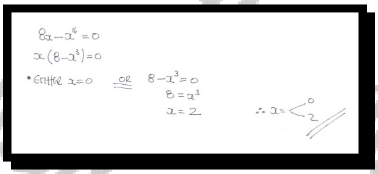
ADVANCED EQUATIONS SOLVING SKILLS

Question 1 **(**)**

Find the solutions of the equation

$$8x - x^4 = 0 .$$

$\boxed{x = 0, 2}$



$8x - x^4 = 0$

$x(8 - x^3) = 0$

• EITHER $x = 0$ $\underline{\underline{\text{OR}}}$ $8 - x^3 = 0$

$8 = x^3$

$x = 2$

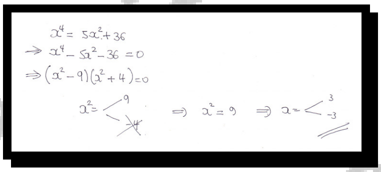
$\therefore x = \Big< \begin{matrix} 0 \\ 2 \end{matrix}$ //

Question 2 **(**)**

Find the solutions of the equation

$$x^4 = 5x^2 + 36 .$$

$\boxed{x = \pm 3}$



$x^4 = 5x^2 + 36$

$\Rightarrow x^4 - 5x^2 - 36 = 0$

$\Rightarrow (x^2 - 9)(x^2 + 4) = 0$

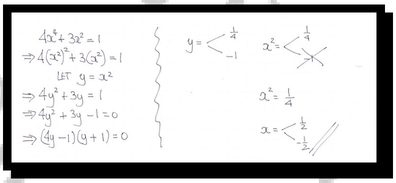
$x^2 = \Big< \begin{matrix} 9 \\ \cancel{-4} \end{matrix}$ $\Rightarrow x^2 = 9$ $\Rightarrow x = \Big< \begin{matrix} 3 \\ -3 \end{matrix}$ //

Question 3 **(**)**

Find the solutions of the equation

$$4x^4 + 3x^2 = 1 .$$

$\boxed{x = \pm \frac{1}{2}}$



$4x^4 + 3x^2 = 1$

$\Rightarrow 4(x^2)^2 + 3(x^2) = 1$

LET $y = x^2$

$\Rightarrow 4y^2 + 3y = 1$

$\Rightarrow 4y^2 + 3y - 1 = 0$

$\Rightarrow (4y - 1)(y + 1) = 0$

$y = \Big< \begin{matrix} \frac{1}{4} \\ -1 \end{matrix}$ $x^2 = \Big< \begin{matrix} \frac{1}{4} \\ \cancel{} \end{matrix}$

$x^2 = \frac{1}{4}$

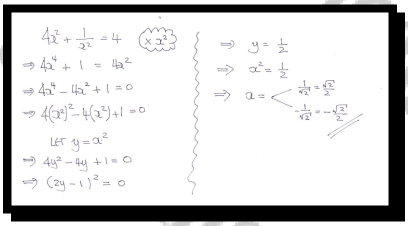
$x = \Big< \begin{matrix} \frac{1}{2} \\ -\frac{1}{2} \end{matrix}$ //

Question 4 (+)**

Find as exact surds the roots of the equation

$$4x^2 + \frac{1}{x^2} = 4, \ x \neq 0 .$$

$$\boxed{x = \pm\frac{\sqrt{2}}{2}}$$



$4x^2 + \frac{1}{x^2} = 4$ $(\times x^2)$

$\Rightarrow 4x^4 + 1 = 4x^2$

$\Rightarrow 4x^4 - 4x^2 + 1 = 0$

$\Rightarrow 4(x^2)^2 - 4(x^2) + 1 = 0$

LET $y = x^2$

$\Rightarrow 4y^2 - 4y + 1 = 0$

$\Rightarrow (2y-1)^2 = 0$

$\Rightarrow y = \frac{1}{2}$

$\Rightarrow x^2 = \frac{1}{2}$

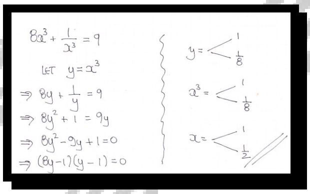
$\Rightarrow x = \begin{cases} \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \\ -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2} \end{cases}$ //

Question 5 (+)**

Solve the equation

$$8x^3 + \frac{1}{x^3} = 9, \ x \neq 0 .$$

$$\boxed{x = 1, \ \ x = \frac{1}{8}}$$



$8x^3 + \frac{1}{x^3} = 9$

LET $y = x^3$

$\Rightarrow 8y + \frac{1}{y} = 9$

$\Rightarrow 8y^2 + 1 = 9y$

$\Rightarrow 8y^2 - 9y + 1 = 0$

$\Rightarrow (8y-1)(y-1) = 0$

$y = \begin{cases} 1 \\ \frac{1}{8} \end{cases}$

$x^3 = \begin{cases} 1 \\ \frac{1}{8} \end{cases}$

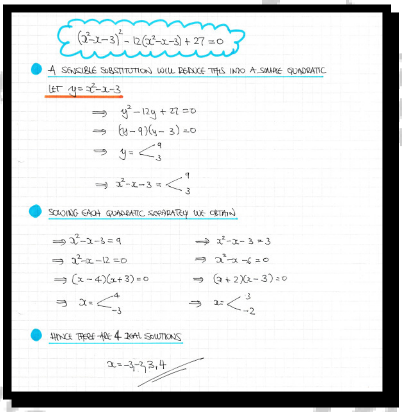
$x = \begin{cases} 1 \\ \frac{1}{2} \end{cases}$ //

Question 6 **(***)**

Find the real solutions of the following equation

$$\left(x^2-x-3\right)^2-12\left(x^2-x-3\right)+27=0\,.$$

$\boxed{\text{[illegible]}}$, $\boxed{x=-3,\ \ x=-2,\ \ x=3,\ \ x=4}$



$\left(x^2-x-3\right)^2-12\left(x^2-x-3\right)+27=0$

- A SENSIBLE SUBSTITUTION WILL REDUCE THIS INTO A SIMPLE QUADRATIC

LET $y=x^2-x-3$

$\Rightarrow y^2-12y+27=0$

$\Rightarrow (y-9)(y-3)=0$

$\Rightarrow y=\begin{cases}9\\3\end{cases}$

$\Rightarrow x^2-x-3=\begin{cases}9\\3\end{cases}$

- SOLVING EACH QUADRATIC SEPARATELY WE OBTAIN

$\Rightarrow x^2-x-3=9$ | $\Rightarrow x^2-x-3=3$

$\Rightarrow x^2-x-12=0$ | $\Rightarrow x^2-x-6=0$

$\Rightarrow (x-4)(x+3)=0$ | $\Rightarrow (x+2)(x-3)=0$

$\Rightarrow x=\begin{cases}4\\-3\end{cases}$ | $\Rightarrow x=\begin{cases}3\\-2\end{cases}$

- HENCE THERE ARE 4 REAL SOLUTIONS

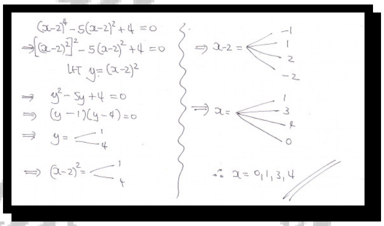
$x=-3,-2,3,4$

Question 7 **(***)**

Determine the four real roots of the equation

$$(x-2)^4-5(x-2)^2+4=0\,.$$

$\boxed{x=0,1,3,4}$



$(x-2)^4-5(x-2)^2+4=0$

$\Rightarrow\left[(x-2)^2\right]^2-5(x-2)^2+4=0$

LET $y=(x-2)^2$

$\Rightarrow y^2-5y+4=0$

$\Rightarrow (y-1)(y-4)=0$

$\Rightarrow y=\begin{cases}1\\4\end{cases}$

$\Rightarrow (x-2)^2=\begin{cases}1\\4\end{cases}$

$\Rightarrow x-2=\begin{cases}-1\\1\\2\\-2\end{cases}$

$\Rightarrow x=\begin{cases}1\\3\\4\\0\end{cases}$

$\therefore x=0,1,3,4$

Question 8 (***)

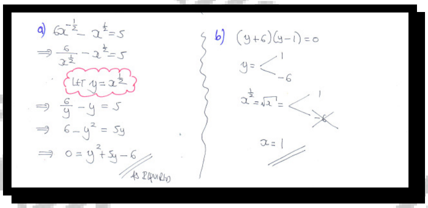
$$6x^{-\frac{1}{2}} - x^{\frac{1}{2}} = 5 .$$

a) Show clearly that the substitution $y = x^{\frac{1}{2}}$ transforms the above irrational equation into the quadratic equation

$$y^2 + 5y - 6 = 0 .$$

b) Solve the quadratic equation and hence find the root of the **irrational** equation.

$\boxed{x=1}$



a) $6x^{-\frac{1}{2}} - x^{\frac{1}{2}} = 5$

$\Rightarrow \frac{6}{x^{\frac{1}{2}}} - x^{\frac{1}{2}} = 5$

LET $y = x^{\frac{1}{2}}$

$\Rightarrow \frac{6}{y} - y = 5$

$\Rightarrow 6 - y^2 = 5y$

$\Rightarrow 0 = y^2 + 5y - 6$

AS REQUIRED

b) $(y+6)(y-1) = 0$

$y = \left\langle \begin{matrix} 1 \\ -6 \end{matrix} \right.$

$x^{\frac{1}{2}} = \sqrt{x} = \left\langle \begin{matrix} 1 \\ \cancel{-6} \end{matrix} \right.$

$x = 1$

Question 9 (***)

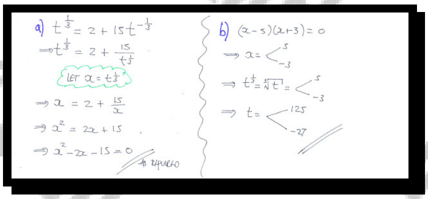
$$t^{\frac{1}{3}} = 2 + 15t^{-\frac{1}{3}}.$$

a) Show that the substitution $x = t^{\frac{1}{3}}$ transforms the above irrational equation into the quadratic equation

$$x^2 - 2x - 15 = 0.$$

b) Solve the quadratic equation and hence find the two solutions of the **irrational** equation.

$\boxed{t = -27, \ \ t = 125}$



a) $t^{\frac{1}{3}} = 2 + 15t^{-\frac{1}{3}}$

$\Rightarrow t^{\frac{1}{3}} = 2 + \frac{15}{t^{\frac{1}{3}}}$

LET $x = t^{\frac{1}{3}}$

$\Rightarrow x = 2 + \frac{15}{x}$

$\Rightarrow x^2 = 2x + 15$

$\Rightarrow x^2 - 2x - 15 = 0$ // AS REQUIRED

b) $(x-5)(x+3) = 0$

$\Rightarrow x = \begin{cases} 5 \\ -3 \end{cases}$

$\Rightarrow t^{\frac{1}{3}} = \sqrt[3]{t^1} = \begin{cases} 5 \\ -3 \end{cases}$

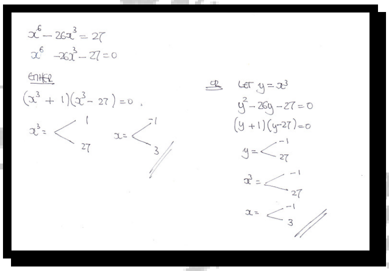
$\Rightarrow t = \begin{cases} 125 \\ -27 \end{cases}$ //

Question 10 (***+)

Find the solutions of the equation

$$x^6 - 26x^3 = 27.$$

$\boxed{x = -1, 3}$



$x^6 - 26x^3 = 27$

$x^6 - 26x^3 - 27 = 0$

EITHER

$(x^3 + 1)(x^3 - 27) = 0$

$x^3 = \begin{cases} 1 \\ 27 \end{cases}$ $x = \begin{cases} -1 \\ 3 \end{cases}$ //

OR LET $y = x^3$

$y^2 - 26y - 27 = 0$

$(y+1)(y-27) = 0$

$y = \begin{cases} -1 \\ 27 \end{cases}$

$x^3 = \begin{cases} -1 \\ 27 \end{cases}$

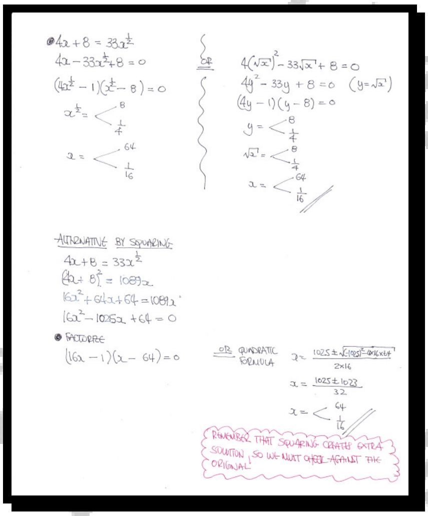
$x = \begin{cases} -1 \\ 3 \end{cases}$ //

Question 11 **(***+)**

Find the roots of the equation

$$4x+8=33x^{\frac{1}{2}}, \ x \geq 0 .$$

$$\boxed{x=\frac{1}{16},64}$$



$4x+8=33x^{\frac{1}{2}}$

$4x-33x^{\frac{1}{2}}+8=0$

$(4x^{\frac{1}{2}}-1)(x^{\frac{1}{2}}-8)=0$

$x^{\frac{1}{2}}= < \begin{array}{l} 8 \\ \frac{1}{4} \end{array}$

$x= < \begin{array}{l} 64 \\ \frac{1}{16} \end{array}$

OR

$4(\sqrt{x})^2-33\sqrt{x}+8=0$

$4y^2-33y+8=0 \quad (y=\sqrt{x})$

$(4y-1)(y-8)=0$

$y= < \begin{array}{l} 8 \\ \frac{1}{4} \end{array}$

$\sqrt{x}= < \begin{array}{l} 8 \\ \frac{1}{4} \end{array}$

$x= < \begin{array}{l} 64 \\ \frac{1}{16} \end{array}$ //

ALTERNATIVE BY SQUARING

$4x+8=33x^{\frac{1}{2}}$

$(4x+8)^2=1089x$

$16x^2+64x+64=1089x$

$16x^2-1025x+64=0$

FACTORISE

$(16x-1)(x-64)=0$

OR QUADRATIC FORMULA

$x=\frac{1025\pm\sqrt{(-1025)^2-4\times16\times64}}{2\times16}$

$x=\frac{1025\pm1023}{32}$

$x= < \begin{array}{l} 64 \\ \frac{1}{16} \end{array}$ //

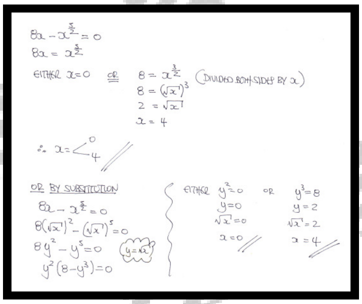
REMEMBER THAT SQUARING CREATES EXTRA SOLUTION, SO WE MUST CHECK AGAINST THE ORIGINAL!

Question 12 **(***+)**

Find the solutions of the equation

$$8x-x^{\frac{5}{2}}=0, \ x \geq 0 .$$

$$\boxed{x=0,\ 4}$$



$8x-x^{\frac{5}{2}}=0$

$8x=x^{\frac{5}{2}}$

EITHER $x=0$ OR $8=x^{\frac{3}{2}}$ (DIVIDED BOTH SIDES BY x)

$8=(\sqrt{x})^3$

$2=\sqrt{x}$

$x=4$

$\therefore x= < \begin{array}{l} 0 \\ 4 \end{array}$ //

OR BY SUBSTITUTION

$8x-x^{\frac{5}{2}}=0$

$8(\sqrt{x})^2-(\sqrt{x})^5=0$

$8y^2-y^5=0 \quad (y=\sqrt{x})$

$y^2(8-y^3)=0$

EITHER $y^2=0$, $y=0$, $\sqrt{x}=0$, $x=0$ // OR $y^3=8$, $y=2$, $\sqrt{x}=2$, $x=4$ //

Question 13 **(***+)**

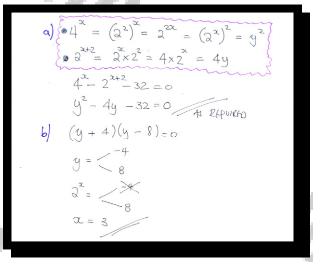
$$4^x - 2^{x+2} = 32 .$$

a) Show that the substitution $y = 2^x$ transforms the above indicial equation into the quadratic equation

$$y^2 - 4y - 32 = 0 .$$

b) Solve the quadratic equation and hence find the root of the **indicial** equation.

$\boxed{x = 3}$



a) • $4^x = (2^2)^x = 2^{2x} = (2^x)^2 = y^2$

• $2^{x+2} = 2^x \times 2^2 = 4 \times 2^x = 4y$

$4^x - 2^{x+2} - 32 = 0$

$y^2 - 4y - 32 = 0$ AS REQUIRED

b) $(y+4)(y-8) = 0$

$y = \begin{cases} -4 \\ 8 \end{cases}$

$2^x = \begin{cases} \cancel{-4} \\ 8 \end{cases}$

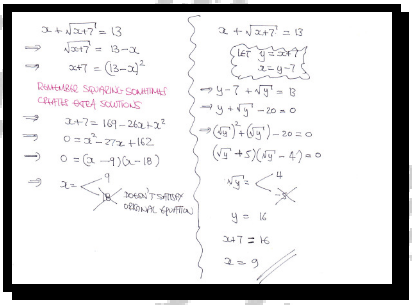
$x = 3$

Question 14 **(***+)**

Find the solution of the equation

$$x + \sqrt{x+7} = 13 , \quad x \geq -7 .$$

$\boxed{x = 9}$



$x + \sqrt{x+7} = 13$

$\Rightarrow \sqrt{x+7} = 13 - x$

$\Rightarrow x + 7 = (13 - x)^2$

REMEMBER SQUARING SOMETIMES CREATES EXTRA SOLUTIONS

$\Rightarrow x + 7 = 169 - 26x + x^2$

$\Rightarrow 0 = x^2 - 27x + 162$

$\Rightarrow 0 = (x - 9)(x - 18)$

$\Rightarrow x = \begin{cases} 9 \\ \cancel{18} \end{cases}$ DOESN'T SATISFY ORIGINAL EQUATION

$x + \sqrt{x+7} = 13$

LET $y = x + 7$

$x = y - 7$

$\Rightarrow y - 7 + \sqrt{y} = 13$

$\Rightarrow y + \sqrt{y} - 20 = 0$

$\Rightarrow (\sqrt{y})^2 + (\sqrt{y}) - 20 = 0$

$(\sqrt{y} + 5)(\sqrt{y} - 4) = 0$

$\sqrt{y} = \begin{cases} 4 \\ \cancel{-5} \end{cases}$

$y = 16$

$x + 7 = 16$

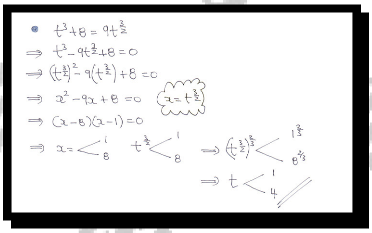
$x = 9$

Question 15 **(***+)**

Solve the equation

$$t^3 + 8 = 9t^{\frac{3}{2}}, \ x \geq 0 .$$

$$\boxed{t = 1,4}$$



- $t^3+8=9t^{\frac{3}{2}}$

$\Rightarrow t^3-9t^{\frac{3}{2}}+8=0$

$\Rightarrow \left(t^{\frac{3}{2}}\right)^2-9\left(t^{\frac{3}{2}}\right)+8=0$

$\Rightarrow x^2-9x+8=0$ $\quad\left(x=t^{\frac{3}{2}}\right)$

$\Rightarrow (x-8)(x-1)=0$

$\Rightarrow x=\begin{cases}1\\8\end{cases} \quad t^{\frac{3}{2}}\begin{cases}1\\8\end{cases} \Rightarrow \left(t^{\frac{3}{2}}\right)^{\frac{2}{3}}\begin{cases}1^{\frac{2}{3}}\\8^{2/3}\end{cases}$

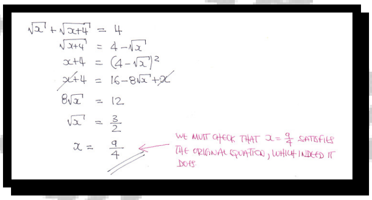
$\Rightarrow t\begin{cases}1\\4\end{cases}$ //

Question 16 **(***+)**

Find the solution of the equation

$$\sqrt{x} + \sqrt{x+4} = 4, \ x \geq 0 .$$

$$\boxed{x = \frac{9}{4}}$$



$\sqrt{x}+\sqrt{x+4}=4$

$\sqrt{x+4}=4-\sqrt{x}$

$x+4=(4-\sqrt{x})^2$

$\cancel{x}+4=16-8\sqrt{x}+\cancel{x}$

$8\sqrt{x}=12$

$\sqrt{x}=\frac{3}{2}$

$x=\frac{9}{4}$ //

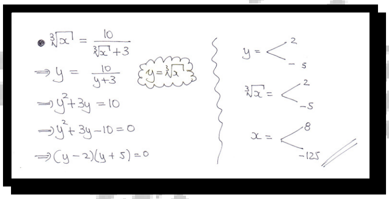
WE MUST CHECK THAT $x=\frac{9}{4}$ SATISFIES THE ORIGINAL EQUATION, WHICH INDEED IT DOES.

Question 17 **(***+)**

Solve the equation

$$\sqrt[3]{x} = \frac{10}{\sqrt[3]{x}+3}, \quad x \neq -\sqrt[3]{x}.$$

$\boxed{x = -125, 8}$



$$\bullet \ \sqrt[3]{x} = \frac{10}{\sqrt[3]{x}+3}$$

$$\Rightarrow y = \frac{10}{y+3} \qquad \boxed{y = \sqrt[3]{x}}$$

$$\Rightarrow y^2 + 3y = 10$$

$$\Rightarrow y^2 + 3y - 10 = 0$$

$$\Rightarrow (y-2)(y+5) = 0$$

$$y = \begin{cases} 2 \\ -5 \end{cases}$$

$$\sqrt[3]{x} = \begin{cases} 2 \\ -5 \end{cases}$$

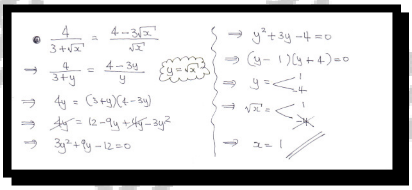
$$x = \begin{cases} 8 \\ -125 \end{cases}$$

Question 18 **(***+)**

Solve the equation

$$\frac{4}{3+\sqrt{x}} = \frac{4-3\sqrt{x}}{\sqrt{x}}, \quad x > 0.$$

$\boxed{x = 1}$



$$\bullet \ \frac{4}{3+\sqrt{x}} = \frac{4-3\sqrt{x}}{\sqrt{x}}$$

$$\Rightarrow \frac{4}{3+y} = \frac{4-3y}{y} \qquad \boxed{y = \sqrt{x}}$$

$$\Rightarrow 4y = (3+y)(4-3y)$$

$$\Rightarrow \cancel{4y} = 12 - 9y + \cancel{4y} - 3y^2$$

$$\Rightarrow 3y^2 + 9y - 12 = 0$$

$$\Rightarrow y^2 + 3y - 4 = 0$$

$$\Rightarrow (y-1)(y+4) = 0$$

$$\Rightarrow y = \begin{cases} 1 \\ -4 \end{cases}$$

$$\Rightarrow \sqrt{x} = \begin{cases} 1 \\ \cancel{-4} \end{cases}$$

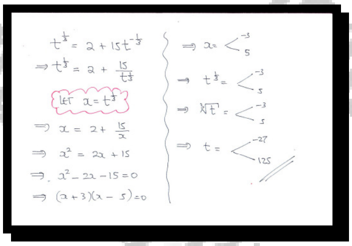
$$\Rightarrow x = 1$$

Question 19 (***+)

$$t^{\frac{1}{3}} = 2 + 15t^{-\frac{1}{3}}, \quad t \neq 0.$$

Use the substitution $x = t^{\frac{1}{3}}$ to solve the above irrational equation.

$\boxed{t = -27, \ \ t = 125}$



$t^{\frac{1}{3}} = 2 + 15t^{-\frac{1}{3}}$

$\Rightarrow t^{\frac{1}{3}} = 2 + \frac{15}{t^{\frac{1}{3}}}$

LET $x = t^{\frac{1}{3}}$

$\Rightarrow x = 2 + \frac{15}{x}$

$\Rightarrow x^2 = 2x + 15$

$\Rightarrow x^2 - 2x - 15 = 0$

$\Rightarrow (x+3)(x-5) = 0$

$\Rightarrow x = \begin{array}{l} -3 \\ 5 \end{array}$

$\Rightarrow t^{\frac{1}{3}} = \begin{array}{l} -3 \\ 5 \end{array}$

$\Rightarrow \sqrt[3]{t} = \begin{array}{l} -3 \\ 5 \end{array}$

$\Rightarrow t = \begin{array}{l} -27 \\ 125 \end{array}$ //

Question 20 (****)

The indicial equation

$$2^{x+1} + 2^{3-x} = 17, \ x \in \mathbb{R},$$

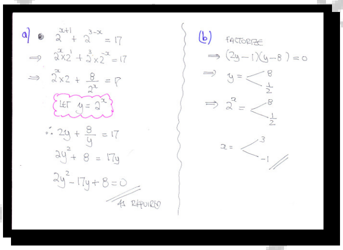
is to be solved by a suitable substitution.

a) Show clearly that the substitution $y = 2^x$ transforms the above indicial equation into the quadratic equation

$$2y^2 - 17y + 8 = 0.$$

b) Solve the quadratic equation by factorization and hence determine the two solutions of the **indicial** equation.

$\boxed{}$, $\boxed{x = -1, \ 3}$



a) • $2^{x+1} + 2^{3-x} = 17$

$\Rightarrow 2^x \times 2^1 + 2^3 \times 2^{-x} = 17$

$\Rightarrow 2^x \times 2 + \frac{8}{2^x} = 17$

LET $y = 2^x$

$\therefore 2y + \frac{8}{y} = 17$

$2y^2 + 8 = 17y$

$2y^2 - 17y + 8 = 0$ //

AS REQUIRED

(b) FACTORIZE

$\Rightarrow (2y-1)(y-8) = 0$

$\Rightarrow y = \begin{array}{l} 8 \\ \frac{1}{2} \end{array}$

$\Rightarrow 2^x = \begin{array}{l} 8 \\ \frac{1}{2} \end{array}$

$x = \begin{array}{l} 3 \\ -1 \end{array}$ //

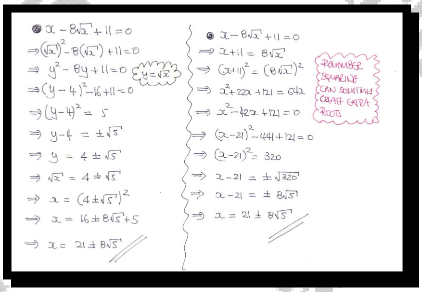
Question 21 **(****)**

Solve the equation

$$x - 8\sqrt{x} + 11 = 0, \; x \geq 0,$$

giving the answers in the form $a + b\sqrt{5}$, where a and b are integers.

$\boxed{x = 21 \pm 8\sqrt{5}}$



$\bullet \; x - 8\sqrt{x} + 11 = 0$

$\Rightarrow (\sqrt{x})^2 - 8(\sqrt{x}) + 11 = 0$

$\Rightarrow y^2 - 8y + 11 = 0$ $y = \sqrt{x}$

$\Rightarrow (y-4)^2 - 16 + 11 = 0$

$\Rightarrow (y-4)^2 = 5$

$\Rightarrow y - 4 = \pm\sqrt{5}$

$\Rightarrow y = 4 \pm \sqrt{5}$

$\Rightarrow \sqrt{x} = 4 \pm \sqrt{5}$

$\Rightarrow x = (4 \pm \sqrt{5})^2$

$\Rightarrow x = 16 \pm 8\sqrt{5} + 5$

$\Rightarrow x = 21 \pm 8\sqrt{5}$ //

$\bullet \; x - 8\sqrt{x} + 11 = 0$

$\Rightarrow x + 11 = 8\sqrt{x}$

$\Rightarrow (x+11)^2 = (8\sqrt{x})^2$

$\Rightarrow x^2 + 22x + 121 = 64x$

$\Rightarrow x^2 - 42x + 121 = 0$

$\Rightarrow (x-21)^2 - 441 + 121 = 0$

$\Rightarrow (x-21)^2 = 320$

$\Rightarrow x - 21 = \pm\sqrt{320}$

$\Rightarrow x - 21 = \pm 8\sqrt{5}$

$\Rightarrow x = 21 \pm 8\sqrt{5}$ //

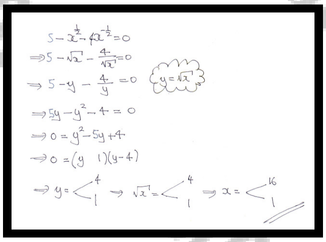
REMEMBER SQUARING CAN SOMETIMES CREATE EXTRA ROOTS

Question 22 **(****)**

Find the two roots of the equation

$$5 - x^{\frac{1}{2}} - 4x^{-\frac{1}{2}} = 0, \; x > 0.$$

$\boxed{x = 1, 16}$



$5 - x^{\frac{1}{2}} - 4x^{-\frac{1}{2}} = 0$

$\Rightarrow 5 - \sqrt{x} - \frac{4}{\sqrt{x}} = 0$

$\Rightarrow 5 - y - \frac{4}{y} = 0$ $y = \sqrt{x}$

$\Rightarrow 5y - y^2 - 4 = 0$

$\Rightarrow 0 = y^2 - 5y + 4$

$\Rightarrow 0 = (y \;\; 1)(y-4)$

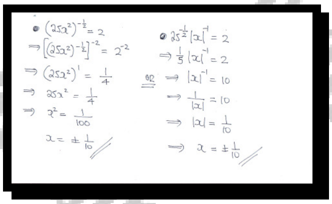
$\Rightarrow y = \left\langle \begin{matrix} 4 \\ 1 \end{matrix} \right. \Rightarrow \sqrt{x} = \left\langle \begin{matrix} 4 \\ 1 \end{matrix} \right. \Rightarrow x = \left\langle \begin{matrix} 16 \\ 1 \end{matrix} \right.$ //

Question 23 **(****)**

Solve the equation

$$\left(25x^2\right)^{-\frac{1}{2}} = 2, \quad x \neq 0.$$

$$\boxed{x = \pm\frac{1}{10}}$$



$\bullet\ (25x^2)^{-\frac{1}{2}} = 2$

$\Rightarrow \left[(25x^2)^{-\frac{1}{2}}\right]^{-2} = 2^{-2}$

$\Rightarrow (25x^2)^1 = \frac{1}{4}$

$\Rightarrow 25x^2 = \frac{1}{4}$

$\Rightarrow x^2 = \frac{1}{100}$

$x = \pm\frac{1}{10}$ //

OR

$\bullet\ 25^{-\frac{1}{2}}|x|^{-1} = 2$

$\Rightarrow \frac{1}{5}|x|^{-1} = 2$

$\Rightarrow |x|^{-1} = 10$

$\Rightarrow \frac{1}{|x|} = 10$

$\Rightarrow |x| = \frac{1}{10}$

$\Rightarrow x = \pm\frac{1}{10}$ //

Question 24 **(****)**

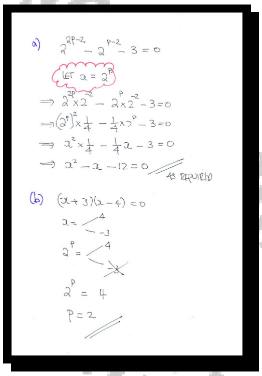
$$2^{2p-2} - 2^{p-2} - 3 = 0, \; p \in \mathbb{R},$$

a) Show clearly that the substitution $x = 2^p$ transforms the above indicial equation into the quadratic equation

$$x^2 - x - 12 = 0.$$

b) Solve the quadratic equation and hence determine the value of p.

$$\boxed{p = 2}$$



a) $2^{2p-2} - 2^{p-2} - 3 = 0$

LET $x = 2^p$

$\Rightarrow 2^{2p} \times 2^{-2} - 2^p \times 2^{-2} - 3 = 0$

$\Rightarrow (2^p)^2 \times \frac{1}{4} - \frac{1}{4} \times 2^p - 3 = 0$

$\Rightarrow x^2 \times \frac{1}{4} - \frac{1}{4}x - 3 = 0$

$\Rightarrow x^2 - x - 12 = 0$ // AS REQUIRED

(b) $(x+3)(x-4) = 0$

$x = 4, -3$

$2^p = 4, ~~-3~~$

$2^p = 4$

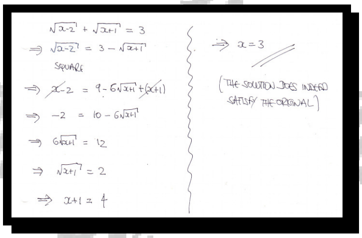
$p = 2$ //

Question 25 **(****)**

Solve the equation

$$\sqrt{x-2}+\sqrt{x+1}=3\,,\ \ x\ge 2\,.$$

$$\boxed{x=3}$$



$\sqrt{x-2}+\sqrt{x+1}=3$

$\Rightarrow \sqrt{x-2}=3-\sqrt{x+1}$

SQUARE

$\Rightarrow \cancel{x}-2 = 9-6\sqrt{x+1}+(\cancel{x}+1)$

$\Rightarrow -2 = 10-6\sqrt{x+1}$

$\Rightarrow 6\sqrt{x+1}=12$

$\Rightarrow \sqrt{x+1}=2$

$\Rightarrow x+1=4$

$\Rightarrow x=3$

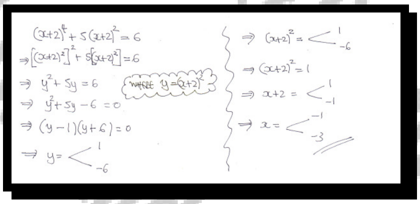
(THE SOLUTION DOES INDEED SATISFY THE ORIGINAL)

Question 26 **(****)**

Determine the two real roots of the equation

$$(x+2)^4+5(x+2)^2=6\,.$$

$$\boxed{x=-1,-3}$$



$(x+2)^4+5(x+2)^2=6$

$\Rightarrow \left[(x+2)^2\right]^2+5\left[(x+2)^2\right]=6$

$\Rightarrow y^2+5y=6$ WHERE $y=(x+2)^2$

$\Rightarrow y^2+5y-6=0$

$\Rightarrow (y-1)(y+6)=0$

$\Rightarrow y = < \begin{array}{l}1\\-6\end{array}$

$\Rightarrow (x+2)^2 = < \begin{array}{l}1\\-6\end{array}$

$\Rightarrow (x+2)^2=1$

$\Rightarrow x+2 = < \begin{array}{l}1\\-1\end{array}$

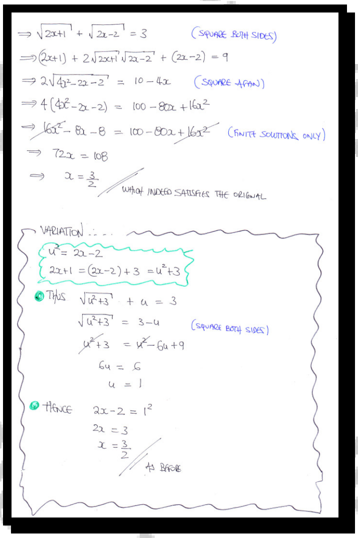
$\Rightarrow x = < \begin{array}{l}-1\\-3\end{array}$

Question 27 (****)

Solve the equation

$$\sqrt{2x+1}+\sqrt{2x-2}=3$$

$$\boxed{x=\frac{3}{2}}$$



$\Rightarrow \sqrt{2x+1}+\sqrt{2x-2}=3$ (SQUARE BOTH SIDES)

$\Rightarrow (2x+1)+2\sqrt{2x+1}\sqrt{2x-2}+(2x-2)=9$

$\Rightarrow 2\sqrt{4x^2-2x-2}=10-4x$ (SQUARE AGAIN)

$\Rightarrow 4(4x^2-2x-2)=100-80x+16x^2$

$\Rightarrow \cancel{16x^2}-8x-8=100-80x+\cancel{16x^2}$ (FINITE SOLUTIONS ONLY)

$\Rightarrow 72x=108$

$\Rightarrow x=\frac{3}{2}$ // WHICH INDEED SATISFIES THE ORIGINAL

VARIATION ....

