DIFFERENTIATION II

EXAM QUESTIONS

Question 1 **(**)**

The curve C has equation

$$y = \frac{x^2}{2x+1}, \; x \neq -\frac{1}{2}$$

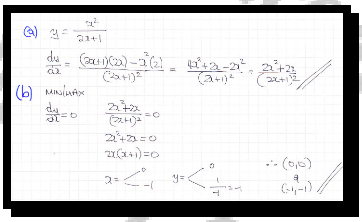
a) Show clearly that

$$\frac{dy}{dx} = \frac{2x^2+2x}{(2x+1)^2}.$$

b) Find the coordinates of the stationary points of C.

[*the nature of these stationary points need not be determined*]

$$\boxed{(0,0) \,\&\, (-1,-1)}$$



(a) $y = \frac{x^2}{2x+1}$

$\frac{dy}{dx} = \frac{(2x+1)(2x) - x^2(2)}{(2x+1)^2} = \frac{4x^2+2x-2x^2}{(2x+1)^2} = \frac{2x^2+2x}{(2x+1)^2}$

(b) MIN/MAX

$\frac{dy}{dx} = 0 \qquad \frac{2x^2+2x}{(2x+1)^2} = 0$

$2x^2+2x = 0$

$2x(x+1) = 0$

$x = \begin{cases} 0 \\ -1 \end{cases} \qquad y = \begin{cases} 0 \\ \frac{1}{-1} = -1 \end{cases} \qquad \therefore (0,0) \; \& \; (-1,-1)$

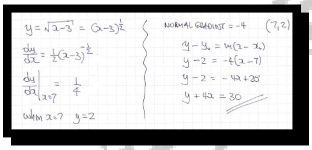
Question 2 **(**)**

A curve C has equation

$$y = \sqrt{x-3}, \; x > 3.$$

Find an equation of the normal to C at the point where $x = 7$

$$\boxed{}, \; \boxed{4x + y = 30}$$



$y = \sqrt{x-3} = (x-3)^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2}(x-3)^{-\frac{1}{2}}$

$\left.\frac{dy}{dx}\right|_{x=7} = \frac{1}{4}$

when $x=7 \quad y=2$

NORMAL GRADIENT $= -4 \qquad (7,2)$

$y - y_0 = m(x - x_0)$

$y - 2 = -4(x-7)$

$y - 2 = -4x + 28$

$y + 4x = 30$

Question 3 (**)

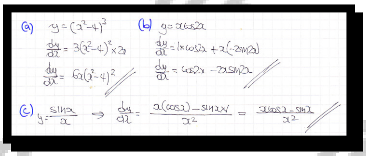
Differentiate each of the following expressions with respect to x, simplifying the final answers as far as possible

a) $y = \left(x^2 - 4\right)^3$

b) $y = x\cos 2x$

c) $y = \dfrac{\sin x}{x}$

$\boxed{\quad}$, $\boxed{\dfrac{dy}{dx} = 6x\left(x^2-4\right)^2}$, $\boxed{\dfrac{dy}{dx} = \cos 2x - 2x\sin 2x}$, $\boxed{\dfrac{dy}{dx} = \dfrac{x\cos x - \sin x}{x^2}$}



(a) $y = (x^2-4)^3$

$\frac{dy}{dx} = 3(x^2-4)^2 \times 2x$

$\frac{dy}{dx} = 6x(x^2-4)^2$

(b) $y = x\cos 2x$

$\frac{dy}{dx} = 1\times\cos 2x + x(-2\sin 2x)$

$\frac{dy}{dx} = \cos 2x - 2x\sin 2x$

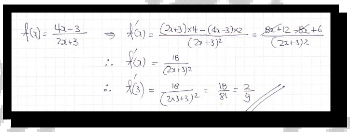
(c) $y = \frac{\sin x}{x} \Rightarrow \frac{dy}{dx} = \frac{x(\cos x) - \sin x \times 1}{x^2} = \frac{x\cos x - \sin x}{x^2}$

Question 4 (**)

$$f(x) = \frac{4x-3}{2x+3}, \quad x \neq -\frac{3}{2}.$$

Evaluate $f'(3)$.

$\boxed{f'(3) = \frac{2}{9}}$



$f(x) = \frac{4x-3}{2x+3} \Rightarrow f'(x) = \frac{(2x+3)\times 4 - (4x-3)\times 2}{(2x+3)^2} = \frac{\cancel{8x}+12\cancel{-8x}+6}{(2x+3)^2}$

$\therefore f'(x) = \frac{18}{(2x+3)^2}$

$\therefore f'(3) = \frac{18}{(2\times 3+3)^2} = \frac{18}{81} = \frac{2}{9}$

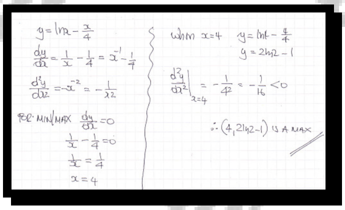
Question 5 (**)

The curve C has equation

$$y = \ln x - \frac{x}{4}, \;\; x > 0 .$$

Find the exact coordinates of the turning point of C, determining by calculation whether it is a maximum or minimum.

$$\boxed{\max\left(4, 2\ln 2 - 1\right)}$$



$y = \ln x - \frac{x}{4}$

$\frac{dy}{dx} = \frac{1}{x} - \frac{1}{4} = x^{-1} - \frac{1}{4}$

$\frac{d^2y}{dx^2} = -x^{-2} = -\frac{1}{x^2}$

FOR MIN/MAX $\frac{dy}{dx} = 0$

$\frac{1}{x} - \frac{1}{4} = 0$

$\frac{1}{x} = \frac{1}{4}$

$x = 4$

when $x = 4$ $y = \ln 4 - \frac{4}{4}$

$y = 2\ln 2 - 1$

$\left.\frac{d^2y}{dx^2}\right|_{x=4} = -\frac{1}{4^2} = -\frac{1}{16} < 0$

$\therefore (4, 2\ln 2 - 1)$ IS A MAX

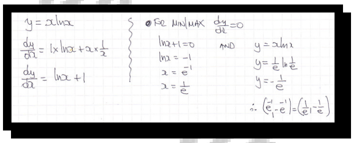
Question 6 (**)

The curve C has equation

$$y = x\ln x, \;\; x > 0 .$$

Find the exact coordinates of the turning point of C.

$$\boxed{\left(\frac{1}{e}, -\frac{1}{e}\right)}$$



$y = x\ln x$

$\frac{dy}{dx} = 1 \times \ln x + x \times \frac{1}{x}$

$\frac{dy}{dx} = \ln x + 1$

FOR MIN/MAX $\frac{dy}{dx} = 0$

$\ln x + 1 = 0$

$\ln x = -1$

$x = e^{-1}$

$x = \frac{1}{e}$

AND

$y = x\ln x$

$y = \frac{1}{e}\ln\frac{1}{e}$

$y = -\frac{1}{e}$

$\therefore \left(e^{-1}, -e^{-1}\right) = \left(\frac{1}{e}, -\frac{1}{e}\right)$

Question 7 **(**)**

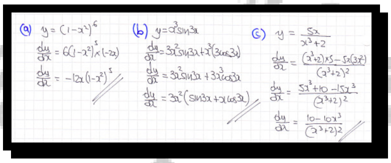
Differentiate each of the following expressions with respect to x, simplifying the final answers as far as possible.

a) $y = \left(1 - x^2\right)^6$

b) $y = x^3 \sin 3x$

c) $y = \dfrac{5x}{x^3 + 2}$

$\boxed{\text{C3F}}, \boxed{\dfrac{dy}{dx} = -12x\left(1 - x^2\right)^5}, \boxed{\dfrac{dy}{dx} = 3x^2\left(\sin 3x + x\cos 3x\right)}, \boxed{\dfrac{dy}{dx} = \dfrac{10\left(1 - x^3\right)}{\left(x^3 + 2\right)^2}}$



(a) $y = (1 - x^2)^6$

$\frac{dy}{dx} = 6(1 - x^2)^5 \times (-2x)$

$\frac{dy}{dx} = -12x(1 - x^2)^5$

(b) $y = x^3 \sin 3x$

$\frac{dy}{dx} = 3x^2 \sin 3x + x^3(3\cos 3x)$

$\frac{dy}{dx} = 3x^2 \sin 3x + 3x^3 \cos 3x$

$\frac{dy}{dx} = 3x^2(\sin 3x + x\cos 3x)$

(c) $y = \frac{5x}{x^3 + 2}$

$\frac{dy}{dx} = \frac{(x^3+2) \times 5 - 5x(3x^2)}{(x^3+2)^2}$

$\frac{dy}{dx} = \frac{5x^3 + 10 - 15x^3}{(x^3+2)^2}$

$\frac{dy}{dx} = \frac{10 - 10x^3}{(x^3+2)^2}$

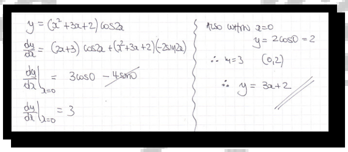
Question 8 **(**+)**

A curve has equation

$$y = \left(x^2 + 3x + 2\right)\cos 2x .$$

Determine an equation of the tangent to the curve at the point where the curve crosses the y axis.

$\boxed{}, \boxed{y = 3x + 2}$



$y = (x^2 + 3x + 2)\cos 2x$

$\frac{dy}{dx} = (2x + 3)\cos 2x + (x^2 + 3x + 2)(-2\sin 2x)$

$\left.\frac{dy}{dx}\right|_{x=0} = 3\cos 0 - \cancel{4\sin 0}$

$\left.\frac{dy}{dx}\right|_{x=0} = 3$

Also when $x = 0$

$y = 2\cos 0 = 2$

$\therefore m = 3 \quad (0, 2)$

$\therefore y = 3x + 2$

Question 9 **(**+)**

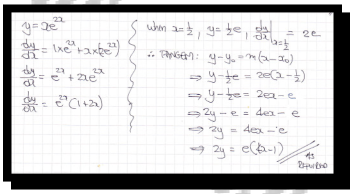
A curve C has equation

$$y = x\mathrm{e}^{2x}, \ x \in \mathbb{R}.$$

Show that an equation of the tangent to C at the point where $x = \frac{1}{2}$ is

$$2y = \mathrm{e}(4x - 1).$$

☐, proof



$y = xe^{2x}$

$\frac{dy}{dx} = 1 \times e^{2x} + x \times (2e^{2x})$

$\frac{dy}{dx} = e^{2x} + 2xe^{2x}$

$\frac{dy}{dx} = e^{2x}(1+2x)$

When $x = \frac{1}{2}$, $y = \frac{1}{2}e$, $\left.\frac{dy}{dx}\right|_{x=\frac{1}{2}} = 2e$

$\therefore$ TANGENT: $y - y_0 = m(x - x_0)$

$\Rightarrow y - \frac{1}{2}e = 2e(x - \frac{1}{2})$

$\Rightarrow y - \frac{1}{2}e = 2ex - e$

$\Rightarrow 2y - e = 4ex - e$

$\Rightarrow 2y = 4ex - e$

$\Rightarrow 2y = e(4x-1)$

AS REQUIRED

Question 10 **(**+)**

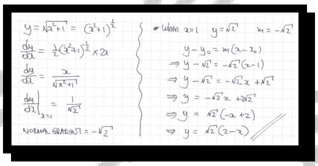
A curve C has equation

$$y = \sqrt{x^2 + 1}, \ x \in \mathbb{R}.$$

Show that an equation of the normal to C at the point where $x = 1$ is given by

$$y = \sqrt{2}(2 - x).$$

☐, proof



$y = \sqrt{x^2+1} = (x^2+1)^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2}(x^2+1)^{-\frac{1}{2}} \times 2x$

$\frac{dy}{dx} = \frac{x}{\sqrt{x^2+1}}$

$\left.\frac{dy}{dx}\right|_{x=1} = \frac{1}{\sqrt{2}}$

NORMAL GRADIENT $= -\sqrt{2}$

- When $x = 1$ $y = \sqrt{2}$ $m = -\sqrt{2}$

$y - y_0 = m(x - x_0)$

$\Rightarrow y - \sqrt{2} = -\sqrt{2}(x-1)$

$\Rightarrow y - \sqrt{2} = -\sqrt{2}x + \sqrt{2}$

$\Rightarrow y = -\sqrt{2}x + 2\sqrt{2}$

$\Rightarrow y = \sqrt{2}(-x+2)$

$\Rightarrow y = \sqrt{2}(2-x)$

Question 11 **(**+)**

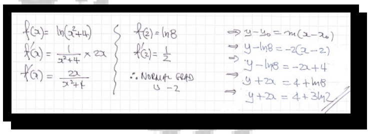
The point P, where $x = 2$, lies on the curve with equation

$$f(x) = \ln\left(x^2 + 4\right).$$

Show that an equation of the normal to the curve at P, is given by

$$y + 2x = 4 + 3\ln 2.$$

$\boxed{\text{proof}}$



$f(x) = \ln(x^2+4)$

$f'(x) = \frac{1}{x^2+4} \times 2x$

$f'(x) = \frac{2x}{x^2+4}$

$f(2) = \ln 8$

$f'(2) = \frac{1}{2}$

∴ NORMAL GRAD is -2

$\Rightarrow y - y_0 = m(x - x_0)$

$\Rightarrow y - \ln 8 = -2(x-2)$

$\Rightarrow y - \ln 8 = -2x + 4$

$\Rightarrow y + 2x = 4 + \ln 8$

$\Rightarrow y + 2x = 4 + 3\ln 2$

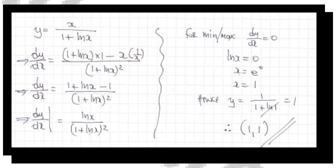
Question 12 **(**+)**

The curve C has equation

$$y = \frac{x}{1 + \ln x}, \quad x > 0, \quad x \neq \mathrm{e}^{-1}.$$

Show that C has a single stationary point and find its coordinates.

$\boxed{}$, $\boxed{(1,1)}$



$y = \frac{x}{1+\ln x}$

$\Rightarrow \frac{dy}{dx} = \frac{(1+\ln x)\times 1 - x\left(\frac{1}{x}\right)}{(1+\ln x)^2}$

$\Rightarrow \frac{dy}{dx} = \frac{1 + \ln x - 1}{(1+\ln x)^2}$

$\Rightarrow \left.\frac{dy}{dx}\right| = \frac{\ln x}{(1+\ln x)^2}$

for min/max $\frac{dy}{dx} = 0$

$\ln x = 0$

$x = e^0$

$x = 1$

hence $y = \frac{1}{1+\ln 1} = 1$

∴ $(1,1)$

Question 13 (+)**

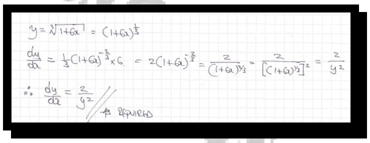
The curve C has equation

$$y = \sqrt[3]{1+6x}\,,\ x \geq -\frac{1}{6}.$$

Show clearly that

$$\frac{dy}{dx} = \frac{2}{y^2}\,.$$

$\boxed{\text{proof}}$



$y = \sqrt[3]{1+6x} = (1+6x)^{\frac{1}{3}}$

$\frac{dy}{dx} = \frac{1}{3}(1+6x)^{-\frac{2}{3}} \times 6 = 2(1+6x)^{-\frac{2}{3}} = \frac{2}{(1+6x)^{2/3}} = \frac{2}{[(1+6x)^{1/3}]^2} = \frac{2}{y^2}$

$\therefore\ \frac{dy}{dx} = \frac{2}{y^2}$ // AS REQUIRED

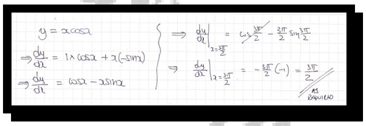
Question 14 (+)**

Given that

$$y = x\cos x\,,\ x \in \mathbb{R}$$

Show clearly that the value of $\frac{dy}{dx}$ at $x = \frac{3\pi}{2}$ is $\frac{3\pi}{2}$.

$\boxed{\text{proof}}$



$y = x\cos x$

$\Rightarrow \frac{dy}{dx} = 1 \times \cos x + x(-\sin x)$

$\Rightarrow \frac{dy}{dx} = \cos x - x\sin x$

$\Rightarrow \left.\frac{dy}{dx}\right|_{x=\frac{3\pi}{2}} = \cancel{\cos\frac{3\pi}{2}} - \frac{3\pi}{2}\sin\frac{3\pi}{2}$

$\Rightarrow \left.\frac{dy}{dx}\right|_{x=\frac{3\pi}{2}} = -\frac{3\pi}{2}(-1) = \frac{3\pi}{2}$ // AS REQUIRED

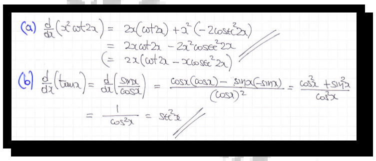
Question 15 **(***)**

a) Find $\frac{d}{dx}\left(x^2 \cot 2x\right)$

b) Show clearly that

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\boxed{\frac{dy}{dx} = 2x\left(\cot 2x - x\operatorname{cosec}^2 2x\right)}$$



(a) $\frac{d}{dx}\left(x^2 \cot 2x\right) = 2x(\cot 2x) + x^2\left(-2\operatorname{cosec}^2 2x\right)$

$= 2x\cot 2x - 2x^2\operatorname{cosec}^2 2x$

$(= 2x\left(\cot 2x - x\operatorname{cosec}^2 2x\right))$ //

(b) $\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x(\cos x) - \sin x(-\sin x)}{(\cos x)^2} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$

$= \frac{1}{\cos^2 x} = \sec^2 x$ //

Question 16 **(***)**

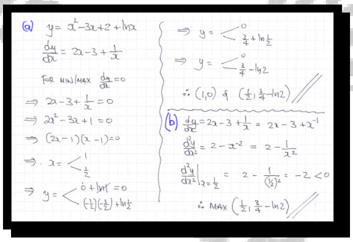
The curve C has equation

$$y = (x-1)(x-2) + \ln x, \quad x > 0.$$

a) Show that one of the turning points of C has coordinates $\left(\frac{1}{2}, \frac{3}{4} - \ln 2\right)$ and find the coordinates of the other.

b) Determine the nature of the turning point with coordinates $\left(\frac{1}{2}, \frac{3}{4} - \ln 2\right)$.

$$\boxed{(1,0)}, \quad \boxed{\max\left(\frac{1}{2}, \frac{3}{4} - \ln 2\right)}$$



(a) $y = x^2 - 3x + 2 + \ln x$

$\frac{dy}{dx} = 2x - 3 + \frac{1}{x}$

FOR MIN/MAX $\frac{dy}{dx} = 0$

$\Rightarrow 2x - 3 + \frac{1}{x} = 0$

$\Rightarrow 2x^2 - 3x + 1 = 0$

$\Rightarrow (2x-1)(x-1) = 0$

$\Rightarrow x = \begin{cases} 1 \\ \frac{1}{2} \end{cases}$

$\Rightarrow y = \begin{cases} 0 + \ln 1 = 0 \\ \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right) + \ln\frac{1}{2} \end{cases}$

$\Rightarrow y = \begin{cases} 0 \\ \frac{3}{4} + \ln\frac{1}{2} \end{cases}$

$\Rightarrow y = \begin{cases} 0 \\ \frac{3}{4} - \ln 2 \end{cases}$

$\therefore (1,0)$ & $\left(\frac{1}{2}, \frac{3}{4} - \ln 2\right)$ //

(b) $\frac{dy}{dx} = 2x - 3 + \frac{1}{x} = 2x - 3 + x^{-1}$

$\frac{d^2y}{dx^2} = 2 - x^{-2} = 2 - \frac{1}{x^2}$

$\left.\frac{d^2y}{dx^2}\right|_{x=\frac{1}{2}} = 2 - \frac{1}{\left(\frac{1}{2}\right)^2} = -2 < 0$

$\therefore$ MAX $\left(\frac{1}{2}, \frac{3}{4} - \ln 2\right)$ //

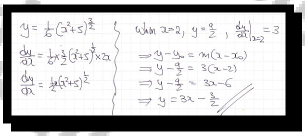
Question 17 (***)

The point P, where $x=2$, lies on the curve with equation

$$y=\frac{1}{6}\left(x^2+5\right)^{\frac{3}{2}},\ x\in\mathbb{R}.$$

Find an equation of the tangent to the curve at P.

☐ , $\boxed{y=3x-\frac{3}{2}}$



$y=\frac{1}{6}(x^2+5)^{\frac{3}{2}}$

$\frac{dy}{dx}=\frac{1}{6}\times\frac{3}{2}(x^2+5)^{\frac{1}{2}}\times 2x$

$\frac{dy}{dx}=\frac{1}{2}x(x^2+5)^{\frac{1}{2}}$

when $x=2$, $y=\frac{9}{2}$, $\left.\frac{dy}{dx}\right|_{x=2}=3$

$\Rightarrow y-y_0=m(x-x_0)$

$\Rightarrow y-\frac{9}{2}=3(x-2)$

$\Rightarrow y-\frac{9}{2}=3x-6$

$\Rightarrow y=3x-\frac{3}{2}$

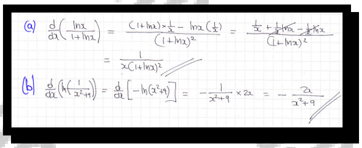
Question 18 (***)

Differentiate each of the following expressions with respect to x, writing the final answers as simplified fractions.

a) $y=\dfrac{\ln x}{1+\ln x}$.

b) $y=\ln\left(\dfrac{1}{x^2+9}\right)$.

☐ , $\boxed{\frac{dy}{dx}=\frac{1}{x(1+\ln x)^2}}$, $\boxed{\frac{dy}{dx}=-\frac{2x}{x^2+9}}$



(a) $\frac{d}{dx}\left(\frac{\ln x}{1+\ln x}\right)=\frac{(1+\ln x)\times\frac{1}{x}-\ln x\left(\frac{1}{x}\right)}{(1+\ln x)^2}=\frac{\frac{1}{x}+\cancel{\frac{1}{x}\ln x}-\cancel{\frac{1}{x}\ln x}}{(1+\ln x)^2}$

$=\frac{1}{x(1+\ln x)^2}$

(b) $\frac{d}{dx}\left(\ln\left(\frac{1}{x^2+9}\right)\right)=\frac{d}{dx}\left[-\ln(x^2+9)\right]=-\frac{1}{x^2+9}\times 2x=-\frac{2x}{x^2+9}$

Question 19 (***)

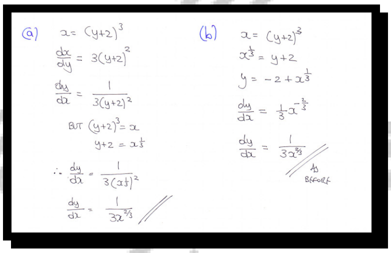
A curve has equation

$$x = (y+2)^3 .$$

a) Find $\frac{dy}{dx}$ in **terms of** $\boldsymbol{x}$, by **first** finding $\frac{dx}{dy}$.

b) By making y the subject of the equation and differentiating the resulting equation, verify the result of part (a).

$$\boxed{\frac{dy}{dx} = \frac{1}{3x^{\frac{2}{3}}}}$$



(a) $x = (y+2)^3$

$\frac{dx}{dy} = 3(y+2)^2$

$\frac{dy}{dx} = \frac{1}{3(y+2)^2}$

BUT $(y+2)^3 = x$

$y+2 = x^{\frac{1}{3}}$

$\therefore \frac{dy}{dx} = \frac{1}{3(x^{\frac{1}{3}})^2}$

$\frac{dy}{dx} = \frac{1}{3x^{2/3}}$ //

(b) $x = (y+2)^3$

$x^{\frac{1}{3}} = y+2$

$y = -2 + x^{\frac{1}{3}}$

$\frac{dy}{dx} = \frac{1}{3}x^{-\frac{2}{3}}$

$\frac{dy}{dx} = \frac{1}{3x^{2/3}}$ // AS BEFORE

Question 20 (***)

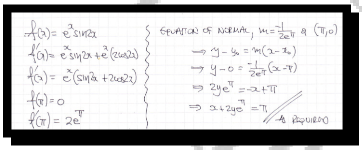
The point P, where $x = \pi$, lies on the curve with equation

$$f(x) = e^x \sin 2x, \ 0 \le x < 2\pi .$$

Show that an equation of the normal to the curve at P, is given by

$$x + 2ye^{\pi} = \pi .$$

$\boxed{}$, $\boxed{\text{proof}}$



$f(x) = e^x \sin 2x$

$f'(x) = e^x \sin 2x + e^x(2\cos 2x)$

$f'(x) = e^x(\sin 2x + 2\cos 2x)$

$f(\pi) = 0$

$f'(\pi) = 2e^{\pi}$

EQUATION OF NORMAL, $m = \frac{-1}{2e^{\pi}}$ & $(\pi, 0)$

$\Rightarrow y - y_0 = m(x - x_0)$

$\Rightarrow y - 0 = \frac{-1}{2e^{\pi}}(x - \pi)$

$\Rightarrow 2ye^{\pi} = -x + \pi$

$\Rightarrow x + 2ye^{\pi} = \pi$ // AS REQUIRED

Question 21 (***)

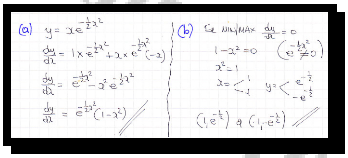
The curve C has equation

$$y = x\mathrm{e}^{-\frac{1}{2}x^2}, \quad x \in \mathbb{R}.$$

a) Find an expression for $\frac{dy}{dx}$.

b) Find the exact coordinates of the turning points of C.

$$\boxed{\frac{dy}{dx} = \left(1-x^2\right)\mathrm{e}^{-\frac{1}{2}x^2}}, \quad \boxed{\left(1, \frac{1}{\sqrt{\mathrm{e}}}\right), \left(-1, -\frac{1}{\sqrt{\mathrm{e}}}\right)}$$



(a) $y = xe^{-\frac{1}{2}x^2}$

$\frac{dy}{dx} = 1 \times e^{-\frac{1}{2}x^2} + x \times e^{-\frac{1}{2}x^2}(-x)$

$\frac{dy}{dx} = e^{-\frac{1}{2}x^2} - x^2 e^{-\frac{1}{2}x^2}$

$\frac{dy}{dx} = e^{-\frac{1}{2}x^2}(1-x^2)$ //

(b) For MIN/MAX $\frac{dy}{dx} = 0$

$1 - x^2 = 0 \quad \left(e^{-\frac{1}{2}x^2} \neq 0\right)$

$x^2 = 1$

$x = \begin{cases} 1 \\ -1 \end{cases} \quad y = \begin{cases} e^{-\frac{1}{2}} \\ -e^{-\frac{1}{2}} \end{cases}$

$(1, e^{-\frac{1}{2}})$ & $(-1, -e^{-\frac{1}{2}})$ //

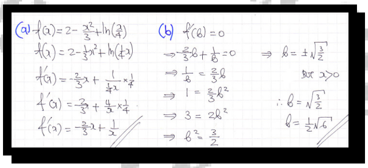
Question 22 (***)

$$f(x) = 2 - \frac{x^2}{3} + \ln\left(\frac{x}{4}\right), \quad x > 0.$$

a) Find an expression for $f'(x)$.

b) Find β in exact surd form, such that $f'(\beta) = 0$.

$$\boxed{f'(x) = \frac{1}{x} - \frac{2}{3}x}, \quad \boxed{\beta = \frac{1}{2}\sqrt{6}}$$



(a) $f(x) = 2 - \frac{x^2}{3} + \ln\left(\frac{x}{4}\right)$

$f(x) = 2 - \frac{1}{3}x^2 + \ln\left(\frac{1}{4}x\right)$

$f'(x) = -\frac{2}{3}x + \frac{1}{\frac{1}{4}x} \times \frac{1}{4}$

$f'(x) = -\frac{2}{3}x + \frac{4}{x} \times \frac{1}{4}$

$f'(x) = -\frac{2}{3}x + \frac{1}{x}$ //

(b) $f'(\beta) = 0$

$\Rightarrow -\frac{2}{3}\beta + \frac{1}{\beta} = 0$

$\Rightarrow \frac{1}{\beta} = \frac{2}{3}\beta$

$\Rightarrow 1 = \frac{2}{3}\beta^2$

$\Rightarrow 3 = 2\beta^2$

$\Rightarrow \beta^2 = \frac{3}{2}$

$\Rightarrow \beta = \pm\sqrt{\frac{3}{2}}$

But $x > 0$

$\therefore \beta = \sqrt{\frac{3}{2}}$

$\beta = \frac{1}{2}\sqrt{6}$ //

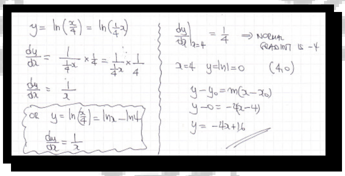
Question 23 (***)

A curve C has equation

$$y = \ln\left(\frac{x}{4}\right), \ x > 0.$$

Find an equation of the normal to C at the point where $x = 4$

$\boxed{}$, $\boxed{y = 16 - 4x}$



$y = \ln\left(\frac{x}{4}\right) = \ln\left(\frac{1}{4}x\right)$

$\frac{dy}{dx} = \frac{1}{\frac{1}{4}x} \times \frac{1}{4} = \frac{1}{\frac{1}{4}x} \times \frac{1}{4}$

$\frac{dy}{dx} = \frac{1}{x}$

OR $y = \ln\left(\frac{x}{4}\right) = \ln x - \ln 4$

$\frac{dy}{dx} = \frac{1}{x}$

$\left.\frac{dy}{dx}\right|_{x=4} = \frac{1}{4} \Rightarrow$ NORMAL GRADIENT IS -4

$x = 4 \quad y = \ln 1 = 0 \qquad (4, 0)$

$y - y_0 = m(x - x_0)$

$y - 0 = -4(x - 4)$

$y = -4x + 16$

Question 24 (***)

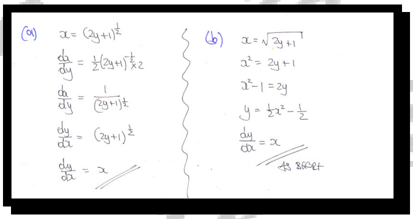
A curve has equation

$$x = \sqrt{2y + 1}, \ y \geq -\frac{1}{2}.$$

a) Find $\frac{dy}{dx}$ **in terms of x**, by **first** finding $\frac{dx}{dy}$.

b) By making y the subject of the above equation and differentiating the resulting equation, verify the result of part **(a)**.

$\boxed{}$, $\boxed{\frac{dy}{dx} = x}$



(a) $x = (2y+1)^{\frac{1}{2}}$

$\frac{dx}{dy} = \frac{1}{2}(2y+1)^{-\frac{1}{2}} \times 2$

$\frac{dx}{dy} = \frac{1}{(2y+1)^{\frac{1}{2}}}$

$\frac{dy}{dx} = (2y+1)^{\frac{1}{2}}$

$\frac{dy}{dx} = x$

(b) $x = \sqrt{2y+1}$

$x^2 = 2y + 1$

$x^2 - 1 = 2y$

$y = \frac{1}{2}x^2 - \frac{1}{2}$

$\frac{dy}{dx} = x$

AS BEFORE

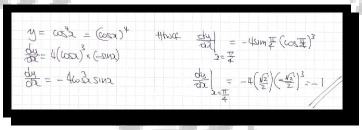
Question 25 (***)

Given that

$$y = \cos^4 x$$

find the value of $\frac{dy}{dx}$ at $x = \frac{\pi}{4}$.

$\boxed{\frac{dy}{dx} = -1}$



$y = \cos^4 x = (\cos x)^4$ thus $\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{4}} = -4\sin\frac{\pi}{4}\left(\cos\frac{\pi}{4}\right)^3$

$\frac{dy}{dx} = 4(\cos x)^3 \times (-\sin x)$

$\frac{dy}{dx} = -4\cos^3 x \sin x$ $\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{4}} = -4\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)^3 = -1$

Question 26 (***)

The point P with $x = \frac{\pi}{4}$ lies on the curve with equation

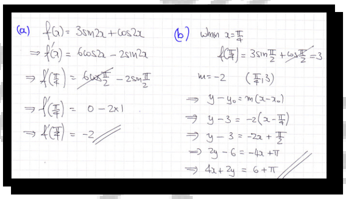
$$f(x) = 3\sin 2x + \cos 2x,\ 0 \le x < 2\pi.$$

a) Find the gradient at P.

b) Show that an equation of the tangent to the curve at P, is given by

$$4x + 2y = 6 + \pi.$$

$\boxed{-2}$



(a) $f(x) = 3\sin 2x + \cos 2x$

$\Rightarrow f'(x) = 6\cos 2x - 2\sin 2x$

$\Rightarrow f'\left(\frac{\pi}{4}\right) = \cancel{6\cos\frac{\pi}{2}} - 2\sin\frac{\pi}{2}$

$\Rightarrow f'\left(\frac{\pi}{4}\right) = 0 - 2 \times 1$

$\Rightarrow f'\left(\frac{\pi}{4}\right) = -2$

(b) when $x = \frac{\pi}{4}$

$f\left(\frac{\pi}{4}\right) = 3\sin\frac{\pi}{2} + \cancel{\cos\frac{\pi}{2}} = 3$

$m = -2$ $\left(\frac{\pi}{4}, 3\right)$

$\Rightarrow y - y_0 = m(x - x_0)$

$\Rightarrow y - 3 = -2\left(x - \frac{\pi}{4}\right)$

$\Rightarrow y - 3 = -2x + \frac{\pi}{2}$

$\Rightarrow 2y - 6 = -4x + \pi$

$\Rightarrow 4x + 2y = 6 + \pi$

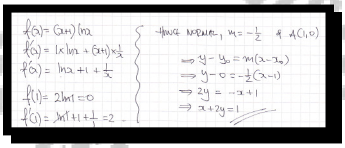
Question 27 (***)

The point A, where $x = 1$, lies on the curve with equation

$$f(x) = (x+1)\ln x, \; x > 0.$$

Find an equation of the normal to the curve at A.

$$\boxed{2y + x = 1}$$



$f(x) = (x+1)\ln x$

$f'(x) = 1 \times \ln x + (x+1) \times \frac{1}{x}$

$f'(x) = \ln x + 1 + \frac{1}{x}$

$f(1) = 2\ln 1 = 0$

$f'(1) = \cancel{\ln 1} + 1 + \frac{1}{1} = 2$

HENCE NORMAL, $m = -\frac{1}{2}$ & $A(1,0)$

$\Rightarrow y - y_0 = m(x - x_0)$

$\Rightarrow y - 0 = -\frac{1}{2}(x-1)$

$\Rightarrow 2y = -x + 1$

$\Rightarrow x + 2y = 1$

Question 28 (***)

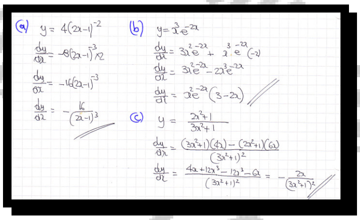
Differentiate each of the following expressions with respect to x, simplifying the final answers as far as possible

a) $y = \dfrac{4}{(2x-1)^2}$.

b) $y = x^3 e^{-2x}$.

c) $y = \dfrac{2x^2+1}{3x^2+1}$.

$\boxed{}$, $\boxed{\dfrac{dy}{dx} = -\dfrac{16}{(2x-1)^3}}$, $\boxed{\dfrac{dy}{dx} = x^2(3-2x)e^{-2x}}$, $\boxed{\dfrac{dy}{dx} = -\dfrac{2x}{\left(3x^2+1\right)^2}}$



(a) $y = 4(2x-1)^{-2}$

$\frac{dy}{dx} = -8(2x-1)^{-3} \times 2$

$\frac{dy}{dx} = -16(2x-1)^{-3}$

$\frac{dy}{dx} = -\frac{16}{(2x-1)^3}$

(b) $y = x^3 e^{-2x}$

$\frac{dy}{dx} = 3x^2 e^{-2x} + x^3 e^{-2x}(-2)$

$\frac{dy}{dx} = 3x^2 e^{-2x} - 2x^3 e^{-2x}$

$\frac{dy}{dx} = x^2 e^{-2x}(3-2x)$

(c) $y = \frac{2x^2+1}{3x^2+1}$

$\frac{dy}{dx} = \frac{(3x^2+1)(4x) - (2x^2+1)(6x)}{(3x^2+1)^2}$

$\frac{dy}{dx} = \frac{4x + 12x^3 - 12x^3 - 6x}{(3x^2+1)^2} = -\frac{2x}{(3x^2+1)^2}$

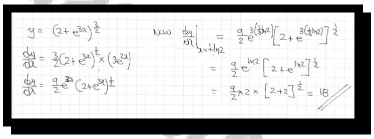
Question 29 **(***)**

Given that

$$y=\left(2+\mathrm{e}^{3x}\right)^{\frac{3}{2}}$$

find the value of $\frac{dy}{dx}$ at $x=\frac{1}{3}\ln 2$.

☐ , $\boxed{\frac{dy}{dx}=18}$



$y=\left(2+e^{3x}\right)^{\frac{3}{2}}$

$\frac{dy}{dx}=\frac{3}{2}\left(2+e^{3x}\right)^{\frac{1}{2}}\times\left(3e^{2x}\right)$

$\frac{dy}{dx}=\frac{9}{2}e^{3x}\left(2+e^{3x}\right)^{\frac{1}{2}}$

Now $\left.\frac{dy}{dx}\right|_{x=\frac{1}{3}\ln 2}=\frac{9}{2}e^{3\left(\frac{1}{3}\ln 2\right)}\left[2+e^{3\left(\frac{1}{3}\ln 2\right)}\right]^{\frac{1}{2}}$

$=\frac{9}{2}e^{\ln 2}\left[2+e^{\ln 2}\right]^{\frac{1}{2}}$

$=\frac{9}{2}\times 2\times\left[2+2\right]^{\frac{1}{2}}=18$ //

Question 30 (***)

A curve has equation

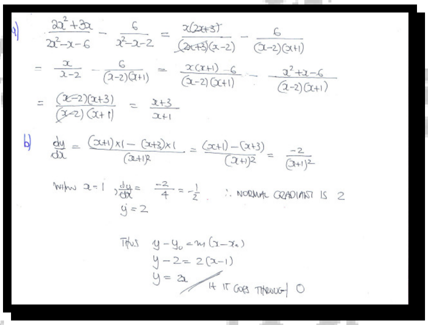
$$y = \frac{2x^2+3x}{2x^2-x-6} - \frac{6}{x^2-x-2}, \; x \in \mathbb{R}, \; 0 < x < 2.$$

a) Show clearly that

$$y = \frac{x+3}{x+1}, \; x \in \mathbb{R}, \; 0 < x < 2.$$

b) Show further that the equation of the normal to the curve at the point where $x = 1$ passes through the origin.

$\boxed{\text{C3E}}$, $\boxed{\text{proof}}$



a) $\frac{2x^2+3x}{2x^2-x-6} - \frac{6}{x^2-x-2} = \frac{x\cancel{(2x+3)}}{\cancel{(2x+3)}(x-2)} - \frac{6}{(x-2)(x+1)}$

$= \frac{x}{x-2} - \frac{6}{(x-2)(x+1)} = \frac{x(x+1)-6}{(x-2)(x+1)} = \frac{x^2+x-6}{(x-2)(x+1)}$

$= \frac{\cancel{(x-2)}(x+3)}{\cancel{(x-2)}(x+1)} = \frac{x+3}{x+1}$

b) $\frac{dy}{dx} = \frac{(x+1)\times 1 - (x+3)\times 1}{(x+1)^2} = \frac{(x+1)-(x+3)}{(x+1)^2} = \frac{-2}{(x+1)^2}$

WHEN $x=1$, $\frac{dy}{dx} = \frac{-2}{4} = -\frac{1}{2}$ $\therefore$ NORMAL GRADIENT IS 2

$y = 2$

THUS $y - y_0 = m(x - x_0)$

$y - 2 = 2(x-1)$

$y = x$ // IT GOES THROUGH O

Question 31 (***)

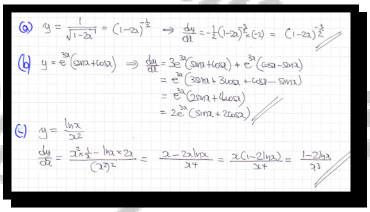
Differentiate each of the following expressions with respect to x, simplifying the final answers where possible.

a) $y = \dfrac{1}{\sqrt{1-2x}}$.

b) $y = \mathrm{e}^{3x}\left(\sin x + \cos x\right)$.

c) $y = \dfrac{\ln x}{x^2}$.

$\boxed{}$, $\boxed{\dfrac{dy}{dx} = \left(1-2x\right)^{-\frac{3}{2}}}$, $\boxed{\dfrac{dy}{dx} = 2\mathrm{e}^{3x}\left(\sin x + 2\cos x\right)}$, $\boxed{\dfrac{dy}{dx} = \dfrac{1-2\ln x}{x^3}}$



(a) $y = \frac{1}{\sqrt{1-2x}} = (1-2x)^{-\frac{1}{2}} \Rightarrow \frac{dy}{dx} = -\frac{1}{2}(1-2x)^{-\frac{3}{2}} \times (-2) = (1-2x)^{-\frac{3}{2}}$ //

(b) $y = e^{3x}(\sin x + \cos x) \Rightarrow \frac{dy}{dx} = 3e^{3x}(\sin x + \cos x) + e^{3x}(\cos x - \sin x)$

$= e^{3x}(3\sin x + 3\cos x + \cos x - \sin x)$

$= e^{3x}(2\sin x + 4\cos x)$

$= 2e^{3x}(\sin x + 2\cos x)$ //

(c) $y = \frac{\ln x}{x^2}$

$\frac{dy}{dx} = \frac{x^2 \times \frac{1}{x} - \ln x \times 2x}{(x^2)^2} = \frac{x - 2x\ln x}{x^4} = \frac{x(1-2\ln x)}{x^4} = \frac{1-2\ln x}{x^3}$ //

Question 32 (***)

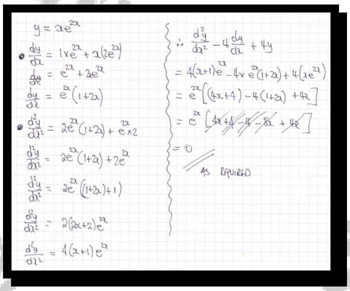
A curve has equation

$$y = x\mathrm{e}^{2x}, \ x \in \mathbb{R}.$$

Show clearly that

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0.$$

$\boxed{\text{proof}}$



$y = xe^{2x}$

• $\frac{dy}{dx} = 1 \times e^{2x} + x(2e^{2x})$

$\frac{dy}{dx} = e^{2x} + 2xe^{2x}$

$\frac{dy}{dx} = e^{2x}(1+2x)$

• $\frac{d^2y}{dx^2} = 2e^{2x}(1+2x) + e^{2x} \times 2$

$\frac{d^2y}{dx^2} = 2e^{2x}(1+2x) + 2e^{2x}$

$\frac{d^2y}{dx^2} = 2e^{2x}((1+2x)+1)$

$\frac{d^2y}{dx^2} = 2(2x+2)e^{2x}$

$\frac{d^2y}{dx^2} = 4(x+1)e^{2x}$

$\therefore \frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y$

$= 4(x+1)e^{2x} - 4 \times e^{2x}(1+2x) + 4(xe^{2x})$

$= e^{2x}\left[(4x+4) - 4(1+2x) + 4x\right]$

$= e^{2x}\left[\cancel{4x} + \cancel{4} - \cancel{4} - \cancel{8x} + \cancel{4x}\right]$

$= 0$

AS REQUIRED

Question 33 (***)

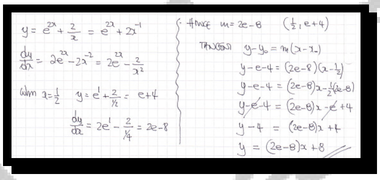
The point A, where $x = \frac{1}{2}$, lies on the curve with equation

$$y = \mathrm{e}^{2x} + \frac{2}{x}, \ x \neq 0.$$

Show that an equation of the tangent to the curve at A is given by

$$y = (2\mathrm{e} - 8)x + 8.$$

$\boxed{\text{proof}}$



$y = e^{2x} + \frac{2}{x} = e^{2x} + 2x^{-1}$

$\frac{dy}{dx} = 2e^{2x} - 2x^{-2} = 2e^{2x} - \frac{2}{x^2}$

when $x = \frac{1}{2}$ $\quad y = e^1 + \frac{2}{1/2} = e+4$

$\frac{dy}{dx} = 2e^1 - \frac{2}{1/4} = 2e - 8$

Hence $m = 2e - 8$ $\quad \left(\frac{1}{2}, e+4\right)$

TANGENT $\quad y - y_0 = m(x - x_0)$

$y - e - 4 = (2e-8)\left(x - \frac{1}{2}\right)$

$y - e - 4 = (2e-8)x - \frac{1}{2}(2e-8)$

$y - \cancel{e} - 4 = (2e-8)x - \cancel{e} + 4$

$y - 4 = (2e-8)x + 4$

$y = (2e-8)x + 8$

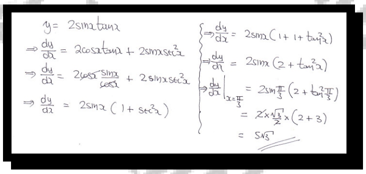
Question 34 (***)

Given that

$$y = 2\sin x \tan x$$

find the exact value of $\frac{dy}{dx}$ at $x = \frac{\pi}{3}$.

$\boxed{\frac{dy}{dx} = 5\sqrt{3}}$



$y = 2\sin x \tan x$

$\Rightarrow \frac{dy}{dx} = 2\cos x \tan x + 2\sin x \sec^2 x$

$\Rightarrow \frac{dy}{dx} = 2\cancel{\cos x}\frac{\sin x}{\cancel{\cos x}} + 2\sin x \sec^2 x$

$\Rightarrow \frac{dy}{dx} = 2\sin x\,(1 + \sec^2 x)$

$\Rightarrow \frac{dy}{dx} = 2\sin x\,(1 + 1 + \tan^2 x)$

$\Rightarrow \frac{dy}{dx} = 2\sin x\,(2 + \tan^2 x)$

$\Rightarrow \left.\frac{dy}{dx}\right|_{x=\frac{\pi}{3}} = 2\sin\frac{\pi}{3}\left(2 + \tan^2\frac{\pi}{3}\right)$

$= \cancel{2} \times \frac{\sqrt{3}}{\cancel{2}} \times (2 + 3)$

$= 5\sqrt{3}$

Question 35 (***)

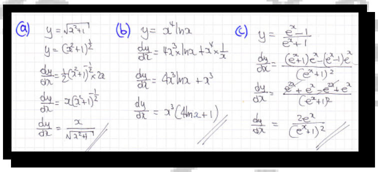
Differentiate each of the following expressions with respect to x, simplifying the final answers where possible.

a) $y = \sqrt{x^2 - 1}$

b) $y = x^4 \ln x$

c) $y = \frac{e^x - 1}{e^x + 1}$

, $\boxed{\frac{dy}{dx} = x\left(x^2 - 1\right)^{-\frac{1}{2}}}$, $\boxed{\frac{dy}{dx} = 4x^3 \ln x + x^3}$, $\boxed{\frac{dy}{dx} = \frac{2e^x}{\left(e^x + 1\right)^2}}$



(a) $y = \sqrt{x^2+1}$

$y = (x^2+1)^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2}(x^2+1)^{-\frac{1}{2}} \times 2x$

$\frac{dy}{dx} = x(x^2+1)^{-\frac{1}{2}}$

$\frac{dy}{dx} = \frac{x}{\sqrt{x^2+1}}$

(b) $y = x^4 \ln x$

$\frac{dy}{dx} = 4x^3 \times \ln x + x^4 \times \frac{1}{x}$

$\frac{dy}{dx} = 4x^3 \ln x + x^3$

$\frac{dy}{dx} = x^3(4\ln x + 1)$

(c) $y = \frac{e^x - 1}{e^x + 1}$

$\frac{dy}{dx} = \frac{(e^x+1)e^x - (e^x-1)e^x}{(e^x+1)^2}$

$\frac{dy}{dx} = \frac{\cancel{e^{2x}} + e^x - \cancel{e^{2x}} + e^x}{(e^x+1)^2}$

$\frac{dy}{dx} = \frac{2e^x}{(e^x+1)^2}$

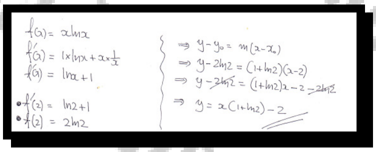
Question 36 **(***)**

The point A , where $x = 2$, lies on the curve with equation

$$f(x) = x \ln x, \; x > 0 .$$

Find an equation of the tangent to the curve at A, giving the answer in the form $y = mx + c$, where m and c are exact constants.

$$\boxed{y = x(1 + \ln 2) - 2}$$



$f(x) = x\ln x$

$f'(x) = 1 \times \ln x + x \times \frac{1}{x}$

$f'(x) = \ln x + 1$

- $f'(2) = \ln 2 + 1$
- $f(2) = 2\ln 2$

$\Rightarrow y - y_0 = m(x - x_0)$

$\Rightarrow y - 2\ln 2 = (1 + \ln 2)(x - 2)$

$\Rightarrow y \cancel{- 2\ln 2} = (1 + \ln 2)x - 2 \cancel{- 2\ln 2}$

$\Rightarrow y = x(1 + \ln 2) - 2$

Question 37 **(***)**

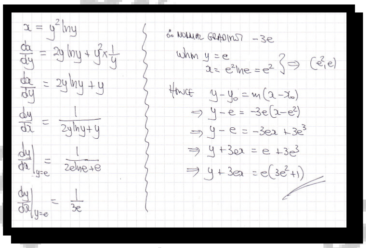
A curve C has equation

$$x = y^2 \ln y, \; y > 0 .$$

Show that an equation of the normal to C at the point where $y = \mathrm{e}$ is

$$y + 3\mathrm{e}x = \mathrm{e}\left(3\mathrm{e}^2 + 1\right).$$

$\boxed{}$, $\boxed{\text{proof}}$



$x = y^2 \ln y$

$\frac{dx}{dy} = 2y\ln y + y^2 \times \frac{1}{y}$

$\frac{dx}{dy} = 2y\ln y + y$

$\frac{dy}{dx} = \frac{1}{2y\ln y + y}$

$\left.\frac{dy}{dx}\right|_{y=e} = \frac{1}{2e\ln e + e}$

$\left.\frac{dy}{dx}\right|_{y=e} = \frac{1}{3e}$

∴ NORMAL GRADIENT $-3e$

WHEN $y = e$

$x = e^2 \ln e = e^2 \Big\} \Rightarrow (e^2, e)$

HENCE $y - y_0 = m(x - x_0)$

$\Rightarrow y - e = -3e(x - e^2)$

$\Rightarrow y - e = -3ex + 3e^3$

$\Rightarrow y + 3ex = e + 3e^3$

$\Rightarrow y + 3ex = e(3e^2 + 1)$

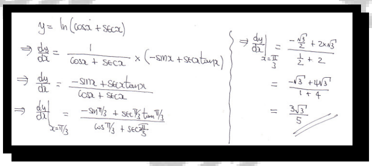
Question 38 **(***)**

Given that

$$y = \ln\left(\cos x + \sec x\right)$$

find the exact value of $\frac{dy}{dx}$ at $x = \frac{\pi}{3}$.

$$\boxed{\frac{dy}{dx} = \frac{3}{5}\sqrt{3}}$$



$y = \ln(\cos x + \sec x)$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos x + \sec x} \times (-\sin x + \sec x \tan x)$

$\Rightarrow \frac{dy}{dx} = \frac{-\sin x + \sec x \tan x}{\cos x + \sec x}$

$\Rightarrow \left.\frac{dy}{dx}\right|_{x=\pi/3} = \frac{-\sin\pi/3 + \sec\pi/3 \tan\pi/3}{\cos\pi/3 + \sec\frac{\pi}{3}}$

$\Rightarrow \left.\frac{dy}{dx}\right|_{x=\frac{\pi}{3}} = \frac{-\frac{\sqrt{3}}{2} + 2\times\sqrt{3}}{\frac{1}{2} + 2}$

$= \frac{-\sqrt{3} + 4\sqrt{3}}{1 + 4}$

$= \frac{3\sqrt{3}}{5}$

Question 39 **(***)**

The curve C has equation

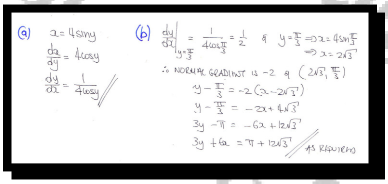
$$x = 4\sin y, \quad x > 0.$$

a) Find $\frac{dy}{dx}$ in terms of y.

b) Show that an equation of the normal to the curve at the point where $y = \frac{\pi}{3}$ is

$$3y + 6x = \pi + 12\sqrt{3}.$$

$$\boxed{\frac{dy}{dx} = \frac{1}{4\cos y}}$$



(a) $x = 4\sin y$

$\frac{dx}{dy} = 4\cos y$

$\frac{dy}{dx} = \frac{1}{4\cos y}$

(b) $\left.\frac{dy}{dx}\right|_{y=\frac{\pi}{3}} = \frac{1}{4\cos\frac{\pi}{3}} = \frac{1}{2}$ & $y = \frac{\pi}{3} \Rightarrow x = 4\sin\frac{\pi}{3}$

$\Rightarrow x = 2\sqrt{3}$

$\therefore$ NORMAL GRADIENT IS -2 & $(2\sqrt{3}, \frac{\pi}{3})$

$y - \frac{\pi}{3} = -2(x - 2\sqrt{3})$

$y - \frac{\pi}{3} = -2x + 4\sqrt{3}$

$3y - \pi = -6x + 12\sqrt{3}$

$3y + 6x = \pi + 12\sqrt{3}$ AS REQUIRED

Question 40 (***)

$$f(x) = 5\ln x + \frac{1}{x}, \quad x > 0.$$

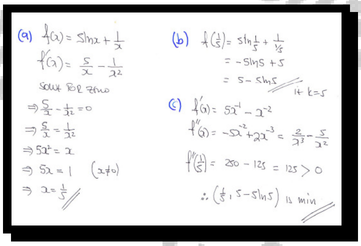
a) Solve the equation

$$f'(x) = 0.$$

b) Hence write down the y coordinate of the turning point of $f(x)$ in the form $k - k\ln k$, where k is an integer.

c) Find $f''(x)$ and use it to determine the nature of the turning point of $f(x)$.

$$\boxed{x = \frac{1}{5}}, \quad \boxed{y = 5 - 5\ln 5} \quad \boxed{f''(x) = \frac{2}{x^3} - \frac{5}{x^2}, f''\left(\frac{1}{5}\right) = 125 > 0 \text{ so minimum}}$$



(a) $f(x) = 5\ln x + \frac{1}{x}$

$f'(x) = \frac{5}{x} - \frac{1}{x^2}$

SOLVE FOR ZERO

$\Rightarrow \frac{5}{x} - \frac{1}{x^2} = 0$

$\Rightarrow \frac{5}{x} = \frac{1}{x^2}$

$\Rightarrow 5x^2 = x$

$\Rightarrow 5x = 1 \quad (x \neq 0)$

$\Rightarrow x = \frac{1}{5}$

(b) $f\left(\frac{1}{5}\right) = 5\ln\frac{1}{5} + \frac{1}{1/5}$

$= -5\ln 5 + 5$

$= 5 - 5\ln 5$

i.e. $k=5$

(c) $f'(x) = 5x^{-1} - x^{-2}$

$f''(x) = -5x^{-2} + 2x^{-3} = \frac{2}{x^3} - \frac{5}{x^2}$

$f''\left(\frac{1}{5}\right) = 250 - 125 = 125 > 0$

$\therefore \left(\frac{1}{5}, 5 - 5\ln 5\right)$ is min

Question 41 (***)

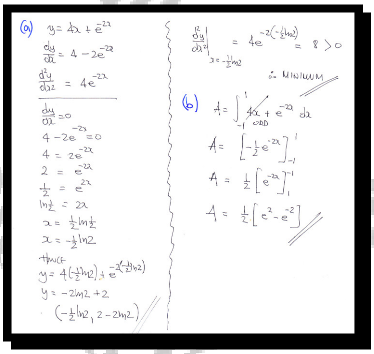
The curve C with equation $y = 4x + e^{-2x}$ has a turning point at A.

a) Find the exact coordinates of A and determine whether it is a local maximum or a local minimum.

The curve C lies entirely above the x axis.

b) Calculate the exact value of area bounded by the curve C, the x axis and the lines $x = 1$ and $x = -1$.

$\boxed{\text{min}\left(-\frac{1}{2}\ln 2, 2 - 2\ln 2\right)}$, $\boxed{\text{area} = \frac{1}{2}\left(e^2 - e^{-2}\right)}$



(a) $y = 4x + e^{-2x}$

$\frac{dy}{dx} = 4 - 2e^{-2x}$

$\frac{d^2y}{dx^2} = 4e^{-2x}$

$\frac{dy}{dx} = 0$

$4 - 2e^{-2x} = 0$

$4 = 2e^{-2x}$

$2 = e^{-2x}$

$\frac{1}{2} = e^{2x}$

$\ln\frac{1}{2} = 2x$

$x = \frac{1}{2}\ln\frac{1}{2}$

$x = -\frac{1}{2}\ln 2$

thus

$y = 4\left(-\frac{1}{2}\ln 2\right) + e^{-2\left(-\frac{1}{2}\ln 2\right)}$

$y = -2\ln 2 + 2$

$\left(-\frac{1}{2}\ln 2, 2 - 2\ln 2\right)$

$\left.\frac{d^2y}{dx^2}\right|_{x=-\frac{1}{2}\ln 2} = 4e^{-2\left(-\frac{1}{2}\ln 2\right)} = 8 > 0$

$\therefore$ MINIMUM

(b) $A = \int_{-1}^{1} \underbrace{\cancel{4x}}_{\text{ODD}} + e^{-2x}\, dx$

$A = \left[-\frac{1}{2}e^{-2x}\right]_{-1}^{1}$

$A = \frac{1}{2}\left[e^{-2x}\right]_{1}^{-1}$

$A = \frac{1}{2}\left[e^{2} - e^{-2}\right]$

Question 42 (***)

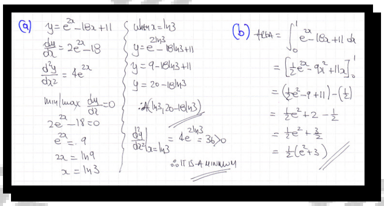
The curve C with equation $y = \mathrm{e}^{2x} - 18x + 11$ has a turning point at A.

a) Find the exact coordinates of A and determine whether it is a local maximum or a local minimum.

The curve C lies entirely above the x axis.

b) Calculate the exact value of area bounded by the curve C, the coordinate axes and the line $x = 1$.

$\boxed{\min\left(\ln 3, 20 - 18\ln 3\right)}$, $\boxed{\text{area} = \frac{1}{2}\left(\mathrm{e}^2 + 3\right)}$



(a) $y = e^{2x} - 18x + 11$

$\frac{dy}{dx} = 2e^{2x} - 18$

$\frac{d^2y}{dx^2} = 4e^{2x}$

min/max $\frac{dy}{dx} = 0$

$2e^{2x} - 18 = 0$

$e^{2x} = 9$

$2x = \ln 9$

$x = \ln 3$

when $x = \ln 3$

$y = e^{2\ln 3} - 18\ln 3 + 11$

$y = 9 - 18\ln 3 + 11$

$y = 20 - 18\ln 3$

$\therefore A(\ln 3, 20 - 18\ln 3)$

$\left.\frac{d^2y}{dx^2}\right|_{x=\ln 3} = 4e^{2\ln 3} = 36 > 0$

$\therefore$ IT IS A MINIMUM

(b) AREA $= \int_0^1 e^{2x} - 18x + 11 \, dx$

$= \left[\frac{1}{2}e^{2x} - 9x^2 + 11x\right]_0^1$

$= \left(\frac{1}{2}e^2 - 9 + 11\right) - \left(\frac{1}{2}\right)$

$= \frac{1}{2}e^2 + 2 - \frac{1}{2}$

$= \frac{1}{2}e^2 + \frac{3}{2}$

$= \frac{1}{2}(e^2 + 3)$

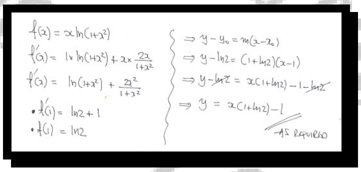
Question 43 (***)

$$f(x) = x\ln\left(1 + x^2\right), \; x \in \mathbb{R}.$$

Show that an equation of the tangent to the curve with equation $y = f(x)$, at the point where $x = 1$, is given by

$$y = x(1 + \ln 2) - 1.$$

$\boxed{\text{C3V}}$, $\boxed{\text{proof}}$



$f(x) = x\ln(1+x^2)$

$f'(x) = 1 \times \ln(1+x^2) + x \times \frac{2x}{1+x^2}$

$f'(x) = \ln(1+x^2) + \frac{2x^2}{1+x^2}$

- $f'(1) = \ln 2 + 1$
- $f(1) = \ln 2$

$\Rightarrow y - y_0 = m(x - x_0)$

$\Rightarrow y - \ln 2 = (1 + \ln 2)(x - 1)$

$\Rightarrow y - \cancel{\ln 2} = x(1 + \ln 2) - 1 - \cancel{\ln 2}$

$\Rightarrow y = x(1 + \ln 2) - 1$

AS REQUIRED

Question 44 (***)

The equation of the curve C is given by

$$y = \mathrm{e}^{2x}\left(\cos x + \sin x\right).$$

a) Find an expression for $\dfrac{dy}{dx}$.

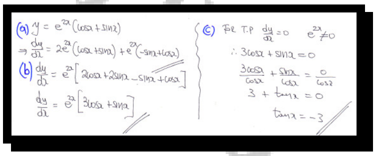
b) Show further that $\dfrac{dy}{dx}$ can be simplified to

$$\frac{dy}{dx} = \mathrm{e}^{2x}\left(\sin x + 3\cos x\right).$$

c) Hence show that the x coordinates of the turning points of C satisfy

$$\tan x = -3$$

$\boxed{\quad}$, $\boxed{\dfrac{dy}{dx} = 2\mathrm{e}^{2x}\left(\cos x + \sin x\right) + \mathrm{e}^{2x}\left(\cos x - \sin x\right)}$



(a) $y = e^{2x}(\cos x + \sin x)$

$\Rightarrow \frac{dy}{dx} = 2e^{2x}(\cos x + \sin x) + e^{2x}(-\sin x + \cos x)$

(b) $\frac{dy}{dx} = e^{2x}\left[2\cos x + 2\sin x - \sin x + \cos x\right]$

$\frac{dy}{dx} = e^{2x}\left[3\cos x + \sin x\right]$

(c) For T.P $\frac{dy}{dx} = 0$ $\quad e^{2x} \neq 0$

$\therefore 3\cos x + \sin x = 0$

$\frac{3\cos x}{\cos x} + \frac{\sin x}{\cos x} = \frac{0}{\cos x}$

$3 + \tan x = 0$

$\tan x = -3$

Question 45 (***)

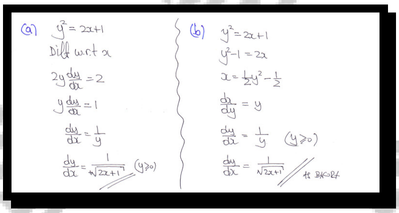
A curve has equation

$$y^2 = 2x+1,\ x \ge -\tfrac{1}{2},\ y \ge 0.$$

a) Use implicit differentiation to find $\dfrac{dy}{dx}$ in terms of x.

b) By making x the subject of the equation and differentiating the resulting equation, verify the result of part **(a)**.

$$\boxed{\frac{dy}{dx} = \frac{1}{\sqrt{2x+1}}}$$



(a) $y^2 = 2x+1$

Diff w.r.t x

$2y\frac{dy}{dx} = 2$

$y\frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{y}$

$\frac{dy}{dx} = \frac{1}{+\sqrt{2x+1}}$ $(y \geqslant 0)$

(b) $y^2 = 2x+1$

$y^2 - 1 = 2x$

$x = \frac{1}{2}y^2 - \frac{1}{2}$

$\frac{dx}{dy} = y$

$\frac{dy}{dx} = \frac{1}{y}$ $(y \geqslant 0)$

$\frac{dy}{dx} = \frac{1}{\sqrt{2x+1}}$ AS BEFORE

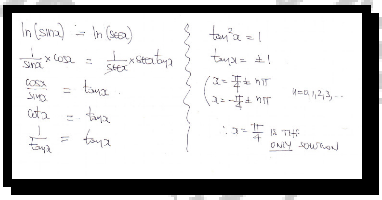
Question 46 (***)

By differentiating both sides of the equation

$$\ln(\sin x) = \ln(\sec x),\ 0 \le x < \frac{\pi}{2},$$

show that the only solution is $x = \dfrac{\pi}{4}$.

$\boxed{\text{proof}}$



$\ln(\sin x) = \ln(\sec x)$

$\frac{1}{\sin x} \times \cos x = \frac{1}{\cancel{\sec x}} \times \cancel{\sec x}\tan x$

$\frac{\cos x}{\sin x} = \tan x$

$\cot x = \tan x$

$\frac{1}{\tan x} = \tan x$

$\tan^2 x = 1$

$\tan x = \pm 1$

$x = \frac{\pi}{4} \pm n\pi$

$x = -\frac{\pi}{4} \pm n\pi$ $\quad n = 0, 1, 2, 3, \ldots$

$\therefore x = \frac{\pi}{4}$ IS THE ONLY SOLUTION

Question 47 (***)

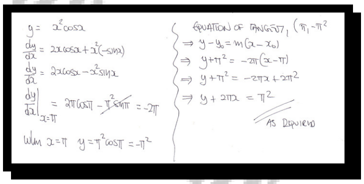
A curve has equation

$$y = x^2 \cos x,\ x \in \mathbb{R}.$$

Show that the tangent to the curve at the point where $x = \pi$ is given by

$$y + 2\pi x = \pi^2.$$

$\boxed{\text{proof}}$



$y = x^2 \cos x$

$\frac{dy}{dx} = 2x\cos x + x^2(-\sin x)$

$\frac{dy}{dx} = 2x\cos x - x^2 \sin x$

$\left.\frac{dy}{dx}\right|_{x=\pi} = 2\pi\cos\pi - \cancel{\pi^2 \sin\pi} = -2\pi$

When $x=\pi$ $y = \pi^2\cos\pi = -\pi^2$

EQUATION OF TANGENT, $(\pi, -\pi^2$

$\Rightarrow y - y_0 = m(x - x_0)$

$\Rightarrow y + \pi^2 = -2\pi(x - \pi)$

$\Rightarrow y + \pi^2 = -2\pi x + 2\pi^2$

$\Rightarrow y + 2\pi x = \pi^2$

AS REQUIRED

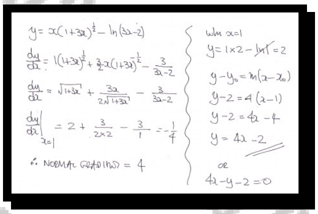
Question 48 (***)

Find an equation of the normal to the curve with equation

$$y = x\sqrt{1+3x} - \ln(3x-2),\ x \in \mathbb{R},\ x > \frac{2}{3},$$

at the point on the curve where $x = 1$.

