $y^3 - x^2y^2 = x^2 + 3x + 1$

• Diff w.r.t x

$\Rightarrow 3y^2\frac{dy}{dx} - 2xy^2 - x^2 \times 2y\frac{dy}{dx} = 2x + 3$

$\Rightarrow 3y^2\frac{dy}{dx} - 2xy^2 - 2x^2y\frac{dy}{dx} = 2x + 3$

$\Rightarrow (3y^2 - 2x^2y)\frac{dy}{dx} = 2x + 3 + 2xy^2$

$\Rightarrow \frac{dy}{dx} = \frac{2x + 3 + 2xy^2}{3y^2 - 2x^2y}$

(b) $8y^2 + x^2y^3 = 10 - x^5$

• Diff w.r.t x

$\Rightarrow 16y\frac{dy}{dx} + 2xy^3 + x^2 \times 3y^2\frac{dy}{dx} = -5x^4$

$\Rightarrow 16y\frac{dy}{dx} + 3x^2y^2\frac{dy}{dx} = -2xy^3 - 5x^4$

$\Rightarrow (16y + 3x^2y^2)\frac{dy}{dx} = -2xy^3 - 5x^4$

$\Rightarrow \frac{dy}{dx} = -\frac{5x^4 + 2xy^3}{16y + 3x^2y^2}$

(c) $(3x - y)(2x + 3y) = 8$

$\Rightarrow 6x^2 + 9xy - 2xy - 3y^2 = 8$

$\Rightarrow 6x^2 + 7xy - 3y^2 = 8$

• Diff w.r.t x

$\Rightarrow 12x + 7y + 7x\frac{dy}{dx} - 6y\frac{dy}{dx} = 0$

$\Rightarrow 12x + 7y = (6y - 7x)\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{12x + 7y}{6y - 7x}$

(d) $y(x^3 + y^3) = (x+1)(x+4)$

$\Rightarrow yx^3 + y^4 = x^2 + 5x + 4$

• Diff w.r.t x

$\Rightarrow \frac{dy}{dx}x^3 + y(3x^2) + 4y^3\frac{dy}{dx} = 2x + 5$

$\Rightarrow \frac{dy}{dx}x^3 + 4y^3\frac{dy}{dx} = 2x + 5 - 3x^2y$

$\Rightarrow \frac{dy}{dx}(x^3 + 4y^3) = 2x + 5 - 3x^2y$

$\Rightarrow \frac{dy}{dx} = \frac{2x + 5 - 3x^2y}{x^3 + 4y^3}$

Question 3

For each of the following implicit relationships, find an expression for $\frac{dy}{dx}$, in terms of x and y.

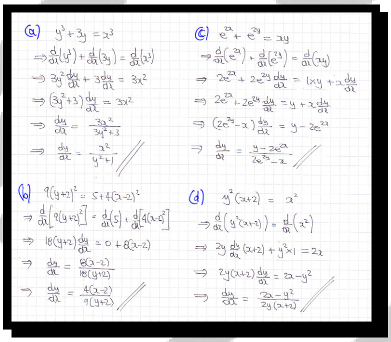
a) $y^3+3y=x^3$

b) $9(y+2)^2=5+4(x-2)^2$

c) $e^{2x}+e^{2y}=xy$

d) $y^2(x+2)=x^2$

$$\boxed{\frac{dy}{dx}=\frac{x^2}{y^2+1}},\ \boxed{\frac{dy}{dx}=\frac{4(x-2)}{9(y+2)}},\ \boxed{\frac{dy}{dx}=\frac{y-2e^{2x}}{2e^{2y}-x}},\ \boxed{\frac{dy}{dx}=\frac{2x-y^2}{2xy+4y}}$$



(a) $y^3+3y=x^3$

$\Rightarrow \frac{d}{dx}(y^3)+\frac{d}{dx}(3y)=\frac{d}{dx}(x^3)$

$\Rightarrow 3y^2\frac{dy}{dx}+3\frac{dy}{dx}=3x^2$

$\Rightarrow (3y^2+3)\frac{dy}{dx}=3x^2$

$\Rightarrow \frac{dy}{dx}=\frac{3x^2}{3y^2+3}$

$\Rightarrow \frac{dy}{dx}=\frac{x^2}{y^2+1}$ //

(c) $e^{2x}+e^{2y}=xy$

$\Rightarrow \frac{d}{dx}(e^{2x})+\frac{d}{dx}(e^{2y})=\frac{d}{dx}(xy)$

$\Rightarrow 2e^{2x}+2e^{2y}\frac{dy}{dx}=1\times y+x\frac{dy}{dx}$

$\Rightarrow 2e^{2x}+2e^{2y}\frac{dy}{dx}=y+x\frac{dy}{dx}$

$\Rightarrow (2e^{2y}-x)\frac{dy}{dx}=y-2e^{2x}$

$\Rightarrow \frac{dy}{dx}=\frac{y-2e^{2x}}{2e^{2y}-x}$ //

(b) $9(y+2)^2=5+4(x-2)^2$

$\Rightarrow \frac{d}{dx}\left[9(y+2)^2\right]=\frac{d}{dx}(5)+\frac{d}{dx}\left[4(x-2)^2\right]$