$u^2=2x-2$

$2x+1=(2x-2)+3=u^2+3$

THUS $\sqrt{u^2+3}+u=3$

$\sqrt{u^2+3}=3-u$ (SQUARE BOTH SIDES)

$\cancel{u^2}+3=\cancel{u^2}-6u+9$

$6u=6$

$u=1$

HENCE $2x-2=1^2$

$2x=3$

$x=\frac{3}{2}$ // AS BEFORE

Question 28 (****)

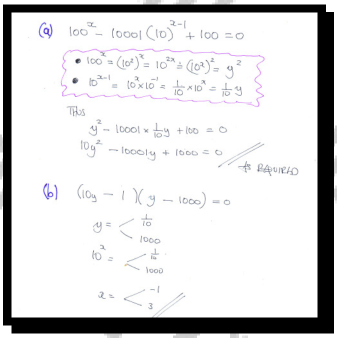
$$100^x - 10001\left(10^{x-1}\right) + 100 = 0.$$

a) Show that the substitution $y = 2^x$ transforms the above indicial equation into the quadratic equation

$$10y^2 - 10001y + 1000 = 0.$$

b) Solve the quadratic equation and hence find the two solutions of the **indicial** equation.

$$\boxed{x = -1, \quad x = 3}$$



(a) $100^x - 10001(10)^{x-1} + 100 = 0$

- $100^x = (10^2)^x = 10^{2x} = (10^2)^2 = y^2$
- $10^{x-1} = 10^x \times 10^{-1} = \frac{1}{10} \times 10^x = \frac{1}{10}y$

Thus

$y^2 - 10001 \times \frac{1}{10}y + 100 = 0$

$10y^2 - 10001y + 1000 = 0$ // AS REQUIRED

(b) $(10y - 1)(y - 1000) = 0$

$y = \frac{1}{10}$, 1000

$10^x = \frac{1}{10}$, 1000

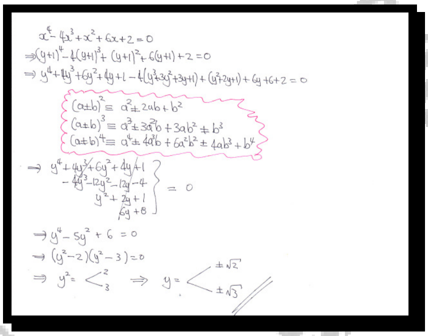
$x = -1$, 3 //

Question 29 **(****)**

By using the substitution $x = y+1$, or otherwise, find in exact surd form the roots of the equation

$$x^4 - 4x^3 + x^2 + 6x + 2 = 0 .$$

$$\boxed{x = \pm\sqrt{2}, \pm\sqrt{3}}$$



$$x^4 - 4x^3 + x^2 + 6x + 2 = 0$$
$$\Rightarrow (y+1)^4 - 4(y+1)^3 + (y+1)^2 + 6(y+1) + 2 = 0$$
$$\Rightarrow y^4 + 4y^3 + 6y^2 + 4y + 1 - 4(y^3 + 3y^2 + 3y + 1) + (y^2 + 2y + 1) + 6y + 6 + 2 = 0$$

$$(a \pm b)^2 \equiv a^2 \pm 2ab + b^2$$
$$(a \pm b)^3 \equiv a^3 \pm 3a^2b + 3ab^2 \pm b^3$$
$$(a \pm b)^4 \equiv a^4 \pm 4a^3b + 6a^2b^2 \pm 4ab^3 + b^4$$

$$\Rightarrow \left.\begin{array}{r} y^4 + \cancel{4y^3} + 6y^2 + \cancel{4y} + 1 \\ -\cancel{4y^3} - 12y^2 - \cancel{12y} - 4 \\ y^2 + \cancel{2y} + 1 \\ \cancel{6y} + 8 \end{array}\right\} = 0$$

$$\Rightarrow y^4 - 5y^2 + 6 = 0$$
$$\Rightarrow (y^2 - 2)(y^2 - 3) = 0$$
$$\Rightarrow y^2 = \left\langle \begin{array}{l} 2 \\ 3 \end{array}\right. \quad \Rightarrow \quad y = \left\langle \begin{array}{l} \pm\sqrt{2} \\ \pm\sqrt{3} \end{array}\right. //$$

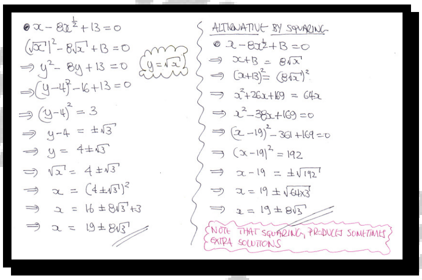
Question 30 **(****)**

Solve the equation

$$x - 8x^{\frac{1}{2}} + 13 = 0, \ x \geq 0,$$

giving the answers in the form $a + b\sqrt{3}$, where a and b are integers.

$$\boxed{x = 19 \pm 8\sqrt{3}}$$



- $x - 8x^{\frac{1}{2}} + 13 = 0$

$$(\sqrt{x})^2 - 8\sqrt{x} + 13 = 0 \qquad y = \sqrt{x}$$
$$\Rightarrow y^2 - 8y + 13 = 0$$
$$\Rightarrow (y-4)^2 - 16 + 13 = 0$$
$$\Rightarrow (y-4)^2 = 3$$
$$\Rightarrow y - 4 = \pm\sqrt{3}$$
$$\Rightarrow y = 4 \pm \sqrt{3}$$
$$\Rightarrow \sqrt{x} = 4 \pm \sqrt{3}$$
$$\Rightarrow x = (4 \pm \sqrt{3})^2$$
$$\Rightarrow x = 16 \pm 8\sqrt{3} + 3$$
$$\Rightarrow x = 19 \pm 8\sqrt{3} //$$

ALTERNATIVE BY SQUARING

- $x - 8x^{\frac{1}{2}} + 13 = 0$

$$\Rightarrow x + 13 = 8\sqrt{x}$$
$$\Rightarrow (x+13)^2 = (8\sqrt{x})^2$$
$$\Rightarrow x^2 + 26x + 169 = 64x$$
$$\Rightarrow x^2 - 38x + 169 = 0$$
$$\Rightarrow (x-19)^2 - 361 + 169 = 0$$
$$\Rightarrow (x-19)^2 = 192$$
$$\Rightarrow x - 19 = \pm\sqrt{192}$$
$$\Rightarrow x = 19 \pm \sqrt{64 \times 3}$$
$$\Rightarrow x = 19 \pm 8\sqrt{3} //$$

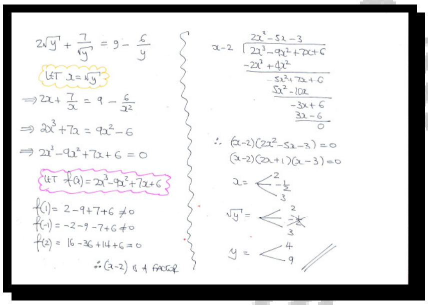
NOTE THAT SQUARING, PRODUCES SOMETIMES EXTRA SOLUTIONS

Question 31 (****)

Find the two real solutions of the equation

$$2\sqrt{y}+\frac{7}{\sqrt{y}}=9-\frac{6}{y},\ \ y>0$$

$\boxed{y=4,\ \ y=9}$



$2\sqrt{y}+\frac{7}{\sqrt{y}}=9-\frac{6}{y}$

LET $x=\sqrt{y}$

$\Rightarrow 2x+\frac{7}{x}=9-\frac{6}{x^2}$

$\Rightarrow 2x^3+7x=9x^2-6$

$\Rightarrow 2x^3-9x^2+7x+6=0$

LET $f(x)=2x^3-9x^2+7x+6$

$f(1)=2-9+7+6\neq 0$

$f(-1)=-2-9-7+6\neq 0$

$f(2)=16-36+14+6=0$

$\therefore (x-2)$ IS A FACTOR

$$\begin{array}{r|l} & 2x^2-5x-3 \\ x-2 & 2x^3-9x^2+7x+6 \\ & -2x^3+4x^2 \\ & -5x^2+7x+6 \\ & 5x^2-10x \\ & -3x+6 \\ & 3x-6 \\ & 0 \end{array}$$

$\therefore (x-2)(2x^2-5x-3)=0$

$(x-2)(2x+1)(x-3)=0$

$x=\begin{cases}2\\-\frac{1}{2}\\3\end{cases}$

$\sqrt{y}=\begin{cases}2\\\cancel{-\frac{1}{2}}\\3\end{cases}$

$y=\begin{cases}4\\9\end{cases}$ //

Question 32 (****)

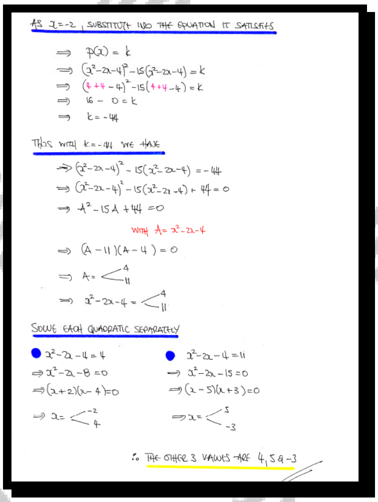
A polynomial $p(x)$ is defined as

$$p(x) \equiv \left(x^2-2x-4\right)^2 - 15\left(x^2-2x-4\right), \quad x \in \mathbb{R}.$$

The equation $p(x)=k$, where k is a constant, is satisfied by $x=-2$.

Determine the other three values of x that satisfy the equation $p(x)=k$.

$\boxed{}$, $\boxed{x=-3 \cup x=4 \cup x=5}$



AS $x=-2$, SUBSTITUTE INTO THE EQUATION IT SATISFIES

$\Rightarrow p(x)=k$

$\Rightarrow (x^2-2x-4)^2-15(x^2-2x-4)=k$

$\Rightarrow (4+4-4)^2-15(4+4-4)=k$

$\Rightarrow 16-0=k$

$\Rightarrow k=-44$

THUS WITH $k=-44$ WE HAVE

$\Rightarrow (x^2-2x-4)^2-15(x^2-2x-4)=-44$

$\Rightarrow (x^2-2x-4)^2-15(x^2-2x-4)+44=0$

$\Rightarrow A^2-15A+44=0$

WITH $A=x^2-2x-4$

$\Rightarrow (A-11)(A-4)=0$

$\Rightarrow A = \begin{cases} 4 \\ 11 \end{cases}$

$\Rightarrow x^2-2x-4 = \begin{cases} 4 \\ 11 \end{cases}$

SOLVE EACH QUADRATIC SEPARATELY

- $x^2-2x-4=4$
  $\Rightarrow x^2-2x-8=0$
  $\Rightarrow (x+2)(x-4)=0$
  $\Rightarrow x = \begin{cases} -2 \\ 4 \end{cases}$

- $x^2-2x-4=11$
  $\Rightarrow x^2-2x-15=0$
  $\Rightarrow (x-5)(x+3)=0$
  $\Rightarrow x = \begin{cases} 5 \\ -3 \end{cases}$

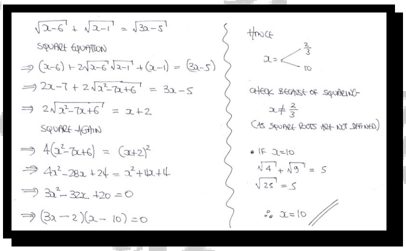
$\therefore$ THE OTHER 3 VALUES ARE $4, 5$ & -3

Question 33 (****+)

Determine the real root of the equation

$$\sqrt{x-6}+\sqrt{x-1}=\sqrt{3x-5}\,.$$

$\boxed{x=10}$



$\sqrt{x-6}+\sqrt{x-1}=\sqrt{3x-5}$

SQUARE EQUATION

$\Rightarrow (x-6)+2\sqrt{x-6}\sqrt{x-1}+(x-1)=(3x-5)$

$\Rightarrow 2x-7+2\sqrt{x^2-7x+6}=3x-5$

$\Rightarrow 2\sqrt{x^2-7x+6}=x+2$

SQUARE AGAIN

$\Rightarrow 4(x^2-7x+6)=(x+2)^2$

$\Rightarrow 4x^2-28x+24=x^2+4x+4$

$\Rightarrow 3x^2-32x+20=0$

$\Rightarrow (3x-2)(x-10)=0$

HENCE

$x=\begin{cases}\frac{2}{3}\\10\end{cases}$

CHECK BECAUSE OF SQUARING

$x\neq\frac{2}{3}$

(AS SQUARE ROOTS ARE NOT DEFINED)

- IF $x=10$

$\sqrt{4}+\sqrt{9}=5$

$\sqrt{25}=5$

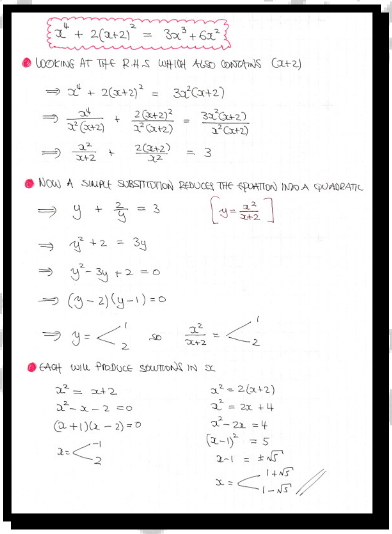
$\therefore x=10$ //

Question 34 (****)

Determine, in exact form where appropriate, the solutions of the following equation.

$$x^4+2(x+2)^2=3x^3+6x^2\,.$$

$\boxed{\quad}$, $\boxed{x=-1}$, $\boxed{x=2}$, $\boxed{x=1+\sqrt{5}}$, $\boxed{x=1-\sqrt{5}}$



$\boxed{x^4+2(x+2)^2=3x^3+6x^2}$

- LOOKING AT THE R.H.S WHICH ALSO CONTAINS $(x+2)$

$\Rightarrow x^4+2(x+2)^2=3x^2(x+2)$

$\Rightarrow \frac{x^4}{x^2(x+2)}+\frac{2(x+2)^2}{x^2(x+2)}=\frac{3x^2(x+2)}{x^2(x+2)}$

$\Rightarrow \frac{x^2}{x+2}+\frac{2(x+2)}{x^2}=3$

- NOW A SIMPLE SUBSTITUTION REDUCES THE EQUATION INTO A QUADRATIC

$\Rightarrow y+\frac{2}{y}=3$ $\left[y=\frac{x^2}{x+2}\right]$

$\Rightarrow y^2+2=3y$

$\Rightarrow y^2-3y+2=0$

$\Rightarrow (y-2)(y-1)=0$

$\Rightarrow y=\begin{cases}1\\2\end{cases}$ so $\frac{x^2}{x+2}=\begin{cases}1\\2\end{cases}$

- EACH WILL PRODUCE SOLUTIONS IN x

$x^2=x+2$

$x^2-x-2=0$

$(x+1)(x-2)=0$

$x=\begin{cases}-1\\2\end{cases}$

$x^2=2(x+2)$

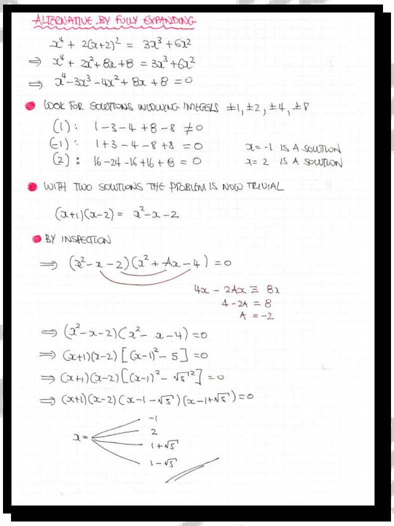
$x^2=2x+4$

$x^2-2x=4$

$(x-1)^2=5$

$x-1=\pm\sqrt{5}$

$x=\begin{cases}1+\sqrt{5}\\1-\sqrt{5}\end{cases}$ //



ALTERNATIVE BY FULLY EXPANDING

$x^4+2(x+2)^2=3x^3+6x^2$

$\Rightarrow x^4+2x^2+8x+8=3x^3+6x^2$

$\Rightarrow x^4-3x^3-4x^2+8x+8=0$

- LOOK FOR SOLUTIONS INVOLVING INTEGERS $\pm1,\pm2,\pm4,\pm8$

$(1):\ 1-3-4+8-8\neq0$

$(-1):\ 1+3-4-8+8=0$ $x=-1$ IS A SOLUTION

$(2):\ 16-24-16+16+8=0$ $x=2$ IS A SOLUTION

- WITH TWO SOLUTIONS THE PROBLEM IS NOW TRIVIAL

$(x+1)(x-2)=x^2-x-2$

- BY INSPECTION

$\Rightarrow (x^2-x-2)(x^2+Ax-4)=0$

$4x-2Ax\equiv8x$

$4-2A=8$

$A=-2$

$\Rightarrow (x^2-x-2)(x^2-x-4)=0$

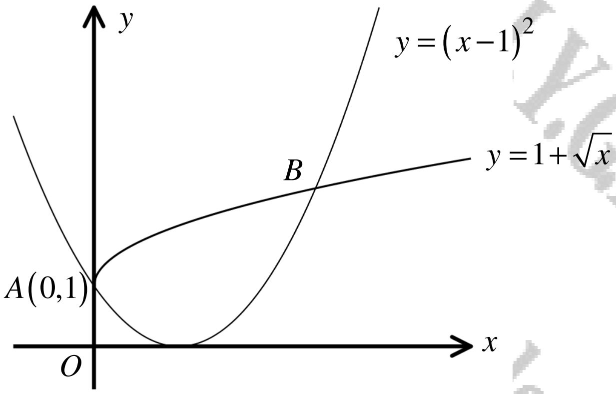
$\Rightarrow (x+1)(x-2)\left[(x-1)^2-5\right]=0$

$\Rightarrow (x+1)(x-2)\left[(x-1)^2-\sqrt{5}^2\right]=0$

$\Rightarrow (x+1)(x-2)(x-1-\sqrt{5})(x-1+\sqrt{5})=0$

$x=\begin{cases}-1\\2\\1+\sqrt{5}\\1-\sqrt{5}\end{cases}$ //

Question 35 (****+)



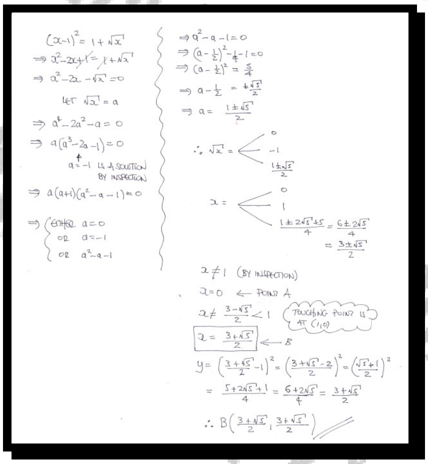
The figure above shows the graphs of the curves with equations

$$y=(x-1)^2 \quad \text{and} \quad y=1+\sqrt{x}.$$

The curves meet at the point $A(0,1)$ and at the point B.

Determine the exact coordinates of Q.

$$\boxed{B\left(\frac{3+\sqrt{5}}{2},\frac{3+\sqrt{5}}{2}\right)}$$



$(x-1)^2 = 1+\sqrt{x}$

$\Rightarrow x^2-2x+\cancel{1} = \cancel{1}+\sqrt{x}$

$\Rightarrow x^2-2x-\sqrt{x}=0$

LET $\sqrt{x}=a$

$\Rightarrow a^4-2a^2-a=0$

$\Rightarrow a(a^3-2a-1)=0$

$a=-1$ IS A SOLUTION BY INSPECTION

$\Rightarrow a(a+1)(a^2-a-1)=0$

$\Rightarrow \begin{cases} \text{EITHER } a=0 \\ \text{OR } a=-1 \\ \text{OR } a^2-a-1 \end{cases}$

$\Rightarrow a^2-a-1=0$

$\Rightarrow (a-\frac{1}{2})^2-\frac{1}{4}-1=0$

$\Rightarrow (a-\frac{1}{2})^2=\frac{5}{4}$

$\Rightarrow a-\frac{1}{2}=\pm\frac{\sqrt{5}}{2}$

$\Rightarrow a=\frac{1\pm\sqrt{5}}{2}$

$\therefore \sqrt{x} = 0,\ -1,\ \frac{1\pm\sqrt{5}}{2}$

$x = 0,\ 1,\ \frac{1\pm2\sqrt{5}+5}{4}=\frac{6\pm2\sqrt{5}}{4}=\frac{3\pm\sqrt{5}}{2}$

$x\neq1$ (BY INSPECTION)

$x=0 \leftarrow$ POINT A

$x\neq\frac{3-\sqrt{5}}{2}<1$ TOUCHING POINT IS AT $(1,0)$

$\boxed{x=\frac{3+\sqrt{5}}{2}} \leftarrow B$

$y=\left(\frac{3+\sqrt{5}}{2}-1\right)^2=\left(\frac{3+\sqrt{5}-2}{2}\right)^2=\left(\frac{\sqrt{5}+1}{2}\right)^2$

$=\frac{5+2\sqrt{5}+1}{4}=\frac{6+2\sqrt{5}}{4}=\frac{3+\sqrt{5}}{2}$

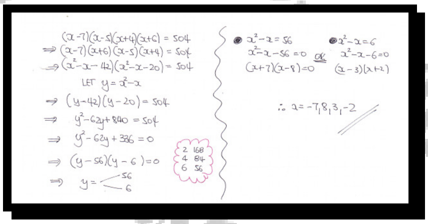
$\therefore B\left(\frac{3+\sqrt{5}}{2},\frac{3+\sqrt{5}}{2}\right)$

Question 36 **(****+)**

By using the substitution $y = x^2 - x$, or otherwise, find the roots of the equation

$$(x-7)(x-5)(x+4)(x+6) = 504.$$

$$\boxed{x = -7, -2, 3, 8}$$



$(x-7)(x-5)(x+4)(x+6) = 504$

$\Rightarrow (x-7)(x+6)(x-5)(x+4) = 504$

$\Rightarrow (x^2-x-42)(x^2-x-20) = 504$

LET $y = x^2 - x$

$\Rightarrow (y-42)(y-20) = 504$

$\Rightarrow y^2 - 62y + 840 = 504$

$\Rightarrow y^2 - 62y + 336 = 0$

$\Rightarrow (y-56)(y-6) = 0$

$\Rightarrow y = \begin{cases} 56 \\ 6 \end{cases}$

2 168
4 84
6 56

● $x^2 - x = 56$
$x^2 - x - 56 = 0$
$(x+7)(x-8) = 0$

or

● $x^2 - x = 6$
$x^2 - x - 6 = 0$
$(x-3)(x+2)$

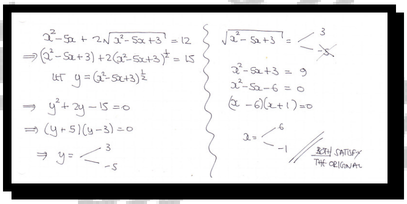
$\therefore x = -7, 8, 3, -2$ //

Question 37 **(****+)**

Determine the two real roots of the equation

$$x^2 - 5x + 2\sqrt{x^2 - 5x + 3} = 12.$$

$$\boxed{x = -1, 6}$$



$x^2 - 5x + 2\sqrt{x^2-5x+3} = 12$

$\Rightarrow (x^2-5x+3) + 2(x^2-5x+3)^{\frac{1}{2}} = 15$

LET $y = (x^2-5x+3)^{\frac{1}{2}}$

$\Rightarrow y^2 + 2y - 15 = 0$

$\Rightarrow (y+5)(y-3) = 0$

$\Rightarrow y = \begin{cases} 3 \\ -5 \end{cases}$

$\sqrt{x^2-5x+3} = \begin{cases} 3 \\ \cancel{-5} \end{cases}$

$x^2 - 5x + 3 = 9$

$x^2 - 5x - 6 = 0$

$(x-6)(x+1) = 0$

$x = \begin{cases} 6 \\ -1 \end{cases}$ //

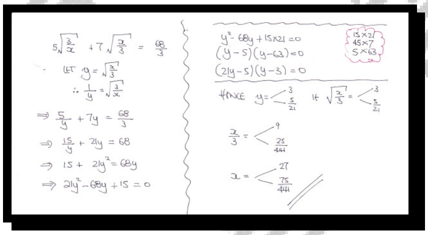
BOTH SATISFY THE ORIGINAL

Question 38 (****+)

By using a suitable substitution, or otherwise, find as exact fractions where appropriate, the solutions of the equation

$$5\sqrt{\frac{3}{x}}+7\sqrt{\frac{x}{3}}=\frac{68}{3}, \ x>0 .$$

$$\boxed{x=27,\frac{75}{441}}$$



$$5\sqrt{\frac{3}{x}}+7\sqrt{\frac{x}{3}}=\frac{68}{3}$$

LET $y=\sqrt{\frac{x}{3}}$

$\therefore \frac{1}{y}=\sqrt{\frac{3}{x}}$

$\Rightarrow \frac{5}{y}+7y=\frac{68}{3}$

$\Rightarrow \frac{15}{y}+21y=68$

$\Rightarrow 15+21y^2=68y$

$\Rightarrow 21y^2-68y+15=0$

$y^2-68y+15\times21=0$

$(y-5)(y-63)=0$

$(21y-5)(y-3)=0$

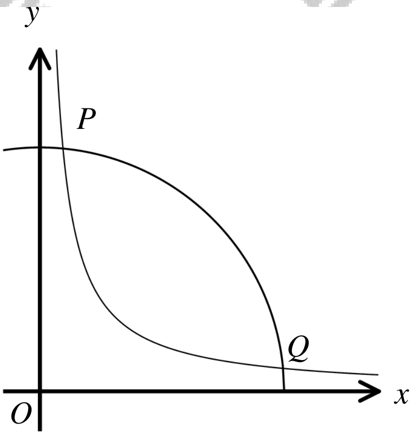
15×21
45×7
5×63

HENCE $y=\left\langle\begin{matrix}3\\ \frac{5}{21}\end{matrix}\right.$ IE $\sqrt{\frac{x}{3}}=\left\langle\begin{matrix}3\\ \frac{5}{21}\end{matrix}\right.$

$\frac{x}{3}=\left\langle\begin{matrix}9\\ \frac{25}{441}\end{matrix}\right.$

$x=\left\langle\begin{matrix}27\\ \frac{75}{441}\end{matrix}\right.$

Question 39 (**+)**



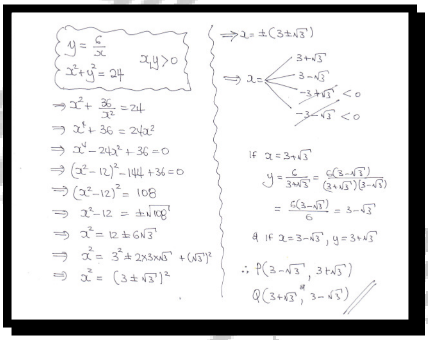
The figure above shows a rectangular hyperbola and a circle with respective Cartesian equations

$$y=\frac{6}{x},\ x>0 \qquad \text{and} \qquad x^2+y^2=8,\ x>0,\ y>0.$$

The points P and Q are the points of intersection between the rectangular hyperbola and the circle.

Find the coordinates of P and Q, in the form $\left(a+\sqrt{a},b+\sqrt{b}\right)$

$$\boxed{P\left(3-\sqrt{3},3+\sqrt{3}\right)},\ \boxed{Q\left(3+\sqrt{3},3-\sqrt{3}\right)}$$



$$\boxed{\begin{array}{l} y=\frac{6}{x} \\ x^2+y^2=24 \end{array} \quad x,y>0}$$

$\Rightarrow x^2+\frac{36}{x^2}=24$

$\Rightarrow x^4+36=24x^2$

$\Rightarrow x^4-24x^2+36=0$

$\Rightarrow (x^2-12)^2-144+36=0$

$\Rightarrow (x^2-12)^2=108$

$\Rightarrow x^2-12=\pm\sqrt{108}$

$\Rightarrow x^2=12\pm6\sqrt{3}$

$\Rightarrow x^2=3^2\pm2\times3\times\sqrt{3}+(\sqrt{3})^2$

$\Rightarrow x^2=(3\pm\sqrt{3})^2$

$\Rightarrow x=\pm(3\pm\sqrt{3})$

$\Rightarrow x = 3+\sqrt{3},\ 3-\sqrt{3},\ \cancel{-3+\sqrt{3}}<0,\ \cancel{-3-\sqrt{3}}<0$

If $x=3+\sqrt{3}$

$y=\frac{6}{3+\sqrt{3}}=\frac{6(3-\sqrt{3})}{(3+\sqrt{3})(3-\sqrt{3})}$

$=\frac{6(3-\sqrt{3})}{6}=3-\sqrt{3}$

& if $x=3-\sqrt{3}$, $y=3+\sqrt{3}$

$\therefore P(3-\sqrt{3},\ 3+\sqrt{3})$

$Q(3+\sqrt{3},\ 3-\sqrt{3})$ //

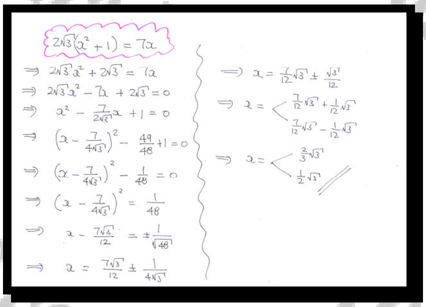
Question 40 **(****+)**

Find the solutions of the quadratic equation

$$2\sqrt{3}\left(x^2+1\right)=7x\,.$$

Give the answers in the form $k\sqrt{3}$, where k is a constant.

$$\boxed{x=\frac{2}{3}\sqrt{3},\ \ x=\frac{1}{2}\sqrt{3}}$$



$\boxed{2\sqrt{3}(x^2+1)=7x}$

$\Rightarrow 2\sqrt{3}x^2+2\sqrt{3}=7x$

$\Rightarrow 2\sqrt{3}x^2-7x+2\sqrt{3}=0$

$\Rightarrow x^2-\frac{7}{2\sqrt{3}}x+1=0$

$\Rightarrow \left(x-\frac{7}{4\sqrt{3}}\right)^2-\frac{49}{48}+1=0$

$\Rightarrow \left(x-\frac{7}{4\sqrt{3}}\right)^2-\frac{1}{48}=0$

$\Rightarrow \left(x-\frac{7}{4\sqrt{3}}\right)^2=\frac{1}{48}$

$\Rightarrow x-\frac{7\sqrt{3}}{12}=\pm\frac{1}{\sqrt{48}}$

$\Rightarrow x=\frac{7\sqrt{3}}{12}\pm\frac{1}{4\sqrt{3}}$

$\Rightarrow x=\frac{7}{12}\sqrt{3}\pm\frac{\sqrt{3}}{12}$

$\Rightarrow x=\left\langle\begin{matrix}\frac{7}{12}\sqrt{3}+\frac{1}{12}\sqrt{3}\\ \frac{7}{12}\sqrt{3}-\frac{1}{12}\sqrt{3}\end{matrix}\right.$

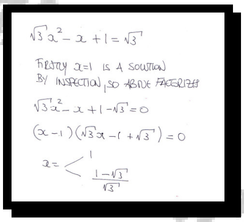
$\Rightarrow x=\left\langle\begin{matrix}\frac{2}{3}\sqrt{3}\\ \frac{1}{2}\sqrt{3}\end{matrix}\right.$ //

Question 41 **(****+)**

Find in exact simplified form where appropriate the solutions of the equation

$$\sqrt{3}x^2-x+1=\sqrt{3}\,.$$

$$\boxed{x=1,\ x=\frac{1-\sqrt{3}}{\sqrt{3}}}$$



$\sqrt{3}x^2-x+1=\sqrt{3}$

FIRSTLY $x=1$ IS A SOLUTION BY INSPECTION, SO ABOVE FACTORIZES

$\sqrt{3}x^2-x+1-\sqrt{3}=0$

$(x-1)(\sqrt{3}x-1+\sqrt{3})=0$

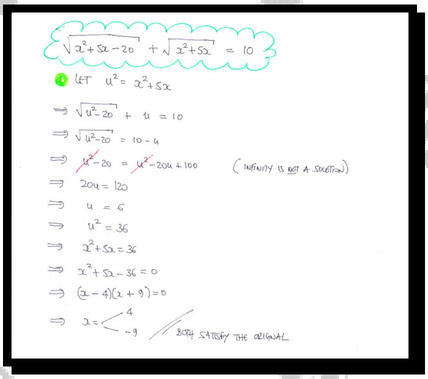
$x=\left\langle\begin{matrix}1\\ \frac{1-\sqrt{3}}{\sqrt{3}}\end{matrix}\right.$

Question 42 (****+)

Solve the equation

$$\sqrt{x^2+5x-20}+\sqrt{x^2+5x}=10$$

$$\boxed{x=4,\ \ x=-9}$$



$$\sqrt{x^2+5x-20}+\sqrt{x^2+5x}=10$$

LET $u^2 = x^2+5x$

$\Rightarrow \sqrt{u^2-20}+u=10$

$\Rightarrow \sqrt{u^2-20}=10-u$

$\Rightarrow \cancel{u^2}-20=\cancel{u^2}-20u+100$ (INFINITY IS NOT A SOLUTION)

$\Rightarrow 20u=120$

$\Rightarrow u=6$

$\Rightarrow u^2=36$

$\Rightarrow x^2+5x=36$

$\Rightarrow x^2+5x-36=0$

$\Rightarrow (x-4)(x+9)=0$

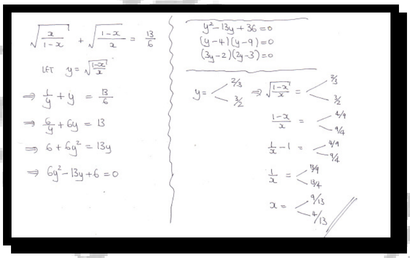
$\Rightarrow x = \left\langle \begin{matrix} 4 \\ -9 \end{matrix} \right.$ // BOTH SATISFY THE ORIGNAL

Question 43 (****+)

By using a suitable substitution, or otherwise, find as exact fractions the solutions of the equation

$$\sqrt{\frac{x}{1-x}}+\sqrt{\frac{1-x}{x}}=\frac{13}{6},\ \ x>0\,.$$

$$\boxed{x=\frac{9}{13},\frac{4}{13}}$$



$\sqrt{\frac{x}{1-x}}+\sqrt{\frac{1-x}{x}}=\frac{13}{6}$

LET $y=\sqrt{\frac{1-x}{x}}$

$\Rightarrow \frac{1}{y}+y=\frac{13}{6}$

$\Rightarrow \frac{6}{y}+6y=13$

$\Rightarrow 6+6y^2=13y$

$\Rightarrow 6y^2-13y+6=0$

$y^2-13y+36=0$

$(y-4)(y-9)=0$

$(3y-2)(2y-3)=0$

$y = \left\langle \begin{matrix} 2/3 \\ 3/2 \end{matrix} \right. \Rightarrow \sqrt{\frac{1-x}{x}} = \left\langle \begin{matrix} 2/3 \\ 3/2 \end{matrix} \right.$

$\frac{1-x}{x} = \left\langle \begin{matrix} 4/9 \\ 9/4 \end{matrix} \right.$

$\frac{1}{x}-1 = \left\langle \begin{matrix} 4/9 \\ 9/4 \end{matrix} \right.$

$\frac{1}{x} = \left\langle \begin{matrix} 13/9 \\ 13/4 \end{matrix} \right.$

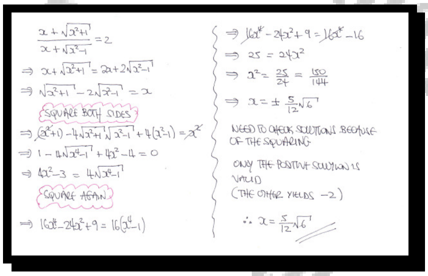
$x = \left\langle \begin{matrix} 9/13 \\ 4/13 \end{matrix} \right.$ //

Question 44 **(****+)**

Determine, in exact surd form, the solution of the equation

$$\frac{x+\sqrt{x^2+1}}{x+\sqrt{x^2-1}} = 2 .$$

$$\boxed{x = \frac{5}{12}\sqrt{6}}$$



$$\frac{x+\sqrt{x^2+1}}{x+\sqrt{x^2-1}} = 2$$

$$\Rightarrow x+\sqrt{x^2+1} = 2x+2\sqrt{x^2-1}$$

$$\Rightarrow \sqrt{x^2+1} - 2\sqrt{x^2-1} = x$$

SQUARE BOTH SIDES

$$\Rightarrow (\cancel{x^2}+1) - 4\sqrt{x^2+1}\sqrt{x^2-1} + 4(x^2-1) = \cancel{x^2}$$

$$\Rightarrow 1 - 4\sqrt{x^4-1} + 4x^2 - 4 = 0$$

$$\Rightarrow 4x^2 - 3 = 4\sqrt{x^4-1}$$

SQUARE AGAIN

$$\Rightarrow 16x^4 - 24x^2 + 9 = 16(x^4-1)$$

$$\Rightarrow \cancel{16x^4} - 24x^2 + 9 = \cancel{16x^4} - 16$$

$$\Rightarrow 25 = 24x^2$$

$$\Rightarrow x^2 = \frac{25}{24} = \frac{150}{144}$$

$$\Rightarrow x = \pm\frac{5}{12}\sqrt{6}$$

NEED TO CHECK SOLUTIONS BECAUSE OF THE SQUARING

ONLY THE POSITIVE SOLUTION IS VALID
(THE OTHER YIELDS −2)

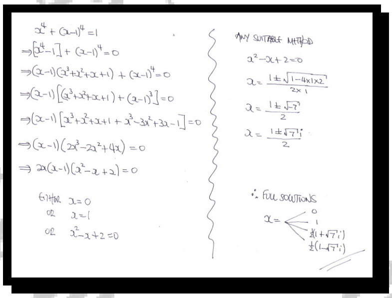
$$\therefore x = \frac{5}{12}\sqrt{6}$$

Question 45 **(****+)**

$$x^4 + (x-1)^4 = 1, \ x \in \mathbb{C} .$$

Determine, in exact form where appropriate, the four roots of the above equation.