$\boxed{y = 4x - 2}$



$y = x(1+3x)^{\frac{1}{2}} - \ln(3x-2)$

$\frac{dy}{dx} = 1(1+3x)^{\frac{1}{2}} + \frac{3}{2}x(1+3x)^{-\frac{1}{2}} - \frac{3}{3x-2}$

$\frac{dy}{dx} = \sqrt{1+3x} + \frac{3x}{2\sqrt{1+3x}} - \frac{3}{3x-2}$

$\left.\frac{dy}{dx}\right|_{x=1} = 2 + \frac{3}{2\times2} - \frac{3}{1} = -\frac{1}{4}$

$\therefore$ NORMAL GRADIENT $= 4$

When $x=1$

$y = 1\times2 - \cancel{\ln 1} = 2$

$y - y_0 = m(x - x_0)$

$y - 2 = 4(x-1)$

$y - 2 = 4x - 4$

$y = 4x - 2$

OR

$4x - y - 2 = 0$

Question 49 **(***+)**

A function is defined by

$$f(x)=\sqrt{e^{x}-1}\,,\ x\geq 0\,.$$

a) Find the values of …

i. … $f(\ln 5)$.

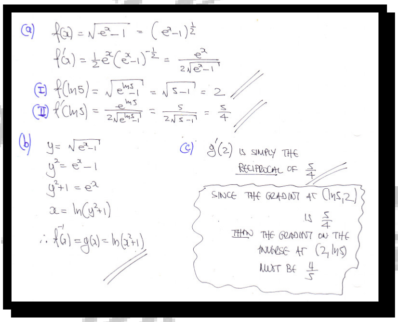
ii. … $f'(\ln 5)$.

The inverse function of $f(x)$ is $g(x)$.

b) Determine an expression for $g(x)$.

c) State the value of $g'(2)$.

$\boxed{f(\ln 5)=2}$, $\boxed{f'(\ln 5)=\frac{5}{4}}$, $\boxed{g(x)=\ln\left(x^{2}+1\right)}$, $\boxed{g'(2)=\frac{4}{5}}$



(a) $f(x)=\sqrt{e^{x}-1}=(e^{x}-1)^{\frac{1}{2}}$

$f'(x)=\frac{1}{2}e^{x}(e^{x}-1)^{-\frac{1}{2}}=\frac{e^{x}}{2\sqrt{e^{x}-1}}$

(I) $f(\ln 5)=\sqrt{e^{\ln 5}-1}=\sqrt{5-1}=2$ //

(II) $f'(\ln 5)=\frac{e^{\ln 5}}{2\sqrt{e^{\ln 5}-1}}=\frac{5}{2\sqrt{5-1}}=\frac{5}{4}$ //

(b) $y=\sqrt{e^{x}-1}$

$y^{2}=e^{x}-1$

$y^{2}+1=e^{x}$

$x=\ln(y^{2}+1)$

$\therefore f^{-1}(x)=g(x)=\ln(x^{2}+1)$ //

(c) $g'(2)$ IS SIMPLY THE RECIPROCAL OF $\frac{5}{4}$ //

SINCE THE GRADINT AT $(\ln 5, 2)$ IS $\frac{5}{4}$
THEN THE GRADINT ON THE INVERSE AT $(2, \ln 5)$ MUST BE $\frac{4}{5}$

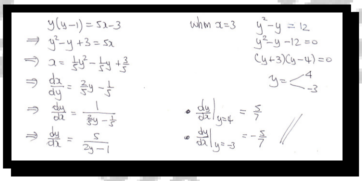
Question 50 **(***+)**

A curve has equation

$$y(y-1)=5x-3.$$

Find the gradient at each of the points on the curve where $x=3$.

$\boxed{}$, $\boxed{\pm\frac{5}{7}}$



$y(y-1)=5x-3$

$\Rightarrow y^2-y+3=5x$

$\Rightarrow x=\frac{1}{5}y^2-\frac{1}{5}y+\frac{3}{5}$

$\Rightarrow \frac{dx}{dy}=\frac{2}{5}y-\frac{1}{5}$

$\Rightarrow \frac{dy}{dx}=\frac{1}{\frac{2}{5}y-\frac{1}{5}}$

$\Rightarrow \frac{dy}{dx}=\frac{5}{2y-1}$

when $x=3$ $y^2-y=12$

$y^2-y-12=0$

$(y+3)(y-4)=0$

$y=\genfrac{}{}{0pt}{}{4}{-3}$

- $\left.\frac{dy}{dx}\right|_{y=4}=\frac{5}{7}$
- $\left.\frac{dy}{dx}\right|_{y=-3}=-\frac{5}{7}$ //

Question 51 **(***+)**

The curve C has equation

$$f(x)=(2x-1)\mathrm{e}^{-2x},\ x\in\mathbb{R}.$$

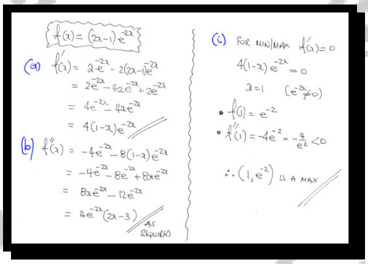
a) Find an expression for $f'(x)$.

b) Show clearly that

$$f''(x)=4(2x-3)\mathrm{e}^{-2x}.$$

c) Hence find the exact coordinates of the stationary point of C and determine its nature.

$\boxed{f'(x)=4(1-x)\mathrm{e}^{-2x}}$ $\boxed{\max\left(1,\mathrm{e}^{-2}\right)}$



$f(x)=(2x-1)e^{-2x}$

(a) $f'(x)=2e^{-2x}-2(2x-1)e^{-2x}$

$=2e^{-2x}-4xe^{-2x}+2e^{-2x}$

$=4e^{-2x}-4xe^{-2x}$

$=4(1-x)e^{-2x}$ //

(b) $f''(x)=-4e^{-2x}-8(1-x)e^{-2x}$

$=-4e^{-2x}-8e^{-2x}+8xe^{-2x}$

$=8xe^{-2x}-12e^{-2x}$

$=4e^{-2x}(2x-3)$ // AS REQUIRED

(c) FOR MIN/MAX $f'(x)=0$

$4(1-x)e^{-2x}=0$

$x=1$ $(e^{-2x}\neq0)$

- $f(1)=e^{-2}$
- $f''(1)=-4e^{-2}=-\frac{4}{e^2}<0$

$\therefore (1,e^{-2})$ IS A MAX //

Question 52 **(***+)**

A curve has equation

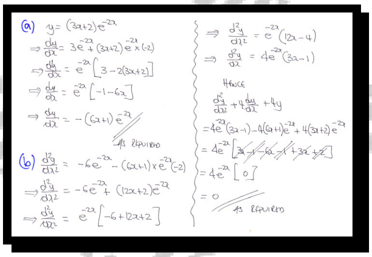
$$y = (3x+2)\mathrm{e}^{-2x}.$$

Show clearly that …

a) … $\dfrac{dy}{dx} = -(6x+1)\mathrm{e}^{-2x}$.

b) … $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 4y = 0$.

$\boxed{\text{proof}}$



(a) $y = (3x+2)e^{-2x}$

$\Rightarrow \frac{dy}{dx} = 3e^{-2x} + (3x+2)e^{-2x} \times (-2)$

$\Rightarrow \frac{dy}{dx} = e^{-2x}\left[3 - 2(3x+2)\right]$

$\Rightarrow \frac{dy}{dx} = e^{-2x}\left[-1-6x\right]$

$\Rightarrow \frac{dy}{dx} = -(6x+1)e^{-2x}$ // AS REQUIRED

(b) $\frac{d^2y}{dx^2} = -6e^{-2x} - (6x+1) \times e^{-2x}(-2)$

$\Rightarrow \frac{d^2y}{dx^2} = -6e^{-2x} + (12x+2)e^{-2x}$

$\Rightarrow \frac{d^2y}{dx^2} = e^{-2x}\left[-6+12x+2\right]$

$\Rightarrow \frac{d^2y}{dx^2} = e^{-2x}(12x-4)$

$\Rightarrow \frac{d^2y}{dx} = 4e^{-2x}(3x-1)$

HENCE

$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y$

$= 4e^{-2x}(3x-1) - 4(6x+1)e^{-2x} + 4(3x+2)e^{-2x}$

$= 4e^{-2x}\left[\cancel{3x} \cancel{-1} \cancel{-6x} \cancel{-1} \cancel{+3x} \cancel{+2}\right]$

$= 4e^{-2x}\left[0\right]$

$= 0$ // AS REQUIRED

Question 53 **(***+)**

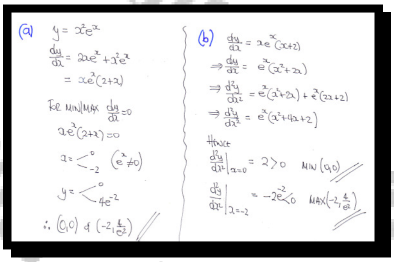
The curve C has equation

$$y = x^2 \mathrm{e}^x, \ x \in \mathbb{R}.$$

a) Find the exact coordinates of the stationary points of C.

b) By considering the sign of $\frac{d^2y}{dx^2}$ at each of these points determine their nature.

$\boxed{}$, $\boxed{\min(0,0)}$, $\boxed{\max\left(-2, \frac{4}{\mathrm{e}^2}\right)}$



(a) $y = x^2e^x$

$\frac{dy}{dx} = 2xe^x + x^2e^x$

$= xe^x(2+x)$

FOR MIN/MAX $\frac{dy}{dx} = 0$

$xe^x(2+x) = 0$

$x = \begin{cases} 0 \\ -2 \end{cases} \quad (e^x \neq 0)$

$y = \begin{cases} 0 \\ 4e^{-2} \end{cases}$

$\therefore (0,0)$ & $\left(-2, \frac{4}{e^2}\right)$

(b) $\frac{dy}{dx} = xe^x(x+2)$

$\Rightarrow \frac{dy}{dx} = e^x(x^2+2x)$

$\Rightarrow \frac{d^2y}{dx^2} = e^x(x^2+2x) + e^x(2x+2)$

$\Rightarrow \frac{d^2y}{dx^2} = e^x(x^2+4x+2)$

HENCE

$\left.\frac{d^2y}{dx^2}\right|_{x=0} = 2 > 0$ MIN $(0,0)$

$\left.\frac{d^2y}{dx^2}\right|_{x=-2} = -2e^{-2} < 0$ MAX $\left(-2, \frac{4}{e^2}\right)$

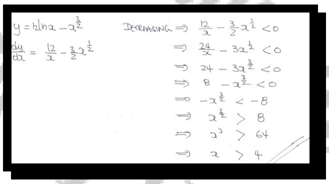
Question 54 **(***+)**

The curve C has equation

$$y = 12\ln x - x^{\frac{3}{2}}, \ x > 0.$$

Determine the range of values of x for which y is decreasing.

$\boxed{x > 4}$



$y = 12\ln x - x^{\frac{3}{2}}$

$\frac{dy}{dx} = \frac{12}{x} - \frac{3}{2}x^{\frac{1}{2}}$

DECREASING $\Rightarrow \frac{12}{x} - \frac{3}{2}x^{\frac{1}{2}} < 0$

$\Rightarrow \frac{24}{x} - 3x^{\frac{1}{2}} < 0$

$\Rightarrow 24 - 3x^{\frac{3}{2}} < 0$

$\Rightarrow 8 - x^{\frac{3}{2}} < 0$

$\Rightarrow -x^{\frac{3}{2}} < -8$

$\Rightarrow x^{\frac{3}{2}} > 8$

$\Rightarrow x^3 > 64$

$\Rightarrow x > 4$

Question 55 (***+)

The curve C has equation

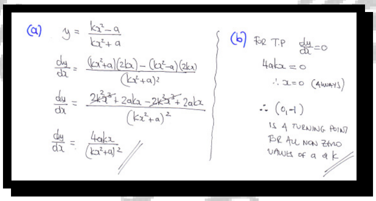
$$y = \frac{kx^2 - a}{kx^2 + a},$$

where k and a are non zero constants.

a) Find a simplified expression for $\frac{dy}{dx}$ in terms of a and k.

b) Hence show that C has a single turning point for all values of a and k, and state its coordinates.

$$\boxed{\quad}, \boxed{\frac{dy}{dx} = \frac{4akx}{\left(kx^2 + a\right)^2}}, \boxed{(0,-1)}$$



(a) $y = \frac{kx^2-a}{kx^2+a}$

$\frac{dy}{dx} = \frac{(kx^2+a)(2kx)-(kx^2-a)(2kx)}{(kx^2+a)^2}$

$\frac{dy}{dx} = \frac{\cancel{2k^2x^3}+2akx-\cancel{2k^2x^3}+2akx}{(kx^2+a)^2}$

$\frac{dy}{dx} = \frac{4akx}{(kx^2+a)^2}$ //

(b) FOR T.P $\frac{dy}{dx}=0$

$4akx=0$

$\therefore x=0$ (ALWAYS)

$\therefore (0,-1)$

IS A TURNING POINT FOR ALL NON ZERO VALUES OF a & k //

Question 56 **(***+)**

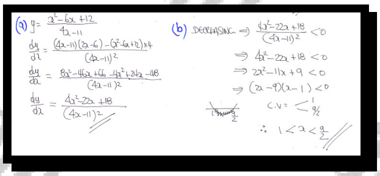
The curve C has equation

$$y=\frac{x^2-6x+12}{4x-11},\ x\in\mathbb{R},\ x\neq\frac{11}{4}\ .$$

a) Find a simplified expression for $\dfrac{dy}{dx}$.

b) Determine the range of values of x, for which y is decreasing.

$\boxed{\quad}$, $\boxed{\dfrac{dy}{dx}=\dfrac{4x^2-22x+18}{(4x-11)^2}}$, $\boxed{1<x<\dfrac{9}{2}}$



(a) $y=\dfrac{x^2-6x+12}{4x-11}$

$\dfrac{dy}{dx}=\dfrac{(4x-11)(2x-6)-(x^2-6x+12)\times4}{(4x-11)^2}$

$\dfrac{dy}{dx}=\dfrac{8x^2-46x+66-4x^2+24x-48}{(4x-11)^2}$

$\dfrac{dy}{dx}=\dfrac{4x^2-22x+18}{(4x-11)^2}$ //

(b) DECREASING $\Rightarrow \dfrac{4x^2-22x+18}{(4x-11)^2}<0$

$\Rightarrow 4x^2-22x+18<0$

$\Rightarrow 2x^2-11x+9<0$

$\Rightarrow (2x-9)(x-1)<0$

C.V. = $<^{1}_{9/2}$

$\therefore\ 1<x<\frac{9}{2}$ //

Question 57 **(***+)**

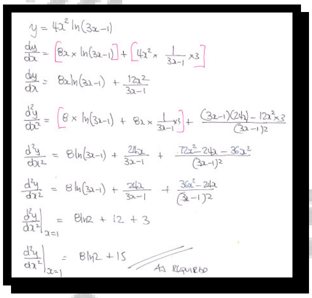
A curve C has equation

$$y = 4x^2 \ln(3x-1), \quad x \in \mathbb{R}, \quad x > \frac{1}{3}.$$

Show that the value of $\frac{d^2y}{dx^2}$ at the point where $x=1$ is

$$15 + 8\ln 2.$$

$\boxed{\text{proof}}$



$y = 4x^2 \ln(3x-1)$

$\frac{dy}{dx} = \left[8x \times \ln(3x-1)\right] + \left[4x^2 \times \frac{1}{3x-1} \times 3\right]$

$\frac{dy}{dx} = 8x\ln(3x-1) + \frac{12x^2}{3x-1}$

$\frac{d^2y}{dx^2} = \left[8 \times \ln(3x-1) + 8x \times \frac{1}{3x-1} \times 3\right] + \frac{(3x-1)(24x) - 12x^2 \times 3}{(3x-1)^2}$

$\frac{d^2y}{dx^2} = 8\ln(3x-1) + \frac{24x}{3x-1} + \frac{72x^2 - 24x - 36x^2}{(3x-1)^2}$

$\frac{d^2y}{dx^2} = 8\ln(3x-1) + \frac{24x}{3x-1} + \frac{36x^2 - 24x}{(3x-1)^2}$

$\left.\frac{d^2y}{dx^2}\right|_{x=1} = 8\ln 2 + 12 + 3$

$\left.\frac{d^2y}{dx^2}\right|_{x=1} = 8\ln 2 + 15$ // AS REQUIRED

Question 58 **(***+)**

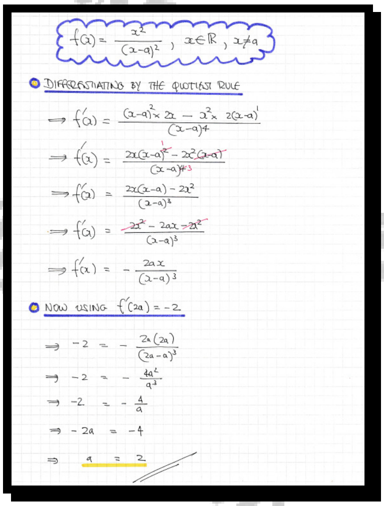
The curve C has equation

$$f(x) = \frac{x^2}{(x-a)^2}, \ x \in \mathbb{R}, \ x \neq a \ ,$$

where a is a non zero constant.

Given that $f'(2a) = -2$, determine the value of a.

$\boxed{}$, $\boxed{a=2}$



$f(x) = \frac{x^2}{(x-a)^2}, \ x \in \mathbb{R}, \ x \neq a$

DIFFERENTIATING BY THE QUOTIENT RULE

$\Rightarrow f'(x) = \frac{(x-a)^2 \times 2x - x^2 \times 2(x-a)^1}{(x-a)^4}$

$\Rightarrow f'(x) = \frac{2x(x-a)^{\cancel{2}\,1} - 2x^2\cancel{(x-a)}}{(x-a)^{\cancel{4}\,3}}$

$\Rightarrow f'(x) = \frac{2x(x-a) - 2x^2}{(x-a)^3}$

$\Rightarrow f'(x) = \frac{\cancel{2x^2} - 2ax \cancel{- 2x^2}}{(x-a)^3}$

$\Rightarrow f'(x) = -\frac{2ax}{(x-a)^3}$

NOW USING $f'(2a) = -2$

$\Rightarrow -2 = -\frac{2a(2a)}{(2a-a)^3}$

$\Rightarrow -2 = -\frac{4a^2}{a^3}$

$\Rightarrow -2 = -\frac{4}{a}$

$\Rightarrow -2a = -4$

$\Rightarrow a = 2$

Question 59 **(***+)**

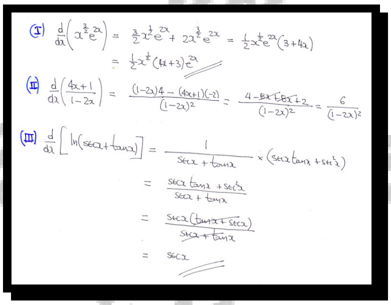
Show clearly that …

i. … $\dfrac{d}{dx}\left(x^{\frac{3}{2}}\,\mathrm{e}^{2x}\right)=\dfrac{1}{2}(4x+3)\,x^{\frac{1}{2}}\,\mathrm{e}^{2x}$.

ii. … $\dfrac{d}{dx}\left(\dfrac{4x+1}{1-2x}\right)=\dfrac{6}{(1-2x)^2}$.

iii. … $\dfrac{d}{dx}\left(\ln(\sec x+\tan x)\right)=\sec x$.

proof



(I) $\frac{d}{dx}\left(x^{\frac{3}{2}}e^{2x}\right) = \frac{3}{2}x^{\frac{1}{2}}e^{2x} + 2x^{\frac{3}{2}}e^{2x} = \frac{1}{2}x^{\frac{1}{2}}e^{2x}(3+4x)$

$= \frac{1}{2}x^{\frac{1}{2}}(4x+3)e^{2x}$

(II) $\frac{d}{dx}\left(\frac{4x+1}{1-2x}\right) = \frac{(1-2x)4-(4x+1)(-2)}{(1-2x)^2} = \frac{4\cancel{-8x}\cancel{+8x}+2}{(1-2x)^2} = \frac{6}{(1-2x)^2}$

(III) $\frac{d}{dx}\left[\ln(\sec x+\tan x)\right] = \frac{1}{\sec x+\tan x}\times(\sec x\tan x+\sec^2 x)$

$= \frac{\sec x\tan x+\sec^2 x}{\sec x+\tan x}$

$= \frac{\sec x\cancel{(\tan x+\sec x)}}{\cancel{\sec x+\tan x}}$

$= \sec x$

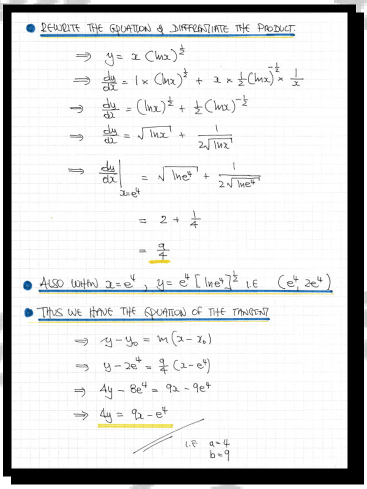
Question 60 **(***+)**

The curve C has equation

$$y = x\sqrt{\ln x}\,,\ \ x > 1\,.$$

Find an equation of the tangent to the curve at the point where $x = \mathrm{e}^4$ giving the answer in the form $ay = bx - \mathrm{e}^4$, where a and b are integers.

$\boxed{\quad\quad}$, $\boxed{4y = 9x - \mathrm{e}^4}$



- REWRITE THE EQUATION & DIFFERENTIATE THE PRODUCT.

$$\Rightarrow\ y = x(\ln x)^{\frac{1}{2}}$$

$$\Rightarrow\ \frac{dy}{dx} = 1 \times (\ln x)^{\frac{1}{2}} + x \times \frac{1}{2}(\ln x)^{-\frac{1}{2}} \times \frac{1}{x}$$

$$\Rightarrow\ \frac{dy}{dx} = (\ln x)^{\frac{1}{2}} + \frac{1}{2}(\ln x)^{-\frac{1}{2}}$$

$$\Rightarrow\ \frac{dy}{dx} = \sqrt{\ln x} + \frac{1}{2\sqrt{\ln x}}$$

$$\Rightarrow\ \left.\frac{dy}{dx}\right|_{x=e^4} = \sqrt{\ln e^4} + \frac{1}{2\sqrt{\ln e^4}}$$

$$= 2 + \frac{1}{4}$$

$$= \frac{9}{4}$$

- ALSO WHEN $x = e^4$, $y = e^4[\ln e^4]^{\frac{1}{2}}$ I.E $(e^4, 2e^4)$
- THUS WE HAVE THE EQUATION OF THE TANGENT

$$\Rightarrow\ y - y_0 = m(x - x_0)$$

$$\Rightarrow\ y - 2e^4 = \frac{9}{4}(x - e^4)$$

$$\Rightarrow\ 4y - 8e^4 = 9x - 9e^4$$

$$\Rightarrow\ 4y = 9x - e^4$$

I.E $a = 4$
$b = 9$

Question 61 **(***+)**

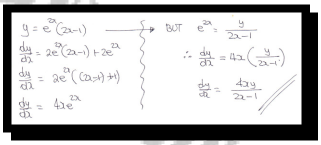
The curve C has equation

$$y = e^{2x}(2x-1), \ x \in \mathbb{R}.$$

Show clearly that

$$\frac{dy}{dx} = \frac{4xy}{2x-1}.$$

proof



$y = e^{2x}(2x-1) \longrightarrow$ BUT $e^{2x} = \frac{y}{2x-1}$

$\frac{dy}{dx} = 2e^{2x}(2x-1) + 2e^{2x}$

$\frac{dy}{dx} = 2e^{2x}((2x\cancel{-1})\cancel{+1})$

$\frac{dy}{dx} = 4xe^{2x}$

$\therefore \frac{dy}{dx} = 4x\left(\frac{y}{2x-1}\right)$

$\frac{dy}{dx} = \frac{4xy}{2x-1}$ //

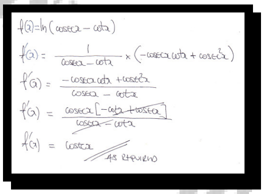
Question 62 **(***+)**

$$f(x) = \ln(\operatorname{cosec} x - \cot x), \ 0 < x < \pi.$$

Show clearly that

$$f'(x) = \operatorname{cosec} x.$$

proof



$f(x) = \ln(\operatorname{cosec}x - \cot x)$

$f'(x) = \frac{1}{\operatorname{cosec}x - \cot x} \times (-\operatorname{cosec}x\cot x + \operatorname{cosec}^2 x)$

$f'(x) = \frac{-\operatorname{cosec}x\cot x + \operatorname{cosec}^2 x}{\operatorname{cosec}x - \cot x}$

$f'(x) = \frac{\operatorname{cosec}x\left[\cancel{-\cot x + \operatorname{cosec}x}\right]}{\cancel{\operatorname{cosec}x - \cot x}}$

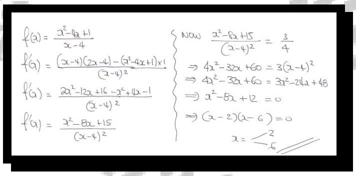
$f'(x) = \operatorname{cosec}x$ // AS REQUIRED

Question 63 **(***+)**

$$f(x) = \frac{x^2 - 4x + 1}{x - 4}, \quad x \in \mathbb{R}, \; x \neq 4 .$$

Solve the equation $f'(x) = \frac{3}{4}$.

$$\boxed{x = 2, \; 6}$$



$f(x) = \frac{x^2-4x+1}{x-4}$

$f'(x) = \frac{(x-4)(2x-4)-(x^2-4x+1)\times 1}{(x-4)^2}$

$f'(x) = \frac{2x^2-12x+16-x^2+4x-1}{(x-4)^2}$

$f'(x) = \frac{x^2-8x+15}{(x-4)^2}$

Now $\frac{x^2-8x+15}{(x-4)^2} = \frac{3}{4}$

$\Rightarrow 4x^2-32x+60 = 3(x-4)^2$

$\Rightarrow 4x^2-32x+60 = 3x^2-24x+48$

$\Rightarrow x^2-8x+12 = 0$

$\Rightarrow (x-2)(x-6) = 0$

$x = \left\langle \begin{array}{l} 2 \\ 6 \end{array} \right.$

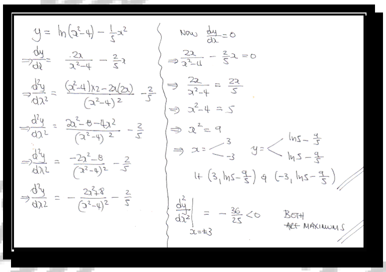
Question 64 **(***+)**

The curve C has equation

$$y = \ln\left(x^2 - 4\right) - \frac{1}{5}x^2 \;, \; |x| > 2 .$$

Find the exact coordinates of the turning points of C and determine their nature.

$$\boxed{\max\left(-3, \ln 5 - \frac{9}{5}\right)}, \; \boxed{\max\left(3, \ln 5 - \frac{9}{5}\right)}$$



$y = \ln(x^2-4) - \frac{1}{5}x^2$

$\Rightarrow \frac{dy}{dx} = \frac{2x}{x^2-4} - \frac{2}{5}x$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{(x^2-4)\times 2 - 2x(2x)}{(x^2-4)^2} - \frac{2}{5}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{2x^2-8-4x^2}{(x^2-4)^2} - \frac{2}{5}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{-2x^2-8}{(x^2-4)^2} - \frac{2}{5}$

$\Rightarrow \frac{d^2y}{dx^2} = -\frac{2x^2+8}{(x^2-4)^2} - \frac{2}{5}$

Now $\frac{dy}{dx} = 0$

$\Rightarrow \frac{2x}{x^2-4} - \frac{2}{5}x = 0$

$\Rightarrow \frac{2x}{x^2-4} = \frac{2x}{5}$

$\Rightarrow x^2-4 = 5$

$\Rightarrow x^2 = 9$

$\Rightarrow x = \left\langle \begin{array}{l} 3 \\ -3 \end{array} \right. \quad y = \left\langle \begin{array}{l} \ln 5 - \frac{9}{5} \\ \ln 5 - \frac{9}{5} \end{array} \right.$

$\therefore \left(3, \ln 5 - \frac{9}{5}\right) \;\&\; \left(-3, \ln 5 - \frac{9}{5}\right)$

$\left.\frac{d^2y}{dx^2}\right|_{x=\pm 3} = -\frac{36}{25} < 0$ BOTH ARE MAXIMUMS

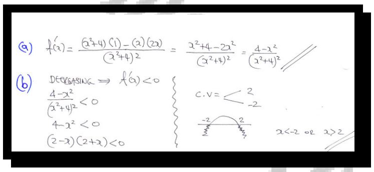
Question 65 (***+)

$$f(x) = \frac{x}{x^2+4}, \ x \in \mathbb{R}.$$

a) Find an expression for $f'(x)$.

b) Determine the range of values of x, for which $f(x)$ is decreasing.

$\boxed{\quad}$, $\boxed{f'(x) = \dfrac{4-x^2}{\left(x^2+4\right)^2}}$, $\boxed{x < -2 \text{ or } x > 2}$



(a) $f'(x) = \frac{(x^2+4)(1)-(x)(2x)}{(x^2+4)^2} = \frac{x^2+4-2x^2}{(x^2+4)^2} = \frac{4-x^2}{(x^2+4)^2}$ //

(b) DECREASING $\Rightarrow f'(x) < 0$

$\frac{4-x^2}{(x^2+4)^2} < 0$

$4-x^2 < 0$

$(2-x)(2+x) < 0$

C.V = 2, −2

$x < -2$ OR $x > 2$ //

Question 66 (***+)

A curve C is defined by the function

$$f(x)=\frac{1+\sin 2x}{4+\cos 2x}\,,\ 0\le x<2\pi\,.$$

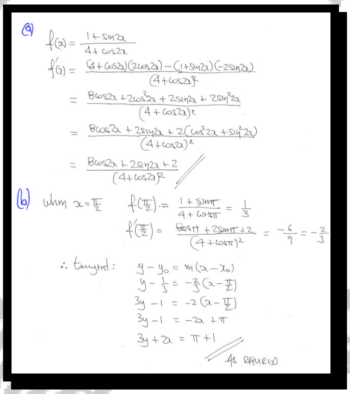
a) Show clearly that

$$f'(x)=\frac{2+2\sin 2x+8\cos 2x}{(4+\cos 2x)^2}\,.$$

b) Show further that the equation of the tangent to C at the point with $x=\frac{\pi}{2}$ is given by

$$2x+3y=\pi+1.$$

proof



(a)

$$f(x)=\frac{1+\sin 2x}{4+\cos 2x}$$

$$f'(x)=\frac{(4+\cos 2x)(2\cos 2x)-(1+\sin 2x)(-2\sin 2x)}{(4+\cos 2x)^2}$$

$$=\frac{8\cos 2x+2\cos^2 2x+2\sin 2x+2\sin^2 2x}{(4+\cos 2x)^2}$$

$$=\frac{8\cos 2x+2\sin 2x+2(\cos^2 2x+\sin^2 2x)}{(4+\cos 2x)^2}$$

$$=\frac{8\cos 2x+2\sin 2x+2}{(4+\cos 2x)^2}$$

(b) when $x=\frac{\pi}{2}$ $f\left(\frac{\pi}{2}\right)=\frac{1+\sin\pi}{4+\cos\pi}=\frac{1}{3}$

$$f'\left(\frac{\pi}{2}\right)=\frac{8\cos\pi+2\sin\pi+2}{(4+\cos\pi)^2}=-\frac{6}{9}=-\frac{2}{3}$$

∴ tangent:

$$y-y_0=m(x-x_0)$$

$$y-\frac{1}{3}=-\frac{2}{3}\left(x-\frac{\pi}{2}\right)$$

$$3y-1=-2\left(x-\frac{\pi}{2}\right)$$

$$3y-1=-2x+\pi$$

$$3y+2x=\pi+1$$

AS REQUIRED

Question 67 **(***+)**

A curve C has equation

$$y = \frac{4x+k}{4x-k}, \quad x \neq \frac{k}{4},$$

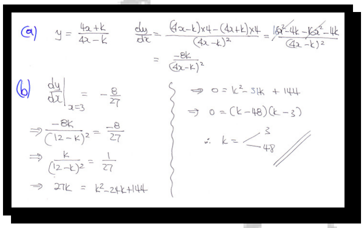
where k is a non zero constant.

a) Find a simplified expression for $\frac{dy}{dx}$, in terms k.

The point P lies on C, where $x = 3$.

b) Given that the gradient at P is $-\frac{8}{27}$, show that one possible value of k is 48 and find the other.

[illegible], $\boxed{\frac{dy}{dx} = -\frac{8k}{(4x-k)^2}}$, $\boxed{k=3}$



(a) $y = \frac{4x+k}{4x-k} \qquad \frac{dy}{dx} = \frac{(4x-k)\times 4-(4x+k)\times 4}{(4x-k)^2} = \frac{\cancel{16x^2}-4k-\cancel{16x^2}-4k}{(4x-k)^2}$

$= \frac{-8k}{(4x-k)^2}$

(b) $\left.\frac{dy}{dx}\right|_{x=3} = -\frac{8}{27}$

$\Rightarrow \frac{-8k}{(12-k)^2} = \frac{-8}{27}$

$\Rightarrow \frac{k}{(12-k)^2} = \frac{1}{27}$

$\Rightarrow 27k = k^2-24k+144$

$\Rightarrow 0 = k^2-51k+144$

$\Rightarrow 0 = (k-48)(k-3)$

$\therefore k = \begin{cases} 3 \\ 48 \end{cases}$ //

Question 68 **(***+)**

A curve C has equation

$$y = e^{2x}\left(x^2 - 4x - 2\right),\ x \in \mathbb{R}.$$

a) Show clearly that

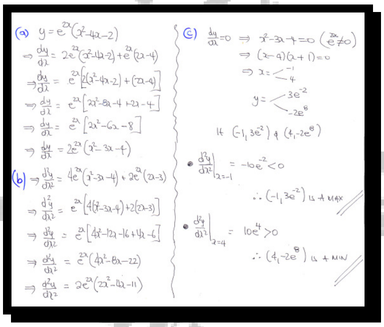
$$\frac{dy}{dx} = 2e^{2x}\left(x^2 - 3x - 4\right).$$

b) Show further that

$$\frac{d^2y}{dx^2} = 2e^{2x}\left(2x^2 - 4x - 11\right).$$

c) Hence find the exact coordinates of the stationary points of C and use $\frac{d^2y}{dx^2}$ to determine their nature.

$\boxed{\quad\quad\quad\quad}$, $\boxed{\min\left(4,-2e^{8}\right)}$, $\boxed{\max\left(-1,3e^{-2}\right)}$



(a) $y = e^{2x}(x^2-4x-2)$

$\Rightarrow \frac{dy}{dx} = 2e^{2x}(x^2-4x-2) + e^{2x}(2x-4)$

$\Rightarrow \frac{dy}{dx} = e^{2x}\left[2(x^2-4x-2) + (2x-4)\right]$

$\Rightarrow \frac{dy}{dx} = e^{2x}\left[2x^2-8x-4+2x-4\right]$

$\Rightarrow \frac{dy}{dx} = e^{2x}\left[2x^2-6x-8\right]$

$\Rightarrow \frac{dy}{dx} = 2e^{2x}(x^2-3x-4)$

(b) $\Rightarrow \frac{d^2y}{dx^2} = 4e^{2x}(x^2-3x-4) + 2e^{2x}(2x-3)$

$\Rightarrow \frac{d^2y}{dx^2} = e^{2x}\left[4(x^2-3x-4)+2(2x-3)\right]$

$\Rightarrow \frac{d^2y}{dx^2} = e^{2x}\left[4x^2-12x-16+4x-6\right]$

$\Rightarrow \frac{d^2y}{dx^2} = e^{2x}(4x^2-8x-22)$

$\Rightarrow \frac{d^2y}{dx^2} = 2e^{2x}(2x^2-4x-11)$

(c) $\frac{dy}{dx} = 0 \Rightarrow x^2-3x-4=0 \quad (e^{2x} \neq 0)$

$\Rightarrow (x-4)(x+1)=0$

$\Rightarrow x = \left\langle \begin{array}{l} -1 \\ 4 \end{array} \right.$

$y = \left\langle \begin{array}{l} 3e^{-2} \\ -2e^{8} \end{array} \right.$

∴ $(-1, 3e^{-2})$ & $(4, -2e^{8})$

• $\left.\frac{d^2y}{dx^2}\right|_{x=-1} = -6e^{-2} < 0$

∴ $(-1, 3e^{-2})$ IS A MAX //

• $\left.\frac{d^2y}{dx^2}\right|_{x=4} = 10e^{4} > 0$

∴ $(4, -2e^{8})$ IS A MIN //

Question 69 **(***+)**

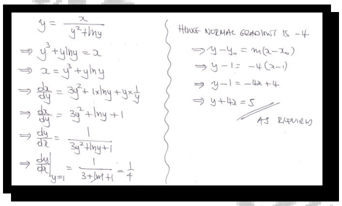
The curve C has equation given by

$$y = \frac{x}{y^2 + \ln y}, \ y > 0.$$

Show that an equation of the normal to C at the point $(1,1)$ is

$$4x + y = 5.$$

$\boxed{}$, $\boxed{\text{proof}}$



$y = \frac{x}{y^2+\ln y}$

$\Rightarrow y^3 + y\ln y = x$

$\Rightarrow x = y^3 + y\ln y$

$\Rightarrow \frac{dx}{dy} = 3y^2 + 1\times\ln y + y\times\frac{1}{y}$

$\Rightarrow \frac{dx}{dy} = 3y^2 + \ln y + 1$

$\Rightarrow \frac{dy}{dx} = \frac{1}{3y^2+\ln y+1}$

$\Rightarrow \left.\frac{dy}{dx}\right|_{y=1} = \frac{1}{3+\cancel{\ln 1}+1} = \frac{1}{4}$

HENCE NORMAL GRADIENT IS -4

$\Rightarrow y - y_0 = m(x - x_0)$

$\Rightarrow y - 1 = -4(x-1)$

$\Rightarrow y - 1 = -4x + 4$

$\Rightarrow y + 4x = 5$

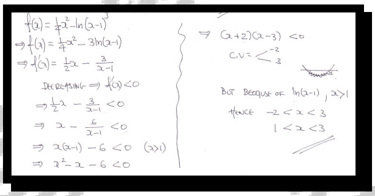
AS REQUIRED

Question 70 **(***+)**

$$f(x) = \frac{1}{4}x^2 - \ln(x-1)^3, \ x \in \mathbb{R}, \ x > 1.$$

Find the range of values of x for which $f(x)$ is a decreasing function.

$\boxed{1 < x < 3}$



$f(x) = \frac{1}{4}x^2 - \ln(x-1)^3$

$\Rightarrow f(x) = \frac{1}{4}x^2 - 3\ln(x-1)$

$\Rightarrow f'(x) = \frac{1}{2}x - \frac{3}{x-1}$

DECREASING $\Rightarrow f'(x) < 0$

$\Rightarrow \frac{1}{2}x - \frac{3}{x-1} < 0$

$\Rightarrow x - \frac{6}{x-1} < 0$

$\Rightarrow x(x-1) - 6 < 0 \quad (x>1)$

$\Rightarrow x^2 - x - 6 < 0$

$\Rightarrow (x+2)(x-3) < 0$

C.V. = -2, 3

BUT BECAUSE OF $\ln(x-1)$, $x>1$

HENCE $-2 < x < 3$

$1 < x < 3$

Question 71 **(***+)**

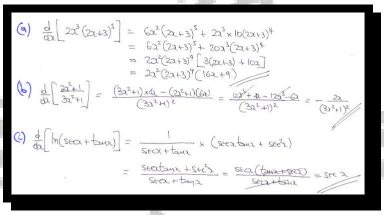
Show clearly that …

i. … $\frac{d}{dx}\left[2x^3(2x+3)^5\right] = 2x^2(16x+9)(2x+3)^4$.

ii. … $\frac{d}{dx}\left[\frac{2x^2+1}{3x^2+1}\right] = -\frac{2x}{\left(3x^2+1\right)^2}$.

iii. … $\frac{d}{dx}\left[\ln(\sec x + \tan x)\right] = \sec x$.

$\boxed{\text{proof}}$



(a) $\frac{d}{dx}\left[2x^3(2x+3)^5\right] = 6x^2(2x+3)^5 + 2x^3 \times 10(2x+3)^4$

$= 6x^2(2x+3)^5 + 20x^3(2x+3)^4$

$= 2x^2(2x+3)^4\left[3(2x+3) + 10x\right]$

$= 2x^2(2x+3)^4(16x+9)$

(b) $\frac{d}{dx}\left[\frac{2x^2+1}{3x^2+1}\right] = \frac{(3x^2+1)\times 4x - (2x^2+1)(6x)}{(3x^2+1)^2} = \frac{\cancel{12x^3} + 4x - \cancel{12x^3} - 6x}{(3x^2+1)^2} = -\frac{2x}{(3x^2+1)^2}$

(c) $\frac{d}{dx}\left[\ln(\sec x + \tan x)\right] = \frac{1}{\sec x + \tan x} \times (\sec x \tan x + \sec^2 x)$

$= \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} = \frac{\sec x\,\cancel{(\tan x + \sec x)}}{\cancel{\sec x + \tan x}} = \sec x$

Question 72 **(***+)**

The curve C has equation

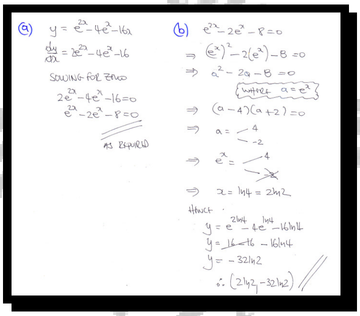
$$y = e^{2x} - 4e^{x} - 16x.$$

a) Show that the x coordinates of the stationary points of C satisfy the equation

$$e^{2x} - 2e^{x} - 8 = 0.$$

b) Hence determine the exact coordinates of the stationary point of C, giving the answer in terms of $\ln 2$.

$\boxed{\quad}$, $\boxed{(2\ln 2, -32\ln 2)}$



ⓐ $y = e^{2x} - 4e^{x} - 16x$

$\frac{dy}{dx} = 2e^{2x} - 4e^{x} - 16$

SOLVING FOR ZERO

$2e^{2x} - 4e^{x} - 16 = 0$

$e^{2x} - 2e^{x} - 8 = 0$ //

AS REQUIRED

ⓑ $e^{2x} - 2e^{x} - 8 = 0$

$\Rightarrow (e^{x})^{2} - 2(e^{x}) - 8 = 0$

$\Rightarrow a^{2} - 2a - 8 = 0$

{WHERE $a = e^{x}$}

$\Rightarrow (a-4)(a+2) = 0$

$\Rightarrow a = \begin{cases} 4 \\ -2 \end{cases}$

$\Rightarrow e^{x} = \begin{cases} 4 \\ \cancel{-2} \end{cases}$

$\Rightarrow x = \ln 4 = 2\ln 2$

HENCE

$y = e^{2\ln 4} - 4e^{\ln 4} - 16\ln 4$

$y = \cancel{16 - 16} - 16\ln 4$

$y = -32\ln 2$

$\therefore (2\ln 2, -32\ln 2)$ //

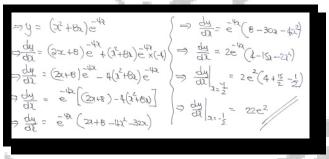
Question 73 **(***+)**

Given that

$$y = \left(x^{2} + 8x\right)e^{-4x},$$

find the exact value of $\frac{dy}{dx}$ at $x = -\frac{1}{2}$.

$$\boxed{\left.\frac{dy}{dx}\right|_{x=-\frac{1}{2}} = 22e^{2}}$$



$\Rightarrow y = (x^{2} + 8x)e^{-4x}$

$\Rightarrow \frac{dy}{dx} = (2x+8)e^{-4x} + (x^{2}+8x)e^{-4x} \times (-4)$

$\Rightarrow \frac{dy}{dx} = (2x+8)e^{-4x} - 4(x^{2}+8x)e^{-4x}$

$\Rightarrow \frac{dy}{dx} = e^{-4x}\left[(2x+8) - 4(x^{2}+8x)\right]$

$\Rightarrow \frac{dy}{dx} = e^{-4x}(2x + 8 - 4x^{2} - 32x)$

$\Rightarrow \frac{dy}{dx} = e^{-4x}(8 - 30x - 4x^{2})$

$\Rightarrow \frac{dy}{dx} = 2e^{-4x}(4 - 15x - 2x^{2})$

$\Rightarrow \left.\frac{dy}{dx}\right|_{x=-\frac{1}{2}} = 2e^{2}\left(4 + \frac{15}{2} - \frac{1}{2}\right)$

$\Rightarrow \left.\frac{dy}{dx}\right|_{x=-\frac{1}{2}} = 22e^{2}$ //

Question 74 (***+)

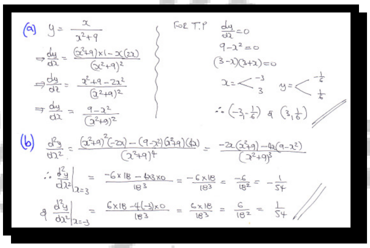
The equation of a curve C is

$$y = \frac{x}{x^2+9}, \quad x \in \mathbb{R} .$$

a) Find the coordinates of the stationary points of C.

b) Evaluate the exact value of $\frac{d^2y}{dx^2}$ at each of these stationary points.

$$\boxed{}, \ \boxed{\left(3,\frac{1}{6}\right), \left(-3,-\frac{1}{6}\right)}, \ \boxed{\left.\frac{d^2y}{dx^2}\right|_{x=\pm3} = \mp\frac{1}{54}}$$



(a) $y = \frac{x}{x^2+9}$

$\Rightarrow \frac{dy}{dx} = \frac{(x^2+9)\times1 - x(2x)}{(x^2+9)^2}$

$\Rightarrow \frac{dy}{dx} = \frac{x^2+9-2x^2}{(x^2+9)^2}$

$\Rightarrow \frac{dy}{dx} = \frac{9-x^2}{(x^2+9)^2}$

FOR T.P $\frac{dy}{dx}=0$

$9-x^2=0$

$(3-x)(3+x)=0$

$x = \begin{cases} -3 \\ 3 \end{cases}$ $\quad y = \begin{cases} -\frac{1}{6} \\ \frac{1}{6} \end{cases}$

$\therefore \left(-3,-\frac{1}{6}\right)$ & $\left(3,\frac{1}{6}\right)$ //

(b) $\frac{d^2y}{dx^2} = \frac{(x^2+9)^2(-2x) - (9-x^2)(x^2+9)(4x)}{(x^2+9)^4} = \frac{-2x(x^2+9)-4x(9-x^2)}{(x^2+9)^3}$

$\therefore \left.\frac{d^2y}{dx^2}\right|_{x=3} = \frac{-6\times18 - 4\times3\times0}{18^3} = -\frac{6\times18}{18^3} = \frac{-6}{18^2} = -\frac{1}{54}$ //

& $\left.\frac{d^2y}{dx^2}\right|_{x=-3} = \frac{6\times18-4(-3)\times0}{18^3} = \frac{6\times18}{18^3} = \frac{6}{18^2} = \frac{1}{54}$ //

Question 75 (***+)

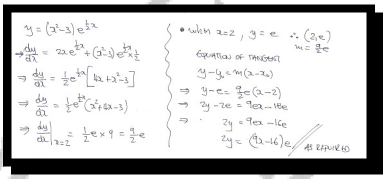
The point A, where $x=2$, lies on the curve with equation

$$y = \left(x^2-3\right)e^{\frac{1}{2}x} .$$

Show that an equation of the tangent to the curve at A is given by

$$2y = (9x-16)e .$$

$$\boxed{\text{proof}}$$



$y = (x^2-3)e^{\frac{1}{2}x}$

$\Rightarrow \frac{dy}{dx} = 2xe^{\frac{1}{2}x} + (x^2-3)e^{\frac{1}{2}x}\times\frac{1}{2}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2}e^{\frac{1}{2}x}\left[4x+x^2-3\right]$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2}e^{\frac{1}{2}x}(x^2+4x-3)$

$\Rightarrow \left.\frac{dy}{dx}\right|_{x=2} = \frac{1}{2}e\times9 = \frac{9}{2}e$

- when $x=2$, $y=e$ $\therefore (2,e)$ $m=\frac{9}{2}e$

EQUATION OF TANGENT

$y-y_0 = m(x-x_0)$

$\Rightarrow y-e = \frac{9}{2}e(x-2)$

$\Rightarrow 2y-2e = 9ex-18e$

$\Rightarrow 2y = 9ex-16e$

$2y = (9x-16)e$ // AS REQUIRED

Question 76 **(***+)**

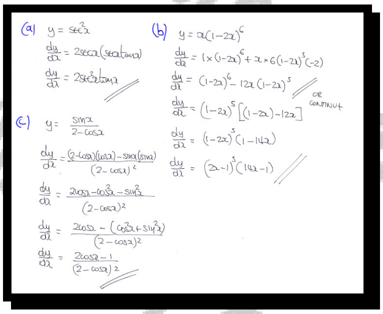
Differentiate each of the following expressions with respect to x, simplifying the final answer as far as possible.

a) $y = \sec^2 x$.

b) $y = x(1-2x)^6$.

c) $y = \dfrac{\sin x}{2-\cos x}$.

$$\boxed{\frac{dy}{dx} = 2\sec^2 x \tan x}, \boxed{\frac{dy}{dx} = (14x-1)(2x-1)^5 = (1-14x)(1-2x)^5}, \boxed{\frac{dy}{dx} = \frac{2\cos x - 1}{(2-\cos x)^2}}$$



(a) $y = \sec^2 x$

$\frac{dy}{dx} = 2\sec x(\sec x \tan x)$

$\frac{dy}{dx} = 2\sec^2 x \tan x$

(b) $y = x(1-2x)^6$

$\frac{dy}{dx} = 1 \times (1-2x)^6 + x \times 6(1-2x)^5(-2)$

$\frac{dy}{dx} = (1-2x)^6 - 12x(1-2x)^5$

OR CONTINUE

$\frac{dy}{dx} = (1-2x)^5\left[(1-2x) - 12x\right]$

$\frac{dy}{dx} = (1-2x)^5(1-14x)$

$\frac{dy}{dx} = (2x-1)^5(14x-1)$

(c) $y = \frac{\sin x}{2-\cos x}$

$\frac{dy}{dx} = \frac{(2-\cos x)(\cos x) - \sin x(\sin x)}{(2-\cos x)^2}$

$\frac{dy}{dx} = \frac{2\cos x - \cos^2 x - \sin^2 x}{(2-\cos x)^2}$

$\frac{dy}{dx} = \frac{2\cos x - (\cos^2 x + \sin^2 x)}{(2-\cos x)^2}$

$\frac{dy}{dx} = \frac{2\cos x - 1}{(2-\cos x)^2}$

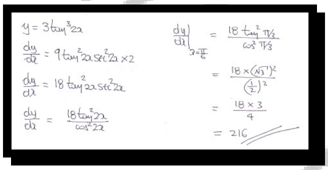
Question 77 **(***+)**

Given that

$$y = 3\tan^3 2x$$

find the value of $\dfrac{dy}{dx}$ at $x = \dfrac{\pi}{6}$.

$$\boxed{\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{6}} = 216}$$



$y = 3\tan^3 2x$

$\frac{dy}{dx} = 9\tan^2 2x \sec^2 2x \times 2$

$\frac{dy}{dx} = 18\tan^2 2x \sec^2 2x$

$\frac{dy}{dx} = \frac{18\tan^2 2x}{\cos^2 2x}$

$\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{6}} = \frac{18\tan^2 \pi/3}{\cos^2 \pi/3}$

$= \frac{18 \times (\sqrt{3})^2}{\left(\frac{1}{2}\right)^2}$

$= \frac{18 \times 3}{4}$

$= 216$

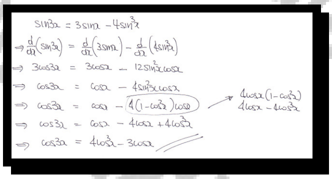
Question 78 **(***+)**

The following trigonometric identity is given

$$\sin 3x \equiv 3\sin x - 4\sin^3 x .$$

By differentiating both sides of the above trigonometric identity with respect to x, find the corresponding identity for $\cos 3x$ in terms of $\cos x$.

, $\boxed{\cos 3x = 4\cos^3 x - 3\cos x}$



$\sin 3x = 3\sin x - 4\sin^3 x$

$\Rightarrow \frac{d}{dx}(\sin 3x) = \frac{d}{dx}(3\sin x) - \frac{d}{dx}(4\sin^3 x)$

$\Rightarrow 3\cos 3x = 3\cos x - 12\sin^2 x\cos x$

$\Rightarrow \cos 3x = \cos x - 4\sin^2 x\cos x$

$\Rightarrow \cos 3x = \cos x - 4(1-\cos^2 x)\cos x \longrightarrow \begin{matrix} 4\cos x(1-\cos^2 x) \\ 4\cos x - 4\cos^3 x \end{matrix}$

$\Rightarrow \cos 3x = \cos x - 4\cos x + 4\cos^3 x$

$\Rightarrow \cos 3x = 4\cos^3 x - 3\cos x$

Question 79 **(***+)**

$$f(x) = 8x^2 + 8x + \ln x, \; x > 0 .$$

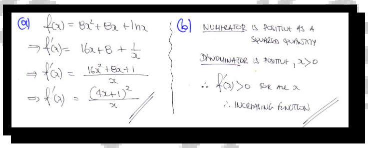
a) Show clearly that

$$f'(x) = \frac{(ax+b)^2}{x},$$

where a and b are integers.

b) Hence show that $f(x)$ is an increasing function.

$\boxed{a = 4, \; b = 1}$



(a) $f(x) = 8x^2 + 8x + \ln x$

$\Rightarrow f'(x) = 16x + 8 + \frac{1}{x}$

$\Rightarrow f'(x) = \frac{16x^2 + 8x + 1}{x}$

$\Rightarrow f'(x) = \frac{(4x+1)^2}{x}$

(b) NUMERATOR IS POSITIVE AS A SQUARED QUANTITY

DENOMINATOR IS POSITIVE, $x > 0$

$\therefore f'(x) > 0$ FOR ALL x

$\therefore$ INCREASING FUNCTION

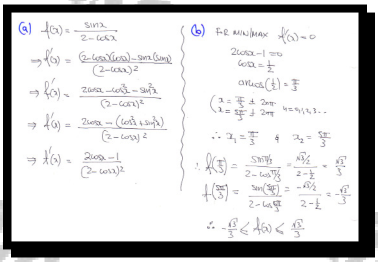
Question 80 **(***+)**

$$f(x)=\frac{\sin x}{2-\cos x},\ 0 \le x < 2\pi \ .$$

a) Find a simplified expression for $f'(x)$.

b) Hence find the minimum and maximum value of $f(x)$.

$\boxed{f(x)=\frac{2\cos x-1}{(2-\cos x)^2}}$, $\boxed{-\frac{\sqrt{3}}{3} \le f(x) \le \frac{\sqrt{3}}{3}}$



(a) $f(x)=\frac{\sin x}{2-\cos x}$

$\Rightarrow f'(x) = \frac{(2-\cos x)(\cos x)-\sin x(\sin x)}{(2-\cos x)^2}$

$\Rightarrow f'(x) = \frac{2\cos x-\cos^2 x-\sin^2 x}{(2-\cos x)^2}$

$\Rightarrow f'(x) = \frac{2\cos x-(\cos^2 x+\sin^2 x)}{(2-\cos x)^2}$

$\Rightarrow f'(x) = \frac{2\cos x-1}{(2-\cos x)^2}$

(b) FOR MIN/MAX $f'(x)=0$

$2\cos x-1=0$

$\cos x=\frac{1}{2}$

$\arccos\left(\frac{1}{2}\right)=\frac{\pi}{3}$

$\begin{pmatrix} x=\frac{\pi}{3} \pm 2n\pi \\ x=\frac{5\pi}{3} \pm 2n\pi \end{pmatrix}$ $n=0,1,2,3\ldots$

$\therefore x_1=\frac{\pi}{3}$ & $x_2=\frac{5\pi}{3}$

$\therefore f\left(\frac{\pi}{3}\right)=\frac{\sin \pi/3}{2-\cos \pi/3}=\frac{\sqrt{3}/2}{2-\frac{1}{2}}=\frac{\sqrt{3}}{3}$

$f\left(\frac{5\pi}{3}\right)=\frac{\sin\left(\frac{5\pi}{3}\right)}{2-\cos\frac{5\pi}{3}}=\frac{-\sqrt{3}/2}{2-\frac{1}{2}}=-\frac{\sqrt{3}}{3}$

$\therefore -\frac{\sqrt{3}}{3} \leqslant f(x) \leqslant \frac{\sqrt{3}}{3}$

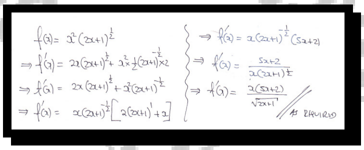
Question 81 **(***+)**

$$f(x)=x^2\sqrt{2x+1},\ x \ge -\tfrac{1}{2} \ .$$

Show clearly that

$$f'(x)=\frac{x(5x+2)}{\sqrt{2x+1}}.$$

$\boxed{\text{proof}}$



$f(x)=x^2(2x+1)^{\frac{1}{2}}$

$\Rightarrow f'(x)=2x(2x+1)^{\frac{1}{2}}+x^2\times\frac{1}{2}(2x+1)^{-\frac{1}{2}}\times 2$

$\Rightarrow f'(x)=2x(2x+1)^{\frac{1}{2}}+x^2(2x+1)^{-\frac{1}{2}}$

$\Rightarrow f'(x)=x(2x+1)^{-\frac{1}{2}}\left[2(2x+1)^{1}+x\right]$

$\Rightarrow f'(x)=x(2x+1)^{-\frac{1}{2}}(5x+2)$

$\Rightarrow f'(x)=\frac{5x+2}{x(2x+1)^{\frac{1}{2}}}$

$\Rightarrow f'(x)=\frac{x(5x+2)}{\sqrt{2x+1}}$ // AS REQUIRED

Question 82 **(***+)**

The curve C has equation

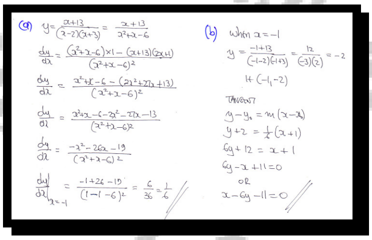
$$y = \frac{x+13}{(x-2)(x+3)}, \quad x \neq -3, 2.$$

The point A lies on C and has $x = -1$.

a) Find the value of $\frac{dy}{dx}$ at A.

b) Find an equation of the tangent to C at A, giving the final answer in the form $ax + by + c = 0$, where a, b and c are integers.

$\boxed{\frac{1}{6}}$, $\boxed{x - 6y - 11 = 0}$



(a) $y = \frac{x+13}{(x-2)(x+3)} = \frac{x+13}{x^2+x-6}$

$\frac{dy}{dx} = \frac{(x^2+x-6)\times 1 - (x+13)(2x+1)}{(x^2+x-6)^2}$

$\frac{dy}{dx} = \frac{x^2+x-6-(2x^2+27x+13)}{(x^2+x-6)^2}$

$\frac{dy}{dx} = \frac{x^2+x-6-2x^2-27x-13}{(x^2+x-6)^2}$

$\frac{dy}{dx} = \frac{-x^2-26x-19}{(x^2+x-6)^2}$

$\left.\frac{dy}{dx}\right|_{x=-1} = \frac{-1+26-19}{(1-1-6)^2} = \frac{6}{36} = \frac{1}{6}$ //

(b) when $x = -1$

$y = \frac{-1+13}{(-1-2)(-1+3)} = \frac{12}{(-3)(2)} = -2$

A$(-1,-2)$

TANGENT

$y - y_0 = m(x - x_0)$

$y + 2 = \frac{1}{6}(x+1)$

$6y + 12 = x + 1$

$6y - x + 11 = 0$

OR

$x - 6y - 11 = 0$ //

Question 83 **(***+)**

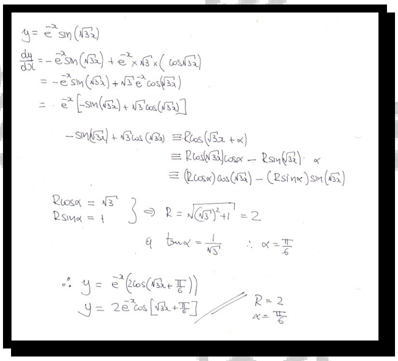
$$y = e^{-x}\sin\left(\sqrt{3}x\right),\ x \in \mathbb{R}.$$

Find the exact value of each of the constants R and α so that

$$\frac{dy}{dx} = R\,e^{-x}\cos\left(\sqrt{3}x+\alpha\right),$$

where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$.

$\boxed{\text{C3H}}$, $\boxed{R=2}$, $\boxed{\alpha = \frac{\pi}{6}}$



$y = e^{-x}\sin(\sqrt{3}x)$

$\frac{dy}{dx} = -e^{-x}\sin(\sqrt{3}x) + e^{-x} \times \sqrt{3} \times (\cos\sqrt{3}x)$

$= -e^{-x}\sin(\sqrt{3}x) + \sqrt{3}e^{-x}\cos(\sqrt{3}x)$

$= e^{-x}\left[-\sin(\sqrt{3}x) + \sqrt{3}\cos(\sqrt{3}x)\right]$

$-\sin(\sqrt{3}x) + \sqrt{3}\cos(\sqrt{3}x) \equiv R\cos(\sqrt{3}x + \alpha)$

$\equiv R\cos(\sqrt{3}x)\cos\alpha - R\sin(\sqrt{3}x)\cdot\alpha$

$\equiv (R\cos\alpha)\cos(\sqrt{3}x) - (R\sin\alpha)\sin(\sqrt{3}x)$

$\left.\begin{array}{l} R\cos\alpha = \sqrt{3} \\ R\sin\alpha = 1 \end{array}\right\} \Rightarrow R = \sqrt{(\sqrt{3})^2+1} = 2$

& $\tan\alpha = \frac{1}{\sqrt{3}}$ $\therefore \alpha = \frac{\pi}{6}$

$\therefore y = e^{-x}\left(2\cos\left(\sqrt{3}x + \frac{\pi}{6}\right)\right)$

$y = 2e^{-x}\cos\left[\sqrt{3}x + \frac{\pi}{6}\right]$

$R = 2$

$\alpha = \frac{\pi}{6}$

Question 84 (***+)

A curve has equation

$$y = \frac{1}{2}\ln\left(\frac{x}{3}\right), \quad x > 0 .$$

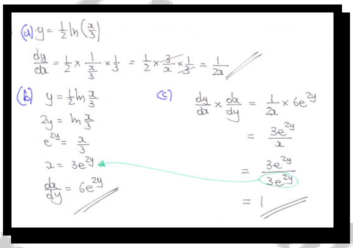
a) Find an expression for $\frac{dy}{dx}$ in terms of x.

b) By making x the subject of the equation and differentiating the resulting equation, find $\frac{dx}{dy}$.

c) Use the results of parts (a) and (b), to deduce that

$$\frac{dy}{dx} \times \frac{dx}{dy} = 1 .$$

$$\boxed{}, \boxed{\frac{dy}{dx} = \frac{1}{2x}}, \boxed{\frac{dx}{dy} = 6e^{2y} = 2x}$$



(a) $y = \frac{1}{2}\ln\left(\frac{x}{3}\right)$

$\frac{dy}{dx} = \frac{1}{2} \times \frac{1}{\frac{x}{3}} \times \frac{1}{3} = \frac{1}{2} \times \cancel{\frac{3}{x}} \times \cancel{\frac{1}{3}} = \frac{1}{2x}$

(b) $y = \frac{1}{2}\ln\frac{x}{3}$

$2y = \ln\frac{x}{3}$

$e^{2y} = \frac{x}{3}$

$x = 3e^{2y}$

$\frac{dx}{dy} = 6e^{2y}$

(c) $\frac{dy}{dx} \times \frac{dx}{dy} = \frac{1}{2x} \times 6e^{2y}$

$= \frac{3e^{2y}}{x}$

$= \frac{3e^{2y}}{3e^{2y}}$

$= 1$

Question 85 **(***+)**

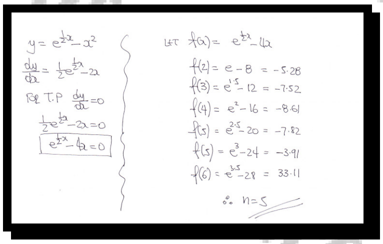
A curve C has equation

$$y = \mathrm{e}^{\frac{1}{2}x} - x^2, \ x \in \mathbb{R}.$$

The curve has a single stationary point at $x = x_0$, such that $n < x_0 < n+1, \ n \in \mathbb{N}$.