$\Rightarrow 18(y+2)\frac{dy}{dx}=0+8(x-2)$

$\Rightarrow \frac{dy}{dx}=\frac{8(x-2)}{18(y+2)}$

$\Rightarrow \frac{dy}{dx}=\frac{4(x-2)}{9(y+2)}$ //

(d) $y^2(x+2)=x^2$

$\Rightarrow \frac{d}{dx}\left(y^2(x+2)\right)=\frac{d}{dx}(x^2)$

$\Rightarrow 2y\frac{dy}{dx}(x+2)+y^2\times 1=2x$

$\Rightarrow 2y(x+2)\frac{dy}{dx}=2x-y^2$

$\Rightarrow \frac{dy}{dx}=\frac{2x-y^2}{2y(x+2)}$ //

Question 4

For each of the following implicit relationships, find an expression for $\frac{dy}{dx}$, in terms of x and y.

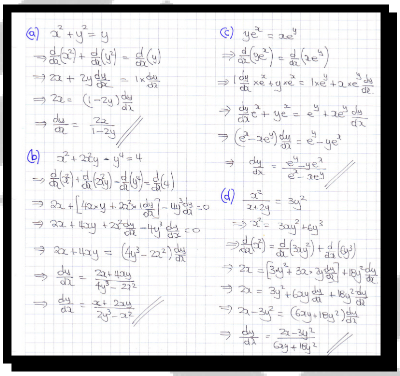
a) $x^2 + y^2 = y$

b) $x^2 + 2x^2 y - y^4 = 4$

c) $y\mathrm{e}^x = x\mathrm{e}^y$

d) $\dfrac{x^2}{x+2y} = 3y^2$

$$\boxed{\frac{dy}{dx} = \frac{2x}{1-2y}}, \quad \boxed{\frac{dy}{dx} = \frac{x+2xy}{2y^3 - x^2}}, \quad \boxed{\frac{dy}{dx} = \frac{\mathrm{e}^y - y\mathrm{e}^x}{\mathrm{e}^x - x\mathrm{e}^y}}, \quad \boxed{\frac{dy}{dx} = \frac{2x - 3y^2}{6y(x+3y)}}$$



(a) $x^2 + y^2 = y$

$\Rightarrow \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(y)$

$\Rightarrow 2x + 2y\frac{dy}{dx} = 1 \times \frac{dy}{dx}$

$\Rightarrow 2x = (1-2y)\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{2x}{1-2y}$

(b) $x^2 + 2x^2y - y^4 = 4$

$\Rightarrow \frac{d}{dx}(x^2) + \frac{d}{dx}(2x^2y) - \frac{d}{dx}(y^4) = \frac{d}{dx}(4)$

$\Rightarrow 2x + \left[4x \times y + 2x^2 \times 1\frac{dy}{dx}\right] - 4y^3\frac{dy}{dx} = 0$

$\Rightarrow 2x + 4xy + 2x^2\frac{dy}{dx} - 4y^3\frac{dy}{dx} = 0$

$\Rightarrow 2x + 4xy = (4y^3 - 2x^2)\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{2x+4xy}{4y^3 - 2x^2}$

$\Rightarrow \frac{dy}{dx} = \frac{x + 2xy}{2y^3 - x^2}$

(c) $ye^x = xe^y$

$\Rightarrow \frac{d}{dx}(ye^x) = \frac{d}{dx}(xe^y)$

$\Rightarrow 1\frac{dy}{dx} \times e^x + y \times e^x = 1 \times e^y + x \times e^y\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx}e^x + ye^x = e^y + xe^y\frac{dy}{dx}$