$$\boxed{x = 0,\ 1,\ \frac{1}{2}\left(1 \pm \mathrm{i}\sqrt{7}\right)}$$



$$x^4 + (x-1)^4 = 1$$

$$\Rightarrow [x^4-1] + (x-1)^4 = 0$$

$$\Rightarrow (x-1)(x^3+x^2+x+1) + (x-1)^4 = 0$$

$$\Rightarrow (x-1)\left[(x^3+x^2+x+1) + (x-1)^3\right] = 0$$

$$\Rightarrow (x-1)\left[x^3+x^2+x+1 + x^3-3x^2+3x-1\right] = 0$$

$$\Rightarrow (x-1)(2x^3-2x^2+4x) = 0$$

$$\Rightarrow 2x(x-1)(x^2-x+2) = 0$$

EITHER $x=0$
OR $x=1$
OR $x^2-x+2=0$

ANY SUITABLE METHOD

$$x^2-x+2=0$$

$$x = \frac{1\pm\sqrt{1-4\times1\times2}}{2\times1}$$

$$x = \frac{1\pm\sqrt{-7}}{2}$$

$$x = \frac{1\pm\sqrt{7}\,i}{2}$$

∴ FULL SOLUTIONS

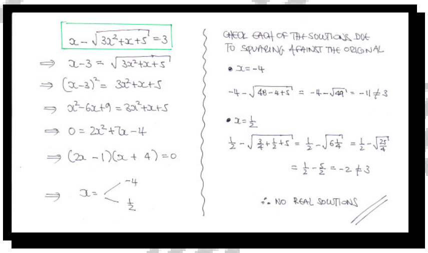
$$x = \begin{cases} 0 \\ 1 \\ \frac{1}{2}(1+\sqrt{7}\,i) \\ \frac{1}{2}(1-\sqrt{7}\,i) \end{cases}$$

Question 46 **(****+)**

Solve the following equation

$$x - \sqrt{3x^2 + x + 5} = 3, \; x \in \mathbb{R}.$$

$\boxed{\text{no solutions}}$



$\boxed{x - \sqrt{3x^2+x+5} = 3}$

$\Rightarrow x-3 = \sqrt{3x^2+x+5}$

$\Rightarrow (x-3)^2 = 3x^2+x+5$

$\Rightarrow x^2-6x+9 = 3x^2+x+5$

$\Rightarrow 0 = 2x^2+7x-4$

$\Rightarrow (2x-1)(x+4) = 0$

$\Rightarrow x = \begin{cases} -4 \\ \frac{1}{2} \end{cases}$

CHECK EACH OF THE SOLUTIONS DUE TO SQUARING AGAINST THE ORIGINAL

- $x = -4$

$-4 - \sqrt{48-4+5} = -4-\sqrt{49} = -11 \neq 3$

- $x = \frac{1}{2}$

$\frac{1}{2} - \sqrt{\frac{3}{4}+\frac{1}{2}+5} = \frac{1}{2} - \sqrt{6\frac{1}{4}} = \frac{1}{2} - \sqrt{\frac{25}{4}}$

$= \frac{1}{2} - \frac{5}{2} = -2 \neq 3$

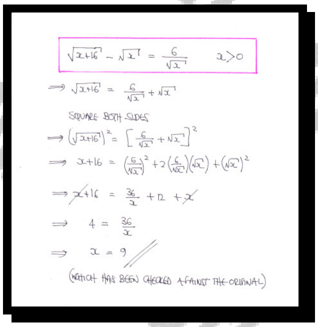
$\therefore$ NO REAL SOLUTIONS //

Question 47 **(****+)**

Solve the following equation.

$$\sqrt{x+16} - \sqrt{x} = \frac{6}{\sqrt{x}}, \quad x \in \mathbb{R}, \; x > 0.$$

$\boxed{x = 9}$



$\boxed{\sqrt{x+16} - \sqrt{x} = \frac{6}{\sqrt{x}} \qquad x>0}$

$\Longrightarrow \sqrt{x+16} = \frac{6}{\sqrt{x}} + \sqrt{x}$

SQUARE BOTH SIDES

$\Longrightarrow \left(\sqrt{x+16}\right)^2 = \left[\frac{6}{\sqrt{x}} + \sqrt{x}\right]^2$

$\Longrightarrow x+16 = \left(\frac{6}{\sqrt{x}}\right)^2 + 2\left(\frac{6}{\sqrt{x}}\right)\left(\sqrt{x}\right) + \left(\sqrt{x}\right)^2$

$\Longrightarrow \cancel{x}+16 = \frac{36}{x} + 12 + \cancel{x}$

$\Longrightarrow 4 = \frac{36}{x}$

$\Longrightarrow x = 9$ //

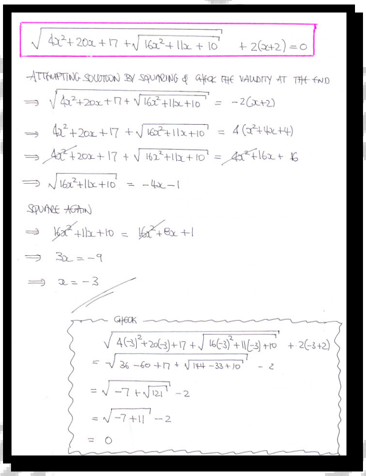
(WHICH HAS BEEN CHECKED AGAINST THE ORIGINAL)

Question 48 **(****+)**

Determine the real root of the equation

$$\sqrt{4x^2+20x+17+\sqrt{16x^2+11x+10}}+2(x+2)=0.$$

$\boxed{x=-3}$



$$\sqrt{4x^2+20x+17+\sqrt{16x^2+11x+10}}+2(x+2)=0$$

ATTEMPTING SOLUTION BY SQUARING & CHECK THE VALIDITY AT THE END

$\Rightarrow \sqrt{4x^2+20x+17+\sqrt{16x^2+11x+10}} = -2(x+2)$

$\Rightarrow 4x^2+20x+17+\sqrt{16x^2+11x+10} = 4(x^2+4x+4)$

$\Rightarrow \cancel{4x^2}+20x+17+\sqrt{16x^2+11x+10} = \cancel{4x^2}+16x+16$

$\Rightarrow \sqrt{16x^2+11x+10} = -4x-1$

SQUARE AGAIN

$\Rightarrow \cancel{16x^2}+11x+10 = \cancel{16x^2}+8x+1$

$\Rightarrow 3x=-9$

$\Rightarrow x=-3$

CHECK

$\sqrt{4(-3)^2+20(-3)+17+\sqrt{16(-3)^2+11(-3)+10}}+2(-3+2)$

$=\sqrt{36-60+17+\sqrt{144-33+10}}-2$

$=\sqrt{-7+\sqrt{121}}-2$

$=\sqrt{-7+11}-2$

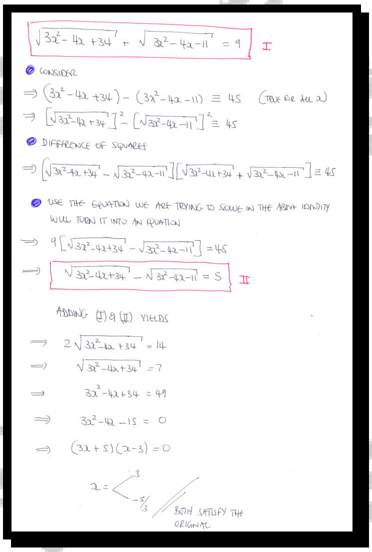
$=0$

Question 49 **(****+)**

Determine the two real roots of the equation

$$\sqrt{3x^2-4x+34}+\sqrt{3x^2-4x-11}=9.$$

$$\boxed{x=3,-\frac{5}{3}}$$



$$\boxed{\sqrt{3x^2-4x+34}+\sqrt{3x^2-4x-11}=9} \quad \text{I}$$

CONSIDER

$\Rightarrow (3x^2-4x+34)-(3x^2-4x-11) \equiv 45$ (TRUE FOR ALL x)

$\Rightarrow \left[\sqrt{3x^2-4x+34}\right]^2-\left[\sqrt{3x^2-4x-11}\right]^2 \equiv 45$

DIFFERENCE OF SQUARES

$\Rightarrow \left[\sqrt{3x^2-4x+34}-\sqrt{3x^2-4x-11}\right]\left[\sqrt{3x^2-4x+34}+\sqrt{3x^2-4x-11}\right] \equiv 45$

USE THE EQUATION WE ARE TRYING TO SOLVE IN THE ABOVE IDENTITY WILL TURN IT INTO AN EQUATION

$\Rightarrow 9\left[\sqrt{3x^2-4x+34}-\sqrt{3x^2-4x-11}\right]=45$

$\Rightarrow \boxed{\sqrt{3x^2-4x+34}-\sqrt{3x^2-4x-11}=5} \quad \text{II}$

ADDING (I) & (II) YIELDS

$\Rightarrow 2\sqrt{3x^2-4x+34}=14$

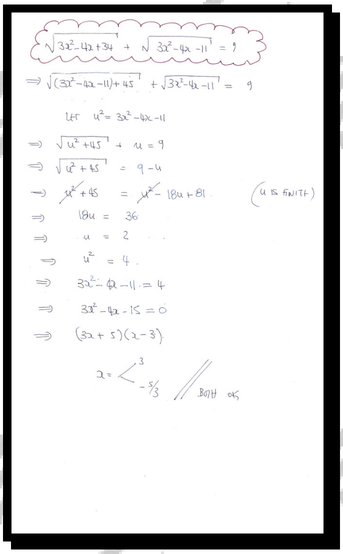
$\Rightarrow \sqrt{3x^2-4x+34}=7$

$\Rightarrow 3x^2-4x+34=49$

$\Rightarrow 3x^2-4x-15=0$

$\Rightarrow (3x+5)(x-3)=0$

$x = \begin{cases} 3 \\ -5/3 \end{cases}$ // BOTH SATISFY THE ORIGINAL.



$$\boxed{\sqrt{3x^2-4x+34}+\sqrt{3x^2-4x-11}=9}$$

$\Rightarrow \sqrt{(3x^2-4x-11)+45}+\sqrt{3x^2-4x-11}=9$

LET $u^2=3x^2-4x-11$

$\Rightarrow \sqrt{u^2+45}+u=9$

$\Rightarrow \sqrt{u^2+45}=9-u$

$\Rightarrow \cancel{u^2}+45=\cancel{u^2}-18u+81$ (u IS FINITE)

$\Rightarrow 18u=36$

$\Rightarrow u=2$

$\Rightarrow u^2=4$

$\Rightarrow 3x^2-4x-11=4$

$\Rightarrow 3x^2-4x-15=0$

$\Rightarrow (3x+5)(x-3)$

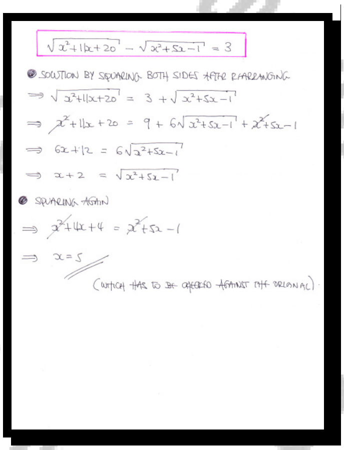
$x = \begin{cases} 3 \\ -5/3 \end{cases}$ // BOTH OK

Question 50 **(****+)**

Solve the following equation.

$$\sqrt{x^2+11x+20}+\sqrt{x^2+5x-1}=3, \quad x\in\mathbb{R}.$$

$\boxed{\quad\quad}$, $\boxed{x=5}$



$$\boxed{\sqrt{x^2+11x+20}-\sqrt{x^2+5x-1}=3}$$

SOLUTION BY SQUARING BOTH SIDES AFTER REARRANGING

$\Rightarrow \sqrt{x^2+11x+20}=3+\sqrt{x^2+5x-1}$

$\Rightarrow \cancel{x^2}+11x+20=9+6\sqrt{x^2+5x-1}+\cancel{x^2}+5x-1$

$\Rightarrow 6x+12=6\sqrt{x^2+5x-1}$

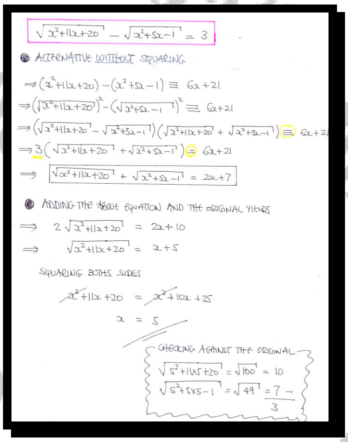
$\Rightarrow x+2=\sqrt{x^2+5x-1}$

SQUARING AGAIN

$\Rightarrow \cancel{x^2}+4x+4=\cancel{x^2}+5x-1$

$\Rightarrow x=5$ //

(WHICH HAS TO BE CHECKED AGAINST THE ORIGINAL).



$$\boxed{\sqrt{x^2+11x+20}-\sqrt{x^2+5x-1}=3}$$

ALTERNATIVE WITHOUT SQUARING

$\Rightarrow (x^2+11x+20)-(x^2+5x-1) \equiv 6x+21$

$\Rightarrow \left(\sqrt{x^2+11x+20}\right)^2-\left(\sqrt{x^2+5x-1}\right)^2 \equiv 6x+21$

$\Rightarrow \left(\sqrt{x^2+11x+20}-\sqrt{x^2+5x-1}\right)\left(\sqrt{x^2+11x+20}+\sqrt{x^2+5x-1}\right) \equiv 6x+21$

$\Rightarrow 3\left(\sqrt{x^2+11x+20}+\sqrt{x^2+5x-1}\right) = 6x+21$

$\Rightarrow \boxed{\sqrt{x^2+11x+20}+\sqrt{x^2+5x-1}=2x+7}$

ADDING THE ABOVE EQUATION AND THE ORIGINAL YIELDS

$\Rightarrow 2\sqrt{x^2+11x+20}=2x+10$

$\Rightarrow \sqrt{x^2+11x+20}=x+5$

SQUARING BOTHS SIDES

$\cancel{x^2}+11x+20=\cancel{x^2}+10x+25$

$x=5$ //

CHECKING AGAINST THE ORIGINAL

$\sqrt{5^2+11\times5+20}=\sqrt{100}=10$

$\sqrt{5^2+5\times5-1}=\sqrt{49}=7$ −

$\overline{\quad 3 \quad}$

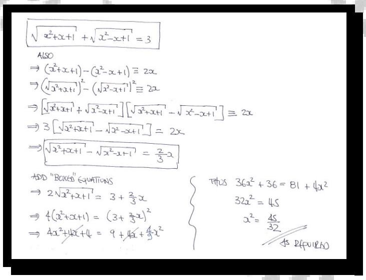
Question 51 (****+)

Given the equation

$$\sqrt{x^2+x+1}+\sqrt{x^2-x+1}=3,\ x\in\mathbb{R},$$

show clearly that $x^2=\dfrac{45}{32}$.

$\boxed{\text{proof}}$



$\boxed{\sqrt{x^2+x+1}+\sqrt{x^2-x+1}=3}$

ALSO

$\Rightarrow (x^2+x+1)-(x^2-x+1)\equiv 2x$

$\Rightarrow \left(\sqrt{x^2+x+1}\right)^2-\left(\sqrt{x^2-x+1}\right)^2\equiv 2x$

$\Rightarrow \left[\sqrt{x^2+x+1}+\sqrt{x^2-x+1}\right]\left[\sqrt{x^2+x+1}-\sqrt{x^2-x+1}\right]\equiv 2x$

$\Rightarrow 3\left[\sqrt{x^2+x+1}-\sqrt{x^2-x+1}\right]=2x$

$\Rightarrow \boxed{\sqrt{x^2+x+1}-\sqrt{x^2-x+1}=\frac{2}{3}x}$

ADD "BOXED" EQUATIONS

$\Rightarrow 2\sqrt{x^2+x+1}=3+\frac{2}{3}x$

$\Rightarrow 4(x^2+x+1)=\left(3+\frac{2}{3}x\right)^2$

$\Rightarrow 4x^2+\cancel{4x}+4=9+\cancel{4x}+\frac{4}{9}x^2$

THUS $36x^2+36=81+4x^2$

$32x^2=45$

$x^2=\frac{45}{32}$

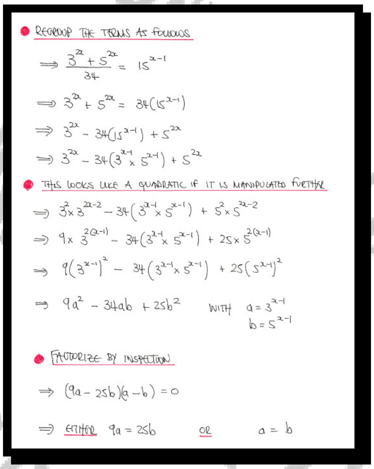
AS REQUIRED

Question 52 (*****)

Solve the exponential equation

$$\frac{3^{2x}+5^{2x}}{34}=15^{x-1}.$$

V, $\boxed{}$, $\boxed{x=\pm1}$



- REGROUP THE TERMS AS FOLLOWS

$\Longrightarrow \frac{3^{2x}+5^{2x}}{34}=15^{x-1}$

$\Longrightarrow 3^{2x}+5^{2x}=34(15^{x-1})$

$\Longrightarrow 3^{2x}-34(15^{x-1})+5^{2x}$

$\Longrightarrow 3^{2x}-34(3^{x-1}\times5^{x-1})+5^{2x}$

- THIS LOOKS LIKE A QUADRATIC IF IT IS MANIPULATED FURTHER

$\Longrightarrow 3^2\times3^{2x-2}-34(3^{x-1}\times5^{x-1})+5^2\times5^{2x-2}$

$\Longrightarrow 9\times3^{2(x-1)}-34(3^{x-1}\times5^{x-1})+25\times5^{2(x-1)}$

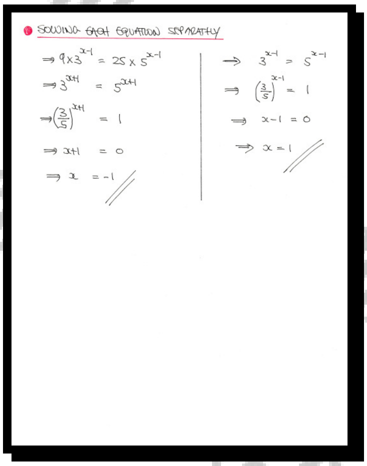
$\Longrightarrow 9(3^{x-1})^2-34(3^{x-1}\times5^{x-1})+25(5^{x-1})^2$

$\Longrightarrow 9a^2-34ab+25b^2$ WITH $a=3^{x-1}$, $b=5^{x-1}$

- FACTORIZE BY INSPECTION

$\Longrightarrow (9a-25b)(a-b)=0$

$\Longrightarrow$ EITHER $9a=25b$ OR $a=b$



- SOLVING EACH EQUATION SEPARATELY

$\Rightarrow 9\times3^{x-1}=25\times5^{x-1}$

$\Rightarrow 3^{x+1}=5^{x+1}$

$\Rightarrow \left(\frac{3}{5}\right)^{x+1}=1$

$\Rightarrow x+1=0$

$\Rightarrow x=-1$

$\Rightarrow 3^{x-1}=5^{x-1}$

$\Rightarrow \left(\frac{3}{5}\right)^{x-1}=1$

$\Rightarrow x-1=0$

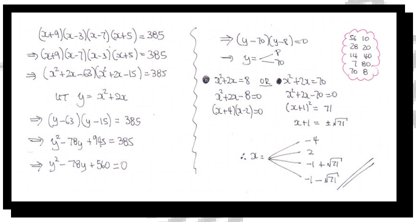
$\Rightarrow x=1$

Question 53 **(*****)**

By using a suitable quadratic substitution, or otherwise, find in exact surd form where appropriate, the four real roots of the equation

$$(x-7)(x-3)(x+5)(x+9)=385.$$

V, $\boxed{x=-4,2,-1\pm\sqrt{71}}$



$(x+9)(x-3)(x-7)(x+5)=385$

$\Rightarrow (x+9)(x-7)(x-3)(x+5)=385$

$\Rightarrow (x^2+2x-63)(x^2+2x-15)=385$

LET $y=x^2+2x$

$\Rightarrow (y-63)(y-15)=385$

$\Rightarrow y^2-78y+945=385$

$\Rightarrow y^2-78y+560=0$

$\Rightarrow (y-70)(y-8)=0$

$\Rightarrow y=\left<\begin{array}{l}8\\70\end{array}\right.$

56 10
28 20
14 40
7 80
70 8

● $x^2+2x=8$ OR ● $x^2+2x=70$

$x^2+2x-8=0$ $x^2+2x-70=0$

$(x+4)(x-2)=0$ $(x+1)^2=71$

$x+1=\pm\sqrt{71}$

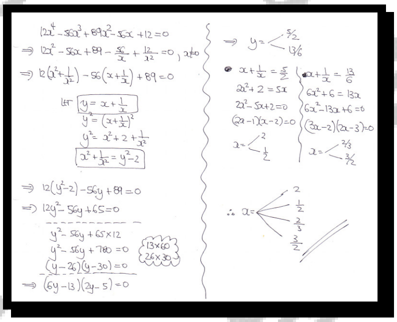
$\therefore x=\left<\begin{array}{l}-4\\2\\-1+\sqrt{71}\\-1-\sqrt{71}\end{array}\right.$ //

Question 54 **(*****)**

Solve the equation

$$12x^4-56x^3+89x^2-56x+12=0.$$

$\boxed{x=\frac{1}{2},\frac{2}{3},\frac{3}{2},2}$



$12x^4-56x^3+89x^2-56x+12=0$

$\Rightarrow 12x^2-56x+89-\frac{56}{x}+\frac{12}{x^2}=0,\ x\neq0$

$\Rightarrow 12\left(x^2+\frac{1}{x^2}\right)-56\left(x+\frac{1}{x}\right)+89=0$

LET $\boxed{y=x+\frac{1}{x}}$

$y^2=\left(x+\frac{1}{x}\right)^2$

$y^2=x^2+2+\frac{1}{x^2}$

$\boxed{x^2+\frac{1}{x^2}=y^2-2}$

$\Rightarrow 12(y^2-2)-56y+89=0$

$\Rightarrow 12y^2-56y+65=0$

$y^2-56y+65\times12$

$y^2-56y+780=0$

$(y-26)(y-30)=0$

13×60
26×30

$\Rightarrow (6y-13)(2y-5)=0$

$\Rightarrow y=\left<\begin{array}{l}5/2\\13/6\end{array}\right.$

● $x+\frac{1}{x}=\frac{5}{2}$

$2x^2+2=5x$

$2x^2-5x+2=0$

$(2x-1)(x-2)=0$

$x=\left<\begin{array}{l}2\\\frac{1}{2}\end{array}\right.$

● $x+\frac{1}{x}=\frac{13}{6}$

$6x^2+6=13x$

$6x^2-13x+6=0$

$(3x-2)(2x-3)=0$

$x=\left<\begin{array}{l}2/3\\3/2\end{array}\right.$

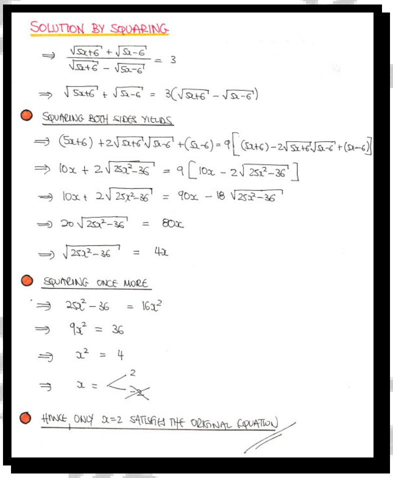
$\therefore x=\left<\begin{array}{l}2\\\frac{1}{2}\\\frac{2}{3}\\\frac{3}{2}\end{array}\right.$ //

Question 55 (*****)

Solve the equation

$$\frac{\sqrt{5x+6}+\sqrt{5x-6}}{\sqrt{5x+6}-\sqrt{5x-6}}=3, \qquad x>\frac{6}{5}.$$

$\boxed{\text{S}\quad\text{L}}$, $\boxed{x=2}$



SOLUTION BY SQUARING

$\Rightarrow \dfrac{\sqrt{5x+6}+\sqrt{5x-6}}{\sqrt{5x+6}-\sqrt{5x-6}}=3$

$\Rightarrow \sqrt{5x+6}+\sqrt{5x-6}=3\left(\sqrt{5x+6}-\sqrt{5x-6}\right)$

- SQUARING BOTH SIDES YIELDS

$\Rightarrow (5x+6)+2\sqrt{5x+6}\sqrt{5x-6}+(5x-6)=9\left[(5x+6)-2\sqrt{5x+6}\sqrt{5x-6}+(5x-6)\right]$

$\Rightarrow 10x+2\sqrt{25x^2-36}=9\left[10x-2\sqrt{25x^2-36}\right]$

$\Rightarrow 10x+2\sqrt{25x^2-36}=90x-18\sqrt{25x^2-36}$

$\Rightarrow 20\sqrt{25x^2-36}=80x$

$\Rightarrow \sqrt{25x^2-36}=4x$

- SQUARING ONCE MORE

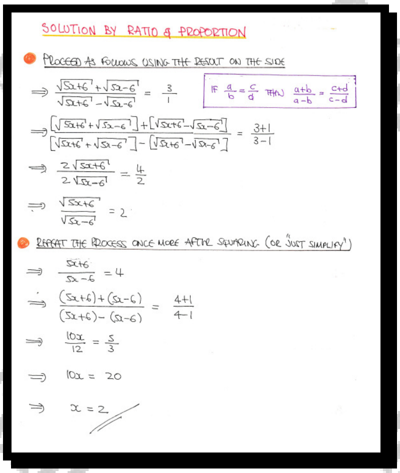
$\Rightarrow 25x^2-36=16x^2$

$\Rightarrow 9x^2=36$

$\Rightarrow x^2=4$

$\Rightarrow x=\begin{cases}2\\ \cancel{-2}\end{cases}$

- HENCE, ONLY $x=2$ SATISFIES THE ORIGINAL EQUATION



SOLUTION BY RATIO & PROPORTION

- PROCEED AS FOLLOWS USING THE RESULT ON THE SIDE

$\Rightarrow \dfrac{\sqrt{5x+6}+\sqrt{5x-6}}{\sqrt{5x+6}-\sqrt{5x-6}}=\dfrac{3}{1}$

IF $\dfrac{a}{b}=\dfrac{c}{d}$ THEN $\dfrac{a+b}{a-b}=\dfrac{c+d}{c-d}$

$\Rightarrow \dfrac{\left[\sqrt{5x+6}+\sqrt{5x-6}\right]+\left[\sqrt{5x+6}-\sqrt{5x-6}\right]}{\left[\sqrt{5x+6}+\sqrt{5x-6}\right]-\left[\sqrt{5x+6}-\sqrt{5x-6}\right]}=\dfrac{3+1}{3-1}$

$\Rightarrow \dfrac{2\sqrt{5x+6}}{2\sqrt{5x-6}}=\dfrac{4}{2}$

$\Rightarrow \dfrac{\sqrt{5x+6}}{\sqrt{5x-6}}=2$

- REPEAT THE PROCESS ONCE MORE AFTER SQUARING (OR "JUST SIMPLIFY")

$\Rightarrow \dfrac{5x+6}{5x-6}=4$

$\Rightarrow \dfrac{(5x+6)+(5x-6)}{(5x+6)-(5x-6)}=\dfrac{4+1}{4-1}$

$\Rightarrow \dfrac{10x}{12}=\dfrac{5}{3}$

$\Rightarrow 10x=20$

$\Rightarrow x=2$

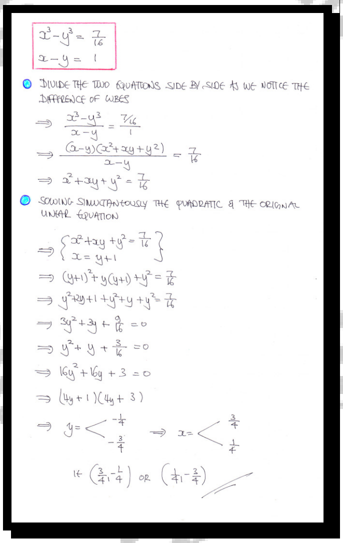
Question 56 (*****)

Use algebra to solve the following simultaneous equations

$$x^3 - y^3 = \frac{7}{16} \qquad \text{and} \qquad x - y = 1,$$

given further that $x \in \mathbb{R}$ and $y \in \mathbb{R}$.

$$\boxed{(x,y) = \left(\frac{3}{4}, -\frac{1}{4}\right) = \left(\frac{1}{4}, -\frac{3}{4}\right)}$$



$$\boxed{\begin{aligned} x^3 - y^3 &= \tfrac{7}{16} \\ x - y &= 1 \end{aligned}}$$

- DIVIDE THE TWO EQUATIONS SIDE BY SIDE AS WE NOTICE THE DIFFERENCE OF CUBES

$\Rightarrow \frac{x^3-y^3}{x-y} = \frac{7/16}{1}$

$\Rightarrow \frac{(x-y)(x^2+xy+y^2)}{x-y} = \frac{7}{16}$

$\Rightarrow x^2+xy+y^2 = \frac{7}{16}$

- SOLVING SIMULTANEOUSLY THE QUADRATIC & THE ORIGINAL LINEAR EQUATION

$\Rightarrow \left\{ \begin{matrix} x^2+xy+y^2 = \frac{7}{16} \\ x = y+1 \end{matrix} \right\}$

$\Rightarrow (y+1)^2 + y(y+1) + y^2 = \frac{7}{16}$

$\Rightarrow y^2+2y+1+y^2+y+y^2 = \frac{7}{16}$

$\Rightarrow 3y^2+3y+\frac{9}{16} = 0$

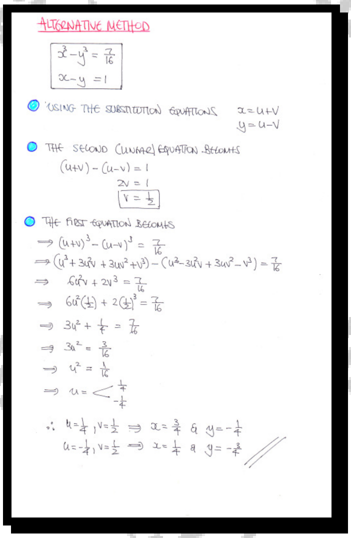
$\Rightarrow y^2+y+\frac{3}{16} = 0$

$\Rightarrow 16y^2+16y+3 = 0$

$\Rightarrow (4y+1)(4y+3)$

$\Rightarrow y = \begin{cases} -\frac{1}{4} \\ -\frac{3}{4} \end{cases} \quad \Rightarrow x = \begin{cases} \frac{3}{4} \\ \frac{1}{4} \end{cases}$

IE $\left(\frac{3}{4}, -\frac{1}{4}\right)$ OR $\left(\frac{1}{4}, -\frac{3}{4}\right)$



ALTERNATIVE METHOD

$$\boxed{\begin{aligned} x^3 - y^3 &= \tfrac{7}{16} \\ x - y &= 1 \end{aligned}}$$

- USING THE SUBSTITUTION EQUATIONS $x = u+v$, $y = u-v$
- THE SECOND (LINEAR) EQUATION BECOMES

$(u+v) - (u-v) = 1$

$2v = 1$

$\boxed{v = \frac{1}{2}}$

- THE FIRST EQUATION BECOMES

$\Rightarrow (u+v)^3 - (u-v)^3 = \frac{7}{16}$

$\Rightarrow (u^3+3u^2v+3uv^2+v^3) - (u^3-3u^2v+3uv^2-v^3) = \frac{7}{16}$

$\Rightarrow 6u^2v + 2v^3 = \frac{7}{16}$

$\Rightarrow 6u^2\left(\frac{1}{2}\right) + 2\left(\frac{1}{2}\right)^3 = \frac{7}{16}$

$\Rightarrow 3u^2 + \frac{1}{4} = \frac{7}{16}$

$\Rightarrow 3u^2 = \frac{3}{16}$

$\Rightarrow u^2 = \frac{1}{16}$

$\Rightarrow u = \begin{cases} \frac{1}{4} \\ -\frac{1}{4} \end{cases}$

$\therefore u = \frac{1}{4}, v = \frac{1}{2} \Rightarrow x = \frac{3}{4}$ & $y = -\frac{1}{4}$

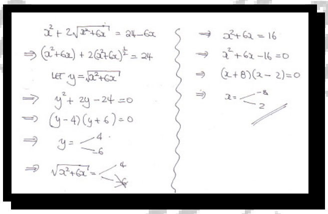
$u = -\frac{1}{4}, v = \frac{1}{2} \Rightarrow x = \frac{1}{4}$ & $y = -\frac{3}{4}$

Question 57 (*****)

Determine the two real roots of the equation

$$x^2 + 2\sqrt{x^2+6x} = 24 - 6x.$$

$$\boxed{x = -8, 2}$$



$x^2 + 2\sqrt{x^2+6x} = 24-6x$

$\Rightarrow (x^2+6x) + 2(x^2+6x)^{\frac{1}{2}} = 24$

LET $y = \sqrt{x^2+6x}$

$\Rightarrow y^2 + 2y - 24 = 0$

$\Rightarrow (y-4)(y+6) = 0$

$\Rightarrow y = \begin{cases} 4 \\ -6 \end{cases}$

$\Rightarrow \sqrt{x^2+6x} = \begin{cases} 4 \\ \cancel{-6} \end{cases}$

$\Rightarrow x^2+6x = 16$

$\Rightarrow x^2+6x-16 = 0$

$\Rightarrow (x+8)(x-2) = 0$

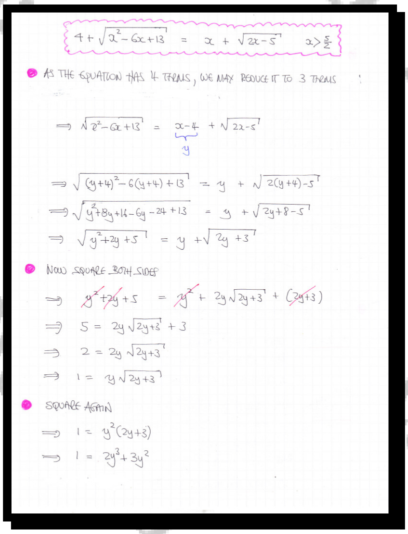
$\Rightarrow x = \begin{cases} -8 \\ 2 \end{cases}$

Question 58 (*****)

Solve following equation

$$4+\sqrt{x^2-6x+13}=x+\sqrt{2x-5}\,,\quad x\in\mathbb{R}\,,\ x\geq\frac{5}{2}\,.$$

$\boxed{\quad\quad}$, $\boxed{x=\frac{9}{2}}$



$$4+\sqrt{x^2-6x+13} = x+\sqrt{2x-5} \qquad x>\frac{5}{2}$$

- AS THE EQUATION HAS 4 TERMS, WE MAY REDUCE IT TO 3 TERMS

$$\Longrightarrow \sqrt{x^2-6x+13} = \underbrace{x-4}_{y} + \sqrt{2x-5}$$

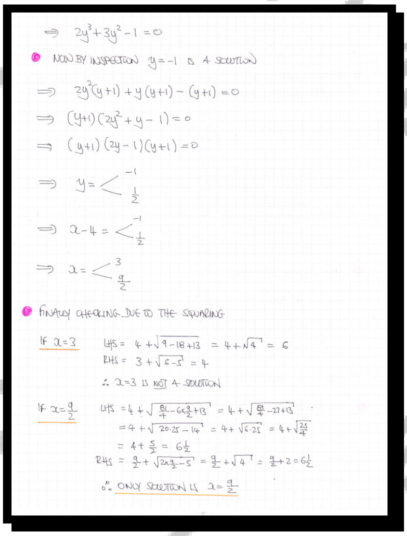
$$\Longrightarrow \sqrt{(y+4)^2-6(y+4)+13} = y + \sqrt{2(y+4)-5}$$
$$\Longrightarrow \sqrt{y^2+8y+16-6y-24+13} = y + \sqrt{2y+8-5}$$
$$\Longrightarrow \sqrt{y^2+2y+5} = y+\sqrt{2y+3}$$

- NOW SQUARE BOTH SIDES

$$\Longrightarrow \cancel{y^2}+\cancel{2y}+5 = \cancel{y^2}+2y\sqrt{2y+3}+(\cancel{2y}+3)$$
$$\Longrightarrow 5 = 2y\sqrt{2y+3}+3$$
$$\Longrightarrow 2 = 2y\sqrt{2y+3}$$
$$\Longrightarrow 1 = y\sqrt{2y+3}$$

- SQUARE AGAIN

$$\Longrightarrow 1 = y^2(2y+3)$$
$$\Longrightarrow 1 = 2y^3+3y^2$$



$$\Longrightarrow 2y^3+3y^2-1=0$$

- NOW BY INSPECTION $y=-1$ IS A SOLUTION

$$\Longrightarrow 2y^2(y+1)+y(y+1)-(y+1)=0$$
$$\Longrightarrow (y+1)(2y^2+y-1)=0$$
$$\Longrightarrow (y+1)(2y-1)(y+1)=0$$
$$\Longrightarrow y = \left\langle\begin{matrix}-1\\ \frac{1}{2}\end{matrix}\right.$$
$$\Longrightarrow x-4 = \left\langle\begin{matrix}-1\\ \frac{1}{2}\end{matrix}\right.$$
$$\Longrightarrow x = \left\langle\begin{matrix}3\\ \frac{9}{2}\end{matrix}\right.$$

- FINALLY CHECKING DUE TO THE SQUARING

IF $x=3$ LHS $= 4+\sqrt{9-18+13} = 4+\sqrt{4} = 6$

RHS $= 3+\sqrt{6-5} = 4$

∴ $x=3$ IS NOT A SOLUTION

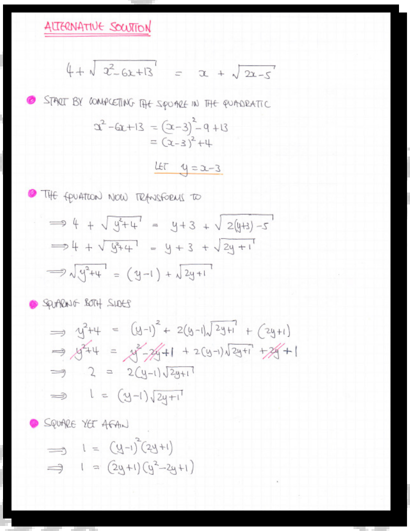
IF $x=\frac{9}{2}$ LHS $= 4+\sqrt{\frac{81}{4}-6\times\frac{9}{2}+13} = 4+\sqrt{\frac{81}{4}-27+13}$

$= 4+\sqrt{20.25-14} = 4+\sqrt{6.25} = 4+\sqrt{\frac{25}{4}}$

$= 4+\frac{5}{2} = 6\frac{1}{2}$

RHS $= \frac{9}{2}+\sqrt{2\times\frac{9}{2}-5} = \frac{9}{2}+\sqrt{4} = \frac{9}{2}+2 = 6\frac{1}{2}$

∴ ONLY SOLUTION IS $x=\frac{9}{2}$



ALTERNATIVE SOLUTION

$$4+\sqrt{x^2-6x+13} = x+\sqrt{2x-5}$$

- START BY COMPLETING THE SQUARE IN THE QUADRATIC

$$\begin{aligned}x^2-6x+13 &= (x-3)^2-9+13\\ &= (x-3)^2+4\end{aligned}$$

LET $y=x-3$

- THE EQUATION NOW TRANSFORMS TO

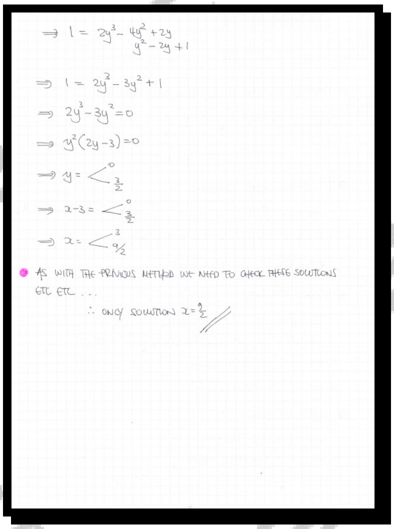
$$\Longrightarrow 4+\sqrt{y^2+4} = y+3+\sqrt{2(y+3)-5}$$
$$\Longrightarrow 4+\sqrt{y^2+4} = y+3+\sqrt{2y+1}$$
$$\Longrightarrow \sqrt{y^2+4} = (y-1)+\sqrt{2y+1}$$

- SQUARING BOTH SIDES

$$\Longrightarrow y^2+4 = (y-1)^2+2(y-1)\sqrt{2y+1}+(2y+1)$$
$$\Longrightarrow \cancel{y^2}+4 = \cancel{y^2}-\cancel{2y}+1+2(y-1)\sqrt{2y+1}+\cancel{2y}+1$$
$$\Longrightarrow 2 = 2(y-1)\sqrt{2y+1}$$
$$\Longrightarrow 1 = (y-1)\sqrt{2y+1}$$

- SQUARE YET AGAIN

$$\Longrightarrow 1 = (y-1)^2(2y+1)$$
$$\Longrightarrow 1 = (2y+1)(y^2-2y+1)$$



$$\begin{aligned}\Longrightarrow 1 = 2y^3-4y^2+2y\\ y^2-2y+1\end{aligned}$$
$$\Longrightarrow 1 = 2y^3-3y^2+1$$
$$\Longrightarrow 2y^3-3y^2=0$$
$$\Longrightarrow y^2(2y-3)=0$$
$$\Longrightarrow y = \left\langle\begin{matrix}0\\ \frac{3}{2}\end{matrix}\right.$$
$$\Longrightarrow x-3 = \left\langle\begin{matrix}0\\ \frac{3}{2}\end{matrix}\right.$$
$$\Longrightarrow x = \left\langle\begin{matrix}3\\ 9/2\end{matrix}\right.$$

- AS WITH THE PREVIOUS METHOD WE NEED TO CHECK THESE SOLUTIONS ETC ETC ...