Determine the value of n.

$$\boxed{n = 5}$$



$y = e^{\frac{1}{2}x} - x^2$

$\frac{dy}{dx} = \frac{1}{2}e^{\frac{1}{2}x} - 2x$

FOR T.P $\frac{dy}{dx} = 0$

$\frac{1}{2}e^{\frac{1}{2}x} - 2x = 0$

$\boxed{e^{\frac{1}{2}x} - 4x = 0}$

LET $f(x) = e^{\frac{1}{2}x} - 4x$

$f(2) = e - 8 = -5.28$

$f(3) = e^{1.5} - 12 = -7.52$

$f(4) = e^{2} - 16 = -8.61$

$f(5) = e^{2.5} - 20 = -7.82$

$f(5) = e^{3} - 24 = -3.91$

$f(6) = e^{3.5} - 28 = 33.11$

$\therefore n = 5$

Question 86 **(***+)**

A curve has equation

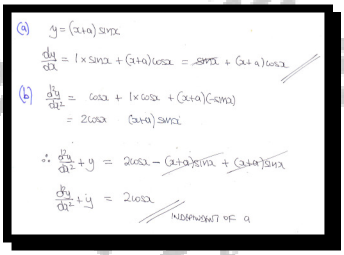
$$y = (x + a)\sin x\,,$$

where a is a non zero constant.

a) Find an expression for $\dfrac{dy}{dx}$.

b) Show that $\dfrac{d^2 y}{dx^2} + y$ is independent of a .

$\boxed{\dfrac{dy}{dx} = (a + x)\cos x + \sin x}$, $\boxed{\dfrac{d^2 y}{dx^2} + y = 2\cos x}$



(a) $y = (x+a)\sin x$

$\frac{dy}{dx} = 1 \times \sin x + (x+a)\cos x = \sin x + (x+a)\cos x$

(b) $\frac{d^2y}{dx^2} = \cos x + 1 \times \cos x + (x+a)(-\sin x)$

$= 2\cos x \quad (x+a)\sin x$

$\therefore \frac{d^2y}{dx^2} + y = 2\cos x - \cancel{(x+a)\sin x} + \cancel{(x+a)\sin x}$

$\frac{d^2y}{dx^2} + y = 2\cos x$

INDEPENDENT OF a

Question 87 (***+)

The curve C has equation

$$y = \frac{(x+3)(x-1)}{x+2} - 3\ln(x+2), \quad x > -2.$$

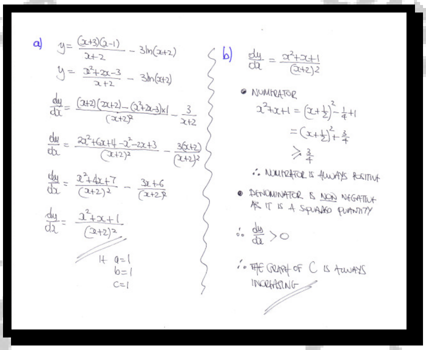
a) Show clearly that

$$\frac{dy}{dx} = \frac{ax^2+bx+c}{(x+2)^2},$$

where a, b and c are constants to be found.

b) Deduce that the graph of C is increasing for all allowable values of x.

$\boxed{a=1}$, $\boxed{b=1}$, $\boxed{c=1}$



a) $y = \frac{(x+3)(x-1)}{x+2} - 3\ln(x+2)$

$y = \frac{x^2+2x-3}{x+2} - 3\ln(x+2)$

$\frac{dy}{dx} = \frac{(x+2)(2x+2)-(x^2+2x-3)\times 1}{(x+2)^2} - \frac{3}{x+2}$

$\frac{dy}{dx} = \frac{2x^2+6x+4-x^2-2x+3}{(x+2)^2} - \frac{3(x+2)}{(x+2)^2}$

$\frac{dy}{dx} = \frac{x^2+4x+7}{(x+2)^2} - \frac{3x+6}{(x+2)^2}$

$\frac{dy}{dx} = \frac{x^2+x+1}{(x+2)^2}$

$\ \ \ a=1$
$\ \ \ b=1$
$\ \ \ c=1$

b) $\frac{dy}{dx} = \frac{x^2+x+1}{(x+2)^2}$

- NUMERATOR
  $x^2+x+1 = \left(x+\frac{1}{2}\right)^2 - \frac{1}{4} + 1$
  $= \left(x+\frac{1}{2}\right)^2 + \frac{3}{4}$
  $\geqslant \frac{3}{4}$

$\therefore$ NUMERATOR IS ALWAYS POSITIVE

- DENOMINATOR IS NON NEGATIVE AS IT IS A SQUARED QUANTITY

$\therefore \frac{dy}{dx} > 0$

$\therefore$ THE GRAPH OF C IS ALWAYS INCREASING

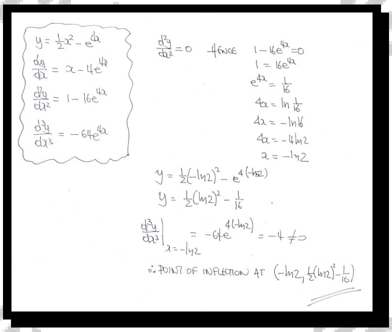
Question 88 (*+)**

The curve C has equation

$$y = \frac{1}{2}x^2 - e^{4x}.$$

Show clearly that C has a point of inflection, determining its exact coordinates.

$$\boxed{\left[-\ln 2, \frac{1}{2}(\ln 2)^2 - \frac{1}{16}\right]}$$



$y = \frac{1}{2}x^2 - e^{4x}$

$\frac{dy}{dx} = x - 4e^{4x}$

$\frac{d^2y}{dx^2} = 1 - 16e^{4x}$

$\frac{d^3y}{dx^3} = -64e^{4x}$

$\frac{d^2y}{dx^2} = 0$ HENCE $1 - 16e^{4x} = 0$

$1 = 16e^{4x}$

$e^{4x} = \frac{1}{16}$

$4x = \ln\frac{1}{16}$

$4x = -\ln 16$

$4x = -4\ln 2$

$x = -\ln 2$

$y = \frac{1}{2}(-\ln 2)^2 - e^{4(-\ln 2)}$

$y = \frac{1}{2}(\ln 2)^2 - \frac{1}{16}$.

$\left.\frac{d^3y}{dx^3}\right|_{x=-\ln 2} = -64e^{4(-\ln 2)} = -4 \neq 0$

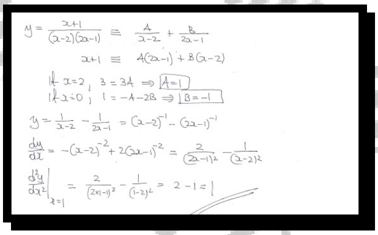
$\therefore$ POINT OF INFLECTION AT $\left(-\ln 2, \frac{1}{2}(\ln 2)^2 - \frac{1}{16}\right)$

Question 89 (*+)**

$$y = \frac{x+1}{(x-2)(2x-1)}, \; x \neq 2, \; x \neq \frac{1}{2}.$$

Find the value of $\frac{dy}{dx}$ at $x = 1$.

$$\boxed{\left.\frac{dy}{dx}\right|_{x=1} = 1}$$



$y = \frac{x+1}{(x-2)(2x-1)} \equiv \frac{A}{x-2} + \frac{B}{2x-1}$

$x+1 \equiv A(2x-1) + B(x-2)$

If $x=2$, $3 = 3A \Rightarrow \boxed{A=1}$

If $x=0$, $1 = -A-2B \Rightarrow \boxed{B=-1}$

$y = \frac{1}{x-2} - \frac{1}{2x-1} = (x-2)^{-1} - (2x-1)^{-1}$

$\frac{dy}{dx} = -(x-2)^{-2} + 2(2x-1)^{-2} = \frac{2}{(2x-1)^2} - \frac{1}{(x-2)^2}$

$\left.\frac{d^2y}{dx^2}\right|_{x=1} = \frac{2}{(2\times1-1)^2} - \frac{1}{(1-2)^2} = 2 - 1 = 1$

Question 90 (***+)

A curve C has equation

$$y = \frac{1}{4}e^{2x} + 3, \quad x \in \mathbb{R}.$$

The point P lies on C where $x = \ln 2$.

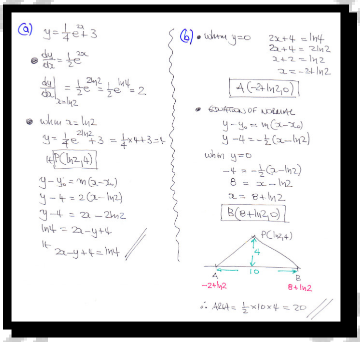
a) Show that the equation of the tangent to the curve at the point P is

$$2x - y + 4 = \ln 4.$$

This tangent meets the x axis at the point A, and the normal to the curve at the point P meets the x axis at the point B.

b) Show that the area of the triangle APB is 20 square units.

proof



(a) $y = \frac{1}{4}e^{2x} + 3$

• $\frac{dy}{dx} = \frac{1}{2}e^{2x}$

$\left.\frac{dy}{dx}\right|_{x=\ln 2} = \frac{1}{2}e^{2\ln 2} = \frac{1}{2}e^{\ln 4} = 2$

• when $x = \ln 2$

$y = \frac{1}{4}e^{2\ln 2} + 3 = \frac{1}{4} \times 4 + 3 = 4$

$\therefore \boxed{P(\ln 2, 4)}$

$y - y_0 = m(x - x_0)$

$y - 4 = 2(x - \ln 2)$

$y - 4 = 2x - 2\ln 2$

$\ln 4 = 2x - y + 4$

$\therefore 2x - y + 4 = \ln 4$ //

(b) • when $y = 0$

$2x + 4 = \ln 4$

$2x + 4 = 2\ln 2$

$x + 2 = \ln 2$

$x = -2 + \ln 2$

$\boxed{A(-2+\ln 2, 0)}$

• EQUATION OF NORMAL

$y - y_0 = m(x - x_0)$

$y - 4 = -\frac{1}{2}(x - \ln 2)$

when $y = 0$

$-4 = -\frac{1}{2}(x - \ln 2)$

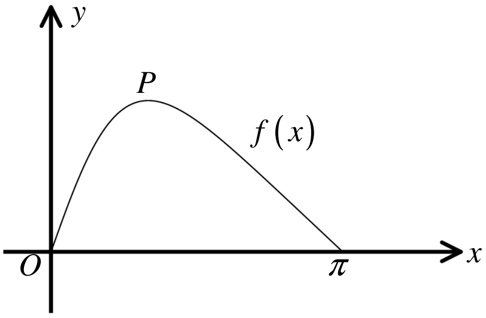
$8 = x - \ln 2$

$x = 8 + \ln 2$

$\boxed{B(8+\ln 2, 0)}$

$\therefore$ AREA $= \frac{1}{2} \times 10 \times 4 = 20$ //

Question 91 **(***+)**



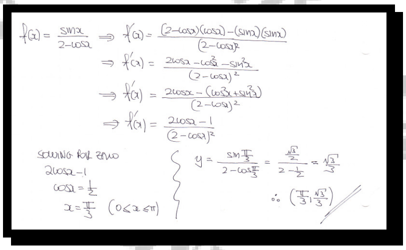
The figure above shows the graph of the curve with equation

$$f(x) = \frac{\sin x}{2 - \cos x}, \ 0 \le x \le \pi .$$

The curve has a stationary point at P.

Determine the exact coordinates of P.

$\boxed{}$, $\boxed{P\left(\frac{1}{3}\pi, \frac{1}{3}\sqrt{3}\right)}$



$f(x) = \frac{\sin x}{2-\cos x} \Rightarrow f'(x) = \frac{(2-\cos x)(\cos x) - (\sin x)(\sin x)}{(2-\cos x)^2}$

$\Rightarrow f'(x) = \frac{2\cos x - \cos^2 x - \sin^2 x}{(2-\cos x)^2}$

$\Rightarrow f'(x) = \frac{2\cos x - (\cos^2 x + \sin^2 x)}{(2-\cos x)^2}$

$\Rightarrow f'(x) = \frac{2\cos x - 1}{(2-\cos x)^2}$

SOLVING FOR ZERO

$2\cos x - 1$

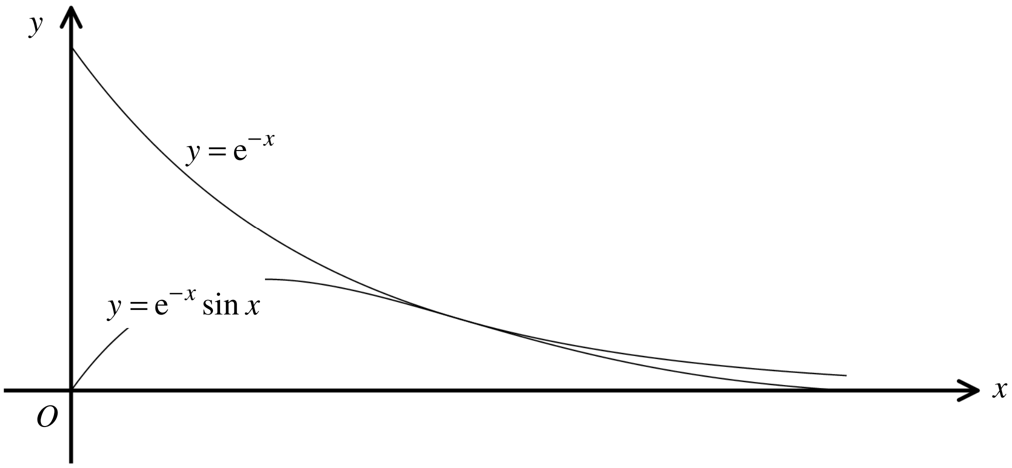
$\cos x = \frac{1}{2}$

$x = \frac{\pi}{3} \quad (0 \le x \le \pi)$

$y = \frac{\sin\frac{\pi}{3}}{2-\cos\frac{\pi}{3}} = \frac{\frac{\sqrt{3}}{2}}{2-\frac{1}{2}} = \frac{\sqrt{3}}{3}$

$\therefore \left(\frac{\pi}{3}, \frac{\sqrt{3}}{3}\right)$

Question 92 **(***+)**



The figure above shows the graph of

$$y = e^{-x} \quad \text{and} \quad y = e^{-x}\sin x,\ \ 0 \le x \le \pi .$$

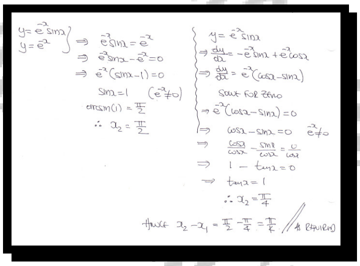
The curve with equation $y = e^{-x}\sin x$ has a local maximum at the point where $x = x_1$.

The curves touch each other at the point where $x = x_2$.

Show clearly that

$$x_2 - x_1 = \frac{\pi}{4} .$$

C3, proof

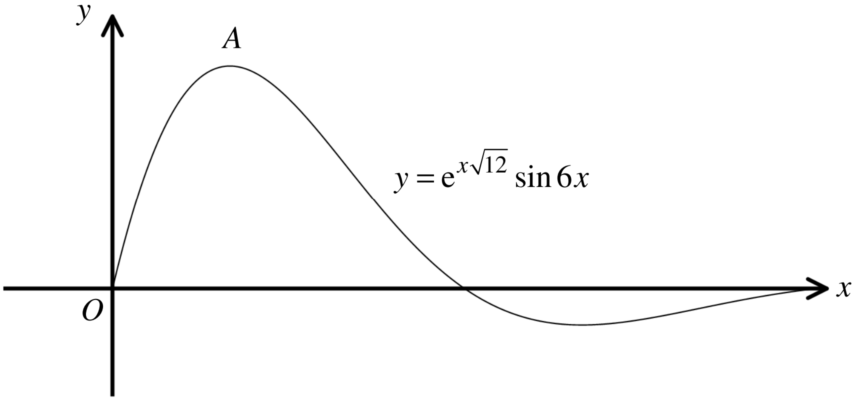


$y = e^{-x}\sin x$
$y = e^{-x}$ $\Rightarrow$ $e^{-x}\sin x = e^{-x}$
$\Rightarrow$ $e^{-x}\sin x - e^{-x} = 0$
$\Rightarrow$ $e^{-x}(\sin x - 1) = 0$
$\sin x = 1$ $(e^{-x} \neq 0)$
$\arcsin(1) = \frac{\pi}{2}$
$\therefore\ x_2 = \frac{\pi}{2}$

$y = e^{-x}\sin x$
$\Rightarrow \frac{dy}{dx} = -e^{-x}\sin x + e^{-x}\cos x$
$\Rightarrow \frac{dy}{dx} = e^{-x}(\cos x - \sin x)$
SOLVE FOR ZERO
$\Rightarrow e^{-x}(\cos x - \sin x) = 0$
$\Rightarrow$ $\cos x - \sin x = 0$ $e^{-x} \neq 0$
$\Rightarrow$ $\frac{\cos x}{\cos x} - \frac{\sin x}{\cos x} = \frac{0}{\cos x}$
$\Rightarrow$ $1 - \tan x = 0$
$\Rightarrow$ $\tan x = 1$
$\therefore\ x_2 = \frac{\pi}{4}$

HENCE $x_2 - x_1 = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$ // AS REQUIRED

Question 93 (***+)



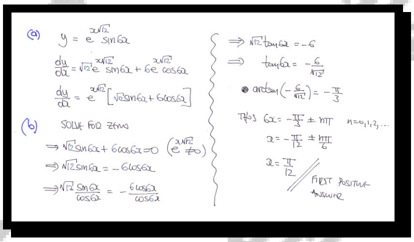
The figure above shows the graph of the curve with equation

$$y = e^{x\sqrt{12}} \sin 6x, \; 0 \le x \le \frac{\pi}{3}.$$

The curve has a local maximum at the point A.

Find, in terms of π, the x coordinate of A.

$\boxed{\text{C3K}}$, $\boxed{\frac{\pi}{9}}$


(a) $y = e^{x\sqrt{12}} \sin 6x$

$\frac{dy}{dx} = \sqrt{12} e^{x\sqrt{12}} \sin 6x + 6e^{x\sqrt{12}} \cos 6x$

$\frac{dy}{dx} = e^{x\sqrt{12}} \left[\sqrt{12} \sin 6x + 6\cos 6x\right]$

(b) SOLVE FOR ZERO

$\Rightarrow \sqrt{12} \sin 6x + 6\cos 6x = 0 \quad (e^{x\sqrt{12}} \neq 0)$

$\Rightarrow \sqrt{12} \sin 6x = -6\cos 6x$

$\Rightarrow \sqrt{12} \frac{\sin 6x}{\cos 6x} = -\frac{6\cos 6x}{\cos 6x}$

$\Rightarrow \sqrt{12} \tan 6x = -6$

$\Rightarrow \tan 6x = -\frac{6}{\sqrt{12}}$

● $\arctan\left(-\frac{6}{\sqrt{12}}\right) = -\frac{\pi}{3}$

THUS $6x = -\frac{\pi}{3} \pm n\pi \quad n = 0, 1, 2, \ldots$

$x = -\frac{\pi}{12} \pm \frac{n\pi}{6}$

$x = \frac{\pi}{12}$ // FIRST POSITIVE ANSWER

Question 94 **(***+)**

$$f(x)=\frac{6x-13}{(x+2)(x-3)},\ x\in\mathbb{R},\ x\neq -2,3.$$

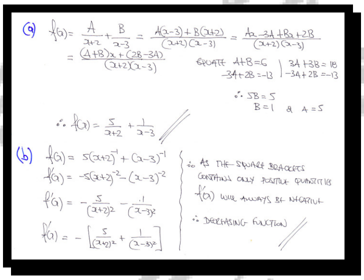
a) Show clearly that

$$f(x)=\frac{A}{(x+2)}+\frac{B}{(x-3)},$$

where A and B are integers.

b) Hence show further that $f(x)$ is a decreasing function.

$\boxed{A=5,\ B=1}$



(a) $f(x)=\frac{A}{x+2}+\frac{B}{x-3}=\frac{A(x-3)+B(x+2)}{(x+2)(x-3)}=\frac{Ax-3A+Bx+2B}{(x+2)(x-3)}$

$=\frac{(A+B)x+(2B-3A)}{(x+2)(x-3)}$

EQUATE $A+B=6$ | $3A+3B=18$

$-3A+2B=-13$ | $-3A+2B=-13$

$\therefore\ 5B=5$

$B=1$ & $A=5$

$\therefore\ f(x)=\frac{5}{x+2}+\frac{1}{x-3}$ //

(b) $f(x)=5(x+2)^{-1}+(x-3)^{-1}$

$f'(x)=-5(x+2)^{-2}-(x-3)^{-2}$

$f'(x)=-\frac{5}{(x+2)^2}-\frac{1}{(x-3)^2}$

$f'(x)=-\left[\frac{5}{(x+2)^2}+\frac{1}{(x-3)^2}\right]$

$\therefore$ AS THE SQUARE BRACKETS CONTAINS ONLY POSITIVE QUANTITIES $f'(x)$ WILL ALWAYS BE NEGATIVE

$\therefore$ DECREASING FUNCTION //

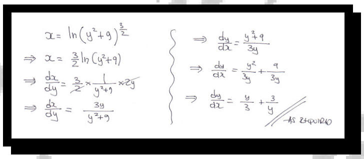
Question 95 **(***+)**

$$x=\ln\left(y^2+9\right)^{\frac{3}{2}}.$$

Show clearly that

$$\frac{dy}{dx}=\frac{y}{3}+\frac{3}{y}.$$

$\boxed{}$, $\boxed{\text{proof}}$



$x=\ln\left(y^2+9\right)^{\frac{3}{2}}$

$\Rightarrow x=\frac{3}{2}\ln(y^2+9)$

$\Rightarrow \frac{dx}{dy}=\frac{3}{\cancel{2}}\times\frac{1}{y^2+9}\times\cancel{2}y$

$\Rightarrow \frac{dx}{dy}=\frac{3y}{y^2+9}$

$\Rightarrow \frac{dy}{dx}=\frac{y^2+9}{3y}$

$\Rightarrow \frac{dy}{dx}=\frac{y^2}{3y}+\frac{9}{3y}$

$\Rightarrow \frac{dy}{dx}=\frac{y}{3}+\frac{3}{y}$ // AS REQUIRED

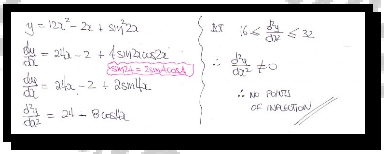
Question 96 (****)

The curve C has equation

$$y = 12x^2 - 2x + \sin^2 2x .$$

Show clearly that C has no points of inflection.

$\boxed{\text{proof}}$



$y = 12x^2 - 2x + \sin^2 2x$

$\frac{dy}{dx} = 24x - 2 + 4\sin 2x \cos 2x$

$\boxed{\sin 2A = 2\sin A \cos A}$

$\frac{dy}{dx} = 24x - 2 + 2\sin 4x$

$\frac{d^2y}{dx^2} = 24 - 8\cos 4x$

BUT $16 \leqslant \frac{d^2y}{dx^2} \leqslant 32$

$\therefore \frac{d^2y}{dx^2} \neq 0$

$\therefore$ NO POINTS OF INFLECTION

Question 97 (****)

$$y = a^x , \ a > 0 , \ a \neq 1 .$$

a) Show clearly that

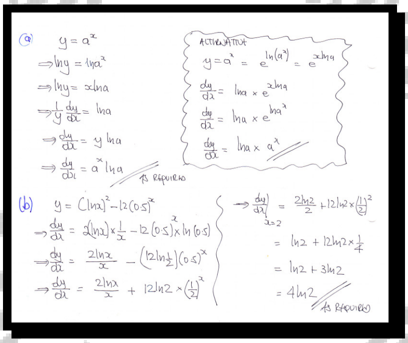
$$\frac{dy}{dx} = a^x \ln a .$$

A curve has equation

$$y = (\ln x)^2 - 12(0.5)^x , \ x > 0 .$$

b) Show that at the point of the curve where $x = 2$, the gradient is $4\ln 2$.

$\boxed{\text{proof}}$



(a) $y = a^x$

$\Rightarrow \ln y = \ln a^x$

$\Rightarrow \ln y = x\ln a$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \ln a$

$\Rightarrow \frac{dy}{dx} = y \ln a$

$\Rightarrow \frac{dy}{dx} = a^x \ln a$

AS REQUIRED

ALTERNATIVE

$y = a^x = e^{\ln(a^x)} = e^{x\ln a}$

$\frac{dy}{dx} = \ln a \times e^{x\ln a}$

$\frac{dy}{dx} = \ln a \times e^{\ln a^x}$

$\frac{dy}{dx} = \ln a \times a^x$

(b) $y = (\ln x)^2 - 12(0.5)^x$

$\Rightarrow \frac{dy}{dx} = 2(\ln x) \times \frac{1}{x} - 12(0.5)^x \times \ln(0.5)$

$\Rightarrow \frac{dy}{dx} = \frac{2\ln x}{x} - \left(12\ln\frac{1}{2}\right)(0.5)^x$

$\Rightarrow \frac{dy}{dx} = \frac{2\ln x}{x} + 12\ln 2 \times \left(\frac{1}{2}\right)^x$

$\Rightarrow \left.\frac{dy}{dx}\right|_{x=2} = \frac{2\ln 2}{2} + 12\ln 2 \times \left(\frac{1}{2}\right)^2$

$= \ln 2 + 12\ln 2 \times \frac{1}{4}$

$= \ln 2 + 3\ln 2$

$= 4\ln 2$

AS REQUIRED

Question 98 (****)

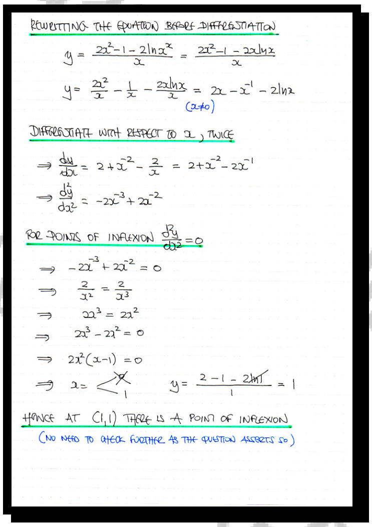
The curve C has equation

$$y = \frac{2x^2 - 1 - 2\ln x^x}{x}, \quad x > 0.$$

The curve has a point of inflection at P.

Show that the straight line with equation $y = x$ is a tangent to C at P.

$\boxed{\quad\quad}$, $\boxed{\text{proof}}$



REWRITTING THE EQUATION BEFORE DIFFERENTIATION

$$y = \frac{2x^2-1-2\ln x^x}{x} = \frac{2x^2-1-2x\ln x}{x}$$

$$y = \frac{2x^2}{x} - \frac{1}{x} - \frac{2x\ln x}{x} = 2x - x^{-1} - 2\ln x$$

$(x \neq 0)$

DIFFERENTIATE WITH RESPECT TO x, TWICE

$$\Rightarrow \frac{dy}{dx} = 2 + x^{-2} - \frac{2}{x} = 2 + x^{-2} - 2x^{-1}$$

$$\Rightarrow \frac{d^2y}{dx^2} = -2x^{-3} + 2x^{-2}$$

FOR POINTS OF INFLEXION $\frac{d^2y}{dx^2} = 0$

$$\Rightarrow -2x^{-3} + 2x^{-2} = 0$$

$$\Rightarrow \frac{2}{x^2} = \frac{2}{x^3}$$

$$\Rightarrow 2x^3 = 2x^2$$

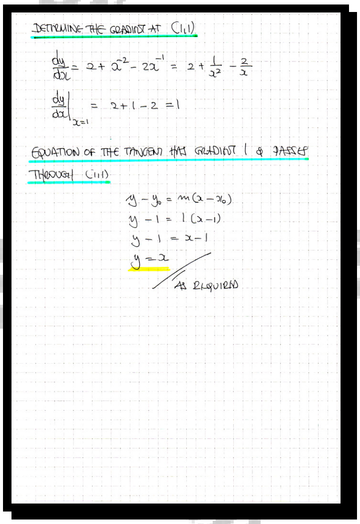
$$\Rightarrow 2x^3 - 2x^2 = 0$$

$$\Rightarrow 2x^2(x-1) = 0$$

$$\Rightarrow x = \begin{cases}\cancel{0} \\ 1\end{cases} \qquad y = \frac{2 - 1 - 2\cancel{\ln 1}}{1} = 1$$

HENCE AT $(1,1)$ THERE IS A POINT OF INFLEXION

(NO NEED TO CHECK FURTHER AS THE QUESTION ASSERTS SO)



DETERMINE THE GRADIENT AT $(1,1)$

$$\frac{dy}{dx} = 2 + x^{-2} - 2x^{-1} = 2 + \frac{1}{x^2} - \frac{2}{x}$$

$$\left.\frac{dy}{dx}\right|_{x=1} = 2 + 1 - 2 = 1$$

EQUATION OF THE TANGENT HAS GRADIENT 1 & PASSES THROUGH $(1,1)$

$$y - y_0 = m(x - x_0)$$

$$y - 1 = 1(x-1)$$

$$y - 1 = x - 1$$

$$y = x$$

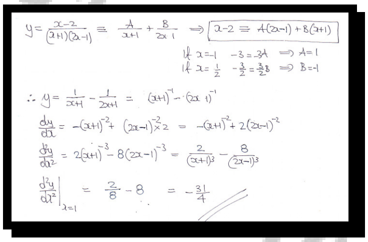
AS REQUIRED

Question 99 (****)

$$y=\frac{x-2}{(x+1)(2x-1)},\ x\neq -1,\ x\neq \frac{1}{2}.$$

Find the value of $\frac{d^2y}{dx^2}$ at $x=1$.

$$\boxed{\left.\frac{d^2y}{dx^2}\right|_{x=1}=-\frac{31}{4}}$$



$$y=\frac{x-2}{(x+1)(2x-1)} \equiv \frac{A}{x+1}+\frac{B}{2x-1} \Rightarrow \boxed{x-2 \equiv A(2x-1)+B(x+1)}$$

If $x=-1$ $\quad -3=-3A \Rightarrow A=1$

If $x=\frac{1}{2}$ $\quad -\frac{3}{2}=\frac{3}{2}B \Rightarrow B=-1$

$$\therefore y=\frac{1}{x+1}-\frac{1}{2x+1}=(x+1)^{-1}-(2x-1)^{-1}$$

$$\frac{dy}{dx}=-(x+1)^{-2}+(2x-1)^{-2}\times 2=-(x+1)^{-2}+2(2x-1)^{-2}$$

$$\frac{d^2y}{dx^2}=2(x+1)^{-3}-8(2x-1)^{-3}=\frac{2}{(x+1)^3}-\frac{8}{(2x-1)^3}$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=1}=\frac{2}{8}-8=-\frac{31}{4}$$

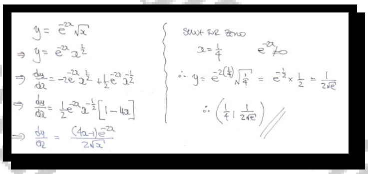
Question 100 (****)

The equation of a curve C is

$$y=e^{-2x}\sqrt{x},\ x\geq 0\ .$$

Determine the exact coordinates of the stationary point of C.

$$\boxed{\left(\frac{1}{4},\frac{1}{2\sqrt{e}}\right)}$$



$$y=e^{-2x}\sqrt{x}$$

$$\Rightarrow y=e^{-2x}x^{\frac{1}{2}}$$

$$\Rightarrow \frac{dy}{dx}=-2e^{-2x}x^{\frac{1}{2}}+\frac{1}{2}e^{-2x}x^{-\frac{1}{2}}$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{2}e^{-2x}x^{-\frac{1}{2}}\left[1-4x\right]$$

$$\Rightarrow \frac{dy}{dx}=\frac{(4x-1)e^{-2x}}{2\sqrt{x}}$$

SOLVE FOR ZERO

$x=\frac{1}{4}$ $\qquad e^{-2x}\neq 0$

$$\therefore y=e^{-2\left(\frac{1}{4}\right)}\sqrt{\frac{1}{4}}=e^{-\frac{1}{2}}\times\frac{1}{2}=\frac{1}{2\sqrt{e}}$$

$$\therefore \left(\frac{1}{4}, \frac{1}{2\sqrt{e}}\right)$$

Question 101 **(****)**

The curve C has equation

$$x = y\sqrt{1-4y}, \quad y \le \frac{1}{4}.$$

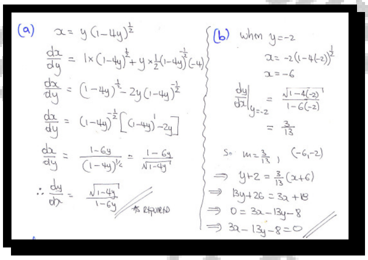
a) Show clearly that

$$\frac{dy}{dx} = \frac{\sqrt{1-4y}}{1-6y}.$$

b) Show further that an equation of the tangent to C at the point where $y = -2$ is

$$3x - 13y - 8 = 0.$$

$\boxed{\text{proof}}$



(a) $x = y(1-4y)^{\frac{1}{2}}$

$\frac{dx}{dy} = 1 \times (1-4y)^{\frac{1}{2}} + y \times \frac{1}{2}(1-4y)^{-\frac{1}{2}}(-4)$

$\frac{dx}{dy} = (1-4y)^{\frac{1}{2}} - 2y(1-4y)^{-\frac{1}{2}}$

$\frac{dx}{dy} = (1-4y)^{-\frac{1}{2}}\left[(1-4y)^{1} - 2y\right]$

$\frac{dx}{dy} = \frac{1-6y}{(1-4y)^{1/2}} = \frac{1-6y}{\sqrt{1-4y}}$

$\therefore \frac{dy}{dx} = \frac{\sqrt{1-4y}}{1-6y}$ // AS REQUIRED

(b) when $y=-2$

$x = -2(1-4(-2))^{\frac{1}{2}}$

$x = -6$

$\left.\frac{dy}{dx}\right|_{y=-2} = \frac{\sqrt{1-4(-2)}}{1-6(-2)}$

$= \frac{3}{13}$

So $m = \frac{3}{13}$, $(-6,-2)$

$\Rightarrow y+2 = \frac{3}{13}(x+6)$

$\Rightarrow 13y+26 = 3x+18$

$\Rightarrow 0 = 3x-13y-8$

$\Rightarrow 3x-13y-8=0$ //

Question 102 **(****)**

Differentiate each the following expressions with respect to x, simplifying the final answers as far as possible.

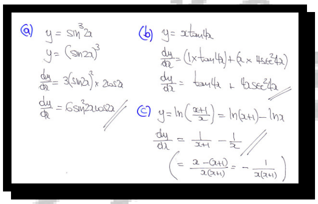
(Fractional answers must not involve double fractions)

a) $y = \sin^3 2x$.

b) $y = x\tan 4x$.

c) $y = \ln\left(\frac{x+1}{x}\right)$.

$\boxed{\frac{dy}{dx} = 6\sin^2 2x\cos 2x}$, $\boxed{\frac{dy}{dx} = \tan 4x + 4x\sec^2 4x}$, $\boxed{\frac{dy}{dx} = -\frac{1}{x^2+x}}$



(a) $y = \sin^3 2x$

$y = (\sin 2x)^3$

$\frac{dy}{dx} = 3(\sin 2x)^2 \times 2\cos 2x$

$\frac{dy}{dx} = 6\sin^2 2x\cos 2x$ //

(b) $y = x\tan 4x$

$\frac{dy}{dx} = (1 \times \tan 4x) + (x \times 4\sec^2 4x)$

$\frac{dy}{dx} = \tan 4x + 4x\sec^2 4x$ //

(c) $y = \ln\left(\frac{x+1}{x}\right) = \ln(x+1) - \ln x$

$\frac{dy}{dx} = \frac{1}{x+1} - \frac{1}{x}$ //

$\left(= \frac{x-(x+1)}{x(x+1)} = -\frac{1}{x(x+1)} \right)$

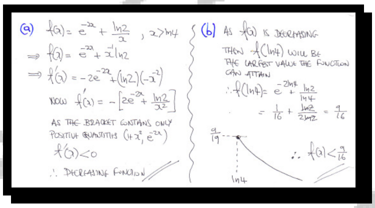
Question 103 **(****)**

$$f(x) = e^{-2x} + \frac{\ln 2}{x}, \quad x \in \mathbb{R}, \quad x > \ln 4.$$

a) Show that $f(x)$ in a decreasing function.

b) Hence find the range of $f(x)$.

$\boxed{\quad}$, $\boxed{f(x) \in \mathbb{R},\ f(x) < \frac{9}{16}}$



(a) $f(x) = e^{-2x} + \frac{\ln 2}{x}$, $x > \ln 4$

$\Rightarrow f(x) = e^{-2x} + x^{-1}\ln 2$

$\Rightarrow f'(x) = -2e^{-2x} + (\ln 2)(-x^{-2})$

NOW $f'(x) = -\left[2e^{-2x} + \frac{\ln 2}{x^2}\right]$

AS THE BRACKET CONTAINS ONLY POSITIVE QUANTITIES (x^2, e^{-2x})

$f'(x) < 0$

$\therefore$ DECREASING FUNCTION //

(b) AS $f(x)$ IS DECREASING THEN $f(\ln 4)$ WILL BE THE LARGEST VALUE THE FUNCTION CAN ATTAIN

$\therefore f(\ln 4) = e^{-2\ln 4} + \frac{\ln 2}{\ln 4}$

$= \frac{1}{16} + \frac{\ln 2}{2\ln 2} = \frac{9}{16}$

$\frac{9}{19}$

$\ln 4$

$\therefore f(x) < \frac{9}{16}$ //

Question 104 **(****)**

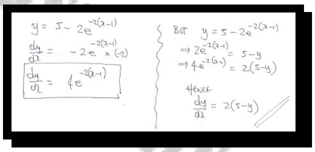
Given that

$$y = 5 - 2e^{-2(x-1)}$$

show clearly that

$$\frac{dy}{dx} = 2(5-y).$$

$\boxed{\text{proof}}$



$y = 5 - 2e^{-2(x-1)}$

$\frac{dy}{dx} = -2e^{-2(x-1)} \times (-2)$

$\boxed{\frac{dy}{dx} = 4e^{-2(x-1)}}$

BUT $y = 5 - 2e^{-2(x-1)}$

$\Rightarrow 2e^{-2(x-1)} = 5 - y$

$\Rightarrow 4e^{-2(x-1)} = 2(5-y)$

HENCE

$\frac{dy}{dx} = 2(5-y)$ //

Question 105 (****)

$$f(x) = x\sqrt{9-4x^2}\ ,\ 0 \le x \le \frac{3}{2}\ .$$

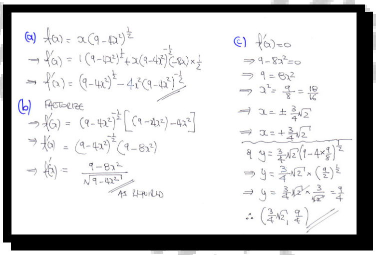
a) Find an expression for $f'(x)$.

b) Show further that

$$f'(x) = \frac{9-8x^2}{\sqrt{9-4x^2}}\ .$$

c) Hence calculate that the exact coordinates of the stationary point of C.

$$\boxed{f'(x) = \left(9-4x^2\right)^{\frac{1}{2}} - 4x^2\left(9-4x^2\right)^{-\frac{1}{2}}}\ ,\ \boxed{\left(\frac{3\sqrt{2}}{4}, \frac{9}{4}\right)}$$



(a) $f(x) = x(9-4x^2)^{\frac{1}{2}}$

$\Rightarrow f'(x) = 1(9-4x^2)^{\frac{1}{2}} + x(9-4x^2)^{-\frac{1}{2}}(-8x) \times \frac{1}{2}$

$\Rightarrow f'(x) = (9-4x^2)^{\frac{1}{2}} - 4x^2(9-4x^2)^{-\frac{1}{2}}$

(b) FACTORIZE

$\Rightarrow f'(x) = (9-4x^2)^{-\frac{1}{2}}\left[(9-4x^2) - 4x^2\right]$

$\Rightarrow f'(x) = (9-4x^2)^{-\frac{1}{2}}(9-8x^2)$

$\Rightarrow f'(x) = \frac{9-8x^2}{\sqrt{9-4x^2}}$

AS REQUIRED

(c) $f'(x) = 0$

$\Rightarrow 9 - 8x^2 = 0$

$\Rightarrow 9 = 8x^2$

$\Rightarrow x^2 = \frac{9}{8} = \frac{18}{16}$

$\Rightarrow x = \pm\frac{3}{4}\sqrt{2}$

$\Rightarrow x = +\frac{3}{4}\sqrt{2}$

& $y = \frac{3}{4}\sqrt{2}\left(9 - 4 \times \frac{9}{8}\right)^{\frac{1}{2}}$

$\Rightarrow y = \frac{3}{4}\sqrt{2} \times \left(\frac{9}{2}\right)^{\frac{1}{2}}$

$\Rightarrow y = \frac{3}{4}\cancel{\sqrt{2}} \times \frac{3}{\cancel{\sqrt{2}}} = \frac{9}{4}$

$\therefore \left(\frac{3}{4}\sqrt{2}, \frac{9}{4}\right)$

Question 106 (****)

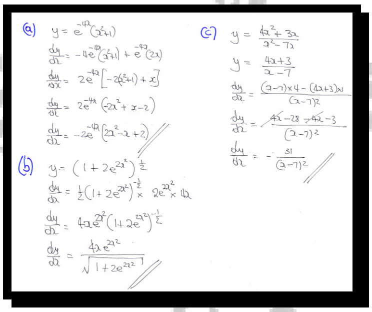
Differentiate each of the following expressions with respect to x, simplifying the answers as far as possible.

a) $y = \mathrm{e}^{-4x}\left(x^2+1\right)$.

b) $y = \sqrt{1+2\mathrm{e}^{2x^2}}$.

c) $y = \dfrac{4x^2+3x}{x^2-7x}$.

$$\boxed{\frac{dy}{dx} = -2\mathrm{e}^{-4x}\left(2x^2-x+2\right)}, \quad \boxed{\frac{dy}{dx} = \frac{4x\mathrm{e}^{2x^2}}{\sqrt{1+2\mathrm{e}^{2x^2}}}}, \quad \boxed{\frac{dy}{dx} = -\frac{31}{(x-7)^2}}$$



(a) $y = e^{-4x}(x^2+1)$

$\frac{dy}{dx} = -4e^{-4x}(x^2+1) + e^{-4x}(2x)$

$\frac{dy}{dx} = 2e^{-4x}\left[-2(x^2+1) + x\right]$

$\frac{dy}{dx} = 2e^{-4x}\left(-2x^2 + x - 2\right)$

$\frac{dy}{dx} = -2e^{-4x}\left(2x^2 - x + 2\right)$ //

(b) $y = \left(1 + 2e^{2x^2}\right)^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2}\left(1+2e^{2x^2}\right)^{-\frac{1}{2}} \times 2e^{2x^2} \times 4x$

$\frac{dy}{dx} = 4xe^{2x^2}\left(1+2e^{2x^2}\right)^{-\frac{1}{2}}$

$\frac{dy}{dx} = \frac{4xe^{2x^2}}{\sqrt{1+2e^{2x^2}}}$ //

(c) $y = \frac{4x^2+3x}{x^2-7x}$

$y = \frac{4x+3}{x-7}$

$\frac{dy}{dx} = \frac{(x-7)\times 4 - (4x+3)\times 1}{(x-7)^2}$

$\frac{dy}{dx} = \frac{\cancel{4x} - 28 - \cancel{4x} - 3}{(x-7)^2}$

$\frac{dy}{dx} = -\frac{31}{(x-7)^2}$ //

Question 107 **(****)**

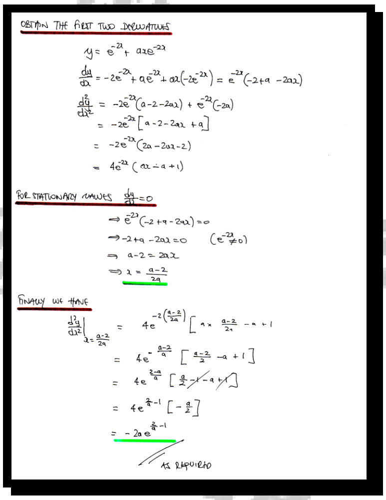
A curve has equation

$$y = e^{-2x} + axe^{-2x} ,$$

where a is a non zero constant.

Show that the value of $\frac{d^2y}{dx^2}$ at the stationary point of the curve is $-2ae^{\frac{2}{a}-1}$.

$\boxed{\quad}$, $\boxed{\text{proof}}$



OBTAIN THE FIRST TWO DERIVATIVES

$$y = e^{-2x} + axe^{-2x}$$

$$\frac{dy}{dx} = -2e^{-2x} + ae^{-2x} + ax(-2e^{-2x}) = e^{-2x}(-2+a-2ax)$$

$$\frac{d^2y}{dx^2} = -2e^{-2x}(a-2-2ax) + e^{-2x}(-2a)$$

$$= -2e^{-2x}[a-2-2ax+a]$$

$$= -2e^{-2x}(2a-2ax-2)$$

$$= 4e^{-2x}(ax-a+1)$$

FOR STATIONARY VALUES $\frac{dy}{dx}=0$

$$\Rightarrow e^{-2x}(-2+a-2ax)=0$$

$$\Rightarrow -2+a-2ax=0 \qquad (e^{-2x} \neq 0)$$

$$\Rightarrow a-2=2ax$$

$$\Rightarrow x = \frac{a-2}{2a}$$

FINALLY WE HAVE

$$\left.\frac{d^2y}{dx^2}\right|_{x=\frac{a-2}{2a}} = 4e^{-2\left(\frac{a-2}{2a}\right)}\left[a \times \frac{a-2}{2a} - a + 1\right.$$

$$= 4e^{-\frac{a-2}{a}}\left[\frac{a-2}{2} - a + 1\right]$$

$$= 4e^{\frac{2-a}{a}}\left[\frac{a}{2}\cancel{-1} - a \cancel{+1}\right]$$

$$= 4e^{\frac{2}{a}-1}\left[-\frac{a}{2}\right]$$

$$= -2ae^{\frac{2}{a}-1}$$

AS REQUIRED

Question 108 (****)

The curve C has equation given by

$$y = \frac{x}{\sqrt{x-2}}, \quad x \geq 2.$$

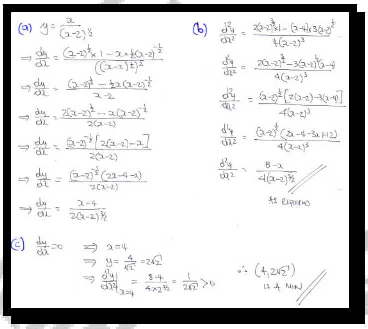
a) Show clearly that …

i. … $\dfrac{dy}{dx} = \dfrac{x-4}{2(x-2)^{\frac{3}{2}}}$.

ii. … $\dfrac{d^2y}{dx^2} = \dfrac{8-x}{4(x-2)^{\frac{5}{2}}}$.

b) Hence find the exact coordinates of the stationary point of C, and determine its nature.

$\boxed{\text{min at } \left(4, 2\sqrt{2}\right)}$



(a) $y = \frac{x}{(x-2)^{1/2}}$

$\Rightarrow \frac{dy}{dx} = \frac{(x-2)^{\frac{1}{2}} \times 1 - x \times \frac{1}{2}(x-2)^{-\frac{1}{2}}}{\left((x-2)^{\frac{1}{2}}\right)^2}$

$\Rightarrow \frac{dy}{dx} = \frac{(x-2)^{\frac{1}{2}} - \frac{1}{2}x(x-2)^{-\frac{1}{2}}}{x-2}$

$\Rightarrow \frac{dy}{dx} = \frac{2(x-2)^{\frac{1}{2}} - x(x-2)^{-\frac{1}{2}}}{2(x-2)}$

$\Rightarrow \frac{dy}{dx} = \frac{(x-2)^{-\frac{1}{2}}\left[2(x-2)-x\right]}{2(x-2)}$

$\Rightarrow \frac{dy}{dx} = \frac{(x-2)^{-\frac{1}{2}}(2x-4-x)}{2(x-2)}$

$\Rightarrow \frac{dy}{dx} = \frac{x-4}{2(x-2)^{3/2}}$

(b) $\frac{d^2y}{dx^2} = \frac{2(x-2)^{\frac{3}{2}} \times 1 - (x-4) \times 3(x-2)^{\frac{1}{2}}}{4(x-2)^3}$

$\frac{d^2y}{dx^2} = \frac{2(x-2)^{\frac{3}{2}} - 3(x-2)^{\frac{1}{2}}(x-4)}{4(x-2)^3}$

$\frac{d^2y}{dx^2} = \frac{(x-2)^{\frac{1}{2}}\left[2(x-2)-3(x-4)\right]}{4(x-2)^3}$

$\frac{d^2y}{dx^2} = \frac{(x-2)^{\frac{1}{2}}(2x-4-3x+12)}{4(x-2)^3}$

$\frac{d^2y}{dx^2} = \frac{8-x}{4(x-2)^{5/2}}$ //

AS REQUIRED

(c) $\frac{dy}{dx} = 0 \Rightarrow x = 4$

$\Rightarrow y = \frac{4}{\sqrt{2}} = 2\sqrt{2}$

$\Rightarrow \left.\frac{d^2y}{dx^2}\right|_{x=4} = \frac{8-4}{4 \times 2^{5/2}} = \frac{1}{2\sqrt{2}} > 0$

$\therefore (4, 2\sqrt{2})$ IS A MIN //

Question 109 (****)

The curve C has equation

$$y = \frac{2\ln x - 1}{2\ln x + 1}, \quad x > 0.$$

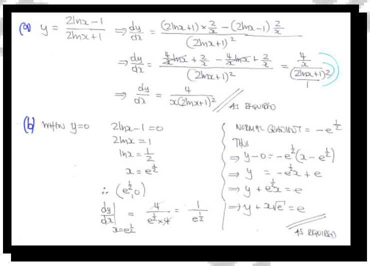
a) Show clearly that

$$\frac{dy}{dx} = \frac{4}{x\left(2\ln x + 1\right)^2}.$$

b) Show that the equation of the normal at the point where the curve crosses the x axis is given by

$$y + x\sqrt{\mathrm{e}} = \mathrm{e}.$$

proof



(a) $y = \frac{2\ln x - 1}{2\ln x + 1} \Rightarrow \frac{dy}{dx} = \frac{(2\ln x + 1)\times\frac{2}{x} - (2\ln x - 1)\frac{2}{x}}{(2\ln x + 1)^2}$

$\Rightarrow \frac{dy}{dx} = \frac{\cancel{\frac{4}{x}\ln x} + \frac{2}{x} - \cancel{\frac{4}{x}\ln x} + \frac{2}{x}}{(2\ln x + 1)^2} = \frac{\frac{4}{x}}{\frac{(2\ln x + 1)^2}{1}}$

$\Rightarrow \frac{dy}{dx} = \frac{4}{x(2\ln x + 1)^2}$ AS REQUIRED

(b) WHEN $y = 0$

$2\ln x - 1 = 0$

$2\ln x = 1$

$\ln x = \frac{1}{2}$

$x = e^{\frac{1}{2}}$

$\therefore (e^{\frac{1}{2}}, 0)$

$\left.\frac{dy}{dx}\right|_{x = e^{\frac{1}{2}}} = \frac{4}{e^{\frac{1}{2}} \times 4} = \frac{1}{e^{\frac{1}{2}}}$

NORMAL GRADIENT $= -e^{\frac{1}{2}}$

THUS

$\Rightarrow y - 0 = -e^{\frac{1}{2}}(x - e^{\frac{1}{2}})$

$\Rightarrow y = -e^{\frac{1}{2}}x + e$

$\Rightarrow y + e^{\frac{1}{2}}x = e$

$\Rightarrow y + x\sqrt{e} = e$

AS REQUIRED

Question 110 (****)

$$f(x) = \frac{2x}{\sqrt{1-x}}, \quad x < 1 .$$

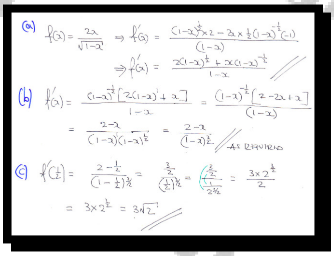
a) Find an expression for $f'(x)$.

b) Show further that

$$f'(x) = \frac{2-x}{(1-x)^{\frac{3}{2}}} .$$

c) Hence calculate that the exact value of $f'\left(\frac{1}{2}\right)$.

$$\boxed{f'(x) = \frac{2(1-x)^{\frac{1}{2}} + x(1-x)^{-\frac{1}{2}}}{1-x}}, \quad \boxed{f'\left(\frac{1}{2}\right) = 3\sqrt{2}}$$



(a) $f(x) = \frac{2x}{\sqrt{1-x}} \Rightarrow f'(x) = \frac{(1-x)^{\frac{1}{2}} \times 2 - 2x \times \frac{1}{2}(1-x)^{-\frac{1}{2}}(-1)}{(1-x)}$

$\Rightarrow f'(x) = \frac{2(1-x)^{\frac{1}{2}} + x(1-x)^{-\frac{1}{2}}}{1-x}$ //

(b) $f'(x) = \frac{(1-x)^{-\frac{1}{2}}\left[2(1-x)^{1} + x\right]}{1-x} = \frac{(1-x)^{-\frac{1}{2}}\left[2-2x+x\right]}{(1-x)}$

$= \frac{2-x}{(1-x)^{1}(1-x)^{\frac{1}{2}}} = \frac{2-x}{(1-x)^{\frac{3}{2}}}$ // AS REQUIRED

(c) $f'\left(\frac{1}{2}\right) = \frac{2-\frac{1}{2}}{\left(1-\frac{1}{2}\right)^{3/2}} = \frac{\frac{3}{2}}{\left(\frac{1}{2}\right)^{3/2}} = \frac{\frac{3}{2}}{\frac{1}{2^{3/2}}} = \frac{3 \times 2^{\frac{3}{2}}}{2}$

$= 3 \times 2^{\frac{1}{2}} = 3\sqrt{2}$ //

Question 111 (****)

A curve C has equation

$$y = x - 2\ln\left(x^2+4\right),\ x \in \mathbb{R}.$$

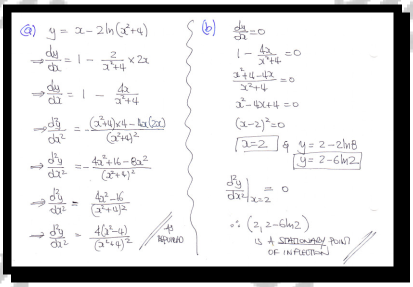
a) Show clearly that

$$\frac{d^2y}{dx^2} = \frac{4\left(x^2-4\right)}{\left(x^2+4\right)^2}.$$

The curve has a single stationary point.

b) Find its exact coordinates and determine its nature.

$\boxed{\quad}$, $\boxed{\text{point of inflection at} \left(2, 2-6\ln 2\right)}$



(a) $y = x - 2\ln(x^2+4)$

$\Rightarrow \frac{dy}{dx} = 1 - \frac{2}{x^2+4} \times 2x$

$\Rightarrow \frac{dy}{dx} = 1 - \frac{4x}{x^2+4}$

$\Rightarrow \frac{d^2y}{dx^2} = -\frac{(x^2+4)\times 4 - 4x(2x)}{(x^2+4)^2}$

$\Rightarrow \frac{d^2y}{dx^2} = -\frac{4x^2+16-8x^2}{(x^2+4)^2}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{4x^2-16}{(x^2+4)^2}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{4(x^2-4)}{(x^2+4)^2}$ // AS REQUIRED

(b) $\frac{dy}{dx} = 0$

$1 - \frac{4x}{x^2+4} = 0$

$\frac{x^2+4-4x}{x^2+4} = 0$

$x^2 - 4x + 4 = 0$

$(x-2)^2 = 0$

$\boxed{x=2}$ & $y = 2 - 2\ln 8$

$\boxed{y = 2 - 6\ln 2}$

$\left.\frac{d^2y}{dx^2}\right|_{x=2} = 0$

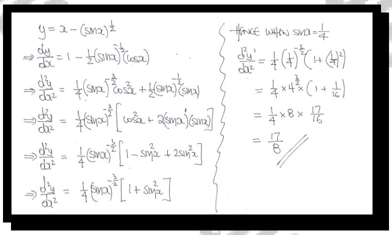
$\therefore (2, 2-6\ln 2)$ IS A STATIONARY POINT OF INFLECTION //

Question 112 **(****)**

$$y = x - \sqrt{\sin x}\,,\ \ 0 < x < \frac{\pi}{2}.$$

Show clearly that when $\sin x = \frac{1}{4}$, the value of $\frac{d^2y}{dx^2}$ is $\frac{17}{8}$.

proof



$y = x - (\sin x)^{\frac{1}{2}}$

$\Rightarrow \frac{dy}{dx} = 1 - \frac{1}{2}(\sin x)^{-\frac{1}{2}}(\cos x)$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{1}{4}(\sin x)^{-\frac{3}{2}}\cos^2 x + \frac{1}{2}(\sin x)^{-\frac{1}{2}}(\sin x)$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{1}{4}(\sin x)^{-\frac{3}{2}}\left[\cos^2 x + 2(\sin x)^{1}(\sin x)\right]$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{1}{4}(\sin x)^{-\frac{3}{2}}\left[1 - \sin^2 x + 2\sin^2 x\right]$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{1}{4}(\sin x)^{-\frac{3}{2}}\left[1 + \sin^2 x\right]$

Hence when $\sin x = \frac{1}{4}$

$\frac{d^2y}{dx^2} = \frac{1}{4}\left(\frac{1}{4}\right)^{-\frac{3}{2}}\left(1 + \left(\frac{1}{4}\right)^2\right)$

$= \frac{1}{4} \times 4^{\frac{3}{2}} \times \left(1 + \frac{1}{16}\right)$

$= \frac{1}{4} \times 8 \times \frac{17}{16}$

$= \frac{17}{8}$

Question 113 (****)

$$f(x) = e^{3x} - 4e^{-3x}, \ x \in \mathbb{R} .$$

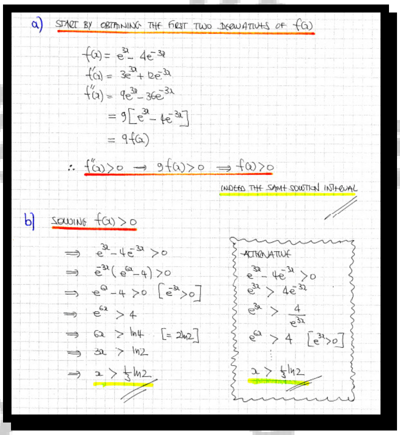
a) Show that the inequalities

$$f(x) > 0 \text{ and } f''(x) > 0$$

have the same solution interval.

b) Determine, in exact form, the common solution interval.

$\boxed{\text{SYN}}$, $\boxed{x > \frac{1}{3}\ln 2}$



a) START BY OBTAINING THE FIRST TWO DERIVATIVES OF $f(x)$

$f(x) = e^{3x} - 4e^{-3x}$

$f'(x) = 3e^{3x} + 12e^{-3x}$

$f''(x) = 9e^{3x} - 36e^{-3x}$

$= 9\left[e^{3x} - 4e^{-3x}\right]$

$= 9f(x)$

$\therefore f''(x) > 0 \rightarrow 9f(x) > 0 \Rightarrow f(x) > 0$

INDEED THE SAME SOLUTION INTERVAL

b) SOLVING $f(x) > 0$

$\Rightarrow e^{3x} - 4e^{-3x} > 0$

$\Rightarrow e^{-3x}(e^{6x} - 4) > 0$

$\Rightarrow e^{6x} - 4 > 0 \quad [e^{-3x} > 0]$

$\Rightarrow e^{6x} > 4$

$\Rightarrow 6x > \ln 4 \quad [= 2\ln 2]$

$\Rightarrow 3x > \ln 2$

$\Rightarrow x > \frac{1}{3}\ln 2$

ALTERNATIVE

$e^{3x} - 4e^{-3x} > 0$

$e^{3x} > 4e^{-3x}$

$e^{3x} > \frac{4}{e^{3x}}$

$e^{6x} > 4 \quad [e^{3x} > 0]$

$\vdots$

$x > \frac{1}{3}\ln 2$

Question 114 (****)

The equation of the curve C is

$$y = (x+2)^2 e^{1-x}, \; x \in \mathbb{R}.$$

a) Show clearly that …

i. … $\dfrac{dy}{dx} = -x(x+2)e^{1-x}$.

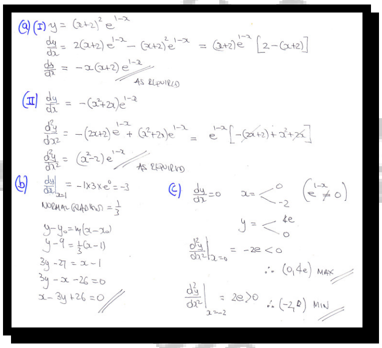
ii. … $\dfrac{d^2y}{dx^2} = (x^2-2)e^{1-x}$.

b) Find an equation of the normal to C at the point $(1,9)$.

The curve has two stationary points at P and Q.

c) Find the exact coordinates of P and Q, further determining the nature of these stationary points.

$\boxed{x-3y+26=0}$, $\boxed{\text{max at } (0,4e)}$, $\boxed{\text{min at } (-2,0)}$



(a) (I) $y = (x+2)^2 e^{1-x}$

$\frac{dy}{dx} = 2(x+2)e^{1-x} - (x+2)^2 e^{1-x} = (x+2)e^{1-x}\left[2-(x+2)\right]$

$\frac{dy}{dx} = -x(x+2)e^{1-x}$ // AS REQUIRED

(II) $\frac{dy}{dx} = -(x^2+2x)e^{1-x}$

$\frac{d^2y}{dx^2} = -(2x+2)e^{1-x} + (x^2+2x)e^{1-x} = e^{1-x}\left[-(\cancel{2x}+2)+x^2+\cancel{2x}\right]$

$\frac{d^2y}{dx^2} = (x^2-2)e^{1-x}$ // AS REQUIRED

(b) $\left.\frac{dy}{dx}\right|_{x=1} = -1\times3\times e^0 = -3$

NORMAL GRADIENT $= \frac{1}{3}$

$y - y_0 = m(x-x_0)$

$y - 9 = \frac{1}{3}(x-1)$

$3y - 27 = x - 1$

$3y - x - 26 = 0$

$x - 3y + 26 = 0$ //

(c) $\frac{dy}{dx} = 0 \quad x = \begin{cases} 0 \\ -2 \end{cases} \quad (e^{1-x} \neq 0)$

$y = \begin{cases} 4e \\ 0 \end{cases}$

$\left.\frac{d^2y}{dx^2}\right|_{x=0} = -2e < 0$

$\therefore (0,4e)$ MAX //

$\left.\frac{d^2y}{dx^2}\right|_{x=-2} = 2e > 0 \quad \therefore (-2,0)$ MIN //

Question 115 (****)

The equation of the curve C is given by

$$y = x\sqrt{x^3+1}\,,\ x \geq -1.$$

a) Find an expression for $\frac{dy}{dx}$.

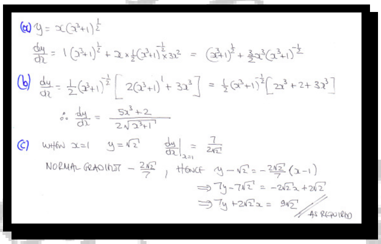
b) Show further that $\frac{dy}{dx}$ can be simplified to

$$\frac{dy}{dx} = \frac{5x^3+2}{2\sqrt{x^3+1}}.$$

c) Hence show that an equation of the normal to C at the point where $x=1$ is

$$7y + 2x\sqrt{2} = 9\sqrt{2}.$$

$$\boxed{\frac{dy}{dx} = \left(x^3+1\right)^{\frac{1}{2}} + \frac{3}{2}x^3\left(x^3+1\right)^{-\frac{1}{2}}}$$



(a) $y = x(x^3+1)^{\frac{1}{2}}$

$\frac{dy}{dx} = 1(x^3+1)^{\frac{1}{2}} + x \times \frac{1}{2}(x^3+1)^{-\frac{1}{2}} \times 3x^2 = (x^3+1)^{\frac{1}{2}} + \frac{3}{2}x^3(x^3+1)^{-\frac{1}{2}}$

(b) $\frac{dy}{dx} = \frac{1}{2}(x^3+1)^{-\frac{1}{2}}\left[2(x^3+1)^1 + 3x^3\right] = \frac{1}{2}(x^3+1)^{-\frac{1}{2}}\left[2x^3+2+3x^3\right]$

$\therefore \frac{dy}{dx} = \frac{5x^3+2}{2\sqrt{x^3+1}}$

(c) WHEN $x=1$ $y=\sqrt{2}$ $\left.\frac{dy}{dx}\right|_{x=1} = \frac{7}{2\sqrt{2}}$

NORMAL GRADIENT $-\frac{2\sqrt{2}}{7}$, HENCE $y-\sqrt{2} = -\frac{2\sqrt{2}}{7}(x-1)$

$\Rightarrow 7y - 7\sqrt{2} = -2\sqrt{2}x + 2\sqrt{2}$

$\Rightarrow 7y + 2\sqrt{2}x = 9\sqrt{2}$ // AS REQUIRED

Question 116 (****)

It is given that

$$\frac{d}{dx}(\sec x) = \sec x \tan x .$$

a) Prove the validity of the above result by writing $\sec x$ as $\dfrac{1}{\cos x}$.

The curve C has equation

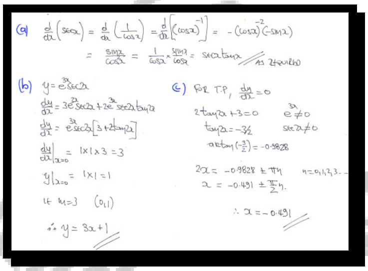
$$y = e^{3x} \sec 2x , \quad -\frac{\pi}{4} < x < \frac{\pi}{4} .$$

b) Find an equation of the tangent to C at the point where C crosses the y axis.

The curve has a single stationary point at P.

c) Find the x coordinate of P, correct to 3 significant figures.

$\boxed{y = 3x+1}$, $\boxed{x \approx -0.491}$



(a) $\frac{d}{dx}(\sec x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \frac{d}{dx}\left[(\cos x)^{-1}\right] = -(\cos x)^{-2}(-\sin x)$

$= \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x} \times \frac{\sin x}{\cos x} = \sec x \tan x$ // AS REQUIRED

(b) $y = e^{3x}\sec 2x$

$\frac{dy}{dx} = 3e^{2x}\sec 2x + 2e^{3x}\sec 2x \tan 2x$

$\frac{dy}{dx} = e^{3x}\sec 2x\left[3 + 2\tan 2x\right]$

$\left.\frac{dy}{dx}\right|_{x=0} = 1 \times 1 \times 3 = 3$

$\left.y\right|_{x=0} = 1 \times 1 = 1$

if $m = 3$ $(0,1)$

$\therefore y = 3x + 1$ //

(c) FOR T.P, $\frac{dy}{dx} = 0$

$2\tan 2x + 3 = 0$ $\quad e^{3x} \neq 0$

$\tan 2x = -3/2$ $\quad \sec 2x \neq 0$

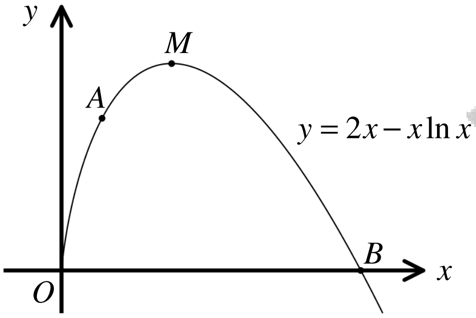
$\arctan\left(-\frac{3}{2}\right) = -0.9828$

$2x = -0.9828 \pm \pi n$ $\quad n = 0,1,2,3\ldots$

$x = -0.491 \pm \frac{\pi}{2}n$

$\therefore x = -0.491$ //

Question 117 **(****)**



The diagram above shows the graph of a curve C with equation

$$y = 2x - x\ln x, \quad x > 0.$$

The curve has a maximum at M.

a) Find the exact coordinates of M.