$\Rightarrow (e^x - xe^y)\frac{dy}{dx} = e^y - ye^x$

$\Rightarrow \frac{dy}{dx} = \frac{e^y - ye^x}{e^x - xe^y}$

(d) $\frac{x^2}{x+2y} = 3y^2$

$\Rightarrow x^2 = 3xy^2 + 6y^3$

$\Rightarrow \frac{d}{dx}(x^2) = \frac{d}{dx}(3xy^2) + \frac{d}{dx}(6y^3)$

$\Rightarrow 2x = \left[3y^2 + 3x \times 2y\frac{dy}{dx}\right] + 18y^2\frac{dy}{dx}$

$\Rightarrow 2x = 3y^2 + 6xy\frac{dy}{dx} + 18y^2\frac{dy}{dx}$

$\Rightarrow 2x - 3y^2 = (6xy + 18y^2)\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{2x - 3y^2}{6xy + 18y^2}$

Question 5

For each of the following implicit relationships, find an expression for $\frac{dy}{dx}$, in terms of x and y.

a) $\frac{(x+2y)^2}{4x-y}+y=x$

b) $\sqrt{xy}-x+y^2=0$

$$\boxed{\frac{dy}{dx}=\frac{2x-3y}{2y+3x}}, \quad \boxed{\frac{dy}{dx}=\frac{2x-2y^2-y}{x+4xy-4y^3}}$$

(a) $\frac{(x+2y)^2}{4x-y}+y=x$

$\Rightarrow \frac{(x+2y)^2}{4x-y}=x-y$

$\Rightarrow (x+2y)^2=(x-y)(4x-y)$

$\Rightarrow x^2+4xy+4y^2=4x^2-xy-4xy+y^2$

$\Rightarrow 3y^2-3x^2+9xy=0$

$\Rightarrow y^2-x^2+3xy=0$

DIFF W.R.T. x

$\Rightarrow 2y\frac{dy}{dx}-2x+3y+3x\frac{dy}{dx}=0$

$\Rightarrow (2y+3x)\frac{dy}{dx}=2x-3y$

$\Rightarrow \frac{dy}{dx}=\frac{2x-3y}{2y+3x}$

(b) $\sqrt{xy}-x+y^2=0$

$\Rightarrow \sqrt{xy}=x-y^2$

$\Rightarrow xy=(x-y^2)^2$

$\Rightarrow xy=x^2-2xy^2+y^4$

Diff w.r.t x

$\Rightarrow y+x\frac{dy}{dx}=2x-2y^2-2x\left(2y\frac{dy}{dx}\right)+4y^3\frac{dy}{dx}$

$\Rightarrow y+x\frac{dy}{dx}=2x-2y^2-4xy\frac{dy}{dx}+4y^3\frac{dy}{dx}$

$\Rightarrow x\frac{dy}{dx}+4xy\frac{dy}{dx}-4y^3\frac{dy}{dx}=2x-2y^2-y$

$\Rightarrow (x+4xy-4y^3)\frac{dy}{dx}=2x-2y^2-y$

$\frac{dy}{dx}=\frac{2x-2y^2-y}{x+4xy-4y^3}$

TANGENTS AND NORMALS

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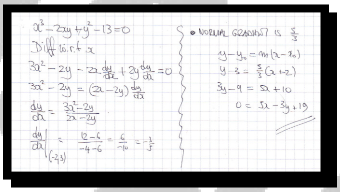
Question 1

A curve C has equation

$$x^3 - 2xy + y^2 - 13 = 0 .$$

Find an equation for the normal to C at the point $P(-2,3)$.

$$\boxed{5x - 3y + 19 = 0}$$


$x^3 - 2xy + y^2 - 13 = 0$
Diff w.r.t x
$3x^2 - 2y - 2x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$
$3x^2 - 2y = (2x - 2y)\frac{dy}{dx}$
$\frac{dy}{dx} = \frac{3x^2 - 2y}{2x - 2y}$
$\left.\frac{dy}{dx}\right|_{(-2,3)} = \frac{12-6}{-4-6} = \frac{6}{-10} = -\frac{3}{5}$
• NORMAL GRADIENT IS $\frac{5}{3}$
$y - y_0 = m(x - x_0)$
$y - 3 = \frac{5}{3}(x + 2)$
$3y - 9 = 5x + 10$
$0 = 5x - 3y + 19$

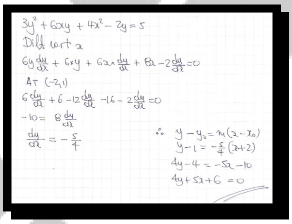
Question 2

A curve is given implicitly by the equation

$$3y^2 + 6xy + 4x^2 - 2y = 5 .$$

Find an equation for the tangent to the curve at the point $P(-2,1)$.