∴ ONLY SOLUTION $x=\frac{9}{2}$ //

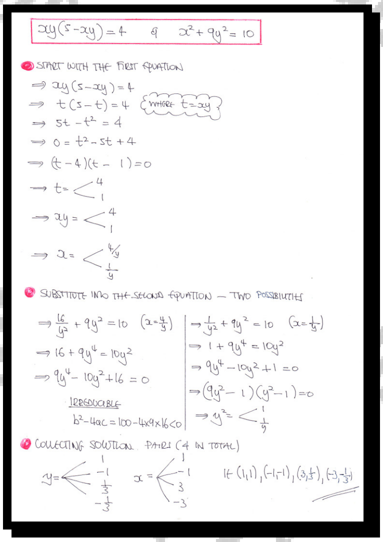
Question 59 (*****)

Use algebra to solve the following simultaneous equations

$$xy(5-xy)=4 \qquad \text{and} \qquad x^2+9y^2=10,$$

given further that $x \in \mathbb{R}$ and $y \in \mathbb{R}$.

$\boxed{}$, $\boxed{(x,y)=(1,1),\,(-1,-1),\,\left(3,\tfrac{1}{3}\right),\,\left(-3,-\tfrac{1}{3}\right)}$



$\boxed{xy(5-xy)=4 \quad \& \quad x^2+9y^2=10}$

● START WITH THE FIRST EQUATION

$\Rightarrow xy(5-xy)=4$

$\Rightarrow t(5-t)=4$ {WHERE $t=xy$}

$\Rightarrow 5t-t^2=4$

$\Rightarrow 0=t^2-5t+4$

$\Rightarrow (t-4)(t-1)=0$

$\Rightarrow t=\left\langle \begin{matrix} 4 \\ 1 \end{matrix}\right.$

$\Rightarrow xy=\left\langle \begin{matrix} 4 \\ 1 \end{matrix}\right.$

$\Rightarrow x=\left\langle \begin{matrix} 4/y \\ \frac{1}{y} \end{matrix}\right.$

● SUBSTITUTE INTO THE SECOND EQUATION — TWO POSSIBILITIES

$\Rightarrow \frac{16}{y^2}+9y^2=10 \quad \left(x=\frac{4}{y}\right)$

$\Rightarrow 16+9y^4=10y^2$

$\Rightarrow 9y^4-10y^2+16=0$

IRREDUCIBLE

$b^2-4ac=100-4\times9\times16<0$

$\Rightarrow \frac{1}{y^2}+9y^2=10 \quad \left(x=\frac{1}{y}\right)$

$\Rightarrow 1+9y^4=10y^2$

$\Rightarrow 9y^4-10y^2+1=0$

$\Rightarrow (9y^2-1)(y^2-1)=0$

$\Rightarrow y^2=\left\langle \begin{matrix} 1 \\ \frac{1}{9} \end{matrix}\right.$

● COLLECTING SOLUTION PAIRS (4 IN TOTAL)

$y=\left\langle \begin{matrix} 1 \\ -1 \\ \frac{1}{3} \\ -\frac{1}{3} \end{matrix}\right. \qquad x=\left\langle \begin{matrix} 1 \\ -1 \\ 3 \\ -3 \end{matrix}\right. \qquad$ i.e. $(1,1),(-1,-1),\left(3,\frac{1}{3}\right),\left(-3,-\frac{1}{3}\right)$

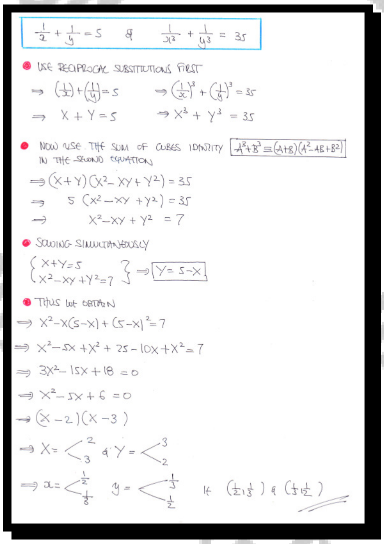
Question 60 (*****)

Use algebra to solve the following simultaneous equations

$$\frac{1}{x}+\frac{1}{y}=5 \qquad \text{and} \qquad \frac{1}{x^3}+\frac{1}{y^3}=35,$$

given further that $x \in \mathbb{R}$ and $y \in \mathbb{R}$.

$\boxed{}$, $\boxed{(x,y)=\left(\frac{1}{2},\frac{1}{3}\right)=\left(\frac{1}{3},\frac{1}{2}\right)}$



$\boxed{\frac{1}{x}+\frac{1}{y}=5 \quad \& \quad \frac{1}{x^3}+\frac{1}{y^3}=35}$

- USE RECIPROCAL SUBSTITUTIONS FIRST

$\Rightarrow \left(\frac{1}{x}\right)+\left(\frac{1}{y}\right)=5 \qquad \Rightarrow \left(\frac{1}{x}\right)^3+\left(\frac{1}{y}\right)^3=35$

$\Rightarrow X+Y=5 \qquad \Rightarrow X^3+Y^3=35$

- NOW USE THE SUM OF CUBES IDENTITY $\boxed{A^3+B^3\equiv(A+B)(A^2-AB+B^2)}$ IN THE SECOND EQUATION

$\Rightarrow (X+Y)(X^2-XY+Y^2)=35$

$\Rightarrow 5(X^2-XY+Y^2)=35$

$\Rightarrow X^2-XY+Y^2=7$

- SOLVING SIMULTANEOUSLY

$\left.\begin{cases} X+Y=5 \\ X^2-XY+Y^2=7 \end{cases}\right\} \Rightarrow \boxed{Y=5-X}$

- THUS WE OBTAIN

$\Rightarrow X^2-X(5-X)+(5-X)^2=7$

$\Rightarrow X^2-5X+X^2+25-10X+X^2=7$

$\Rightarrow 3X^2-15X+18=0$

$\Rightarrow X^2-5X+6=0$

$\Rightarrow (X-2)(X-3)$

$\Rightarrow X=\begin{cases}2\\3\end{cases} \; \& \; Y=\begin{cases}3\\2\end{cases}$

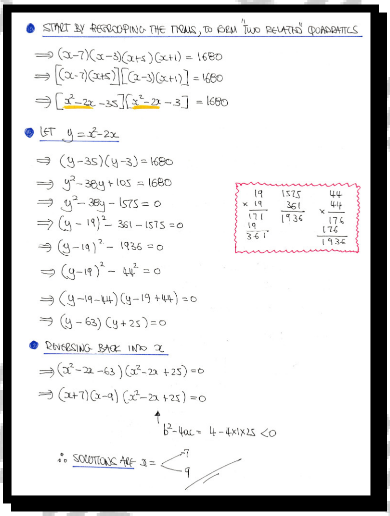
$\Rightarrow x=\begin{cases}\frac{1}{2}\\ \frac{1}{3}\end{cases} \quad y=\begin{cases}\frac{1}{3}\\ \frac{1}{2}\end{cases} \quad$ ie $\left(\frac{1}{2},\frac{1}{3}\right)$ & $\left(\frac{1}{3},\frac{1}{2}\right)$

Question 61 (*****)

Determine the two real roots of the equation

$$(x-7)(x-3)(x+5)(x+1)=1680\,.$$

5P , $\boxed{x=-7,9}$



- START BY REGROUPING THE TERMS, TO FORM "TWO RELATED" QUADRATICS

$\Rightarrow (x-7)(x-3)(x+5)(x+1)=1680$

$\Rightarrow [(x-7)(x+5)][(x-3)(x+1)]=1680$

$\Rightarrow [x^2-2x-35][x^2-2x-3]=1680$

- LET $y=x^2-2x$

$\Rightarrow (y-35)(y-3)=1680$

$\Rightarrow y^2-38y+105=1680$

$\Rightarrow y^2-38y-1575=0$

$\Rightarrow (y-19)^2-361-1575=0$

$\Rightarrow (y-19)^2-1936=0$

$\Rightarrow (y-19)^2-44^2=0$

$\Rightarrow (y-19-44)(y-19+44)=0$

$\Rightarrow (y-63)(y+25)=0$

$$\begin{array}{r} 19\\ \times\ 19\\ \hline 171\\ 19\\ \hline 361\end{array}\qquad \begin{array}{r}1575\\ 361\\ \hline 1936\end{array}\qquad \begin{array}{r}44\\ \times\ 44\\ \hline 176\\ 176\\ \hline 1936\end{array}$$

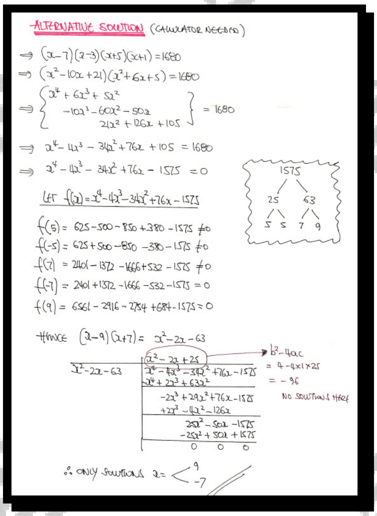
- REVERSING BACK INTO x

$\Rightarrow (x^2-2x-63)(x^2-2x+25)=0$

$\Rightarrow (x+7)(x-9)(x^2-2x+25)=0$

$\uparrow$ $b^2-4ac=4-4\times1\times25<0$

$\therefore$ SOLUTIONS ARE $x=\begin{cases}-7\\9\end{cases}$



ALTERNATIVE SOLUTION (CALCULATOR NEEDED)

$\Rightarrow (x-7)(x-3)(x+5)(x+1)=1680$

$\Rightarrow (x^2-10x+21)(x^2+6x+5)=1680$

$\Rightarrow \left\{\begin{array}{l} x^4+6x^3+5x^2\\ \quad -10x^3-60x^2-50x\\ \qquad 21x^2+126x+105\end{array}\right\}=1680$

$\Rightarrow x^4-4x^3-34x^2+76x+105=1680$

$\Rightarrow x^4-4x^3-34x^2+76x-1575=0$

LET $f(x)=x^4-4x^3-34x^2+76x-1575$

$f(5)=625-500-850+380-1575\neq0$

$f(-5)=625+500-850-380-1575\neq0$

$f(7)=2401-1372-1666+532-1575\neq0$

$f(-7)=2401+1372-1666-532-1575=0$

$f(9)=6561-2916-2754+684-1575=0$

1575 → 25, 63; 25 → 5, 5; 63 → 7, 9

HENCE $(x-9)(x+7)=x^2-2x-63$

$$\begin{array}{r|l} & x^2-2x+25\\ \hline x^2-2x-63 & x^4-4x^3-34x^2+76x-1575\\ & -x^4+2x^3+63x^2\\ \hline & -2x^3+29x^2+76x-1575\\ & +2x^3-4x^2-126x\\ \hline & 25x^2-50x-1575\\ & -25x^2+50x+1575\\ \hline & 0\quad 0\quad 0\end{array}$$

$\rightarrow b^2-4ac$

$=4-4\times1\times25$

$=-96$

NO SOLUTIONS HERE

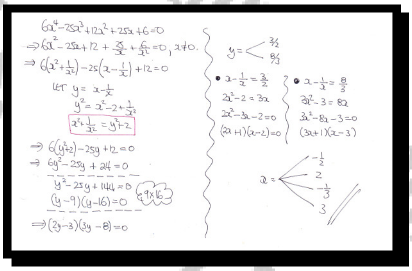
$\therefore$ ONLY SOLUTIONS $x=\begin{cases}9\\-7\end{cases}$

Question 62 (*****)

Solve the equation

$$6x^4-25x^3+12x^2+25x+6=0\,.$$

V , $\boxed{x=-\frac{1}{2},-\frac{1}{3},2,3}$



$6x^4-25x^3+12x^2+25x+6=0$

$\Rightarrow 6x^2-25x+12+\frac{25}{x}+\frac{6}{x^2}=0,\ x\neq0.$

$\Rightarrow 6\left(x^2+\frac{1}{x^2}\right)-25\left(x-\frac{1}{x}\right)+12=0$

LET $y=x-\frac{1}{x}$

$y^2=x^2-2+\frac{1}{x^2}$

$\boxed{x^2+\frac{1}{x^2}=y^2+2}$

$\Rightarrow 6(y^2+2)-25y+12=0$

$\Rightarrow 6y^2-25y+24=0$

$y^2-25y+144=0$

$(y-9)(y-16)=0$ 9×16

$\Rightarrow (2y-3)(3y-8)=0$

$y=\begin{cases}3/2\\8/3\end{cases}$

- $x-\frac{1}{x}=\frac{3}{2}$

$2x^2-2=3x$

$2x^2-3x-2=0$

$(2x+1)(x-2)=0$

- $x-\frac{1}{x}=\frac{8}{3}$

$3x^2-3=8x$

$3x^2-8x-3=0$

$(3x+1)(x-3)$

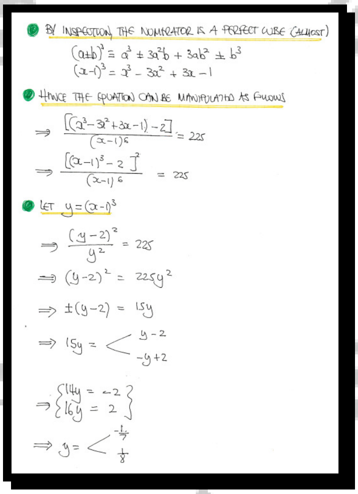
$x=\begin{cases}-\frac{1}{2}\\2\\-\frac{1}{3}\\3\end{cases}$

Question 63 (*****)

Find in exact form the two real solutions of the equation

$$\frac{\left(x^3-3x^2+3x-3\right)^2}{(x-1)^6}=225.$$

$$\boxed{x=\frac{3}{2}, \qquad x=1-\frac{1}{7}\sqrt[3]{49}}$$



- BY INSPECTION, THE NUMERATOR IS A PERFECT CUBE (ALMOST)

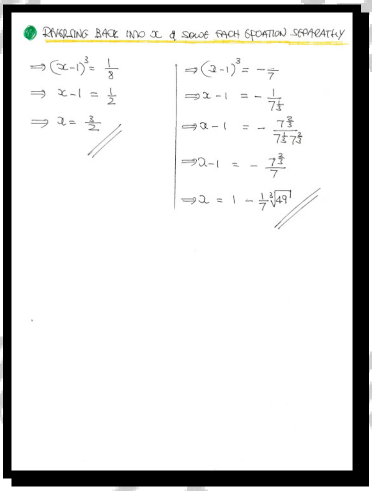
$$(a\pm b)^3 \equiv a^3 \pm 3a^2b + 3ab^2 \pm b^3$$
$$(x-1)^3 = x^3 - 3x^2 + 3x - 1$$

- HENCE THE EQUATION CAN BE MANIPULATED AS FOLLOWS

$$\Rightarrow \frac{\left[\left(x^3-3x^2+3x-1\right)-2\right]}{(x-1)^6}=225$$
$$\Rightarrow \frac{\left[(x-1)^3-2\right]^2}{(x-1)^6}=225$$

- LET $y=(x-1)^3$

$$\Rightarrow \frac{(y-2)^2}{y^2}=225$$
$$\Rightarrow (y-2)^2 = 225y^2$$
$$\Rightarrow \pm(y-2) = 15y$$
$$\Rightarrow 15y = \begin{cases} y-2 \\ -y+2 \end{cases}$$
$$\Rightarrow \begin{Bmatrix} 14y = -2 \\ 16y = 2 \end{Bmatrix}$$
$$\Rightarrow y = \begin{cases} -\frac{1}{7} \\ \frac{1}{8} \end{cases}$$



- REVERSING BACK INTO x & SOLVE EACH EQUATION SEPARATELY

$$\Rightarrow (x-1)^3 = \frac{1}{8}$$
$$\Rightarrow x-1 = \frac{1}{2}$$
$$\Rightarrow x = \frac{3}{2}$$

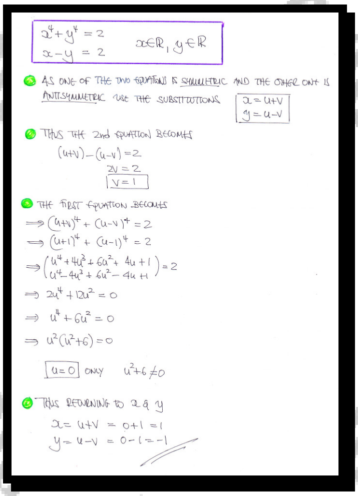
$$\Rightarrow (x-1)^3 = -\frac{1}{7}$$
$$\Rightarrow x-1 = -\frac{1}{7^{\frac{1}{3}}}$$
$$\Rightarrow x-1 = -\frac{7^{\frac{2}{3}}}{7^{\frac{1}{3}}7^{\frac{2}{3}}}$$
$$\Rightarrow x-1 = -\frac{7^{\frac{2}{3}}}{7}$$
$$\Rightarrow x = 1-\frac{1}{7}\sqrt[3]{49}$$

Question 64 (*****)

Use an algebraic method to show that $x = 1$ and $y = -1$ is the only real solution pair of the following simultaneous equations

$$x^4 + y^4 = 2 \quad \text{and} \quad x - y = 2.$$

V, S, proof



$x^4 + y^4 = 2$
$x - y = 2$ $\quad x \in \mathbb{R},\ y \in \mathbb{R}$

- AS ONE OF THE TWO EQUATIONS IS SYMMETRIC AND THE OTHER ONE IS ANTISYMMETRIC USE THE SUBSTITUTIONS $\boxed{\begin{array}{l} x = u + v \\ y = u - v \end{array}}$

- THUS THE 2nd EQUATION BECOMES

$$(u+v) - (u-v) = 2$$
$$2v = 2$$
$$\boxed{v = 1}$$

- THE FIRST EQUATION BECOMES

$$\Rightarrow (u+v)^4 + (u-v)^4 = 2$$
$$\Rightarrow (u+1)^4 + (u-1)^4 = 2$$
$$\Rightarrow \left(\begin{array}{l} u^4 + 4u^3 + 6u^2 + 4u + 1 \\ u^4 - 4u^3 + 6u^2 - 4u + 1 \end{array}\right) = 2$$
$$\Rightarrow 2u^4 + 12u^2 = 0$$
$$\Rightarrow u^4 + 6u^2 = 0$$
$$\Rightarrow u^2(u^2 + 6) = 0$$

$\boxed{u = 0}$ ONLY $\quad u^2 + 6 \neq 0$

- THUS RETURNING TO x & y

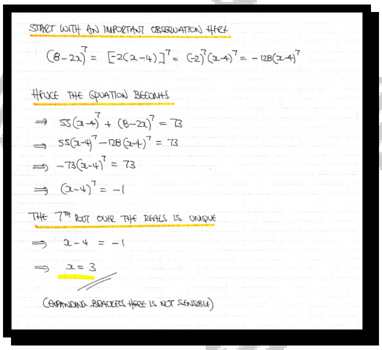
$$x = u + v = 0 + 1 = 1$$
$$y = u - v = 0 - 1 = -1$$

Question 65 (*****)

Use algebra to solve the equation

$$55(x-4)^7+(8-2x)^7=73, \quad x\in\mathbb{R}.$$

V, $\boxed{}$, $\boxed{x=3}$



START WITH AN IMPORTANT OBSERVATION HERE

$(8-2x)^7 = [-2(x-4)]^7 = (-2)^7(x-4)^7 = -128(x-4)^7$

HENCE THE EQUATION BECOMES

$\Rightarrow 55(x-4)^7 + (8-2x)^7 = 73$

$\Rightarrow 55(x-4)^7 - 128(x-4)^7 = 73$

$\Rightarrow -73(x-4)^7 = 73$

$\Rightarrow (x-4)^7 = -1$

THE 7TH ROOT OVER THE REALS IS UNIQUE

$\Rightarrow x-4 = -1$

$\Rightarrow x = 3$

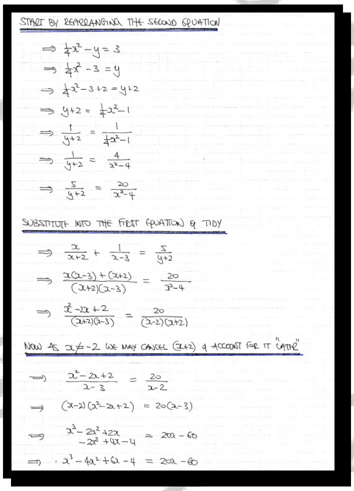
(EXPANDING BRACKETS HERE IS NOT SENSIBLE)

Question 66 **(*****)**

Use algebra to solve following simultaneous equations over the set of real numbers.

$$\frac{x}{x+2}+\frac{1}{x-3}=\frac{5}{y+2} \quad \text{and} \quad \frac{1}{4}x^2-y=3.$$

V, ☐, $\boxed{(4,1)\cup(-2,-2)\cup\left(\sqrt{14},\tfrac{1}{2}\right)\cup\left(-\sqrt{14},\tfrac{1}{2}\right)}$



START BY REARRANGING THE SECOND EQUATION

$\Rightarrow \frac{1}{4}x^2-y=3$

$\Rightarrow \frac{1}{4}x^2-3=y$

$\Rightarrow \frac{1}{4}x^2-3+2=y+2$

$\Rightarrow y+2=\frac{1}{4}x^2-1$

$\Rightarrow \frac{1}{y+2}=\frac{1}{\frac{1}{4}x^2-1}$

$\Rightarrow \frac{1}{y+2}=\frac{4}{x^2-4}$

$\Rightarrow \frac{5}{y+2}=\frac{20}{x^2-4}$

SUBSTITUTE INTO THE FIRST EQUATION & TIDY

$\Rightarrow \frac{x}{x+2}+\frac{1}{x-3}=\frac{5}{y+2}$

$\Rightarrow \frac{x(x-3)+(x+2)}{(x+2)(x-3)}=\frac{20}{x^2-4}$

$\Rightarrow \frac{x^2-2x+2}{(x+2)(x-3)}=\frac{20}{(x-2)(x+2)}$

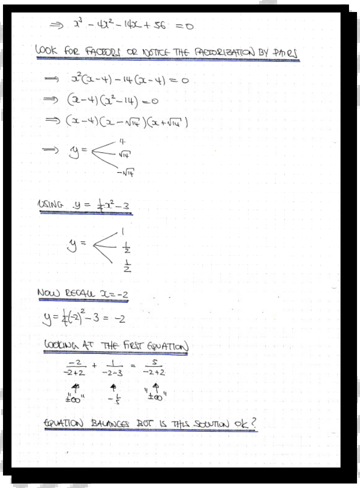
NOW AS $x \neq -2$ WE MAY CANCEL $(x+2)$ & ACCOUNT FOR IT "LATER"

$\Rightarrow \frac{x^2-2x+2}{x-3}=\frac{20}{x-2}$

$\Rightarrow (x-2)(x^2-2x+2)=20(x-3)$

$\Rightarrow \begin{array}{l} x^3-2x^2+2x \\ \quad -2x^2+4x-4 \end{array} = 20x-60$

$\Rightarrow x^3-4x^2+6x-4=20x-60$



$\Rightarrow x^3-4x^2-14x+56=0$

LOOK FOR FACTORS OR NOTICE THE FACTORIZATION BY PAIRS

$\Rightarrow x^2(x-4)-14(x-4)=0$

$\Rightarrow (x-4)(x^2-14)=0$

$\Rightarrow (x-4)(x-\sqrt{14})(x+\sqrt{14})$

$\Rightarrow y= \begin{cases} 4 \\ \sqrt{14} \\ -\sqrt{14} \end{cases}$

USING $y=\frac{1}{4}x^2-3$

$y= \begin{cases} 1 \\ \frac{1}{2} \\ \frac{1}{2} \end{cases}$

NOW RECALL $x=-2$

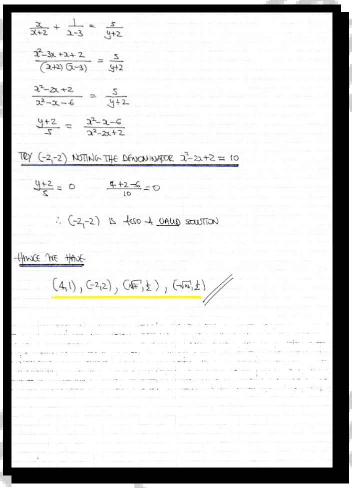
$y=\frac{1}{4}(-2)^2-3=-2$

LOOKING AT THE FIRST EQUATION

$\frac{-2}{-2+2}+\frac{1}{-2-3}=\frac{5}{-2+2}$

$\uparrow$ "$\pm\infty$" $\quad \uparrow$ $-\frac{1}{5}$ $\quad \uparrow$ "$\pm\infty$"

EQUATION BALANCES BUT IS THIS SOLUTION OK?



$\frac{x}{x+2}+\frac{1}{x-3}=\frac{5}{y+2}$

$\frac{x^2-3x+x+2}{(x+2)(x-3)}=\frac{5}{y+2}$

$\frac{x^2-2x+2}{x^2-x-6}=\frac{5}{y+2}$

$\frac{y+2}{5}=\frac{x^2-x-6}{x^2-2x+2}$

TRY $(-2,-2)$ NOTING THE DENOMINATOR $x^2-2x+2=10$

$\frac{y+2}{5}=0 \qquad \frac{4+2-6}{10}=0$

$\therefore (-2,-2)$ IS ALSO A VALID SOLUTION

HENCE WE HAVE

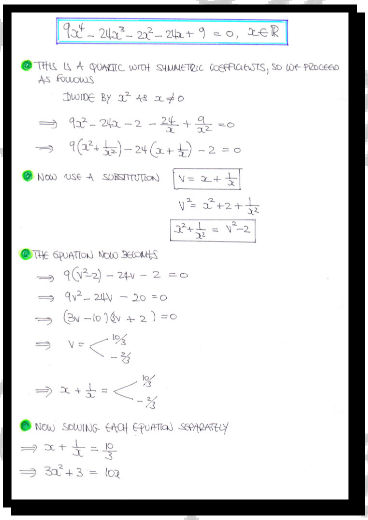
$(4,1),\ (-2,2),\ (\sqrt{14},\frac{1}{2}),\ (-\sqrt{14},\frac{1}{2})$ //

Question 67 (*****)

Solve the equation

$$9x^4-24x^3-2x^2-24x+9=0,\ x\in\mathbb{R}.$$

$\boxed{x=\frac{1}{3},3}$



$\boxed{9x^4-24x^3-2x^2-24x+9=0,\ x\in\mathbb{R}}$

THIS IS A QUARTIC WITH SYMMETRIC COEFFICIENTS, SO WE PROCEED AS FOLLOWS

DIVIDE BY x^2 AS $x\neq 0$

$\Rightarrow 9x^2-24x-2-\frac{24}{x}+\frac{9}{x^2}=0$

$\Rightarrow 9\left(x^2+\frac{1}{x^2}\right)-24\left(x+\frac{1}{x}\right)-2=0$

NOW USE A SUBSTITUTION $\boxed{v=x+\frac{1}{x}}$

$v^2=x^2+2+\frac{1}{x^2}$

$\boxed{x^2+\frac{1}{x^2}=v^2-2}$

THE EQUATION NOW BECOMES

$\Rightarrow 9(v^2-2)-24v-2=0$

$\Rightarrow 9v^2-24v-20=0$

$\Rightarrow (3v-10)(3v+2)=0$

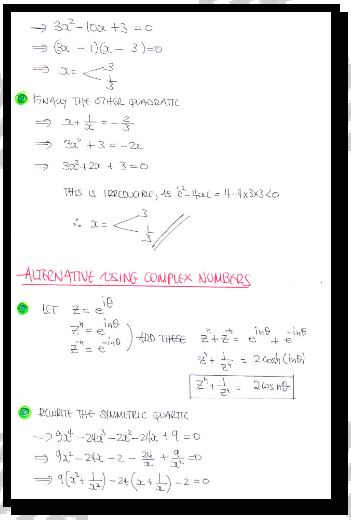
$\Rightarrow v=\left\langle\begin{matrix}10/3\\-2/3\end{matrix}\right.$

$\Rightarrow x+\frac{1}{x}=\left\langle\begin{matrix}10/3\\-2/3\end{matrix}\right.$

NOW SOLVING EACH EQUATION SEPARATELY

$\Rightarrow x+\frac{1}{x}=\frac{10}{3}$

$\Rightarrow 3x^2+3=10x$



$\Rightarrow 3x^2-10x+3=0$

$\Rightarrow (3x-1)(x-3)=0$

$\Rightarrow x=\left\langle\begin{matrix}3\\\frac{1}{3}\end{matrix}\right.$

FINALLY THE OTHER QUADRATIC

$\Rightarrow x+\frac{1}{x}=-\frac{2}{3}$

$\Rightarrow 3x^2+3=-2x$

$\Rightarrow 3x^2+2x+3=0$

THIS IS IRREDUCIBLE, AS $b^2-4ac=4-4\times3\times3<0$

$\therefore x=\left\langle\begin{matrix}3\\\frac{1}{3}\end{matrix}\right.$ //

ALTERNATIVE USING COMPLEX NUMBERS

LET $z=e^{i\theta}$

$\left.\begin{matrix}z^n=e^{in\theta}\\z^{-n}=e^{-in\theta}\end{matrix}\right)$ ADD THESE $z^n+z^{-n}=e^{in\theta}+e^{-in\theta}$

$z^n+\frac{1}{z^n}=2\cosh(in\theta)$

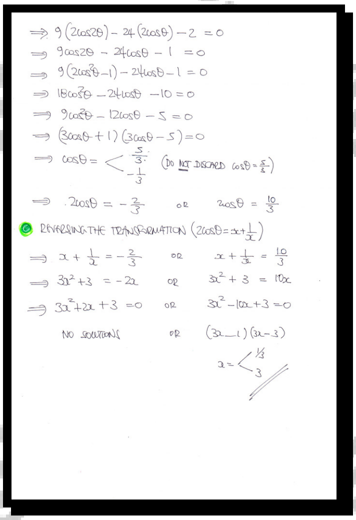
$\boxed{z^n+\frac{1}{z^n}=2\cos n\theta}$

REWRITE THE SYMMETRIC QUARTIC

$\Rightarrow 9x^4-24x^3-2x^2-24x+9=0$

$\Rightarrow 9x^2-24x-2-\frac{24}{x}+\frac{9}{x^2}=0$

$\Rightarrow 9\left(x^2+\frac{1}{x^2}\right)-24\left(x+\frac{1}{x}\right)-2=0$



$\Rightarrow 9(2\cos2\theta)-24(2\cos\theta)-2=0$

$\Rightarrow 9\cos2\theta-24\cos\theta-1=0$

$\Rightarrow 9(2\cos^2\theta-1)-24\cos\theta-1=0$

$\Rightarrow 18\cos^2\theta-24\cos\theta-10=0$

$\Rightarrow 9\cos^2\theta-12\cos\theta-5=0$

$\Rightarrow (3\cos\theta+1)(3\cos\theta-5)=0$

$\Rightarrow \cos\theta=\left\langle\begin{matrix}\frac{5}{3}\\-\frac{1}{3}\end{matrix}\right.$ (DO NOT DISCARD $\cos\theta=\frac{5}{3}$)

$\Rightarrow 2\cos\theta=-\frac{2}{3}$ OR $2\cos\theta=\frac{10}{3}$

REVERSING THE TRANSFORMATION $\left(2\cos\theta=x+\frac{1}{x}\right)$

$\Rightarrow x+\frac{1}{x}=-\frac{2}{3}$ OR $x+\frac{1}{x}=\frac{10}{3}$

$\Rightarrow 3x^2+3=-2x$ OR $3x^2+3=10x$

$\Rightarrow 3x^2+2x+3=0$ OR $3x^2-10x+3=0$

NO SOLUTIONS OR $(3x-1)(3x-3)$

$x=\left\langle\begin{matrix}1/3\\3\end{matrix}\right.$ //

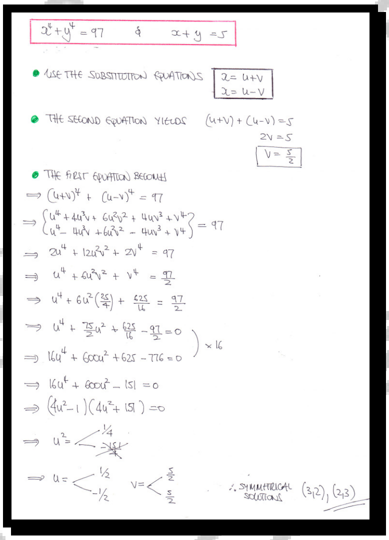
Question 68 (*****)

Use algebra to solve the following simultaneous equations

$$x^4 + y^4 = 97 \qquad \text{and} \qquad x + y = 5,$$

given further that $x \in \mathbb{R}$ and $y \in \mathbb{R}$.