The point A lies on C where $x = 1$.

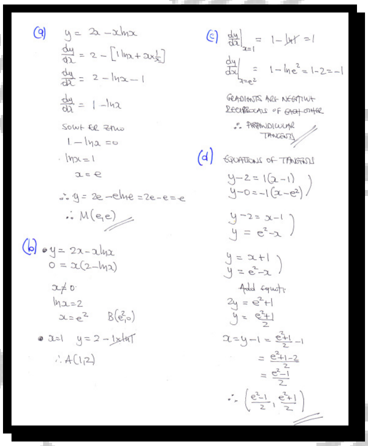
The curve crosses the x axis at the point B.

b) Determine the coordinates of A and the exact coordinates B.

c) Show that the tangents to the curve at A and B are perpendicular to each other.

d) Show that these tangents intersect at the point $Q\left(\frac{e^2-1}{2}, \frac{e^2+1}{2}\right)$.

$\boxed{M(e,e)}$, $\boxed{A(1,2),\ B\left(e^2,0\right)}$



(a) $y = 2x - x\ln x$

$\frac{dy}{dx} = 2 - [1\ln x + x \times \frac{1}{x}]$

$\frac{dy}{dx} = 2 - \ln x - 1$

$\frac{dy}{dx} = 1 - \ln x$

SOLVE FOR ZERO

$1 - \ln x = 0$

$\ln x = 1$

$x = e$

$\therefore y = 2e - e\ln e = 2e - e = e$

$\therefore M(e,e)$

(b) • $y = 2x - x\ln x$

$0 = x(2 - \ln x)$

$x \neq 0$

$\ln x = 2$

$x = e^2 \quad B(e^2,0)$

• $x=1 \quad y = 2 - \cancel{1 \times \ln 1}$

$\therefore A(1,2)$

(c) $\left.\frac{dy}{dx}\right|_{x=1} = 1 - \cancel{\ln 1} = 1$

$\left.\frac{dy}{dx}\right|_{x=e^2} = 1 - \ln e^2 = 1 - 2 = -1$

GRADIENTS ARE NEGATIVE RECIPROCALS OF EACH OTHER

$\therefore$ PERPENDICULAR TANGENTS

(d) EQUATIONS OF TANGENTS

$y - 2 = 1(x-1)$
$y - 0 = -1(x - e^2)$

$y - 2 = x - 1$
$y = e^2 - x$

$y = x + 1$
$y = e^2 - x$

Add equations

$2y = e^2 + 1$

$y = \frac{e^2+1}{2}$

$x = y - 1 = \frac{e^2+1}{2} - 1$

$= \frac{e^2+1-2}{2}$

$= \frac{e^2-1}{2}$

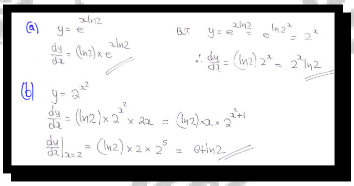
$\therefore \left(\frac{e^2-1}{2}, \frac{e^2+1}{2}\right)$

Question 118 (****)

a) By differentiating $y = e^{x \ln 2}$, find the derivative of $y = 2^x$.

b) Hence find the exact value of the gradient on the curve with equation $y = 2^{x^2}$ at the point where $x = 2$.

$$\boxed{\frac{d}{dx}\left(2^x\right) = 2^x \ln 2}, \ \boxed{64\ln 2}$$



(a) $y = e^{x\ln 2}$

$\frac{dy}{dx} = (\ln 2) \times e^{x \ln 2}$

BUT $y = e^{x\ln 2} = e^{\ln 2^x} = 2^x$

$\therefore \frac{dy}{dx} = (\ln 2) 2^x = 2^x \ln 2$

(b) $y = 2^{x^2}$

$\frac{dy}{dx} = (\ln 2) \times 2^{x^2} \times 2x = (\ln 2) \times x \times 2^{x^2+1}$

$\left.\frac{dy}{dx}\right|_{x=2} = (\ln 2) \times 2 \times 2^5 = 64 \ln 2$

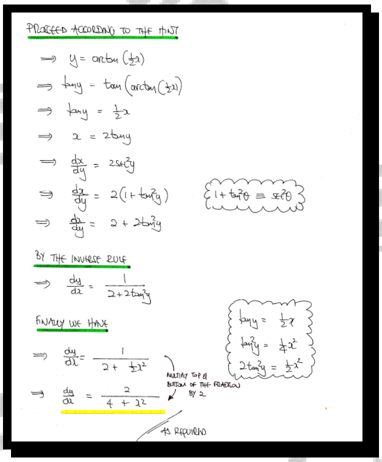
Question 119 (****)

$$y = \arctan\left(\tfrac{1}{2}x\right), \ x \in \mathbb{R}.$$

By writing $y = \arctan\left(\tfrac{1}{2}x\right)$ as $x = f(y)$, show that

$$\frac{dy}{dx} = \frac{2}{x^2 + 4}.$$

$$\boxed{\quad\quad}, \ \boxed{\text{proof}}$$



PROCEED ACCORDING TO THE HINT

$\Rightarrow y = \arctan\left(\tfrac{1}{2}x\right)$

$\Rightarrow \tan y = \tan\left(\arctan\left(\tfrac{1}{2}x\right)\right)$

$\Rightarrow \tan y = \tfrac{1}{2}x$

$\Rightarrow x = 2\tan y$

$\Rightarrow \frac{dx}{dy} = 2\sec^2 y$

$\Rightarrow \frac{dx}{dy} = 2(1 + \tan^2 y)$ $1 + \tan^2\theta \equiv \sec^2\theta$

$\Rightarrow \frac{dx}{dy} = 2 + 2\tan^2 y$

BY THE INVERSE RULE

$\Rightarrow \frac{dy}{dx} = \frac{1}{2 + 2\tan^2 y}$

FINALLY WE HAVE

$\tan y = \tfrac{1}{2}x$

$\tan^2 y = \tfrac{1}{4}x^2$

$2\tan^2 y = \tfrac{1}{2}x^2$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2 + \frac{1}{2}x^2}$

MULTIPLY TOP & BOTTOM OF THE FRACTION BY 2

$\Rightarrow \frac{dy}{dx} = \frac{2}{4 + x^2}$

AS REQUIRED

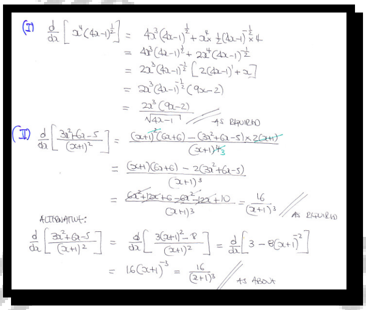
Question 120 **(****)**

Prove that …

i. … $\dfrac{d}{dx}\left(x^4\sqrt{4x-1}\right)=\dfrac{2x^3(9x-2)}{\sqrt{4x-1}}$.

ii. … $\dfrac{d}{dx}\left(\dfrac{3x^2+6x-5}{(x+1)^2}\right)=\dfrac{16}{(x+1)^3}$.

$\boxed{\text{proof}}$



(I) $\frac{d}{dx}\left[x^4(4x-1)^{\frac{1}{2}}\right] = 4x^3(4x-1)^{\frac{1}{2}} + x^4 \times \frac{1}{2}(4x-1)^{-\frac{1}{2}} \times 4$

$= 4x^3(4x-1)^{\frac{1}{2}} + 2x^4(4x-1)^{-\frac{1}{2}}$

$= 2x^3(4x-1)^{-\frac{1}{2}}\left[2(4x-1)^1 + x\right]$

$= 2x^3(4x-1)^{-\frac{1}{2}}(9x-2)$

$= \frac{2x^3(9x-2)}{\sqrt{4x-1}}$ // AS REQUIRED

(II) $\frac{d}{dx}\left[\frac{3x^2+6x-5}{(x+1)^2}\right] = \frac{(x+1)^{\cancel{2}}(6x+6) - (3x^2+6x-5)\times 2\cancel{(x+1)}}{(x+1)^{\cancel{4}3}}$

$= \frac{(x+1)(6x+6) - 2(3x^2+6x-5)}{(x+1)^3}$

$= \frac{\cancel{6x^2}+\cancel{12x}+6-\cancel{6x^2}-\cancel{12x}+10}{(x+1)^3} = \frac{16}{(x+1)^3}$ // AS REQUIRED

ALTERNATIVE:

$\frac{d}{dx}\left[\frac{3x^2+6x-5}{(x+1)^2}\right] = \frac{d}{dx}\left[\frac{3(x+1)^2-8}{(x+1)^2}\right] = \frac{d}{dx}\left[3-8(x+1)^{-2}\right]$

$= 16(x+1)^{-3} = \frac{16}{(x+1)^3}$ // AS ABOVE

Question 121 (****)

The curve C has equation

$$f(x) = e^{2x}\sin 2x, \quad 0 \le x \le \pi .$$

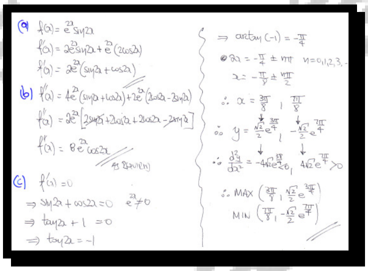
a) Find an expression for $f'(x)$.

b) Show clearly that

$$f''(x) = 8e^{2x}\cos 2x .$$

c) Hence find the exact coordinates of the stationary points of C and determine their nature.

$\boxed{f'(x) = 2e^{2x}(\sin 2x + \cos 2x)}$, $\boxed{\max\left(\frac{3}{8}\pi, \frac{\sqrt{2}}{2}e^{\frac{3\pi}{4}}\right)}$, $\boxed{\min\left(\frac{7}{8}\pi, -\frac{\sqrt{2}}{2}e^{\frac{7\pi}{4}}\right)}$



(a) $f(x) = e^{2x}\sin 2x$

$f'(x) = 2e^{2x}\sin 2x + e^{2x}(2\cos 2x)$

$f'(x) = 2e^{2x}(\sin 2x + \cos 2x)$

(b) $f''(x) = 4e^{2x}(\sin 2x + \cos 2x) + 2e^{2x}(2\cos 2x - 2\sin 2x)$

$f''(x) = 2e^{2x}[\cancel{2\sin 2x} + 2\cos 2x + 2\cos 2x - \cancel{2\sin 2x}]$

$f''(x) = 8e^{2x}\cos 2x$

AS REQUIRED

(c) $f'(x) = 0$

$\Rightarrow \sin 2x + \cos 2x = 0 \quad e^{2x} \neq 0$

$\Rightarrow \tan 2x + 1 = 0$

$\Rightarrow \tan 2x = -1$

$\Rightarrow \arctan(-1) = -\frac{\pi}{4}$

$2x = -\frac{\pi}{4} \pm n\pi \quad n = 0,1,2,3,\ldots$

$x = -\frac{\pi}{8} \pm \frac{n\pi}{2}$

$\therefore x = \frac{3\pi}{8}, \frac{7\pi}{8}$

$\therefore y = \frac{\sqrt{2}}{2}e^{\frac{3\pi}{4}}, -\frac{\sqrt{2}}{2}e^{\frac{7\pi}{4}}$

$\therefore \frac{d^2y}{dx^2} = -4\sqrt{2}e^{\frac{3\pi}{4}} < 0, \; 4\sqrt{2}e^{\frac{7\pi}{4}} > 0$

$\therefore$ MAX $\left(\frac{3\pi}{8}, \frac{\sqrt{2}}{2}e^{\frac{3\pi}{4}}\right)$

MIN $\left(\frac{7\pi}{8}, -\frac{\sqrt{2}}{2}e^{\frac{7\pi}{4}}\right)$

Question 122 (****)

The curve C has equation given by

$$y = x\sqrt{1-2x}\ ,\ x \le \frac{1}{2}\ .$$

a) Show clearly that

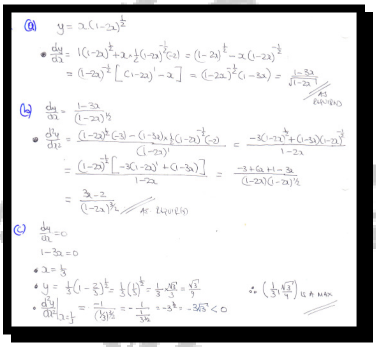
$$\frac{dy}{dx} = \frac{1-3x}{\sqrt{1-2x}}\ .$$

b) Show further that

$$\frac{d^2y}{dx^2} = \frac{3x-2}{(1-2x)^{\frac{3}{2}}}\ .$$

c) Hence find the exact coordinates of the stationary point of C, and determine its nature.

$$\boxed{\text{max at } \left(\frac{1}{3}, \frac{1}{9}\sqrt{3}\right)}$$



(a) $y = x(1-2x)^{\frac{1}{2}}$

- $\frac{dy}{dx} = 1(1-2x)^{\frac{1}{2}} + x \times \frac{1}{2}(1-2x)^{-\frac{1}{2}}(-2) = (1-2x)^{\frac{1}{2}} - x(1-2x)^{-\frac{1}{2}}$

$= (1-2x)^{-\frac{1}{2}}\left[(1-2x)^{1} - x\right] = (1-2x)^{-\frac{1}{2}}(1-3x) = \frac{1-3x}{\sqrt{1-2x}}$ // AS REQUIRED

(b) $\frac{dy}{dx} = \frac{1-3x}{(1-2x)^{1/2}}$

- $\frac{d^2y}{dx^2} = \frac{(1-2x)^{\frac{1}{2}}(-3) - (1-3x) \times \frac{1}{2}(1-2x)^{-\frac{1}{2}}(-2)}{(1-2x)^{1}} = \frac{-3(1-2x)^{\frac{1}{2}} + (1-3x)(1-2x)^{-\frac{1}{2}}}{1-2x}$

$= \frac{(1-2x)^{-\frac{1}{2}}\left[-3(1-2x)^{1} + (1-3x)\right]}{1-2x} = \frac{-3+6x+1-3x}{(1-2x)(1-2x)^{1/2}}$

$= \frac{3x-2}{(1-2x)^{3/2}}$ // AS REQUIRED

(c) $\frac{dy}{dx} = 0$

$1 - 3x = 0$

- $x = \frac{1}{3}$
- $y = \frac{1}{3}\left(1-\frac{2}{3}\right)^{\frac{1}{2}} = \frac{1}{3}\left(\frac{1}{3}\right)^{\frac{1}{2}} = \frac{1}{3} \times \frac{\sqrt{3}}{3} = \frac{\sqrt{3}}{9}$
- $\left.\frac{d^2y}{dx^2}\right|_{x=\frac{1}{3}} = \frac{-1}{(1/3)^{3/2}} = -\frac{1}{\frac{1}{3^{3/2}}} = -3^{\frac{3}{2}} = -3\sqrt{3} < 0$

$\therefore \left(\frac{1}{3}, \frac{\sqrt{3}}{4}\right)$ IS A MAX //

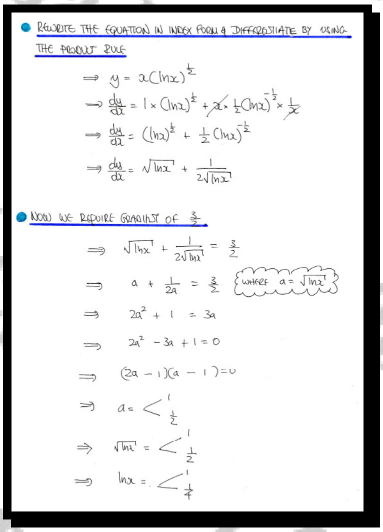
Question 123 (****)

A curve has equation

$$y = x\sqrt{\ln x}\,, \qquad x \in \mathbb{R}\,, \qquad x > 0\,.$$

Find the exact coordinates of the two points on the curve which have gradient $\frac{3}{2}$.

$\boxed{}$, $\boxed{(\mathrm{e},\mathrm{e}),\ \left(\mathrm{e}^{\frac{1}{4}},\frac{1}{2}\mathrm{e}^{\frac{1}{4}}\right)}$

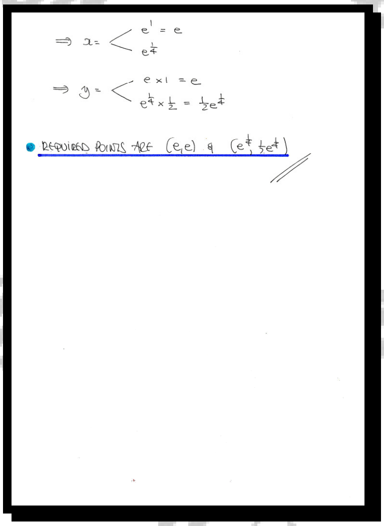


- REWRITE THE EQUATION IN INDEX FORM & DIFFERENTIATE BY USING THE PRODUCT RULE

$$\Rightarrow y = x(\ln x)^{\frac{1}{2}}$$
$$\Rightarrow \frac{dy}{dx} = 1 \times (\ln x)^{\frac{1}{2}} + \cancel{x} \times \tfrac{1}{2}(\ln x)^{-\frac{1}{2}} \times \frac{1}{\cancel{x}}$$
$$\Rightarrow \frac{dy}{dx} = (\ln x)^{\frac{1}{2}} + \tfrac{1}{2}(\ln x)^{-\frac{1}{2}}$$
$$\Rightarrow \frac{dy}{dx} = \sqrt{\ln x} + \frac{1}{2\sqrt{\ln x}}$$

- NOW WE REQUIRE GRADIENT OF $\frac{3}{2}$

$$\Rightarrow \sqrt{\ln x} + \frac{1}{2\sqrt{\ln x}} = \frac{3}{2}$$
$$\Rightarrow a + \frac{1}{2a} = \frac{3}{2} \qquad \text{WHERE } a = \sqrt{\ln x}$$
$$\Rightarrow 2a^2 + 1 = 3a$$
$$\Rightarrow 2a^2 - 3a + 1 = 0$$
$$\Rightarrow (2a-1)(a-1) = 0$$
$$\Rightarrow a = \left\langle \begin{matrix} 1 \\ \frac{1}{2} \end{matrix} \right.$$
$$\Rightarrow \sqrt{\ln x} = \left\langle \begin{matrix} 1 \\ \frac{1}{2} \end{matrix} \right.$$
$$\Rightarrow \ln x = \left\langle \begin{matrix} 1 \\ \frac{1}{4} \end{matrix} \right.$$



$$\Rightarrow x = \left\langle \begin{matrix} e^1 = e \\ e^{\frac{1}{4}} \end{matrix} \right.$$
$$\Rightarrow y = \left\langle \begin{matrix} e \times 1 = e \\ e^{\frac{1}{4}} \times \frac{1}{2} = \frac{1}{2}e^{\frac{1}{4}} \end{matrix} \right.$$

- REQUIRED POINTS ARE (e,e) & $\left(e^{\frac{1}{4}}, \frac{1}{2}e^{\frac{1}{4}}\right)$ //

Question 124 (****)

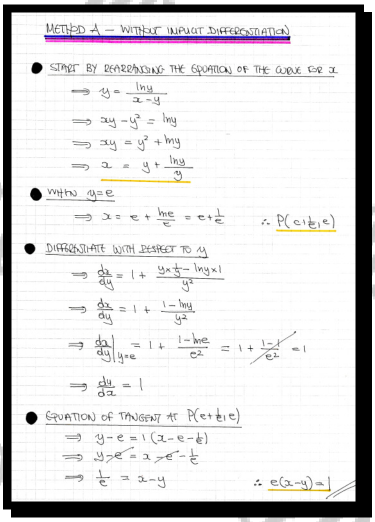
The curve C has equation

$$y = \frac{\ln y}{x - y}, \quad y > 0.$$

Show that the equation of the tangent to C at the point where $y = \mathrm{e}$ can be written as

$$\mathrm{e}(x - y) = 1.$$

$\boxed{}$, $\boxed{\text{proof}}$



METHOD A – WITHOUT IMPLICIT DIFFERENTIATION

- START BY REARRANGING THE EQUATION OF THE CURVE FOR x

$$\Rightarrow y = \frac{\ln y}{x-y}$$
$$\Rightarrow xy - y^2 = \ln y$$
$$\Rightarrow xy = y^2 + \ln y$$
$$\Rightarrow x = y + \frac{\ln y}{y}$$

- WHEN $y=e$

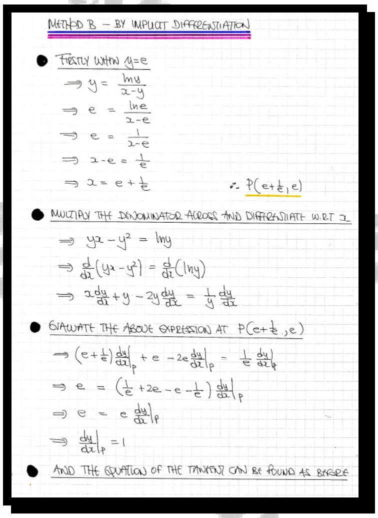
$$\Rightarrow x = e + \frac{\ln e}{e} = e + \frac{1}{e} \qquad \therefore P\left(e+\frac{1}{e}, e\right)$$

- DIFFERENTIATE WITH RESPECT TO y

$$\Rightarrow \frac{dx}{dy} = 1 + \frac{y \times \frac{1}{y} - \ln y \times 1}{y^2}$$
$$\Rightarrow \frac{dx}{dy} = 1 + \frac{1 - \ln y}{y^2}$$
$$\Rightarrow \left.\frac{dx}{dy}\right|_{y=e} = 1 + \frac{1 - \ln e}{e^2} = 1 + \cancel{\frac{1-1}{e^2}} = 1$$
$$\Rightarrow \frac{dy}{dx} = 1$$

- EQUATION OF TANGENT AT $P\left(e+\frac{1}{e}, e\right)$

$$\Rightarrow y - e = 1\left(x - e - \frac{1}{e}\right)$$
$$\Rightarrow y \cancel{-e} = x \cancel{-e} - \frac{1}{e}$$
$$\Rightarrow \frac{1}{e} = x - y \qquad \therefore e(x-y) = 1$$



METHOD B – BY IMPLICIT DIFFERENTIATION

- FIRSTLY WHEN $y=e$

$$\Rightarrow y = \frac{\ln y}{x-y}$$
$$\Rightarrow e = \frac{\ln e}{x-e}$$
$$\Rightarrow e = \frac{1}{x-e}$$
$$\Rightarrow x - e = \frac{1}{e}$$
$$\Rightarrow x = e + \frac{1}{e} \qquad \therefore P\left(e+\frac{1}{e}, e\right)$$

- MULTIPLY THE DENOMINATOR ACROSS AND DIFFERENTIATE W.R.T x

$$\Rightarrow yx - y^2 = \ln y$$
$$\Rightarrow \frac{d}{dx}(yx - y^2) = \frac{d}{dx}(\ln y)$$
$$\Rightarrow x\frac{dy}{dx} + y - 2y\frac{dy}{dx} = \frac{1}{y}\frac{dy}{dx}$$

- EVALUATE THE ABOVE EXPRESSION AT $P\left(e+\frac{1}{e}, e\right)$

$$\Rightarrow \left(e+\frac{1}{e}\right)\left.\frac{dy}{dx}\right|_P + e - 2e\left.\frac{dy}{dx}\right|_P = \frac{1}{e}\left.\frac{dy}{dx}\right|_P$$
$$\Rightarrow e = \left(\frac{1}{e} + 2e - e - \frac{1}{e}\right)\left.\frac{dy}{dx}\right|_P$$
$$\Rightarrow e = e\left.\frac{dy}{dx}\right|_P$$
$$\Rightarrow \left.\frac{dy}{dx}\right|_P = 1$$

- AND THE EQUATION OF THE TANGENT CAN BE FOUND AS BEFORE

Question 125 (****)

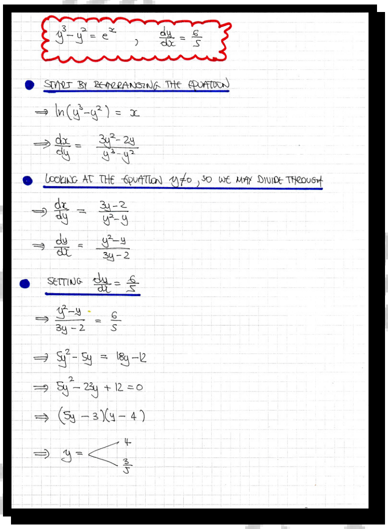
At the point P, which lies on the curve with equation

$$y^3 - y^2 = \mathrm{e}^x,$$

the gradient is $\frac{6}{5}$.

Determine the possible coordinates of P.

$\boxed{}$, $\boxed{P(\ln 48, 4)}$



$$\boxed{y^3 - y^2 = e^x \quad , \quad \frac{dy}{dx} = \frac{6}{5}}$$

- START BY REARRANGING THE EQUATION

$\Rightarrow \ln(y^3 - y^2) = x$

$\Rightarrow \frac{dx}{dy} = \frac{3y^2 - 2y}{y^3 - y^2}$

- LOOKING AT THE EQUATION $y \neq 0$, SO WE MAY DIVIDE THROUGH

$\Rightarrow \frac{dx}{dy} = \frac{3y - 2}{y^2 - y}$

$\Rightarrow \frac{dy}{dx} = \frac{y^2 - y}{3y - 2}$

- SETTING $\frac{dy}{dx} = \frac{6}{5}$

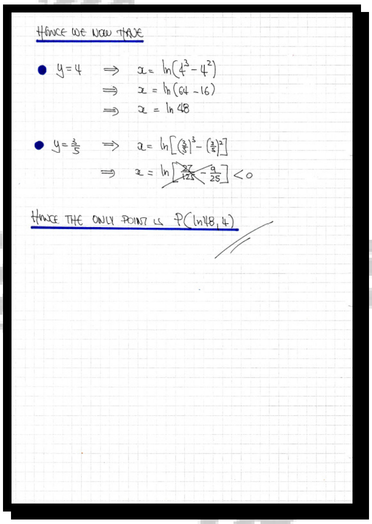
$\Rightarrow \frac{y^2 - y}{3y - 2} = \frac{6}{5}$

$\Rightarrow 5y^2 - 5y = 18y - 12$

$\Rightarrow 5y^2 - 23y + 12 = 0$

$\Rightarrow (5y - 3)(y - 4)$

$\Rightarrow y = \begin{cases} 4 \\ \frac{3}{5} \end{cases}$



HENCE WE NOW HAVE

- $y = 4 \quad \Rightarrow \quad x = \ln(4^3 - 4^2)$

$\Rightarrow \quad x = \ln(64 - 16)$

$\Rightarrow \quad x = \ln 48$

- $y = \frac{3}{5} \quad \Rightarrow \quad x = \ln\left[\left(\frac{3}{5}\right)^3 - \left(\frac{3}{5}\right)^2\right]$

$\Rightarrow \quad x = \ln \cancel{\left[\frac{27}{125} - \frac{9}{25}\right]} < 0$

HENCE THE ONLY POINT IS $P(\ln 48, 4)$

Question 126 (****)

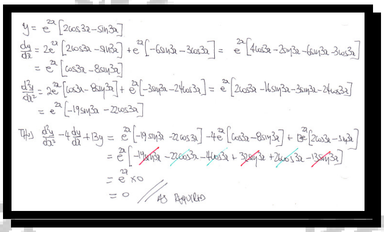
A curve has equation

$$y = \mathrm{e}^{2x}\left(2\cos 3x - \sin 3x\right).$$

Show that

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 13y = 0.$$

$\boxed{}$, $\boxed{\text{proof}}$



$y = e^{2x}\left[2\cos 3x - \sin 3x\right]$

$\frac{dy}{dx} = 2e^{2x}\left[2\cos 3x - \sin 3x\right] + e^{2x}\left[-6\sin 3x - 3\cos 3x\right] = e^{2x}\left[4\cos 3x - 2\sin 3x - 6\sin 3x - 3\cos 3x\right]$

$= e^{2x}\left[\cos 3x - 8\sin 3x\right]$

$\frac{d^2y}{dx^2} = 2e^{2x}\left[\cos 3x - 8\sin 3x\right] + e^{2x}\left[-3\sin 3x - 24\cos 3x\right] = e^{2x}\left[2\cos 3x - 16\sin 3x - 3\sin 3x - 24\cos 3x\right]$

$= e^{2x}\left[-19\sin 3x - 22\cos 3x\right]$

THUS $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 13y = e^{2x}\left[-19\sin 3x - 22\cos 3x\right] - 4e^{2x}\left[\cos 3x - 8\sin 3x\right] + 13e^{2x}\left[2\cos 3x - \sin 3x\right]$

$= e^{2x}\left[\cancel{-19\sin 3x} \cancel{- 22\cos 3x} \cancel{- 4\cos 3x} \cancel{+ 32\sin 3x} \cancel{+ 26\cos 3x} \cancel{- 13\sin 3x}\right]$

$= e^{2x} \times 0$

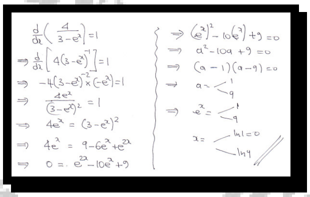
$= 0$ // AS REQUIRED

Question 127 (****)

Find, in exact form where appropriate, the solutions of the equation

$$\frac{d}{dx}\left(\frac{4}{3 - \mathrm{e}^{x}}\right) = 1.$$

$\boxed{x = 0,\ \ x = \ln 9}$



$\frac{d}{dx}\left(\frac{4}{3-e^x}\right) = 1$

$\Rightarrow \frac{d}{dx}\left[4(3-e^x)^{-1}\right] = 1$

$\Rightarrow -4(3-e^x)^{-2} \times (-e^x) = 1$

$\Rightarrow \frac{4e^x}{(3-e^x)^2} = 1$

$\Rightarrow 4e^x = (3-e^x)^2$

$\Rightarrow 4e^x = 9 - 6e^x + e^{2x}$

$\Rightarrow 0 = e^{2x} - 10e^x + 9$

$\Rightarrow (e^x)^2 - 10(e^x) + 9 = 0$

$\Rightarrow a^2 - 10a + 9 = 0$

$\Rightarrow (a-1)(a-9) = 0$

$\Rightarrow a = \begin{cases} 1 \\ 9 \end{cases}$

$\Rightarrow e^x = \begin{cases} 1 \\ 9 \end{cases}$

$x = \begin{cases} \ln 1 = 0 \\ \ln 9 \end{cases}$ //

Question 128 (****)

The point $P(\ln 2, 5b-3a)$ lies on the curve with equation

$$y = a + b\mathrm{e}^{x},$$

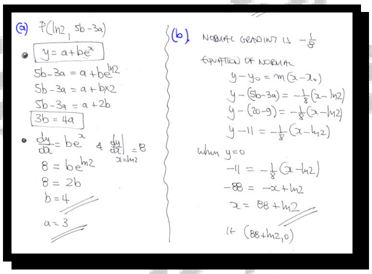
where a and b are non zero constants.

The gradient at P is 8.

a) Find the value of a and the value of b.

b) Find the exact coordinates of the point where the normal to the curve at P crosses the x axis.

$\boxed{a = 3,\ b = 4}$, $\boxed{(88 + \ln 2, 0)}$



(a) $P(\ln 2, 5b-3a)$

- $\boxed{y = a + be^{x}}$

$5b - 3a = a + be^{\ln 2}$

$5b - 3a = a + b \times 2$

$5b - 3a = a + 2b$

$\boxed{3b = 4a}$

- $\frac{dy}{dx} = be^{x}$ & $\left.\frac{dy}{dx}\right|_{x=\ln 2} = 8$

$8 = be^{\ln 2}$

$8 = 2b$

$b = 4$

$a = 3$

(b) NORMAL GRADINT IS $-\frac{1}{8}$

EQUATION OF NORMAL

$y - y_0 = m(x - x_0)$

$y - (5b - 3a) = -\frac{1}{8}(x - \ln 2)$

$y - (20 - 9) = -\frac{1}{8}(x - \ln 2)$

$y - 11 = -\frac{1}{8}(x - \ln 2)$

when $y = 0$

$-11 = -\frac{1}{8}(x - \ln 2)$

$-88 = -x + \ln 2$

$x = 88 + \ln 2$

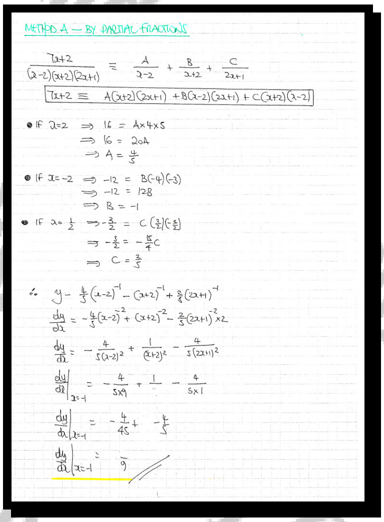
IE $(88 + \ln 2, 0)$

Question 129 **(****)**

$$y = \frac{7x+2}{(x-2)(x+2)(2x+1)}, \quad x \neq \pm 2, \quad x \neq -\frac{1}{2}.$$

Find the exact value of $\frac{dy}{dx}$ at $x = -1$.

$$\boxed{}, \quad \boxed{\left.\frac{dy}{dx}\right|_{x=-1} = \frac{1}{9}}$$



METHOD A – BY PARTIAL FRACTIONS

$$\frac{7x+2}{(x-2)(x+2)(2x+1)} \equiv \frac{A}{x-2} + \frac{B}{x+2} + \frac{C}{2x+1}$$

$$\boxed{7x+2 \equiv A(x+2)(2x+1) + B(x-2)(2x+1) + C(x+2)(x-2)}$$

- IF $x=2 \Rightarrow 16 = A \times 4 \times 5$
  $\Rightarrow 16 = 20A$
  $\Rightarrow A = \frac{4}{5}$
- IF $x=-2 \Rightarrow -12 = B(-4)(-3)$
  $\Rightarrow -12 = 12B$
  $\Rightarrow B = -1$
- IF $x = \frac{1}{2} \Rightarrow -\frac{3}{2} = C\left(\frac{3}{2}\right)\left(-\frac{5}{2}\right)$
  $\Rightarrow -\frac{3}{2} = -\frac{15}{4}C$
  $\Rightarrow C = \frac{2}{5}$

$$\therefore \quad y = \frac{4}{5}(x-2)^{-1} - (x+2)^{-1} + \frac{2}{5}(2x+1)^{-1}$$

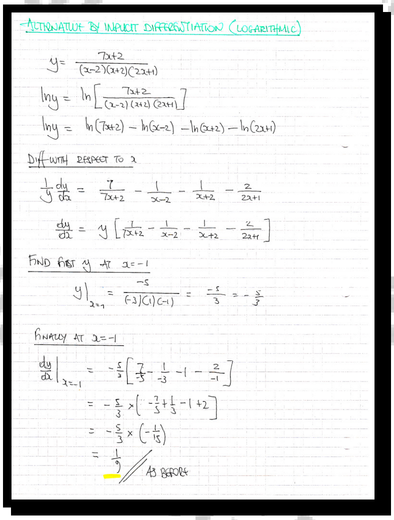
$$\frac{dy}{dx} = -\frac{4}{5}(x-2)^{-2} + (x+2)^{-2} - \frac{2}{5}(2x+1)^{-2} \times 2$$

$$\frac{dy}{dx} = -\frac{4}{5(x-2)^2} + \frac{1}{(x+2)^2} - \frac{4}{5(2x+1)^2}$$

$$\left.\frac{dy}{dx}\right|_{x=-1} = -\frac{4}{5 \times 9} + \frac{1}{\ } - \frac{4}{5 \times 1}$$

$$\left.\frac{dy}{dx}\right|_{x=-1} = -\frac{4}{45} + \quad -\frac{4}{5}$$

$$\left.\frac{dy}{dx}\right|_{x=-1} = \frac{\ }{9} \quad //$$



ALTERNATIVE BY IMPLICIT DIFFERENTIATION (LOGARITHMIC)

$$y = \frac{7x+2}{(x-2)(x+2)(2x+1)}$$

$$\ln y = \ln\left[\frac{7x+2}{(x-2)(x+2)(2x+1)}\right]$$

$$\ln y = \ln(7x+2) - \ln(x-2) - \ln(x+2) - \ln(2x+1)$$

DIFF WITH RESPECT TO x

$$\frac{1}{y}\frac{dy}{dx} = \frac{7}{7x+2} - \frac{1}{x-2} - \frac{1}{x+2} - \frac{2}{2x+1}$$

$$\frac{dy}{dx} = y\left[\frac{7}{7x+2} - \frac{1}{x-2} - \frac{1}{x+2} - \frac{2}{2x+1}\right]$$

FIND FIRST y AT $x=-1$

$$y\Big|_{x=1} = \frac{-5}{(-3)(1)(-1)} = \frac{-5}{3} = -\frac{5}{3}$$

FINALLY AT $x=-1$

$$\left.\frac{dy}{dx}\right|_{x=-1} = -\frac{5}{3}\left[\frac{7}{-5} - \frac{1}{-3} - 1 - \frac{2}{-1}\right]$$

$$= -\frac{5}{3} \times \left(-\frac{7}{5} + \frac{1}{3} - 1 + 2\right]$$

$$= -\frac{5}{3} \times \left(-\frac{1}{15}\right)$$

$$= \frac{1}{9} \quad //$$

AS BEFORE

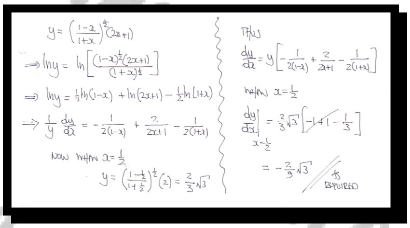
Question 130 **(****)**

The curve C has equation

$$y=\left(\frac{1-x}{1+x}\right)^{\frac{1}{2}}(2x+1),\ -1<x\leq 1.$$

By taking logarithms on both sides of this equation, or otherwise, show that at the point on C where $x=\frac{1}{2}$, the gradient is $-\frac{2}{9}\sqrt{3}$.

$\boxed{}$, $\boxed{\text{proof}}$



$y=\left(\frac{1-x}{1+x}\right)^{\frac{1}{2}}(2x+1)$

$\Rightarrow \ln y = \ln\left[\frac{(1-x)^{\frac{1}{2}}(2x+1)}{(1+x)^{\frac{1}{2}}}\right]$

$\Rightarrow \ln y = \frac{1}{2}\ln(1-x)+\ln(2x+1)-\frac{1}{2}\ln(1+x)$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = -\frac{1}{2(1-x)}+\frac{2}{2x+1}-\frac{1}{2(1+x)}$

Now when $x=\frac{1}{2}$

$y=\left(\frac{1-\frac{1}{2}}{1+\frac{1}{2}}\right)^{\frac{1}{2}}(2)=\frac{2}{3}\sqrt{3}$

Thus

$\frac{dy}{dx}=y\left[-\frac{1}{2(1-x)}+\frac{2}{2x+1}-\frac{1}{2(1+x)}\right]$

when $x=\frac{1}{2}$

$\left.\frac{dy}{dx}\right|_{x=\frac{1}{2}}=\frac{2}{3}\sqrt{3}\left[\cancel{-1+1}-\frac{1}{3}\right]$

$=-\frac{2}{9}\sqrt{3}$

as required

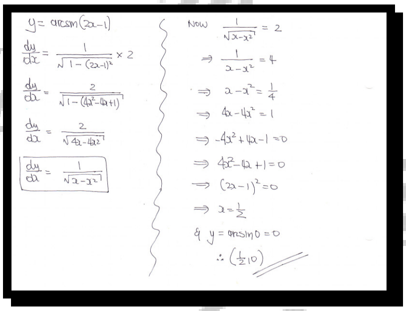
Question 131 **(****)**

The curve C has equation

$$y=\arcsin(2x-1),\ 0\leq x\leq 1.$$

Find the coordinates of the point on C, whose gradient is 2.

$$\boxed{\left(\frac{1}{2},0\right)}$$



$y=\arcsin(2x-1)$

$\frac{dy}{dx}=\frac{1}{\sqrt{1-(2x-1)^2}}\times 2$

$\frac{dy}{dx}=\frac{2}{\sqrt{1-(4x^2-4x+1)}}$

$\frac{dy}{dx}=\frac{2}{\sqrt{4x-4x^2}}$

$\boxed{\frac{dy}{dx}=\frac{1}{\sqrt{x-x^2}}}$

Now $\frac{1}{\sqrt{x-x^2}}=2$

$\Rightarrow \frac{1}{x-x^2}=4$

$\Rightarrow x-x^2=\frac{1}{4}$

$\Rightarrow 4x-4x^2=1$

$\Rightarrow -4x^2+4x-1=0$

$\Rightarrow 4x^2-4x+1=0$

$\Rightarrow (2x-1)^2=0$

$\Rightarrow x=\frac{1}{2}$

& $y=\arcsin 0=0$

$\therefore \left(\frac{1}{2},0\right)$

Question 132 (****)

A curve C has equation

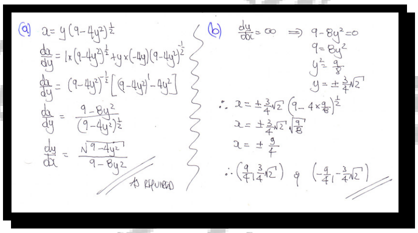
$$x = y\sqrt{9-4y^2}\ ,\ |y| \le \frac{3}{2}.$$

a) Show clearly that

$$\frac{dy}{dx} = \frac{\sqrt{9-4y^2}}{9-8y^2}.$$

b) Find the exact coordinates of the two points on C, with infinite gradient.

$\boxed{}$, $\boxed{\left(\frac{9}{4},\frac{3}{4}\sqrt{2}\right),\left(-\frac{9}{4},-\frac{3}{4}\sqrt{2}\right)}$



(a) $x = y(9-4y^2)^{\frac{1}{2}}$

$\frac{dx}{dy} = 1\times(9-4y^2)^{\frac{1}{2}} + y\times(-4y)(9-4y^2)^{-\frac{1}{2}}$

$\frac{dx}{dy} = (9-4y^2)^{-\frac{1}{2}}\left[(9-4y^2)^1 - 4y^2\right]$

$\frac{dx}{dy} = \frac{9-8y^2}{(9-4y^2)^{\frac{1}{2}}}$

$\frac{dy}{dx} = \frac{\sqrt{9-4y^2}}{9-8y^2}$

AS REQUIRED

(b) $\frac{dy}{dx} = \infty \Rightarrow 9-8y^2=0$

$9 = 8y^2$

$y^2 = \frac{9}{8}$

$y = \pm\frac{3}{4}\sqrt{2}$

$\therefore x = \pm\frac{3}{4}\sqrt{2}\left(9-4\times\frac{9}{8}\right)^{\frac{1}{2}}$

$x = \pm\frac{3}{4}\sqrt{2}\sqrt{\frac{9}{8}}$

$x = \pm\frac{9}{4}$

$\therefore \left(\frac{9}{4},\frac{3}{4}\sqrt{2}\right)$ & $\left(-\frac{9}{4},-\frac{3}{4}\sqrt{2}\right)$

Question 133 (****)

Given that

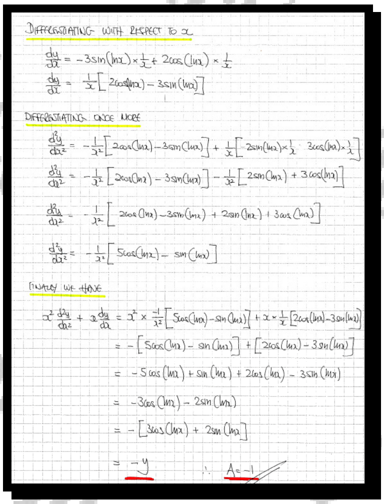
$$y = 3\cos(\ln x) + 2\sin(\ln x), \ \ x > 0,$$

show clearly that

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} = Ay,$$

stating the value of the constant A.

$\boxed{}$, $\boxed{A = -1}$



DIFFERENTIATING WITH RESPECT TO x

$$\frac{dy}{dx} = -3\sin(\ln x)\times\frac{1}{x} + 2\cos(\ln x)\times\frac{1}{x}$$

$$\frac{dy}{dx} = \frac{1}{x}\left[2\cos(\ln x) - 3\sin(\ln x)\right]$$

DIFFERENTIATING ONCE MORE

$$\frac{d^2y}{dx^2} = -\frac{1}{x^2}\left[2\cos(\ln x) - 3\sin(\ln x)\right] + \frac{1}{x}\left[-2\sin(\ln x)\times\frac{1}{x} \quad 3\cos(\ln x)\times\frac{1}{x}\right]$$

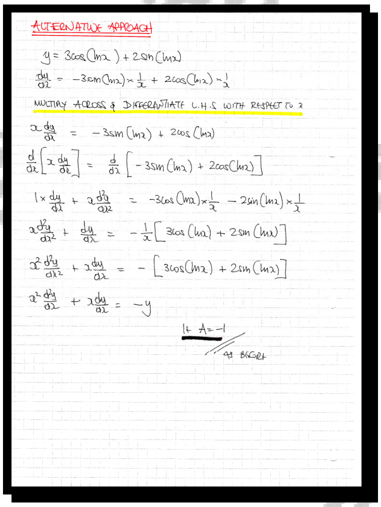
$$\frac{d^2y}{dx^2} = -\frac{1}{x^2}\left[2\cos(\ln x) - 3\sin(\ln x)\right] - \frac{1}{x^2}\left[2\sin(\ln x) + 3\cos(\ln x)\right]$$

$$\frac{d^2y}{dx^2} = -\frac{1}{x^2}\left[2\cos(\ln x) - 3\sin(\ln x) + 2\sin(\ln x) + 3\cos(\ln x)\right]$$

$$\frac{d^2y}{dx^2} = -\frac{1}{x^2}\left[5\cos(\ln x) - \sin(\ln x)\right]$$

FINALLY WE HAVE

$$\begin{aligned} x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} &= x^2\times\frac{-1}{x^2}\left[5\cos(\ln x) - \sin(\ln x)\right] + x\times\frac{1}{x}\left[2\cos(\ln x) - 3\sin(\ln x)\right] \\ &= -\left[5\cos(\ln x) - \sin(\ln x)\right] + \left[2\cos(\ln x) - 3\sin(\ln x)\right] \\ &= -5\cos(\ln x) + \sin(\ln x) + 2\cos(\ln x) - 3\sin(\ln x) \\ &= -3\cos(\ln x) - 2\sin(\ln x) \\ &= -\left[3\cos(\ln x) + 2\sin(\ln x)\right] \\ &= -y \qquad \therefore \ A = -1 \end{aligned}$$



ALTERNATIVE APPROACH

$$y = 3\cos(\ln x) + 2\sin(\ln x)$$

$$\frac{dy}{dx} = -3\sin(\ln x)\times\frac{1}{x} + 2\cos(\ln x)\times\frac{1}{x}$$

MULTIPLY ACROSS & DIFFERENTIATE L.H.S WITH RESPECT TO x

$$x\frac{dy}{dx} = -3\sin(\ln x) + 2\cos(\ln x)$$

$$\frac{d}{dx}\left[x\frac{dy}{dx}\right] = \frac{d}{dx}\left[-3\sin(\ln x) + 2\cos(\ln x)\right]$$

$$1\times\frac{dy}{dx} + x\frac{d^2y}{dx^2} = -3\cos(\ln x)\times\frac{1}{x} - 2\sin(\ln x)\times\frac{1}{x}$$

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = -\frac{1}{x}\left[3\cos(\ln x) + 2\sin(\ln x)\right]$$

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -\left[3\cos(\ln x) + 2\sin(\ln x)\right]$$

$$x^2\frac{d^2y}{dx} + x\frac{dy}{dx} = -y$$

ie $A = -1$

AS BEFORE

Question 134 (****)

The equation of a curve C is

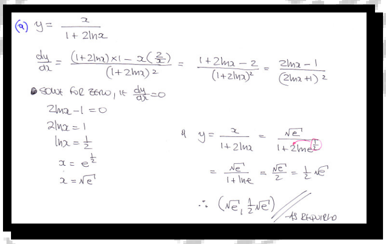
$$y = \frac{x}{1+2\ln x}, \ x \in \mathbb{R}, \ x > 0 \ .$$

The curve has a single turning point at P.

a) Show that the coordinates of P are $\left(\sqrt{e}, \frac{1}{2}\sqrt{e}\right)$.

b) Evaluate the exact value of $\frac{d^2y}{dx^2}$ at P and hence determine its nature.

$\boxed{}$, $\boxed{\frac{d^2y}{dx^2} = \frac{1}{2\sqrt{e}} > 0, \text{ so minimum}}$



(a) $y = \frac{x}{1+2\ln x}$

$\frac{dy}{dx} = \frac{(1+2\ln x)\times 1 - x\left(\frac{2}{x}\right)}{(1+2\ln x)^2} = \frac{1+2\ln x - 2}{(1+2\ln x)^2} = \frac{2\ln x - 1}{(2\ln x + 1)^2}$

- SOLVE FOR ZERO, IE $\frac{dy}{dx} = 0$

$2\ln x - 1 = 0$

$2\ln x = 1$

$\ln x = \frac{1}{2}$

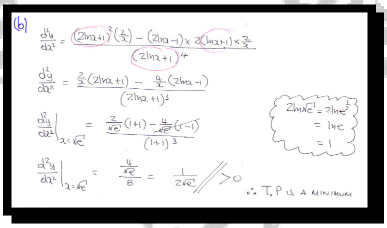
$x = e^{\frac{1}{2}}$

$x = \sqrt{e}$

& $y = \frac{x}{1+2\ln x} = \frac{\sqrt{e}}{1+2\ln e^{\frac{1}{2}}}$

$= \frac{\sqrt{e}}{1+\ln e} = \frac{\sqrt{e}}{2} = \frac{1}{2}\sqrt{e}$

$\therefore \left(\sqrt{e}, \frac{1}{2}\sqrt{e}\right)$ // AS REQUIRED



(b)

$\frac{d^2y}{dx^2} = \frac{(2\ln x+1)^2\left(\frac{2}{x}\right) - (2\ln x - 1)\times 2(\ln x + 1)\times\frac{2}{x}}{(2\ln x+1)^4}$

$\frac{d^2y}{dx^2} = \frac{\frac{2}{x}(2\ln x + 1) - \frac{4}{x}(2\ln x - 1)}{(2\ln x+1)^3}$

$\left.\frac{d^2y}{dx^2}\right|_{x=\sqrt{e}} = \frac{\frac{2}{\sqrt{e}}(1+1) - \cancel{\frac{4}{\sqrt{e}}(1-1)}}{(1+1)^3}$

$2\ln\sqrt{e} = 2\ln e^{\frac{1}{2}}$
$= \ln e$
$= 1$

$\left.\frac{d^2y}{dx^2}\right|_{x=\sqrt{e}} = \frac{\frac{4}{\sqrt{e}}}{8} = \frac{1}{2\sqrt{e}}$ // >0

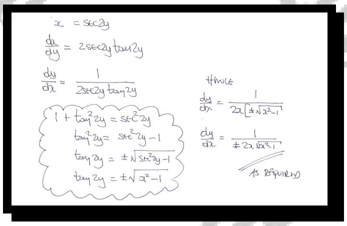
$\therefore$ T.P IS A MINIMUM

Question 135 (****)

Show that if $x = \sec 2y$, then

$$\frac{dy}{dx} = \pm\frac{1}{2x\sqrt{x^2-1}} .$$

$\boxed{\text{proof}}$



$x = \sec 2y$

$\frac{dx}{dy} = 2\sec 2y \tan 2y$

$\frac{dy}{dx} = \frac{1}{2\sec 2y \tan 2y}$

$1 + \tan^2 2y = \sec^2 2y$
$\tan^2 2y = \sec^2 2y - 1$
$\tan 2y = \pm\sqrt{\sec^2 2y - 1}$
$\tan 2y = \pm\sqrt{x^2 - 1}$

HENCE

$\frac{dy}{dx} = \frac{1}{2x\left[\pm\sqrt{x^2-1}\right]}$

$\frac{dy}{dx} = \frac{1}{\pm 2x\sqrt{x^2-1}}$ // AS REQUIRED

Question 136 **(****)**

A curve has equation given by

$$y = x\sqrt{x+1}\,,\ x \geq -1.$$

a) Show clearly that

$$\frac{dy}{dx} = \frac{1}{2}(3x+2)(x+1)^{-\frac{1}{2}}.$$

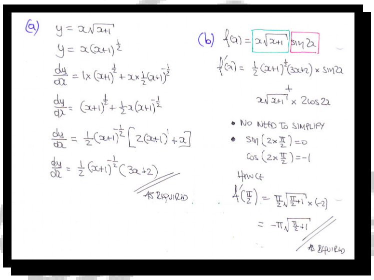
The function f is defined as

$$f(x) = x\sqrt{x+1}\sin 2x\,,\ x \geq 1.$$

b) Show further that

$$f'\left(\frac{\pi}{2}\right) = -\pi\sqrt{\frac{\pi}{2}+1}\ .$$

$\boxed{}$, $\boxed{\text{proof}}$



(a) $y = x\sqrt{x+1}$

$y = x(x+1)^{\frac{1}{2}}$

$\frac{dy}{dx} = 1\times(x+1)^{\frac{1}{2}} + x\times\frac{1}{2}(x+1)^{-\frac{1}{2}}$

$\frac{dy}{dx} = (x+1)^{\frac{1}{2}} + \frac{1}{2}x(x+1)^{-\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2}(x+1)^{-\frac{1}{2}}\left[2(x+1)^{1} + x\right]$

$\frac{dy}{dx} = \frac{1}{2}(x+1)^{-\frac{1}{2}}(3x+2)$

AS REQUIRED

(b) $f(x) = \boxed{x\sqrt{x+1}}\boxed{\sin 2x}$

$f'(x) = \frac{1}{2}(x+1)^{\frac{1}{2}}(3x+2)\times\sin 2x$

$+$

$x\sqrt{x+1}\times 2\cos 2x$

- NO NEED TO SIMPLIFY
- $\sin\left(2\times\frac{\pi}{2}\right) = 0$
  $\cos\left(2\times\frac{\pi}{2}\right) = -1$

HENCE

$f'\left(\frac{\pi}{2}\right) = \frac{\pi}{2}\sqrt{\frac{\pi}{2}+1}\times(-2)$

$= -\pi\sqrt{\frac{\pi}{2}+1}$

AS REQUIRED

Question 137 (****)

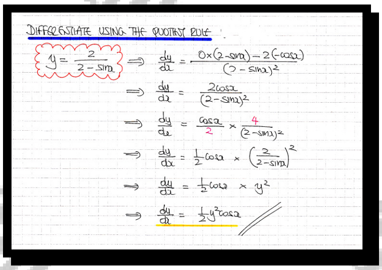
The curve C has equation

$$y = \frac{2}{2 - \sin x}.$$

Show clearly that

$$\frac{dy}{dx} = \frac{1}{2} y^2 \cos x \ .$$

$\boxed{}$, $\boxed{\text{proof}}$



DIFFERENTIATE USING THE QUOTIENT RULE

$$\boxed{y = \frac{2}{2 - \sin x}} \Longrightarrow \frac{dy}{dx} = \frac{0 \times (2 - \sin x) - 2(-\cos x)}{(2 - \sin x)^2}$$

$$\Longrightarrow \frac{dy}{dx} = \frac{2\cos x}{(2 - \sin x)^2}$$

$$\Longrightarrow \frac{dy}{dx} = \frac{\cos x}{2} \times \frac{4}{(2 - \sin x)^2}$$

$$\Longrightarrow \frac{dy}{dx} = \frac{1}{2}\cos x \times \left(\frac{2}{2 - \sin x}\right)^2$$

$$\Longrightarrow \frac{dy}{dx} = \frac{1}{2}\cos x \times y^2$$

$$\Longrightarrow \frac{dy}{dx} = \frac{1}{2} y^2 \cos x$$

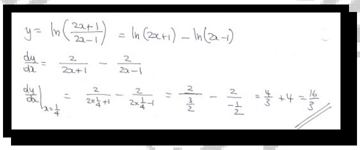
Question 138 (****)

Given that

$$y = \ln\left(\frac{2x+1}{2x-1}\right)$$

find the exact value of $\dfrac{dy}{dx}$ at $x = \dfrac{1}{4}$.

$$\boxed{\frac{dy}{dx} = \frac{16}{3}}$$



$$y = \ln\left(\frac{2x+1}{2x-1}\right) = \ln(2x+1) - \ln(2x-1)$$

$$\frac{dy}{dx} = \frac{2}{2x+1} - \frac{2}{2x-1}$$

$$\left.\frac{dy}{dx}\right|_{x=\frac{1}{4}} = \frac{2}{2 \times \frac{1}{4} + 1} - \frac{2}{2 \times \frac{1}{4} - 1} = \frac{2}{\frac{3}{2}} - \frac{2}{-\frac{1}{2}} = \frac{4}{3} + 4 = \frac{16}{3}$$

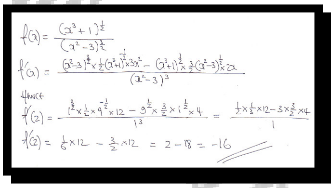
Question 139 **(****)**

The function f is defined by

$$f(x) = \frac{\left(x^3+1\right)^{\frac{1}{2}}}{\left(x^2-3\right)^{\frac{3}{2}}}, \quad x \in \mathbb{R}, \quad x > -1 .$$

Show clearly that $f'(2) = -16$.

proof



$f(x) = \frac{(x^3+1)^{\frac12}}{(x^2-3)^{\frac32}}$

$f'(x) = \frac{(x^2-3)^{\frac32}\times\frac12(x^3+1)^{-\frac12}\times3x^2 - (x^3+1)^{\frac12}\times\frac32(x^2-3)^{\frac12}\times2x}{(x^2-3)^3}$

Hence

$f'(2) = \frac{1^{\frac32}\times\frac12\times9^{-\frac12}\times12 - 9^{\frac12}\times\frac32\times1^{\frac12}\times4}{1^3} = \frac{\frac12\times\frac13\times12-3\times\frac32\times4}{1}$

$f'(2) = \frac16\times12 - \frac32\times12 = 2-18 = -16$

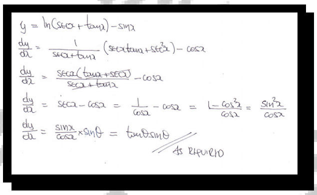
Question 140 **(****)**

$$y = \ln\left(\sec x + \tan x\right) - \sin x .$$

Show clearly that

$$\frac{dy}{dx} = \sin x \tan x \ .$$

proof



$y = \ln(\sec x+\tan x)-\sin x$

$\frac{dy}{dx} = \frac{1}{\sec x+\tan x}(\sec x\tan x+\sec^2 x)-\cos x$

$\frac{dy}{dx} = \frac{\sec x\cancel{(\tan x+\sec x)}}{\cancel{\sec x+\tan x}}-\cos x$

$\frac{dy}{dx} = \sec x-\cos x = \frac{1}{\cos x}-\cos x = \frac{1-\cos^2 x}{\cos x} = \frac{\sin^2 x}{\cos x}$

$\frac{dy}{dx} = \frac{\sin x}{\cos x}\times\sin\theta = \tan\theta\sin\theta$

AS REQUIRED

Question 141 (****)

It is given that

$$\frac{d}{dx}(\tan 2x) = 2\sec^2 2x\,.$$

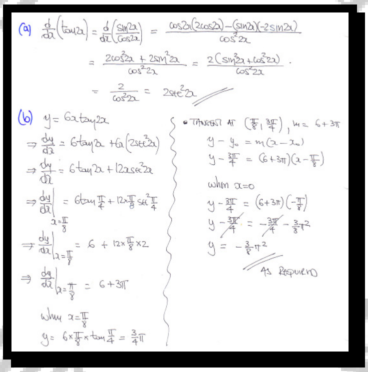
a) Prove the validity of the above result by considering the derivatives of $\sin 2x$ and $\cos 2x$.

A curve has equation

$$y = 6x\tan 2x\,,\ x \in \mathbb{R}\,.$$

b) Show that the tangent to the curve at the point where $x = \frac{1}{8}\pi$ meets the y axis at the point with coordinates $\left(0, -\frac{3}{8}\pi^2\right)$.

$\boxed{}$, $\boxed{\text{proof}}$



(a) $\frac{d}{dx}(\tan 2x) = \frac{d}{dx}\left(\frac{\sin 2x}{\cos 2x}\right) = \frac{\cos 2x(2\cos 2x) - (\sin 2x)(-2\sin 2x)}{\cos^2 2x}$

$= \frac{2\cos^2 2x + 2\sin^2 2x}{\cos^2 2x} = \frac{2(\sin^2 2x + \cos^2 2x)}{\cos^2 2x}$.

$= \frac{2}{\cos^2 2x} = 2\sec^2 2x$

(b) $y = 6x\tan 2x$

$\Rightarrow \frac{dy}{dx} = 6\tan 2x + 6x(2\sec^2 2x)$

$\Rightarrow \frac{dy}{dx} = 6\tan 2x + 12x\sec^2 2x$

$\Rightarrow \left.\frac{dy}{dx}\right|_{x=\frac{\pi}{8}} = 6\tan\frac{\pi}{4} + 12\times\frac{\pi}{8}\sec^2\frac{\pi}{4}$

$\Rightarrow \left.\frac{dy}{dx}\right|_{x=\frac{\pi}{8}} = 6 + 12\times\frac{\pi}{8}\times 2$

$\Rightarrow \left.\frac{dy}{dx}\right|_{x=\frac{\pi}{8}} = 6 + 3\pi$

when $x = \frac{\pi}{8}$

$y = 6\times\frac{\pi}{8}\times\tan\frac{\pi}{4} = \frac{3}{4}\pi$

- TANGENT AT $\left(\frac{\pi}{8}, \frac{3\pi}{4}\right)$, $m = 6 + 3\pi$

$y - y_0 = m(x - x_0)$

$y - \frac{3\pi}{4} = (6 + 3\pi)\left(x - \frac{\pi}{8}\right)$

when $x = 0$

$y - \frac{3\pi}{4} = (6 + 3\pi)\left(-\frac{\pi}{8}\right)$

$y - \cancel{\frac{3\pi}{4}} = -\cancel{\frac{3\pi}{4}} - \frac{3}{8}\pi^2$

$y = -\frac{3}{8}\pi^2$

AS REQUIRED

Question 142 (****)

$$y = x - \frac{12}{x^2+2x-3} + \frac{3}{x-1}, \ x \in \mathbb{R}, \ x \neq -3, \ x \neq 1.$$

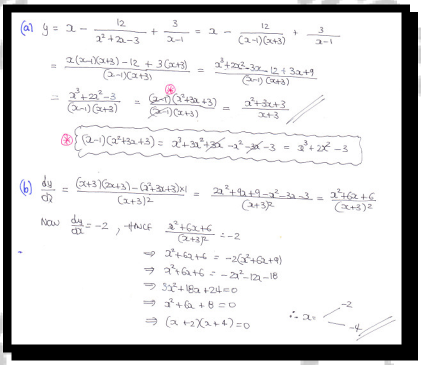
a) Show clearly that

$$y = \frac{x^2+3x+3}{x+3}.$$

b) Solve the equation

$$\frac{dy}{dx} = -2.$$

$$\boxed{x=-2, \ x=4}$$



(a) $y = x - \frac{12}{x^2+2x-3} + \frac{3}{x-1} = x - \frac{12}{(x-1)(x+3)} + \frac{3}{x-1}$

$= \frac{x(x-1)(x+3) - 12 + 3(x+3)}{(x-1)(x+3)} = \frac{x^3+2x^2-3x-12+3x+9}{(x-1)(x+3)}$

$= \frac{x^3+2x^2-3}{(x-1)(x+3)} \overset{\circledast}{=} \frac{\cancel{(x-1)}(x^2+3x+3)}{\cancel{(x-1)}(x+3)} = \frac{x^2+3x+3}{x+3}$ //

$\circledast$ $(x-1)(x^2+3x+3) = x^3+3x^2+\cancel{3x}-x^2-\cancel{3x}-3 = x^3+2x^2-3$

(b) $\frac{dy}{dx} = \frac{(x+3)(2x+3)-(x^2+3x+3)\times 1}{(x+3)^2} = \frac{2x^2+9x+9-x^2-3x-3}{(x+3)^2} = \frac{x^2+6x+6}{(x+3)^2}$

Now $\frac{dy}{dx} = -2$, hence $\frac{x^2+6x+6}{(x+3)^2} = -2$

$\Rightarrow x^2+6x+6 = -2(x^2+6x+9)$

$\Rightarrow x^2+6x+6 = -2x^2-12x-18$

$\Rightarrow 3x^2+18x+24=0$

$\Rightarrow x^2+6x+8=0$

$\Rightarrow (x+2)(x+4)=0$

$\therefore x = \begin{cases} -2 \\ -4 \end{cases}$ //

Question 143 (****)

A curve has equation

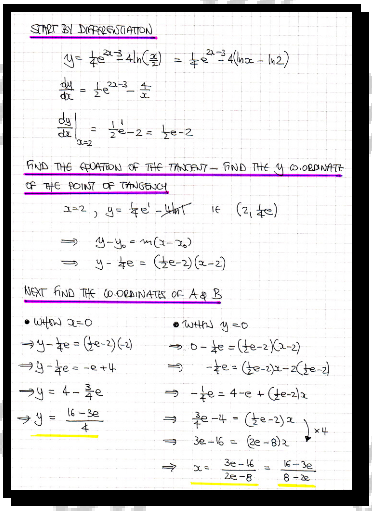
$$y = \frac{1}{4}e^{2x-3} - 4\ln\left(\frac{1}{2}x\right), \quad x > 0 .$$

The tangent to the curve, at the point where $x = 2$, crosses the coordinate axes at the points A and B.