$$\boxed{4y + 5x + 6 = 0}$$


$3y^2 + 6xy + 4x^2 - 2y = 5$
Diff w.r.t x
$6y\frac{dy}{dx} + 6xy + 6x\frac{dy}{dx} + 8x - 2\frac{dy}{dx} = 0$
AT $(-2,1)$
$6\frac{dy}{dx} + 6 - 12\frac{dy}{dx} - 16 - 2\frac{dy}{dx} = 0$
$-10 = 8\frac{dy}{dx}$
$\frac{dy}{dx} = -\frac{5}{4}$
$\therefore\ y - y_0 = m(x - x_0)$
$y - 1 = -\frac{5}{4}(x + 2)$
$4y - 4 = -5x - 10$
$4y + 5x + 6 = 0$

Question 3

A curve has implicit equation

$$4y^2 - 2xy - x^2 + 11 = 0 .$$

Find an equation of the normal to the curve at the point $P(-3,-1)$.

$$\boxed{4y + x + 7 = 0}$$

$$4y^2 - 2xy - x^2 + 11 = 0$$

Diff w.r.t x

$$\Rightarrow 8y\frac{dy}{dx} - 2y - 2x\frac{dy}{dx} - 2x = 0$$

when $x=-3$, $y=-1$

$$\Rightarrow -8\frac{dy}{dx} + 2 + 6\frac{dy}{dx} + 6 = 0$$

$$\Rightarrow 8 = 2\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = 4$$

NORMAL GRADIENT $= -\frac{1}{4}$, $P(-3,-1)$

$$\Rightarrow y - y_0 = m(x - x_0)$$

$$\Rightarrow y + 1 = -\frac{1}{4}(x+3)$$

$$\Rightarrow 4y + 4 = -x - 3$$

$$\Rightarrow 4y + x + 7 = 0$$

Question 4

A curve is given implicitly by the equation

$$3y^2 - 2x^2 - 3x + 2y + 5 = 0 .$$

Find an equation for the tangent to the curve at the point $P(1,0)$.

$$\boxed{2y = 7x - 7}$$

$$3y^2 - 2x^2 - 3x + 2y + 5 = 0$$

Diff w.r.t x

$$\Rightarrow 6y\frac{dy}{dx} - 4x - 3 + 2\frac{dy}{dx} = 0$$

AT $P(1,0)$

$$\Rightarrow -4 - 3 + 2\frac{dy}{dx} = 0$$

$$\Rightarrow 2\frac{dy}{dx} = 7$$

$$\Rightarrow \frac{dy}{dx} = \frac{7}{2}$$

TANGENT AT $P(1,0)$

$$\Rightarrow y - y_0 = m(x - x_0)$$

$$\Rightarrow y - 0 = \frac{7}{2}(x - 1)$$

$$\Rightarrow 2y = 7x - 7.$$

Question 5

A curve has implicit equation

$$x^3 + y^3 + 3y^2 + 3y - 6x = 50 + 2xy\ .$$

Find an equation of the normal to the curve at the point $P(4,2)$.

$\boxed{x = 2y}$

$x^3 + y^3 + 3y^2 + 3y - 6x = 50 + 2xy$

Diff w.r.t x

$\Rightarrow 3x^2 + 3y^2\frac{dy}{dx} + 6y\frac{dy}{dx} + 3\frac{dy}{dx} - 6 = 2y + 2x\frac{dy}{dx}$

$\Rightarrow 3x^2 - 6 - 2y = (2x - 3y^2 - 6y - 3)\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{3x^2 - 6 - 2y}{2x - 3y^2 - 6y - 3}$

$\left.\frac{dy}{dx}\right|_{(4,2)} = \frac{48 - 6 - 4}{8 - 12 - 12 - 3} = \frac{38}{-19} = -2$

NORMAL GRADIENT IS $\frac{1}{2}$, $P(4,2)$

$\Rightarrow y - y_0 = m(x - x_0)$

$\Rightarrow y - 2 = \frac{1}{2}(x - 4)$

$\Rightarrow y - \cancel{2} = \frac{1}{2}x - \cancel{2}$

$\therefore y = \frac{1}{2}x$

Question 6

A curve is described by the implicit relationship

$$4x^2 + 3xy - y^2 = 21\ .$$

Find an equation of the tangent to the curve at the point $(2,1)$.