$\boxed{}$, $\boxed{(x, y) = (3,2) = (2,3)}$



$\boxed{x^4+y^4=97 \quad \& \quad x+y=5}$

- USE THE SUBSTITUTION EQUATIONS $\boxed{\begin{array}{l} x=u+v \\ x=u-v \end{array}}$
- THE SECOND EQUATION YIELDS $(u+v)+(u-v)=5$

$2v=5$

$\boxed{v=\frac{5}{2}}$

- THE FIRST EQUATION BECOMES

$\Rightarrow (u+v)^4 + (u-v)^4 = 97$

$\Rightarrow \left\{\begin{array}{l} u^4+4u^3v+6u^2v^2+4uv^3+v^4 \\ u^4-4u^3v+6u^2v^2-4uv^3+v^4 \end{array}\right\} = 97$

$\Rightarrow 2u^4 + 12u^2v^2 + 2v^4 = 97$

$\Rightarrow u^4 + 6u^2v^2 + v^4 = \frac{97}{2}$

$\Rightarrow u^4 + 6u^2\left(\frac{25}{4}\right) + \frac{625}{16} = \frac{97}{2}$

$\Rightarrow u^4 + \frac{75}{2}u^2 + \frac{625}{16} - \frac{97}{2} = 0$

$\Rightarrow 16u^4 + 600u^2 + 625 - 776 = 0$) $\times 16$

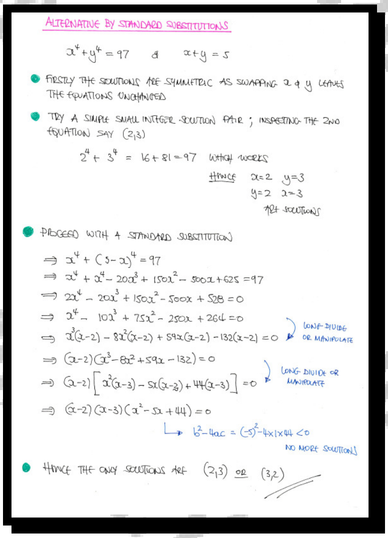
$\Rightarrow 16u^4 + 600u^2 - 151 = 0$

$\Rightarrow (4u^2-1)(4u^2+151) = 0$

$\Rightarrow u^2 = \left\langle \begin{array}{l} 1/4 \\ \cancel{-151/4} \end{array}\right.$

$\Rightarrow u = \left\langle \begin{array}{l} 1/2 \\ -1/2 \end{array}\right. \quad v = \left\langle \begin{array}{l} \frac{5}{2} \\ \frac{5}{2} \end{array}\right.$

$\therefore$ SYMMETRICAL SOLUTIONS $(3,2), (2,3)$



ALTERNATIVE BY STANDARD SUBSTITUTIONS

$x^4+y^4=97 \quad \& \quad x+y=5$

- FIRSTLY THE SOLUTIONS ARE SYMMETRIC AS SWAPPING x & y LEAVES THE EQUATIONS UNCHANGED
- TRY A SIMPLE SMALL INTEGER SOLUTION PAIR; INSPECTING THE 2ND EQUATION SAY $(2,3)$

$2^4 + 3^4 = 16 + 81 = 97$ WHICH WORKS

HENCE $x=2 \quad y=3$

$y=2 \quad x=3$

ARE SOLUTIONS

- PROCEED WITH A STANDARD SUBSTITUTION

$\Rightarrow x^4 + (5-x)^4 = 97$

$\Rightarrow x^4 + x^4 - 20x^3 + 150x^2 - 500x + 625 = 97$

$\Rightarrow 2x^4 - 20x^3 + 150x^2 - 500x + 528 = 0$

$\Rightarrow x^4 - 10x^3 + 75x^2 - 250x + 264 = 0$

$\Rightarrow x^3(x-2) - 8x^2(x-2) + 59x(x-2) - 132(x-2) = 0$ LONG DIVIDE OR MANIPULATE

$\Rightarrow (x-2)(x^3-8x^2+59x-132) = 0$

$\Rightarrow (x-2)\left[x^2(x-3) - 5x(x-3) + 44(x-3)\right] = 0$ LONG DIVIDE OR MANIPULATE

$\Rightarrow (x-2)(x-3)(x^2-5x+44) = 0$

$\hookrightarrow b^2-4ac = (-5)^2 - 4\times1\times44 < 0$

NO MORE SOLUTIONS

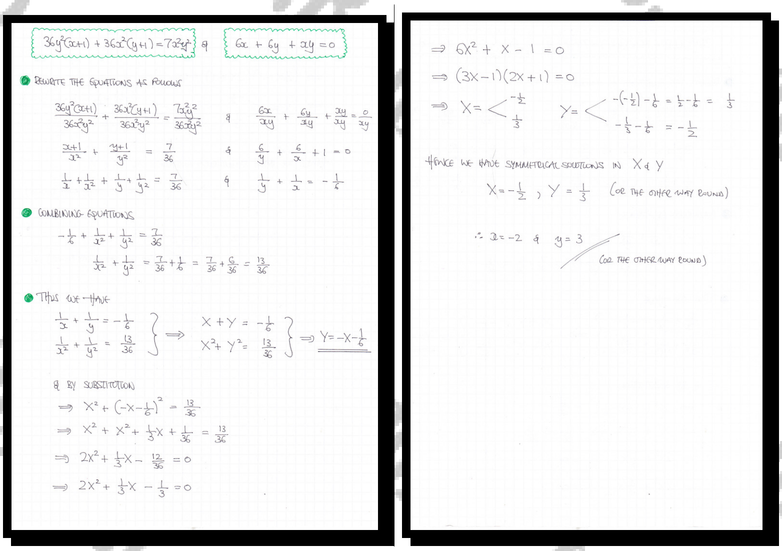
- HENCE THE ONLY SOLUTIONS ARE $(2,3)$ OR $(3,2)$

Question 69 **(*****)**

If $x \in \mathbb{R}$, $y \in \mathbb{R}$, find the non-trivial solution the following simultaneous equations.

$$36y^2(x+1)+36x^2(y+1)=7x^2y^2 \qquad \text{and} \qquad 6x+6y+xy=0\,.$$

$\boxed{}$, $\boxed{(x,y)=(-2,3)=(3,-2)}$



$\boxed{36y^2(x+1)+36x^2(y+1)=7x^2y^2}$ & $\boxed{6x+6y+xy=0}$

● REWRITE THE EQUATIONS AS FOLLOWS

$$\frac{36y^2(x+1)}{36x^2y^2}+\frac{36x^2(y+1)}{36x^2y^2}=\frac{7x^2y^2}{36x^2y^2} \quad \& \quad \frac{6x}{xy}+\frac{6y}{xy}+\frac{xy}{xy}=\frac{0}{xy}$$

$$\frac{x+1}{x^2}+\frac{y+1}{y^2}=\frac{7}{36} \quad \& \quad \frac{6}{y}+\frac{6}{x}+1=0$$

$$\frac{1}{x}+\frac{1}{x^2}+\frac{1}{y}+\frac{1}{y^2}=\frac{7}{36} \quad \& \quad \frac{1}{y}+\frac{1}{x}=-\frac{1}{6}$$

● COMBINING EQUATIONS

$$-\frac{1}{6}+\frac{1}{x^2}+\frac{1}{y^2}=\frac{7}{36}$$

$$\frac{1}{x^2}+\frac{1}{y^2}=\frac{7}{36}+\frac{1}{6}=\frac{7}{36}+\frac{6}{36}=\frac{13}{36}$$

● THUS WE HAVE

$$\left.\begin{aligned}\frac{1}{x}+\frac{1}{y}&=-\frac{1}{6}\\ \frac{1}{x^2}+\frac{1}{y^2}&=\frac{13}{36}\end{aligned}\right\} \Longrightarrow \left.\begin{aligned}X+Y&=-\frac{1}{6}\\ X^2+Y^2&=\frac{13}{36}\end{aligned}\right\} \Rightarrow \underline{\underline{Y=-X-\frac{1}{6}}}$$

& BY SUBSTITUTION

$$\Rightarrow X^2+\left(-X-\frac{1}{6}\right)^2=\frac{13}{36}$$

$$\Rightarrow X^2+X^2+\frac{1}{3}X+\frac{1}{36}=\frac{13}{36}$$

$$\Rightarrow 2X^2+\frac{1}{3}X-\frac{12}{36}=0$$

$$\Rightarrow 2X^2+\frac{1}{3}X-\frac{1}{3}=0$$

$$\Rightarrow 6X^2+X-1=0$$

$$\Rightarrow (3X-1)(2X+1)=0$$

$$\Rightarrow X=\left\langle \begin{matrix}-\frac{1}{2}\\ \frac{1}{3}\end{matrix}\right. \qquad Y=\left\langle \begin{matrix}-\left(-\frac{1}{2}\right)-\frac{1}{6}=\frac{1}{2}-\frac{1}{6}=\frac{1}{3}\\ -\frac{1}{3}-\frac{1}{6}=-\frac{1}{2}\end{matrix}\right.$$

HENCE WE HAVE SYMMETRICAL SOLUTIONS IN X & Y

$X=-\frac{1}{2}$, $Y=\frac{1}{3}$ (OR THE OTHER WAY ROUND)

$\therefore x=-2$ & $y=3$

(OR THE OTHER WAY ROUND)

Question 70 (*****)

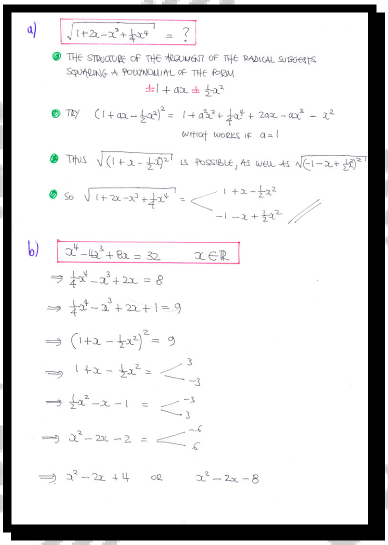
$$f(x) = 1 + 2x - x^3 + \tfrac{1}{4}x^4 \;, \quad x \in \mathbb{R} .$$

a) Extract the square roots of $f(x)$.

b) Hence, or otherwise, solve the equation

$$x^4 - 4x^3 + 8x = 32 \,, \;\; x \in \mathbb{R} .$$

$\boxed{\text{SP}}$, $\boxed{\sqrt{f(x)} = \pm\left(1 + x - \frac{1}{2}x^2\right)}$, $\boxed{x = 4, \quad x = -2}$



a) $\boxed{\sqrt{1+2x-x^3+\frac{1}{4}x^4} = ?}$

- THE STRUCTURE OF THE ARGUMENT OF THE RADICAL SUGGESTS SQUARING A POLYNOMIAL OF THE FORM

$$\pm 1 + ax \pm \tfrac{1}{2}x^2$$

- TRY $(1+ax-\frac{1}{2}x^2)^2 = 1 + a^2x^2 + \frac{1}{4}x^4 + 2ax - ax^3 - x^2$

WHICH WORKS IF $a = 1$

- THUS $\sqrt{(1+x-\frac{1}{2}x^2)^2}$ IS POSSIBLE, AS WELL AS $\sqrt{(-1-x+\frac{1}{2}x^2)^2}$

- SO $\sqrt{1+2x-x^3+\frac{1}{4}x^4} = \begin{cases} 1+x-\frac{1}{2}x^2 \\ -1-x+\frac{1}{2}x^2 \end{cases}$ //

b) $\boxed{x^4 - 4x^3 + 8x = 32 \qquad x \in \mathbb{R}}$

$\Rightarrow \frac{1}{4}x^4 - x^3 + 2x = 8$

$\Rightarrow \frac{1}{4}x^4 - x^3 + 2x + 1 = 9$

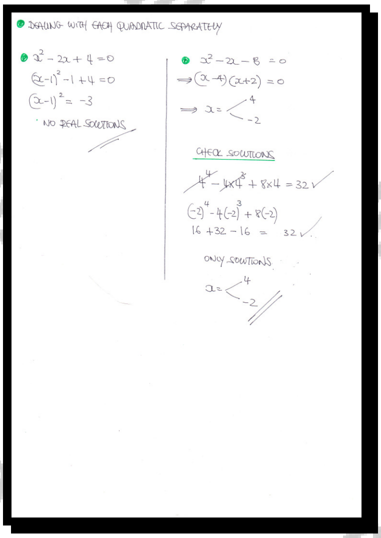
$\Rightarrow (1+x-\frac{1}{2}x^2)^2 = 9$

$\Rightarrow 1+x-\frac{1}{2}x^2 = \begin{cases} 3 \\ -3 \end{cases}$

$\Rightarrow \frac{1}{2}x^2 - x - 1 = \begin{cases} -3 \\ 3 \end{cases}$

$\Rightarrow x^2 - 2x - 2 = \begin{cases} -6 \\ 6 \end{cases}$

$\Rightarrow x^2 - 2x + 4 \quad \text{OR} \quad x^2 - 2x - 8$



- DEALING WITH EACH QUADRATIC SEPARATELY

- $x^2 - 2x + 4 = 0$

$(x-1)^2 - 1 + 4 = 0$

$(x-1)^2 = -3$

∴ NO REAL SOLUTIONS //

- $x^2 - 2x - 8 = 0$

$\Rightarrow (x-4)(x+2) = 0$

$\Rightarrow x = \begin{cases} 4 \\ -2 \end{cases}$

CHECK SOLUTIONS

$\cancel{4^4 - 4\times4^3} + 8\times4 = 32$ ✓

$(-2)^4 - 4(-2)^3 + 8(-2)$

$16 + 32 - 16 = 32$ ✓

ONLY SOLUTIONS

$x = \begin{cases} 4 \\ -2 \end{cases}$ //

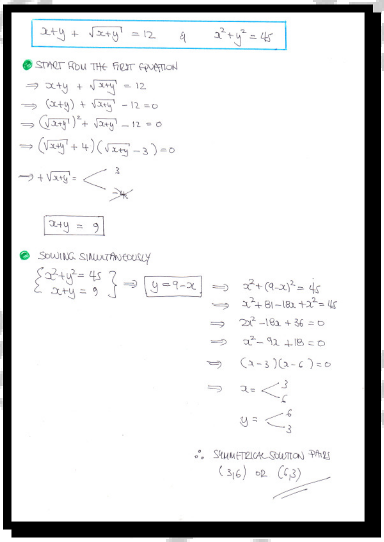
Question 71 **(*****)**

Use algebra to solve the following simultaneous equations

$$x + y + \sqrt{x+y} = 12 \qquad \text{and} \qquad x^2 + y^2 = 45,$$

given further that $x \in \mathbb{R}$ and $y \in \mathbb{R}$.

$\boxed{}$, $\boxed{(x, y) = (3,6) = (6,3)}$



$\boxed{x+y + \sqrt{x+y} = 12 \quad \& \quad x^2+y^2=45}$

START FROM THE FIRST EQUATION

$\Rightarrow x+y + \sqrt{x+y} = 12$

$\Rightarrow (x+y) + \sqrt{x+y} - 12 = 0$

$\Rightarrow \left(\sqrt{x+y}\right)^2 + \sqrt{x+y} - 12 = 0$

$\Rightarrow \left(\sqrt{x+y} + 4\right)\left(\sqrt{x+y} - 3\right) = 0$

$\Rightarrow +\sqrt{x+y} = \begin{cases} 3 \\ \cancel{-4} \end{cases}$

$\boxed{x+y = 9}$

SOLVING SIMULTANEOUSLY

$\left\{\begin{array}{c} x^2+y^2=45 \\ x+y=9 \end{array}\right\} \Rightarrow \boxed{y = 9-x} \Rightarrow x^2+(9-x)^2 = 45$

$\Rightarrow x^2+81-18x+x^2=45$

$\Rightarrow 2x^2-18x+36=0$

$\Rightarrow x^2-9x+18=0$

$\Rightarrow (x-3)(x-6)=0$

$\Rightarrow x = \begin{cases} 3 \\ 6 \end{cases}$

$y = \begin{cases} 6 \\ 3 \end{cases}$

$\therefore$ SYMMETRICAL SOLUTION PAIRS

$(3,6)$ OR $(6,3)$

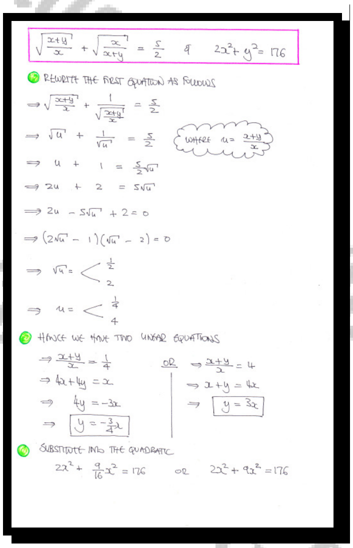
Question 72 (*****)

Use algebra to solve the following simultaneous equations

$$\sqrt{\frac{x+y}{x}}+\sqrt{\frac{x}{x+y}}=\frac{5}{2} \qquad \text{and} \qquad 2x^2+y^2=176,$$

given further that $x \in \mathbb{R}$ and $y \in \mathbb{R}$.

$\boxed{}$, $\boxed{(x,y)=(4,12),(-4,-12),\left(16\sqrt{\frac{11}{41}},-12\sqrt{\frac{11}{41}}\right),\left(-16\sqrt{\frac{11}{41}},-16\sqrt{\frac{11}{41}}\right)}$



$\boxed{\sqrt{\frac{x+y}{x}}+\sqrt{\frac{x}{x+y}}=\frac{5}{2} \quad \& \quad 2x^2+y^2=176}$

REWRITE THE FIRST EQUATION AS FOLLOWS

$\Rightarrow \sqrt{\frac{x+y}{x}}+\frac{1}{\sqrt{\frac{x+y}{x}}}=\frac{5}{2}$

$\Rightarrow \sqrt{u}+\frac{1}{\sqrt{u}}=\frac{5}{2}$ WHERE $u=\frac{x+y}{x}$

$\Rightarrow u+1=\frac{5}{2}\sqrt{u}$

$\Rightarrow 2u+2=5\sqrt{u}$

$\Rightarrow 2u-5\sqrt{u}+2=0$

$\Rightarrow (2\sqrt{u}-1)(\sqrt{u}-2)=0$

$\Rightarrow \sqrt{u}= \begin{cases}\frac{1}{2}\\ 2\end{cases}$

$\Rightarrow u= \begin{cases}\frac{1}{4}\\ 4\end{cases}$

HENCE WE HAVE TWO LINEAR EQUATIONS

$\Rightarrow \frac{x+y}{x}=\frac{1}{4}$

$\Rightarrow 4x+4y=x$

$\Rightarrow 4y=-3x$

$\Rightarrow \boxed{y=-\frac{3}{4}x}$

OR

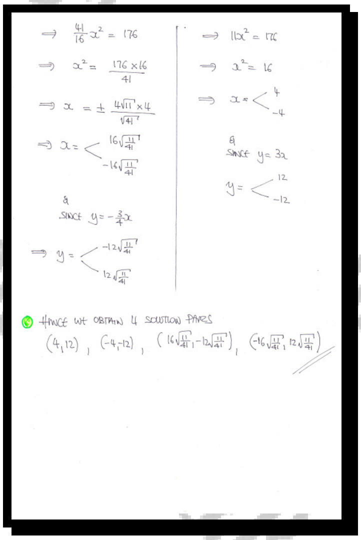
$\Rightarrow \frac{x+y}{x}=4$

$\Rightarrow x+y=4x$

$\Rightarrow \boxed{y=3x}$

SUBSTITUTE INTO THE QUADRATIC

$2x^2+\frac{9}{16}x^2=176$ OR $2x^2+9x^2=176$



$\Rightarrow \frac{41}{16}x^2=176$

$\Rightarrow x^2=\frac{176\times 16}{41}$

$\Rightarrow x=\pm\frac{4\sqrt{11}\times 4}{\sqrt{41}}$

$\Rightarrow x= \begin{cases}16\sqrt{\frac{11}{41}}\\ -16\sqrt{\frac{11}{41}}\end{cases}$

&

SINCE $y=-\frac{3}{4}x$

$\Rightarrow y= \begin{cases}-12\sqrt{\frac{11}{41}}\\ 12\sqrt{\frac{11}{41}}\end{cases}$

$\Rightarrow 11x^2=176$

$\Rightarrow x^2=16$

$\Rightarrow x= \begin{cases}4\\ -4\end{cases}$

&

SINCE $y=3x$

$y= \begin{cases}12\\ -12\end{cases}$

HENCE WE OBTAIN 4 SOLUTION PAIRS

$(4,12)$, $(-4,-12)$, $\left(16\sqrt{\frac{11}{41}},-12\sqrt{\frac{11}{41}}\right)$, $\left(-16\sqrt{\frac{11}{41}},12\sqrt{\frac{11}{41}}\right)$

Question 73 **(*****)**

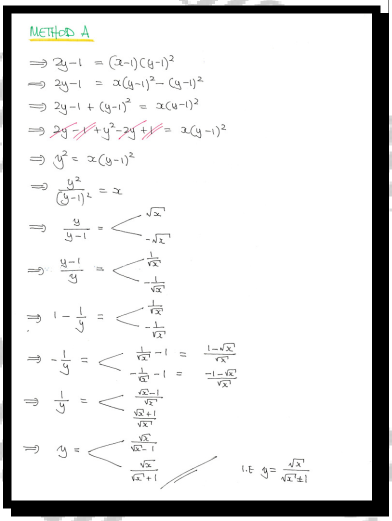
A curve has Cartesian equation

$$2y-1=(x-1)(y-1)^2,\ x\geq 0\,.$$

Make y the subject of the above equation, to show that

$$y=\frac{\sqrt{x}}{\sqrt{x}\pm 1}\,.$$

, $\boxed{y=\frac{\sqrt{x}}{\pm 1+\sqrt{x}}}$



METHOD A

$\Rightarrow 2y-1=(x-1)(y-1)^2$

$\Rightarrow 2y-1=x(y-1)^2-(y-1)^2$

$\Rightarrow 2y-1+(y-1)^2=x(y-1)^2$

$\Rightarrow \cancel{2y}\cancel{-1}+y^2\cancel{-2y}\cancel{+1}=x(y-1)^2$

$\Rightarrow y^2=x(y-1)^2$

$\Rightarrow \frac{y^2}{(y-1)^2}=x$

$\Rightarrow \frac{y}{y-1}=<\begin{array}{l}\sqrt{x}\\ -\sqrt{x}\end{array}$

$\Rightarrow \frac{y-1}{y}=<\begin{array}{l}\frac{1}{\sqrt{x}}\\ -\frac{1}{\sqrt{x}}\end{array}$

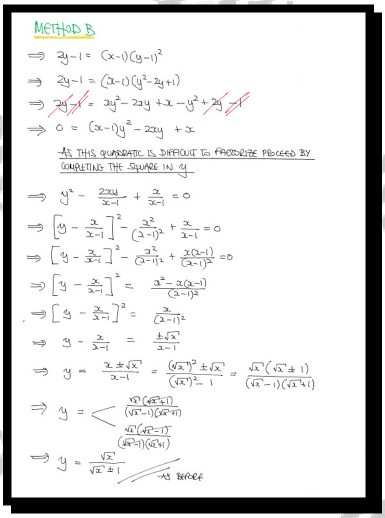
$\Rightarrow 1-\frac{1}{y}=<\begin{array}{l}\frac{1}{\sqrt{x}}\\ -\frac{1}{\sqrt{x}}\end{array}$

$\Rightarrow -\frac{1}{y}=<\begin{array}{l}\frac{1}{\sqrt{x}}-1=\frac{1-\sqrt{x}}{\sqrt{x}}\\ -\frac{1}{\sqrt{x}}-1=\frac{-1-\sqrt{x}}{\sqrt{x}}\end{array}$

$\Rightarrow \frac{1}{y}=<\begin{array}{l}\frac{\sqrt{x}-1}{\sqrt{x}}\\ \frac{\sqrt{x}+1}{\sqrt{x}}\end{array}$

$\Rightarrow y=<\begin{array}{l}\frac{\sqrt{x}}{\sqrt{x}-1}\\ \frac{\sqrt{x}}{\sqrt{x}+1}\end{array}$

I.E $y=\frac{\sqrt{x}}{\sqrt{x}\pm 1}$



METHOD B

$\Rightarrow 2y-1=(x-1)(y-1)^2$

$\Rightarrow 2y-1=(x-1)(y^2-2y+1)$

$\Rightarrow \cancel{2y}\cancel{-1}=xy^2-2xy+x-y^2\cancel{+2y}\cancel{-1}$

$\Rightarrow 0=(x-1)y^2-2xy+x$

AS THIS QUADRATIC IS DIFFICULT TO FACTORIZE PROCEED BY COMPLETING THE SQUARE IN y

$\Rightarrow y^2-\frac{2xy}{x-1}+\frac{x}{x-1}=0$

$\Rightarrow \left[y-\frac{x}{x-1}\right]^2-\frac{x^2}{(x-1)^2}+\frac{x}{x-1}=0$

$\Rightarrow \left[y-\frac{x}{x-1}\right]^2-\frac{x^2}{(x-1)^2}+\frac{x(x-1)}{(x-1)^2}=0$

$\Rightarrow \left[y-\frac{x}{x-1}\right]^2=\frac{x^2-x(x-1)}{(x-1)^2}$

$\Rightarrow \left[y-\frac{x}{x-1}\right]^2=\frac{x}{(x-1)^2}$

$\Rightarrow y-\frac{x}{x-1}=\frac{\pm\sqrt{x}}{x-1}$

$\Rightarrow y=\frac{x\pm\sqrt{x}}{x-1}=\frac{(\sqrt{x})^2\pm\sqrt{x}}{(\sqrt{x})^2-1}=\frac{\sqrt{x}(\sqrt{x}\pm 1)}{(\sqrt{x}-1)(\sqrt{x}+1)}$

$\Rightarrow y=<\begin{array}{l}\frac{\sqrt{x}(\sqrt{x}+1)}{(\sqrt{x}-1)(\sqrt{x}+1)}\\ \frac{\sqrt{x}(\sqrt{x}-1)}{(\sqrt{x}-1)(\sqrt{x}+1)}\end{array}$

$\Rightarrow y=\frac{\sqrt{x}}{\sqrt{x}\pm 1}$

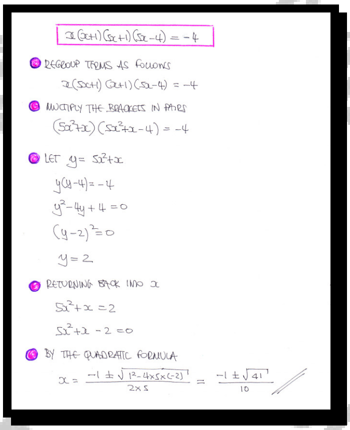
AS BEFORE

Question 74 (*****)

Find as exact surds the solutions of the equation

$$x(x+1)(5x+1)(5x-4)=-4,\ x\in\mathbb{R}.$$

$$\boxed{},\ \boxed{x=\frac{-1\pm\sqrt{41}}{10}}$$



$\boxed{x(x+1)(5x+1)(5x-4)=-4}$

- REGROUP TERMS AS FOLLOWS

$x(5x+1)(x+1)(5x-4)=-4$

- MULTIPLY THE BRACKETS IN PAIRS

$(5x^2+x)(5x^2+x-4)=-4$

- LET $y=5x^2+x$

$y(y-4)=-4$

$y^2-4y+4=0$

$(y-2)^2=0$

$y=2$

- RETURNING BACK INTO x

$5x^2+x=2$

$5x^2+x-2=0$

- BY THE QUADRATIC FORMULA

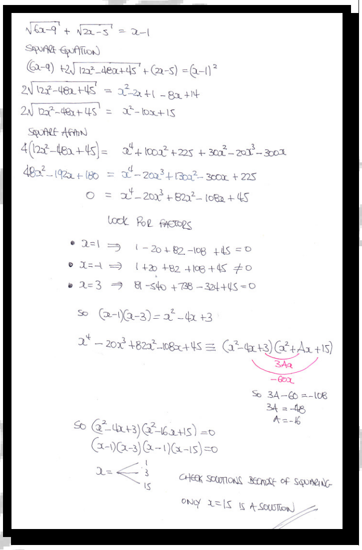
$x=\frac{-1\pm\sqrt{1^2-4\times5\times(-2)}}{2\times5}=\frac{-1\pm\sqrt{41}}{10}$

Question 75 (*****)

Determine the real root of the equation

$$\sqrt{6x-9}+\sqrt{2x-5}=x-1.$$

$$\boxed{x=15}$$



$\sqrt{6x-9}+\sqrt{2x-5}=x-1$

SQUARE EQUATION

$(6x-9)+2\sqrt{12x^2-48x+45}+(2x-5)=(x-1)^2$

$2\sqrt{12x^2-48x+45}=x^2-2x+1-8x+14$

$2\sqrt{12x^2-48x+45}=x^2-10x+15$

SQUARE AGAIN

$4(12x^2-48x+45)=x^4+100x^2+225+30x^2-20x^3-300x$

$48x^2-192x+180=x^4-20x^3+130x^2-300x+225$

$0=x^4-20x^3+82x^2-108x+45$

LOOK FOR FACTORS

- $x=1 \Rightarrow 1-20+82-108+45=0$
- $x=-1 \Rightarrow 1+20+82+108+45\neq0$
- $x=3 \Rightarrow 81-540+738-324+45=0$

SO $(x-1)(x-3)=x^2-4x+3$

$x^4-20x^3+82x^2-108x+45\equiv(x^2-4x+3)(x^2+Ax+15)$

$3Ax$

$-60x$

SO $3A-60=-108$

$3A=-48$

$A=-16$

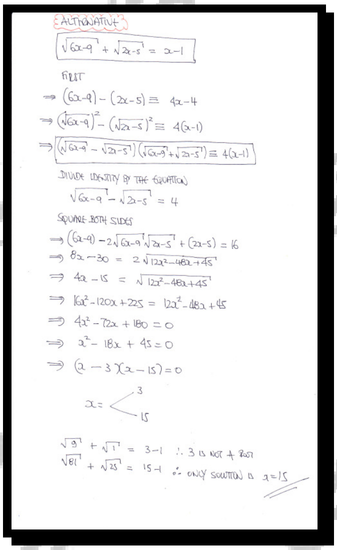
SO $(x^2-4x+3)(x^2-16x+15)=0$

$(x-1)(x-3)(x-1)(x-15)=0$

$x=\left\langle\begin{matrix}1\\3\\15\end{matrix}\right.$

CHECK SOLUTIONS BECAUSE OF SQUARING

ONLY $x=15$ IS A SOLUTION



ALTERNATIVE

$\boxed{\sqrt{6x-9}+\sqrt{2x-5}=x-1}$

FIRST

$\Rightarrow (6x-9)-(2x-5)\equiv4x-4$

$\Rightarrow (\sqrt{6x-9})^2-(\sqrt{2x-5})^2\equiv4(x-1)$

$\Rightarrow \boxed{(\sqrt{6x-9}-\sqrt{2x-5})(\sqrt{6x-9}+\sqrt{2x-5})\equiv4(x-1)}$

DIVIDE IDENTITY BY THE EQUATION

$\sqrt{6x-9}-\sqrt{2x-5}=4$

SQUARE BOTH SIDES

$\Rightarrow (6x-9)-2\sqrt{6x-9}\sqrt{2x-5}+(2x-5)=16$

$\Rightarrow 8x-30=2\sqrt{12x^2-48x+45}$

$\Rightarrow 4x-15=\sqrt{12x^2-48x+45}$

$\Rightarrow 16x^2-120x+225=12x^2-48x+45$

$\Rightarrow 4x^2-72x+180=0$

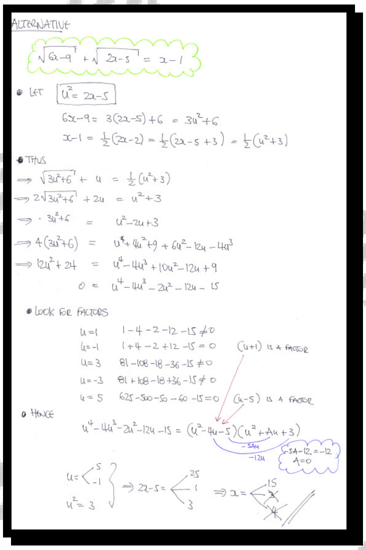
$\Rightarrow x^2-18x+45=0$

$\Rightarrow (x-3)(x-15)=0$

$x=\left\langle\begin{matrix}3\\15\end{matrix}\right.$

$\sqrt{9}+\sqrt{1}=3-1$ $\therefore$ 3 IS NOT A ROOT

$\sqrt{81}+\sqrt{25}=15-1$ $\therefore$ ONLY SOLUTION IS $x=15$



ALTERNATIVE

$\sqrt{6x-9}+\sqrt{2x-5}=x-1$

- LET $\boxed{u^2=2x-5}$

$6x-9=3(2x-5)+6=3u^2+6$

$x-1=\frac{1}{2}(2x-2)=\frac{1}{2}(2x-5+3)=\frac{1}{2}(u^2+3)$

- THUS

$\Rightarrow \sqrt{3u^2+6}+u=\frac{1}{2}(u^2+3)$

$\Rightarrow 2\sqrt{3u^2+6}+2u=u^2+3$

$\Rightarrow 3u^2+6=u^2-2u+3$

$\Rightarrow 4(3u^2+6)=u^4+4u^2+9+6u^2-12u-4u^3$

$\Rightarrow 12u^2+24=u^4-4u^3+10u^2-12u+9$

$0=u^4-4u^3-2u^2-12u-15$

- LOOK FOR FACTORS

$u=1$ $1-4-2-12-15\neq0$

$u=-1$ $1+4-2+12-15=0$ $(u+1)$ IS A FACTOR

$u=3$ $81-108-18-36-15\neq0$

$u=-3$ $81+108-18+36-15\neq0$

$u=5$ $625-500-50-60-15=0$ $(u-5)$ IS A FACTOR

- HENCE

$u^4-4u^3-2u^2-12u-15=(u^2-4u-5)(u^2+Au+3)$

$-5Au$

$-12u$

$-5A-12=-12$

$A=0$

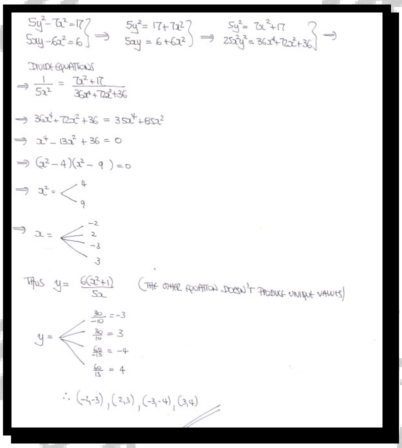
$\left.\begin{matrix}u=\left\langle\begin{matrix}5\\-1\end{matrix}\right.\\u^2=3\end{matrix}\right\}\Rightarrow 2x-5=\left\langle\begin{matrix}25\\1\\3\end{matrix}\right.\Rightarrow x=\left\langle\begin{matrix}15\\\cancel{3}\\\cancel{4}\end{matrix}\right.$

Question 76 (*****)

Find the real solutions for the following system of simultaneous equations

$$5y^2 - 7x^2 = 17$$
$$5xy - 6x^2 = 6.$$

$$\boxed{(-2,-3),\ (2,3),\ (3,4),\ (-3,-4)}$$



$$\left.\begin{array}{l}5y^2-7x^2=17\\5xy-6x^2=6\end{array}\right\} \Rightarrow \left.\begin{array}{l}5y^2=17+7x^2\\5xy=6+6x^2\end{array}\right\} \Rightarrow \left.\begin{array}{l}5y^2=7x^2+17\\25x^2y^2=36x^4+72x^2+36\end{array}\right\} \Rightarrow$$

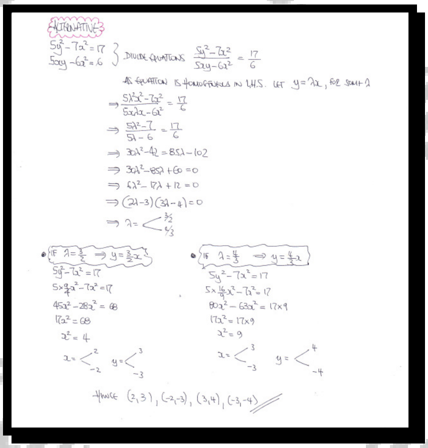
DIVIDE EQUATIONS

$$\Rightarrow \frac{1}{5x^2} = \frac{7x^2+17}{36x^4+72x^2+36}$$
$$\Rightarrow 36x^4+72x^2+36 = 35x^4+85x^2$$
$$\Rightarrow x^4-13x^2+36=0$$
$$\Rightarrow (x^2-4)(x^2-9)=0$$
$$\Rightarrow x^2 = \begin{cases}4\\9\end{cases}$$
$$\Rightarrow x = \begin{cases}-2\\2\\-3\\3\end{cases}$$

Thus $y = \frac{6(x^2+1)}{5x}$ (THE OTHER EQUATION DOESN'T PRODUCE UNIQUE VALUES)

$$y = \begin{cases}\frac{30}{-10}=-3\\\frac{30}{10}=3\\\frac{60}{-15}=-4\\\frac{60}{15}=4\end{cases}$$

$\therefore (-2,-3), (2,3), (-3,-4), (3,4)$



ALTERNATIVE

$$\left.\begin{array}{l}5y^2-7x^2=17\\5xy-6x^2=6\end{array}\right\}$$ DIVIDE EQUATIONS $\frac{5y^2-7x^2}{5xy-6x^2}=\frac{17}{6}$

AS EQUATION IS HOMOGENEOUS IN L.H.S. LET $y=\lambda x$, FOR SOME λ

$$\Rightarrow \frac{5\lambda^2x^2-7x^2}{5x\lambda x-6x^2}=\frac{17}{6}$$
$$\Rightarrow \frac{5\lambda^2-7}{5\lambda-6}=\frac{17}{6}$$
$$\Rightarrow 30\lambda^2-42=85\lambda-102$$
$$\Rightarrow 30\lambda^2-85\lambda+60=0$$
$$\Rightarrow 6\lambda^2-17\lambda+12=0$$
$$\Rightarrow (2\lambda-3)(3\lambda-4)=0$$
$$\Rightarrow \lambda = \begin{cases}3/2\\4/3\end{cases}$$