Show that the area of the triangle OAB, where O is the origin, is given by

$$\frac{(16-3e)^2}{16(4-e)} .$$

proof



START BY DIFFERENTIATION

$$y = \frac{1}{4}e^{2x-3} - 4\ln\left(\frac{x}{2}\right) = \frac{1}{4}e^{2x-3} - 4(\ln x - \ln 2)$$

$$\frac{dy}{dx} = \frac{1}{2}e^{2x-3} - \frac{4}{x}$$

$$\left.\frac{dy}{dx}\right|_{x=2} = \frac{1}{2}e^{1} - 2 = \frac{1}{2}e - 2$$

FIND THE EQUATION OF THE TANGENT – FIND THE y CO.ORDINATE OF THE POINT OF TANGENCY

$$x=2, \quad y = \frac{1}{4}e^{1} - \cancel{4\ln 1} \quad \text{ie} \quad \left(2, \frac{1}{4}e\right)$$

$$\Rightarrow \quad y - y_0 = m(x - x_0)$$

$$\Rightarrow \quad y - \frac{1}{4}e = \left(\frac{1}{2}e - 2\right)(x-2)$$

NEXT FIND THE CO.ORDINATES OF A & B

- WHEN $x = 0$

$$\Rightarrow y - \frac{1}{4}e = \left(\frac{1}{2}e-2\right)(-2)$$

$$\Rightarrow y - \frac{1}{4}e = -e + 4$$

$$\Rightarrow y = 4 - \frac{3}{4}e$$

$$\Rightarrow y = \frac{16-3e}{4}$$

- WHEN $y = 0$

$$\Rightarrow 0 - \frac{1}{4}e = \left(\frac{1}{2}e-2\right)(x-2)$$

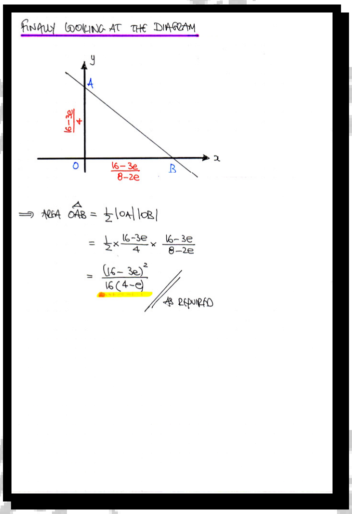
$$\Rightarrow -\frac{1}{4}e = \left(\frac{1}{2}e-2\right)x - 2\left(\frac{1}{2}e-2\right)$$

$$\Rightarrow -\frac{1}{4}e = 4 - e + \left(\frac{1}{2}e-2\right)x$$

$$\Rightarrow \frac{3}{4}e - 4 = \left(\frac{1}{2}e-2\right)x \quad \times 4$$

$$\Rightarrow 3e - 16 = (2e-8)x$$

$$\Rightarrow x = \frac{3e-16}{2e-8} = \frac{16-3e}{8-2e}$$



FINALLY LOOKING AT THE DIAGRAM

$$\Rightarrow \text{AREA } \overset{\triangle}{OAB} = \frac{1}{2}|OA||OB|$$

$$= \frac{1}{2} \times \frac{16-3e}{4} \times \frac{16-3e}{8-2e}$$

$$= \frac{(16-3e)^2}{16(4-e)}$$

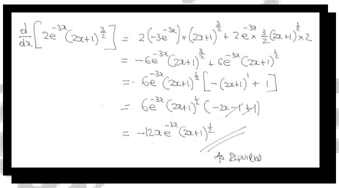
// AS REQUIRED

Question 144 **(****)**

Show clearly that

$$\frac{d}{dx}\left(2\mathrm{e}^{-3x}(2x+1)^{\frac{3}{2}}\right)=-12x\mathrm{e}^{-3x}(2x+1)^{\frac{1}{2}}.$$

$\boxed{\text{proof}}$



$$\frac{d}{dx}\left[2e^{-3x}(2x+1)^{\frac{3}{2}}\right]=2\left(-3e^{-3x}\right)\times(2x+1)^{\frac{3}{2}}+2e^{-3x}\times\frac{3}{2}(2x+1)^{\frac{1}{2}}\times 2$$

$$=-6e^{-3x}(2x+1)^{\frac{3}{2}}+6e^{-3x}(2x+1)^{\frac{1}{2}}$$

$$=6e^{-3x}(2x+1)^{\frac{1}{2}}\left[-(2x+1)^{1}+1\right]$$

$$=6e^{-3x}(2x+1)^{\frac{1}{2}}(-2x\cancel{-1}\cancel{+1})$$

$$=-12xe^{-3x}(2x+1)^{\frac{1}{2}}$$

As Required

Question 145 **(****)**

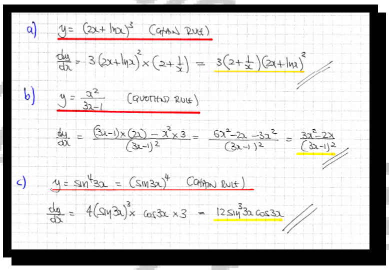
Differentiate each of the following expressions with respect to x.

a) $y=(2x+\ln x)^{3}$.

b) $y=\dfrac{x^{2}}{3x-1}$.

c) $y=\sin^{4}3x$.

$\boxed{\text{MP2-E}}$, $\boxed{\dfrac{dy}{dx}=3(2x+\ln x)^{2}\left(2+\dfrac{1}{x}\right)}$, $\boxed{\dfrac{dy}{dx}=\dfrac{3x^{2}-2x}{(3x-1)^{2}}}$, $\boxed{\dfrac{dy}{dx}=12\sin^{3}3x\cos 3x}$



a) $y=(2x+\ln x)^{3}$ (CHAIN RULE)

$\dfrac{dy}{dx}=3(2x+\ln x)^{2}\times\left(2+\dfrac{1}{x}\right)=3\left(2+\dfrac{1}{x}\right)(2x+\ln x)^{2}$

b) $y=\dfrac{x^{2}}{3x-1}$ (QUOTIENT RULE)

$\dfrac{dy}{dx}=\dfrac{(3x-1)\times(2x)-x^{2}\times 3}{(3x-1)^{2}}=\dfrac{6x^{2}-2x-3x^{2}}{(3x-1)^{2}}=\dfrac{3x^{2}-2x}{(3x-1)^{2}}$

c) $y=\sin^{4}3x=(\sin 3x)^{4}$ (CHAIN RULE)

$\dfrac{dy}{dx}=4(\sin 3x)^{3}\times\cos 3x\times 3=12\sin^{3}3x\cos 3x$

Question 146 (****)

The functions f and g are defined by

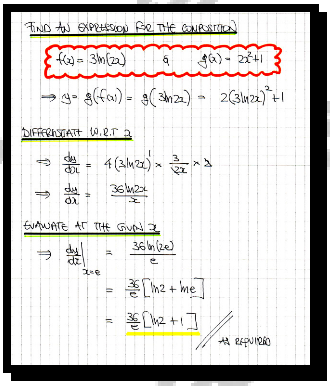
$$f(x) = 3\ln 2x, \ x \in \mathbb{R}, \ x > 0$$

$$g(x) = 2x^2 + 1, \ x \in \mathbb{R} .$$

Show that the value of the gradient on the curve $y = gf(x)$ at the point where $x = \mathrm{e}$ is

$$\frac{36}{\mathrm{e}}(1 + \ln 2) .$$

$\boxed{\quad}$, $\boxed{\text{proof}}$



FIND AN EXPRESSION FOR THE COMPOSITION

$$\boxed{f(x) = 3\ln(2x) \quad \& \quad g(x) = 2x^2+1}$$

$$\Rightarrow y = g(f(x)) = g(3\ln 2x) = 2(3\ln 2x)^2+1$$

DIFFERENTIATE W.R.T x

$$\Rightarrow \frac{dy}{dx} = 4(3\ln 2x)^1 \times \frac{3}{\cancel{2}x} \times \cancel{2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{36\ln 2x}{x}$$

EVALUATE AT THE GIVEN x

$$\Rightarrow \left.\frac{dy}{dx}\right|_{x=e} = \frac{36\ln(2e)}{e}$$

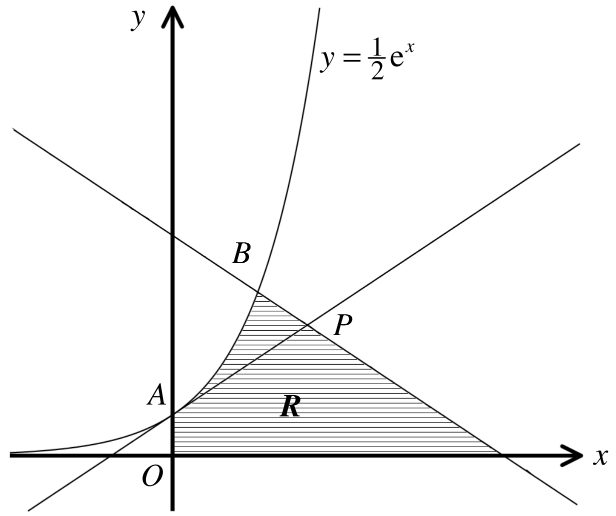
$$= \frac{36}{e}\left[\ln 2 + \ln e\right]$$

$$= \frac{36}{e}\left[\ln 2 + 1\right]$$

// AS REQUIRED

Question 147 (****)

The graph of the curve with equation $y = \frac{1}{2}e^x$ is shown below.



The points A and B lie on the curve, where $x = 0$ and $x = \ln 4$, respectively.

a) Show that the equation of the tangent to the curve at the point A is

$$2y - x - 1 = 0.$$

This tangent meets the normal to the curve at the point B at the point P.

b) Show that the coordinates of P are given by

$$\left(\frac{3}{2} + \ln 2, \frac{5}{4} + \ln\sqrt{2}\right).$$

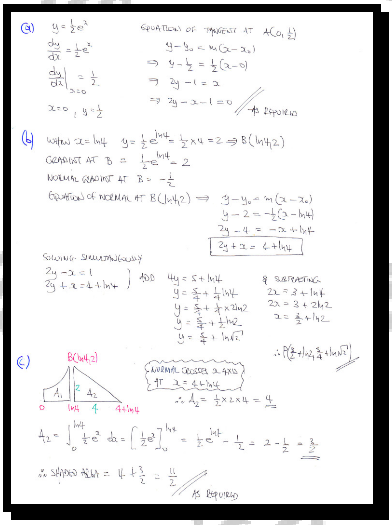
[continues overleaf]

[continued from overleaf]

The region R bounded by the curve, the normal to the curve at B and coordinate axes is shown shaded in the diagram above.

c) Show that the area of R is $\frac{11}{2}$ square units.

proof



(a) $y = \frac{1}{2}e^x$

$\frac{dy}{dx} = \frac{1}{2}e^x$

$\left.\frac{dy}{dx}\right|_{x=0} = \frac{1}{2}$

$x=0\ ,\ y = \frac{1}{2}$

EQUATION OF TANGENT AT $A(0, \frac{1}{2})$

$y - y_0 = m(x - x_0)$

$\Rightarrow y - \frac{1}{2} = \frac{1}{2}(x - 0)$

$\Rightarrow 2y - 1 = x$

$\Rightarrow 2y - x - 1 = 0$ // AS REQUIRED

(b) WHEN $x = \ln 4$ $\ y = \frac{1}{2}e^{\ln 4} = \frac{1}{2} \times 4 = 2 \Rightarrow B(\ln 4, 2)$

GRADIENT AT $B = \frac{1}{2}e^{\ln 4} = 2$

NORMAL GRADIENT AT $B = -\frac{1}{2}$

EQUATION OF NORMAL AT $B(\ln 4, 2) \Rightarrow$ $y - y_0 = m(x - x_0)$

$y - 2 = -\frac{1}{2}(x - \ln 4)$

$2y - 4 = -x + \ln 4$

$\boxed{2y + x = 4 + \ln 4}$

SOLVING SIMULTANEOUSLY

$\left.\begin{array}{l} 2y - x = 1 \\ 2y + x = 4 + \ln 4 \end{array}\right)$ ADD $4y = 5 + \ln 4$

$y = \frac{5}{4} + \frac{1}{4}\ln 4$

$y = \frac{5}{4} + \frac{1}{4} \times 2\ln 2$

$y = \frac{5}{4} + \frac{1}{2}\ln 2$

$y = \frac{5}{4} + \ln\sqrt{2}$

& SUBTRACTING

$2x = 3 + \ln 4$

$2x = 3 + 2\ln 2$

$x = \frac{3}{2} + \ln 2$

$\therefore P(\frac{3}{2} + \ln 2, \frac{5}{4} + \ln\sqrt{2})$ //

(c) B(ln4,2)

A₁ A₂ 2

0 ln4 4 4+ln4

NORMAL CROSSES x AXIS AT $x = 4 + \ln 4$

$\therefore A_2 = \frac{1}{2} \times 2 \times 4 = \underline{4}$

$A_2 = \int_0^{\ln 4} \frac{1}{2}e^x \, dx = \left[\frac{1}{2}e^x\right]_0^{\ln 4} = \frac{1}{2}e^{\ln 4} - \frac{1}{2} = 2 - \frac{1}{2} = \underline{\frac{3}{2}}$

$\therefore$ SHADED AREA $= 4 + \frac{3}{2} = \frac{11}{2}$ // AS REQUIRED

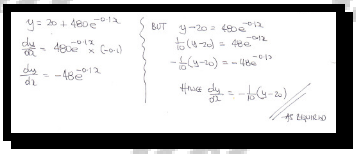
Question 148 **(****)**

$$y = 20 + 480\mathrm{e}^{-0.1x}, \ x \in \mathbb{R}.$$

Show clearly that

$$\frac{dy}{dx} = -\frac{1}{10}(y-20).$$

$\boxed{\text{proof}}$



$y = 20 + 480e^{-0.1x}$

$\frac{dy}{dx} = 480e^{-0.1x} \times (-0.1)$

$\frac{dy}{dx} = -48e^{-0.1x}$

BUT $y - 20 = 480e^{-0.1x}$

$\frac{1}{10}(y-20) = 48e^{-0.1x}$

$-\frac{1}{10}(y-20) = -48e^{-0.1x}$

HENCE $\frac{dy}{dx} = -\frac{1}{10}(y-20)$

AS REQUIRED

Question 149 **(****)**

A curve has equation

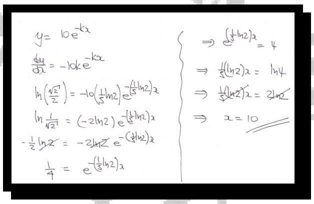
$$y = 10\mathrm{e}^{-kx}, \ x \in \mathbb{R}$$

where $k = \frac{1}{5}\ln 2$.

Find the value of x that satisfies the equation

$$\frac{dy}{dx} = \ln\left(\frac{\sqrt{2}}{2}\right).$$

$\boxed{\text{C3}}$, $\boxed{x = 10}$



$y = 10e^{-kx}$

$\frac{dy}{dx} = -10ke^{-kx}$

$\ln\left(\frac{\sqrt{2}}{2}\right) = -10\left(\frac{1}{5}\ln 2\right)e^{-\left(\frac{1}{5}\ln 2\right)x}$

$\ln\frac{1}{\sqrt{2}} = (-2\ln 2)e^{-\left(\frac{1}{5}\ln 2\right)x}$

$-\frac{1}{2}\cancel{\ln 2} = -2\cancel{\ln 2}\, e^{-\left(\frac{1}{5}\ln 2\right)x}$

$\frac{1}{4} = e^{-\left(\frac{1}{5}\ln 2\right)x}$

$\Rightarrow e^{\left(\frac{1}{5}\ln 2\right)x} = 4$

$\Rightarrow \frac{1}{5}(\ln 2)x = \ln 4$

$\Rightarrow \cancel{\frac{1}{5}(\ln 2)}x = 2\cancel{\ln 2}$

$\Rightarrow x = 10$

Question 150 (****)

A curve C has equation

$$y = \frac{1}{2}e^{2x} - 4x + 1, \; x \in \mathbb{R}.$$

The point P lies on C where $x = \ln 4$.

a) Show that the equation of the tangent to the curve at the point P is

$$y = 12x + 9 - 32\ln 2.$$

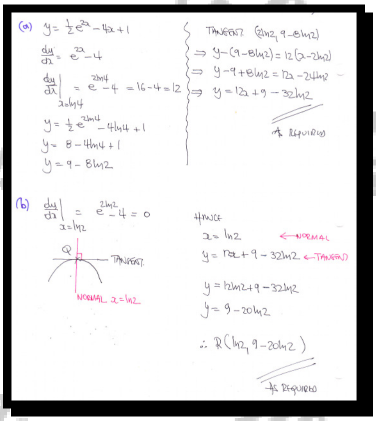
The point Q lies on C where $x = \ln 2$.

The normal to the curve at the point Q meets the tangent to the curve at the point P, at the point R.

b) Show that the coordinates of R are

$$(\ln 2, 9 - 20\ln 2).$$

$\boxed{\text{proof}}$



(a) $y = \frac{1}{2}e^{2x} - 4x + 1$

$\frac{dy}{dx} = e^{2x} - 4$

$\left.\frac{dy}{dx}\right|_{x=\ln 4} = e^{2\ln 4} - 4 = 16 - 4 = 12$

$y = \frac{1}{2}e^{2\ln 4} - 4\ln 4 + 1$

$y = 8 - 4\ln 4 + 1$

$y = 9 - 8\ln 2$

TANGENT $(2\ln 2, 9 - 8\ln 2)$

$\Rightarrow y - (9 - 8\ln 2) = 12(x - 2\ln 2)$

$\Rightarrow y - 9 + 8\ln 2 = 12x - 24\ln 2$

$\Rightarrow y = 12x + 9 - 32\ln 2$

AS REQUIRED

(b) $\left.\frac{dy}{dx}\right|_{x=\ln 2} = e^{2\ln 2} - 4 = 0$

HENCE

$x = \ln 2$ ← NORMAL

$y = 12x + 9 - 32\ln 2$ ← TANGENT

$y = 12\ln 2 + 9 - 32\ln 2$

$y = 9 - 20\ln 2$

$\therefore R(\ln 2, 9 - 20\ln 2)$

AS REQUIRED

Question 151 (****)

A curve C has equation

$$y = (x+1)^2 e^{2x}, \ x \in \mathbb{R}.$$

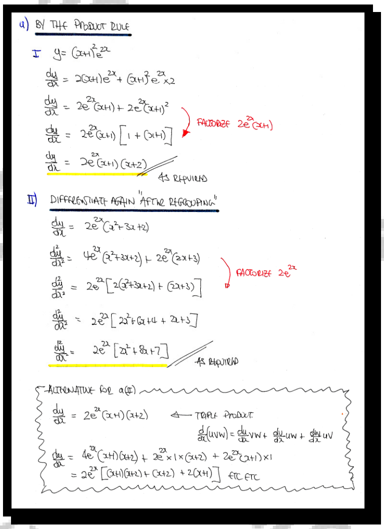
a) Show that

i. $\dfrac{dy}{dx} = 2(x+1)(x+2)e^{2x}$.

ii. $\dfrac{d^2y}{dx^2} = 2\left(2x^2+8x+7\right)e^{2x}$.

b) Hence, or otherwise, find the exact coordinates of the stationary points of C and determine their nature.

$\boxed{\text{[illegible]}}$, $\boxed{\min(-1,0)}$, $\boxed{\max\left(-2, e^{-4}\right)}$



a) BY THE PRODUCT RULE

I $y = (x+1)^2 e^{2x}$

$\frac{dy}{dx} = 2(x+1)e^{2x} + (x+1)^2 e^{2x} \times 2$

$\frac{dy}{dx} = 2e^{2x}(x+1) + 2e^{2x}(x+1)^2$

$\frac{dy}{dx} = 2e^{2x}(x+1)\left[1 + (x+1)\right]$

FACTORIZE $2e^{2x}(x+1)$

$\frac{dy}{dx} = 2e^{2x}(x+1)(x+2)$

AS REQUIRED

II) DIFFERENTIATE AGAIN "AFTER REGROUPING"

$\frac{dy}{dx} = 2e^{2x}(x^2+3x+2)$

$\frac{d^2y}{dx^2} = 4e^{2x}(x^2+3x+2) + 2e^{2x}(2x+3)$

$\frac{d^2y}{dx^2} = 2e^{2x}\left[2(x^2+3x+2) + (2x+3)\right]$

FACTORIZE $2e^{2x}$

$\frac{d^2y}{dx^2} = 2e^{2x}\left[2x^2+6x+4 + 2x+3\right]$

$\frac{d^2y}{dx^2} = 2e^{2x}\left[2x^2+8x+7\right]$

AS REQUIRED

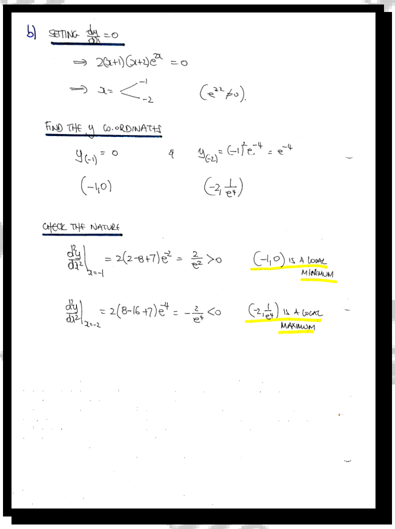
ALTERNATIVE FOR a(II)

$\frac{dy}{dx} = 2e^{2x}(x+1)(x+2)$ ← TRIPLE PRODUCT

$\frac{d}{dx}(uvw) = \frac{du}{dx}vw + \frac{dv}{dx}uw + \frac{dw}{dx}uv$

$\frac{dy}{dx} = 4e^{2x}(x+1)(x+2) + 2e^{2x} \times 1 \times (x+2) + 2e^{2x}(x+1) \times 1$

$= 2e^{2x}\left[(x+1)(x+2) + (x+2) + 2(x+1)\right]$ ETC ETC



b) SETTING $\frac{dy}{dx} = 0$

$\Rightarrow 2(x+1)(x+2)e^{2x} = 0$

$\Rightarrow x = \begin{cases} -1 \\ -2 \end{cases}$ $(e^{2x} \neq 0)$.

FIND THE y CO-ORDINATES

$y_{(-1)} = 0$ & $y_{(-2)} = (-1)^2 e^{-4} = e^{-4}$

$(-1,0)$ $\left(-2, \frac{1}{e^4}\right)$

CHECK THE NATURE

$\left.\frac{d^2y}{dx^2}\right|_{x=-1} = 2(2-8+7)e^{-2} = \frac{2}{e^2} > 0$ $(-1,0)$ IS A LOCAL MINIMUM

$\left.\frac{d^2y}{dx^2}\right|_{x=-2} = 2(8-16+7)e^{-4} = -\frac{2}{e^4} < 0$ $\left(-2, \frac{1}{e^4}\right)$ IS A LOCAL MAXIMUM

Question 152 (****)

A curve C is defined by the equation

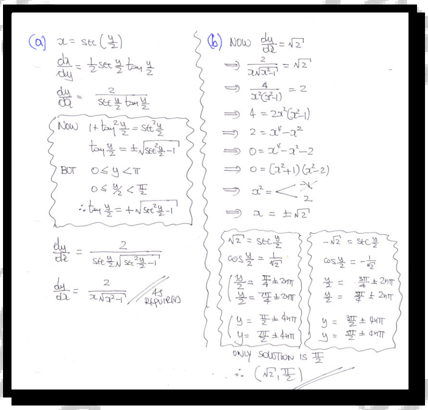
$$x = \sec\left(\frac{y}{2}\right),\ 0 \le y < \pi .$$

a) Show clearly that

$$\frac{dy}{dx} = \frac{2}{x\sqrt{x^2-1}} .$$

b) Hence find the exact coordinates of the point of C, where $\frac{dy}{dx} = \sqrt{2}$.

$\boxed{\text{C3I}}$, $\boxed{\left(\sqrt{2},\frac{\pi}{2}\right)}$



(a) $x = \sec\left(\frac{y}{2}\right)$

$\frac{dx}{dy} = \frac{1}{2}\sec\frac{y}{2}\tan\frac{y}{2}$

$\frac{dy}{dx} = \frac{2}{\sec\frac{y}{2}\tan\frac{y}{2}}$

NOW $1+\tan^2\frac{y}{2} = \sec^2\frac{y}{2}$

$\tan\frac{y}{2} = \pm\sqrt{\sec^2\frac{y}{2}-1}$

BUT $0 \le y < \pi$

$0 \le y/2 < \frac{\pi}{2}$

$\therefore \tan\frac{y}{2} = +\sqrt{\sec^2\frac{y}{2}-1}$

$\frac{dy}{dx} = \frac{2}{\sec\frac{y}{2}\sqrt{\sec^2\frac{y}{2}-1}}$

$\frac{dy}{dx} = \frac{2}{x\sqrt{x^2-1}}$ // AS REQUIRED

(b) NOW $\frac{dy}{dx} = \sqrt{2}$

$\Rightarrow \frac{2}{x\sqrt{x^2-1}} = \sqrt{2}$

$\Rightarrow \frac{4}{x^2(x^2-1)} = 2$

$\Rightarrow 4 = 2x^2(x^2-1)$

$\Rightarrow 2 = x^4 - x^2$

$\Rightarrow 0 = x^4 - x^2 - 2$

$\Rightarrow 0 = (x^2+1)(x^2-2)$

$\Rightarrow x^2 = \begin{cases} \cancel{-1} \\ 2 \end{cases}$

$\Rightarrow x = \pm\sqrt{2}$

$\sqrt{2} = \sec\frac{y}{2}$

$\cos\frac{y}{2} = \frac{1}{\sqrt{2}}$

$\frac{y}{2} = \frac{\pi}{4} \pm 2n\pi$

$\frac{y}{2} = \frac{7\pi}{4} \pm 2n\pi$

$y = \frac{\pi}{2} \pm 4n\pi$

$y = \frac{7\pi}{2} \pm 4n\pi$

$-\sqrt{2} = \sec\frac{y}{2}$

$\cos\frac{y}{2} = -\frac{1}{\sqrt{2}}$

$\frac{y}{2} = \frac{3\pi}{4} \pm 2n\pi$

$\frac{y}{2} = \frac{5\pi}{4} \pm 2n\pi$

$y = \frac{3\pi}{2} \pm 4n\pi$

$y = \frac{5\pi}{2} \pm 4n\pi$

ONLY SOLUTION IS $\frac{\pi}{2}$

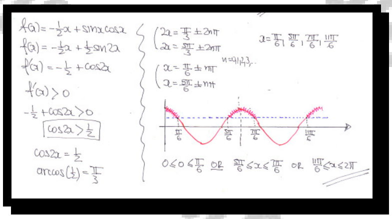
$\therefore \left(\sqrt{2}, \frac{\pi}{2}\right)$ //

Question 153 (****)

$$f(x) = -\frac{1}{2}x + \sin x \cos x, \ 0 \le x < 2\pi .$$

Use differentiation to find the range of values of x for which $f(x)$ is increasing.

$$\boxed{0 \le x < \frac{\pi}{6} \ \cup \ \frac{5\pi}{6} < x < \frac{7\pi}{6} \ \cup \ \frac{11\pi}{6} < x < 2\pi}$$



$f(x) = -\frac{1}{2}x + \sin x \cos x$

$f(x) = -\frac{1}{2}x + \frac{1}{2}\sin 2x$

$f'(x) = -\frac{1}{2} + \cos 2x$

$f'(x) > 0$

$-\frac{1}{2} + \cos 2x > 0$

$\boxed{\cos 2x > \frac{1}{2}}$

$\cos 2x = \frac{1}{2}$

$\arccos(\frac{1}{2}) = \frac{\pi}{3}$

$\begin{cases} 2x = \frac{\pi}{3} \pm 2n\pi \\ 2x = \frac{5\pi}{3} \pm 2n\pi \end{cases}$ $n = 0,1,2,3,\ldots$

$\begin{cases} x = \frac{\pi}{6} \pm n\pi \\ x = \frac{5\pi}{6} \pm n\pi \end{cases}$

$x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

$0 \leqslant 0 \leqslant \frac{\pi}{6}$ OR $\frac{5\pi}{6} \leqslant x \leqslant \frac{7\pi}{6}$ OR $\frac{11\pi}{6} \leqslant x \leqslant 2\pi$

Question 154 (****)

A curve has equation

$$y = \arcsin 2x\,,\ -\frac{1}{2} \le x \le \frac{1}{2}\,,\ -\frac{\pi}{2} \le y \le \frac{\pi}{2}\,.$$

a) By finding $\frac{dx}{dy}$ and using an appropriate trigonometric identity show that

$$\frac{dy}{dx} = \frac{2}{\sqrt{1-4x^2}}\,.$$

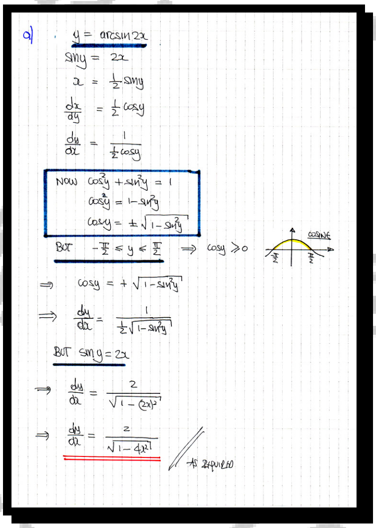
b) Show further that …

i. … $\dfrac{d^2y}{dx^2} = \dfrac{Ax}{\left(1-4x^2\right)^{\frac{3}{2}}}$,

ii. … $\dfrac{d^3y}{dx^3} = \dfrac{Bx^2+C}{\left(1-4x^2\right)^{\frac{5}{2}}}$,

where A, B and C are constants to be found.

proof



a) $y = \arcsin 2x$

$\sin y = 2x$

$x = \frac{1}{2}\sin y$

$\frac{dx}{dy} = \frac{1}{2}\cos y$

$\frac{dy}{dx} = \frac{1}{\frac{1}{2}\cos y}$

NOW $\cos^2 y + \sin^2 y = 1$

$\cos^2 y = 1 - \sin^2 y$

$\cos y = \pm\sqrt{1-\sin^2 y}$

BUT $-\frac{\pi}{2} \le y \le \frac{\pi}{2} \Rightarrow \cos y \geqslant 0$

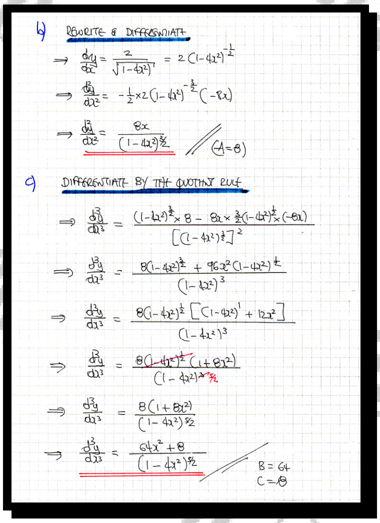
$\Rightarrow \cos y = +\sqrt{1-\sin^2 y}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\frac{1}{2}\sqrt{1-\sin^2 y}}$

BUT $\sin y = 2x$

$\Rightarrow \frac{dy}{dx} = \frac{2}{\sqrt{1-(2x)^2}}$

$\Rightarrow \frac{dy}{dx} = \frac{2}{\sqrt{1-4x^2}}$ // AS REQUIRED



b) REWRITE & DIFFERENTIATE

$\Rightarrow \frac{dy}{dx} = \frac{2}{\sqrt{1-4x^2}} = 2(1-4x^2)^{-\frac{1}{2}}$

$\Rightarrow \frac{d^2y}{dx^2} = -\frac{1}{2}\times 2(1-4x^2)^{-\frac{3}{2}}(-8x)$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{8x}{(1-4x^2)^{3/2}}$ // $(A=8)$

c) DIFFERENTIATE BY THE QUOTIENT RULE

$\Rightarrow \frac{d^3y}{dx^3} = \frac{(1-4x^2)^{\frac{3}{2}}\times 8 - 8x\times\frac{3}{2}(1-4x^2)^{\frac{1}{2}}\times(-8x)}{\left[(1-4x^2)^{\frac{3}{2}}\right]^2}$

$\Rightarrow \frac{d^3y}{dx^3} = \frac{8(1-4x^2)^{\frac{3}{2}} + 96x^2(1-4x^2)^{\frac{1}{2}}}{(1-4x^2)^3}$

$\Rightarrow \frac{d^3y}{dx^3} = \frac{8(1-4x^2)^{\frac{1}{2}}\left[(1-4x^2)^1 + 12x^2\right]}{(1-4x^2)^3}$

$\Rightarrow \frac{d^3y}{dx^3} = \frac{8\cancel{(1-4x^2)^{\frac{1}{2}}}(1+8x^2)}{(1-4x^2)^{\cancel{3}\,5/2}}$

$\Rightarrow \frac{d^3y}{dx^3} = \frac{8(1+8x^2)}{(1-4x^2)^{5/2}}$

$\Rightarrow \frac{d^3y}{dx^3} = \frac{64x^2+8}{(1-4x^2)^{5/2}}$ //

$B = 64$
$C = 8$

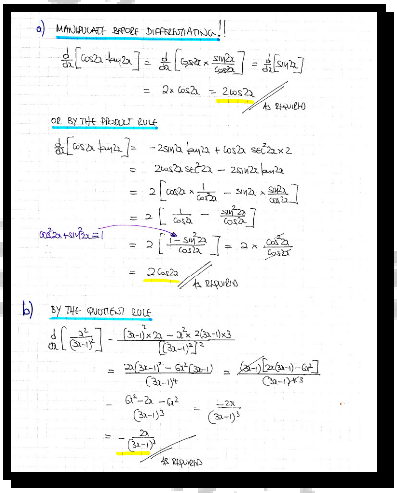
Question 155 (****)

Show, with detailed workings, that

a) $\frac{d}{dx}\left(\cos 2x \tan 2x\right) = 2\cos 2x$.

b) $\frac{d}{dx}\left(\frac{x^2}{(3x-1)^2}\right) = -\frac{2x}{(3x-1)^3}$.

$\boxed{}$, $\boxed{\text{proof}}$



a) MANIPULATE BEFORE DIFFERENTIATING!!

$$\frac{d}{dx}\left[\cos 2x \tan 2x\right] = \frac{d}{dx}\left[\cancel{\cos 2x} \times \frac{\sin 2x}{\cancel{\cos 2x}}\right] = \frac{d}{dx}\left[\sin 2x\right]$$

$$= 2 \times \cos 2x = 2\cos 2x$$

AS REQUIRED

OR BY THE PRODUCT RULE

$$\frac{d}{dx}\left[\cos 2x \tan 2x\right] = -2\sin 2x \tan 2x + \cos 2x \sec^2 2x \times 2$$

$$= 2\cos 2x \sec^2 2x - 2\sin 2x \tan 2x$$

$$= 2\left[\cos 2x \times \frac{1}{\cos^2 2x} - \sin 2x \times \frac{\sin 2x}{\cos 2x}\right]$$

$$= 2\left[\frac{1}{\cos 2x} - \frac{\sin^2 2x}{\cos 2x}\right]$$

$\cos^2 2x + \sin^2 2x \equiv 1$

$$= 2\left[\frac{1 - \sin^2 2x}{\cos 2x}\right] = 2 \times \frac{\cos^{\cancel{2}} 2x}{\cancel{\cos 2x}}$$

$$= 2\cos 2x$$

AS REQUIRED

b) BY THE QUOTIENT RULE

$$\frac{d}{dx}\left[\frac{x^2}{(3x-1)^2}\right] = \frac{(3x-1)^2 \times 2x - x^2 \times 2(3x-1) \times 3}{\left[(3x-1)^2\right]^2}$$

$$= \frac{2x(3x-1)^2 - 6x^2(3x-1)}{(3x-1)^4} = \frac{\cancel{(3x-1)}\left[2x(3x-1) - 6x^2\right]}{(3x-1)^{\cancel{4}3}}$$

$$= \frac{6x^2 - 2x - 6x^2}{(3x-1)^3} = \frac{-2x}{(3x-1)^3}$$

$$= -\frac{2x}{(3x-1)^3}$$

AS REQUIRED

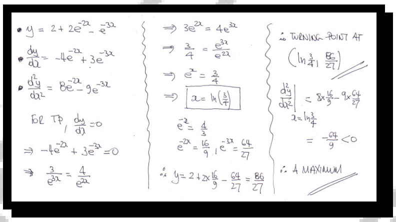
Question 156 (****)

A curve C has equation

$$y = 2 + 2e^{-2x} - e^{-3x}, \ x \in \mathbb{R}.$$

Find the exact coordinates of the stationary point of C and determine its nature.

$$\boxed{\max\left(\ln\left(\tfrac{3}{4}\right), \tfrac{86}{27}\right)}$$



- $y = 2 + 2e^{-2x} - e^{-3x}$
- $\frac{dy}{dx} = -4e^{-2x} + 3e^{-3x}$
- $\frac{d^2y}{dx^2} = 8e^{-2x} - 9e^{-3x}$

FOR TP, $\frac{dy}{dx} = 0$

$\Rightarrow -4e^{-2x} + 3e^{-3x} = 0$

$\Rightarrow \frac{3}{e^{3x}} = \frac{4}{e^{2x}}$

$\Rightarrow 3e^{2x} = 4e^{3x}$

$\Rightarrow \frac{3}{4} = \frac{e^{3x}}{e^{2x}}$

$\Rightarrow e^{x} = \frac{3}{4}$

$\Rightarrow \boxed{x = \ln\left(\frac{3}{4}\right)}$

$e^{-x} = \frac{4}{3}$

$e^{-2x} = \frac{16}{9}, \ e^{-3x} = \frac{64}{27}$

$\therefore y = 2 + 2 \times \frac{16}{9} - \frac{64}{27} = \frac{86}{27}$

$\therefore$ TURNING POINT AT

$\left(\ln\frac{3}{4}, \frac{86}{27}\right)$

$$\left.\frac{d^2y}{dx^2}\right|_{x = \ln\frac{3}{4}} = 8 \times \frac{16}{9} - 9 \times \frac{64}{27}$$

$$= -\frac{64}{9} < 0$$

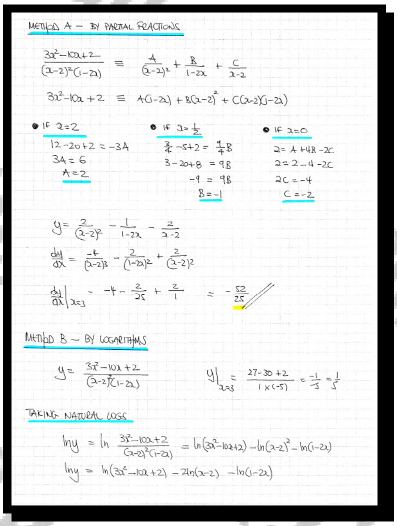
$\therefore$ A MAXIMUM

Question 157 (****)

$$y=\frac{3x^2-10x+2}{(1-2x)(x-2)^2}, \quad x\in\mathbb{R}, \quad x>2.$$

Find the exact value of $\frac{dy}{dx}$ at $x=3$.

$\boxed{}$, $\boxed{\left.\frac{dy}{dx}\right|_{x=3}=-\frac{52}{25}}$



METHOD A – BY PARTIAL FRACTIONS

$$\frac{3x^2-10x+2}{(x-2)^2(1-2x)} \equiv \frac{A}{(x-2)^2}+\frac{B}{1-2x}+\frac{C}{x-2}$$

$$3x^2-10x+2 \equiv A(1-2x)+B(x-2)^2+C(x-2)(1-2x)$$

- IF $x=2$
  $12-20+2=-3A$
  $3A=6$
  $A=2$
- IF $x=\frac{1}{2}$
  $\frac{3}{4}-5+2=\frac{9}{4}B$
  $3-20+8=9B$
  $-9=9B$
  $B=-1$
- IF $x=0$
  $2=A+4B-2C$
  $2=2-4-2C$
  $2C=-4$
  $C=-2$

$$y=\frac{2}{(x-2)^2}-\frac{1}{1-2x}-\frac{2}{x-2}$$

$$\frac{dy}{dx}=\frac{-4}{(x-2)^3}-\frac{2}{(1-2x)^2}+\frac{2}{(x-2)^2}$$

$$\left.\frac{dy}{dx}\right|_{x=3}=-4-\frac{2}{25}+\frac{2}{1} \quad = -\frac{52}{25}$$

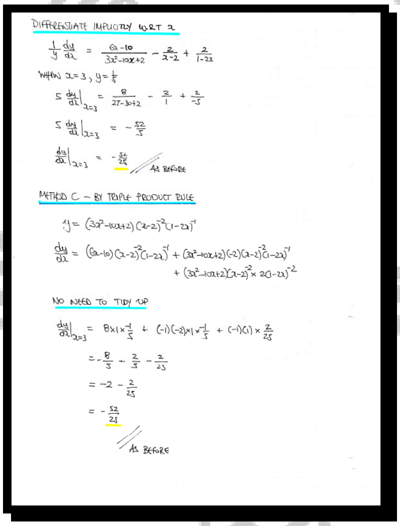
METHOD B – BY LOGARITHMS

$$y=\frac{3x^2-10x+2}{(x-2)^2(1-2x)} \qquad y\Big|_{x=3}=\frac{27-30+2}{1\times(-5)}=\frac{-1}{-5}=\frac{1}{5}$$

TAKING NATURAL LOGS

$$\ln y=\ln\frac{3x^2-10x+2}{(x-2)^2(1-2x)}=\ln(3x^2-10x+2)-\ln(x-2)^2-\ln(1-2x)$$

$$\ln y=\ln(3x^2-10x+2)-2\ln(x-2)-\ln(1-2x)$$



DIFFERENTIATE IMPLICITLY W.R.T x

$$\frac{1}{y}\frac{dy}{dx}=\frac{6x-10}{3x^2-10x+2}-\frac{2}{x-2}+\frac{2}{1-2x}$$

WHEN $x=3$, $y=\frac{1}{5}$

$$5\left.\frac{dy}{dx}\right|_{x=3}=\frac{8}{27-30+2}-\frac{2}{1}+\frac{2}{-5}$$

$$5\left.\frac{dy}{dx}\right|_{x=3}=-\frac{52}{5}$$

$$\left.\frac{dy}{dx}\right|_{x=3}=-\frac{52}{25}$$

AS BEFORE

METHOD C – BY TRIPLE PRODUCT RULE

$$y=(3x^2-10x+2)(x-2)^{-2}(1-2x)^{-1}$$

$$\frac{dy}{dx}=(6x-10)(x-2)^{-2}(1-2x)^{-1}+(3x^2-10x+2)(-2)(x-2)^{-2}(1-2x)^{-1}$$
$$+(3x^2-10x+2)(x-2)^{-2}\times 2(1-2x)^{-2}$$

NO NEED TO TIDY UP

$$\left.\frac{dy}{dx}\right|_{x=3}=8\times1\times\frac{-1}{5}+(-1)(-2)\times1\times\frac{-1}{5}+(-1)(1)\times\frac{2}{25}$$

$$=-\frac{8}{5}+\frac{2}{5}-\frac{2}{25}$$

$$=-2-\frac{2}{25}$$

$$=-\frac{52}{25}$$

AS BEFORE

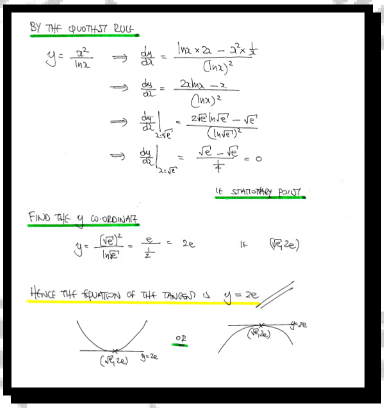
Question 158 **(****)**

A curve C has equation

$$y = \frac{x^2}{\ln x}, \quad x \in \mathbb{R}, \quad x > 0 .$$

Find, in exact form, the equation of the tangent to C at the point where $x = \sqrt{\mathrm{e}}$.

$\boxed{}$, $\boxed{y = 2\mathrm{e}}$



BY THE QUOTIENT RULE

$$y = \frac{x^2}{\ln x} \implies \frac{dy}{dx} = \frac{\ln x \times 2x - x^2 \times \frac{1}{x}}{(\ln x)^2}$$

$$\implies \frac{dy}{dx} = \frac{2x\ln x - x}{(\ln x)^2}$$

$$\implies \left.\frac{dy}{dx}\right|_{x=\sqrt{e}} = \frac{2\sqrt{e}\ln\sqrt{e} - \sqrt{e}}{(\ln\sqrt{e})^2}$$

$$\implies \left.\frac{dy}{dx}\right|_{x=\sqrt{e}} = \frac{\sqrt{e} - \sqrt{e}}{\frac{1}{4}} = 0$$

IE STATIONARY POINT

FIND THE y CO-ORDINATE

$$y = \frac{(\sqrt{e})^2}{\ln\sqrt{e}} = \frac{e}{\frac{1}{2}} = 2e \qquad \text{IE} \qquad (\sqrt{e}, 2e)$$

HENCE THE EQUATION OF THE TANGENT IS $y = 2e$

Question 159 (****)

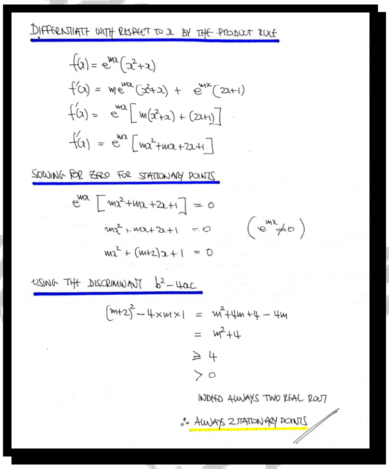
The function f is given by

$$f(x) \equiv \mathrm{e}^{mx}\left(x^2 + x\right),\ x \in \mathbb{R},$$

where m is a non zero constant.

Show that f has two stationary points, for all non zero values of m.

$\boxed{}$, $\boxed{\text{proof}}$



DIFFERENTIATE WITH RESPECT TO x BY THE PRODUCT RULE

$$f(x) = e^{mx}(x^2+x)$$
$$f'(x) = me^{mx}(x^2+x) + e^{mx}(2x+1)$$
$$f'(x) = e^{mx}\left[m(x^2+x) + (2x+1)\right]$$
$$f'(x) = e^{mx}\left[mx^2+mx+2x+1\right]$$

SOLVING FOR ZERO FOR STATIONARY POINTS

$$e^{mx}\left[mx^2+mx+2x+1\right] = 0$$
$$mx^2+mx+2x+1 = 0 \qquad (e^{mx} \neq 0)$$
$$mx^2+(m+2)x+1 = 0$$

USING THE DISCRIMINANT b^2-4ac

$$(m+2)^2 - 4 \times m \times 1 = m^2+4m+4-4m$$
$$= m^2+4$$
$$\geqslant 4$$
$$> 0$$

INDEED ALWAYS TWO REAL ROOT

$\therefore$ ALWAYS 2 STATIONARY POINTS

Question 160 (****)

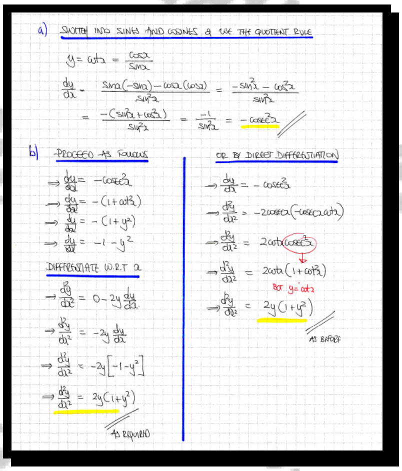
$$y = \cot x, \ 0 < x < \tfrac{\pi}{2}.$$

Show, with detailed workings, that

a) $\dfrac{dy}{dx} = -\operatorname{cosec}^2 x$.

b) $\dfrac{d^2y}{dx^2} = 2y\left(y^2+1\right)$.

$\boxed{}$, $\boxed{\text{proof}}$



a) SWITCH INTO SINES AND COSINES & USE THE QUOTIENT RULE

$y = \cot x = \dfrac{\cos x}{\sin x}$

$\dfrac{dy}{dx} = \dfrac{\sin x(-\sin x) - \cos x(\cos x)}{\sin^2 x} = \dfrac{-\sin^2 x - \cos^2 x}{\sin^2 x}$

$= \dfrac{-(\sin^2 x + \cos^2 x)}{\sin^2 x} = \dfrac{-1}{\sin^2 x} = -\operatorname{cosec}^2 x$ //

b) PROCEED AS FOLLOWS

$\Rightarrow \dfrac{dy}{dx} = -\operatorname{cosec}^2 x$

$\Rightarrow \dfrac{dy}{dx} = -(1+\cot^2 x)$

$\Rightarrow \dfrac{dy}{dx} = -(1+y^2)$

$\Rightarrow \dfrac{dy}{dx} = -1-y^2$

DIFFERENTIATE W.R.T x

$\Rightarrow \dfrac{d^2y}{dx^2} = 0 - 2y\dfrac{dy}{dx}$

$\Rightarrow \dfrac{d^2y}{dx^2} = -2y\dfrac{dy}{dx}$

$\Rightarrow \dfrac{d^2y}{dx^2} = -2y\left[-1-y^2\right]$

$\Rightarrow \dfrac{d^2y}{dx^2} = 2y(1+y^2)$ //

AS REQUIRED

OR BY DIRECT DIFFERENTIATION

$\Rightarrow \dfrac{dy}{dx} = -\operatorname{cosec}^2 x$

$\Rightarrow \dfrac{d^2y}{dx^2} = -2\operatorname{cosec} x(-\operatorname{cosec} x\cot x)$

$\Rightarrow \dfrac{d^2y}{dx^2} = 2\cot x\operatorname{cosec}^2 x$

$\Rightarrow \dfrac{d^2y}{dx^2} = 2\cot x(1+\cot^2 x)$

BUT $y = \cot x$

$\Rightarrow \dfrac{d^2y}{dx^2} = 2y(1+y^2)$ //

AS BEFORE

Question 161 (****)

The curve C has equation

$$x = \sec^2 y + \tan y\,,\ 0 \le y < \frac{\pi}{2}\,.$$

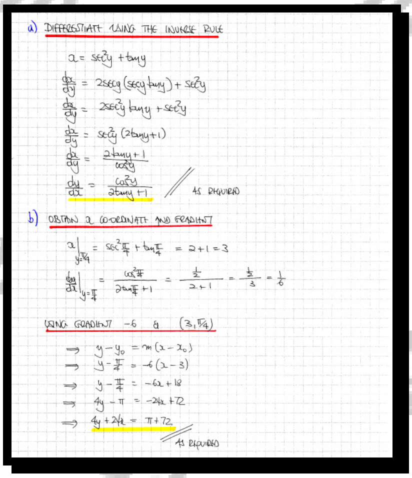
a) Show that

$$\frac{dy}{dx} = \frac{\cos^2 y}{2\tan y + 1}\,.$$

b) Hence show that the equation of the normal to C at the point where $y = \frac{\pi}{4}$ is

$$4y + 24x = \pi + 72\,.$$

☐ , proof



a) DIFFERENTIATE USING THE INVERSE RULE

$$x = \sec^2 y + \tan y$$

$$\frac{dx}{dy} = 2\sec y(\sec y \tan y) + \sec^2 y$$

$$\frac{dx}{dy} = 2\sec^2 y \tan y + \sec^2 y$$

$$\frac{dx}{dy} = \sec^2 y(2\tan y + 1)$$

$$\frac{dx}{dy} = \frac{2\tan y + 1}{\cos^2 y}$$

$$\frac{dy}{dx} = \frac{\cos^2 y}{2\tan y + 1} \quad // \quad \text{AS REQUIRED}$$

b) OBTAIN x COORDINATE AND GRADIENT

$$x\Big|_{y=\frac{\pi}{4}} = \sec^2\frac{\pi}{4} + \tan\frac{\pi}{4} = 2 + 1 = 3$$

$$\frac{dy}{dx}\Big|_{y=\frac{\pi}{4}} = \frac{\cos^2\frac{\pi}{4}}{2\tan\frac{\pi}{4} + 1} = \frac{\frac{1}{2}}{2+1} = \frac{\frac{1}{2}}{3} = \frac{1}{6}$$

USING GRADIENT -6 & $(3, \frac{\pi}{4})$

$$\Rightarrow y - y_0 = m(x - x_0)$$

$$\Rightarrow y - \frac{\pi}{4} = -6(x - 3)$$

$$\Rightarrow y - \frac{\pi}{4} = -6x + 18$$

$$\Rightarrow 4y - \pi = -24x + 72$$

$$\Rightarrow 4y + 24x = \pi + 72 \quad // \quad \text{AS REQUIRED}$$

Question 162 (****)

$$y = \arcsin x,\ -\frac{1}{2} \le x \le \frac{1}{2},\ -\frac{\pi}{2} \le y \le \frac{\pi}{2}.$$

a) By finding $\frac{dx}{dy}$ and using an appropriate trigonometric identity show that

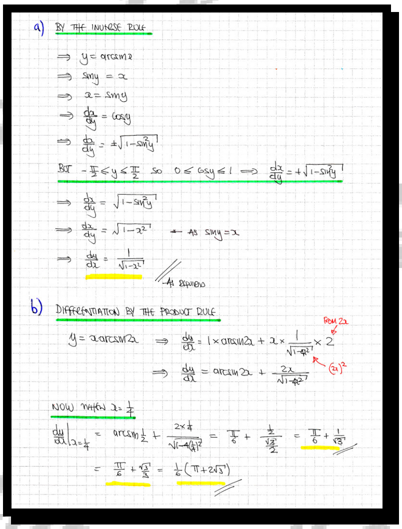
$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}.$$

A curve C has equation

$$y = x \arcsin 2x,\ -\frac{1}{2} \le x \le \frac{1}{2},\ -\frac{\pi}{2} \le y \le \frac{\pi}{2}.$$

b) Find the exact value of $\frac{dy}{dx}$ at the point on C where $x = \frac{1}{4}$.

$\boxed{}$, $\boxed{\frac{1}{6}\left(\pi + 2\sqrt{3}\right)}$



a) BY THE INVERSE RULE

$\Rightarrow y = \arcsin x$

$\Rightarrow \sin y = x$

$\Rightarrow x = \sin y$

$\Rightarrow \frac{dx}{dy} = \cos y$

$\Rightarrow \frac{dx}{dy} = \pm\sqrt{1-\sin^2 y}$

BUT $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ SO $0 \le \cos y \le 1 \Rightarrow \frac{dx}{dy} = +\sqrt{1-\sin^2 y}$

$\Rightarrow \frac{dx}{dy} = \sqrt{1-\sin^2 y}$

$\Rightarrow \frac{dx}{dy} = \sqrt{1-x^2}$ ← AS $\sin y = x$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$ // AS REQUIRED

b) DIFFERENTIATION BY THE PRODUCT RULE

$y = x\arcsin 2x \Rightarrow \frac{dy}{dx} = 1 \times \arcsin 2x + x \times \frac{1}{\sqrt{1-4x^2}} \times 2$ FROM $2x$ $(2x)^2$

$\Rightarrow \frac{dy}{dx} = \arcsin 2x + \frac{2x}{\sqrt{1-4x^2}}$

NOW WHEN $x = \frac{1}{4}$

$\left.\frac{dy}{dx}\right|_{x=\frac{1}{4}} = \arcsin\frac{1}{2} + \frac{2 \times \frac{1}{4}}{\sqrt{1-4\left(\frac{1}{4}\right)^2}} = \frac{\pi}{6} + \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{\pi}{6} + \frac{1}{\sqrt{3}}$ //

$= \frac{\pi}{6} + \frac{\sqrt{3}}{3} = \frac{1}{6}\left(\pi + 2\sqrt{3}\right)$ //

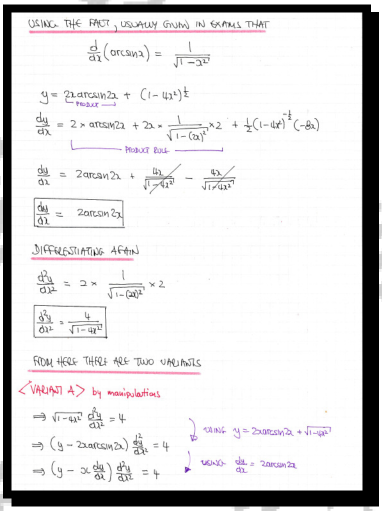
Question 163 (****)

$$y = 2x\arcsin 2x + \sqrt{1-4x^2}\ ,\ -\tfrac{1}{2} \le x \le \tfrac{1}{2}.$$

Show clearly that

$$\frac{d^3y}{dx^3}\left(y - x\frac{dy}{dx}\right) = x\left(\frac{d^2y}{dx^2}\right)^2 .$$

$\boxed{}$, $\boxed{\text{proof}}$



USING THE FACT, USUALLY GIVEN IN EXAMS THAT

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$$

$$y = \underbrace{2x\arcsin 2x}_{\text{PRODUCT}} + (1-4x^2)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \underbrace{2 \times \arcsin 2x + 2x \times \frac{1}{\sqrt{1-(2x)^2}} \times 2}_{\text{PRODUCT RULE}} + \tfrac{1}{2}(1-4x^2)^{-\frac{1}{2}}(-8x)$$

$$\frac{dy}{dx} = 2\arcsin 2x + \cancel{\frac{4x}{\sqrt{1-4x^2}}} - \cancel{\frac{4x}{\sqrt{1-4x^2}}}$$

$$\boxed{\frac{dy}{dx} = 2\arcsin 2x}$$

DIFFERENTIATING AGAIN

$$\frac{d^2y}{dx^2} = 2 \times \frac{1}{\sqrt{1-(2x)^2}} \times 2$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{4}{\sqrt{1-4x^2}}}$$

FROM HERE THERE ARE TWO VARIANTS

<VARIANT A> by manipulations

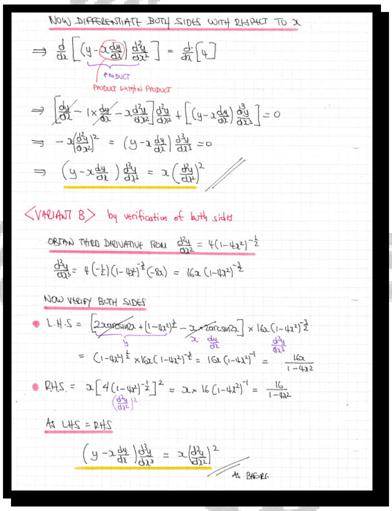
$$\Rightarrow \sqrt{1-4x^2}\,\frac{d^2y}{dx^2} = 4$$

USING $y = 2x\arcsin 2x + \sqrt{1-4x^2}$

$$\Rightarrow (y - 2x\arcsin 2x)\frac{d^2y}{dx^2} = 4$$

USING $\frac{dy}{dx} = 2\arcsin 2x$

$$\Rightarrow \left(y - x\frac{dy}{dx}\right)\frac{d^2y}{dx^2} = 4$$



NOW DIFFERENTIATE BOTH SIDES WITH RESPECT TO x

$$\Rightarrow \frac{d}{dx}\left[\underbrace{\left(y - x\frac{dy}{dx}\right)\frac{d^2y}{dx^2}}_{\text{PRODUCT}}\right] = \frac{d}{dx}[4]$$

PRODUCT WITHIN PRODUCT

$$\Rightarrow \left[\cancel{\frac{dy}{dx}} - 1 \times \cancel{\frac{dy}{dx}} - x\frac{d^2y}{dx^2}\right]\frac{d^2y}{dx^2} + \left[\left(y - x\frac{dy}{dx}\right)\frac{d^3y}{dx^3}\right] = 0$$

$$\Rightarrow -x\left(\frac{d^2y}{dx^2}\right)^2 = \left(y - x\frac{dy}{dx}\right)\frac{d^3y}{dx^3} = 0$$

$$\Rightarrow \left(y - x\frac{dy}{dx}\right)\frac{d^3y}{dx^3} = x\left(\frac{d^2y}{dx^2}\right)^2 \quad //$$

<VARIANT B> by verification of both sides

OBTAIN THIRD DERIVATIVE FROM $\frac{d^2y}{dx^2} = 4(1-4x^2)^{-\frac{1}{2}}$

$$\frac{d^3y}{dx^3} = 4\left(-\tfrac{1}{2}\right)(1-4x^2)^{-\frac{3}{2}}(-8x) = 16x(1-4x^2)^{-\frac{3}{2}}$$

NOW VERIFY BOTH SIDES

- $\text{L.H.S} = \Big[\underbrace{\cancel{2x\arcsin 2x} + (1-4x^2)^{\frac{1}{2}}}_{y} - \cancel{\underset{x}{x} \times \underset{\frac{dy}{dx}}{2\arcsin 2x}}\Big] \times \underset{\frac{d^3y}{dx^3}}{16x(1-4x^2)^{-\frac{3}{2}}}$

  $= (1-4x^2)^{\frac{1}{2}} \times 16x(1-4x^2)^{-\frac{3}{2}} = 16x(1-4x^2)^{-1} = \frac{16x}{1-4x^2}$

- $\text{R.H.S.} = x\Big[\underset{\left(\frac{d^2y}{dx^2}\right)^2}{4(1-4x^2)^{-\frac{1}{2}}}\Big]^2 = x \times 16(1-4x^2)^{-1} = \frac{16}{1-4x^2}$

AS LHS = RHS

$$\left(y - x\frac{dy}{dx}\right)\frac{d^3y}{dx^3} = x\left(\frac{d^2y}{dx^2}\right)^2 \quad //$$

AS BEFORE

Question 164 (****+)

The curve C has equation

$$y = x\sqrt{\ln x}\,,\ x > 0\,.$$

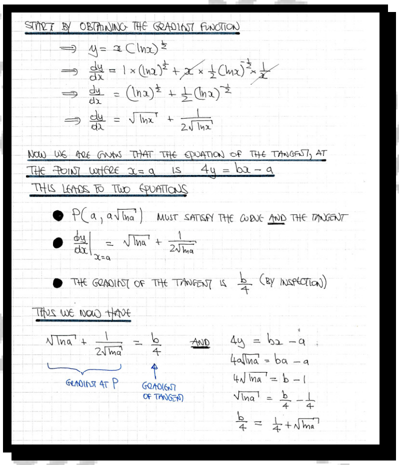
The equation of the tangent to C at the point where $x = a$ is

$$4y = bx - a\,,$$

where a and b are non zero constants.

Determine the exact value of a.