$\boxed{4y + 19x = 42}$

$4x^2 + 3xy - y^2 = 21$

Diff w.r.t

$8x + 3y + 3x\frac{dy}{dx} - 2y\frac{dy}{dx} = 0$

AT $(2,1) \Rightarrow 16 + 3 + 6\frac{dy}{dx} - 2\frac{dy}{dx} = 0$

$\Rightarrow 4\frac{dy}{dx} = -19$

$\Rightarrow \frac{dy}{dx} = -\frac{19}{4}$

EQUATION OF TANGENT

$\Rightarrow y - y_0 = m(x - x_0)$

$\Rightarrow y - 1 = -\frac{19}{4}(x - 2)$

$\Rightarrow 4y - 4 = -19x + 38$

$\Rightarrow 4y + 19x = 42$

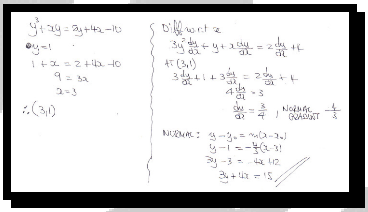
Question 7

A curve is described by the implicit relationship

$$y^3 + xy = 2y + 4x - 10 \, .$$

Find an equation of the normal to the curve at the point where $y = 1$.

$$\boxed{3y + 4x = 15}$$



$y^3 + xy = 2y + 4x - 10$

$y = 1$

$1 + x = 2 + 4x - 10$

$9 = 3x$

$x = 3$

$\therefore (3,1)$

Diff w.r.t x

$3y^2\frac{dy}{dx} + y + x\frac{dy}{dx} = 2\frac{dy}{dx} + 4$

AT $(3,1)$

$3\frac{dy}{dx} + 1 + 3\frac{dy}{dx} = 2\frac{dy}{dx} + 4$

$4\frac{dy}{dx} = 3$

$\frac{dy}{dx} = \frac{3}{4}$, NORMAL GRADIENT $-\frac{4}{3}$

NORMAL: $y - y_0 = m(x - x_0)$

$y - 1 = -\frac{4}{3}(x - 3)$

$3y - 3 = -4x + 12$

$3y + 4x = 15$

Question 8

A curve has equation

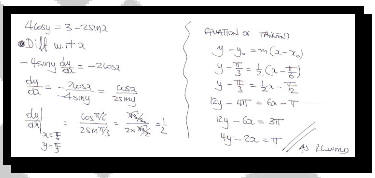
$$4\cos y = 3 - 2\sin x \, , \ x \in \mathbb{R} \, , \ y \in \mathbb{R} \, .$$

Show clearly that

$$4y - 2x = \pi$$

is the equation of the tangent to the curve at the point with coordinates $\left(\frac{\pi}{6}, \frac{\pi}{3}\right)$.

$$\boxed{\text{proof}}$$



$4\cos y = 3 - 2\sin x$

Diff w.r.t x

$-4\sin y \frac{dy}{dx} = -2\cos x$

$\frac{dy}{dx} = \frac{-2\cos x}{-4\sin y} = \frac{\cos x}{2\sin y}$

$\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{6}, y=\frac{\pi}{3}} = \frac{\cos\pi/6}{2\sin\pi/3} = \frac{\cancel{\sqrt{3}/2}}{2 \times \cancel{\sqrt{3}/2}} = \frac{1}{2}$

EQUATION OF TANGENT

$y - y_0 = m(x - x_0)$

$y - \frac{\pi}{3} = \frac{1}{2}\left(x - \frac{\pi}{6}\right)$

$y - \frac{\pi}{3} = \frac{1}{2}x - \frac{\pi}{12}$

$12y - 4\pi = 6x - \pi$

$12y - 6x = 3\pi$

$4y - 2x = \pi$ AS REQUIRED

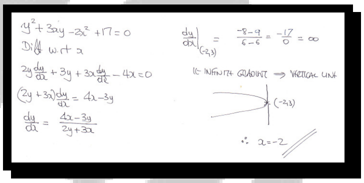
Question 9

A curve has implicit equation

$$y^2 + 3xy - 2x^2 + 17 = 0 .$$

Find an equation of the tangent to the curve at the point $(-2,3)$.

$$\boxed{x = -2}$$



$y^2 + 3xy - 2x^2 + 17 = 0$

Diff w.r.t x

$2y\frac{dy}{dx} + 3y + 3x\frac{dy}{dx} - 4x = 0$

$(2y+3x)\frac{dy}{dx} = 4x - 3y$

$\frac{dy}{dx} = \frac{4x-3y}{2y+3x}$

$\left.\frac{dy}{dx}\right|_{(-2,3)} = \frac{-8-9}{6-6} = \frac{-17}{0} = \infty$