- IF $\lambda=\frac{3}{2} \Rightarrow y=\frac{3}{2}x$

$5y^2-7x^2=17$

$5\times\frac{9}{4}x^2-7x^2=17$

$45x^2-28x^2=68$

$17x^2=68$

$x^2=4$

$x=\begin{cases}2\\-2\end{cases} \quad y=\begin{cases}3\\-3\end{cases}$

- IF $\lambda=\frac{4}{3} \Rightarrow y=\frac{4}{3}x$

$5y^2-7x^2=17$

$5\times\frac{16}{9}x^2-7x^2=17$

$80x^2-63x^2=17\times9$

$17x^2=17\times9$

$x^2=9$

$x=\begin{cases}3\\-3\end{cases} \quad y=\begin{cases}4\\-4\end{cases}$

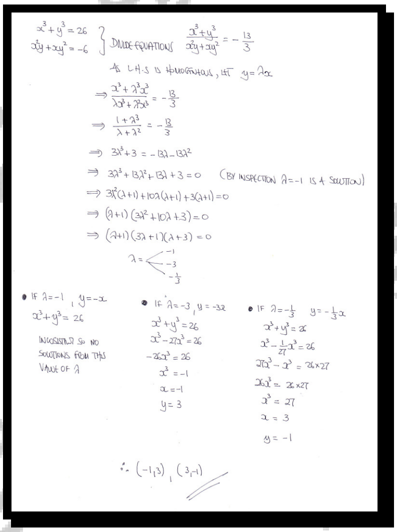
HENCE $(2,3), (-2,-3), (3,4), (-3,-4)$

Question 77 (*****)

Find the real solutions for the following system of simultaneous equations

$$x^3 + y^3 = 26$$
$$x^2y + xy^2 = -6.$$

V, $\boxed{(x,y)=(-1,3) \text{ in any order}}$



$$\left.\begin{array}{l}x^3+y^3=26\\x^2y+xy^2=-6\end{array}\right\}$$ DIVIDE EQUATIONS $\frac{x^3+y^3}{x^2y+xy^2}=-\frac{13}{3}$

AS L.H.S IS HOMOGENEOUS, LET $y=\lambda x$

$$\Rightarrow \frac{x^3+\lambda^3x^3}{\lambda x^3+\lambda^2x^3}=-\frac{13}{3}$$
$$\Rightarrow \frac{1+\lambda^3}{\lambda+\lambda^2}=-\frac{13}{3}$$
$$\Rightarrow 3\lambda^3+3=-13\lambda-13\lambda^2$$
$$\Rightarrow 3\lambda^3+13\lambda^2+13\lambda+3=0$$ (BY INSPECTION $\lambda=-1$ IS A SOLUTION)
$$\Rightarrow 3\lambda^2(\lambda+1)+10\lambda(\lambda+1)+3(\lambda+1)=0$$
$$\Rightarrow (\lambda+1)(3\lambda^2+10\lambda+3)=0$$
$$\Rightarrow (\lambda+1)(3\lambda+1)(\lambda+3)=0$$
$$\lambda=\begin{cases}-1\\-3\\-\frac{1}{3}\end{cases}$$

- IF $\lambda=-1$, $y=-x$

$x^3+y^3=26$

INCONSISTENT SO NO SOLUTIONS FROM THIS VALUE OF λ

- IF $\lambda=-3$, $y=-3x$

$x^3+y^3=26$

$x^3-27x^3=26$

$-26x^3=26$

$x^3=-1$

$x=-1$

$y=3$

- IF $\lambda=-\frac{1}{3}$ $y=-\frac{1}{3}x$

$x^3+y^3=26$

$x^3-\frac{1}{27}x^3=26$

$27x^3-x^3=26\times27$

$26x^3=26\times27$

$x^3=27$

$x=3$

$y=-1$

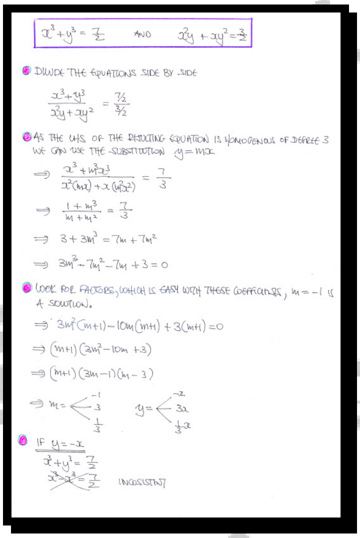
$\therefore (-1,3), (3,-1)$

Question 78 **(*****)**

Find the real solutions for the following system of simultaneous equations

$$x^3 + y^3 = \frac{7}{2}$$
$$x^2 y + xy^2 = \frac{3}{2}.$$

, $\boxed{(x, y) = \left(\frac{1}{2}, \frac{3}{2}\right) \text{ in any order}}$



$\boxed{x^3+y^3=\frac{7}{2} \quad \text{AND} \quad x^2y+xy^2=\frac{3}{2}}$

- DIVIDE THE EQUATIONS SIDE BY SIDE

$$\frac{x^3+y^3}{x^2y+xy^2}=\frac{7/2}{3/2}$$

- AS THE LHS OF THE RESULTING EQUATION IS HOMOGENOUS OF DEGREE 3 WE CAN USE THE SUBSTITUTION $y=mx$

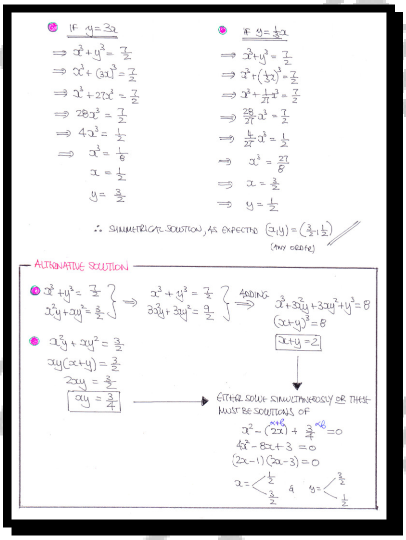
$$\Rightarrow \frac{x^3+m^3x^3}{x^2(mx)+x(m^2x^2)}=\frac{7}{3}$$
$$\Rightarrow \frac{1+m^3}{m+m^2}=\frac{7}{3}$$
$$\Rightarrow 3+3m^3=7m+7m^2$$
$$\Rightarrow 3m^3-7m^2-7m+3=0$$

- LOOK FOR FACTORS, WHICH IS EASY WITH THESE COEFFICIENTS, $m=-1$ IS A SOLUTION.

$$\Rightarrow 3m^2(m+1)-10m(m+1)+3(m+1)=0$$
$$\Rightarrow (m+1)(3m^2-10m+3)$$
$$\Rightarrow (m+1)(3m-1)(m-3)$$
$$\Rightarrow m=\begin{cases}-1\\3\\\frac{1}{3}\end{cases} \qquad y=\begin{cases}-x\\3x\\\frac{1}{3}x\end{cases}$$

- IF $y=-x$

$$x^3+y^3=\frac{7}{2}$$
$$\cancel{x^3-x^3=\frac{7}{2}} \quad \text{INCONSISTENT}$$



- IF $y=3x$

$$\Rightarrow x^3+y^3=\frac{7}{2}$$
$$\Rightarrow x^3+(3x)^3=\frac{7}{2}$$
$$\Rightarrow x^3+27x^3=\frac{7}{2}$$
$$\Rightarrow 28x^3=\frac{7}{2}$$
$$\Rightarrow 4x^3=\frac{1}{2}$$
$$\Rightarrow x^3=\frac{1}{8}$$
$$x=\frac{1}{2}$$
$$y=\frac{3}{2}$$

- IF $y=\frac{1}{3}x$

$$\Rightarrow x^3+y^3=\frac{7}{2}$$
$$\Rightarrow x^3+\left(\tfrac{1}{3}x\right)^3=\frac{7}{2}$$
$$\Rightarrow x^3+\frac{1}{27}x^3=\frac{7}{2}$$
$$\Rightarrow \frac{28}{27}x^3=\frac{7}{2}$$
$$\Rightarrow \frac{4}{27}x^3=\frac{1}{2}$$
$$\Rightarrow x^3=\frac{27}{8}$$
$$\Rightarrow x=\frac{3}{2}$$
$$\Rightarrow y=\frac{1}{2}$$

$\therefore$ SYMMETRICAL SOLUTION, AS EXPECTED $(x,y)=\left(\frac{3}{2},\frac{1}{2}\right)$ //
(ANY ORDER)

ALTERNATIVE SOLUTION

- $\left.\begin{array}{l}x^3+y^3=\frac{7}{2}\\x^2y+xy^2=\frac{3}{2}\end{array}\right\} \Rightarrow \left.\begin{array}{l}x^3+y^3=\frac{7}{2}\\3x^2y+3xy^2=\frac{9}{2}\end{array}\right\} \xrightarrow{\text{ADDING}} \begin{array}{l}x^3+3x^2y+3xy^2+y^3=8\\(x+y)^3=8\\\boxed{x+y=2}\end{array}$

- $x^2y+xy^2=\frac{3}{2}$

$$xy(x+y)=\frac{3}{2}$$
$$2xy=\frac{3}{2}$$
$$\boxed{xy=\frac{3}{4}}$$

EITHER SOLVE SIMULTANEOUSLY OR THESE MUST BE SOLUTIONS OF

$$x^2-(\overset{\alpha+\beta}{2x})+\overset{\alpha\beta}{\frac{3}{4}}=0$$
$$4x^2-8x+3=0$$
$$(2x-1)(2x-3)=0$$
$$x=\begin{cases}\frac{1}{2}\\\frac{3}{2}\end{cases} \quad \& \quad y=\begin{cases}\frac{3}{2}\\\frac{1}{2}\end{cases}$$

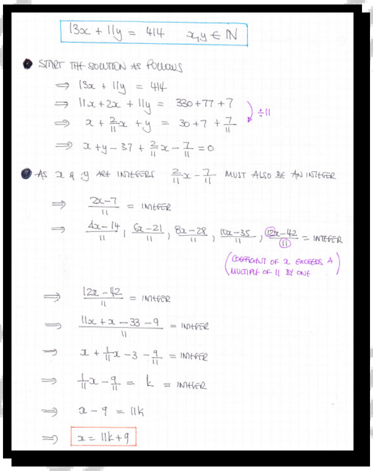
Question 79 (*****)

Solve the equation

$$13x + 11y = 414,$$

given further that $x \in \mathbb{N}$, $y \in \mathbb{N}$.

V, ☐, SP, $\boxed{(x,y) = (9,27),\ (20,14),\ (31,1)}$



$\boxed{13x + 11y = 414 \qquad x,y \in \mathbb{N}}$

- START THE SOLUTION AS FOLLOWS

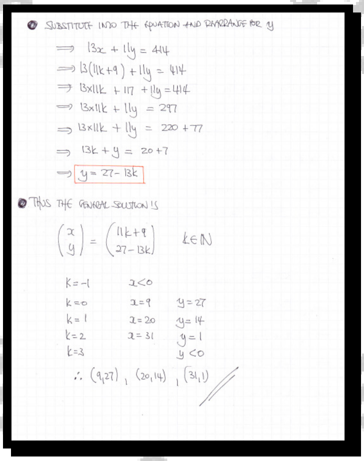
$$\Rightarrow 13x + 11y = 414$$
$$\Rightarrow 11x + 2x + 11y = 330 + 77 + 7 \quad \Big) \div 11$$
$$\Rightarrow x + \tfrac{2}{11}x + y = 30 + 7 + \tfrac{7}{11}$$
$$\Rightarrow x + y - 37 + \tfrac{2}{11}x - \tfrac{7}{11} = 0$$

- AS x & y ARE INTEGERS $\frac{2}{11}x - \frac{7}{11}$ MUST ALSO BE AN INTEGER

$$\Rightarrow \frac{2x-7}{11} = \text{INTEGER}$$
$$\Rightarrow \frac{4x-14}{11},\ \frac{6x-21}{11},\ \frac{8x-28}{11},\ \frac{10x-35}{11},\ \frac{\textcircled{12}x-42}{\textcircled{11}} = \text{INTEGER}$$

(COEFFICIENT OF x EXCEEDS A MULTIPLE OF 11 BY ONE)

$$\Rightarrow \frac{12x-42}{11} = \text{INTEGER}$$
$$\Rightarrow \frac{11x + x - 33 - 9}{11} = \text{INTEGER}$$
$$\Rightarrow x + \tfrac{1}{11}x - 3 - \tfrac{9}{11} = \text{INTEGER}$$
$$\Rightarrow \tfrac{1}{11}x - \tfrac{9}{11} = k = \text{INTEGER}$$
$$\Rightarrow x - 9 = 11k$$
$$\Rightarrow \boxed{x = 11k + 9}$$



- SUBSTITUTE INTO THE EQUATION AND REARRANGE FOR y

$$\Rightarrow 13x + 11y = 414$$
$$\Rightarrow 13(11k+9) + 11y = 414$$
$$\Rightarrow 13 \times 11k + 117 + 11y = 414$$
$$\Rightarrow 13 \times 11k + 11y = 297$$
$$\Rightarrow 13 \times 11k + 11y = 220 + 77$$
$$\Rightarrow 13k + y = 20 + 7$$
$$\Rightarrow \boxed{y = 27 - 13k}$$

- THUS THE GENERAL SOLUTION IS

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 11k+9 \\ 27-13k \end{pmatrix} \qquad k \in \mathbb{N}$$

$k=-1$	$x<0$	
$k=0$	$x=9$	$y=27$
$k=1$	$x=20$	$y=14$
$k=2$	$x=31$	$y=1$
$k=3$		$y<0$

$\therefore (9,27),\ (20,14),\ (31,1)$ //

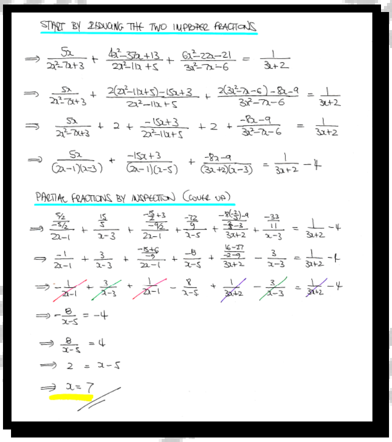
Question 80 (*****)

Solve the following rational equation, over the set of real numbers.

$$\frac{5x}{2x^2-7x+3}+\frac{4x^2-37x+13}{2x^2-11x+5}+\frac{6x^2-22x-21}{3x^2-7x-6}=\frac{1}{3x+2}.$$

You may ignore non finite solutions.

$\boxed{}$, $\boxed{x=7}$



START BY REDUCING THE TWO IMPROPER FRACTIONS

$$\Rightarrow \frac{5x}{2x^2-7x+3}+\frac{4x^2-37x+13}{2x^2-11x+5}+\frac{6x^2-22x-21}{3x^2-7x-6}=\frac{1}{3x+2}$$

$$\Rightarrow \frac{5x}{2x^2-7x+3}+\frac{2(2x^2-11x+5)-15x+3}{2x^2-11x+5}+\frac{2(3x^2-7x-6)-8x-9}{3x^2-7x-6}=\frac{1}{3x+2}$$

$$\Rightarrow \frac{5x}{2x^2-7x+3}+2+\frac{-15x+3}{2x^2-11x+5}+2+\frac{-8x-9}{3x^2-7x-6}=\frac{1}{3x+2}$$

$$\Rightarrow \frac{5x}{(2x-1)(x-3)}+\frac{-15x+3}{(2x-1)(x-5)}+\frac{-8x-9}{(3x+2)(x-3)}=\frac{1}{3x+2}-4$$

PARTIAL FRACTIONS BY INSPECTION (COVER UP)

$$\Rightarrow \frac{\frac{5/2}{-5/2}}{2x-1}+\frac{\frac{15}{5}}{x-3}+\frac{\frac{-\frac{15}{2}+3}{-9/2}}{2x-1}+\frac{\frac{-72}{9}}{x-5}+\frac{\frac{-8(-\frac{2}{3})-9}{-\frac{2}{3}-3}}{3x+2}+\frac{\frac{-33}{11}}{x-3}=\frac{1}{3x+2}-4$$

$$\Rightarrow \frac{-1}{2x-1}+\frac{3}{x-3}+\frac{\frac{-15+6}{-9}}{2x-1}+\frac{-8}{x-5}+\frac{\frac{16-27}{-2-9}}{3x+2}-\frac{3}{x-3}=\frac{1}{3x+2}-4$$

$$\Rightarrow \cancel{-\frac{1}{2x-1}}+\cancel{\frac{3}{x-3}}+\cancel{\frac{1}{2x-1}}-\frac{8}{x-5}+\cancel{\frac{1}{3x+2}}-\cancel{\frac{3}{x-3}}=\cancel{\frac{1}{3x+2}}-4$$

$$\Rightarrow -\frac{8}{x-5}=-4$$

$$\Rightarrow \frac{8}{x-5}=4$$

$$\Rightarrow 2=x-5$$

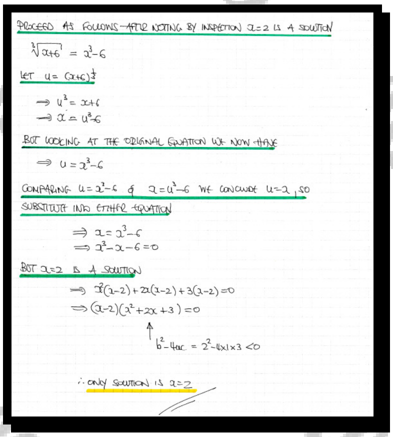
$$\Rightarrow x=7$$

Question 81 (*****)

Show that $x=2$ is the only real solution of the following equation.

$$x^3-6 = \sqrt[3]{x+6}\,.$$

V, , proof



PROCEED AS FOLLOWS – AFTER NOTING BY INSPECTION $x=2$ IS A SOLUTION

$$\sqrt[3]{x+6} = x^3-6$$

LET $u=(x+6)^{\frac{1}{3}}$

$$\Rightarrow u^3 = x+6$$
$$\Rightarrow x = u^3-6$$

BUT LOOKING AT THE ORIGINAL EQUATION WE NOW HAVE

$$\Rightarrow u = x^3-6$$

COMPARING $u=x^3-6$ & $x=u^3-6$ WE CONCLUDE $u=x$, SO SUBSTITUTE INTO EITHER EQUATION

$$\Rightarrow x = x^3-6$$
$$\Rightarrow x^3-x-6=0$$

BUT $x=2$ IS A SOLUTION

$$\Rightarrow x^2(x-2)+2x(x-2)+3(x-2)=0$$
$$\Rightarrow (x-2)(x^2+2x+3)=0$$

$$\uparrow$$
$$b^2-4ac = 2^2-4\times1\times3 < 0$$

$\therefore$ ONLY SOLUTION IS $x=2$

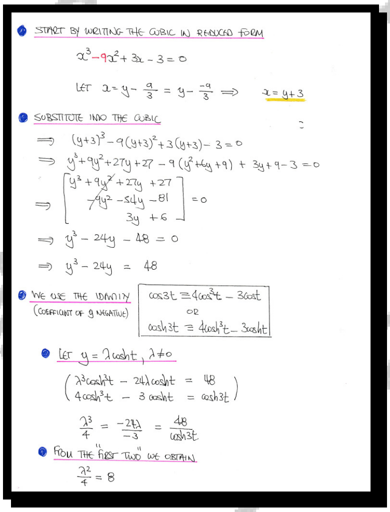
Question 82 (*****)

Solve the cubic equation

$$x^3 - 9x^2 + 3x - 3 = 0, \ x \in \mathbb{R}.$$

You may assume that this cubic equation only has one real root.

V, $\boxed{}$, $\boxed{x = 1 + 2^{-\frac{2}{3}} - 2^{-\frac{4}{3}}}$



● START BY WRITING THE CUBIC IN REDUCED FORM

$x^3 - 9x^2 + 3x - 3 = 0$

LET $x = y - \frac{a}{3} = y - \frac{-9}{3} \Longrightarrow \quad x = y + 3$

● SUBSTITUTE INTO THE CUBIC

$\Longrightarrow (y+3)^3 - 9(y+3)^2 + 3(y+3) - 3 = 0$

$\Longrightarrow y^3 + 9y^2 + 27y + 27 - 9(y^2 + 6y + 9) + 3y + 9 - 3 = 0$

$$\Longrightarrow \left[\begin{array}{rrrr} y^3 + \cancel{9y^2} & +27y & +27 \\ \cancel{-9y^2} & -54y & -81 \\ & 3y & +6 \end{array}\right] = 0$$

$\Longrightarrow y^3 - 24y - 48 = 0$

$\Longrightarrow y^3 - 24y = 48$

● WE USE THE IDENTITY (COEFFICIENT OF y NEGATIVE)

$$\boxed{\begin{array}{c} \cos 3t \equiv 4\cos^3 t - 3\cos t \\ \text{OR} \\ \cosh 3t \equiv 4\cosh^3 t - 3\cosh t \end{array}}$$

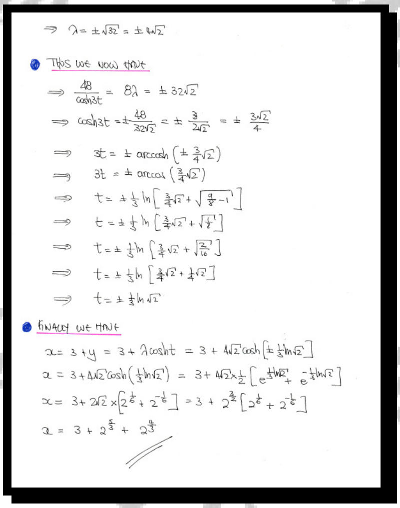
● LET $y = \lambda \cosh t, \ \lambda \neq 0$

$$\left(\begin{array}{l} \lambda^3 \cosh^3 t - 24\lambda \cosh t = 48 \\ 4\cosh^3 t - 3\cosh t = \cosh 3t \end{array}\right)$$

$$\frac{\lambda^3}{4} = \frac{-24\lambda}{-3} = \frac{48}{\cosh 3t}$$

● FROM THE "FIRST TWO" WE OBTAIN

$\frac{\lambda^2}{4} = 8$



$\Longrightarrow \lambda = \pm\sqrt{32} = \pm 4\sqrt{2}$

● THUS WE NOW THAT

$\Longrightarrow \frac{48}{\cosh 3t} = 8\lambda = \pm 32\sqrt{2}$

$\Longrightarrow \cosh 3t = \pm\frac{48}{32\sqrt{2}} = \pm\frac{3}{2\sqrt{2}} = \pm\frac{3\sqrt{2}}{4}$

$\Longrightarrow 3t = \pm \operatorname{arccosh}\left(\pm\frac{3}{4}\sqrt{2}\right)$

$\Longrightarrow 3t = \pm \operatorname{arccos}\left(\frac{3}{4}\sqrt{2}\right)$

$\Longrightarrow t = \pm\frac{1}{3}\ln\left[\frac{3}{4}\sqrt{2} + \sqrt{\frac{9}{8} - 1}\right]$

$\Longrightarrow t = \pm\frac{1}{3}\ln\left[\frac{3}{4}\sqrt{2} + \sqrt{\frac{1}{8}}\right]$

$\Longrightarrow t = \pm\frac{1}{3}\ln\left[\frac{3}{4}\sqrt{2} + \sqrt{\frac{2}{16}}\right]$

$\Longrightarrow t = \pm\frac{1}{3}\ln\left[\frac{3}{4}\sqrt{2} + \frac{1}{4}\sqrt{2}\right]$

$\Longrightarrow t = \pm\frac{1}{3}\ln\sqrt{2}$

● FINALLY WE HAVE

$x = 3 + y = 3 + \lambda\cosh t = 3 + 4\sqrt{2}\cosh\left[\pm\frac{1}{3}\ln\sqrt{2}\right]$

$x = 3 + 4\sqrt{2}\cosh\left(\frac{1}{3}\ln\sqrt{2}\right) = 3 + 4\sqrt{2} \times \frac{1}{2}\left[e^{\frac{1}{3}\ln\sqrt{2}} + e^{-\frac{1}{3}\ln\sqrt{2}}\right]$

$x = 3 + 2\sqrt{2} \times \left[2^{\frac{1}{6}} + 2^{-\frac{1}{6}}\right] = 3 + 2^{\frac{3}{2}}\left[2^{\frac{1}{6}} + 2^{-\frac{1}{6}}\right]$

$x = 3 + 2^{\frac{5}{3}} + 2^{\frac{4}{3}}$ //

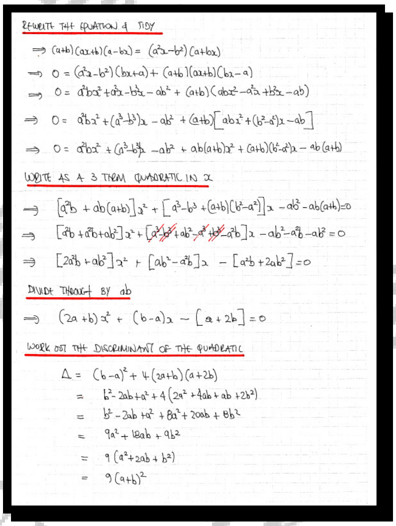
Question 83 (*****)

Solve the following equation

$$(a+b)(ax+b)(a-bx)=\left(a^2x-b^2\right)(a+bx),\ x\in\mathbb{R}.$$

Give the solutions in terms of a and b, where appropriate.

$$\boxed{x=1 \ \cup \ x=-\frac{a+2b}{2a+b}}$$



REWRITE THE EQUATION & TIDY

$\Rightarrow (a+b)(ax+b)(a-bx)=(a^2x-b^2)(a+bx)$

$\Rightarrow 0=(a^2x-b^2)(bx+a)+(a+b)(ax+b)(bx-a)$

$\Rightarrow 0=a^2bx^2+a^3x-b^3x-ab^2+(a+b)(abx^2-a^2x+b^2x-ab)$

$\Rightarrow 0=a^2bx^2+(a^3-b^3)x-ab^2+(a+b)\left[abx^2+(b^2-a^2)x-ab\right]$

$\Rightarrow 0=a^2bx^2+(a^3-b^3)x-ab^2+ab(a+b)x^2+(a+b)(b^2-a^2)x-ab(a+b)$

WRITE AS A 3 TERM QUADRATIC IN x

$\Rightarrow \left[a^2b+ab(a+b)\right]x^2+\left[a^3-b^3+(a+b)(b^2-a^2)\right]x-ab^2-ab(a+b)=0$

$\Rightarrow \left[a^2b+a^2b+ab^2\right]x^2+\left[\cancel{a^3}-\cancel{b^3}+ab^2-\cancel{a^3}+\cancel{b^3}-a^2b\right]x-ab^2-a^2b-ab^2=0$

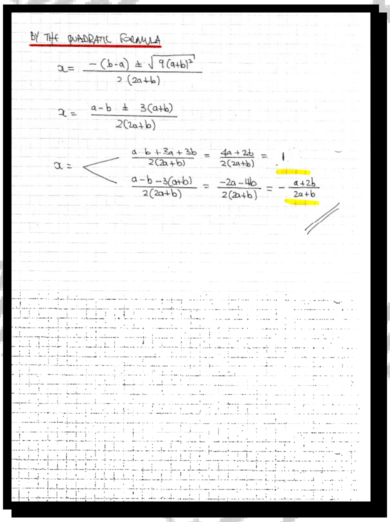
$\Rightarrow \left[2a^2b+ab^2\right]x^2+\left[ab^2-a^2b\right]x-\left[a^2b+2ab^2\right]=0$

DIVIDE THROUGH BY ab

$\Rightarrow (2a+b)x^2+(b-a)x-\left[a+2b\right]=0$

WORK OUT THE DISCRIMINANT OF THE QUADRATIC

$$\begin{aligned}\Delta &= (b-a)^2+4(2a+b)(a+2b)\\ &= b^2-2ab+a^2+4(2a^2+4ab+ab+2b^2)\\ &= b^2-2ab+a^2+8a^2+20ab+8b^2\\ &= 9a^2+18ab+9b^2\\ &= 9(a^2+2ab+b^2)\\ &= 9(a+b)^2\end{aligned}$$



BY THE QUADRATIC FORMULA

$$x=\frac{-(b-a)\pm\sqrt{9(a+b)^2}}{2(2a+b)}$$

$$x=\frac{a-b\pm3(a+b)}{2(2a+b)}$$

$$x=\begin{cases}\dfrac{a-b+3a+3b}{2(2a+b)}=\dfrac{4a+2b}{2(2a+b)}=1\\[2mm] \dfrac{a-b-3(a+b)}{2(2a+b)}=\dfrac{-2a-4b}{2(2a+b)}=-\dfrac{a+2b}{2a+b}\end{cases}$$

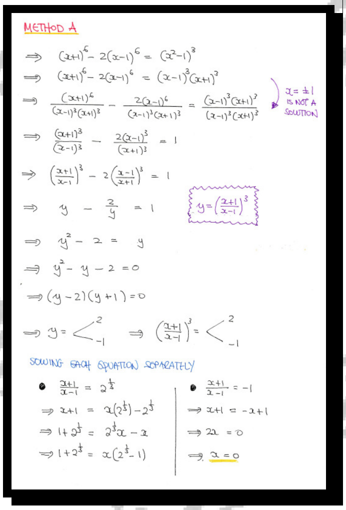
//

Question 84 (*****)

Determine, in exact form where appropriate, the two real roots of the equation

$$(x+1)^6 - 2(x-1)^6 = (x^2-1)^3 .$$

$\boxed{}$, $\boxed{x=0}$, $\boxed{x = \dfrac{2^{\frac{1}{3}}-1}{2^{\frac{1}{3}}+1}}$



METHOD A

$\Rightarrow (x+1)^6 - 2(x-1)^6 = (x^2-1)^3$

$\Rightarrow (x+1)^6 - 2(x-1)^6 = (x-1)^3(x+1)^3$

$\Rightarrow \frac{(x+1)^6}{(x-1)^3(x+1)^3} - \frac{2(x-1)^6}{(x-1)^3(x+1)^3} = \frac{(x-1)^3(x+1)^3}{(x-1)^3(x+1)^3}$

$x = \pm 1$ IS NOT A SOLUTION

$\Rightarrow \frac{(x+1)^3}{(x-1)^3} - \frac{2(x-1)^3}{(x+1)^3} = 1$

$\Rightarrow \left(\frac{x+1}{x-1}\right)^3 - 2\left(\frac{x-1}{x+1}\right)^3 = 1$

$\Rightarrow y - \frac{2}{y} = 1$

$y = \left(\frac{x+1}{x-1}\right)^3$

$\Rightarrow y^2 - 2 = y$

$\Rightarrow y^2 - y - 2 = 0$

$\Rightarrow (y-2)(y+1) = 0$

$\Rightarrow y = \begin{cases} 2 \\ -1 \end{cases} \Rightarrow \left(\frac{x+1}{x-1}\right)^3 = \begin{cases} 2 \\ -1 \end{cases}$

SOLVING EACH EQUATION SEPARATELY

- $\frac{x+1}{x-1} = 2^{\frac{1}{3}}$

$\Rightarrow x+1 = x(2^{\frac{1}{3}}) - 2^{\frac{1}{3}}$

$\Rightarrow 1 + 2^{\frac{1}{3}} = 2^{\frac{1}{3}}x - x$

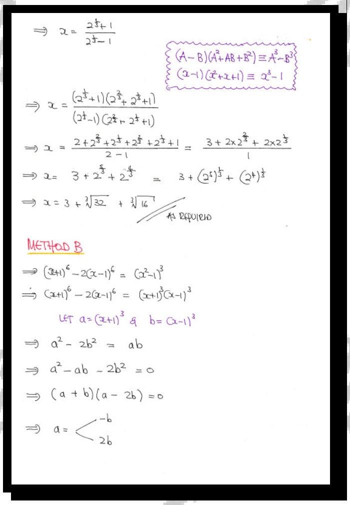
$\Rightarrow 1 + 2^{\frac{1}{3}} = x(2^{\frac{1}{3}} - 1)$

- $\frac{x+1}{x-1} = -1$

$\Rightarrow x+1 = -x+1$

$\Rightarrow 2x = 0$

$\Rightarrow x = 0$



$\Rightarrow x = \frac{2^{\frac{1}{3}}+1}{2^{\frac{1}{3}}-1}$

$(A-B)(A^2+AB+B^2) \equiv A^3 - B^3$

$(x-1)(x^2+x+1) \equiv x^3 - 1$

$\Rightarrow x = \frac{(2^{\frac{1}{3}}+1)(2^{\frac{2}{3}}+2^{\frac{1}{3}}+1)}{(2^{\frac{1}{3}}-1)(2^{\frac{2}{3}}+2^{\frac{1}{3}}+1)}$

$\Rightarrow x = \frac{2+2^{\frac{2}{3}}+2^{\frac{1}{3}}+2^{\frac{2}{3}}+2^{\frac{1}{3}}+1}{2-1} = \frac{3+2\times2^{\frac{2}{3}}+2\times2^{\frac{1}{3}}}{1}$

$\Rightarrow x = 3 + 2^{\frac{5}{3}} + 2^{\frac{4}{3}} = 3 + (2^5)^{\frac{1}{3}} + (2^4)^{\frac{1}{3}}$

$\Rightarrow x = 3 + \sqrt[3]{32} + \sqrt[3]{16}$ // AS REQUIRED

METHOD B

$\Rightarrow (x+1)^6 - 2(x-1)^6 = (x^2-1)^3$

$\Rightarrow (x+1)^6 - 2(x-1)^6 = (x+1)^3(x-1)^3$

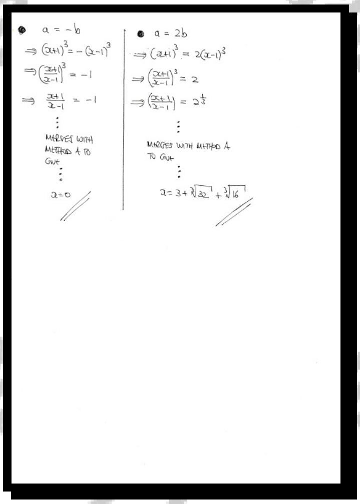
LET $a = (x+1)^3$ & $b = (x-1)^3$

$\Rightarrow a^2 - 2b^2 = ab$

$\Rightarrow a^2 - ab - 2b^2 = 0$

$\Rightarrow (a+b)(a-2b) = 0$

$\Rightarrow a = \begin{cases} -b \\ 2b \end{cases}$



- $a = -b$

$\Rightarrow (x+1)^3 = -(x-1)^3$

$\Rightarrow \left(\frac{x+1}{x-1}\right)^3 = -1$

$\Rightarrow \frac{x+1}{x-1} = -1$

⋮

MERGES WITH METHOD A TO GIVE

⋮

$x = 0$ //

- $a = 2b$

$\Rightarrow (x+1)^3 = 2(x-1)^3$

$\Rightarrow \left(\frac{x+1}{x-1}\right)^3 = 2$

$\Rightarrow \left(\frac{x+1}{x-1}\right) = 2^{\frac{1}{3}}$

⋮

MERGES WITH METHOD A TO GIVE

⋮

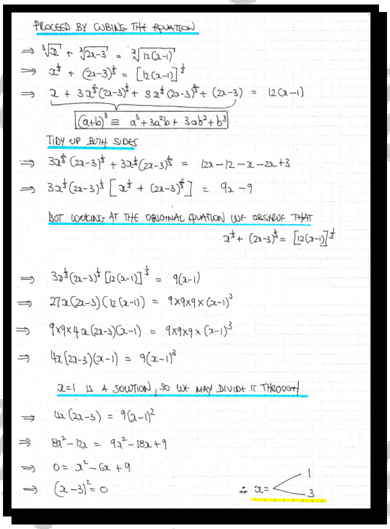
$x = 3 + \sqrt[3]{32} + \sqrt[3]{16}$ //

Question 85 (*****)

Solve the following equation

$$\sqrt[3]{x} + \sqrt[3]{2x-3} = \sqrt[3]{12(x-1)}, \; x \in \mathbb{R}.$$

$\boxed{}$, $\boxed{x=1 \;\cup\; x=3}$



PROCEED BY CUBING THE EQUATION

$\Rightarrow \sqrt[3]{x} + \sqrt[3]{2x-3} = \sqrt[3]{12(x-1)}$

$\Rightarrow x^{\frac{1}{3}} + (2x-3)^{\frac{1}{3}} = [12(x-1)]^{\frac{1}{3}}$

$$\Rightarrow \underbrace{x + 3x^{\frac{2}{3}}(2x-3)^{\frac{1}{3}} + 3x^{\frac{1}{3}}(2x-3)^{\frac{2}{3}} + (2x-3)}_{\boxed{(a+b)^3 \equiv a^3+3a^2b+3ab^2+b^3}} = 12(x-1)$$

TIDY UP BOTH SIDES

$\Rightarrow 3x^{\frac{2}{3}}(2x-3)^{\frac{1}{3}} + 3x^{\frac{1}{3}}(2x-3)^{\frac{2}{3}} = 12x-12-x-2x+3$

$\Rightarrow 3x^{\frac{1}{3}}(2x-3)^{\frac{1}{3}}\left[x^{\frac{1}{3}} + (2x-3)^{\frac{2}{3}}\right] = 9x-9$

BUT LOOKING AT THE ORIGINAL EQUATION WE OBSERVE THAT

$x^{\frac{1}{3}} + (2x-3)^{\frac{1}{3}} = [12(x-1)]^{\frac{1}{3}}$

$\Rightarrow 3x^{\frac{1}{3}}(2x-3)^{\frac{1}{3}}[12(x-1)]^{\frac{1}{3}} = 9(x-1)$

$\Rightarrow 27x(2x-3)(12(x-1)) = 9\times9\times9\times(x-1)^3$

$\Rightarrow 9\times9\times4x(2x-3)(x-1) = 9\times9\times9\times(x-1)^3$

$\Rightarrow 4x(2x-3)(x-1) = 9(x-1)^3$

$x=1$ IS A SOLUTION, SO WE MAY DIVIDE IT THROUGH

$\Rightarrow 4x(2x-3) = 9(x-1)^2$

$\Rightarrow 8x^2-12x = 9x^2-18x+9$

$\Rightarrow 0 = x^2-6x+9$

$\Rightarrow (x-3)^2=0$

$\therefore x = \begin{cases} 1 \\ 3 \end{cases}$

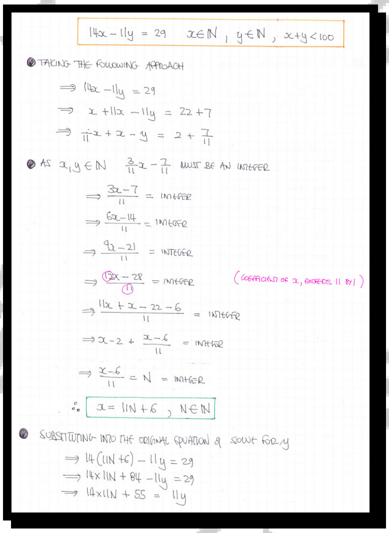
Question 86 (*****)

Solve the equation

$$14x-11y=29,$$

given further that $x \in \mathbb{N}$, $y \in \mathbb{N}$, and $x+y<100$.