$\boxed{\text{SY}}$, $\boxed{a = \mathrm{e}^4}$



START BY OBTAINING THE GRADIENT FUNCTION

$$\Rightarrow y = x(\ln x)^{\frac{1}{2}}$$

$$\Rightarrow \frac{dy}{dx} = 1 \times (\ln x)^{\frac{1}{2}} + \cancel{x} \times \frac{1}{2}(\ln x)^{-\frac{1}{2}} \times \cancel{\frac{1}{x}}$$

$$\Rightarrow \frac{dy}{dx} = (\ln x)^{\frac{1}{2}} + \frac{1}{2}(\ln x)^{-\frac{1}{2}}$$

$$\Rightarrow \frac{dy}{dx} = \sqrt{\ln x} + \frac{1}{2\sqrt{\ln x}}$$

NOW WE ARE GIVEN THAT THE EQUATION OF THE TANGENT, AT THE POINT WHERE $x = a$ IS $4y = bx - a$

THIS LEADS TO TWO EQUATIONS

- $P(a, a\sqrt{\ln a})$ MUST SATISFY THE CURVE AND THE TANGENT
- $\left.\frac{dy}{dx}\right|_{x=a} = \sqrt{\ln a} + \frac{1}{2\sqrt{\ln a}}$
- THE GRADIENT OF THE TANGENT IS $\frac{b}{4}$ (BY INSPECTION)

THUS WE NOW HAVE

$$\underbrace{\sqrt{\ln a} + \frac{1}{2\sqrt{\ln a}}}_{\text{GRADIENT AT } P} = \overset{\uparrow}{\frac{b}{4}}$$

GRADIENT OF TANGENT

AND

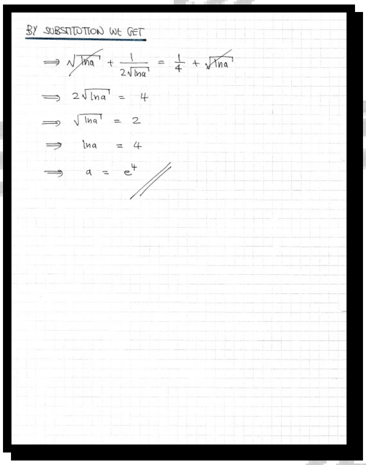
$$4y = bx - a$$

$$4a\sqrt{\ln a} = ba - a$$

$$4\sqrt{\ln a} = b - 1$$

$$\sqrt{\ln a} = \frac{b}{4} - \frac{1}{4}$$

$$\frac{b}{4} = \frac{1}{4} + \sqrt{\ln a}$$



BY SUBSTITUTION WE GET

$$\Rightarrow \cancel{\sqrt{\ln a}} + \frac{1}{2\sqrt{\ln a}} = \frac{1}{4} + \cancel{\sqrt{\ln a}}$$

$$\Rightarrow 2\sqrt{\ln a} = 4$$

$$\Rightarrow \sqrt{\ln a} = 2$$

$$\Rightarrow \ln a = 4$$

$$\Rightarrow a = e^4 \ //$$

Question 165 (****+)

$$y = \arccos x, \quad -1 \le x \le 1, \; 0 \le y \le \pi .$$

a) Prove that

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}} .$$

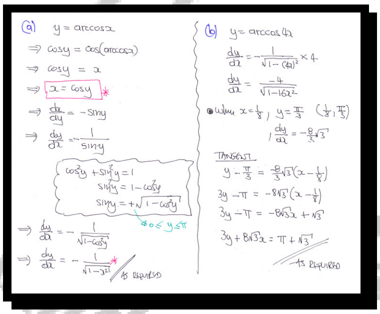
The curve C has equation given by

$$y = \arccos 4x, \; -\tfrac{1}{4} \le x \le \tfrac{1}{4} .$$

b) Show that an equation of the tangent to C where $x = \frac{1}{8}$, is given by

$$3y + 8\sqrt{3}x = \pi + \sqrt{3} .$$

$\boxed{\text{proof}}$



(a) $y = \arccos x$

$\Rightarrow \cos y = \cos(\arccos x)$

$\Rightarrow \cos y = x$

$\Rightarrow \boxed{x = \cos y} *$

$\Rightarrow \frac{dx}{dy} = -\sin y$

$\Rightarrow \frac{dy}{dx} = -\frac{1}{\sin y}$

$\cos^2 y + \sin^2 y = 1$

$\sin^2 y = 1 - \cos^2 y$

$\sin y = +\sqrt{1-\cos^2 y}$

$\uparrow 0 \le y \le \pi$

$\Rightarrow \frac{dy}{dx} = -\frac{1}{\sqrt{1-\cos^2 y}}$

$\Rightarrow \frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}} *$ // AS REQUIRED

(b) $y = \arccos 4x$

$\frac{dy}{dx} = -\frac{1}{\sqrt{1-(4x)^2}} \times 4$

$\frac{dy}{dx} = \frac{-4}{\sqrt{1-16x^2}}$

● when $x = \frac{1}{8}$, $y = \frac{\pi}{3}$ $\left(\frac{1}{8}, \frac{\pi}{3}\right)$

, $\frac{dy}{dx} = -\frac{8}{3}\sqrt{3}$

TANGENT

$y - \frac{\pi}{3} = -\frac{8}{3}\sqrt{3}\left(x - \frac{1}{8}\right)$

$3y - \pi = -8\sqrt{3}\left(x - \frac{1}{8}\right)$

$3y - \pi = -8\sqrt{3}x + \sqrt{3}$

$3y + 8\sqrt{3}x = \pi + \sqrt{3}$

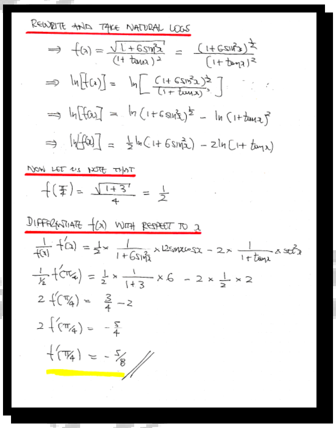
// AS REQUIRED

Question 166 (****+)

$$f(x) \equiv \frac{\sqrt{1+6\sin^2 x}}{(1+\tan x)^2}, \ \tan x \neq -1.$$

By using logarithmic differentiation, or otherwise, determine the value of $f'\left(\frac{\pi}{4}\right)$.

$$\boxed{}, \ \boxed{f'\left(\frac{\pi}{4}\right) = -\frac{5}{8}}$$



REWRITE AND TAKE NATURAL LOGS

$$\Rightarrow f(x) = \frac{\sqrt{1+6\sin^2 x}}{(1+\tan x)^2} = \frac{(1+6\sin^2 x)^{\frac{1}{2}}}{(1+\tan x)^2}$$

$$\Rightarrow \ln[f(x)] = \ln\left[\frac{(1+6\sin^2 x)^{\frac{1}{2}}}{(1+\tan x)^2}\right]$$

$$\Rightarrow \ln[f(x)] = \ln(1+6\sin^2 x)^{\frac{1}{2}} - \ln(1+\tan x)^2$$

$$\Rightarrow \ln[f(x)] = \frac{1}{2}\ln(1+6\sin^2 x) - 2\ln(1+\tan x)$$

NOW LET US NOTE THAT

$$f\left(\frac{\pi}{4}\right) = \frac{\sqrt{1+3}}{4} = \frac{1}{2}$$

DIFFERENTIATE $f(x)$ WITH RESPECT TO x

$$\frac{1}{f(x)} f'(x) = \frac{1}{2} \times \frac{1}{1+6\sin^2 x} \times 12\sin x\cos x - 2 \times \frac{1}{1+\tan x} \times \sec^2 x$$

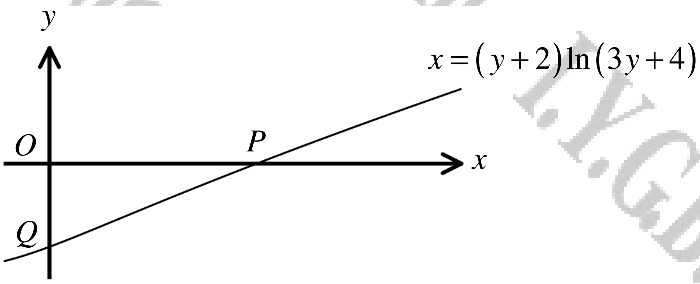
$$\frac{1}{\frac{1}{2}} f'(\pi/4) = \frac{1}{2} \times \frac{1}{1+3} \times 6 - 2 \times \frac{1}{2} \times 2$$

$$2f'(\pi/4) = \frac{3}{4} - 2$$

$$2f'(\pi/4) = -\frac{5}{4}$$

$$f'(\pi/4) = -\frac{5}{8}$$

Question 167 (****+)



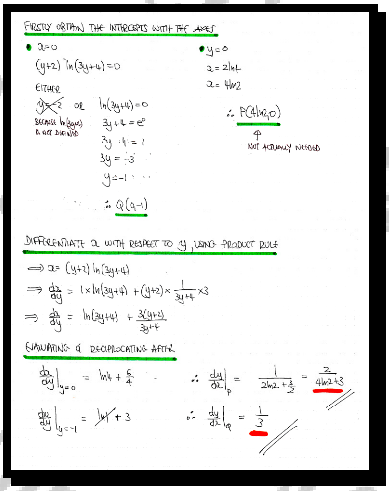
The figure above shows the graph of the curve with equation

$$x = (y+2)\ln(3y+4).$$

The curve meets the coordinate axes at the point P and at the point Q.

Determine the gradient, in exact from where appropriate, at P and at Q.

$\boxed{\quad}$, $\boxed{\left.\frac{dy}{dx}\right|_P = \frac{2}{3+4\ln 2}}$, $\boxed{\left.\frac{dy}{dx}\right|_Q = \frac{1}{3}}$



FIRSTLY OBTAIN THE INTERCEPTS WITH THE AXES

- $x=0$

$(y+2)\ln(3y+4)=0$

EITHER

$\cancel{y=-2}$ OR $\ln(3y+4)=0$

BECAUSE $\ln(3y+4)$ IS NOT DEFINED

$3y+4=e^0$

$3y+4=1$

$3y=-3$

$y=-1$

$\therefore Q(0,-1)$

- $y=0$

$x=2\ln 4$

$x=4\ln 2$

$\therefore P(4\ln 2, 0)$

NOT ACTUALLY NEEDED

DIFFERENTIATE x WITH RESPECT TO y, USING PRODUCT RULE

$\Rightarrow x=(y+2)\ln(3y+4)$

$\Rightarrow \frac{dx}{dy} = 1\times\ln(3y+4) + (y+2)\times\frac{1}{3y+4}\times 3$

$\Rightarrow \frac{dx}{dy} = \ln(3y+4) + \frac{3(y+2)}{3y+4}$

EVALUATING & RECIPROCATING AFTER

$\left.\frac{dx}{dy}\right|_{y=0} = \ln 4 + \frac{6}{4}$ $\therefore \left.\frac{dy}{dx}\right|_P = \frac{1}{2\ln 2+\frac{3}{2}} = \frac{2}{4\ln 2+3}$

$\left.\frac{dx}{dy}\right|_{y=-1} = \cancel{\ln 1} + 3$ $\therefore \left.\frac{dy}{dx}\right|_Q = \frac{1}{3}$

Question 168 (****+)

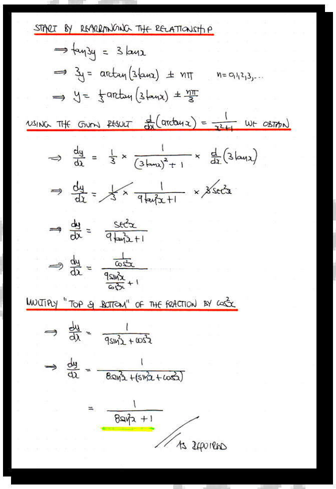
It is given that

$$\frac{d}{dx}\left(\arctan x\right) = \frac{1}{1+x^2}.$$

If $\tan 3y = 3\tan x$ show that

$$\frac{dy}{dx} = \frac{1}{1+8\sin^2 x}.$$

$\boxed{}$, $\boxed{\text{proof}}$



START BY REARRANGING THE RELATIONSHIP

$$\Rightarrow \tan 3y = 3\tan x$$

$$\Rightarrow 3y = \arctan(3\tan x) \pm n\pi \qquad n=0,1,2,3,\ldots$$

$$\Rightarrow y = \frac{1}{3}\arctan(3\tan x) \pm \frac{n\pi}{3}$$

USING THE GIVEN RESULT $\frac{d}{dx}(\arctan x) = \frac{1}{x^2+1}$ WE OBTAIN

$$\Rightarrow \frac{dy}{dx} = \frac{1}{3} \times \frac{1}{(3\tan x)^2+1} \times \frac{d}{dx}(3\tan x)$$

$$\Rightarrow \frac{dy}{dx} = \cancel{\frac{1}{3}} \times \frac{1}{9\tan^2 x+1} \times \cancel{3}\sec^2 x$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sec^2 x}{9\tan^2 x+1}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{1}{\cos^2 x}}{\frac{9\sin^2 x}{\cos^2 x}+1}$$

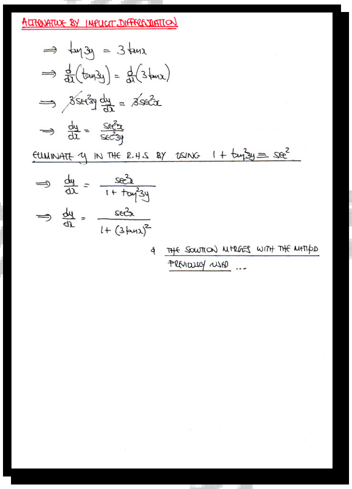
MULTIPLY "TOP & BOTTOM" OF THE FRACTION BY $\cos^2 x$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{9\sin^2 x+\cos^2 x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{8\sin^2 x+(\sin^2 x+\cos^2 x)}$$

$$= \frac{1}{8\sin^2 x+1}$$

// AS REQUIRED



ALTERNATIVE BY IMPLICIT DIFFERENTIATION

$$\Rightarrow \tan 3y = 3\tan x$$

$$\Rightarrow \frac{d}{dx}(\tan 3y) = \frac{d}{dx}(3\tan x)$$

$$\Rightarrow \cancel{3}\sec^2 3y \frac{dy}{dx} = \cancel{3}\sec^2 x$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sec^2 x}{\sec^2 3y}$$

ELIMINATE y IN THE R.H.S BY USING $1+\tan^2 3y \equiv \sec^2$

$$\Rightarrow \frac{dy}{dx} = \frac{\sec^2 x}{1+\tan^2 3y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sec^2 x}{1+(3\tan x)^2}$$

& THE SOLUTION MERGES WITH THE METHOD PREVIOUSLY USED ...

Question 169 **(****+)**

The curve C has equation

$$y = 4e^{-x} - 2e^{-2x} - e^{-3x}, \ x \in \mathbb{R}.$$

a) Show clearly that …

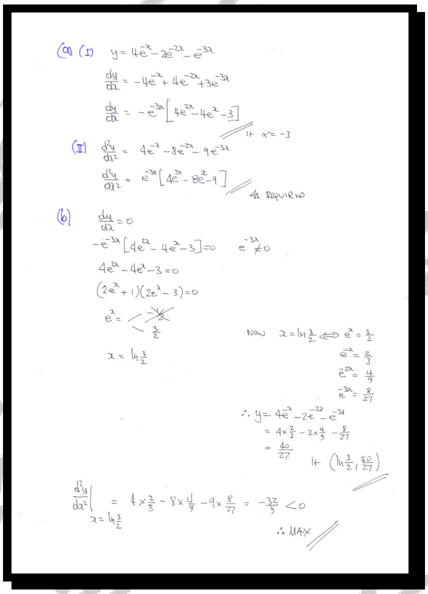
i. … $\frac{dy}{dx} = -e^{\alpha x}\left(4e^{2x} - 4e^{x} - 3\right)$,

ii. … $\frac{d^2y}{dx^2} = e^{\alpha x}\left(4e^{2x} - 8e^{x} - 9\right)$,

where α is a constant to be found.

b) Hence find the exact coordinates of the stationary point of C and determine its nature.

$$\boxed{\max\left(\ln\left(\frac{3}{2}\right), \frac{40}{27}\right)}$$



(a) (I) $y = 4e^{-x} - 2e^{-2x} - e^{-3x}$

$\frac{dy}{dx} = -4e^{-x} + 4e^{-2x} + 3e^{-3x}$

$\frac{dy}{dx} = -e^{-3x}\left[4e^{2x} - 4e^{x} - 3\right]$

It $\alpha = -3$

(II) $\frac{d^2y}{dx^2} = 4e^{-x} - 8e^{-2x} - 9e^{-3x}$

$\frac{d^2y}{dx^2} = e^{-3x}\left[4e^{2x} - 8e^{x} - 9\right]$

AS REQUIRED

(b) $\frac{dy}{dx} = 0$

$-e^{-3x}\left[4e^{2x} - 4e^{x} - 3\right] = 0 \qquad e^{-3x} \neq 0$

$4e^{2x} - 4e^{x} - 3 = 0$

$(2e^{x} + 1)(2e^{x} - 3) = 0$

$e^{x} = \cancel{-\frac{1}{2}}, \ \frac{3}{2}$

$x = \ln\frac{3}{2}$

Now $x = \ln\frac{3}{2} \Longleftrightarrow e^{x} = \frac{3}{2}$

$e^{-x} = \frac{2}{3}$

$e^{-2x} = \frac{4}{9}$

$e^{-3x} = \frac{8}{27}$

$\therefore y = 4e^{-x} - 2e^{-2x} - e^{-3x}$

$= 4 \times \frac{2}{3} - 2 \times \frac{4}{9} - \frac{8}{27}$

$= \frac{40}{27}$

It $\left(\ln\frac{3}{2}, \frac{40}{27}\right)$

$\left.\frac{d^2y}{dx^2}\right|_{x = \ln\frac{3}{2}} = 4 \times \frac{2}{3} - 8 \times \frac{4}{9} - 9 \times \frac{8}{27} = -\frac{32}{9} < 0$

$\therefore$ MAX

Question 170 (****+)

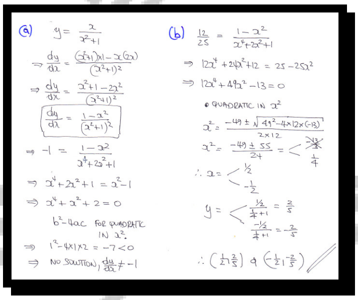
The curve C has equation

$$y = \frac{x}{x^2+1}, \ x \in \mathbb{R} \ .$$

a) Show that there is no point on C where the gradient is -1.

b) Find the coordinates of the points on C where the gradient is $\frac{12}{25}$.

$\boxed{\text{C3N}}$, $\boxed{\left(\frac{1}{2},\frac{2}{5}\right),\left(-\frac{1}{2},-\frac{2}{5}\right)}$



(a) $y = \frac{x}{x^2+1}$

$\Rightarrow \frac{dy}{dx} = \frac{(x^2+1)\times 1 - x(2x)}{(x^2+1)^2}$

$\Rightarrow \frac{dy}{dx} = \frac{x^2+1-2x^2}{(x^2+1)^2}$

$\boxed{\frac{dy}{dx} = \frac{1-x^2}{(x^2+1)^2}}$

$\Rightarrow -1 = \frac{1-x^2}{x^4+2x^2+1}$

$\Rightarrow x^4+2x^2+1 = x^2-1$

$\Rightarrow x^4+x^2+2=0$

b^2-4ac FOR QUADRATIC IN x^2.

$\Rightarrow 1^2-4\times1\times2 = -7<0$

$\Rightarrow$ NO SOLUTION, $\frac{dy}{dx} \neq -1$

(b) $\frac{12}{25} = \frac{1-x^2}{x^4+2x^2+1}$

$\Rightarrow 12x^4+24x^2+12 = 25-25x^2$

$\Rightarrow 12x^4+49x^2-13=0$

• QUADRATIC IN x^2

$x^2 = \frac{-49 \pm \sqrt{49^2-4\times12\times(-13)}}{2\times12}$

$x^2 = \frac{-49\pm55}{24} = \begin{cases} \cancel{-\frac{13}{3}} \\ \frac{1}{4} \end{cases}$

$\therefore x = \begin{cases} \frac{1}{2} \\ -\frac{1}{2} \end{cases}$

$y = \begin{cases} \frac{1/2}{\frac{1}{4}+1} = \frac{2}{5} \\ \frac{-1/2}{\frac{1}{4}+1} = -\frac{2}{5} \end{cases}$

$\therefore \left(\frac{1}{2},\frac{2}{5}\right)$ & $\left(-\frac{1}{2},-\frac{2}{5}\right)$ //

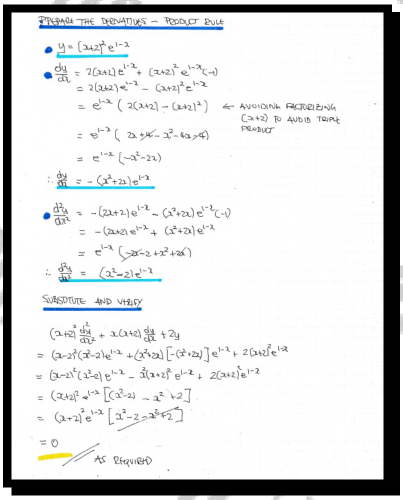
Question 171 (****+)

$$y = (x+2)^2 \mathrm{e}^{1-x}, \ x \in \mathbb{R}$$

Show clearly that

$$(x+2)^2 \frac{d^2y}{dx^2} + x(x+2)\frac{dy}{dx} + 2y = 0 .$$

$\boxed{}$, $\boxed{\text{proof}}$



PREPARE THE DERIVATIVES – PRODUCT RULE

- $y = (x+2)^2 e^{1-x}$
- $\frac{dy}{dx} = 2(x+2)e^{1-x} + (x+2)^2 e^{1-x}(-1)$

$= 2(x+2)e^{1-x} - (x+2)^2 e^{1-x}$

$= e^{1-x}\left(2(x+2) - (x+2)^2\right)$ ← AVOIDING FACTORIZING $(x+2)$ TO AVOID TRIPLE PRODUCT

$= e^{1-x}\left(2x + \cancel{4} - x^2 - 4x - \cancel{4}\right)$

$= e^{1-x}\left(-x^2 - 2x\right)$

$\therefore \frac{dy}{dx} = -(x^2+2x)e^{1-x}$

- $\frac{d^2y}{dx^2} = -(2x+2)e^{1-x} - (x^2+2x)e^{1-x}(-1)$

$= -(2x+2)e^{1-x} + (x^2+2x)e^{1-x}$

$= e^{1-x}\left(\cancel{-2x} - 2 + x^2 + \cancel{2x}\right)$

$\therefore \frac{d^2y}{dx^2} = (x^2-2)e^{1-x}$

SUBSTITUTE AND VERIFY

$(x+2)^2\frac{d^2y}{dx^2} + x(x+2)\frac{dy}{dx} + 2y$

$= (x-2)^2(x^2-2)e^{1-x} + (x^2+2x)\left[-(x^2+2x)\right]e^{1-x} + 2(x+2)^2 e^{1-x}$

$= (x-2)^2(x^2-2)e^{1-x} - x^2(x+2)^2 e^{1-x} + 2(x+2)^2 e^{1-x}$

$= (x+2)^2 e^{1-x}\left[(x^2-2) - x^2 + 2\right]$

$= (x+2)^2 e^{1-x}\left[\cancel{x^2 - 2 - x^2 + 2}\right]$

$= 0$ // AS REQUIRED

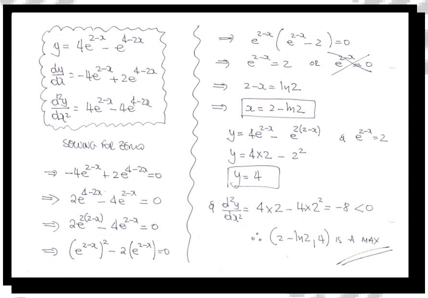
Question 172 **(****+)**

A curve has equation

$$y = 4e^{2-x} - e^{4-2x}, \; x \in \mathbb{R}.$$

Use differentiation to find the exact coordinates of the stationary point of the curve, and further determine its nature.

$\boxed{}$, $\boxed{\max\left(2-\ln 2, 4\right)}$



$y = 4e^{2-x} - e^{4-2x}$

$\frac{dy}{dx} = -4e^{2-x} + 2e^{4-2x}$

$\frac{d^2y}{dx^2} = 4e^{2-x} - 4e^{4-2x}$

SOLVING FOR ZERO

$\Rightarrow -4e^{2-x} + 2e^{4-2x} = 0$

$\Rightarrow 2e^{4-2x} - 4e^{2-x} = 0$

$\Rightarrow 2e^{2(2-x)} - 4e^{2-x} = 0$

$\Rightarrow \left(e^{2-x}\right)^2 - 2\left(e^{2-x}\right) = 0$

$\Rightarrow e^{2-x}\left(e^{2-x} - 2\right) = 0$

$\Rightarrow e^{2-x} = 2$ or $\cancel{e^{2-x} = 0}$

$\Rightarrow 2 - x = \ln 2$

$\Rightarrow \boxed{x = 2 - \ln 2}$

$y = 4e^{2-x} - e^{2(2-x)}$ & $e^{2-x} = 2$

$y = 4 \times 2 - 2^2$

$\boxed{y = 4}$

& $\frac{d^2y}{dx^2} = 4 \times 2 - 4 \times 2^2 = -8 < 0$

$\therefore (2 - \ln 2, 4)$ IS A MAX

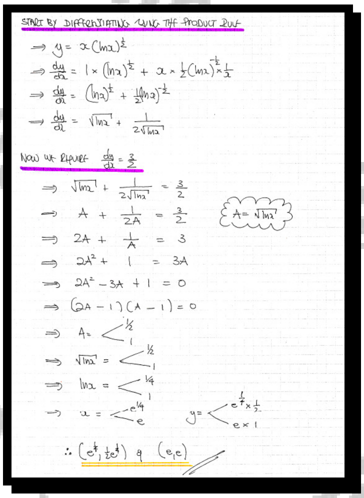
Question 173 **(****+)**

The point P lies on the curve with equation

$$y = x\sqrt{\ln x}, \; x > 1.$$

Determine the two possible sets of coordinates of P given further that the gradient of the curve at P is $\frac{3}{2}$.

$\boxed{}$, $\boxed{P\left(e^{\frac{1}{4}}, \frac{1}{2}e^{\frac{1}{4}}\right) \; \cup \; P(e, e)}$



START BY DIFFERENTIATING USING THE PRODUCT RULE

$\Rightarrow y = x(\ln x)^{\frac{1}{2}}$

$\Rightarrow \frac{dy}{dx} = 1 \times (\ln x)^{\frac{1}{2}} + x \times \frac{1}{2}(\ln x)^{-\frac{1}{2}} \times \frac{1}{x}$

$\Rightarrow \frac{dy}{dx} = (\ln x)^{\frac{1}{2}} + \frac{1}{2}(\ln x)^{-\frac{1}{2}}$

$\Rightarrow \frac{dy}{dx} = \sqrt{\ln x} + \frac{1}{2\sqrt{\ln x}}$

NOW WE REQUIRE $\frac{dy}{dx} = \frac{3}{2}$

$\Rightarrow \sqrt{\ln x} + \frac{1}{2\sqrt{\ln x}} = \frac{3}{2}$

$\Rightarrow A + \frac{1}{2A} = \frac{3}{2}$ $\quad A = \sqrt{\ln x}$

$\Rightarrow 2A + \frac{1}{A} = 3$

$\Rightarrow 2A^2 + 1 = 3A$

$\Rightarrow 2A^2 - 3A + 1 = 0$

$\Rightarrow (2A - 1)(A - 1) = 0$

$\Rightarrow A = \begin{cases} \frac{1}{2} \\ 1 \end{cases}$

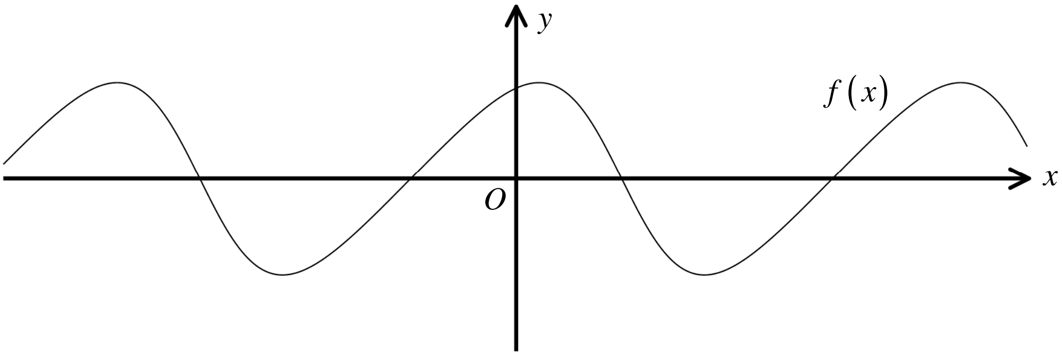
$\Rightarrow \sqrt{\ln x} = \begin{cases} \frac{1}{2} \\ 1 \end{cases}$

$\Rightarrow \ln x = \begin{cases} \frac{1}{4} \\ 1 \end{cases}$

$\Rightarrow x = \begin{cases} e^{\frac{1}{4}} \\ e \end{cases}$ $\quad y = \begin{cases} e^{\frac{1}{4}} \times \frac{1}{2} \\ e \times 1 \end{cases}$

$\therefore \left(e^{\frac{1}{4}}, \frac{1}{2}e^{\frac{1}{4}}\right)$ & (e, e)

Question 174 (****+)



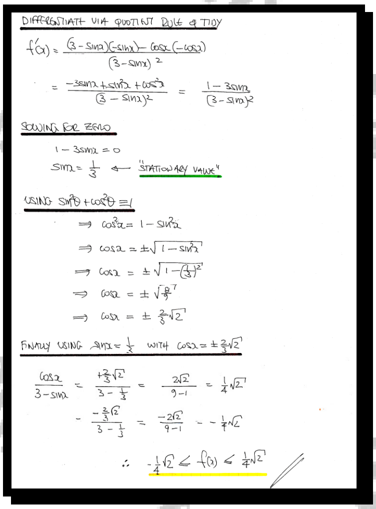
The figure above shows part of the graph of the curve with equation

$$f(x)=\frac{\cos x}{3-\sin x},\ x\in\mathbb{R}.$$

Use differentiation to show that

$$-\tfrac{1}{4}\sqrt{2}\le f(x)\le \tfrac{1}{4}\sqrt{2}.$$

, $\boxed{\text{proof}}$


DIFFERENTIATE VIA QUOTIENT RULE & TIDY

$$f'(x)=\frac{(3-\sin x)(-\sin x)-\cos x(-\cos x)}{(3-\sin x)^2}$$

$$=\frac{-3\sin x+\sin^2 x+\cos^2 x}{(3-\sin x)^2}=\frac{1-3\sin x}{(3-\sin x)^2}$$

SOLVING FOR ZERO

$$1-3\sin x=0$$

$$\sin x=\frac{1}{3}\ \leftarrow\ \text{"STATIONARY VALUE"}$$

USING $\sin^2\theta+\cos^2\theta\equiv 1$

$$\Rightarrow \cos^2 x=1-\sin^2 x$$

$$\Rightarrow \cos x=\pm\sqrt{1-\sin^2 x}$$

$$\Rightarrow \cos x=\pm\sqrt{1-\left(\tfrac{1}{3}\right)^2}$$

$$\Rightarrow \cos x=\pm\sqrt{\tfrac{8}{9}}$$

$$\Rightarrow \cos x=\pm\tfrac{2}{3}\sqrt{2}$$

FINALLY USING $\sin x=\frac{1}{3}$ WITH $\cos x=\pm\frac{2}{3}\sqrt{2}$

$$\frac{\cos x}{3-\sin x}=\frac{+\frac{2}{3}\sqrt{2}}{3-\frac{1}{3}}=\frac{2\sqrt{2}}{9-1}=\frac{1}{4}\sqrt{2}$$

$$-\frac{-\frac{2}{3}\sqrt{2}}{3-\frac{1}{3}}=\frac{-2\sqrt{2}}{9-1}=-\frac{1}{4}\sqrt{2}$$

$$\therefore\ -\tfrac{1}{4}\sqrt{2}\le f(x)\le \tfrac{1}{4}\sqrt{2}$$

Question 175 **(****+)**

The curve C has equation given by

$$y = \frac{e^x}{\sin x}, \quad 0 < x < \pi .$$

a) Show clearly that

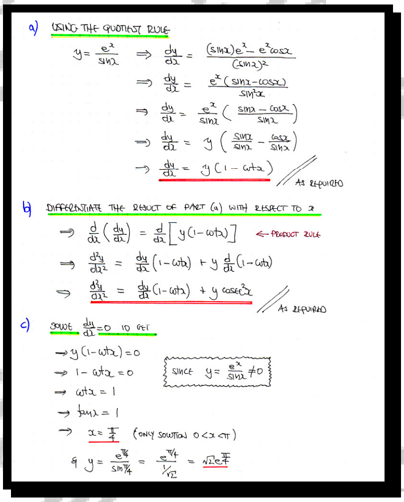
$$\frac{dy}{dx} = y(1 - \cot x).$$

b) Show further that

$$\frac{d^2y}{dx^2} = \frac{dy}{dx}(1 - \cot x) + y\,\text{cosec}^2 x .$$

c) Use the above results to find the exact coordinates of the turning point of C, and determine its nature.

$\boxed{}$, $\boxed{\text{min at } \left(\frac{\pi}{4}, \sqrt{2}\,e^{\frac{\pi}{4}}\right)}$



a) USING THE QUOTIENT RULE

$$y = \frac{e^x}{\sin x} \Rightarrow \frac{dy}{dx} = \frac{(\sin x)e^x - e^x\cos x}{(\sin x)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^x(\sin x - \cos x)}{\sin^2 x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^x}{\sin x}\left(\frac{\sin x - \cos x}{\sin x}\right)$$

$$\Rightarrow \frac{dy}{dx} = y\left(\frac{\sin x}{\sin x} - \frac{\cos x}{\sin x}\right)$$

$$\Rightarrow \frac{dy}{dx} = y(1 - \cot x)$$

// AS REQUIRED

b) DIFFERENTIATE THE RESULT OF PART (a) WITH RESPECT TO x

$$\Rightarrow \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left[y(1-\cot x)\right] \quad \leftarrow \text{PRODUCT RULE}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{dy}{dx}(1-\cot x) + y\frac{d}{dx}(1-\cot x)$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{dy}{dx}(1-\cot x) + y\,\text{cosec}^2 x$$

// AS REQUIRED

c) SOLVE $\frac{dy}{dx} = 0$ TO GET

$$\rightarrow y(1-\cot x) = 0$$

$$\Rightarrow 1 - \cot x = 0$$

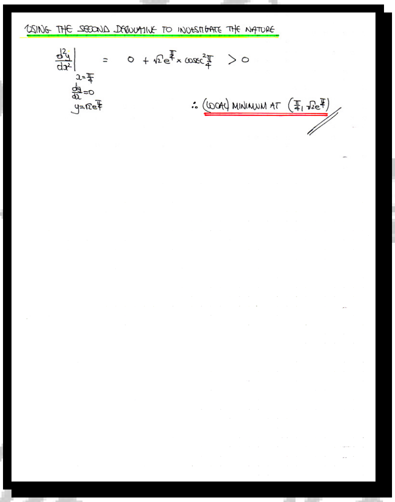
$$\Rightarrow \cot x = 1$$

$$\Rightarrow \tan x = 1$$

$$\Rightarrow x = \frac{\pi}{4} \quad (\text{ONLY SOLUTION } 0 < x < \pi)$$

SINCE $y = \frac{e^x}{\sin x} \neq 0$

$$\& \; y = \frac{e^{\pi/4}}{\sin \pi/4} = \frac{e^{\pi/4}}{1/\sqrt{2}} = \sqrt{2}e^{\frac{\pi}{4}}$$



USING THE SECOND DERIVATIVE TO INVESTIGATE THE NATURE

$$\left.\frac{d^2y}{dx^2}\right|_{\substack{x=\frac{\pi}{4} \\ \frac{dy}{dx}=0 \\ y=\sqrt{2}e^{\frac{\pi}{4}}}} = 0 + \sqrt{2}e^{\frac{\pi}{4}} \times \text{cosec}^2\frac{\pi}{4} > 0$$

$\therefore$ (LOCAL) MINIMUM AT $\left(\frac{\pi}{4}, \sqrt{2}e^{\frac{\pi}{4}}\right)$ //

Question 176 (**+)**

The curve C has equation

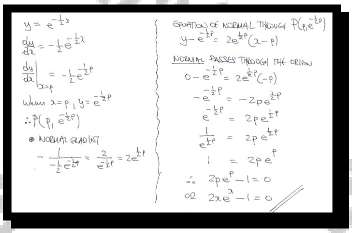
$$y = \mathrm{e}^{-\frac{1}{2}x}, \ x \in \mathbb{R}.$$

The normal to the curve at the point P where $x = p$ passes through the origin.

Show that $x = p$ is a solution of the equation

$$2x\mathrm{e}^{x} - 1 = 0.$$

$\boxed{}$, $\boxed{\text{proof}}$



$y = e^{-\frac{1}{2}x}$

$\frac{dy}{dx} = -\frac{1}{2}e^{-\frac{1}{2}x}$

$\left.\frac{dy}{dx}\right|_{x=p} = -\frac{1}{2}e^{-\frac{1}{2}p}$

when $x = p$, $y = e^{-\frac{1}{2}p}$

$\therefore P(p, e^{-\frac{1}{2}p})$

• NORMAL GRADIENT

$-\frac{1}{-\frac{1}{2}e^{-\frac{1}{2}p}} = \frac{2}{e^{-\frac{1}{2}p}} = 2e^{\frac{1}{2}p}$

EQUATION OF NORMAL THROUGH $P(p, e^{-\frac{1}{2}p})$

$y - e^{-\frac{1}{2}p} = 2e^{\frac{1}{2}p}(x - p)$

NORMAL PASSES THROUGH THE ORIGIN

$0 - e^{-\frac{1}{2}p} = 2e^{\frac{1}{2}p}(-p)$

$-e^{-\frac{1}{2}p} = -2pe^{\frac{1}{2}p}$

$e^{-\frac{1}{2}p} = 2pe^{\frac{1}{2}p}$

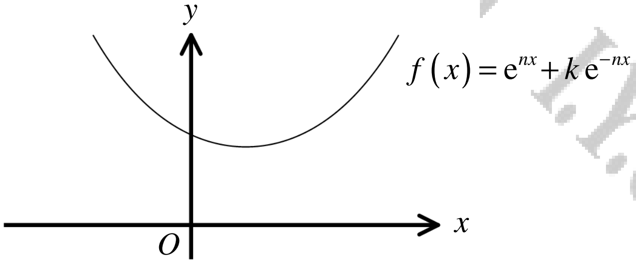
$\frac{1}{e^{\frac{1}{2}p}} = 2pe^{\frac{1}{2}p}$

$1 = 2pe^{p}$

$\therefore 2pe^{p} - 1 = 0$

OR $2xe^{x} - 1 = 0$ //

Question 177 (****+)

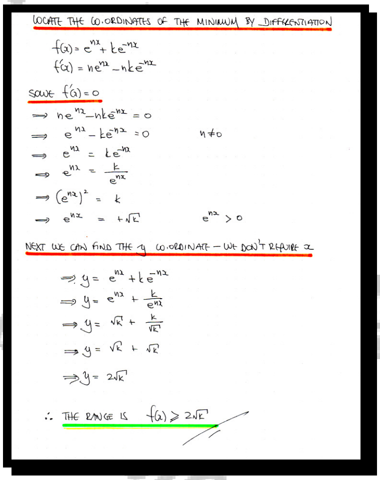


The figure above shows the graph of the curve with equation

$$f(x) = e^{nx} + k\,e^{-nx}, \; x \in \mathbb{R} \;, \; k > 1, \; n > 0 .$$

Find the range of $f(x)$ in exact form.

$\boxed{}$, $\boxed{f(x) \geq 2\sqrt{k}}$



LOCATE THE CO-ORDINATES OF THE MINIMUM BY DIFFERENTIATION

$f(x) = e^{nx} + ke^{-nx}$
$f'(x) = ne^{nx} - nke^{-nx}$

SOLVE $f'(x) = 0$

$\Rightarrow ne^{nx} - nke^{-nx} = 0$
$\Rightarrow e^{nx} - ke^{-nx} = 0 \qquad n \neq 0$
$\Rightarrow e^{nx} = ke^{-nx}$
$\Rightarrow e^{nx} = \frac{k}{e^{nx}}$
$\Rightarrow (e^{nx})^2 = k$
$\Rightarrow e^{nx} = +\sqrt{k} \qquad e^{nx} > 0$

NEXT WE CAN FIND THE y CO-ORDINATE – WE DON'T REQUIRE x

$\Rightarrow y = e^{nx} + ke^{-nx}$
$\Rightarrow y = e^{nx} + \frac{k}{e^{nx}}$
$\Rightarrow y = \sqrt{k} + \frac{k}{\sqrt{k}}$
$\Rightarrow y = \sqrt{k} + \sqrt{k}$
$\Rightarrow y = 2\sqrt{k}$

$\therefore$ THE RANGE IS $f(x) \geqslant 2\sqrt{k}$

Question 178 (****+)

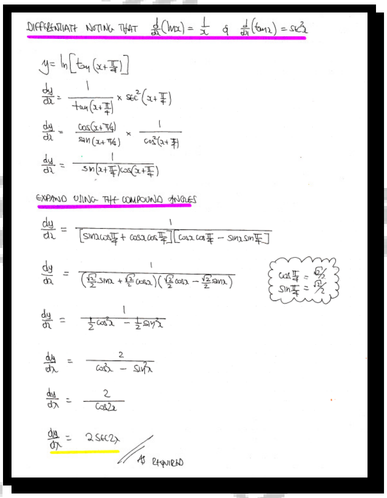
A curve has equation

$$y = \ln\left[\tan\left(x + \frac{\pi}{4}\right)\right], \text{ where } \tan\left(x + \frac{\pi}{4}\right) > 0 .$$

Show that

$$\frac{dy}{dx} = 2\sec 2x \ .$$

$\boxed{}$, $\boxed{\text{proof}}$



DIFFERENTIATE NOTING THAT $\frac{d}{dx}(\ln x) = \frac{1}{x}$ & $\frac{d}{dx}(\tan x) = \sec^2 x$

$$y = \ln\left[\tan\left(x + \frac{\pi}{4}\right)\right]$$

$$\frac{dy}{dx} = \frac{1}{\tan\left(x + \frac{\pi}{4}\right)} \times \sec^2\left(x + \frac{\pi}{4}\right)$$

$$\frac{dy}{dx} = \frac{\cos(x + \pi/4)}{\sin(x + \pi/4)} \times \frac{1}{\cos^2\left(x + \frac{\pi}{4}\right)}$$

$$\frac{dy}{dx} = \frac{1}{\sin\left(x + \frac{\pi}{4}\right)\cos\left(x + \frac{\pi}{4}\right)}$$

EXPAND USING THE COMPOUND ANGLES

$$\frac{dy}{dx} = \frac{1}{\left[\sin x \cos\frac{\pi}{4} + \cos x \cos\frac{\pi}{4}\right]\left[\cos x \cos\frac{\pi}{4} - \sin x \sin\frac{\pi}{4}\right]}$$

$$\frac{dy}{dx} = \frac{1}{\left(\frac{\sqrt{2}}{2}\sin x + \frac{\sqrt{2}}{2}\cos x\right)\left(\frac{\sqrt{2}}{2}\cos x - \frac{\sqrt{2}}{2}\sin x\right)}$$

$\cos\frac{\pi}{4} = \sqrt{2}/2$

$\sin\frac{\pi}{4} = \sqrt{2}/2$

$$\frac{dy}{dx} = \frac{1}{\frac{1}{2}\cos^2 x - \frac{1}{2}\sin^2 x}$$

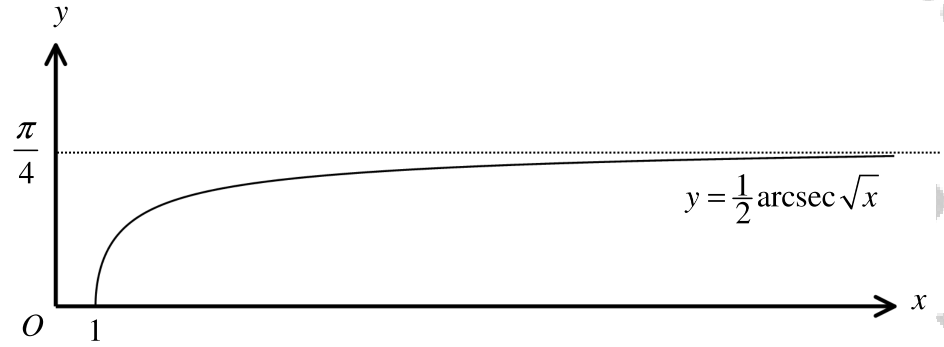
$$\frac{dy}{dx} = \frac{2}{\cos^2 x - \sin^2 x}$$

$$\frac{dy}{dx} = \frac{2}{\cos 2x}$$

$$\frac{dy}{dx} = 2\sec 2x$$

// AS REQUIRED

Question 179 (****+)



The figure above shows the graph of the curve with equation

$$y = \frac{1}{2}\text{arcsec}\sqrt{x}\,,\;\; x \ge 1,\;\; 0 \le y < \frac{1}{4}\pi\,,$$

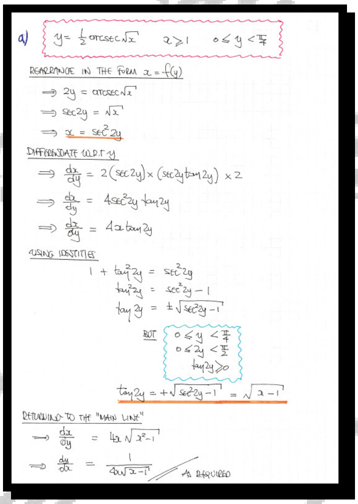
where $\text{arcsec}(u)$ is the inverse function of $\sec(u)$.

Show clearly that …

a) … $\dfrac{dy}{dx} = \dfrac{1}{4x\sqrt{x-1}}$.

b) … $\dfrac{d^2y}{dx^2} = \dfrac{2-3x}{8x^2(x-1)^{\frac{3}{2}}}$.

☐, proof



a) $y = \frac{1}{2}\text{arcsec}\sqrt{x} \quad x \geqslant 1 \quad 0 \leqslant y < \frac{\pi}{4}$

REARRANGE IN THE FORM $x = f(y)$

$\Rightarrow 2y = \text{arcsec}\sqrt{x}$

$\Rightarrow \sec 2y = \sqrt{x}$

$\Rightarrow x = \sec^2 2y$

DIFFERENTIATE W.R.T y

$\Rightarrow \frac{dx}{dy} = 2(\sec 2y) \times (\sec 2y \tan 2y) \times 2$

$\Rightarrow \frac{dx}{dy} = 4\sec^2 2y \tan 2y$

$\Rightarrow \frac{dx}{dy} = 4x \tan 2y$

USING IDENTITIES

$1 + \tan^2 2y = \sec^2 2y$

$\tan^2 2y = \sec^2 2y - 1$

$\tan 2y = \pm\sqrt{\sec^2 2y - 1}$

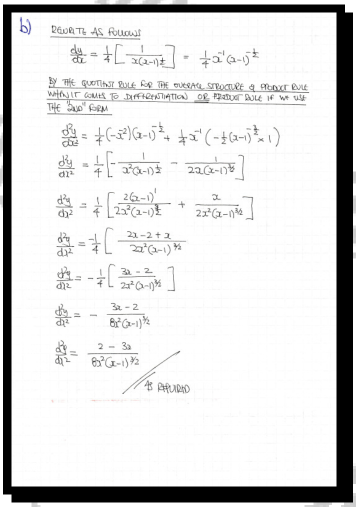
BUT $0 \leqslant y < \frac{\pi}{4}$
$0 \leqslant 2y < \frac{\pi}{2}$
$\tan 2y \geqslant 0$

$\tan 2y = +\sqrt{\sec^2 2y - 1} = \sqrt{x-1}$

RETURNING TO THE "MAIN LINE"

$\rightarrow \frac{dx}{dy} = 4x\sqrt{x^2-1}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{4x\sqrt{x-1}}$ // AS REQUIRED



b) REWRITE AS FOLLOWS

$\frac{dy}{dx} = \frac{1}{4}\left[\frac{1}{x(x-1)^{\frac{1}{2}}}\right] = \frac{1}{4}x^{-1}(x-1)^{-\frac{1}{2}}$

BY THE QUOTIENT RULE FOR THE OVERALL STRUCTURE & PRODUCT RULE WHEN IT COMES TO DIFFERENTIATION OR PRODUCT RULE IF WE USE THE "2ND" FORM

$\frac{d^2y}{dx^2} = \frac{1}{4}(-x^{-2})(x-1)^{-\frac{1}{2}} + \frac{1}{4}x^{-1}\left(-\frac{1}{2}(x-1)^{-\frac{3}{2}} \times 1\right)$

$\frac{d^2y}{dx^2} = \frac{1}{4}\left[-\frac{1}{x^2(x-1)^{\frac{1}{2}}} - \frac{1}{2x(x-1)^{3/2}}\right]$

$\frac{d^2y}{dx^2} = \frac{1}{4}\left[\frac{2(x-1)^1}{2x^2(x-1)^{\frac{3}{2}}} + \frac{x}{2x^2(x-1)^{3/2}}\right]$

$\frac{d^2y}{dx^2} = \frac{-1}{4}\left[\frac{2x-2+x}{2x^2(x-1)^{3/2}}\right.$

$\frac{d^2y}{dx^2} = -\frac{1}{4}\left[\frac{3x-2}{2x^2(x-1)^{3/2}}\right]$

$\frac{d^2y}{dx^2} = -\frac{3x-2}{8x^2(x-1)^{3/2}}$

$\frac{d^2y}{dx^2} = \frac{2-3x}{8x^2(x-1)^{3/2}}$ // AS REQUIRED

Question 180 (****+)

a) Differentiate the following expressions with respect to x, fully simplifying the answers.

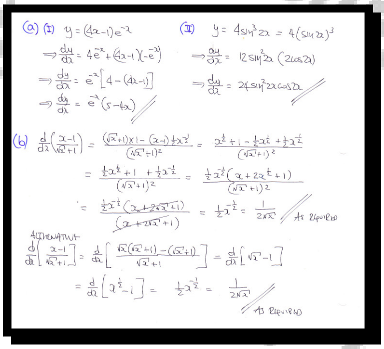
i. $y=(4x-1)e^{-x}$.

ii. $y=4\sin^3 2x$.

b) Prove that

$$\frac{d}{dx}\left(\frac{x-1}{\sqrt{x}+1}\right)=\frac{1}{2\sqrt{x}}.$$

$$\boxed{\frac{dy}{dx}=e^{-x}(5-4x)}, \quad \boxed{\frac{dy}{dx}=24\sin^2 2x\cos 2x}$$



(a) (I) $y=(4x-1)e^{-x}$

$\Rightarrow \frac{dy}{dx}=4e^{-x}+(4x-1)(-e^{-x})$

$\Rightarrow \frac{dy}{dx}=e^{-x}[4-(4x-1)]$

$\Rightarrow \frac{dy}{dx}=e^{-x}(5-4x)$ //

(II) $y=4\sin^3 2x=4(\sin 2x)^3$

$\Rightarrow \frac{dy}{dx}=12\sin^2 2x\,(2\cos 2x)$

$\Rightarrow \frac{dy}{dx}=24\sin^2 2x\cos 2x$ //

(b) $\frac{d}{dx}\left(\frac{x-1}{\sqrt{x}+1}\right)=\frac{(\sqrt{x}+1)\times 1-(x-1)\frac{1}{2}x^{-\frac{1}{2}}}{(\sqrt{x}+1)^2}=\frac{x^{\frac{1}{2}}+1-\frac{1}{2}x^{\frac{1}{2}}+\frac{1}{2}x^{-\frac{1}{2}}}{(\sqrt{x}+1)^2}$

$=\frac{\frac{1}{2}x^{\frac{1}{2}}+1+\frac{1}{2}x^{-\frac{1}{2}}}{(\sqrt{x}+1)^2}=\frac{\frac{1}{2}x^{-\frac{1}{2}}(x+2x^{\frac{1}{2}}+1)}{(\sqrt{x}+1)^2}$

$=\frac{\frac{1}{2}x^{-\frac{1}{2}}\cancel{(x+2\sqrt{x}+1)}}{\cancel{(x+2\sqrt{x}+1)}}=\frac{1}{2}x^{-\frac{1}{2}}=\frac{1}{2\sqrt{x}}$ // AS REQUIRED

ALTERNATIVE

$\frac{d}{dx}\left[\frac{x-1}{\sqrt{x}+1}\right]=\frac{d}{dx}\left[\frac{\sqrt{x}(\sqrt{x}+1)-(\sqrt{x}+1)}{\sqrt{x}+1}\right]=\frac{d}{dx}\left[\sqrt{x}-1\right]$

$=\frac{d}{dx}\left[x^{\frac{1}{2}}-1\right]=\frac{1}{2}x^{-\frac{1}{2}}=\frac{1}{2\sqrt{x}}$ //

AS REQUIRED

Question 181 (****+)

The curve C has equation

$$y = \sqrt{e^{2x} - 2x}, \quad e^{2x} > 2x .$$

The tangent to the curve at the point P where $x = p$ passes through the origin.

a) Show that $x = p$ is a solution of the equation

$$(1-x)e^{2x} = x .$$

b) Show that the equation $(1-x)e^{2x} = x$ has root between 0.8 and 1.

The iterative formula

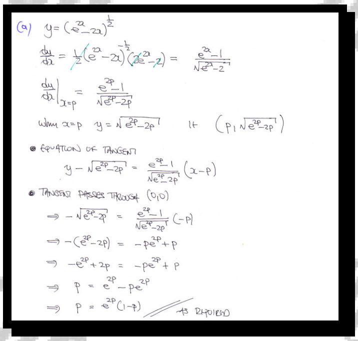
$$x_{n+1} = 1 - x_n e^{-2x_n}$$

with $x_0 = 0.8$ is used to find this root.

c) Find, to 3 decimal places, the value of x_1, x_2, x_3 and x_4.

d) Hence show that the value of p is 0.8439, correct to 4 decimal places.

$\boxed{}$, $\boxed{x_1 = 0.838, \quad x_2 = 0.843, \quad x_3 = 0.844, \quad x_4 = 0.844}$



(a) $y = (e^{2x} - 2x)^{\frac{1}{2}}$

$\frac{dy}{dx} = \cancel{\frac{1}{2}}(e^{2x} - 2x)^{-\frac{1}{2}}(\cancel{2}e^{2x} - \cancel{2}) = \frac{e^{2x} - 1}{\sqrt{e^{2x} - 2}}$

$\left.\frac{dy}{dx}\right|_{x=p} = \frac{e^{2p} - 1}{\sqrt{e^{2p} - 2p}}$

when $x = p$ $y = \sqrt{e^{2p} - 2p}$ It $(p, \sqrt{e^{2p} - 2p})$

- EQUATION OF TANGENT

$y - \sqrt{e^{2p} - 2p} = \frac{e^{2p} - 1}{\sqrt{e^{2p} - 2p}}(x - p)$

- TANGENT PASSES THROUGH $(0,0)$

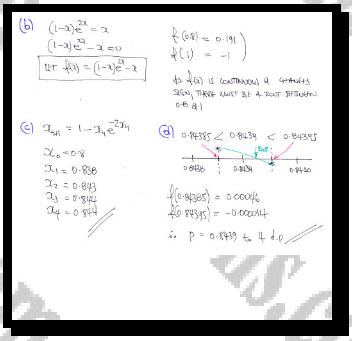
$\Rightarrow -\sqrt{e^{2p} - 2p} = \frac{e^{2p} - 1}{\sqrt{e^{2p} - 2p}}(-p)$

$\Rightarrow -(e^{2p} - 2p) = -pe^{2p} + p$

$\Rightarrow -e^{2p} + 2p = -pe^{2p} + p$

$\Rightarrow p = e^{2p} - pe^{2p}$

$\Rightarrow p = e^{2p}(1 - p)$ // AS REQUIRED



(b) $(1-x)e^{2x} = x$

$(1-x)e^{2x} - x = 0$

$\boxed{\text{LET } f(x) = (1-x)e^{2x} - x}$

$f(0.8) = 0.191$

$f(1) = -1$

AS $f(x)$ IS CONTINUOUS & CHANGES SIGN, THERE MUST BE A ROOT BETWEEN 0.8 & 1

(c) $x_{n+1} = 1 - x_n e^{-2x_n}$

$x_0 = 0.8$

$x_1 = 0.838$

$x_2 = 0.843$

$x_3 = 0.844$

$x_4 = 0.844$ //

(d) $0.84385 < 0.8439 < 0.84395$

$f(0.84385) = 0.00046$

$f(0.84345) = -0.000014$

$\therefore$ $p = 0.8439$ to 4 d.p //

Question 182 (****+)

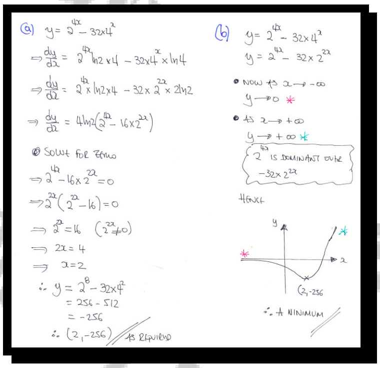
The curve C has equation

$$y = 2^{4x} - 32 \times 4^{x}, \; x \geq 0 .$$

a) Show that C has a turning point at $(2, -256)$.

b) Without further differentiation explain why this turning point must be a minimum.

$\boxed{}$, $\boxed{\text{proof}}$



ⓐ $y = 2^{4x} - 32 \times 4^{x}$

$\Rightarrow \frac{dy}{dx} = 2^{4x} \ln 2 \times 4 - 32 \times 4^{x} \times \ln 4$

$\Rightarrow \frac{dy}{dx} = 2^{4x} \times \ln 2 \times 4 - 32 \times 2^{2x} \times 2 \ln 2$

$\Rightarrow \frac{dy}{dx} = 4 \ln 2 \left(2^{4x} - 16 \times 2^{2x}\right)$

- SOLVE FOR ZERO

$\Rightarrow 2^{4x} - 16 \times 2^{2x} = 0$

$\Rightarrow 2^{2x}\left(2^{2x} - 16\right) = 0$

$\Rightarrow 2^{2x} = 16 \quad \left(2^{2x} \neq 0\right)$

$\Rightarrow 2x = 4$

$\Rightarrow x = 2$

$\therefore y = 2^{8} - 32 \times 4^{2}$

$= 256 - 512$

$= -256$

$\therefore (2, -256)$ // AS REQUIRED

ⓑ $y = 2^{4x} - 32 \times 4^{x}$

$y = 2^{4x} - 32 \times 2^{2x}$

- NOW AS $x \rightarrow -\infty$
  $y \rightarrow 0$ *
- AS $x \rightarrow +\infty$
  $y \rightarrow +\infty$ *

2^{4x} IS DOMINANT OVER -32×2^{2x}

HENCE

$\therefore$ A MINIMUM //

Question 183 (****+)

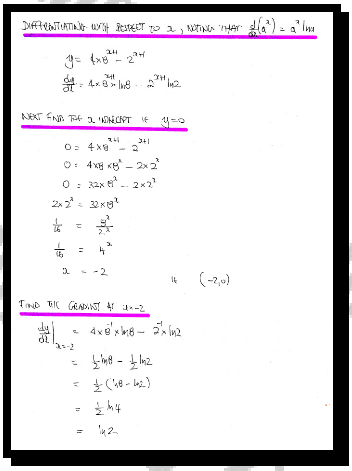
The curve C has equation

$$y = 4 \times 8^{x+1} - 2^{x+1} .$$

Show that an equation of the tangent to the curve, at the point where C crosses the x axis is given by

$$y = (x+2)\ln 2 .$$

$\boxed{}$, $\boxed{\text{proof}}$



DIFFERENTIATING WITH RESPECT TO x, NOTING THAT $\frac{d}{dx}(a^x) = a^x \ln a$

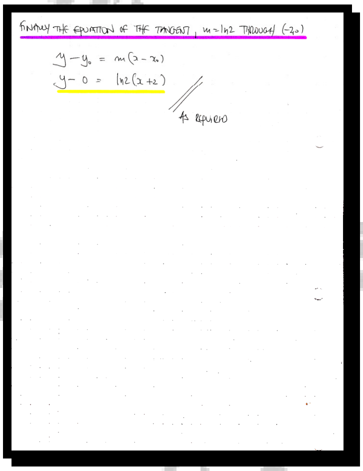
$$y = 4 \times 8^{x+1} - 2^{x+1}$$
$$\frac{dy}{dx} = 4 \times 8^{x+1} \times \ln 8 - 2^{x+1} \ln 2$$

NEXT FIND THE x INTERCEPT IE $y=0$

$$0 = 4 \times 8^{x+1} - 2^{x+1}$$
$$0 = 4 \times 8 \times 8^x - 2 \times 2^x$$
$$0 = 32 \times 8^x - 2 \times 2^x$$
$$2 \times 2^x = 32 \times 8^x$$
$$\frac{1}{16} = \frac{8^x}{2^x}$$
$$\frac{1}{16} = 4^x$$
$$x = -2 \qquad \text{IE} \quad (-2,0)$$

FIND THE GRADIENT AT $x=-2$

$$\left.\frac{dy}{dx}\right|_{x=-2} = 4 \times 8^{-1} \times \ln 8 - 2^{-1} \times \ln 2$$
$$= \frac{1}{2}\ln 8 - \frac{1}{2}\ln 2$$
$$= \frac{1}{2}(\ln 8 - \ln 2)$$
$$= \frac{1}{2}\ln 4$$
$$= \ln 2$$



FINALLY THE EQUATION OF THE TANGENT, $m = \ln 2$ THROUGH $(-2,0)$

$$y - y_0 = m(x - x_0)$$
$$y - 0 = \ln 2(x+2)$$

// AS REQUIRED

Question 184 (****+)

$$y = \arccos x, \; -1 \le x \le 1, \; 0 \le y \le \pi .$$

a) By writing $y = \arccos x$ as $x = \cos y$, show that

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}} .$$

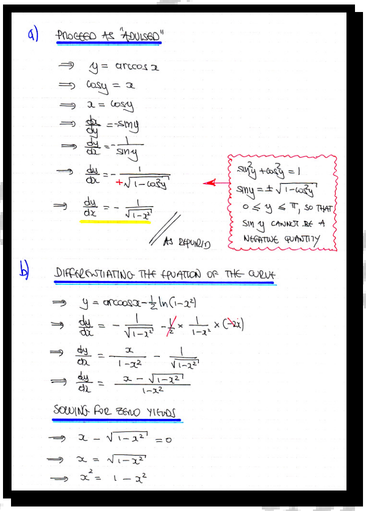
The curve C has equation

$$y = \arccos x - \tfrac{1}{2}\ln\left(1-x^2\right), \; x > 0 .$$

b) Show that the y coordinate of the stationary point of C is

$$\frac{1}{4}\left(\pi + \ln 4\right).$$

$\boxed{}$, $\boxed{\text{proof}}$



a) PROCEED AS "ADVISED"

$$\Rightarrow y = \arccos x$$
$$\Rightarrow \cos y = x$$
$$\Rightarrow x = \cos y$$
$$\Rightarrow \frac{dx}{dy} = -\sin y$$
$$\Rightarrow \frac{dy}{dx} = -\frac{1}{\sin y}$$
$$\Rightarrow \frac{dy}{dx} = -\frac{1}{+\sqrt{1-\cos^2 y}}$$
$$\Rightarrow \frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

// AS REQUIRED

$\sin^2 y + \cos^2 y = 1$

$\sin y = \pm\sqrt{1-\cos^2 y}$

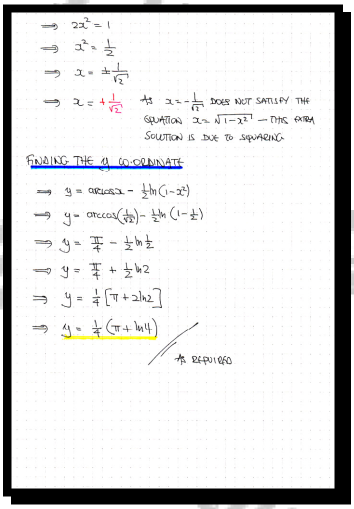
$0 \leqslant y \leqslant \pi$, SO THAT $\sin y$ CANNOT BE A NEGATIVE QUANTITY

b) DIFFERENTIATING THE EQUATION OF THE CURVE

$$\Rightarrow y = \arccos x - \tfrac{1}{2}\ln(1-x^2)$$
$$\Rightarrow \frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}} - \cancel{\tfrac{1}{2}} \times \frac{1}{1-x^2} \times (-\cancel{2}x)$$
$$\Rightarrow \frac{dy}{dx} = \frac{x}{1-x^2} - \frac{1}{\sqrt{1-x^2}}$$
$$\Rightarrow \frac{dy}{dx} = \frac{x - \sqrt{1-x^2}}{1-x^2}$$

SOLVING FOR ZERO YIELDS

$$\Rightarrow x - \sqrt{1-x^2} = 0$$
$$\Rightarrow x = \sqrt{1-x^2}$$
$$\Rightarrow x^2 = 1 - x^2$$



$$\Rightarrow 2x^2 = 1$$
$$\Rightarrow x^2 = \frac{1}{2}$$
$$\Rightarrow x = \pm\frac{1}{\sqrt{2}}$$
$$\Rightarrow x = +\frac{1}{\sqrt{2}}$$

AS $x = -\frac{1}{\sqrt{2}}$ DOES NOT SATISFY THE EQUATION $x = \sqrt{1-x^2}$ — THIS EXTRA SOLUTION IS DUE TO SQUARING

FINDING THE y CO-ORDINATE

$$\Rightarrow y = \arccos x - \tfrac{1}{2}\ln(1-x^2)$$
$$\Rightarrow y = \arccos\left(\tfrac{1}{\sqrt{2}}\right) - \tfrac{1}{2}\ln\left(1-\tfrac{1}{2}\right)$$
$$\Rightarrow y = \frac{\pi}{4} - \frac{1}{2}\ln\frac{1}{2}$$
$$\Rightarrow y = \frac{\pi}{4} + \frac{1}{2}\ln 2$$
$$\Rightarrow y = \frac{1}{4}\left[\pi + 2\ln 2\right]$$
$$\Rightarrow y = \frac{1}{4}\left(\pi + \ln 4\right)$$

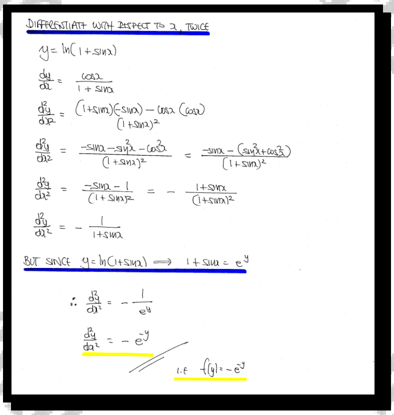
// AS REQUIRED

Question 185 **(****+)**

$$y=\ln\left(1+\sin x\right),\ \sin x\neq -1.$$

Show clearly that $\dfrac{d^2y}{dx^2}=f\left(y\right)$, where $f\left(y\right)$ is a function to be found.

$\boxed{\quad\quad}$, $\boxed{y=-\mathrm{e}^{-y}}$



DIFFERENTIATE WITH RESPECT TO x, TWICE

$y=\ln(1+\sin x)$

$\dfrac{dy}{dx}=\dfrac{\cos x}{1+\sin x}$

$\dfrac{d^2y}{dx^2}=\dfrac{(1+\sin x)(-\sin x)-\cos x(\cos x)}{(1+\sin x)^2}$

$\dfrac{d^2y}{dx^2}=\dfrac{-\sin x-\sin^2x-\cos^2x}{(1+\sin x)^2}=\dfrac{-\sin x-(\sin^2x+\cos^2x)}{(1+\sin x)^2}$

$\dfrac{d^2y}{dx^2}=\dfrac{-\sin x-1}{(1+\sin x)^2}=-\dfrac{1+\sin x}{(1+\sin x)^2}$

$\dfrac{d^2y}{dx^2}=-\dfrac{1}{1+\sin x}$

BUT SINCE $y=\ln(1+\sin x)\implies 1+\sin x=e^y$

$\therefore\ \dfrac{d^2y}{dx^2}=-\dfrac{1}{e^y}$

$\dfrac{d^2y}{dx^2}=-e^{-y}$

i.e. $f(y)=-e^{-y}$

Question 186 **(****+)**

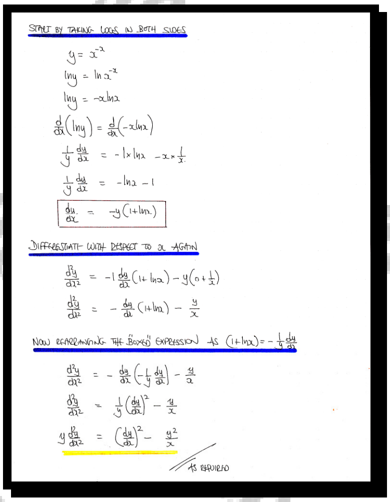
A curve C has equation

$$y = x^{-x}, \quad x \in \mathbb{R}, \quad x > 0 .$$

Show that y is a solution of the equation

$$y\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2 - \frac{y^2}{x} .$$

$\boxed{}$, $\boxed{\text{proof}}$



START BY TAKING LOGS IN BOTH SIDES

$$y = x^{-x}$$
$$\ln y = \ln x^{-x}$$
$$\ln y = -x\ln x$$
$$\frac{d}{dx}(\ln y) = \frac{d}{dx}(-x\ln x)$$
$$\frac{1}{y}\frac{dy}{dx} = -1 \times \ln x - x \times \frac{1}{x}$$
$$\frac{1}{y}\frac{dy}{dx} = -\ln x - 1$$
$$\boxed{\frac{dy}{dx} = -y(1+\ln x)}$$

DIFFERENTIATE WITH RESPECT TO x AGAIN

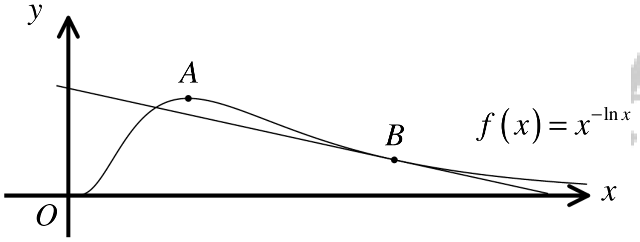
$$\frac{d^2y}{dx^2} = -1\frac{dy}{dx}(1+\ln x) - y\left(0+\frac{1}{x}\right)$$
$$\frac{d^2y}{dx^2} = -\frac{dy}{dx}(1+\ln x) - \frac{y}{x}$$

NOW REARRANGING THE "BOXED" EXPRESSION AS $(1+\ln x) = -\frac{1}{y}\frac{dy}{dx}$

$$\frac{d^2y}{dx^2} = -\frac{dy}{dx}\left(-\frac{1}{y}\frac{dy}{dx}\right) - \frac{y}{x}$$
$$\frac{d^2y}{dx^2} = \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{y}{x}$$
$$y\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2 - \frac{y^2}{x}$$

// AS REQUIRED

Question 187 (****+)



The figure above shows that the graph of

$$f(x)=x^{-\ln x}, \quad x\in\mathbb{R}, \quad x>0 .$$

The curve has a turning point at A.