IE INFINITE GRADIENT $\Rightarrow$ VERTICAL LINE

$(-2,3)$

$\therefore x = -2$

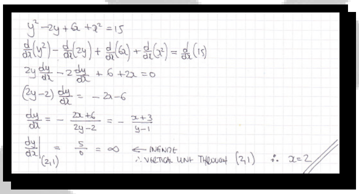
Question 10

A curve is described by the implicit relationship

$$y^2 - 2y + 6x + x^2 = 15 .$$

Find an equation for the tangent to the curve at the point $P(2,1)$.

$$\boxed{x = 2}$$



$y^2 - 2y + 6x + x^2 = 15$

$\frac{d}{dx}(y^2) - \frac{d}{dx}(2y) + \frac{d}{dx}(6x) + \frac{d}{dx}(x^2) = \frac{d}{dx}(15)$

$2y\frac{dy}{dx} - 2\frac{dy}{dx} + 6 + 2x = 0$

$(2y-2)\frac{dy}{dx} = -2x - 6$

$\frac{dy}{dx} = -\frac{2x+6}{2y-2} = -\frac{x+3}{y-1}$

$\left.\frac{dy}{dx}\right|_{(2,1)} = \frac{5}{0} = \infty \leftarrow$ INFINITE

$\therefore$ VERTICAL LINE THROUGH $(2,1)$ $\therefore x = 2$

TURNING POINTS

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Question 1

A curve has implicit equation

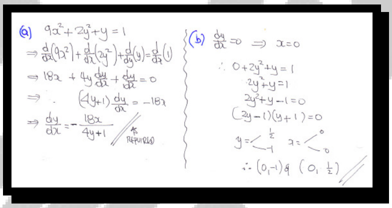
$$9x^2 + 2y^2 + y = 1.$$

a) Show clearly that

$$\frac{dy}{dx} = -\frac{18x}{4y+1}.$$

b) Hence find the coordinates of the points on the curve where $\frac{dy}{dx} = 0$.

$$\boxed{(0,-1),\left(0,\tfrac{1}{2}\right)}$$


(a) $9x^2 + 2y^2 + y = 1$

$\Rightarrow \frac{d}{dx}(9x^2) + \frac{d}{dx}(2y^2) + \frac{d}{dy}(y) = \frac{d}{dx}(1)$

$\Rightarrow 18x + 4y\frac{dy}{dx} + \frac{dy}{dx} = 0$

$\Rightarrow (4y+1)\frac{dy}{dx} = -18x$

$\Rightarrow \frac{dy}{dx} = -\frac{18x}{4y+1}$ // AS REQUIRED

(b) $\frac{dy}{dx} = 0 \Rightarrow x = 0$

$\therefore 0 + 2y^2 + y = 1$

$2y^2 + y = 1$

$2y^2 + y - 1 = 0$

$(2y-1)(y+1) = 0$

$y = \frac{1}{2}, -1 \quad x = 0, 0$

$\therefore (0,-1)$ & $(0, \frac{1}{2})$ //

Question 2

A curve has equation

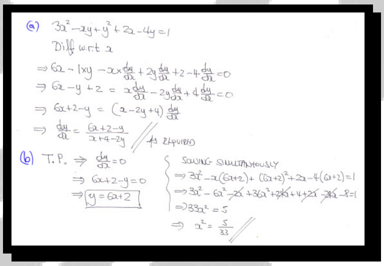
$$3x^2 - xy + y^2 + 2x - 4y = 1.$$

a) Show that

$$\frac{dy}{dx} = \frac{2 + 6x - y}{4 - 2y + x}.$$

b) Hence show that at the turning points of the curve, $x^2 = \dfrac{5}{33}$.

$\boxed{\text{proof}}$



(a) $3x^2 - xy + y^2 + 2x - 4y = 1$

Diff w.r.t x

$\Rightarrow 6x - 1 \times y - x \times \frac{dy}{dx} + 2y\frac{dy}{dx} + 2 - 4\frac{dy}{dx} = 0$

$\Rightarrow 6x - y + 2 = x\frac{dy}{dx} - 2y\frac{dy}{dx} + 4\frac{dy}{dx} = 0$

$\Rightarrow 6x + 2 - y = (x - 2y + 4)\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{6x+2-y}{x+4-2y}$ // AS REQUIRED

(b) T.P. $\Rightarrow \frac{dy}{dx} = 0$

$\Rightarrow 6x + 2 - y = 0$

$\Rightarrow \boxed{y = 6x + 2}$

SOLVING SIMULTANEOUSLY

$\Rightarrow 3x^2 - x(6x+2) + (6x+2)^2 + 2x - 4(6x+2) = 1$

$\Rightarrow 3x^2 - 6x^2 \cancel{-2x} + 36x^2 \cancel{+24x} + 4 \cancel{+2x} \cancel{-24x} - 8 = 1$