V, ☐, $\boxed{(x,y)=(6,5),\ (17,19),\ (28,33),\ (39,47)}$



$\boxed{14x-11y=29 \quad x\in\mathbb{N},\ y\in\mathbb{N},\ x+y<100}$

- TAKING THE FOLLOWING APPROACH

$$\Rightarrow 14x-11y=29$$
$$\Rightarrow x+11x-11y=22+7$$
$$\Rightarrow \frac{3}{11}x+x-y=2+\frac{7}{11}$$

- AS $x,y\in\mathbb{N}$ $\frac{3}{11}x-\frac{7}{11}$ MUST BE AN INTEGER

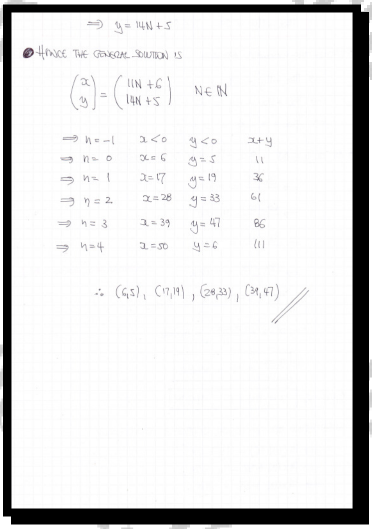
$$\Rightarrow \frac{3x-7}{11}=\text{INTEGER}$$
$$\Rightarrow \frac{6x-14}{11}=\text{INTEGER}$$
$$\Rightarrow \frac{9x-21}{11}=\text{INTEGER}$$
$$\Rightarrow \frac{⑫x-28}{⑪}=\text{INTEGER}$$

(COEFFICIENT OF x, EXCEEDS 11 BY 1)

$$\Rightarrow \frac{11x+x-22-6}{11}=\text{INTEGER}$$
$$\Rightarrow x-2+\frac{x-6}{11}=\text{INTEGER}$$
$$\Rightarrow \frac{x-6}{11}=N=\text{INTEGER}$$
$$\therefore \boxed{x=11N+6,\ N\in\mathbb{N}}$$

- SUBSTITUTING INTO THE ORIGINAL EQUATION & SOLVE FOR y

$$\Rightarrow 14(11N+6)-11y=29$$
$$\Rightarrow 14\times11N+84-11y=29$$
$$\Rightarrow 14\times11N+55=11y$$



$$\Rightarrow y=14N+5$$

- HENCE THE GENERAL SOLUTION IS

$$\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}11N+6\\14N+5\end{pmatrix} \quad N\in\mathbb{N}$$

$\Rightarrow n=-1$	$x<0$	$y<0$	$x+y$
$\Rightarrow n=0$	$x=6$	$y=5$	11
$\Rightarrow n=1$	$x=17$	$y=19$	36
$\Rightarrow n=2$	$x=28$	$y=33$	61
$\Rightarrow n=3$	$x=39$	$y=47$	86
$\Rightarrow n=4$	$x=50$	$y=6$	111

$\therefore (6,5),\ (17,19),\ (28,33),\ (39,47)$ //

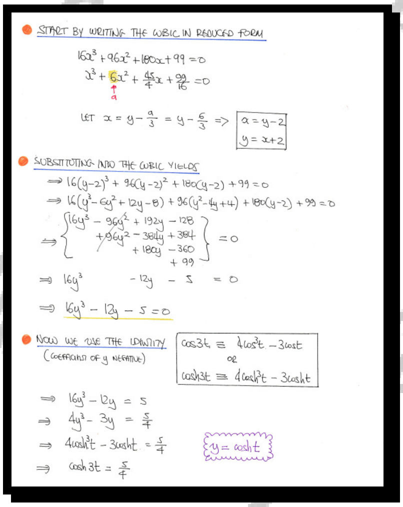
Question 87 (*****)

Solve the cubic equation

$$16x^3 + 96x^2 + 180x + 99 = 0, \ x \in \mathbb{R}.$$

You may assume that this cubic equation only has one real root.

V, ☐, $\boxed{x = -2 + 2^{-\frac{2}{3}} - 2^{-\frac{4}{3}}}$



- START BY WRITING THE CUBIC IN REDUCED FORM

$16x^3 + 96x^2 + 180x + 99 = 0$

$x^3 + \underset{\uparrow \atop a}{6}x^2 + \frac{45}{4}x + \frac{99}{16} = 0$

LET $x = y - \frac{a}{3} = y - \frac{6}{3} \Rightarrow \boxed{\begin{array}{l} x = y-2 \\ y = x+2 \end{array}}$

- SUBSTITUTING INTO THE CUBIC YIELDS

$\Longrightarrow 16(y-2)^3 + 96(y-2)^2 + 180(y-2) + 99 = 0$

$\Longrightarrow 16(y^3 - 6y^2 + 12y - 8) + 96(y^2 - 4y + 4) + 180(y-2) + 99 = 0$

$\Longrightarrow \left\{ \begin{array}{r} 16y^3 - \cancel{96y^2} + 192y - 128 \\ + \cancel{96y^2} - 384y + 384 \\ + 180y - 360 \\ + 99 \end{array} \right\} = 0$

$\Longrightarrow 16y^3 \quad - 12y - 5 = 0$

$\Longrightarrow 16y^3 - 12y - 5 = 0$

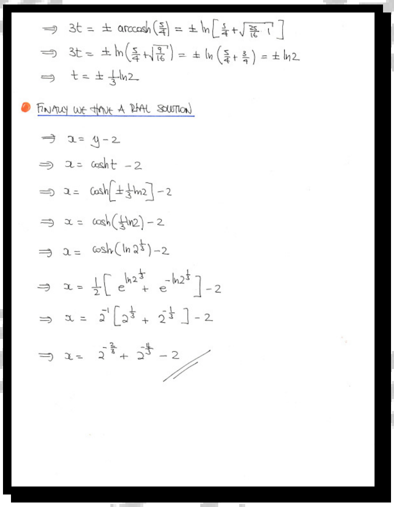
- NOW WE USE THE IDENTITY (COEFFICIENT OF y NEGATIVE) $\boxed{\begin{array}{c} \cos 3t \equiv 4\cos^3 t - 3\cos t \\ \text{OR} \\ \cosh 3t \equiv 4\cosh^3 t - 3\cosh t \end{array}}$

$\Longrightarrow 16y^3 - 12y = 5$

$\Longrightarrow 4y^3 - 3y = \frac{5}{4}$

$\Longrightarrow 4\cosh^3 t - 3\cosh t = \frac{5}{4}$ $\quad y = \cosh t$

$\Longrightarrow \cosh 3t = \frac{5}{4}$



$\Longrightarrow 3t = \pm \operatorname{arccosh}\left(\frac{5}{4}\right) = \pm \ln\left[\frac{5}{4} + \sqrt{\frac{25}{16} - 1}\right]$

$\Longrightarrow 3t = \pm \ln\left(\frac{5}{4} + \sqrt{\frac{9}{16}}\right) = \pm \ln\left(\frac{5}{4} + \frac{3}{4}\right) = \pm \ln 2$

$\Longrightarrow t = \pm \frac{1}{3}\ln 2$

- FINALLY WE HAVE A REAL SOLUTION

$\Longrightarrow x = y - 2$

$\Longrightarrow x = \cosh t - 2$

$\Longrightarrow x = \cosh\left[\pm \frac{1}{3}\ln 2\right] - 2$

$\Longrightarrow x = \cosh\left(\frac{1}{3}\ln 2\right) - 2$

$\Longrightarrow x = \cosh\left(\ln 2^{\frac{1}{3}}\right) - 2$

$\Longrightarrow x = \frac{1}{2}\left[e^{\ln 2^{\frac{1}{3}}} + e^{-\ln 2^{\frac{1}{3}}}\right] - 2$

$\Longrightarrow x = 2^{-1}\left[2^{\frac{1}{3}} + 2^{-\frac{1}{3}}\right] - 2$

$\Longrightarrow x = 2^{-\frac{2}{3}} + 2^{-\frac{4}{3}} - 2$ //

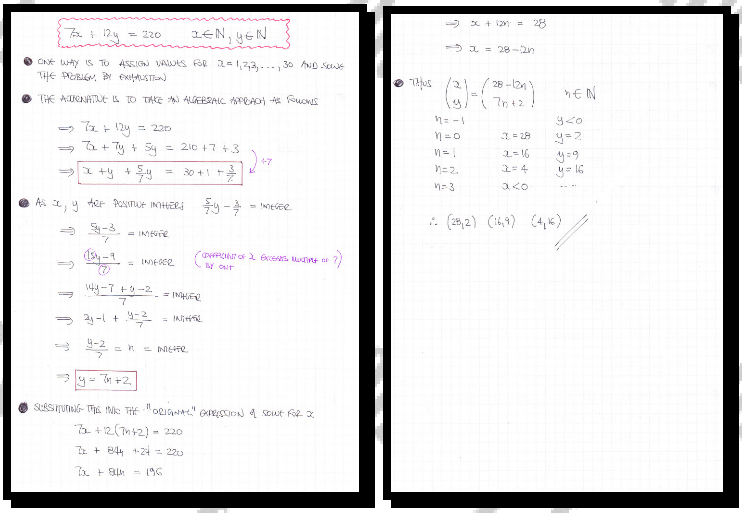
Question 88 (*****)

Solve the equation

$$7x + 12y = 220,$$

given further that $x \in \mathbb{N}$, $y \in \mathbb{N}$.

$\boxed{\text{SPX-A}}$, $\boxed{(x, y) = (28,2),\ (16,9),\ (4,16)}$



$\boxed{7x + 12y = 220 \qquad x \in \mathbb{N},\ y \in \mathbb{N}}$

- ONE WAY IS TO ASSIGN VALUES FOR $x = 1, 2, 3, \dots, 30$ AND SOLVE THE PROBLEM BY EXHAUSTION
- THE ALTERNATIVE IS TO TAKE AN ALGEBRAIC APPROACH AS FOLLOWS

$$\Rightarrow 7x + 12y = 220$$
$$\Rightarrow 7x + 7y + 5y = 210 + 7 + 3$$
$$\Rightarrow \boxed{x + y + \frac{5}{7}y = 30 + 1 + \frac{3}{7}} \quad \div 7$$

- AS x, y ARE POSITIVE INTEGERS $\frac{5}{7}y - \frac{3}{7} = \text{INTEGER}$.

$$\Rightarrow \frac{5y-3}{7} = \text{INTEGER}$$
$$\Rightarrow \frac{15y-9}{7} = \text{INTEGER} \quad \text{(COEFFICIENT OF } x \text{ EXCEEDS MULTIPLE OF 7 BY ONE)}$$
$$\Rightarrow \frac{14y-7+y-2}{7} = \text{INTEGER}$$
$$\Rightarrow 2y - 1 + \frac{y-2}{7} = \text{INTEGER}$$
$$\Rightarrow \frac{y-2}{7} = n = \text{INTEGER}$$
$$\Rightarrow \boxed{y = 7n + 2}$$

- SUBSTITUTING THIS INTO THE "ORIGINAL" EXPRESSION & SOLVE FOR x

$$7x + 12(7n+2) = 220$$
$$7x + 84n + 24 = 220$$
$$7x + 84n = 196$$
$$\Rightarrow x + 12n = 28$$
$$\Rightarrow x = 28 - 12n$$

- THUS $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 28 - 12n \\ 7n + 2 \end{pmatrix} \quad n \in \mathbb{N}$

$n = -1$		$y < 0$
$n = 0$	$x = 28$	$y = 2$
$n = 1$	$x = 16$	$y = 9$
$n = 2$	$x = 4$	$y = 16$
$n = 3$	$x < 0$	...

$\therefore (28,2) \quad (16,9) \quad (4,16)$ //

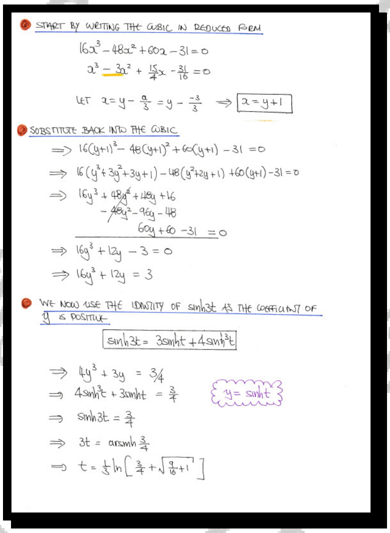
Question 89 (*****)

Solve the cubic equation

$$16x^3 - 48x^2 + 60x - 31 = 0,\ x \in \mathbb{R}.$$

You may assume that this cubic equation only has one real root.

$\boxed{}$, $\boxed{x = 3 + 2^{\frac{5}{3}} + 2^{\frac{4}{3}}}$



START BY WRITING THE CUBIC IN REDUCED FORM

$$16x^3 - 48x^2 + 60x - 31 = 0$$

$$x^3 - 3x^2 + \frac{15}{4}x - \frac{31}{16} = 0$$

LET $x = y - \frac{a}{3} = y - \frac{-3}{3} \Rightarrow \boxed{x = y + 1}$

SUBSTITUTE BACK INTO THE CUBIC

$\Longrightarrow 16(y+1)^3 - 48(y+1)^2 + 60(y+1) - 31 = 0$

$\Longrightarrow 16(y^3 + 3y^2 + 3y + 1) - 48(y^2 + 2y + 1) + 60(y+1) - 31 = 0$

$\Longrightarrow 16y^3 + \cancel{48y^2} + 48y + 16$

$\quad - \cancel{48y^2} - 96y - 48$

$\quad 60y + 60 - 31 = 0$

$\Longrightarrow 16y^3 + 12y - 3 = 0$

$\Longrightarrow 16y^3 + 12y = 3$

WE NOW USE THE IDENTITY OF sinh3t AS THE COEFFICIENT OF y IS POSITIVE

$$\boxed{\sinh 3t = 3\sinh t + 4\sinh^3 t}$$

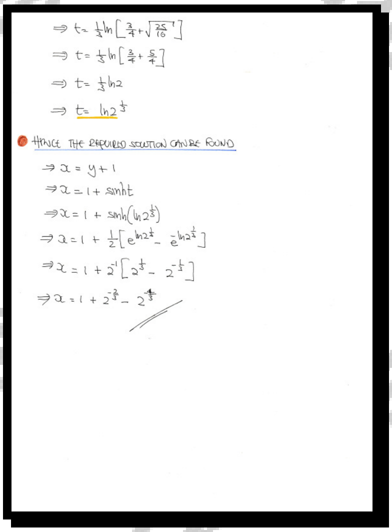
$\Longrightarrow 4y^3 + 3y = 3/4$

$\Longrightarrow 4\sinh^3 t + 3\sinh t = \frac{3}{4}$ $\quad y = \sinh t$

$\Longrightarrow \sinh 3t = \frac{3}{4}$

$\Longrightarrow 3t = \operatorname{arsinh}\frac{3}{4}$

$\Longrightarrow t = \frac{1}{3}\ln\left[\frac{3}{4} + \sqrt{\frac{9}{16} + 1}\right]$



$\Rightarrow t = \frac{1}{3}\ln\left[\frac{3}{4} + \sqrt{\frac{25}{16}}\right]$

$\Rightarrow t = \frac{1}{3}\ln\left[\frac{3}{4} + \frac{5}{4}\right]$

$\Rightarrow t = \frac{1}{3}\ln 2$

$\Rightarrow t = \ln 2^{\frac{1}{3}}$

HENCE THE REQUIRED SOLUTION CAN BE FOUND

$\Rightarrow x = y + 1$

$\Rightarrow x = 1 + \sinh t$

$\Rightarrow x = 1 + \sinh(\ln 2^{\frac{1}{3}})$

$\Rightarrow x = 1 + \frac{1}{2}\left[e^{\ln 2^{\frac{1}{3}}} - e^{-\ln 2^{\frac{1}{3}}}\right]$

$\Rightarrow x = 1 + 2^{-1}\left[2^{\frac{1}{3}} - 2^{-\frac{1}{3}}\right]$

$\Rightarrow x = 1 + 2^{-\frac{2}{3}} - 2^{-\frac{4}{3}}$

Question 90 (*****)

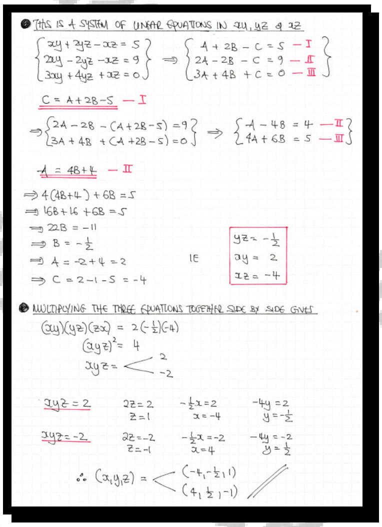
Solve the following system of equations for $x \in \mathbb{R}$, $y \in \mathbb{R}$, $z \in \mathbb{R}$.

$$xy + 2yz - xz = 5,$$

$$2xy - 2yz - xz = 9,$$

$$3xy + 4yz + xz = 0.$$

☐ , $\boxed{(x,y,z) = \left(-4, -\frac{1}{2}, 1\right), \quad (x,y,z) = \left(4, \frac{1}{2}, -1\right)}$



● THIS IS A SYSTEM OF LINEAR EQUATIONS IN xy, yz & xz

$$\left\{\begin{array}{l} xy + 2yz - xz = 5 \\ 2xy - 2yz - xz = 9 \\ 3xy + 4yz + xz = 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{ll} A + 2B - C = 5 & \text{— I} \\ 2A - 2B - C = 9 & \text{— II} \\ 3A + 4B + C = 0 & \text{— III} \end{array}\right\}$$

$$\underline{\underline{C = A + 2B - 5}} \quad \text{— I}$$

$$\Rightarrow \left\{\begin{array}{l} 2A - 2B - (A + 2B - 5) = 9 \\ 3A + 4B + (A + 2B - 5) = 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{ll} A - 4B = 4 & \text{— II} \\ 4A + 6B = 5 & \text{— III} \end{array}\right\}$$

$$\underline{\underline{A = 4B + 4}} \quad \text{— II}$$

$\Rightarrow 4(4B+4) + 6B = 5$

$\Rightarrow 16B + 16 + 6B = 5$

$\Rightarrow 22B = -11$

$\Rightarrow B = -\frac{1}{2}$

$\Rightarrow A = -2 + 4 = 2$ IE $\boxed{\begin{array}{l} yz = -\frac{1}{2} \\ xy = 2 \\ xz = -4 \end{array}}$

$\Rightarrow C = 2 - 1 - 5 = -4$

● MULTIPLYING THE THREE EQUATIONS TOGETHER SIDE BY SIDE GIVES

$$(xy)(yz)(zx) = 2\left(-\tfrac{1}{2}\right)(-4)$$

$$(xyz)^2 = 4$$

$$xyz = \begin{cases} 2 \\ -2 \end{cases}$$

$\underline{xyz = 2}$ $2z = 2$, $z = 1$ $-\frac{1}{2}x = 2$, $x = -4$ $-4y = 2$, $y = -\frac{1}{2}$

$\underline{xyz = -2}$ $2z = -2$, $z = -1$ $-\frac{1}{2}x = -2$, $x = 4$ $-4y = -2$, $y = \frac{1}{2}$

$$\therefore (x,y,z) = \begin{cases} \left(-4, -\frac{1}{2}, 1\right) \\ \left(4, \frac{1}{2}, -1\right) \end{cases} \quad //$$

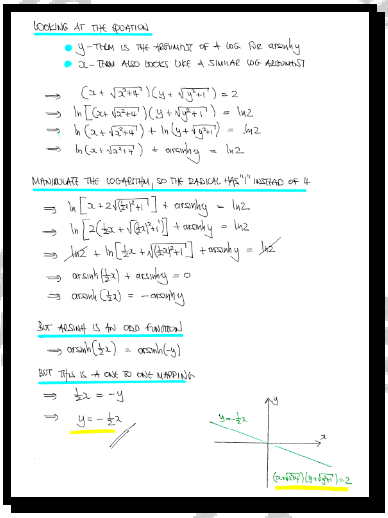
Question 91 (*****)

Sketch the graph of

$$\left[x+\sqrt{x^2+4}\right]\left[y+\sqrt{y^2+1}\right]=2\,,\quad x\in(-\infty,\infty)\,,\ y\in(-\infty,\infty)$$

You must show a detailed method in this question

V, , proof



LOOKING AT THE EQUATION

- y – TERM IS THE ARGUMENT OF A LOG FOR $\text{arsinh}\,y$
- x – TERM ALSO LOOKS LIKE A SIMILAR LOG ARGUMENT

$\Rightarrow (x+\sqrt{x^2+4})(y+\sqrt{y^2+1})=2$

$\Rightarrow \ln\left[(x+\sqrt{x^2+4})(y+\sqrt{y^2+1})\right]=\ln 2$

$\Rightarrow \ln(x+\sqrt{x^2+4})+\ln(y+\sqrt{y^2+1})=\ln 2$

$\Rightarrow \ln(x+\sqrt{x^2+4})+\text{arsinh}\,y=\ln 2$

MANIPULATE THE LOGARITHM, SO THE RADICAL HAS "1" INSTEAD OF 4

$\Rightarrow \ln\left[x+2\sqrt{(\tfrac{1}{2}x)^2+1}\right]+\text{arsinh}\,y=\ln 2$

$\Rightarrow \ln\left[2\left(\tfrac{1}{2}x+\sqrt{(\tfrac{1}{2}x)^2+1}\right)\right]+\text{arsinh}\,y=\ln 2$

$\Rightarrow \cancel{\ln 2}+\ln\left[\tfrac{1}{2}x+\sqrt{(\tfrac{1}{2}x)^2+1}\right]+\text{arsinh}\,y=\cancel{\ln 2}$

$\Rightarrow \text{arsinh}(\tfrac{1}{2}x)+\text{arsinh}\,y=0$

$\Rightarrow \text{arsinh}(\tfrac{1}{2}x)=-\text{arsinh}\,y$

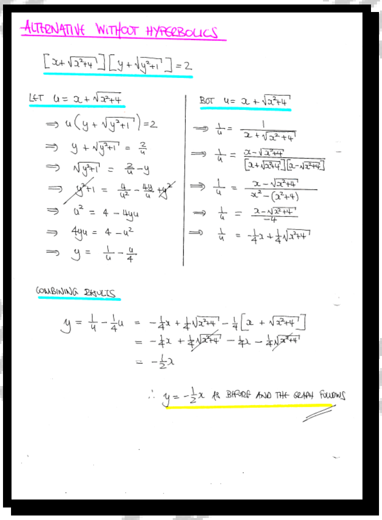
BUT ARSINH IS AN ODD FUNCTION

$\Rightarrow \text{arsinh}(\tfrac{1}{2}x)=\text{arsinh}(-y)$

BUT THIS IS A ONE TO ONE MAPPING

$\Rightarrow \tfrac{1}{2}x=-y$

$\Rightarrow y=-\tfrac{1}{2}x$



ALTERNATIVE WITHOUT HYPERBOLICS

$\left[x+\sqrt{x^2+4}\right]\left[y+\sqrt{y^2+1}\right]=2$

LET $u=x+\sqrt{x^2+4}$

$\Rightarrow u(y+\sqrt{y^2+1})=2$

$\Rightarrow y+\sqrt{y^2+1}=\frac{2}{u}$

$\Rightarrow \sqrt{y^2+1}=\frac{2}{u}-y$

$\Rightarrow \cancel{y^2}+1=\frac{4}{u^2}-\frac{4y}{u}+\cancel{y^2}$

$\Rightarrow u^2=4-4yu$

$\Rightarrow 4yu=4-u^2$

$\Rightarrow y=\frac{1}{u}-\frac{u}{4}$

BUT $u=x+\sqrt{x^2+4}$

$\Rightarrow \frac{1}{u}=\frac{1}{x+\sqrt{x^2+4}}$

$\Rightarrow \frac{1}{u}=\frac{x-\sqrt{x^2+4}}{\left[x+\sqrt{x^2+4}\right]\left[x-\sqrt{x^2+4}\right]}$

$\Rightarrow \frac{1}{u}=\frac{x-\sqrt{x^2+4}}{x^2-(x^2+4)}$

$\Rightarrow \frac{1}{u}=\frac{x-\sqrt{x^2+4}}{-4}$

$\Rightarrow \frac{1}{u}=-\frac{1}{4}x+\frac{1}{4}\sqrt{x^2+4}$

COMBINING RESULTS

$$\begin{aligned}y=\frac{1}{u}-\frac{1}{4}u&=-\tfrac{1}{4}x+\tfrac{1}{4}\sqrt{x^2+4}-\tfrac{1}{4}\left[x+\sqrt{x^2+4}\right]\\&=-\tfrac{1}{4}x+\cancel{\tfrac{1}{4}\sqrt{x^2+4}}-\tfrac{1}{4}x-\cancel{\tfrac{1}{4}\sqrt{x^2+4}}\\&=-\tfrac{1}{2}x\end{aligned}$$

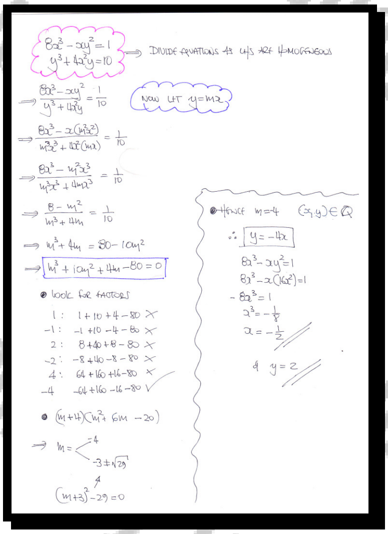
$\therefore\ y=-\tfrac{1}{2}x$ AS BEFORE AND THE GRAPH FOLLOWS

Question 92 **(*****)**

Given that $x \in \mathbb{Q}$ and $y \in \mathbb{Q}$, find the solution pair for the following system of simultaneous equations

$$8x^3 - xy^2 = 1$$
$$y^3 + 4x^2y = 10.$$

V, $\boxed{(x,y) = \left(-\frac{1}{2}, 2\right)}$



$\begin{cases} 8x^3 - xy^2 = 1 \\ y^3 + 4x^2y = 10 \end{cases} \Longrightarrow$ DIVIDE EQUATIONS AS LHS ARE HOMOGENEOUS

$\Longrightarrow \frac{8x^3 - xy^2}{y^3 + 4x^2y} = \frac{1}{10}$ NOW LET $y = mx$

$\Longrightarrow \frac{8x^3 - x(m^2x^2)}{m^3x^3 + 4x^2(mx)} = \frac{1}{10}$

$\Longrightarrow \frac{8x^3 - m^2x^3}{m^3x^3 + 4mx^3} = \frac{1}{10}$

$\Longrightarrow \frac{8 - m^2}{m^3 + 4m} = \frac{1}{10}$

$\Longrightarrow m^3 + 4m = 80 - 10m^2$

$\Longrightarrow \boxed{m^3 + 10m^2 + 4m - 80 = 0}$

- LOOK FOR FACTORS

$1: \quad 1 + 10 + 4 - 80 \times$
$-1: \quad -1 + 10 - 4 - 80 \times$
$2: \quad 8 + 40 + 8 - 80 \times$
$-2: \quad -8 + 40 - 8 - 80 \times$
$4: \quad 64 + 160 + 16 - 80 \times$
$-4 \quad -64 + 160 - 16 - 80 \checkmark$

- $(m+4)(m^2 + 6m - 20)$

$\Longrightarrow m = \begin{cases} -4 \\ -3 \pm \sqrt{29} \end{cases}$

$(m+3)^2 - 29 = 0$

- HENCE $m = -4$ $(x,y) \in \mathbb{Q}$

$\therefore \boxed{y = -4x}$

$8x^3 - xy^2 = 1$
$8x^3 - x(16x^2) = 1$
$-8x^3 = 1$
$x^3 = -\frac{1}{8}$
$x = -\frac{1}{2}$ //

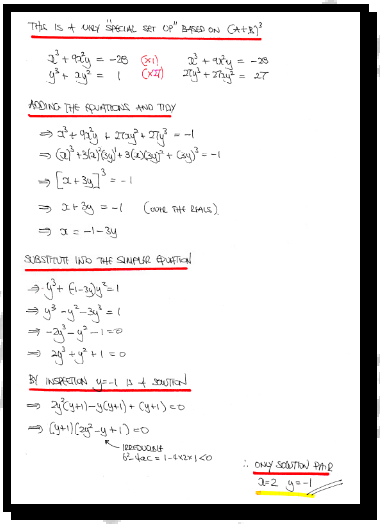
& $y = 2$ //

Question 93 (*****)

Given that $x \in \mathbb{R}$ and $y \in \mathbb{R}$, find the solution pair for the following system of simultaneous equations

$$x^3 + 9x^2y = -28$$
$$y^3 + xy^2 = 1 .$$

$\boxed{}$, $\boxed{(x,y) = (2,-1)}$



THIS IS A VERY "SPECIAL SET UP" BASED ON $(A+B)^3$

$$x^3 + 9x^2y = -28 \quad (\times 1) \qquad x^3 + 9x^2y = -28$$
$$y^3 + xy^2 = 1 \quad (\times 27) \qquad 27y^3 + 27xy^2 = 27$$

ADDING THE EQUATIONS AND TIDY

$$\Rightarrow x^3 + 9x^2y + 27xy^2 + 27y^3 = -1$$
$$\Rightarrow (x)^3 + 3(x)^2(3y)^1 + 3(x)(3y)^2 + (3y)^3 = -1$$
$$\Rightarrow [x+3y]^3 = -1$$
$$\Rightarrow x + 3y = -1 \quad \text{(OVER THE REALS)}$$
$$\Rightarrow x = -1-3y$$

SUBSTITUTE INTO THE SIMPLER EQUATION

$$\Rightarrow y^3 + (-1-3y)y^2 = 1$$
$$\Rightarrow y^3 - y^2 - 3y^3 = 1$$
$$\Rightarrow -2y^3 - y^2 - 1 = 0$$
$$\Rightarrow 2y^3 + y^2 + 1 = 0$$

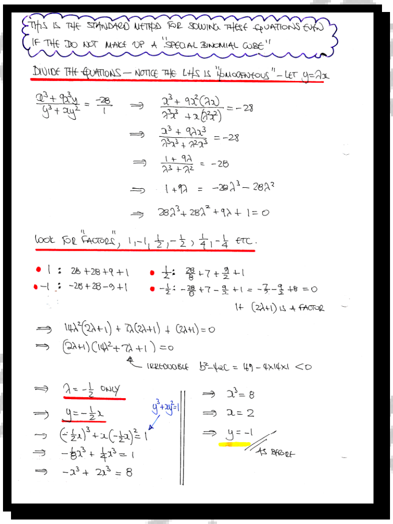
BY INSPECTION $y=-1$ IS A SOLUTION

$$\Rightarrow 2y^2(y+1) - y(y+1) + (y+1) = 0$$
$$\Rightarrow (y+1)(2y^2 - y + 1) = 0$$

IRREDUCIBLE $b^2-4ac = 1-4\times2\times1<0$

$\therefore$ ONLY SOLUTION PAIR

$x=2 \quad y=-1$



THIS IS THE STANDARD METHOD FOR SOLVING THESE EQUATIONS EVEN IF THE DO NOT MAKE UP A "SPECIAL BINOMIAL CUBE"

DIVIDE THE EQUATIONS — NOTICE THE LHS IS "HOMOGENEOUS" — LET $y=\lambda x$

$$\frac{x^3+9x^3y}{y^3+xy^2} = \frac{-28}{1} \Rightarrow \frac{x^3+9x^2(\lambda x)}{\lambda^3x^3 + x(\lambda^2x^2)} = -28$$
$$\Rightarrow \frac{x^3+9\lambda x^3}{\lambda^3x^3+\lambda^2x^3} = -28$$
$$\Rightarrow \frac{1+9\lambda}{\lambda^3+\lambda^2} = -28$$
$$\Rightarrow 1+9\lambda = -28\lambda^3 - 28\lambda^2$$
$$\Rightarrow 28\lambda^3 + 28\lambda^2 + 9\lambda + 1 = 0$$

LOOK FOR "FACTORS", $1, -1, \frac{1}{2}, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{4}$ ETC.

- 1 : $28+28+9+1$
- -1 : $-28+28-9+1$
- $\frac{1}{2}$: $\frac{28}{8}+7+\frac{9}{2}+1$
- $-\frac{1}{2}$: $-\frac{28}{8}+7-\frac{9}{2}+1 = -\frac{7}{2}-\frac{9}{2}+8 = 0$

IT $(2\lambda+1)$ IS A FACTOR

$$\Rightarrow 14\lambda^2(2\lambda+1) + 7\lambda(2\lambda+1) + (2\lambda+1) = 0$$
$$\Rightarrow (2\lambda+1)(14\lambda^2+7\lambda+1) = 0$$

IRREDUCIBLE $b^2-4ac = 49-4\times14\times1<0$

$$\Rightarrow \lambda = -\frac{1}{2} \text{ ONLY}$$
$$\Rightarrow y = -\frac{1}{2}x$$

$y^3+xy^2=1$

$$\Rightarrow \left(-\frac{1}{2}x\right)^3 + x\left(-\frac{1}{2}x\right)^2 = 1$$
$$\Rightarrow -\frac{1}{8}x^3 + \frac{1}{4}x^3 = 1$$
$$\Rightarrow -x^3 + 2x^3 = 8$$
$$\Rightarrow x^3 = 8$$
$$\Rightarrow x = 2$$
$$\Rightarrow y = -1$$

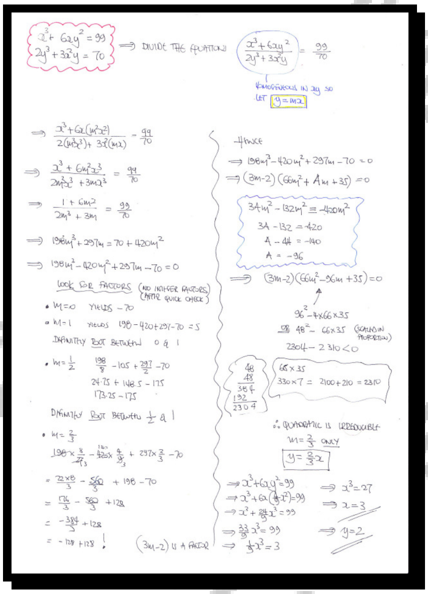
AS BEFORE

Question 94 (*****)

Use an algebraic method to find the real solutions for the following system of simultaneous equations

$$x^3 + 6xy^2 = 99$$
$$2y^3 + 3x^2y = 70.$$

$$\boxed{(x, y) = (3, 2)}$$



$$\boxed{\begin{aligned} x^3 + 6xy^2 &= 99 \\ 2y^3 + 3x^2y &= 70 \end{aligned}} \Longrightarrow \text{DIVIDE THE EQUATIONS } \boxed{\frac{x^3+6xy^2}{2y^3+3x^2y}} = \frac{99}{70}$$

HOMOGENEOUS IN xy SO LET $\boxed{y = mx}$

$$\Longrightarrow \frac{x^3 + 6x(m^2x^2)}{2(m^3x^3) + 3x^2(mx)} = \frac{99}{70}$$

$$\Longrightarrow \frac{x^3 + 6m^2x^3}{2m^3x^3 + 3mx^3} = \frac{99}{70}$$

$$\Longrightarrow \frac{1 + 6m^2}{2m^3 + 3m} = \frac{99}{70}$$

$$\Longrightarrow 198m^3 + 297m = 70 + 420m^2$$

$$\Longrightarrow 198m^2 - 420m^2 + 297m - 70 = 0$$

LOOK FOR FACTORS (NO INTEGER FACTORS) (AFTER QUICK CHECK)

- $m = 0$ YIELDS -70
- $m = 1$ YIELDS $198 - 420 + 297 - 70 = 5$

DEFINITELY ROOT BETWEEN 0 & 1

- $m = \frac{1}{2}$ $\frac{198}{8} - 105 + \frac{297}{2} - 70$

$24.75 + 148.5 - 175$

$173.25 - 175$

DEFINITELY ROOT BETWEEN $\frac{1}{2}$ & 1

- $m = \frac{2}{3}$

$$\cancel{198} \times \frac{8}{\cancel{27}_3} - \overset{140}{\cancel{420}} \times \frac{4}{\cancel{9}_3} + 297 \times \frac{2}{3} - 70$$

$$= \frac{22 \times 8}{3} - \frac{560}{3} + 198 - 70$$

$$= \frac{176}{3} - \frac{560}{3} + 128$$

$$= -\frac{384}{3} + 128$$

$$= -128 + 128 \,!$$

$(3m-2)$ IS A FACTOR

HENCE

$$\Longrightarrow 198m^3 - 420m^2 + 297m - 70 = 0$$

$$\Longrightarrow (3m-2)(66m^2 + Am + 35) = 0$$

$$\boxed{\begin{aligned} 3Am^2 - 132m^2 &\equiv -420m^2 \\ 3A - 132 &= -420 \\ A - 44 &= -140 \\ A &= -96 \end{aligned}}$$

$$\Longrightarrow (3m-2)(66m^2 - 96m + 35) = 0$$

$96^2 - 4 \times 66 \times 35$

OR $48^2 - 66 \times 35$ (SCALED IN PROPORTION)

$2304 - 2310 < 0$

$$\begin{array}{r} 48 \\ 48 \\ \hline 384 \\ 192 \\ \hline 2304 \end{array}$$

66×35

$330 \times 7 = 2100 + 210 = 2310$

$\therefore$ QUADRATIC IS IRREDUCIBLE

$m = \frac{2}{3}$ ONLY

$$\boxed{y = \frac{2}{3}x}$$

$$\Longrightarrow x^3 + 6xy^2 = 99$$

$$\Longrightarrow x^3 + 6x\left(\frac{4}{9}x^2\right) = 99$$

$$\Longrightarrow x^3 + \frac{24}{9}x^3 = 99$$

$$\Longrightarrow \frac{33}{9}x^3 = 99$$

$$\Longrightarrow \frac{1}{9}x^3 = 3$$

$$\Longrightarrow x^3 = 27$$

$$\Longrightarrow x = 3$$

$$\Longrightarrow y = 2$$

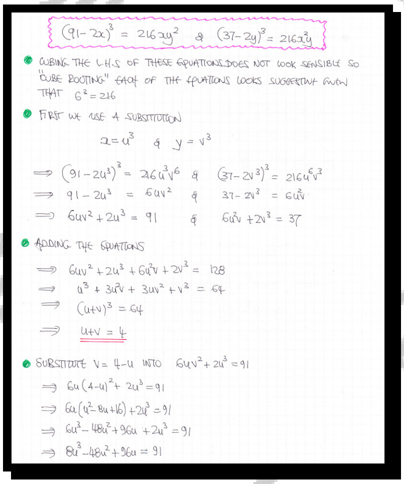
Question 95 (*****)

Solve the following simultaneous equations

$$(91-2x)^3 = 216xy^2 \qquad \text{and} \qquad (37-2y)^3 = 216x^2y\,,$$

where $x \in \mathbb{R}$, $y \in \mathbb{R}$.