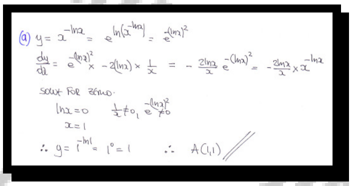
a) Find the coordinates of A.

The point B lies on the curve where $x=\mathrm{e}$.

b) Show that the equation of the tangent to the curve at B is given by

$$\mathrm{e}^2 y+2x=3\mathrm{e} .$$

$\boxed{}$, $\boxed{A(1,1)}$


(a) $y = x^{-\ln x} = e^{\ln(x^{-\ln x})} = e^{-(\ln x)^2}$

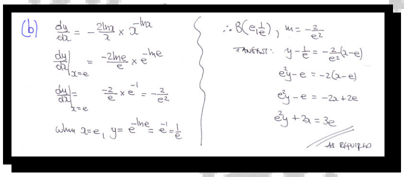
$\frac{dy}{dx} = e^{-(\ln x)^2} \times -2(\ln x) \times \frac{1}{x} = -\frac{2\ln x}{x} e^{-(\ln x)^2} = -\frac{2\ln x}{x} \times x^{-\ln x}$

SOLVE FOR ZERO.

$\ln x = 0 \qquad \frac{1}{x} \neq 0, \; e^{-(\ln x)^2} \neq 0$

$x = 1$

$\therefore y = 1^{-\ln 1} = 1^0 = 1 \qquad \therefore A(1,1)$


(b) $\frac{dy}{dx} = -\frac{2\ln x}{x} \times x^{-\ln x}$

$\left.\frac{dy}{dx}\right|_{x=e} = \frac{-2\ln e}{e} \times e^{-\ln e}$

$\left.\frac{dy}{dx}\right|_{x=e} = \frac{-2}{e} \times e^{-1} = -\frac{2}{e^2}$

when $x=e$, $y = e^{-\ln e} = e^{-1} = \frac{1}{e}$

$\therefore B(e,\frac{1}{e})$, $m = -\frac{2}{e^2}$

TANGENT: $y - \frac{1}{e} = -\frac{2}{e^2}(x-e)$

$e^2y - e = -2(x-e)$

$e^2y - e = -2x+2e$

$e^2y + 2x = 3e$

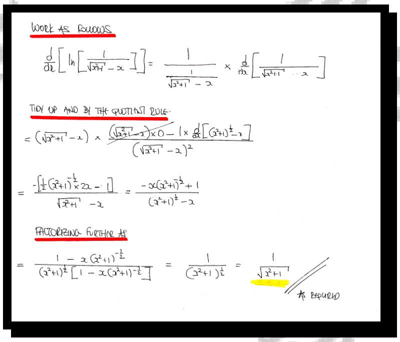
AS REQUIRED

Question 188 **(****+)**

Show, with a detailed method, that

$$\frac{d}{dx}\left[\ln\left(\frac{1}{\sqrt{x^2+1}-x}\right)\right]=\frac{1}{\sqrt{x^2+1}}.$$

[illegible] , proof



WORK AS FOLLOWS

$$\frac{d}{dx}\left[\ln\left[\frac{1}{\sqrt{x^2+1}-x}\right]\right]=\frac{1}{\frac{1}{\sqrt{x^2+1}-x}}\times\frac{d}{dx}\left[\frac{1}{\sqrt{x^2+1}-x}\right]$$

TIDY UP AND BY THE QUOTIENT RULE

$$=\left(\sqrt{x^2+1}-x\right)\times\frac{\cancel{\left(\sqrt{x^2+1}-x\right)\times 0}-1\times\frac{d}{dx}\left[(x^2+1)^{\frac{1}{2}}-x\right]}{\left(\sqrt{x^2+1}-x\right)^2}$$

$$=\frac{-\left[\frac{1}{2}(x^2+1)^{-\frac{1}{2}}\times 2x-1\right]}{\sqrt{x^2+1}-x}=\frac{-x(x^2+1)^{-\frac{1}{2}}+1}{(x^2+1)^{\frac{1}{2}}-x}$$

FACTORIZING FURTHER AS

$$=\frac{1-x(x^2+1)^{-\frac{1}{2}}}{(x^2+1)^{\frac{1}{2}}\left[1-x(x^2+1)^{-\frac{1}{2}}\right]}=\frac{1}{(x^2+1)^{\frac{1}{2}}}=\frac{1}{\sqrt{x^2+1}}$$

AS REQUIRED

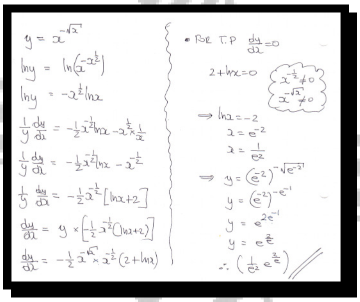
Question 189 **(****+)**

A curve C has equation

$$y=x^{-\sqrt{x}}, \quad x\in\mathbb{R}, \quad x>0.$$

Show that the coordinates of turning point of C are $\left(\frac{1}{e^2}, e^{\frac{2}{e}}\right)$.

proof



$$y=x^{-\sqrt{x}}$$
$$\ln y=\ln\left(x^{-x^{\frac{1}{2}}}\right)$$
$$\ln y=-x^{\frac{1}{2}}\ln x$$
$$\frac{1}{y}\frac{dy}{dx}=-\frac{1}{2}x^{-\frac{1}{2}}\ln x-x^{\frac{1}{2}}\times\frac{1}{x}$$
$$\frac{1}{y}\frac{dy}{dx}=-\frac{1}{2}x^{-\frac{1}{2}}\ln x-x^{-\frac{1}{2}}$$
$$\frac{1}{y}\frac{dy}{dx}=-\frac{1}{2}x^{-\frac{1}{2}}\left[\ln x+2\right]$$
$$\frac{dy}{dx}=y\times\left[-\frac{1}{2}x^{-\frac{1}{2}}(\ln x+2)\right]$$
$$\frac{dy}{dx}=-\frac{1}{2}x^{-\sqrt{x}}\times x^{-\frac{1}{2}}(2+\ln x)$$

- FOR T.P $\frac{dy}{dx}=0$

$2+\ln x=0$

$x^{-\frac{1}{2}}\neq 0$
$x^{-\sqrt{x}}\neq 0$

$$\Rightarrow \ln x=-2$$
$$x=e^{-2}$$
$$x=\frac{1}{e^2}$$
$$\Rightarrow y=\left(e^{-2}\right)^{-\sqrt{e^{-2}}}$$
$$y=\left(e^{-2}\right)^{-e^{-1}}$$
$$y=e^{2e^{-1}}$$
$$y=e^{\frac{2}{e}}$$
$$\therefore\left(\frac{1}{e^2}, e^{\frac{2}{e}}\right)$$

Question 190 **(****+)**

A curve C has equation

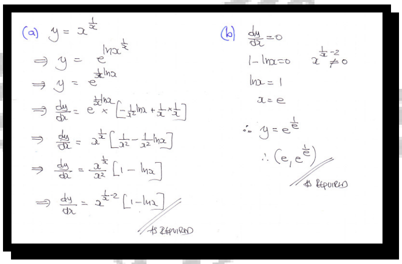
$$y = x^{\frac{1}{x}}, \quad x \in \mathbb{R}, \quad x > 0 .$$

Show clearly that …

a) … $\dfrac{dy}{dx} = x^{\frac{1}{x}-2}\left(1 - \ln x\right)$.

b) … the coordinates of turning point of C are $\left(\mathrm{e}, \mathrm{e}^{\frac{1}{\mathrm{e}}}\right)$.

$\boxed{\text{proof}}$



(a) $y = x^{\frac{1}{x}}$

$\Rightarrow y = e^{\ln x^{\frac{1}{x}}}$

$\Rightarrow y = e^{\frac{1}{x}\ln x}$

$\Rightarrow \frac{dy}{dx} = e^{\frac{1}{x}\ln x} \times \left[-\frac{1}{x^2}\ln x + \frac{1}{x} \times \frac{1}{x}\right]$

$\Rightarrow \frac{dy}{dx} = x^{\frac{1}{x}}\left[\frac{1}{x^2} - \frac{1}{x^2}\ln x\right]$

$\Rightarrow \frac{dy}{dx} = \frac{x^{\frac{1}{x}}}{x^2}\left[1 - \ln x\right]$

$\Rightarrow \frac{dy}{dx} = x^{\frac{1}{x}-2}\left[1 - \ln x\right]$

AS REQUIRED

(b) $\frac{dy}{dx} = 0$

$1 - \ln x = 0 \qquad x^{\frac{1}{x}-2} \neq 0$

$\ln x = 1$

$x = e$

$\therefore y = e^{\frac{1}{e}}$

$\therefore \left(e, e^{\frac{1}{e}}\right)$

AS REQUIRED

Question 191 **(****+)**

A curve C has equation

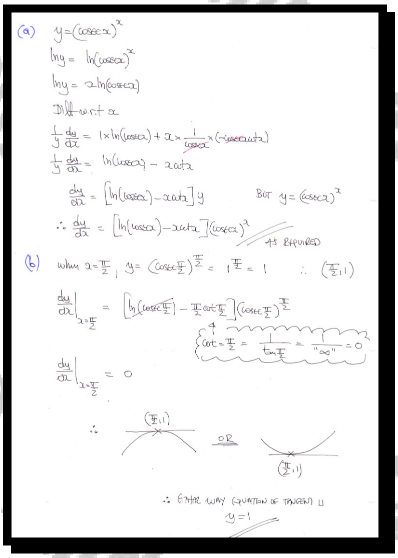
$$y = (\operatorname{cosec} x)^x, \quad x \in \mathbb{R}, \quad x > 0.$$

a) Show that

$$\frac{dy}{dx} = \left[\ln(\operatorname{cosec} x) - x\cot x\right](\operatorname{cosec} x)^x.$$

b) Find the equation of the tangent to C at the point where $x = \frac{\pi}{2}$.

$\boxed{y=1}$



(a) $y = (\operatorname{cosec} x)^x$

$\ln y = \ln(\operatorname{cosec} x)^x$

$\ln y = x\ln(\operatorname{cosec} x)$

Diff w.r.t x

$\frac{1}{y}\frac{dy}{dx} = 1 \times \ln(\operatorname{cosec} x) + x \times \frac{1}{\cancel{\operatorname{cosec} x}} \times (-\cancel{\operatorname{cosec} x}\cot x)$

$\frac{1}{y}\frac{dy}{dx} = \ln(\operatorname{cosec} x) - x\cot x$

$\frac{dy}{dx} = \left[\ln(\operatorname{cosec} x) - x\cot x\right]y$ BUT $y = (\operatorname{cosec} x)^x$

$\therefore \frac{dy}{dx} = \left[\ln(\operatorname{cosec} x) - x\cot x\right](\operatorname{cosec} x)^x$ AS REQUIRED

(b) when $x = \frac{\pi}{2}$, $y = \left(\operatorname{cosec}\frac{\pi}{2}\right)^{\frac{\pi}{2}} = 1^{\frac{\pi}{2}} = 1$ $\therefore \left(\frac{\pi}{2}, 1\right)$

$\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{2}} = \left[\ln\left(\cancel{\operatorname{cosec}\frac{\pi}{2}}\right) - \frac{\pi}{2}\cot\frac{\pi}{2}\right]\left(\operatorname{cosec}\frac{\pi}{2}\right)^{\frac{\pi}{2}}$

$\left\{\cot\frac{\pi}{2} = \frac{1}{\tan\frac{\pi}{2}} = \frac{1}{\text{"}\infty\text{"}} = 0\right\}$

$\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{2}} = 0$

$\therefore$ $\left(\frac{\pi}{2}, 1\right)$ OR $\left(\frac{\pi}{2}, 1\right)$

$\therefore$ EITHER WAY EQUATION OF TANGENT IS

$y = 1$

Question 192 **(****+)**

$$y = \arctan x \,, \quad x \in \mathbb{R} \,.$$

a) By writing the above equation in the from $x = g(y)$, show that

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2} \,.$$

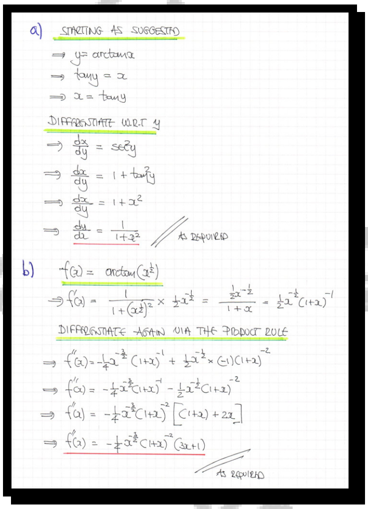
The function f is defined as

$$f(x) = \arctan \sqrt{x} \,, \quad x \in \mathbb{R} \,, \; x \geq 0 \,.$$

b) Show further that

$$f''(x) = -\frac{1}{4} x^{-\frac{3}{2}} (3x+1)(x+1)^{-2} \,.$$

$\boxed{}$, $\boxed{\text{proof}}$



a) STARTING AS SUGGESTED

$\Rightarrow y = \arctan x$

$\Rightarrow \tan y = x$

$\Rightarrow x = \tan y$

DIFFERENTIATE W.R.T y

$\Rightarrow \frac{dx}{dy} = \sec^2 y$

$\Rightarrow \frac{dx}{dy} = 1 + \tan^2 y$

$\Rightarrow \frac{dx}{dy} = 1 + x^2$

$\Rightarrow \frac{dy}{dx} = \frac{1}{1+x^2}$ // AS REQUIRED

b) $f(x) = \arctan(x^{\frac{1}{2}})$

$\Rightarrow f'(x) = \frac{1}{1+(x^{\frac{1}{2}})^2} \times \frac{1}{2}x^{-\frac{1}{2}} = \frac{\frac{1}{2}x^{-\frac{1}{2}}}{1+x} = \frac{1}{2}x^{-\frac{1}{2}}(1+x)^{-1}$

DIFFERENTIATE AGAIN VIA THE PRODUCT RULE

$\Rightarrow f''(x) = -\frac{1}{4}x^{-\frac{3}{2}}(1+x)^{-1} + \frac{1}{2}x^{-\frac{1}{2}} \times (-1)(1+x)^{-2}$

$\Rightarrow f''(x) = -\frac{1}{4}x^{-\frac{3}{2}}(1+x)^{-1} - \frac{1}{2}x^{-\frac{1}{2}}(1+x)^{-2}$

$\Rightarrow f''(x) = -\frac{1}{4}x^{-\frac{3}{2}}(1+x)^{-2}\left[(1+x) + 2x\right]$

$\Rightarrow f''(x) = -\frac{1}{4}x^{-\frac{3}{2}}(1+x)^{-2}(3x+1)$

// AS REQUIRED

Question 193 (****+)

The function f is defined as

$$f(x) = x^x \mathrm{e}^{-2x}, \quad x \in \mathbb{R}, \quad x > 0.$$

a) Find an expression for $f'(x)$.

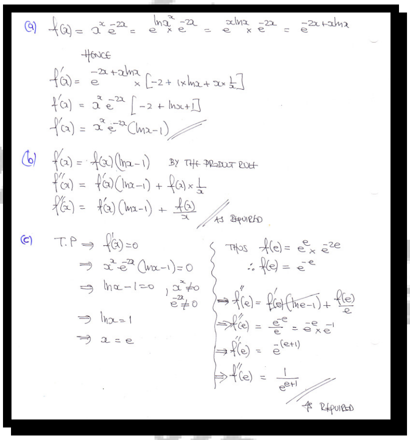
b) Show clearly that

$$f''(x) = f'(x)(\ln x - 1) + \frac{f(x)}{x}.$$

c) Show that the value of $f''(x)$ at the turning point of the function is

$$\frac{1}{\mathrm{e}^{\mathrm{e}+1}}.$$

$$\boxed{f'(x) = x^x \mathrm{e}^{-2x}(\ln x - 1)}$$



(a) $f(x) = x^x e^{-2x} = e^{\ln x^x} \times e^{-2x} = e^{x\ln x} \times e^{-2x} = e^{-2x+x\ln x}$

HENCE

$f'(x) = e^{-2x+x\ln x} \times \left[-2 + 1\times\ln x + x\times\frac{1}{x}\right]$

$f'(x) = x^x e^{-2x}\left[-2 + \ln x + 1\right]$

$f'(x) = x^x e^{-2x}(\ln x - 1)$

(b) $f'(x) = f(x)(\ln x - 1)$ BY THE PRODUCT RULE

$f''(x) = f'(x)(\ln x - 1) + f(x)\times\frac{1}{x}$

$f''(x) = f'(x)(\ln x - 1) + \frac{f(x)}{x}$ AS REQUIRED

(c) T.P $\Rightarrow f'(x) = 0$

$\Rightarrow x^x e^{-2x}(\ln x - 1) = 0$

$\Rightarrow \ln x - 1 = 0$, $x^x \neq 0$, $e^{-2x} \neq 0$

$\Rightarrow \ln x = 1$

$\Rightarrow x = e$

THUS $f(e) = e^e \times e^{-2e}$

$\therefore f(e) = e^{-e}$

$\Rightarrow f''(e) = \cancel{f'(e)(\ln e - 1)} + \frac{f(e)}{e}$

$\Rightarrow f''(e) = \frac{e^{-e}}{e} = e^{-e} \times e^{-1}$

$\Rightarrow f''(e) = e^{-(e+1)}$

$\Rightarrow f''(e) = \frac{1}{e^{e+1}}$

AS REQUIRED

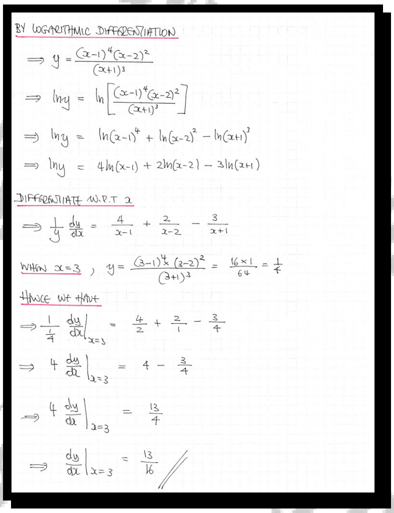
Question 194 **(****+)**

Given that

$$y=\frac{(x-1)^4(x-2)^2}{(x+1)^3},\ x\in\mathbb{R},\ x\neq -1,$$

find the value of $\frac{dy}{dx}$ at $x=3$.

$\boxed{}$, $\boxed{\left.\frac{dy}{dx}\right|_{x=3}=\frac{13}{16}}$



BY LOGARITHMIC DIFFERENTIATION

$$\Rightarrow y=\frac{(x-1)^4(x-2)^2}{(x+1)^3}$$

$$\Rightarrow \ln y=\ln\left[\frac{(x-1)^4(x-2)^2}{(x+1)^3}\right]$$

$$\Rightarrow \ln y=\ln(x-1)^4+\ln(x-2)^2-\ln(x+1)^3$$

$$\Rightarrow \ln y=4\ln(x-1)+2\ln(x-2)-3\ln(x+1)$$

DIFFERENTIATE W.R.T x

$$\Rightarrow \frac{1}{y}\frac{dy}{dx}=\frac{4}{x-1}+\frac{2}{x-2}-\frac{3}{x+1}$$

WHEN $x=3$, $y=\frac{(3-1)^4\times(3-2)^2}{(3+1)^3}=\frac{16\times1}{64}=\frac{1}{4}$

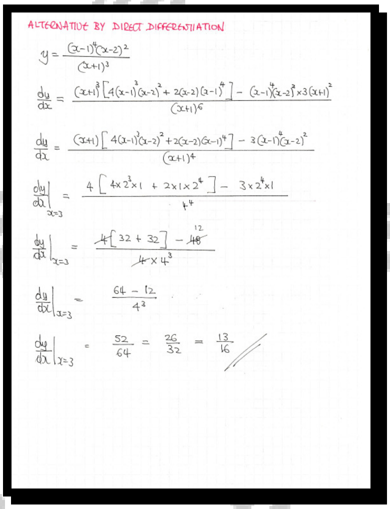
HENCE WE HAVE

$$\Rightarrow \frac{1}{\frac{1}{4}}\left.\frac{dy}{dx}\right|_{x=3}=\frac{4}{2}+\frac{2}{1}-\frac{3}{4}$$

$$\Rightarrow 4\left.\frac{dy}{dx}\right|_{x=3}=4-\frac{3}{4}$$

$$\Rightarrow 4\left.\frac{dy}{dx}\right|_{x=3}=\frac{13}{4}$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{x=3}=\frac{13}{16}\ //$$



ALTERNATIVE BY DIRECT DIFFERENTIATION

$$y=\frac{(x-1)^4(x-2)^2}{(x+1)^3}$$

$$\frac{dy}{dx}=\frac{(x+1)^3\left[4(x-1)^3(x-2)^2+2(x-2)(x-1)^4\right]-(x-1)^4(x-2)^2\times3(x+1)^2}{(x+1)^6}$$

$$\frac{dy}{dx}=\frac{(x+1)\left[4(x-1)^3(x-2)^2+2(x-2)(x-1)^4\right]-3(x-1)^4(x-2)^2}{(x+1)^4}$$

$$\left.\frac{dy}{dx}\right|_{x=3}=\frac{4\left[4\times2^3\times1+2\times1\times2^4\right]-3\times2^4\times1}{4^4}$$

$$\left.\frac{dy}{dx}\right|_{x=3}=\frac{\cancel{4}\left[32+32\right]-\overset{12}{\cancel{48}}}{\cancel{4}\times4^3}$$

$$\left.\frac{dy}{dx}\right|_{x=3}=\frac{64-12}{4^3}$$

$$\left.\frac{dy}{dx}\right|_{x=3}=\frac{52}{64}=\frac{26}{32}=\frac{13}{16}\ //$$

Question 195 (****+)

$$y = x^x, \; x > 0.$$

a) Show by using logarithms that

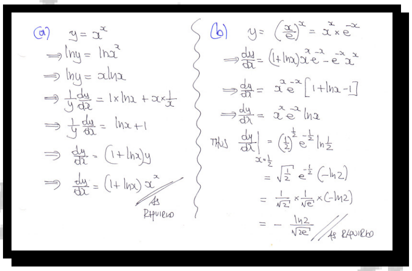
$$\frac{dy}{dx} = (1 + \ln x)x^x.$$

A curve C has equation

$$y = \left(\frac{x}{\mathrm{e}}\right)^x, \; x > 0.$$

b) Show that at the point on C where $x = \frac{1}{2}$, the gradient is $-\frac{\ln 2}{\sqrt{2\,\mathrm{e}}}$.

$\boxed{\text{proof}}$



(a) $y = x^x$

$\Rightarrow \ln y = \ln x^x$

$\Rightarrow \ln y = x\ln x$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = 1\times\ln x + x\times\frac{1}{x}$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \ln x + 1$

$\Rightarrow \frac{dy}{dx} = (1+\ln x)y$

$\Rightarrow \frac{dy}{dx} = (1+\ln x)x^x$ // AS REQUIRED

(b) $y = \left(\frac{x}{e}\right)^x = x^x \times e^{-x}$

$\Rightarrow \frac{dy}{dx} = (1+\ln x)x^x e^{-x} - e^{-x}x^x$

$\Rightarrow \frac{dy}{dx} = x^x e^{-x}[1+\ln x-1]$

$\Rightarrow \frac{dy}{dx} = x^x e^{-x}\ln x$

THUS $\left.\frac{dy}{dx}\right|_{x=\frac{1}{2}} = \left(\frac{1}{2}\right)^{\frac{1}{2}} e^{-\frac{1}{2}} \ln\frac{1}{2}$

$= \sqrt{\frac{1}{2}}\, e^{-\frac{1}{2}}(-\ln 2)$

$= \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{e}} \times (-\ln 2)$

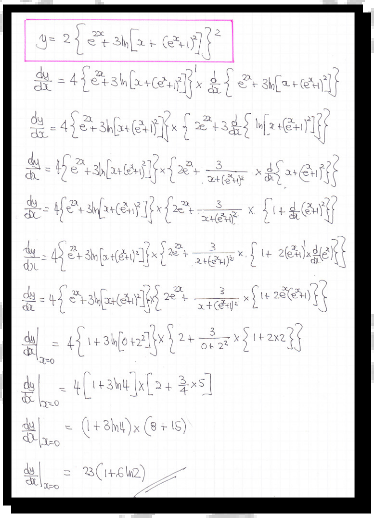
$= -\frac{\ln 2}{\sqrt{2e}}$ // AS REQUIRED

Question 196 (****+)

$$y = 2\left\{ e^{2x} + 3\ln\left[x + \left(e^{x} + 1 \right)^{2} \right] \right\}^{2}.$$

Show that the value of $\frac{dy}{dx}$ at $x = 0$ is $23\left(1 + 6\ln 2\right)$

$\boxed{}$, $\boxed{\text{proof}}$



$$\boxed{y = 2\left\{e^{2x} + 3\ln\left[x + (e^{x}+1)^{2}\right]\right\}^{2}}$$

$$\frac{dy}{dx} = 4\left\{e^{2x} + 3\ln\left[x + (e^{x}+1)^{2}\right]\right\}^{1} \times \frac{d}{dx}\left\{e^{2x} + 3\ln\left[x + (e^{x}+1)^{2}\right]\right\}$$

$$\frac{dy}{dx} = 4\left\{e^{2x} + 3\ln\left[x + (e^{x}+1)^{2}\right]\right\} \times \left\{2e^{2x} + 3\frac{d}{dx}\left\{\ln\left[x + (e^{x}+1)^{2}\right]\right\}\right\}$$

$$\frac{dy}{dx} = 4\left\{e^{2x} + 3\ln\left[x + (e^{x}+1)^{2}\right]\right\} \times \left\{2e^{2x} + \frac{3}{x + (e^{x}+1)^{2}} \times \frac{d}{dx}\left\{x + (e^{x}+1)^{2}\right\}\right\}$$

$$\frac{dy}{dx} = 4\left\{e^{2x} + 3\ln\left[x + (e^{x}+1)^{2}\right]\right\} \times \left\{2e^{2x} + \frac{3}{x + (e^{x}+1)^{2}} \times \left\{1 + \frac{d}{dx}(e^{x}+1)^{2}\right\}\right\}$$

$$\frac{dy}{dx} = 4\left\{e^{2x} + 3\ln\left[x + (e^{x}+1)^{2}\right]\right\} \times \left\{2e^{2x} + \frac{3}{x + (e^{x}+1)^{2}} \times \left\{1 + 2(e^{x}+1)^{1} \times \frac{d}{dx}(e^{x})\right\}\right\}$$

$$\frac{dy}{dx} = 4\left\{e^{2x} + 3\ln\left[x + (e^{x}+1)^{2}\right]\right\} \times \left\{2e^{2x} + \frac{3}{x + (e^{x}+1)^{2}} \times \left\{1 + 2e^{x}(e^{x}+1)\right\}\right\}$$

$$\left.\frac{dy}{dx}\right|_{x=0} = 4\left\{1 + 3\ln\left[0 + 2^{2}\right]\right\} \times \left\{2 + \frac{3}{0 + 2^{2}} \times \left\{1 + 2 \times 2\right\}\right\}$$

$$\left.\frac{dy}{dx}\right|_{x=0} = 4\left[1 + 3\ln 4\right] \times \left[2 + \frac{3}{4} \times 5\right]$$

$$\left.\frac{dy}{dx}\right|_{x=0} = (1 + 3\ln 4) \times (8 + 15)$$

$$\left.\frac{dy}{dx}\right|_{x=0} = 23(1 + 6\ln 2)$$

Question 197 (****+)

A curve has equation

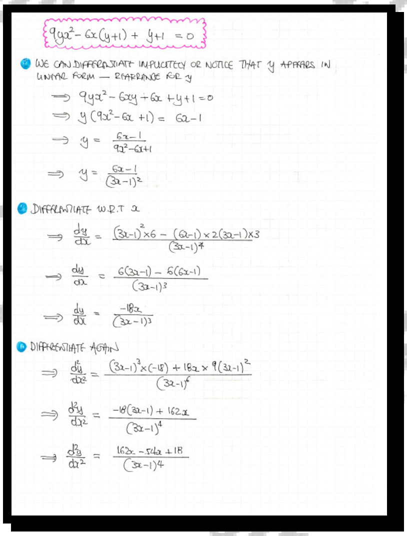
$$9yx^2 - 6x(y+1) + y + 1 = 0, \quad x \in \mathbb{R}, \quad x \neq \frac{1}{3}.$$

Find, in exact form where appropriate, the three solutions of the equation

$$2\frac{d^2y}{dx^2} = 6x + 1,$$

where $\frac{d^2y}{dx^2}$ represents the second derivative of the above equation.

$\boxed{}$, $\boxed{x = -\frac{1}{6}, \ \frac{1}{3}\left(1 \pm \sqrt{6}\right)}$



$\boxed{9yx^2 - 6x(y+1) + y+1 = 0}$

WE CAN DIFFERENTIATE IMPLICITLY OR NOTICE THAT y APPEARS IN LINEAR FORM — REARRANGE FOR y

$\Longrightarrow 9yx^2 - 6xy + 6x + y + 1 = 0$

$\Longrightarrow y(9x^2 - 6x + 1) = 6x - 1$

$\Longrightarrow y = \frac{6x-1}{9x^2-6x+1}$

$\Longrightarrow y = \frac{6x-1}{(3x-1)^2}$

DIFFERENTIATE W.R.T x

$\Longrightarrow \frac{dy}{dx} = \frac{(3x-1)^2 \times 6 - (6x-1) \times 2(3x-1) \times 3}{(3x-1)^4}$

$\Longrightarrow \frac{dy}{dx} = \frac{6(3x-1) - 6(6x-1)}{(3x-1)^3}$

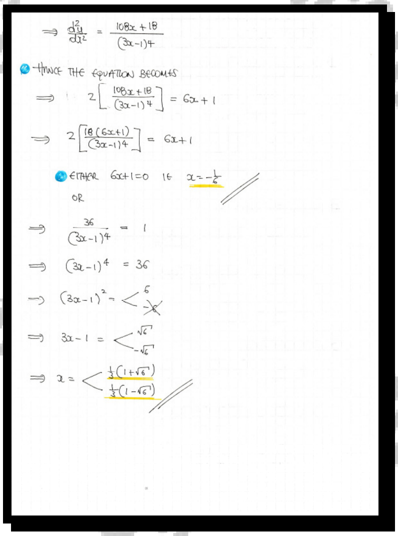
$\Longrightarrow \frac{dy}{dx} = \frac{-18x}{(3x-1)^3}$

DIFFERENTIATE AGAIN

$\Longrightarrow \frac{d^2y}{dx^2} = \frac{(3x-1)^3 \times (-18) + 18x \times 9(3x-1)^2}{(3x-1)^6}$

$\Longrightarrow \frac{d^2y}{dx^2} = \frac{-18(3x-1) + 162x}{(3x-1)^4}$

$\Longrightarrow \frac{d^2y}{dx^2} = \frac{162x - 54x + 18}{(3x-1)^4}$



$\Longrightarrow \frac{d^2y}{dx^2} = \frac{108x + 18}{(3x-1)^4}$

HENCE THE EQUATION BECOMES

$\Longrightarrow 2\left[\frac{108x+18}{(3x-1)^4}\right] = 6x + 1$

$\Longrightarrow 2\left[\frac{18(6x+1)}{(3x-1)^4}\right] = 6x + 1$

EITHER $6x+1=0$ IE $x = -\frac{1}{6}$ //

OR

$\Longrightarrow \frac{36}{(3x-1)^4} = 1$

$\Longrightarrow (3x-1)^4 = 36$

$\Longrightarrow (3x-1)^2 = < \begin{matrix} 6 \\ \cancel{-6} \end{matrix}$

$\Longrightarrow 3x - 1 = < \begin{matrix} \sqrt{6} \\ -\sqrt{6} \end{matrix}$

$\Longrightarrow x = < \begin{matrix} \frac{1}{3}(1+\sqrt{6}) \\ \frac{1}{3}(1-\sqrt{6}) \end{matrix}$ //

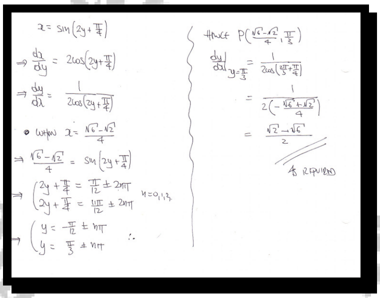
Question 198 **(****+)**

The point P, with x coordinate $\frac{\sqrt{6}-\sqrt{2}}{4}$, lies on the curve with equation

$$x = \sin\left(2y + \frac{\pi}{4}\right),\ 0 \le y \le \frac{\pi}{2}.$$

Show that the value of the gradient at P is $\frac{\sqrt{2}-\sqrt{6}}{2}$.

$\boxed{\text{proof}}$



$$x = \sin\left(2y + \frac{\pi}{4}\right)$$

$$\Rightarrow \frac{dx}{dy} = 2\cos\left(2y + \frac{\pi}{4}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2\cos\left(2y + \frac{\pi}{4}\right)}$$

- WHEN $x = \frac{\sqrt{6}-\sqrt{2}}{4}$

$$\Rightarrow \frac{\sqrt{6}-\sqrt{2}}{4} = \sin\left(2y + \frac{\pi}{4}\right)$$

$$\Rightarrow \begin{cases} 2y + \frac{\pi}{4} = \frac{\pi}{12} \pm 2n\pi \\ 2y + \frac{\pi}{4} = \frac{11\pi}{12} \pm 2n\pi \end{cases} \quad n = 0, 1, 2, 3 \ldots$$

$$\Rightarrow \begin{cases} y = -\frac{\pi}{12} \pm n\pi \\ y = \frac{\pi}{3} \pm n\pi \end{cases} \quad \therefore$$

HENCE $P\left(\frac{\sqrt{6}-\sqrt{2}}{4}, \frac{\pi}{3}\right)$

$$\left.\frac{dy}{dx}\right|_{y=\frac{\pi}{3}} = \frac{1}{2\cos\left(\frac{2\pi}{3}+\frac{\pi}{4}\right)} = \frac{1}{2\left(-\frac{\sqrt{6}+\sqrt{2}}{4}\right)} = \frac{\sqrt{2}-\sqrt{6}}{2}$$

AS REQUIRED

Question 199 **(****+)**

A curve C_1 has equation

$$y = \ln\sqrt{x} + \sqrt{\ln x}\,, \ \ x > 1\,.$$

a) Differentiate y with respect to x, simplifying the answer as far as possible.

A different curve C_2 has equation

$$y = (2x+1)^{\frac{1}{2}}(1-4x)^{-\frac{1}{2}}\ ,\ -\frac{1}{2} \le x < \frac{1}{4}.$$

b) Show that C_2 has no turning points.

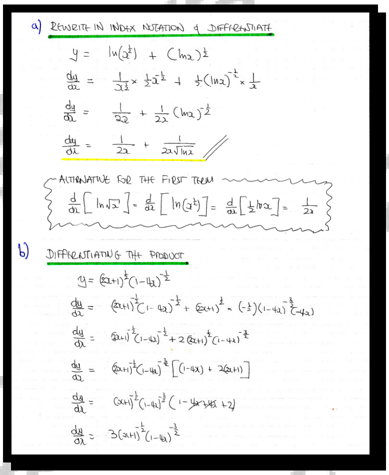
A third curve C_3 has equation

$$y = \frac{2x-1}{\sqrt{2x+1}},\ x \ge -\frac{1}{2}\ .$$

c) Show that

$$\frac{d}{dx}\left(\frac{2x-1}{\sqrt{2x+1}}\right) = \frac{2x+3}{(2x+1)^{\frac{3}{2}}}\,.$$

$$\boxed{}\ ,\ \boxed{\frac{dy}{dx} = \frac{1}{2x} + \frac{1}{2x\ln\sqrt{x}}}$$



a) REWRITE IN INDEX NOTATION & DIFFERENTIATE

$$y = \ln(x^{\frac{1}{2}}) + (\ln x)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{x^{\frac{1}{2}}} \times \frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}(\ln x)^{-\frac{1}{2}} \times \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{1}{2x} + \frac{1}{2x}(\ln x)^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2x} + \frac{1}{2x\sqrt{\ln x}}$$

ALTERNATIVE FOR THE FIRST TERM

$$\frac{d}{dx}\left[\ln\sqrt{x}\right] = \frac{d}{dx}\left[\ln(x^{\frac{1}{2}})\right] = \frac{d}{dx}\left[\frac{1}{2}\ln x\right] = \frac{1}{2x}$$

b) DIFFERENTIATING THE PRODUCT

$$y = (2x+1)^{\frac{1}{2}}(1-4x)^{-\frac{1}{2}}$$

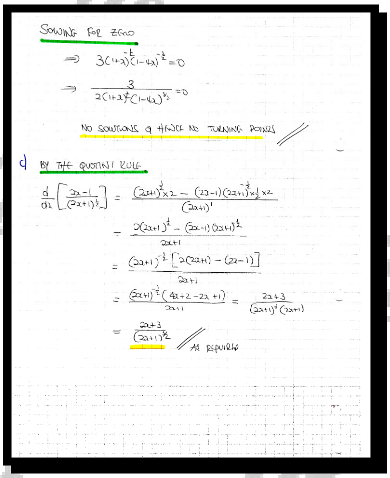
$$\frac{dy}{dx} = (2x+1)^{-\frac{1}{2}}(1-4x)^{-\frac{1}{2}} + (2x+1)^{\frac{1}{2}} \times (-\tfrac{1}{2})(1-4x)^{-\frac{3}{2}}(-4x)$$

$$\frac{dy}{dx} = (2x+1)^{-\frac{1}{2}}(1-4x)^{-\frac{1}{2}} + 2(2x+1)^{\frac{1}{2}}(1-4x)^{-\frac{3}{2}}$$

$$\frac{dy}{dx} = (2x+1)^{-\frac{1}{2}}(1-4x)^{-\frac{3}{2}}\left[(1-4x) + 2(2x+1)\right]$$

$$\frac{dy}{dx} = (x+1)^{-\frac{1}{2}}(1-4x)^{-\frac{3}{2}}(1 - \cancel{4x} + \cancel{4x} + 2)$$

$$\frac{dy}{dx} = 3(x+1)^{-\frac{1}{2}}(1-4x)^{-\frac{3}{2}}$$



SOLVING FOR ZERO

$$\Rightarrow 3(1+x)^{-\frac{1}{2}}(1-4x)^{-\frac{3}{2}} = 0$$

$$\Rightarrow \frac{3}{2(1+x)^{\frac{1}{2}}(1-4x)^{\frac{3}{2}}} = 0$$

NO SOLUTIONS & HENCE NO TURNING POINTS

c) BY THE QUOTIENT RULE

$$\frac{d}{dx}\left[\frac{2x-1}{(2x+1)^{\frac{1}{2}}}\right] = \frac{(2x+1)^{\frac{1}{2}} \times 2 - (2x-1)(2x+1)^{-\frac{1}{2}} \times \frac{1}{2} \times 2}{(2x+1)^{1}}$$

$$= \frac{2(2x+1)^{\frac{1}{2}} - (2x-1)(2x+1)^{-\frac{1}{2}}}{2x+1}$$

$$= \frac{(2x+1)^{-\frac{1}{2}}\left[2(2x+1) - (2x-1)\right]}{2x+1}$$

$$= \frac{(2x+1)^{-\frac{1}{2}}(4x+2-2x+1)}{2x+1} = \frac{2x+3}{(2x+1)^{\frac{1}{2}}(2x+1)}$$

$$= \frac{2x+3}{(2x+1)^{\frac{3}{2}}}$$

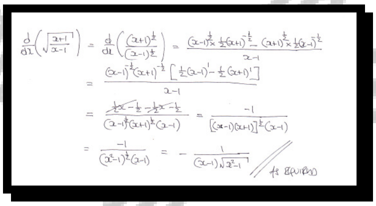
AS REQUIRED

Question 200 **(****+)**

Show clearly that

$$\frac{d}{dx}\left(\sqrt{\frac{x+1}{x-1}}\right) = -\frac{1}{(x-1)\sqrt{x^2-1}}.$$

$\boxed{\text{proof}}$



$$\frac{d}{dx}\left(\sqrt{\frac{x+1}{x-1}}\right) = \frac{d}{dx}\left(\frac{(x+1)^{\frac{1}{2}}}{(x-1)^{\frac{1}{2}}}\right) = \frac{(x-1)^{\frac{1}{2}}\times\frac{1}{2}(x+1)^{-\frac{1}{2}} - (x+1)^{\frac{1}{2}}\times\frac{1}{2}(x-1)^{-\frac{1}{2}}}{x-1}$$

$$= \frac{(x-1)^{-\frac{1}{2}}(x+1)^{-\frac{1}{2}}\left[\frac{1}{2}(x-1)^{1} - \frac{1}{2}(x+1)^{1}\right]}{x-1}$$

$$= \frac{\cancel{\frac{1}{2}x}-\frac{1}{2}-\cancel{\frac{1}{2}x}-\frac{1}{2}}{(x-1)^{\frac{1}{2}}(x+1)^{\frac{1}{2}}(x-1)} = \frac{-1}{\left[(x-1)(x+1)\right]^{\frac{1}{2}}(x-1)}$$

$$= \frac{-1}{(x^2-1)^{\frac{1}{2}}(x-1)} = -\frac{1}{(x-1)\sqrt{x^2-1}}$$

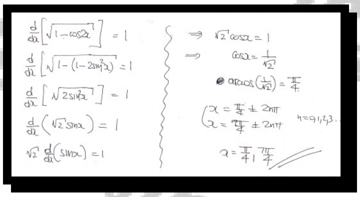
// AS REQUIRED

Question 201 **(****+)**

Solve the equation

$$\frac{d}{dx}\left(\sqrt{1-\cos 2x}\right) = 1,\ 0 \le x < 2\pi.$$

$$\boxed{x = \frac{\pi}{4}, \frac{5\pi}{4}}$$



$$\frac{d}{dx}\left[\sqrt{1-\cos 2x}\right] = 1$$

$$\frac{d}{dx}\left[\sqrt{1-(1-2\sin^2 x)}\right] = 1$$

$$\frac{d}{dx}\left[\sqrt{2\sin^2 x}\right] = 1$$

$$\frac{d}{dx}\left(\sqrt{2}\sin x\right) = 1$$

$$\sqrt{2}\frac{d}{dx}(\sin x) = 1$$

$$\Rightarrow \sqrt{2}\cos x = 1$$

$$\Rightarrow \cos x = \frac{1}{\sqrt{2}}$$

$$\arccos\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$$

$$\begin{cases} x = \frac{\pi}{4} \pm 2n\pi \\ x = \frac{7\pi}{4} \pm 2n\pi \end{cases} \quad n = 0,1,2,3\ldots$$

$$x = \frac{\pi}{4}, \frac{7\pi}{4}$$ //

Question 202 (****+)

Given that

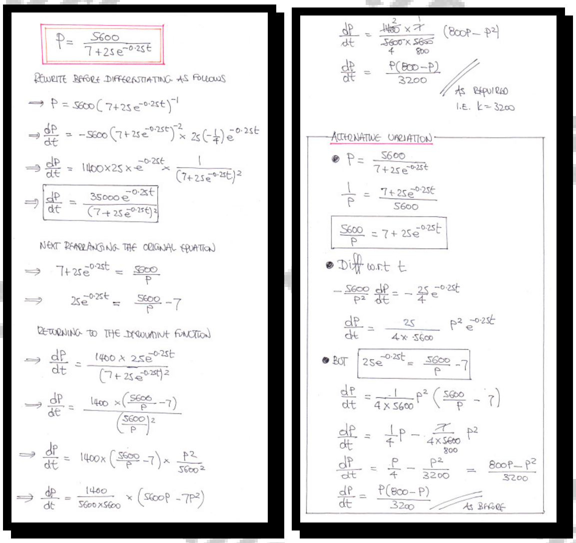
$$P = \frac{5600}{7+25\,\mathrm{e}^{-0.25t}},$$

show by a detailed method that

$$\frac{dP}{dt} = \frac{P(800-P)}{k},$$

where k is an integer to be found.

$\boxed{\text{SP}}$, $\boxed{k = 3200}$



$$\boxed{P = \frac{5600}{7+25e^{-0.25t}}}$$

REWRITE BEFORE DIFFERENTIATING AS FOLLOWS

$$\Rightarrow P = 5600\left(7+25e^{-0.25t}\right)^{-1}$$

$$\Rightarrow \frac{dP}{dt} = -5600\left(7+25e^{-0.25t}\right)^{-2} \times 25\left(-\frac{1}{4}\right)e^{-0.25t}$$

$$\Rightarrow \frac{dP}{dt} = 1400 \times 25 \times e^{-0.25t} \times \frac{1}{\left(7+25e^{-0.25t}\right)^2}$$

$$\Rightarrow \boxed{\frac{dP}{dt} = \frac{35000e^{-0.25t}}{\left(7+25e^{-0.25t}\right)^2}}$$

NEXT REARRANGING THE ORIGINAL EQUATION

$$\Rightarrow 7+25e^{-0.25t} = \frac{5600}{P}$$

$$\Rightarrow 25e^{-0.25t} = \frac{5600}{P} - 7$$

RETURNING TO THE DERIVATIVE FUNCTION

$$\Rightarrow \frac{dP}{dt} = \frac{1400 \times 25e^{-0.25t}}{\left(7+25e^{-0.25t}\right)^2}$$

$$\Rightarrow \frac{dP}{dt} = \frac{1400 \times \left(\frac{5600}{P} - 7\right)}{\left(\frac{5600}{P}\right)^2}$$

$$\Rightarrow \frac{dP}{dt} = 1400 \times \left(\frac{5600}{P} - 7\right) \times \frac{P^2}{5600^2}$$

$$\Rightarrow \frac{dP}{dt} = \frac{1400}{5600 \times 5600} \times \left(5600P - 7P^2\right)$$

$$\frac{dP}{dt} = \frac{\overset{2}{\cancel{1400}} \times \overset{1}{\cancel{7}}}{\underset{4}{\cancel{5600}} \times \underset{800}{\cancel{5600}}}\left(800P - P^2\right)$$

$$\frac{dP}{dt} = \frac{P(800-P)}{3200}$$

AS REQUIRED

I.E. $k = 3200$

ALTERNATIVE VARIATION

- $P = \dfrac{5600}{7+25e^{-0.25t}}$

$$\frac{1}{P} = \frac{7+25e^{-0.25t}}{5600}$$

$$\boxed{\frac{5600}{P} = 7+25e^{-0.25t}}$$

- Diff w.r.t t

$$-\frac{5600}{P^2}\frac{dP}{dt} = -\frac{25}{4}e^{-0.25t}$$

$$\frac{dP}{dt} = \frac{25}{4 \times 5600}P^2e^{-0.25t}$$

- BUT $\boxed{25e^{-0.25t} = \frac{5600}{P} - 7}$

$$\frac{dP}{dt} = \frac{1}{4 \times 5600}P^2\left(\frac{5600}{P} - 7\right)$$

$$\frac{dP}{dt} = \frac{1}{4}P - \frac{\cancel{7}}{4 \times \underset{800}{\cancel{5600}}}P^2$$

$$\frac{dP}{dt} = \frac{P}{4} - \frac{P^2}{3200} = \frac{800P - P^2}{3200}$$

$$\frac{dP}{dt} = \frac{P(800-P)}{3200}$$

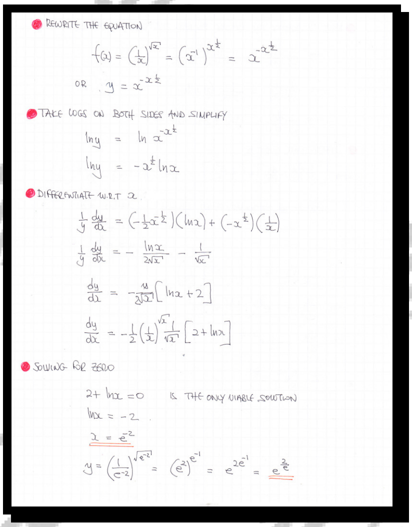
AS BEFORE

Question 203 **(****+)**

$$f(x) = \left(\frac{1}{x}\right)^{\sqrt{x}}, \quad x \in \mathbb{R}, \ x > 0 .$$

Determine in exact simplified form the coordinates of the stationary point of $f(x)$, fully justifying its nature.

$\boxed{}$, $\boxed{\text{local maximum at } \left(\frac{1}{e^2}, e^{\frac{2}{e}}\right)}$



● REWRITE THE EQUATION

$f(x) = \left(\frac{1}{x}\right)^{\sqrt{x}} = \left(x^{-1}\right)^{x^{\frac{1}{2}}} = x^{-x^{\frac{1}{2}}}$

OR $y = x^{-x^{\frac{1}{2}}}$

● TAKE LOGS ON BOTH SIDES AND SIMPLIFY

$\ln y = \ln x^{-x^{\frac{1}{2}}}$

$\ln y = -x^{\frac{1}{2}} \ln x$

● DIFFERENTIATE W.R.T x

$\frac{1}{y}\frac{dy}{dx} = \left(-\frac{1}{2}x^{-\frac{1}{2}}\right)(\ln x) + \left(-x^{\frac{1}{2}}\right)\left(\frac{1}{x}\right)$

$\frac{1}{y}\frac{dy}{dx} = -\frac{\ln x}{2\sqrt{x}} - \frac{1}{\sqrt{x}}$

$\frac{dy}{dx} = -\frac{y}{2\sqrt{x}}\left[\ln x + 2\right]$

$\frac{dy}{dx} = -\frac{1}{2}\left(\frac{1}{x}\right)^{\sqrt{x}}\frac{1}{\sqrt{x}}\left[2 + \ln x\right]$

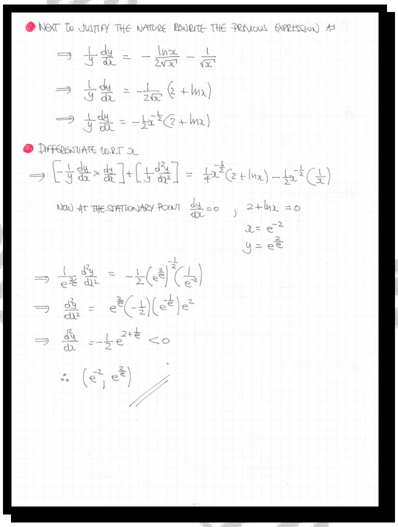
● SOLVING FOR ZERO

$2 + \ln x = 0$ IS THE ONLY VIABLE SOLUTION

$\ln x = -2$

$\underline{\underline{x = e^{-2}}}$

$y = \left(\frac{1}{e^{-2}}\right)^{\sqrt{e^{-2}}} = \left(e^{2}\right)^{e^{-1}} = e^{2e^{-1}} = \underline{\underline{e^{\frac{2}{e}}}}$



● NEXT TO JUSTIFY THE NATURE REWRITE THE PREVIOUS EXPRESSION AS

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = -\frac{\ln x}{2\sqrt{x}} - \frac{1}{\sqrt{x}}$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = -\frac{1}{2\sqrt{x}}\left(2 + \ln x\right)$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = -\frac{1}{2}x^{-\frac{1}{2}}\left(2 + \ln x\right)$

● DIFFERENTIATE W.R.T x

$\Rightarrow \left[-\frac{1}{y}\frac{dy}{dx} \times \frac{dy}{dx}\right] + \left[\frac{1}{y}\frac{d^2y}{dx^2}\right] = \frac{1}{4}x^{-\frac{3}{2}}\left(2 + \ln x\right) - \frac{1}{2}x^{-\frac{1}{2}}\left(\frac{1}{x}\right)$

NOW AT THE STATIONARY POINT $\frac{dy}{dx} = 0$, $2 + \ln x = 0$

$x = e^{-2}$

$y = e^{\frac{2}{e}}$

$\Rightarrow \frac{1}{e^{2/e}}\frac{d^2y}{dx^2} = -\frac{1}{2}\left(e^{\frac{2}{e}}\right)^{-\frac{1}{2}}\left(\frac{1}{e^{-2}}\right)$

$\Rightarrow \frac{d^2y}{dx^2} = e^{\frac{2}{e}}\left(-\frac{1}{2}\right)\left(e^{-\frac{1}{e}}\right)e^{2}$

$\Rightarrow \frac{d^2y}{dx} = -\frac{1}{2}e^{2+\frac{1}{e}} < 0$

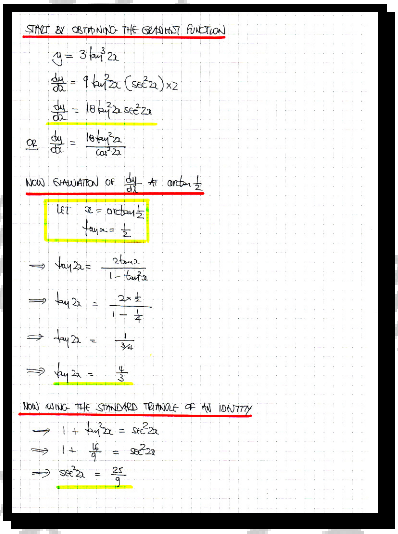
$\therefore \left(e^{-2}, e^{\frac{2}{e}}\right)$ //

Question 204 (****+)

$$y = 3\tan^3 2x\ ,\ x \in \mathbb{R}\ ,\ x \neq \frac{1}{4}n\pi\ .$$

Determine, by showing detailed workings, the value of $\frac{dy}{dx}$, at $x = \arctan\frac{1}{2}$.

$$\boxed{\quad}\ ,\ \boxed{\left.\frac{dy}{dx}\right|_{x=\arctan\frac{1}{2}} = \frac{800}{9}}$$



START BY OBTAINING THE GRADIENT FUNCTION

$$y = 3\tan^3 2x$$

$$\frac{dy}{dx} = 9\tan^2 2x\left(\sec^2 2x\right)\times 2$$

$$\frac{dy}{dx} = 18\tan^2 2x\sec^2 2x$$

OR $$\frac{dy}{dx} = \frac{18\tan^2 2x}{\cos^2 2x}$$

NOW EVALUATION OF $\frac{dy}{dx}$ AT $\arctan\frac{1}{2}$

$$\boxed{\begin{aligned}\text{LET}\quad x &= \arctan\tfrac{1}{2}\\ \tan x &= \tfrac{1}{2}\end{aligned}}$$

$$\Rightarrow \tan 2x = \frac{2\tan x}{1-\tan^2 x}$$

$$\Rightarrow \tan 2x = \frac{2\times\frac{1}{2}}{1-\frac{1}{4}}$$

$$\Rightarrow \tan 2x = \frac{1}{3/4}$$

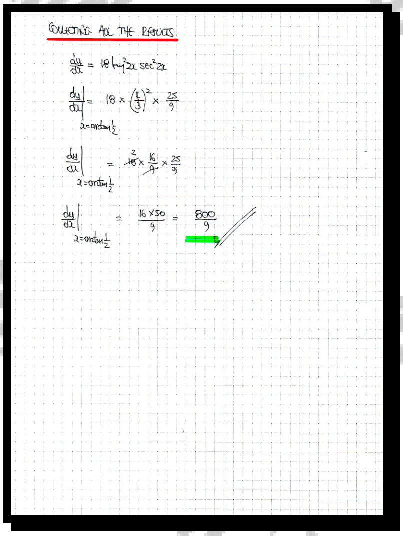
$$\Rightarrow \tan 2x = \frac{4}{3}$$

NOW USING THE STANDARD TRIANGLE OR AN IDENTITY

$$\Rightarrow 1 + \tan^2 2x = \sec^2 2x$$

$$\Rightarrow 1 + \frac{16}{9} = \sec^2 2x$$

$$\Rightarrow \sec^2 2x = \frac{25}{9}$$



COLLECTING ALL THE RESULTS

$$\frac{dy}{dx} = 18\tan^2 2x\sec^2 2x$$

$$\left.\frac{dy}{dx}\right|_{x=\arctan\frac{1}{2}} = 18\times\left(\frac{4}{3}\right)^2\times\frac{25}{9}$$

$$\left.\frac{dy}{dx}\right|_{x=\arctan\frac{1}{2}} = \overset{2}{\cancel{18}}\times\frac{16}{\cancel{9}}\times\frac{25}{9}$$

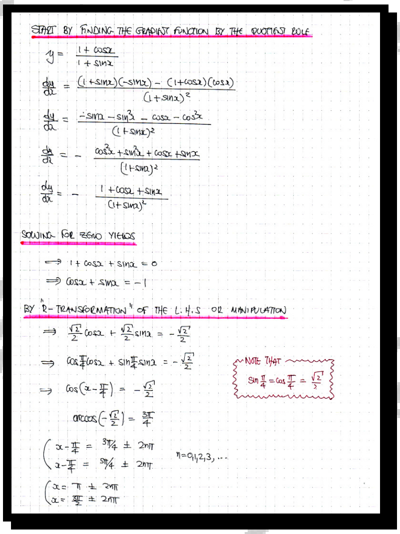
$$\left.\frac{dy}{dx}\right|_{x=\arctan\frac{1}{2}} = \frac{16\times 50}{9} = \frac{800}{9}$$

Question 205 (****+)

$$y=\frac{1+\cos x}{1+\sin x},\ 0\le x<2\pi,\ x\ne\frac{3}{2}\pi .$$

Determine, with full justification, the coordinates of the minimum point of y.

$\boxed{\quad}$, $\boxed{(\pi,0)}$



START BY FINDING THE GRADIENT FUNCTION BY THE QUOTIENT RULE

$$y=\frac{1+\cos x}{1+\sin x}$$

$$\frac{dy}{dx}=\frac{(1+\sin x)(-\sin x)-(1+\cos x)(\cos x)}{(1+\sin x)^2}$$

$$\frac{dy}{dx}=\frac{-\sin x-\sin^2 x-\cos x-\cos^2 x}{(1+\sin x)^2}$$

$$\frac{dy}{dx}=-\frac{\cos^2 x+\sin^2 x+\cos x+\sin x}{(1+\sin x)^2}$$

$$\frac{dy}{dx}=-\frac{1+\cos x+\sin x}{(1+\sin x)^2}$$

SOLVING FOR ZERO YIELDS

$$\Rightarrow 1+\cos x+\sin x=0$$

$$\Rightarrow \cos x+\sin x=-1$$

BY "R-TRANSFORMATION" OF THE L.H.S OR MANIPULATION

$$\Rightarrow \frac{\sqrt{2}}{2}\cos x+\frac{\sqrt{2}}{2}\sin x=-\frac{\sqrt{2}}{2}$$

$$\Rightarrow \cos\frac{\pi}{4}\cos x+\sin\frac{\pi}{4}\sin x=-\frac{\sqrt{2}}{2}$$

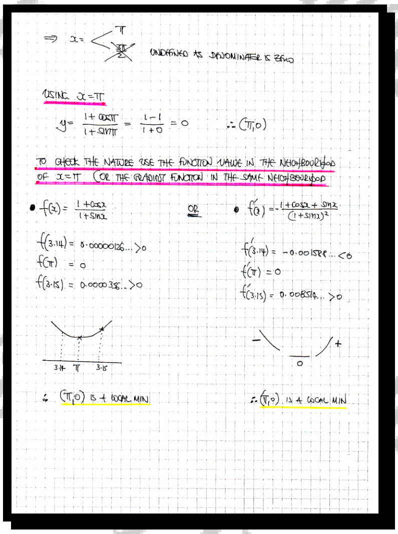
NOTE THAT $\sin\frac{\pi}{4}=\cos\frac{\pi}{4}=\frac{\sqrt{2}}{2}$

$$\Rightarrow \cos\left(x-\frac{\pi}{4}\right)=-\frac{\sqrt{2}}{2}$$

$$\arccos\left(-\frac{\sqrt{2}}{2}\right)=\frac{3\pi}{4}$$

$$\left(\begin{array}{l} x-\frac{\pi}{4}=3\pi/4\pm 2n\pi \\ x-\frac{\pi}{4}=5\pi/4\pm 2n\pi \end{array}\right.\qquad n=0,1,2,3,\ldots$$

$$\left(\begin{array}{l} x=\pi\pm 2n\pi \\ x=\frac{3\pi}{2}\pm 2n\pi \end{array}\right.$$



$$\Rightarrow x=\begin{cases}\pi\\ \cancel{\frac{3\pi}{2}}\end{cases}$$ UNDEFINED AS DENOMINATOR IS ZERO

USING $x=\pi$

$$y=\frac{1+\cos\pi}{1+\sin\pi}=\frac{1-1}{1+0}=0 \qquad \therefore(\pi,0)$$

TO CHECK THE NATURE USE THE FUNCTION VALUE IN THE NEIGHBOURHOOD OF $x=\pi$ (OR THE GRADIENT FUNCTION IN THE SAME NEIGHBOURHOOD

- $f(x)=\frac{1+\cos x}{1+\sin x}$

$f(3.14)=0.00000126\ldots>0$
$f(\pi)=0$
$f(3.15)=0.0000358\ldots>0$

$\therefore (\pi,0)$ IS A LOCAL MIN

OR

- $f'(x)=-\frac{1+\cos x+\sin x}{(1+\sin x)^2}$

$f'(3.14)=-0.001588\ldots<0$
$f'(\pi)=0$
$f'(3.15)=0.008514\ldots>0$

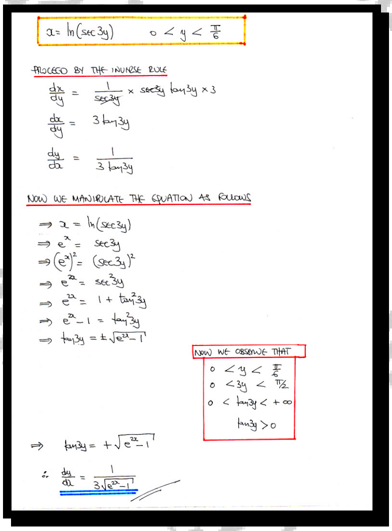
$\therefore (\pi,0)$ IS A LOCAL MIN

Question 206 (****+)

$$x = \ln\left(\sec 3y\right),\ 0 < y < \frac{1}{6}\pi .$$

Determine, with full justification, an expression for $\frac{dy}{dx}$, in terms of x.

$\boxed{\quad}$, $\boxed{\frac{dy}{dx} = \frac{1}{3\sqrt{e^{2x}-1}}}$



$\boxed{x = \ln(\sec 3y) \qquad 0 < y < \frac{\pi}{6}}$

PROCEED BY THE INVERSE RULE

$$\frac{dx}{dy} = \frac{1}{\cancel{\sec 3y}} \times \cancel{\sec 3y} \tan 3y \times 3$$

$$\frac{dx}{dy} = 3\tan 3y$$

$$\frac{dy}{dx} = \frac{1}{3\tan 3y}$$

NOW WE MANIPULATE THE EQUATION AS FOLLOWS

$$\Rightarrow x = \ln(\sec 3y)$$

$$\Rightarrow e^{x} = \sec 3y$$

$$\Rightarrow (e^{x})^2 = (\sec 3y)^2$$

$$\Rightarrow e^{2x} = \sec^2 3y$$

$$\Rightarrow e^{2x} = 1 + \tan^2 3y$$

$$\Rightarrow e^{2x} - 1 = \tan^2 3y$$

$$\Rightarrow \tan 3y = \pm\sqrt{e^{2x}-1}$$

NOW WE OBSERVE THAT

$$0 < y < \frac{\pi}{6}$$

$$0 < 3y < \pi/2$$

$$0 < \tan 3y < +\infty$$

$$\tan 3y > 0$$

$$\Rightarrow \tan 3y = +\sqrt{e^{2x}-1}$$

$$\therefore \frac{dy}{dx} = \frac{1}{3\sqrt{e^{2x}-1}}$$

Question 207 **(****+)**

A curve C has equation

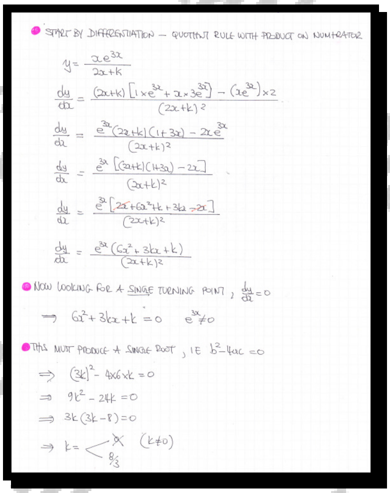
$$y = \frac{x e^{3x}}{2x+k}, \quad x \in \mathbb{R}, \quad x \neq k,$$

where k is a non zero constant.

It is given that C has a single turning point at P.

Find the exact coordinates of P.