$\Rightarrow 33x^2 = 5$

$\Rightarrow x^2 = \frac{5}{33}$ //

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Question 3

A curve has equation

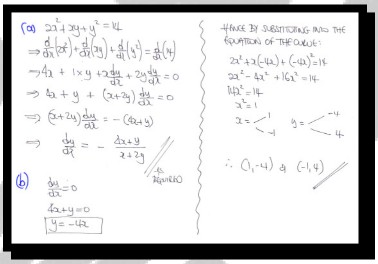
$$2x^2 + xy + y^2 = 14.$$

a) Show clearly that

$$\frac{dy}{dx} = -\frac{4x+y}{x+2y}.$$

b) Hence, find the coordinates of the turning points of the curve.

$$\boxed{(1,-4),(-1,4)}$$



(a) $2x^2 + xy + y^2 = 14$

$\Rightarrow \frac{d}{dx}(2x^2) + \frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = \frac{d}{dx}(14)$

$\Rightarrow 4x + 1 \times y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$

$\Rightarrow 4x + y + (x+2y)\frac{dy}{dx} = 0$

$\Rightarrow (x+2y)\frac{dy}{dx} = -(4x+y)$

$\Rightarrow \frac{dy}{dx} = -\frac{4x+y}{x+2y}$ // AS REQUIRED

(b) $\frac{dy}{dx} = 0$

$4x + y = 0$

$\boxed{y = -4x}$

HENCE BY SUBSTITUTING INTO THE EQUATION OF THE CURVE:

$2x^2 + x(-4x) + (-4x)^2 = 14$

$2x^2 - 4x^2 + 16x^2 = 14$

$14x^2 = 14$

$x^2 = 1$

$x = \begin{cases} 1 \\ -1 \end{cases} \quad y = \begin{cases} -4 \\ 4 \end{cases}$

$\therefore (1,-4)$ & $(-1,4)$ //

Question 4

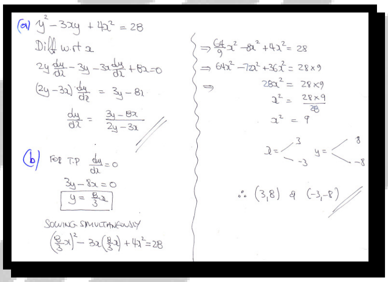
A curve C is defined implicitly by

$$y^2 - 3xy + 4x^2 = 28, \quad x, y \in \mathbb{R}.$$

a) Find an expression for $\frac{dy}{dx}$, in terms of x and y.

b) Determine the coordinates of the turning points of C.

$\boxed{\frac{dy}{dx} = \frac{3y-8x}{2y-3x}}$, $\boxed{(-3,-8),(3,8)}$



(a) $y^2 - 3xy + 4x^2 = 28$

Diff w.r.t x

$2y\frac{dy}{dx} - 3y - 3x\frac{dy}{dx} + 8x = 0$

$(2y-3x)\frac{dy}{dx} = 3y - 8x$

$\frac{dy}{dx} = \frac{3y-8x}{2y-3x}$

(b) FOR T.P $\frac{dy}{dx} = 0$

$3y - 8x = 0$

$\boxed{y = \frac{8}{3}x}$

SOLVING SIMULTANEOUSLY

$\left(\frac{8}{3}x\right)^2 - 3x\left(\frac{8}{3}x\right) + 4x^2 = 28$

$\Rightarrow \frac{64}{9}x^2 - 8x^2 + 4x^2 = 28$

$\Rightarrow 64x^2 - 72x^2 + 36x^2 = 28 \times 9$

$\Rightarrow 28x^2 = 28 \times 9$

$x^2 = \frac{28 \times 9}{28}$

$x^2 = 9$

$x = 3, -3 \quad y = 8, -8$

$\therefore (3,8)$ & $(-3,-8)$

Question 5

The equation of a curve is given by

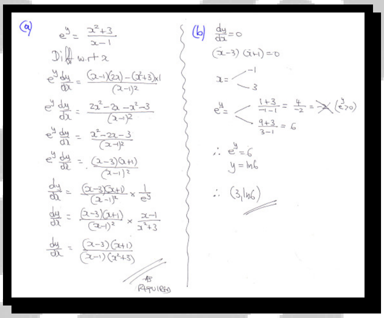
$$e^y = \frac{x^2+3}{x-1}, \ x > 1.$$

a) Show clearly that

$$\frac{dy}{dx} = \frac{(x-3)(x+1)}{\left(x^2+3\right)(x-1)}.$$

b) Find the exact coordinates of the turning point of the curve.