$\boxed{}$, $\boxed{x = \frac{343}{8},\ y = \frac{1}{8}}$



$$\boxed{(91-2x)^3 = 216xy^2 \quad \& \quad (37-2y)^3 = 216x^2y}$$

- CUBING THE L.H.S OF THESE EQUATIONS DOES NOT LOOK SENSIBLE SO "CUBE ROOTING" EACH OF THE EQUATIONS LOOKS SUGGESTIVE GIVEN THAT $6^3 = 216$
- FIRST WE USE A SUBSTITUTION

$$x = u^3 \quad \& \quad y = v^3$$

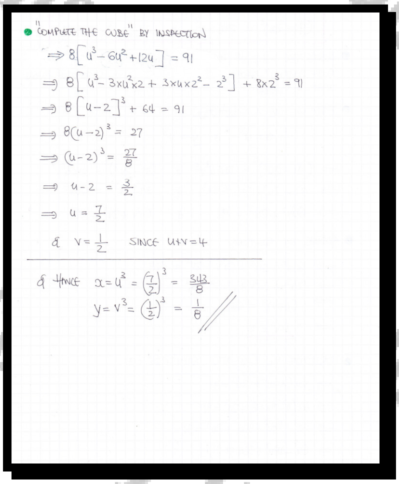
$$\Rightarrow (91-2u^3)^3 = 216u^3v^6 \quad \& \quad (37-2v^3)^3 = 216u^6v^3$$
$$\Rightarrow 91-2u^3 = 6uv^2 \quad \& \quad 37-2v^3 = 6u^2v$$
$$\Rightarrow 6uv^2+2u^3 = 91 \quad \& \quad 6u^2v+2v^3 = 37$$

- ADDING THE EQUATIONS

$$\Rightarrow 6uv^2+2u^3+6u^2v+2v^3 = 128$$
$$\Rightarrow u^3+3u^2v+3uv^2+v^3 = 64$$
$$\Rightarrow (u+v)^3 = 64$$
$$\Rightarrow \underline{\underline{u+v=4}}$$

- SUBSTITUTE $v = 4-u$ INTO $6uv^2+2u^3 = 91$

$$\Rightarrow 6u(4-u)^2+2u^3 = 91$$
$$\Rightarrow 6u(u^2-8u+16)+2u^3 = 91$$
$$\Rightarrow 6u^3-48u^2+96u+2u^3 = 91$$
$$\Rightarrow 8u^3-48u^2+96u = 91$$



- "COMPLETE THE CUBE" BY INSPECTION

$$\Rightarrow 8[u^3-6u^2+12u] = 91$$
$$\Rightarrow 8[u^3-3\times u^2\times 2+3\times u\times 2^2-2^3]+8\times 2^3 = 91$$
$$\Rightarrow 8[u-2]^3+64 = 91$$
$$\Rightarrow 8(u-2)^3 = 27$$
$$\Rightarrow (u-2)^3 = \frac{27}{8}$$
$$\Rightarrow u-2 = \frac{3}{2}$$
$$\Rightarrow u = \frac{7}{2}$$
$$\& \quad v = \frac{1}{2} \quad \text{SINCE } u+v=4$$

& HENCE $x = u^3 = \left(\frac{7}{2}\right)^3 = \frac{343}{8}$

$y = v^3 = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$ //

Question 96 (*****)

A system of simultaneous equations is given below

$$x + y + z = 1$$
$$x^2 + y^2 + z^2 = 21$$
$$x^3 + y^3 + z^3 = 55.$$

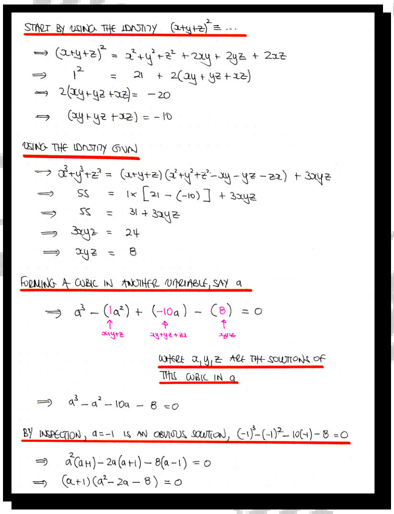
By forming an auxiliary cubic equation find the solution to the above system.

You may find the identity

$$x^3 + y^3 + z^3 \equiv (x + y + z)\left(x^2 + y^2 + z^2 - xy - yz - zx\right) + 3xyz,$$

useful in this question.

$\boxed{\text{P.X.F}}$, $\boxed{x, y, z = -2, -1, 4 \text{ in any order}}$



START BY USING THE IDENTITY $(x+y+z)^2 \equiv \dots$

$\Rightarrow (x+y+z)^2 = x^2+y^2+z^2+2xy+2yz+2xz$

$\Rightarrow 1^2 = 21 + 2(xy+yz+xz)$

$\Rightarrow 2(xy+yz+xz) = -20$

$\Rightarrow (xy+yz+xz) = -10$

USING THE IDENTITY GIVEN

$\rightarrow x^3+y^3+z^3 = (x+y+z)(x^2+y^2+z^2-xy-yz-zx)+3xyz$

$\Rightarrow 55 = 1 \times [21-(-10)] + 3xyz$

$\Rightarrow 55 = 31 + 3xyz$

$\Rightarrow 3xyz = 24$

$\Rightarrow xyz = 8$

FORMING A CUBIC IN ANOTHER VARIABLE, SAY a

$\Rightarrow a^3 - (\underset{\substack{\uparrow\\ x+y+z}}{1}a^2) + (\underset{\substack{\uparrow\\ xy+yz+zx}}{-10}a) - (\underset{\substack{\uparrow\\ xyz}}{8}) = 0$

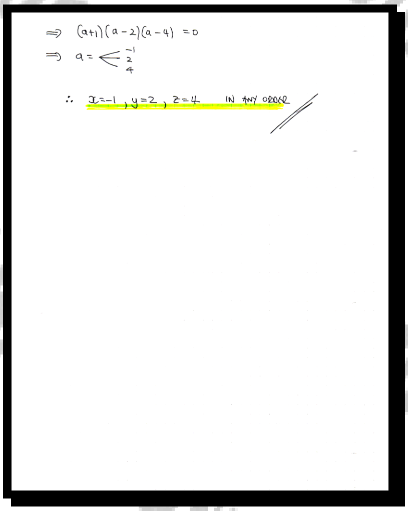
WHERE x, y, z ARE THE SOLUTIONS OF THIS CUBIC IN a

$\Rightarrow a^3 - a^2 - 10a - 8 = 0$

BY INSPECTION, $a=-1$ IS AN OBVIOUS SOLUTION, $(-1)^3-(-1)^2-10(-1)-8=0$

$\Rightarrow a^2(a+1) - 2a(a+1) - 8(a-1) = 0$

$\Rightarrow (a+1)(a^2-2a-8) = 0$



$\Rightarrow (a+1)(a-2)(a-4) = 0$

$\Rightarrow a = \begin{cases} -1 \\ 2 \\ 4 \end{cases}$

$\therefore x=-1,\ y=2,\ z=4$ IN ANY ORDER

Question 97 (*****)

The real numbers a, b, c and d satisfy

$$\frac{a}{b}=\frac{c}{d}, \quad a \neq b \neq c \neq d \neq 0 .$$

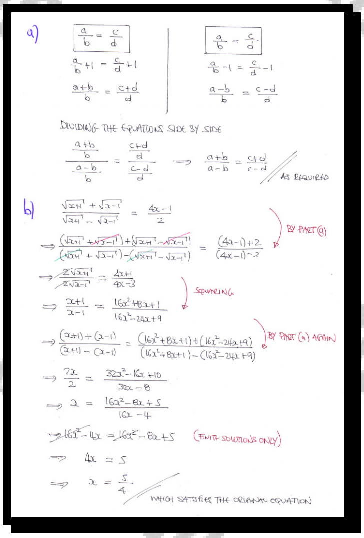
a) Show that

$$\frac{a+b}{a-b}=\frac{c+d}{c-d} .$$

b) By using the result of part **(a)** or otherwise solve the equation

$$\frac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}}=\frac{4x-1}{2}, \quad x>1 .$$

$\boxed{\quad\quad}$, $\boxed{x=\frac{5}{4}}$



a)

$$\boxed{\frac{a}{b}=\frac{c}{d}} \qquad \boxed{\frac{a}{b}=\frac{c}{d}}$$

$$\frac{a}{b}+1=\frac{c}{d}+1 \qquad \frac{a}{b}-1=\frac{c}{d}-1$$

$$\frac{a+b}{b}=\frac{c+d}{d} \qquad \frac{a-b}{b}=\frac{c-d}{d}$$

DIVIDING THE EQUATIONS SIDE BY SIDE

$$\frac{\frac{a+b}{b}}{\frac{a-b}{b}}=\frac{\frac{c+d}{d}}{\frac{c-d}{d}} \Longrightarrow \frac{a+b}{a-b}=\frac{c+d}{c-d}$$

AS REQUIRED

b)

$$\frac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}}=\frac{4x-1}{2}$$

BY PART (a)

$$\Longrightarrow \frac{(\sqrt{x+1}+\cancel{\sqrt{x-1}})+(\sqrt{x+1}-\cancel{\sqrt{x-1}})}{(\cancel{\sqrt{x+1}}+\sqrt{x-1})-(\cancel{\sqrt{x+1}}-\sqrt{x-1})}=\frac{(4x-1)+2}{(4x-1)-2}$$

$$\Longrightarrow \frac{\cancel{2}\sqrt{x+1}}{\cancel{2}\sqrt{x-1}}=\frac{4x+1}{4x-3}$$

SQUARING

$$\Longrightarrow \frac{x+1}{x-1}=\frac{16x^2+8x+1}{16x^2-24x+9}$$

BY PART (a) AGAIN

$$\Longrightarrow \frac{(x+1)+(x-1)}{(x+1)-(x-1)}=\frac{(16x^2+8x+1)+(16x^2-24x+9)}{(16x^2+8x+1)-(16x^2-24x+9)}$$

$$\Longrightarrow \frac{2x}{2}=\frac{32x^2-16x+10}{32x-8}$$

$$\Longrightarrow x=\frac{16x^2-8x+5}{16x-4}$$

$$\Longrightarrow \cancel{16x^2}-4x=\cancel{16x^2}-8x+5$$

(FINITE SOLUTIONS ONLY)

$$\Longrightarrow 4x=5$$

$$\Longrightarrow x=\frac{5}{4}$$

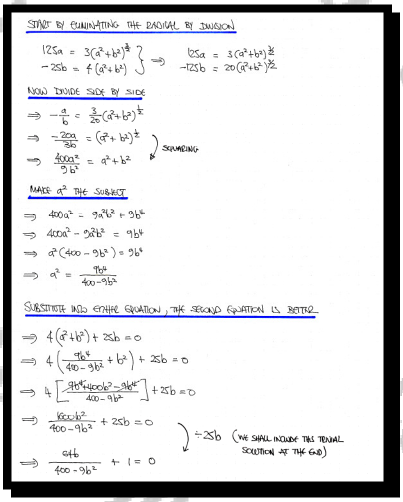
WHICH SATISFIES THE ORIGINAL EQUATION

Question 98 **(*****)**

If $a \in \mathbb{R}$, $b \in \mathbb{R}$, solve the following simultaneous equations.

$$3\left(a^2+b^2\right)^{\frac{3}{2}} - 125a = 0 \qquad \text{and} \qquad 4\left(a^2+b^2\right) + 25b = 0 .$$

$\boxed{}$, $\boxed{(a,b)=(3,-4)}$



START BY ELIMINATING THE RADICAL BY DIVISION

$$\left.\begin{aligned}125a &= 3(a^2+b^2)^{\frac{3}{2}}\\ -25b &= 4(a^2+b^2)\end{aligned}\right\} \Rightarrow \begin{aligned}125a &= 3(a^2+b^2)^{\frac{3}{2}}\\ -125b &= 20(a^2+b^2)^{\frac{3}{2}}\end{aligned}$$

NOW DIVIDE SIDE BY SIDE

$$\Rightarrow -\frac{a}{b} = \frac{3}{20}(a^2+b^2)^{\frac{1}{2}}$$

$$\Rightarrow \frac{-20a}{3b} = (a^2+b^2)^{\frac{1}{2}}$$

SQUARING

$$\Rightarrow \frac{400a^2}{9b^2} = a^2+b^2$$

MAKE a^2 THE SUBJECT

$$\Rightarrow 400a^2 = 9a^2b^2 + 9b^4$$

$$\Rightarrow 400a^2 - 9a^2b^2 = 9b^4$$

$$\Rightarrow a^2(400-9b^2) = 9b^4$$

$$\Rightarrow a^2 = \frac{9b^4}{400-9b^2}$$

SUBSTITUTE INTO EITHER EQUATION, THE SECOND EQUATION IS BETTER

$$\Rightarrow 4(a^2+b^2) + 25b = 0$$

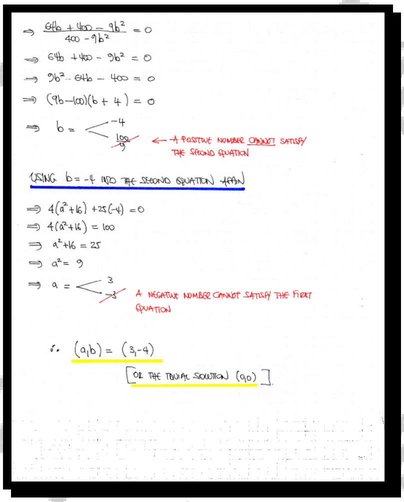
$$\Rightarrow 4\left(\frac{9b^4}{400-9b^2} + b^2\right) + 25b = 0$$

$$\Rightarrow 4\left[\frac{\cancel{9b^4}+400b^2-\cancel{9b^4}}{400-9b^2}\right] + 25b = 0$$

$$\Rightarrow \frac{1600b^2}{400-9b^2} + 25b = 0$$

$\div 25b$ (WE SHALL INCLUDE THIS TRIVIAL SOLUTION AT THE END)

$$\Rightarrow \frac{64b}{400-9b^2} + 1 = 0$$



$$\Rightarrow \frac{64b+400-9b^2}{400-9b^2} = 0$$

$$\Rightarrow 64b + 400 - 9b^2 = 0$$

$$\Rightarrow 9b^2 - 64b - 400 = 0$$

$$\Rightarrow (9b-100)(b+4) = 0$$

$$\Rightarrow b = \begin{cases} -4 \\ \cancel{\frac{100}{9}} \end{cases}$$

← A POSITIVE NUMBER CANNOT SATISFY THE SECOND EQUATION

USING $b=-4$ INTO THE SECOND EQUATION AGAIN

$$\Rightarrow 4(a^2+16) + 25(-4) = 0$$

$$\Rightarrow 4(a^2+16) = 100$$

$$\Rightarrow a^2 + 16 = 25$$

$$\Rightarrow a^2 = 9$$

$$\Rightarrow a = \begin{cases} 3 \\ \cancel{-3} \end{cases}$$

A NEGATIVE NUMBER CANNOT SATISFY THE FIRST EQUATION

$$\therefore \ (a,b) = (3,-4)$$

[OR THE TRIVIAL SOLUTION (0,0)]

Question 99 (*****)

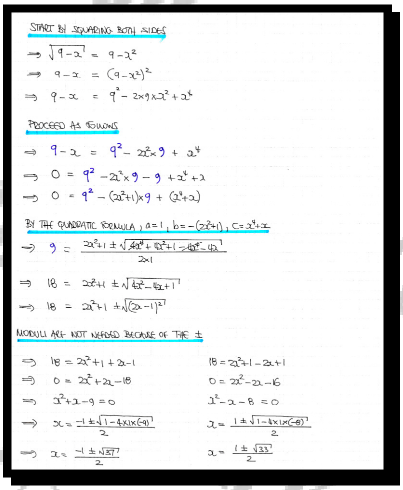
It is required to find the real solutions of the equation

$$\sqrt{9-x} = 9-x^2 .$$

Solve the equation by considering a quadratic equation with variable coefficients in x.

You must fully justify the validity of your answers.

V, ☐, proof



START BY SQUARING BOTH SIDES

$\Rightarrow \sqrt{9-x} = 9-x^2$

$\Rightarrow 9-x = (9-x^2)^2$

$\Rightarrow 9-x = 9^2 - 2\times9\times x^2 + x^4$

PROCEED AS FOLLOWS

$\Rightarrow 9-x = 9^2 - 2x^2\times9 + x^4$

$\Rightarrow 0 = 9^2 - 2x^2\times9 - 9 + x^4 + x$

$\Rightarrow 0 = 9^2 - (2x^2+1)\times9 + (x^4+x)$

BY THE QUADRATIC FORMULA, $a=1$, $b=-(2x^2+1)$, $c=x^4+x$

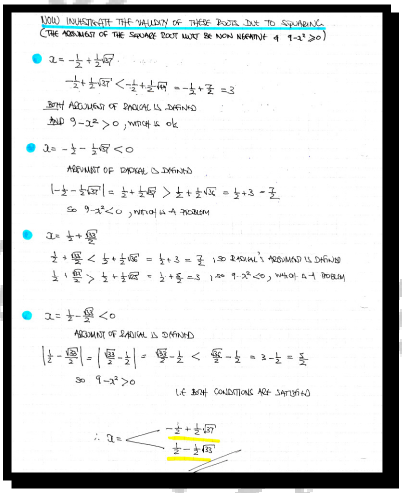
$\Rightarrow 9 = \dfrac{2x^2+1 \pm \sqrt{\cancel{4x^4}+4x^2+1-\cancel{4x^4}-4x}}{2\times1}$

$\Rightarrow 18 = 2x^2+1 \pm \sqrt{4x^2-4x+1}$

$\Rightarrow 18 = 2x^2+1 \pm \sqrt{(2x-1)^2}$

MODULI ARE NOT NEEDED BECAUSE OF THE ±

$\Rightarrow 18 = 2x^2+1+2x-1$	$18 = 2x^2+1-2x+1$
$\Rightarrow 0 = 2x^2+2x-18$	$0 = 2x^2-2x-16$
$\Rightarrow x^2+x-9=0$	$x^2-x-8=0$
$\Rightarrow x = \dfrac{-1\pm\sqrt{1-4\times1\times(-9)}}{2}$	$x = \dfrac{1\pm\sqrt{1-4\times1\times(-8)}}{2}$
$\Rightarrow x = \dfrac{-1\pm\sqrt{37}}{2}$	$x = \dfrac{1\pm\sqrt{33}}{2}$



NOW INVESTIGATE THE VALIDITY OF THESE ROOTS DUE TO SQUARING
(THE ARGUMENT OF THE SQUARE ROOT MUST BE NON NEGATIVE & $9-x^2 \geqslant 0$)

- $x = -\frac{1}{2}+\frac{1}{2}\sqrt{37}$

  $-\frac{1}{2}+\frac{1}{2}\sqrt{37} < -\frac{1}{2}+\frac{1}{2}\sqrt{49} = -\frac{1}{2}+\frac{7}{2} = 3$

  BOTH ARGUMENT OF RADICAL IS DEFINED
  AND $9-x^2>0$, WHICH IS OK

- $x = -\frac{1}{2}-\frac{1}{2}\sqrt{37} < 0$

  ARGUMENT OF RADICAL IS DEFINED

  $\left|-\frac{1}{2}-\frac{1}{2}\sqrt{37}\right| = \frac{1}{2}+\frac{1}{2}\sqrt{37} > \frac{1}{2}+\frac{1}{2}\sqrt{36} = \frac{1}{2}+3 = \frac{7}{2}$

  SO $9-x^2<0$, WHICH IS A PROBLEM

- $x = \frac{1}{2}+\frac{\sqrt{33}}{2}$

  $\frac{1}{2}+\frac{\sqrt{33}}{2} < \frac{1}{2}+\frac{1}{2}\sqrt{36} = \frac{1}{2}+3 = \frac{7}{2}$, SO RADICAL'S ARGUMENT IS DEFINED
  $\frac{1}{2}+\frac{\sqrt{33}}{2} > \frac{1}{2}+\frac{1}{2}\sqrt{25} = \frac{1}{2}+\frac{5}{2} = 3$, SO $9-x^2<0$, WHICH IS A PROBLEM

- $x = \frac{1}{2}-\frac{\sqrt{33}}{2} < 0$

  ARGUMENT OF RADICAL IS DEFINED

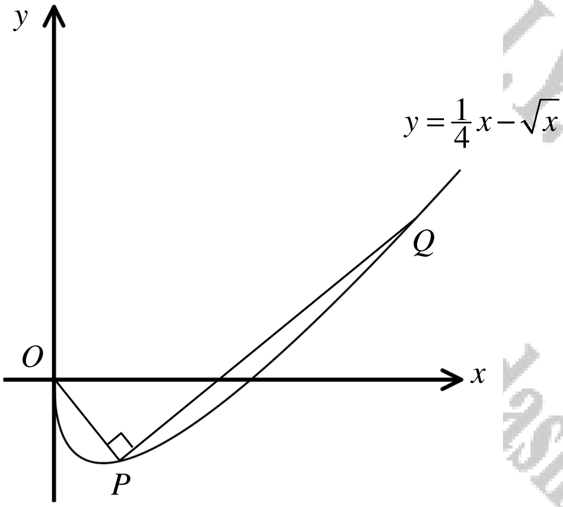
  $\left|\frac{1}{2}-\frac{\sqrt{33}}{2}\right| = \left|\frac{\sqrt{33}}{2}-\frac{1}{2}\right| = \frac{\sqrt{33}}{2}-\frac{1}{2} < \frac{\sqrt{36}}{2}-\frac{1}{2} = 3-\frac{1}{2} = \frac{5}{2}$

  SO $9-x^2>0$

  I.E BOTH CONDITIONS ARE SATISFIED

$$\therefore x = \begin{cases} -\frac{1}{2}+\frac{1}{2}\sqrt{37} \\ \frac{1}{2}-\frac{1}{2}\sqrt{33} \end{cases}$$

Question 100 (*****)



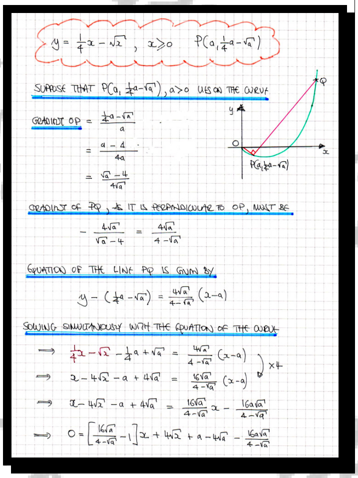
The figure above show the curve with equation

$$y=\frac{1}{4}x-\sqrt{x}\,,\; x\in\mathbb{R}\,,\; x\geq 0\,.$$

The points P and Q lie on the curve, so that $\measuredangle OPQ = 90°$, where O is the origin.

Determine the range of possible values of the x coordinate of P.

$\boxed{}$, $\boxed{0<x<\frac{16}{289}}$



$y=\frac{1}{4}x-\sqrt{x}\,,\quad x\geq 0 \qquad P\left(a,\frac{1}{4}a-\sqrt{a}\right)$

SUPPOSE THAT $P\left(a,\frac{1}{4}a-\sqrt{a}\right)$, $a>0$ LIES ON THE CURVE

GRADIENT OP $=\dfrac{\frac{1}{4}a-\sqrt{a}}{a}$

$=\dfrac{a-4}{4a}$

$=\dfrac{\sqrt{a}-4}{4\sqrt{a}}$

GRADIENT OF PQ, AS IT IS PERPENDICULAR TO OP, MUST BE

$-\dfrac{4\sqrt{a}}{\sqrt{a}-4}=\dfrac{4\sqrt{a}}{4-\sqrt{a}}$

EQUATION OF THE LINE PQ IS GIVEN BY

$y-\left(\frac{1}{4}a-\sqrt{a}\right)=\dfrac{4\sqrt{a}}{4-\sqrt{a}}(x-a)$

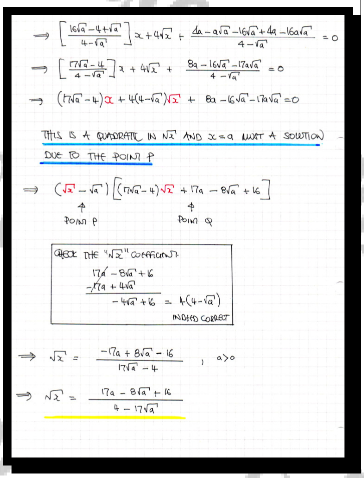
SOLVING SIMULTANEOUSLY WITH THE EQUATION OF THE CURVE

$\Longrightarrow \frac{1}{4}x-\sqrt{x}-\frac{1}{4}a+\sqrt{a}=\dfrac{4\sqrt{a}}{4-\sqrt{a}}(x-a)$ $\quad\times 4$

$\Longrightarrow x-4\sqrt{x}-a+4\sqrt{a}=\dfrac{16\sqrt{a}}{4-\sqrt{a}}(x-a)$

$\Longrightarrow x-4\sqrt{x}-a+4\sqrt{a}=\dfrac{16\sqrt{a}}{4-\sqrt{a}}x-\dfrac{16a\sqrt{a}}{4-\sqrt{a}}$

$\Longrightarrow 0=\left[\dfrac{16\sqrt{a}}{4-\sqrt{a}}-1\right]x+4\sqrt{x}+a-4\sqrt{a}-\dfrac{16a\sqrt{a}}{4-\sqrt{a}}$



$\Longrightarrow \left[\dfrac{16\sqrt{a}-4+\sqrt{a}}{4-\sqrt{a}}\right]x+4\sqrt{x}+\dfrac{4a-a\sqrt{a}-16\sqrt{a}+4a-16a\sqrt{a}}{4-\sqrt{a}}=0$

$\Longrightarrow \left[\dfrac{17\sqrt{a}-4}{4-\sqrt{a}}\right]x+4\sqrt{x}+\dfrac{8a-16\sqrt{a}-17a\sqrt{a}}{4-\sqrt{a}}=0$

$\Longrightarrow (17\sqrt{a}-4)x+4(4-\sqrt{a})\sqrt{x}+8a-16\sqrt{a}-17a\sqrt{a}=0$

THIS IS A QUADRATIC IN $\sqrt{x}$ AND $x=a$ MUST A SOLUTION DUE TO THE POINT P

$\Longrightarrow \underset{\substack{\uparrow\\ \text{POINT P}}}{(\sqrt{x}-\sqrt{a})}\underset{\substack{\uparrow\\ \text{POINT Q}}}{\left[(17\sqrt{a}-4)\sqrt{x}+17a-8\sqrt{a}+16\right]}$

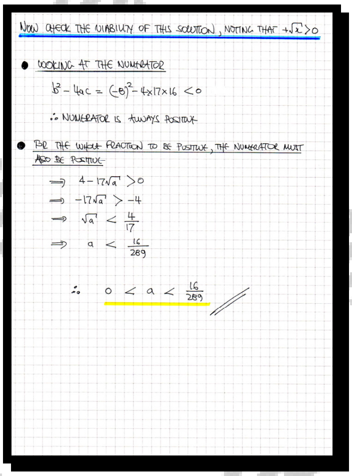
CHECK THE "$\sqrt{x}$" COEFFICIENT.

$\cancel{17a}-8\sqrt{a}+16$
$-\cancel{17a}+4\sqrt{a}$
$-4\sqrt{a}+16=4(4-\sqrt{a})$

INDEED CORRECT

$\Longrightarrow \sqrt{x}=\dfrac{-17a+8\sqrt{a}-16}{17\sqrt{a}-4}\,,\quad a>0$

$\Longrightarrow \sqrt{x}=\dfrac{17a-8\sqrt{a}+16}{4-17\sqrt{a}}$



NOW CHECK THE VIABILITY OF THIS SOLUTION, NOTING THAT $+\sqrt{x}>0$

- LOOKING AT THE NUMERATOR

  $b^2-4ac=(-8)^2-4\times17\times16<0$

  $\therefore$ NUMERATOR IS ALWAYS POSITIVE

- FOR THE WHOLE FRACTION TO BE POSITIVE, THE NUMERATOR MUST ALSO BE POSITIVE

  $\Longrightarrow 4-17\sqrt{a}>0$

  $\Longrightarrow -17\sqrt{a}>-4$

  $\Longrightarrow \sqrt{a}<\frac{4}{17}$

  $\Longrightarrow a<\frac{16}{289}$

$\therefore \quad 0<a<\frac{16}{289}$ //

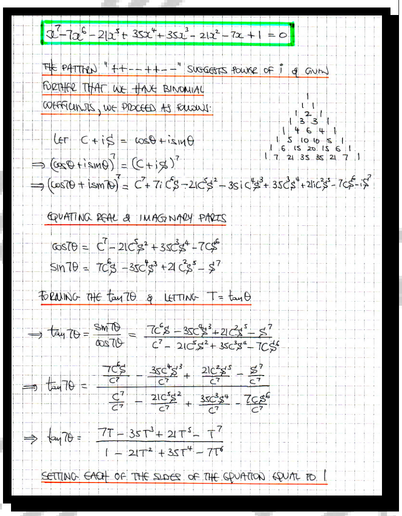
Question 101 (*****)

Find, in exact trigonometric form where appropriate, the real solutions of the following polynomial equation

$$x^7 - 7x^6 - 21x^5 + 35x^4 + 35x^3 - 21x^2 - 7x + 1 = 0 .$$

, $\boxed{x = \tan\left(\frac{\pi}{28}\right)}$, $\boxed{x = \tan\left(\frac{5\pi}{28}\right)}$, $\boxed{x = \tan\left(\frac{9\pi}{28}\right)}$, $\boxed{x = \tan\left(\frac{13\pi}{28}\right)}$,

$\boxed{x = \tan\left(\frac{17\pi}{28}\right)}$, $\boxed{x = \tan\left(\frac{3\pi}{4}\right) = -1}$, $\boxed{x = \tan\left(\frac{25\pi}{28}\right)}$



$\boxed{x^7 - 7x^6 - 21x^5 + 35x^4 + 35x^3 - 21x^2 - 7x + 1 = 0}$

THE PATTERN "++--++--" SUGGESTS POWER OF i & GIVEN FURTHER THAT WE HAVE BINOMIAL COEFFICIENTS, WE PROCEED AS FOLLOWS:

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

LET $C + iS = \cos\theta + i\sin\theta$

$\Rightarrow (\cos\theta + i\sin\theta)^7 = (C + iS)^7$

$\Rightarrow (\cos 7\theta + i\sin 7\theta)^7 = C^7 + 7iC^6S - 21C^5S^2 - 35iC^4S^3 + 35C^3S^4 + 21iC^2S^5 - 7CS^6 - iS^7$

EQUATING REAL & IMAGINARY PARTS

$\cos 7\theta = C^7 - 21C^5S^2 + 35C^3S^4 - 7CS^6$

$\sin 7\theta = 7C^6S - 35C^4S^3 + 21C^2S^5 - S^7$

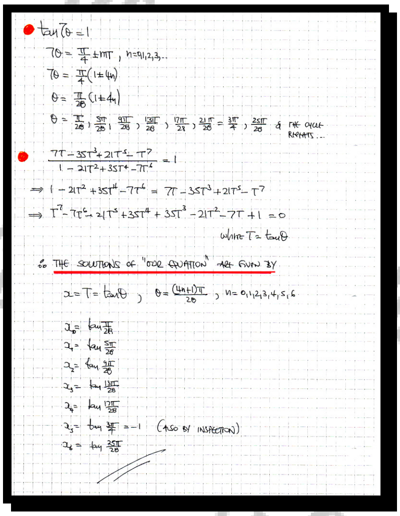
FORMING THE $\tan 7\theta$ & LETTING $T = \tan\theta$

$\Rightarrow \tan 7\theta = \frac{\sin 7\theta}{\cos 7\theta} = \frac{7C^6S - 35C^4S^3 + 21C^2S^5 - S^7}{C^7 - 21C^5S^2 + 35C^3S^4 - 7CS^6}$

$\Rightarrow \tan 7\theta = \frac{\frac{7C^6S}{C^7} - \frac{35C^4S^3}{C^7} + \frac{21C^2S^5}{C^7} - \frac{S^7}{C^7}}{\frac{C^7}{C^7} - \frac{21C^5S^2}{C^7} + \frac{35C^3S^4}{C^7} - \frac{7CS^6}{C^7}}$

$\Rightarrow \tan 7\theta = \frac{7T - 35T^3 + 21T^5 - T^7}{1 - 21T^2 + 35T^4 - 7T^6}$

SETTING EACH OF THE SIDES OF THE EQUATION EQUAL TO 1



● $\tan 7\theta = 1$

$7\theta = \frac{\pi}{4} \pm n\pi$, $n = 0, 1, 2, 3, \ldots$

$7\theta = \frac{\pi}{4}(1 \pm 4n)$

$\theta = \frac{\pi}{28}(1 \pm 4n)$

$\theta = \frac{\pi}{28}, \frac{5\pi}{28}, \frac{9\pi}{28}, \frac{13\pi}{28}, \frac{17\pi}{28}, \frac{21\pi}{28} = \frac{3\pi}{4}, \frac{25\pi}{28}$ & THE CYCLE REPEATS ...

● $\frac{7T - 35T^3 + 21T^5 - T^7}{1 - 21T^2 + 35T^4 - 7T^6} = 1$

$\Rightarrow 1 - 21T^2 + 35T^4 - 7T^6 = 7T - 35T^3 + 21T^5 - T^7$

$\Rightarrow T^7 - 7T^6 - 21T^5 + 35T^4 + 35T^3 - 21T^2 - 7T + 1 = 0$

where $T = \tan\theta$

$\therefore$ THE SOLUTIONS OF "OUR EQUATION" ARE GIVEN BY

$x = T = \tan\theta$, $\theta = \frac{(4n+1)\pi}{28}$, $n = 0, 1, 2, 3, 4, 5, 6$

$x_0 = \tan\frac{\pi}{28}$

$x_1 = \tan\frac{5\pi}{28}$

$x_2 = \tan\frac{9\pi}{28}$

$x_3 = \tan\frac{13\pi}{28}$

$x_4 = \tan\frac{17\pi}{28}$

$x_5 = \tan\frac{3\pi}{4} = -1$ (ALSO BY INSPECTION)

$x_6 = \tan\frac{25\pi}{28}$