$\boxed{\left(3, \ln 6\right)}$


(a)
$e^y = \frac{x^2+3}{x-1}$
Diff w.r.t x
$e^y \frac{dy}{dx} = \frac{(x-1)(2x)-(x^2+3)\times 1}{(x-1)^2}$
$e^y \frac{dy}{dx} = \frac{2x^2-2x-x^2-3}{(x-1)^2}$
$e^y \frac{dy}{dx} = \frac{x^2-2x-3}{(x-1)^2}$
$e^y \frac{dy}{dx} = \frac{(x-3)(x+1)}{(x-1)^2}$
$\frac{dy}{dx} = \frac{(x-3)(x+1)}{(x-1)^2} \times \frac{1}{e^y}$
$\frac{dy}{dx} = \frac{(x-3)(x+1)}{(x-1)^2} \times \frac{x-1}{x^2+3}$
$\frac{dy}{dx} = \frac{(x-3)(x+1)}{(x-1)(x^2+3)}$
AS REQUIRED

(b)
$\frac{dy}{dx} = 0$
$(x-3)(x+1) = 0$
$x = -1, \ 3$
$e^y = \frac{1+3}{-1-1} = \frac{4}{-2} = \cancel{-2} \ (e^y > 0)$
$e^y = \frac{9+3}{3-1} = 6$
$\therefore e^y = 6$
$y = \ln 6$
$\therefore (3, \ln 6)$

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Question 6

The equation of a curve is given by

$$x^2 - 2y^2 - xy - x + 5y + 34 = 0.$$

a) Show clearly that

$$\frac{dy}{dx} = \frac{2x - y - 1}{x + 4y - 5}.$$

b) Find the exact value of gradient at the point on the curve with coordinates

$$\left(1 + 4\sqrt{2}, -5 - \sqrt{2}\right).$$

c) Determine the coordinates of the turning point of the curve.

$$\boxed{-\frac{1}{8}\left(2 + 3\sqrt{2}\right)}, \quad \boxed{(3,5),(-1,-3)}$$

(a) $x^2 - 2y^2 - xy - x + 5y + 34 = 0$

Diff w.r.t. x

$\Rightarrow 2x - 4y\frac{dy}{dx} - 1 \times y - x\frac{dy}{dx} - 1 + 5\frac{dy}{dx} = 0$

$\Rightarrow 2x - y - 1 = (4y + x - 5)\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{2x - y - 1}{4y + x - 5}$ // AS REQUIRED

(b) $\left.\frac{dy}{dx}\right|_{\substack{x = 1+4\sqrt{2} \\ y = -5-\sqrt{2}}} = \frac{2(1+4\sqrt{2}) - (-5-\sqrt{2}) - 1}{4(-5-\sqrt{2}) + (1+4\sqrt{2}) - 5} = \frac{2 + 8\sqrt{2} + 5 + \sqrt{2} - 1}{-20 \cancel{-4\sqrt{2}} + 1 \cancel{+4\sqrt{2}} - 5} = \frac{6 + 9\sqrt{2}}{-24}$

$= \frac{2 + 3\sqrt{2}}{-8} = -\frac{1}{8}\left(2 + 3\sqrt{2}\right)$ //

(c) T.P. $\Rightarrow \frac{dy}{dx} = 0$

$\Rightarrow 2x - y - 1 = 0$

$\Rightarrow \boxed{y = 2x - 1}$

SOLVING SIMULTANEOUSLY

$\Rightarrow x^2 - 2(2x-1)^2 - x(2x-1) - x + 5(2x-1) + 34 = 0$

$\Rightarrow x^2 - 2(4x^2 - 4x + 1) - 2x^2 \cancel{+x} \cancel{-x} + 10x - 5 + 34 = 0$

$\Rightarrow x^2 - 8x^2 + 8x - 2 - 2x^2 + 10x - 5 + 34 = 0$

$\Rightarrow -9x^2 + 18x + 27 = 0$

$\Rightarrow x^2 - 2x - 3 = 0$

$\Rightarrow (x + 1)(x - 3) = 0$

$\Rightarrow x = \begin{cases} -1 \\ 3 \end{cases} \quad y = \begin{cases} -3 \\ 5 \end{cases}$ $\therefore (-1,-3)$ & $(3,5)$ //