$\boxed{\text{SP-M}}$, $\boxed{P\left(-\frac{2}{3}, -\frac{1}{2}e^{-2}\right)}$



● START BY DIFFERENTIATION – QUOTIENT RULE WITH PRODUCT ON NUMERATOR

$$y = \frac{xe^{3x}}{2x+k}$$

$$\frac{dy}{dx} = \frac{(2x+k)\left[1 \times e^{3x} + x \times 3e^{3x}\right] - \left(xe^{3x}\right) \times 2}{(2x+k)^2}$$

$$\frac{dy}{dx} = \frac{e^{3x}(2x+k)(1+3x) - 2xe^{3x}}{(2x+k)^2}$$

$$\frac{dy}{dx} = \frac{e^{3x}\left[(2x+k)(1+3x) - 2x\right]}{(2x+k)^2}$$

$$\frac{dy}{dx} = \frac{e^{3x}\left[\cancel{2x} + 6x^2 + k + 3kx \cancel{-2x}\right]}{(2x+k)^2}$$

$$\frac{dy}{dx} = \frac{e^{3x}\left(6x^2 + 3kx + k\right)}{(2x+k)^2}$$

● NOW LOOKING FOR A SINGLE TURNING POINT, $\frac{dy}{dx} = 0$

$$\Rightarrow \quad 6x^2 + 3kx + k = 0 \qquad e^{3x} \neq 0$$

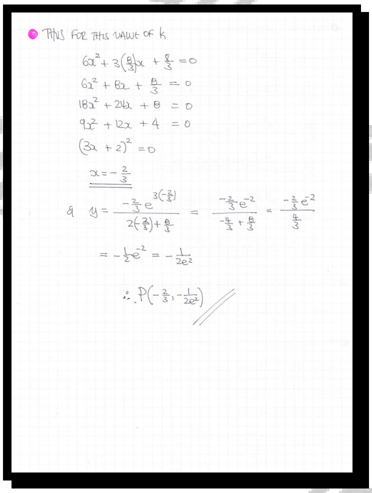
● THIS MUST PRODUCE A SINGLE ROOT, IE $b^2 - 4ac = 0$

$$\Rightarrow \quad (3k)^2 - 4 \times 6 \times k = 0$$

$$\Rightarrow \quad 9k^2 - 24k = 0$$

$$\Rightarrow \quad 3k(3k-8) = 0$$

$$\Rightarrow \quad k = \begin{cases} \cancel{0} & (k \neq 0) \\ \frac{8}{3} \end{cases}$$



● THUS FOR THIS VALUE OF k

$$6x^2 + 3\left(\frac{8}{3}\right)x + \frac{8}{3} = 0$$

$$6x^2 + 8x + \frac{8}{3} = 0$$

$$18x^2 + 24x + 8 = 0$$

$$9x^2 + 12x + 4 = 0$$

$$(3x+2)^2 = 0$$

$$x = -\frac{2}{3}$$

$$\& \quad y = \frac{-\frac{2}{3}e^{3\left(-\frac{2}{3}\right)}}{2\left(-\frac{2}{3}\right) + \frac{8}{3}} = \frac{-\frac{2}{3}e^{-2}}{-\frac{4}{3} + \frac{8}{3}} = \frac{-\frac{2}{3}e^{-2}}{\frac{4}{3}}$$

$$= -\frac{1}{2}e^{-2} = -\frac{1}{2e^2}$$

$$\therefore P\left(-\frac{2}{3}, -\frac{1}{2e^2}\right)$$

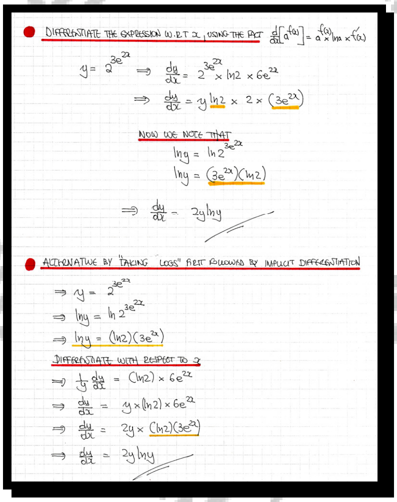
Question 208 (****+)

A curve has equation

$$2^{3e^{2x}}, \quad x \in \mathbb{R}.$$

Express $\frac{dy}{dx}$ in terms of y .

$\boxed{}$, $\boxed{\frac{dy}{dx} = 2y\ln y}$



● DIFFERENTIATE THE EXPRESSION W.R.T x, USING THE FACT $\frac{d}{dx}\left[a^{f(x)}\right] = a^{f(x)} \times \ln a \times f'(x)$

$y = 2^{3e^{2x}} \Rightarrow \frac{dy}{dx} = 2^{3e^{2x}} \times \ln 2 \times 6e^{2x}$

$\Rightarrow \frac{dy}{dx} = y \ln 2 \times 2 \times (3e^{2x})$

NOW WE NOTE THAT

$\ln y = \ln 2^{3e^{2x}}$

$\ln y = (3e^{2x})(\ln 2)$

$\Rightarrow \frac{dy}{dx} = 2y\ln y$

● ALTERNATIVE BY "TAKING LOGS" FIRST FOLLOWED BY IMPLICIT DIFFERENTIATION

$\Rightarrow y = 2^{3e^{2x}}$

$\Rightarrow \ln y = \ln 2^{3e^{2x}}$

$\Rightarrow \ln y = (\ln 2)(3e^{2x})$

DIFFERENTIATE WITH RESPECT TO x

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = (\ln 2) \times 6e^{2x}$

$\Rightarrow \frac{dy}{dx} = y \times (\ln 2) \times 6e^{2x}$

$\Rightarrow \frac{dy}{dx} = 2y \times (\ln 2)(3e^{2x})$

$\Rightarrow \frac{dy}{dx} = 2y\ln y$

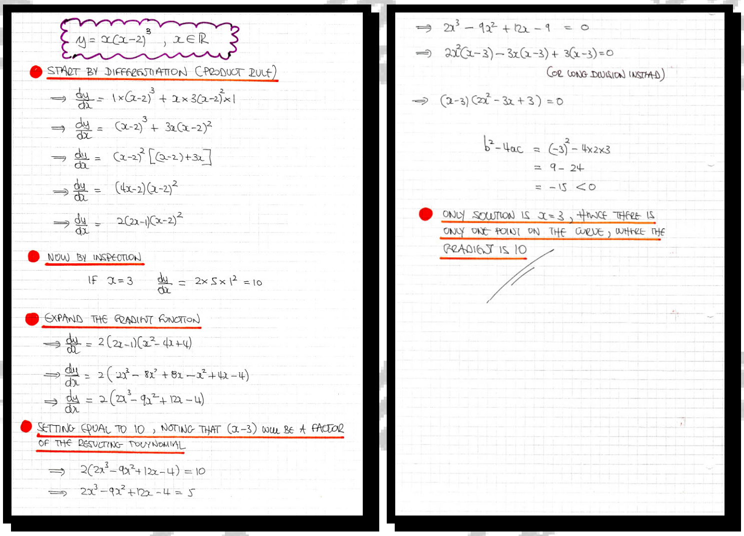
Question 209 (****+)

A quartic curve C has equation

$$y = x(x-2)^3, \; x \in \mathbb{R}.$$

Show that there is only one point on C where the gradient is 10.

proof



$$y = x(x-2)^3, \quad x \in \mathbb{R}$$

● START BY DIFFERENTIATION (PRODUCT RULE)

$$\Rightarrow \frac{dy}{dx} = 1\times(x-2)^3 + x\times 3(x-2)^2\times 1$$

$$\Rightarrow \frac{dy}{dx} = (x-2)^3 + 3x(x-2)^2$$

$$\Rightarrow \frac{dy}{dx} = (x-2)^2\left[(x-2)+3x\right]$$

$$\Rightarrow \frac{dy}{dx} = (4x-2)(x-2)^2$$

$$\Rightarrow \frac{dy}{dx} = 2(2x-1)(x-2)^2$$

● NOW BY INSPECTION

IF $x=3$ $\quad \frac{dy}{dx} = 2\times 5\times 1^2 = 10$

● EXPAND THE GRADIENT FUNCTION

$$\Rightarrow \frac{dy}{dx} = 2(2x-1)(x^2-4x+4)$$

$$\Rightarrow \frac{dy}{dx} = 2(2x^3-8x^2+8x-x^2+4x-4)$$

$$\Rightarrow \frac{dy}{dx} = 2(2x^3-9x^2+12x-4)$$

● SETTING EQUAL TO 10, NOTING THAT $(x-3)$ WILL BE A FACTOR OF THE RESULTING POLYNOMIAL

$$\Rightarrow 2(2x^3-9x^2+12x-4) = 10$$

$$\Rightarrow 2x^3-9x^2+12x-4 = 5$$

$$\Rightarrow 2x^3-9x^2+12x-9 = 0$$

$$\Rightarrow 2x^2(x-3)-3x(x-3)+3(x-3)=0$$

(OR LONG DIVISION INSTEAD)

$$\Rightarrow (x-3)(2x^2-3x+3)=0$$

$$\begin{aligned} b^2-4ac &= (-3)^2-4\times2\times3 \\ &= 9-24 \\ &= -15 < 0 \end{aligned}$$

● ONLY SOLUTION IS $x=3$, HENCE THERE IS ONLY ONE POINT ON THE CURVE, WHERE THE GRADIENT IS 10

Question 210 (****+)

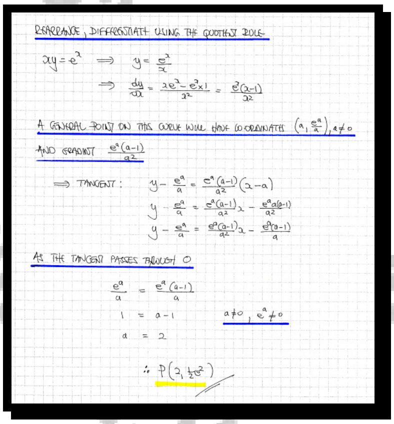
The point P lies on the curve with equation

$$xy = \mathrm{e}^{x}, \qquad xy > 0.$$

The tangent to the curve at P passes through the origin O .

Determine the coordinates of P .

$\boxed{\quad\quad}$, $\boxed{P\left(2, \frac{1}{2}\mathrm{e}^{2}\right)}$



REARRANGE, DIFFERENTIATE USING THE QUOTIENT RULE

$$xy = e^{x} \implies y = \frac{e^{x}}{x}$$

$$\implies \frac{dy}{dx} = \frac{xe^{x} - e^{x} \times 1}{x^{2}} = \frac{e^{x}(x-1)}{x^{2}}$$

A GENERAL POINT ON THIS CURVE WILL HAVE COORDINATES $\left(a, \frac{e^{a}}{a}\right), a \neq 0$

AND GRADIENT $\frac{e^{a}(a-1)}{a^{2}}$

$$\implies \text{TANGENT}: \quad y - \frac{e^{a}}{a} = \frac{e^{a}(a-1)}{a^{2}}(x-a)$$

$$y - \frac{e^{a}}{a} = \frac{e^{a}(a-1)}{a^{2}}x - \frac{e^{a}a(a-1)}{a^{2}}$$

$$y - \frac{e^{a}}{a} = \frac{e^{a}(a-1)}{a^{2}}x - \frac{e^{a}(a-1)}{a}$$

AS THE TANGENT PASSES THROUGH O

$$\frac{e^{a}}{a} = \frac{e^{a}(a-1)}{a}$$

$$1 = a - 1 \qquad a \neq 0, \; e^{a} \neq 0$$

$$a = 2$$

$$\therefore P\left(2, \tfrac{1}{2}e^{2}\right)$$

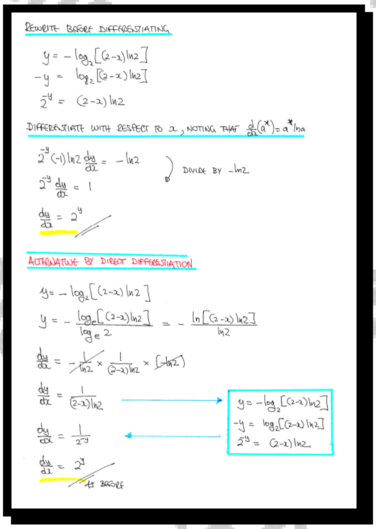
Question 211 (****+)

A curve has equation

$$y = -\log_2\left[(2-x)\ln 2\right], \quad x \in \mathbb{R}, \quad x < 2.$$

Determine a simplified expression for $\frac{dy}{dx}$ in terms of y.

☐, $\boxed{\frac{dy}{dx} = 2^y}$



REWRITE BEFORE DIFFERENTIATING

$y = -\log_2[(2-x)\ln 2]$

$-y = \log_2[(2-x)\ln 2]$

$2^{-y} = (2-x)\ln 2$

DIFFERENTIATE WITH RESPECT TO x, NOTING THAT $\frac{d}{dx}(a^x) = a^x \ln a$

$2^{-y}(-1)\ln 2 \frac{dy}{dx} = -\ln 2$

DIVIDE BY $-\ln 2$

$2^{-y}\frac{dy}{dx} = 1$

$\frac{dy}{dx} = 2^y$

ALTERNATIVE BY DIRECT DIFFERENTIATION

$y = -\log_2[(2-x)\ln 2]$

$y = -\frac{\log_e[(2-x)\ln 2]}{\log_e 2} = -\frac{\ln[(2-x)\ln 2]}{\ln 2}$

$\frac{dy}{dx} = -\cancel{\frac{1}{\ln 2}} \times \frac{1}{(2-x)\ln 2} \times \cancel{(-\ln 2)}$

$\frac{dy}{dx} = \frac{1}{(2-x)\ln 2}$

$y = -\log_2[(2-x)\ln 2]$

$-y = \log_2[(2-x)\ln 2]$

$2^{-y} = (2-x)\ln 2$

$\frac{dy}{dx} = \frac{1}{2^{-y}}$

$\frac{dy}{dx} = 2^y$

AS BEFORE

Question 212 (****+)

$$y = \arccos x\,,\ x \in \mathbb{R}\,,\ -1 \le x \le 1\,.$$

a) By writing the above equation in the from $x = f(y)$, show that

$$\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}\,.$$

A curve has equation

$$y = \arccos\left(1-x^2\right),\ \ x \in \mathbb{R}\,,\ 0 < x \le \sqrt{2}\,.$$

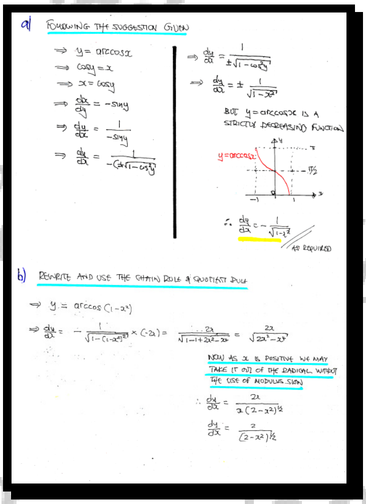
b) Show further that

$$\frac{d^2y}{dx^2} = \frac{2x}{\left(2-x^2\right)^{\frac{3}{2}}}\,.$$

c) Show clearly that

$$16\frac{d^3y}{dx^3} = 4x\frac{d^2y}{dx^2}\left(\frac{dy}{dx}\right)^2 + \left(2+x^2\right)\left(\frac{dy}{dx}\right)^5\,.$$

☐ , proof



a) FOLLOWING THE SUGGESTION GIVEN

$$\Rightarrow y = \arccos x$$
$$\Rightarrow \cos y = x$$
$$\Rightarrow x = \cos y$$
$$\Rightarrow \frac{dx}{dy} = -\sin y$$
$$\Rightarrow \frac{dy}{dx} = \frac{1}{-\sin y}$$
$$\Rightarrow \frac{dy}{dx} = \frac{1}{-(\pm\sqrt{1-\cos^2 y})}$$
$$\Rightarrow \frac{dy}{dx} = \frac{1}{\pm\sqrt{1-\cos^2 y}}$$
$$\Rightarrow \frac{dy}{dx} = \pm\frac{1}{\sqrt{1-x^2}}$$

BUT $y = \arccos x$ IS A STRICTLY DECREASING FUNCTION

$$\therefore \frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

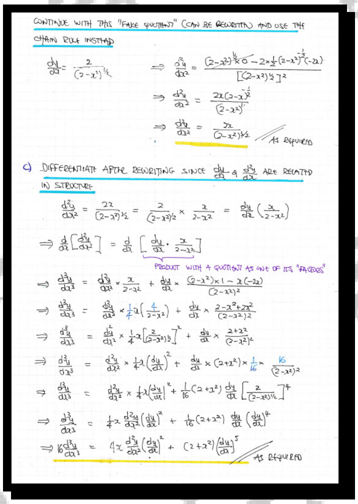
AS REQUIRED

b) REWRITE AND USE THE CHAIN RULE & QUOTIENT RULE

$$\Rightarrow y = \arccos(1-x^2)$$
$$\Rightarrow \frac{dy}{dx} = -\frac{1}{\sqrt{1-(1-x^2)^2}} \times (-2x) = \frac{2x}{\sqrt{1-1+2x^2-x^4}} = \frac{2x}{\sqrt{2x^2-x^4}}$$

NOW AS x IS POSITIVE WE MAY TAKE IT OUT OF THE RADICAL WITHOUT THE USE OF MODULUS SIGN

$$\therefore \frac{dy}{dx} = \frac{2x}{x(2-x^2)^{\frac{1}{2}}}$$
$$\frac{dy}{dx} = \frac{2}{(2-x^2)^{\frac{1}{2}}}$$



CONTINUE WITH THIS "FAKE QUOTIENT" (CAN BE REWRITTEN) AND USE THE CHAIN RULE INSTEAD

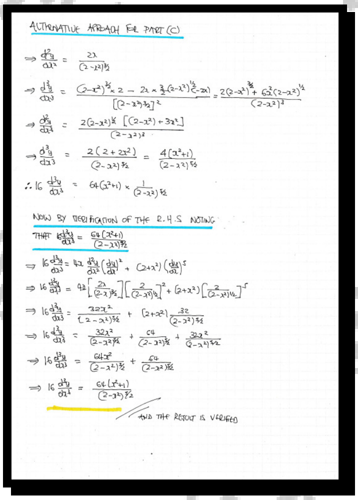
$$\frac{dy}{dx} = \frac{2}{(2-x^2)^{\frac{1}{2}}} \qquad \Rightarrow \frac{d^2y}{dx^2} = \frac{\cancel{(2-x^2)^{\frac{1}{2}}\times 0} - 2\times\frac{1}{2}(2-x^2)^{-\frac{1}{2}}(-2x)}{\left[(2-x^2)^{\frac{1}{2}}\right]^2}$$
$$\Rightarrow \frac{d^2y}{dx^2} = \frac{2x(2-x^2)^{-\frac{1}{2}}}{(2-x^2)^1}$$
$$\Rightarrow \frac{d^2y}{dx^2} = \frac{2x}{(2-x^2)^{\frac{3}{2}}}$$

AS REQUIRED

c) DIFFERENTIATE AFTER REWRITING SINCE $\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$ ARE RELATED IN STRUCTURE

$$\frac{d^2y}{dx^2} = \frac{2x}{(2-x^2)^{\frac{3}{2}}} = \frac{2}{(2-x^2)^{\frac{1}{2}}} \times \frac{x}{2-x^2} = \frac{dy}{dx}\left(\frac{x}{2-x^2}\right)$$
$$\Rightarrow \frac{d}{dx}\left[\frac{d^2y}{dx^2}\right] = \frac{d}{dx}\left[\underbrace{\frac{dy}{dx}\cdot\frac{x}{2-x^2}}_{\text{PRODUCT WITH A QUOTIENT AS ONE OF ITS "FACTORS"}}\right]$$
$$\Rightarrow \frac{d^3y}{dx^3} = \frac{d^2y}{dx^2}\times\frac{x}{2-x^2} + \frac{dy}{dx}\times\frac{(2-x^2)\times 1 - x(-2x)}{(2-x^2)^2}$$
$$\Rightarrow \frac{d^3y}{dx^3} = \frac{d^2y}{dx^2}\times\frac{1}{4}x\left(\frac{4}{2-x^2}\right) + \frac{dy}{dx}\times\frac{2-x^2+2x^2}{(2-x^2)^2}$$
$$\Rightarrow \frac{d^3y}{dx^3} = \frac{d^2y}{dx^2}\times\frac{1}{4}x\left[\frac{2}{(2-x^2)^{\frac{1}{2}}}\right]^2 + \frac{dy}{dx}\times\frac{2+x^2}{(2-x^2)^2}$$
$$\Rightarrow \frac{d^3y}{dx^3} = \frac{d^2y}{dx^2}\times\frac{1}{4}x\left(\frac{dy}{dx}\right)^2 + \frac{dy}{dx}\times(2+x^2)\times\frac{1}{16}\times\frac{16}{(2-x^2)^2}$$
$$\Rightarrow \frac{d^3y}{dx^3} = \frac{d^2y}{dx^2}\times\frac{1}{4}x\left(\frac{dy}{dx}\right)^2 + \frac{1}{16}(2+x^2)\frac{dy}{dx}\left[\frac{2}{(2-x^2)^{\frac{1}{2}}}\right]^4$$
$$\Rightarrow \frac{d^3y}{dx^3} = \frac{1}{4}x\frac{d^2y}{dx^2}\left(\frac{dy}{dx}\right)^2 + \frac{1}{16}(2+x^2)\frac{dy}{dx}\left(\frac{dy}{dx}\right)^4$$
$$\Rightarrow 16\frac{d^3y}{dx^3} = 4x\frac{d^2y}{dx^2}\left(\frac{dy}{dx}\right)^2 + (2+x^2)\left(\frac{dy}{dx}\right)^5$$

AS REQUIRED



ALTERNATIVE APPROACH FOR PART (C)

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{2x}{(2-x^2)^{\frac{3}{2}}}$$
$$\Rightarrow \frac{d^3y}{dx^3} = \frac{(2-x^2)^{\frac{3}{2}}\times 2 - 2x\times\frac{3}{2}(2-x^2)^{\frac{1}{2}}(-2x)}{\left[(2-x^2)^{\frac{3}{2}}\right]^2} = \frac{2(2-x^2)^{\frac{3}{2}} + 6x^2(2-x^2)^{\frac{1}{2}}}{(2-x^2)^3}$$
$$\Rightarrow \frac{d^3y}{dx^3} = \frac{2(2-x^2)^{\frac{1}{2}}\left[(2-x^2)+3x^2\right]}{(2-x^2)^3}$$
$$\Rightarrow \frac{d^3y}{dx^3} = \frac{2(2+2x^2)}{(2-x^2)^{\frac{5}{2}}} = \frac{4(x^2+1)}{(2-x^2)^{\frac{5}{2}}}$$
$$\therefore 16\frac{d^3y}{dx^3} = 64(x^2+1)\times\frac{1}{(2-x^2)^{\frac{5}{2}}}$$

NOW BY VERIFICATION OF THE R.H.S NOTING THAT $16\frac{d^3y}{dx^3} = \frac{64(x^2+1)}{(2-x^2)^{\frac{5}{2}}}$

$$\Rightarrow 16\frac{d^3y}{dx^3} = 4x\frac{d^2y}{dx^2}\left(\frac{dy}{dx}\right)^2 + (2+x^2)\left(\frac{dy}{dx}\right)^5$$
$$\Rightarrow 16\frac{d^3y}{dx^3} = 4x\left[\frac{2x}{(2-x^2)^{\frac{3}{2}}}\right]\left[\frac{2}{(2-x^2)^{\frac{1}{2}}}\right]^2 + (2+x^2)\left[\frac{2}{(2-x^2)^{\frac{1}{2}}}\right]^5$$
$$\Rightarrow 16\frac{d^3y}{dx^3} = \frac{32x^2}{(2-x^2)^{\frac{5}{2}}} + (2+x^2)\frac{32}{(2-x^2)^{\frac{5}{2}}}$$
$$\Rightarrow 16\frac{d^3y}{dx^3} = \frac{32x^2}{(2-x^2)^{\frac{5}{2}}} + \frac{64}{(2-x^2)^{\frac{5}{2}}} + \frac{32x^2}{(2-x^2)^{\frac{5}{2}}}$$
$$\Rightarrow 16\frac{d^3y}{dx^3} = \frac{64x^2}{(2-x^2)^{\frac{5}{2}}} + \frac{64}{(2-x^2)^{\frac{5}{2}}}$$
$$\Rightarrow 16\frac{d^3y}{dx^3} = \frac{64(x^2+1)}{(2-x^2)^{\frac{5}{2}}}$$

AND THE RESULT IS VERIFIED

Question 213 (*****)

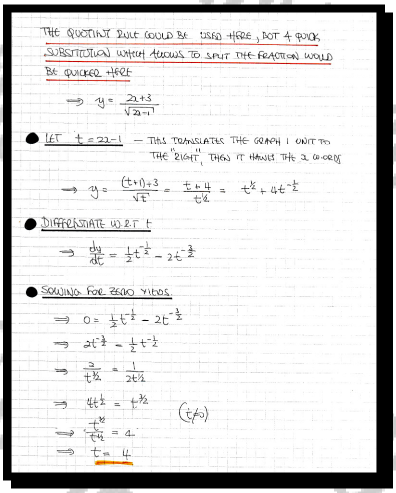
A curve C has equation

$$y = \frac{2x+3}{\sqrt{2x-1}}, \ x \in \mathbb{R}, \ x > \frac{1}{2}.$$

Find the coordinates of the stationary point of C, further determining the nature of this point.

You may not use the product rule, the quotient rule or logarithmic differentiation in this question.

$\boxed{}$, $\boxed{\text{min } \left(\frac{5}{2}, 4\right)}$



THE QUOTIENT RULE COULD BE USED HERE, BUT A QUICK SUBSTITUTION WHICH ALLOWS TO SPLIT THE FRACTION WOULD BE QUICKER HERE

$$\Longrightarrow y = \frac{2x+3}{\sqrt{2x-1}}$$

- LET $t = 2x-1$ — THIS TRANSLATES THE GRAPH 1 UNIT TO THE "RIGHT", THEN IT HALVES THE x CO-ORDS

$$\Longrightarrow y = \frac{(t+1)+3}{\sqrt{t}} = \frac{t+4}{t^{\frac{1}{2}}} = t^{\frac{1}{2}} + 4t^{-\frac{1}{2}}$$

- DIFFERENTIATE W.R.T t

$$\Longrightarrow \frac{dy}{dt} = \frac{1}{2}t^{-\frac{1}{2}} - 2t^{-\frac{3}{2}}$$

- SOLVING FOR ZERO YIELDS.

$$\Longrightarrow 0 = \frac{1}{2}t^{-\frac{1}{2}} - 2t^{-\frac{3}{2}}$$

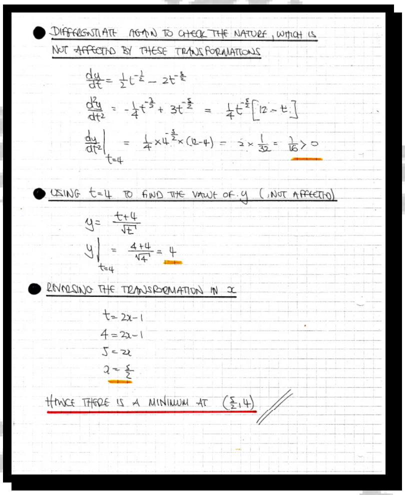
$$\Longrightarrow 2t^{-\frac{3}{2}} = \frac{1}{2}t^{-\frac{1}{2}}$$

$$\Longrightarrow \frac{2}{t^{3/2}} = \frac{1}{2t^{1/2}}$$

$$\Longrightarrow 4t^{\frac{1}{2}} = t^{3/2} \qquad (t \neq 0)$$

$$\Longrightarrow \frac{t^{3/2}}{t^{1/2}} = 4$$

$$\Longrightarrow t = 4$$



- DIFFERENTIATE AGAIN TO CHECK THE NATURE, WHICH IS NOT AFFECTED BY THESE TRANSFORMATIONS

$$\frac{dy}{dt} = \frac{1}{2}t^{-\frac{1}{2}} - 2t^{-\frac{3}{2}}$$

$$\frac{d^2y}{dt^2} = -\frac{1}{4}t^{-\frac{3}{2}} + 3t^{-\frac{5}{2}} = \frac{1}{4}t^{-\frac{5}{2}}\left[12 - t\right]$$

$$\left.\frac{dy}{dt^2}\right|_{t=4} = \frac{1}{4} \times 4^{-\frac{5}{2}} \times (12-4) = 2 \times \frac{1}{32} = \frac{1}{16} > 0$$

- USING $t=4$ TO FIND THE VALUE OF y (NOT AFFECTED)

$$y = \frac{t+4}{\sqrt{t}}$$

$$\left.y\right|_{t=4} = \frac{4+4}{\sqrt{4}} = 4$$

- REVERSING THE TRANSFORMATION IN x

$$t = 2x-1$$

$$4 = 2x-1$$

$$5 = 2x$$

$$x = \frac{5}{2}$$

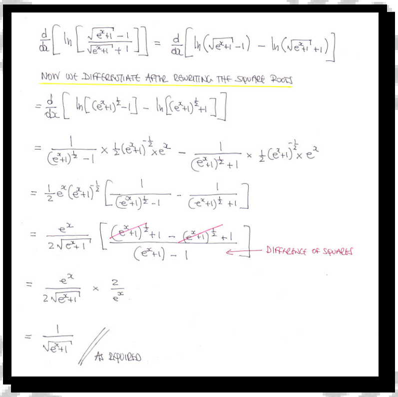
HENCE THERE IS A MINIMUM AT $\left(\frac{5}{2}, 4\right)$ //

Question 214 (*****)

Show with a detailed method that

$$\frac{d}{dx}\left[\ln\left(\frac{\sqrt{e^x+1}-1}{\sqrt{e^x+1}+1}\right)\right]=\frac{1}{\sqrt{e^x+1}}.$$

SP-R, proof



$$\frac{d}{dx}\left[\ln\left[\frac{\sqrt{e^x+1}-1}{\sqrt{e^x+1}+1}\right]\right]=\frac{d}{dx}\left[\ln\left(\sqrt{e^x+1}-1\right)-\ln\left(\sqrt{e^x+1}+1\right)\right]$$

NOW WE DIFFERENTIATE AFTER REWRITING THE SQUARE ROOTS

$$=\frac{d}{dx}\left[\ln\left[(e^x+1)^{\frac{1}{2}}-1\right]-\ln\left[(e^x+1)^{\frac{1}{2}}+1\right]\right]$$

$$=\frac{1}{(e^x+1)^{\frac{1}{2}}-1}\times\frac{1}{2}(e^x+1)^{-\frac{1}{2}}\times e^x-\frac{1}{(e^x+1)^{\frac{1}{2}}+1}\times\frac{1}{2}(e^x+1)^{-\frac{1}{2}}\times e^x$$

$$=\frac{1}{2}e^x(e^x+1)^{-\frac{1}{2}}\left[\frac{1}{(e^x+1)^{\frac{1}{2}}-1}-\frac{1}{(e^x+1)^{\frac{1}{2}}+1}\right]$$

$$=\frac{e^x}{2\sqrt{e^x+1}}\left[\frac{\cancel{(e^x+1)^{\frac{1}{2}}}+1-\cancel{(e^x+1)^{\frac{1}{2}}}+1}{(e^x+1)-1}\right]\longleftarrow\text{DIFFERENCE OF SQUARES}$$

$$=\frac{e^x}{2\sqrt{e^x+1}}\times\frac{2}{e^x}$$

$$=\frac{1}{\sqrt{e^x+1}}\quad //\ \text{AS REQUIRED}$$

Question 215 (*****)

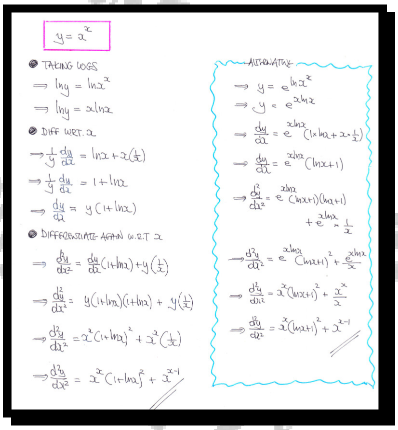
A curve C has equation

$$y = x^x, \quad x \in \mathbb{R}, \quad x > 0.$$

Show that y is a solution of the equation

$$\frac{d^2y}{dx^2} = x^x(1+\ln x)^2 + x^{x-1}.$$

$\boxed{}$, $\boxed{\text{proof}}$



$\boxed{y = x^x}$

- TAKING LOGS

$\Rightarrow \ln y = \ln x^x$

$\Rightarrow \ln y = x\ln x$

- DIFF WRT. x

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \ln x + x\left(\frac{1}{x}\right)$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = 1 + \ln x$

$\Rightarrow \frac{dy}{dx} = y(1+\ln x)$

- DIFFERENTIATE AGAIN W.R.T x

$\Rightarrow \frac{d^2y}{dx^2} = \frac{dy}{dx}(1+\ln x) + y\left(\frac{1}{x}\right)$

$\Rightarrow \frac{d^2y}{dx^2} = y(1+\ln x)(1+\ln x) + y\left(\frac{1}{x}\right)$

$\Rightarrow \frac{d^2y}{dx^2} = x^x(1+\ln x)^2 + x^x\left(\frac{1}{x}\right)$

$\Rightarrow \frac{d^2y}{dx^2} = x^x(1+\ln x)^2 + x^{x-1}$

ALTERNATIVE

$\Rightarrow y = e^{\ln x^x}$

$\Rightarrow y = e^{x\ln x}$

$\Rightarrow \frac{dy}{dx} = e^{x\ln x}\left(1\times\ln x + x\times\frac{1}{x}\right)$

$\Rightarrow \frac{dy}{dx} = e^{x\ln x}(\ln x+1)$

$\Rightarrow \frac{d^2y}{dx^2} = e^{x\ln x}(\ln x+1)(\ln x+1) + e^{x\ln x}\times\frac{1}{x}$

$\Rightarrow \frac{d^2y}{dx^2} = e^{x\ln x}(\ln x+1)^2 + \frac{e^{x\ln x}}{x}$

$\Rightarrow \frac{d^2y}{dx^2} = x^x(\ln x+1)^2 + \frac{x^x}{x}$

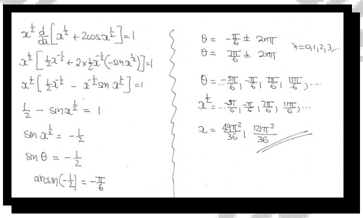
$\Rightarrow \frac{d^2y}{dx^2} = x^x(\ln x+1)^2 + x^{x-1}$

Question 216 (*****)

Find, in terms of π, the solutions of the equation

$$\sqrt{x}\frac{d}{dx}\left(\sqrt{x} + 2\cos\sqrt{x}\right) = 1, \quad 0 \le x < 4\pi^2.$$

$$\boxed{x = \frac{49\pi^2}{36}, \frac{121\pi}{36}}$$



$x^{\frac{1}{2}}\frac{d}{dx}\left[x^{\frac{1}{2}} + 2\cos x^{\frac{1}{2}}\right] = 1$

$x^{\frac{1}{2}}\left[\frac{1}{2}x^{-\frac{1}{2}} + 2\times\frac{1}{2}x^{-\frac{1}{2}}\left(-\sin x^{\frac{1}{2}}\right)\right] = 1$

$x^{\frac{1}{2}}\left[\frac{1}{2}x^{-\frac{1}{2}} - x^{-\frac{1}{2}}\sin x^{\frac{1}{2}}\right] = 1$

$\frac{1}{2} - \sin x^{\frac{1}{2}} = 1$

$\sin x^{\frac{1}{2}} = -\frac{1}{2}$

$\sin\theta = -\frac{1}{2}$

$\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$

$\theta = -\frac{\pi}{6} \pm 2n\pi$

$\theta = \frac{7\pi}{6} \pm 2n\pi$ $\quad n = 0, 1, 2, 3, \ldots$

$\theta = \ldots -\frac{5\pi}{6}, -\frac{\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}, \ldots$

$x^{\frac{1}{2}} = \ldots -\frac{5\pi}{6}, -\frac{\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}, \ldots$

$x = \frac{49\pi^2}{36}, \frac{121\pi^2}{36}$

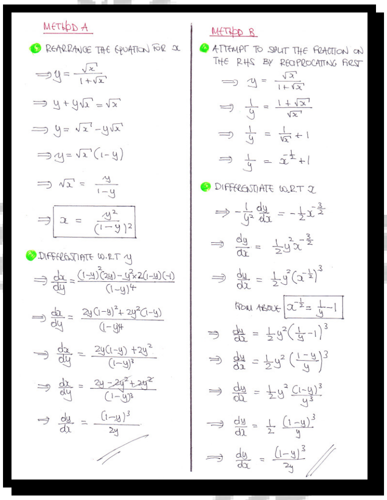
Question 217 **(*****)**

Given that

$$y = \frac{\sqrt{x}}{1+\sqrt{x}},$$

show that $\frac{dy}{dx} = f(y)$, where $f(y)$ is a function to be determined.

$\boxed{\quad}$, $\boxed{f(y) = \frac{(1-y)^3}{2y}}$



METHOD A

● REARRANGE THE EQUATION FOR x

$\Rightarrow y = \frac{\sqrt{x}}{1+\sqrt{x}}$

$\Rightarrow y + y\sqrt{x} = \sqrt{x}$

$\Rightarrow y = \sqrt{x} - y\sqrt{x}$

$\Rightarrow y = \sqrt{x}(1-y)$

$\Rightarrow \sqrt{x} = \frac{y}{1-y}$

$\Rightarrow \boxed{x = \frac{y^2}{(1-y)^2}}$

● DIFFERENTIATE W.R.T y

$\Rightarrow \frac{dx}{dy} = \frac{(1-y)^2(2y) - y^2 \times 2(1-y)(-1)}{(1-y)^4}$

$\Rightarrow \frac{dx}{dy} = \frac{2y(1-y)^2 + 2y^2(1-y)}{(1-y)^4}$

$\Rightarrow \frac{dx}{dy} = \frac{2y(1-y) + 2y^2}{(1-y)^3}$

$\Rightarrow \frac{dx}{dy} = \frac{2y \cancel{-2y^2} \cancel{+2y^2}}{(1-y)^3}$

$\Rightarrow \frac{dy}{dx} = \frac{(1-y)^3}{2y}$ //

METHOD B

● ATTEMPT TO SPLIT THE FRACTION ON THE RHS BY RECIPROCATING FIRST

$\Rightarrow y = \frac{\sqrt{x}}{1+\sqrt{x}}$

$\Rightarrow \frac{1}{y} = \frac{1+\sqrt{x}}{\sqrt{x}}$

$\Rightarrow \frac{1}{y} = \frac{1}{\sqrt{x}} + 1$

$\Rightarrow \frac{1}{y} = x^{-\frac{1}{2}} + 1$

● DIFFERENTIATE W.R.T x

$\Rightarrow -\frac{1}{y^2}\frac{dy}{dx} = -\frac{1}{2}x^{-\frac{3}{2}}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2}y^2 x^{-\frac{3}{2}}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2}y^2\left(x^{-\frac{1}{2}}\right)^3$

FROM ABOVE $\boxed{x^{-\frac{1}{2}} = \frac{1}{y} - 1}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2}y^2\left(\frac{1}{y} - 1\right)^3$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2}y^2\left(\frac{1-y}{y}\right)^3$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2}y^2\frac{(1-y)^3}{y^3}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\frac{(1-y)^3}{y}$

$\Rightarrow \frac{dy}{dx} = \frac{(1-y)^3}{2y}$ //

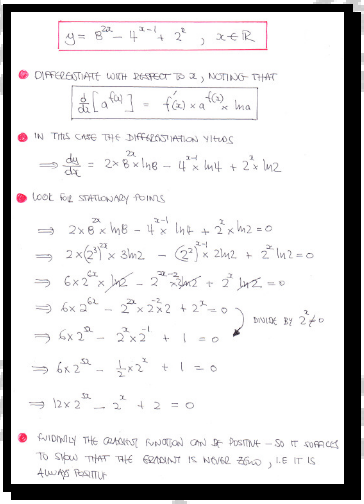
Question 218 (*****)

A curve C has equation

$$y = 8^{2x} - 4^{x-1} + 2^{x}, \ x \in \mathbb{R}$$

Show that C has no stationary points.

[illegible], $\boxed{\text{proof}}$



$\boxed{y = 8^{2x} - 4^{x-1} + 2^{x}, \ x \in \mathbb{R}}$

- DIFFERENTIATE WITH RESPECT TO x, NOTING THAT

$$\boxed{\frac{d}{dx}\left[a^{f(x)}\right] = f'(x) \times a^{f(x)} \times \ln a}$$

- IN THIS CASE THE DIFFERENTIATION YIELDS

$$\Rightarrow \frac{dy}{dx} = 2 \times 8^{2x} \times \ln 8 - 4^{x-1} \times \ln 4 + 2^{x} \times \ln 2$$

- LOOK FOR STATIONARY POINTS

$$\Rightarrow 2 \times 8^{2x} \times \ln 8 - 4^{x-1} \times \ln 4 + 2^{x} \times \ln 2 = 0$$

$$\Rightarrow 2 \times (2^{3})^{2x} \times 3\ln 2 - (2^{2})^{x-1} \times 2\ln 2 + 2^{x} \ln 2 = 0$$

$$\Rightarrow 6 \times 2^{6x} \times \cancel{\ln 2} - 2^{2x-2} \times 2\cancel{\ln 2} + 2^{x} \cancel{\ln 2} = 0$$

$$\Rightarrow 6 \times 2^{6x} - 2^{2x} \times 2^{-2} \times 2 + 2^{x} = 0$$

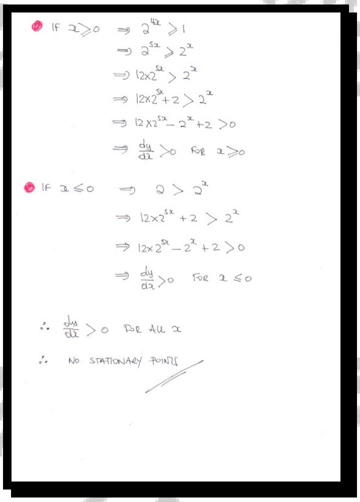
DIVIDE BY $2^{x} \neq 0$

$$\Rightarrow 6 \times 2^{5x} - 2^{x} \times 2^{-1} + 1 = 0$$

$$\Rightarrow 6 \times 2^{5x} - \frac{1}{2} \times 2^{x} + 1 = 0$$

$$\Rightarrow 12 \times 2^{5x} - 2^{x} + 2 = 0$$

- EVIDENTLY THE GRADIENT FUNCTION CAN BE POSITIVE – SO IT SUFFICES TO SHOW THAT THE GRADIENT IS NEVER ZERO, I.E IT IS ALWAYS POSITIVE



- IF $x \geqslant 0$ $\Rightarrow 2^{4x} \geqslant 1$

$$\Rightarrow 2^{5x} \geqslant 2^{x}$$

$$\Rightarrow 12 \times 2^{5x} > 2^{x}$$

$$\Rightarrow 12 \times 2^{5x} + 2 > 2^{x}$$

$$\Rightarrow 12 \times 2^{5x} - 2^{x} + 2 > 0$$

$$\Rightarrow \frac{dy}{dx} > 0 \text{ FOR } x \geqslant 0$$

- IF $x \leqslant 0$ $\Rightarrow 2 > 2^{x}$

$$\Rightarrow 12 \times 2^{5x} + 2 > 2^{x}$$

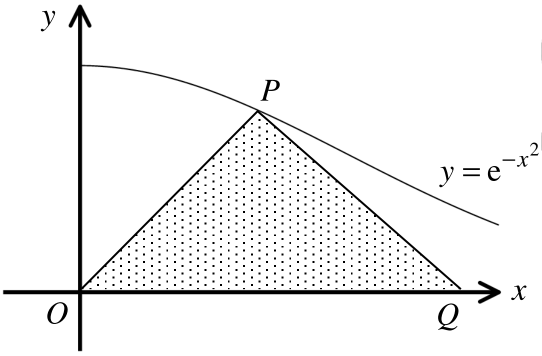
$$\Rightarrow 12 \times 2^{5x} - 2^{x} + 2 > 0$$

$$\Rightarrow \frac{dy}{dx} > 0 \text{ FOR } x \leqslant 0$$

$\therefore \frac{dy}{dx} > 0$ FOR ALL x

$\therefore$ NO STATIONARY POINTS

Question 219 (*****)



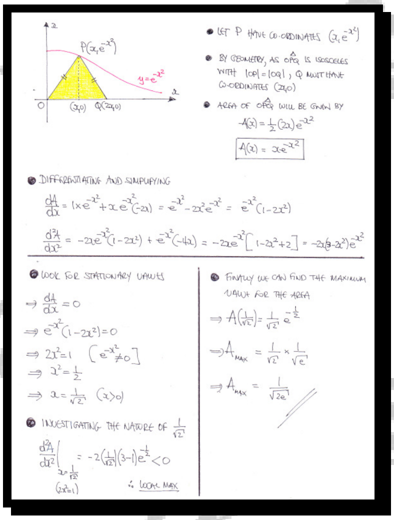
The figure above shows the graph of the curve with equation

$$y = \mathrm{e}^{-x^2}, \ x \geq 0 .$$

The point P lies on the curve and the point Q lies on the positive x axis so that $|OP| = |PQ|$ where O is the origin.

Show with full justification that the largest area of the triangle OPQ is $\dfrac{1}{\sqrt{2\mathrm{e}}}$.

, proof



- LET P HAVE CO-ORDINATES (x, e^{-x^2})
- BY GEOMETRY, AS $\overset{\triangle}{OPQ}$ IS ISOSCELES WITH $|OP| = |OQ|$, Q MUST HAVE CO-ORDINATES $(2x, 0)$
- AREA OF $\overset{\triangle}{OPQ}$ WILL BE GIVEN BY

$$A(x) = \frac{1}{2}(2x)e^{-x^2}$$

$$\boxed{A(x) = xe^{-x^2}}$$

- DIFFERENTIATING AND SIMPLIFYING

$$\frac{dA}{dx} = 1 \times e^{-x^2} + xe^{-x^2}(-2x) = e^{-x^2} - 2x^2e^{-x^2} = e^{-x^2}(1-2x^2)$$

$$\frac{d^2A}{dx^2} = -2xe^{-x^2}(1-2x^2) + e^{-x^2}(-4x) = -2xe^{-x^2}\left[1-2x^2+2\right] = -2x(3-2x^2)e^{-x^2}$$

- LOOK FOR STATIONARY VALUES

$$\Rightarrow \frac{dA}{dx} = 0$$

$$\Rightarrow e^{-x^2}(1-2x^2) = 0$$

$$\Rightarrow 2x^2 = 1 \quad \left[e^{-x^2} \neq 0\right]$$

$$\Rightarrow x^2 = \frac{1}{2}$$

$$\Rightarrow x = \frac{1}{\sqrt{2}} \quad (x>0)$$

- INVESTIGATING THE NATURE OF $\frac{1}{\sqrt{2}}$

$$\left.\frac{d^2A}{dx^2}\right|_{\substack{x=\frac{1}{\sqrt{2}} \\ (2x^2=1)}} = -2\left(\frac{1}{\sqrt{2}}\right)(3-1)e^{-\frac{1}{2}} < 0$$

$\therefore$ LOCAL MAX

- FINALLY WE CAN FIND THE MAXIMUM VALUE FOR THE AREA

$$\Rightarrow A\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}} e^{-\frac{1}{2}}$$

$$\Rightarrow A_{\text{MAX}} = \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{e}}$$

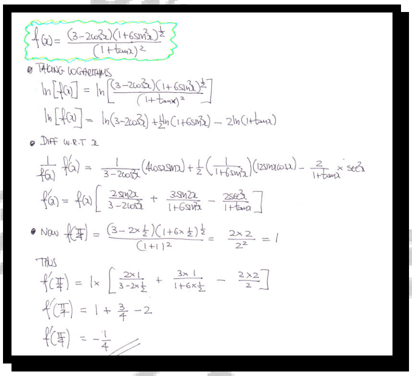
$$\Rightarrow A_{\text{MAX}} = \frac{1}{\sqrt{2e}}$$

Question 220 (*****)

$$f(x) \equiv \frac{\left(3-2\cos^2 x\right)\left(1+6\sin^2 x\right)^{\frac{1}{2}}}{\left(1+\tan x\right)^2}, \quad \tan x \neq -1.$$

Determine the value of $f'\left(\frac{\pi}{4}\right)$.

$\boxed{}$, $\boxed{f'\left(\frac{\pi}{4}\right) = -\frac{1}{4}}$



$$\boxed{f(x) = \frac{(3-2\cos^2 x)(1+6\sin^2 x)^{\frac{1}{2}}}{(1+\tan x)^2}}$$

- TAKING LOGARITHMS

$$\ln[f(x)] = \ln\left[\frac{(3-2\cos^2 x)(1+6\sin^2 x)^{\frac{1}{2}}}{(1+\tan x)^2}\right]$$

$$\ln[f(x)] = \ln(3-2\cos^2 x) + \tfrac{1}{2}\ln(1+6\sin^2 x) - 2\ln(1+\tan x)$$

- DIFF W.R.T x

$$\frac{1}{f(x)}f'(x) = \frac{1}{3-2\cos^2 x}(4\cos x\sin x) + \frac{1}{2}\left(\frac{1}{1+6\sin^2 x}\right)(12\sin x\cos x) - \frac{2}{1+\tan x}\times\sec^2 x$$

$$f'(x) = f(x)\left[\frac{2\sin 2x}{3-2\cos^2 x} + \frac{3\sin 2x}{1+6\sin^2 x} - \frac{2\sec^2 x}{1+\tan x}\right]$$

- NOW $f\left(\frac{\pi}{4}\right) = \frac{(3-2\times\frac{1}{2})(1+6\times\frac{1}{2})^{\frac{1}{2}}}{(1+1)^2} = \frac{2\times 2}{2^2} = 1$

THUS

$$f'\left(\tfrac{\pi}{4}\right) = 1\times\left[\frac{2\times 1}{3-2\times\frac{1}{2}} + \frac{3\times 1}{1+6\times\frac{1}{2}} - \frac{2\times 2}{2}\right]$$

$$f'\left(\tfrac{\pi}{4}\right) = 1 + \tfrac{3}{4} - 2$$

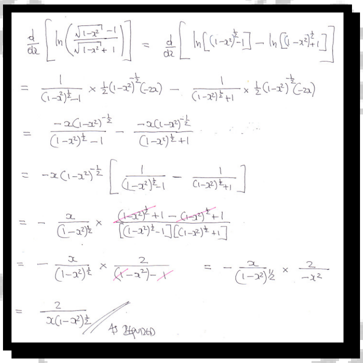
$$f'\left(\tfrac{\pi}{4}\right) = -\tfrac{1}{4}$$

Question 221 (*****)

Show clearly that

$$\frac{d}{dx}\left[\ln\left(\frac{\sqrt{1-x^2}-1}{\sqrt{1-x^2}+1}\right)\right] = \frac{2}{x\sqrt{1-x^2}}.$$

$\boxed{\text{proof}}$



$$\frac{d}{dx}\left[\ln\left(\frac{\sqrt{1-x^2}-1}{\sqrt{1-x^2}+1}\right)\right] = \frac{d}{dx}\left[\ln\left[(1-x^2)^{\frac{1}{2}}-1\right] - \ln\left[(1-x^2)^{\frac{1}{2}}+1\right]\right]$$

$$= \frac{1}{(1-x^2)^{\frac{1}{2}}-1}\times\tfrac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x) - \frac{1}{(1-x^2)^{\frac{1}{2}}+1}\times\tfrac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x)$$

$$= \frac{-x(1-x^2)^{-\frac{1}{2}}}{(1-x^2)^{\frac{1}{2}}-1} - \frac{-x(1-x^2)^{-\frac{1}{2}}}{(1-x^2)^{\frac{1}{2}}+1}$$

$$= -x(1-x^2)^{-\frac{1}{2}}\left[\frac{1}{(1-x^2)^{\frac{1}{2}}-1} - \frac{1}{(1-x^2)^{\frac{1}{2}}+1}\right]$$

$$= -\frac{x}{(1-x^2)^{\frac{1}{2}}}\times\frac{\cancel{(1-x^2)^{\frac{1}{2}}}+1-\cancel{(1-x^2)^{\frac{1}{2}}}+1}{\left[(1-x^2)^{\frac{1}{2}}-1\right]\left[(1-x^2)^{\frac{1}{2}}+1\right]}$$

$$= -\frac{x}{(1-x^2)^{\frac{1}{2}}}\times\frac{2}{(\cancel{1}-x^2)-\cancel{1}} \qquad = -\frac{x}{(1-x^2)^{\frac{1}{2}}}\times\frac{2}{-x^2}$$

$$= \frac{2}{x(1-x^2)^{\frac{1}{2}}}$$

AS REQUIRED

Question 222 (*****)

A curve is defined over the largest real domain by the equation

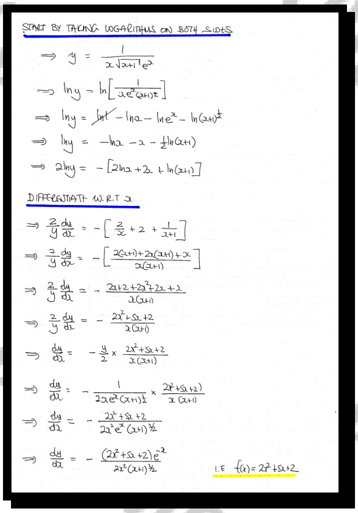
$$y = \frac{1}{x\mathrm{e}^{x}\sqrt{x+1}}$$

Show that

$$\frac{dy}{dx} = \frac{f(x)\ \mathrm{e}^{-x}}{2x^{2}(x+1)^{\frac{3}{2}}},$$

where $f(x)$ is a quadratic expression and hence find, in exact form, the coordinates of any stationary points of the curve.

$\boxed{\quad\quad}$, $\boxed{f(x) = 2x^{2} + 5x + 2}$, $\boxed{\left(-\frac{1}{2}, -2\sqrt{2\mathrm{e}}\right)}$



START BY TAKING LOGARITHMS ON BOTH SIDES

$$\Rightarrow y = \frac{1}{x\sqrt{x+1}\,e^{x}}$$

$$\Rightarrow \ln y = \ln\left[\frac{1}{xe^{x}(x+1)^{\frac{1}{2}}}\right]$$

$$\Rightarrow \ln y = \cancel{\ln 1} - \ln x - \ln e^{x} - \ln(x+1)^{\frac{1}{2}}$$

$$\Rightarrow \ln y = -\ln x - x - \tfrac{1}{2}\ln(x+1)$$

$$\Rightarrow 2\ln y = -\left[2\ln x + 2x + \ln(x+1)\right]$$

DIFFERENTIATE W.R.T x

$$\Rightarrow \frac{2}{y}\frac{dy}{dx} = -\left[\frac{2}{x} + 2 + \frac{1}{x+1}\right]$$

$$\Rightarrow \frac{2}{y}\frac{dy}{dx} = -\left[\frac{2(x+1) + 2x(x+1) + x}{x(x+1)}\right]$$

$$\Rightarrow \frac{2}{y}\frac{dy}{dx} = -\frac{2x+2+2x^{2}+2x+x}{x(x+1)}$$

$$\Rightarrow \frac{2}{y}\frac{dy}{dx} = -\frac{2x^{2}+5x+2}{x(x+1)}$$

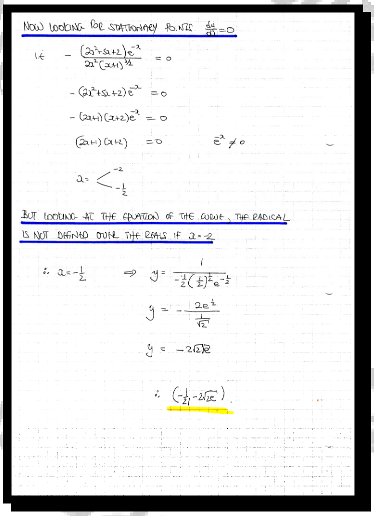
$$\Rightarrow \frac{dy}{dx} = -\frac{y}{2} \times \frac{2x^{2}+5x+2}{x(x+1)}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{2xe^{x}(x+1)^{\frac{1}{2}}} \times \frac{2x^{2}+5x+2)}{x(x+1)}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{2x^{2}+5x+2}{2x^{2}e^{x}(x+1)^{3/2}}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{(2x^{2}+5x+2)e^{-x}}{2x^{2}(x+1)^{3/2}}$$

I.E $f(x) = 2x^{2}+5x+2$



NOW LOOKING FOR STATIONARY POINTS $\frac{dy}{dx}=0$

$$\text{i.e.}\quad -\frac{(2x^{2}+5x+2)e^{-x}}{2x^{2}(x+1)^{3/2}} = 0$$

$$-(2x^{2}+5x+2)e^{-x} = 0$$

$$-(2x+1)(x+2)e^{-x} = 0$$

$$(2x+1)(x+2) = 0 \qquad e^{-x} \neq 0$$

$$x = \begin{cases} -2 \\ -\frac{1}{2} \end{cases}$$

BUT LOOKING AT THE EQUATION OF THE CURVE, THE RADICAL IS NOT DEFINED OVER THE REALS IF $x=-2$

$$\therefore x = -\frac{1}{2} \quad \Rightarrow \quad y = \frac{1}{-\frac{1}{2}\left(\frac{1}{2}\right)^{\frac{1}{2}}e^{-\frac{1}{2}}}$$

$$y = -\frac{2e^{\frac{1}{2}}}{\frac{1}{\sqrt{2}}}$$

$$y = -2\sqrt{2}\sqrt{e}$$

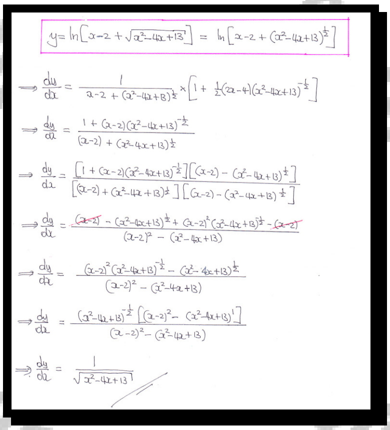
$$\therefore \left(-\tfrac{1}{2}, -2\sqrt{2e}\right)$$

Question 223 **(*****)**

Show clearly that

$$\frac{d}{dx}\left[\ln\left(x-2+\sqrt{x^2-4x+13}\right)\right]=\frac{1}{\sqrt{x^2-4x+13}}.$$

$\boxed{\text{[illegible]}}$, $\boxed{\text{proof}}$



$$\boxed{y=\ln\left[x-2+\sqrt{x^2-4x+13}\right]=\ln\left[x-2+(x^2-4x+13)^{\frac{1}{2}}\right]}$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{x-2+(x^2-4x+13)^{\frac{1}{2}}}\times\left[1+\frac{1}{2}(2x-4)(x^2-4x+13)^{-\frac{1}{2}}\right]$$

$$\Rightarrow \frac{dy}{dx}=\frac{1+(x-2)(x^2-4x+13)^{-\frac{1}{2}}}{(x-2)+(x^2-4x+13)^{\frac{1}{2}}}$$

$$\Rightarrow \frac{dy}{dx}=\frac{\left[1+(x-2)(x^2-4x+13)^{-\frac{1}{2}}\right]\left[(x-2)-(x^2-4x+13)^{\frac{1}{2}}\right]}{\left[(x-2)+(x^2-4x+13)^{\frac{1}{2}}\right]\left[(x-2)-(x^2-4x+13)^{\frac{1}{2}}\right]}$$

$$\Rightarrow \frac{dy}{dx}=\frac{\cancel{(x-2)}-(x^2-4x+13)^{\frac{1}{2}}+(x-2)^2(x^2-4x+13)^{-\frac{1}{2}}-\cancel{(x-2)}}{(x-2)^2-(x^2-4x+13)}$$

$$\Rightarrow \frac{dy}{dx}=\frac{(x-2)^2(x^2-4x+13)^{-\frac{1}{2}}-(x^2-4x+13)^{\frac{1}{2}}}{(x-2)^2-(x^2-4x+13)}$$

$$\Rightarrow \frac{dy}{dx}=\frac{(x^2-4x+13)^{-\frac{1}{2}}\left[(x-2)^2-(x^2-4x+13)^1\right]}{(x-2)^2-(x^2-4x+13)}$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{\sqrt{x^2-4x+13}}$$

Question 224 (*****)

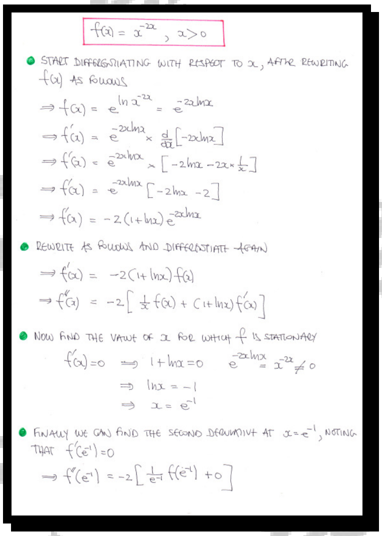
The function f is defined as

$$f(x) = x^{-2x}, \quad x \in \mathbb{R}, \quad x > 0 .$$

Show that the value of $f''(x)$ at the stationary point of the function is

$$-2\,\mathrm{e}^{\frac{\mathrm{e}+2}{\mathrm{e}}} .$$

$\boxed{}$, $\boxed{\text{proof}}$



$\boxed{f(x) = x^{-2x}, \quad x>0}$

- START DIFFERENTIATING WITH RESPECT TO x, AFTER REWRITING $f(x)$ AS FOLLOWS

$\Rightarrow f(x) = e^{\ln x^{-2x}} = e^{-2x\ln x}$

$\Rightarrow f'(x) = e^{-2x\ln x} \times \frac{d}{dx}\left[-2x\ln x\right]$

$\Rightarrow f'(x) = e^{-2x\ln x} \times \left[-2\ln x - 2x \times \frac{1}{x}\right]$

$\Rightarrow f'(x) = e^{-2x\ln x}\left[-2\ln x - 2\right]$

$\Rightarrow f'(x) = -2(1+\ln x)e^{-2x\ln x}$

- REWRITE AS FOLLOWS AND DIFFERENTIATE AGAIN

$\Rightarrow f'(x) = -2(1+\ln x)f(x)$

$\Rightarrow f''(x) = -2\left[\frac{1}{x}f(x) + (1+\ln x)f'(x)\right]$

- NOW FIND THE VALUE OF x FOR WHICH f IS STATIONARY

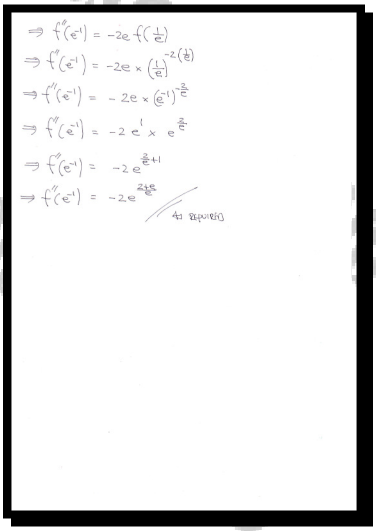
$f'(x)=0 \Rightarrow 1+\ln x = 0 \qquad e^{-2x\ln x} = x^{-2x} \neq 0$

$\Rightarrow \ln x = -1$

$\Rightarrow x = e^{-1}$

- FINALLY WE CAN FIND THE SECOND DERIVATIVE AT $x=e^{-1}$, NOTING THAT $f'(e^{-1})=0$

$\Rightarrow f''(e^{-1}) = -2\left[\frac{1}{e^{-1}}f(e^{-1}) + 0\right]$



$\Rightarrow f''(e^{-1}) = -2e\,f\left(\frac{1}{e}\right)$

$\Rightarrow f''(e^{-1}) = -2e \times \left(\frac{1}{e}\right)^{-2\left(\frac{1}{e}\right)}$

$\Rightarrow f''(e^{-1}) = -2e \times \left(e^{-1}\right)^{-\frac{2}{e}}$

$\Rightarrow f''(e^{-1}) = -2e^{1} \times e^{\frac{2}{e}}$

$\Rightarrow f''(e^{-1}) = -2e^{\frac{2}{e}+1}$

$\Rightarrow f''(e^{-1}) = -2e^{\frac{2+e}{e}}$ // AS REQUIRED

Question 225 (*****)

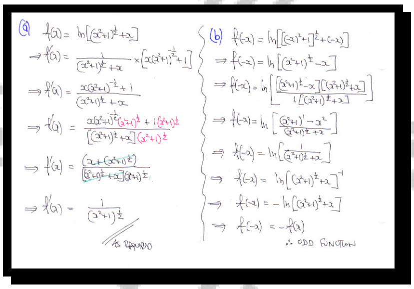
$$f(x)=\ln\left[\left(x^2+1\right)^{\frac{1}{2}}+x\right],\ x\in\mathbb{R}.$$

Show clearly that …

a) … $f'(x)=\dfrac{1}{\sqrt{x^2+1}}$.

b) … $f(x)$ is an odd function.

proof



(a) $f(x)=\ln\left[(x^2+1)^{\frac{1}{2}}+x\right]$

$\Rightarrow f'(x)=\dfrac{1}{(x^2+1)^{\frac{1}{2}}+x}\times\left[x(x^2+1)^{-\frac{1}{2}}+1\right]$

$\Rightarrow f'(x)=\dfrac{x(x^2+1)^{-\frac{1}{2}}+1}{(x^2+1)^{\frac{1}{2}}+x}$

$\Rightarrow f'(x)=\dfrac{x(x^2+1)^{-\frac{1}{2}}(x^2+1)^{\frac{1}{2}}+1(x^2+1)^{\frac{1}{2}}}{\left[(x^2+1)^{\frac{1}{2}}+x\right](x^2+1)^{\frac{1}{2}}}$

$\Rightarrow f'(x)=\dfrac{\cancel{\left(x+(x^2+1)^{\frac{1}{2}}\right)}}{\cancel{\left[(x^2+1)^{\frac{1}{2}}+x\right]}(x^2+1)^{\frac{1}{2}}}$

$\Rightarrow f'(x)=\dfrac{1}{(x^2+1)^{\frac{1}{2}}}$

// AS REQUIRED

(b) $f(-x)=\ln\left[\left[(-x)^2+1\right]^{\frac{1}{2}}+(-x)\right]$

$\Rightarrow f(-x)=\ln\left[(x^2+1)^{\frac{1}{2}}-x\right]$

$\Rightarrow f(-x)=\ln\left[\dfrac{\left[(x^2+1)^{\frac{1}{2}}-x\right]\left[(x^2+1)^{\frac{1}{2}}+x\right]}{1\left[(x^2+1)^{\frac{1}{2}}+x\right]}\right]$

$\Rightarrow f(-x)=\ln\left[\dfrac{(x^2+1)^1-x^2}{(x^2+1)^{\frac{1}{2}}+x}\right]$

$\Rightarrow f(-x)=\ln\left[\dfrac{1}{(x^2+1)^{\frac{1}{2}}+x}\right]$

$\Rightarrow f(-x)=\ln\left[(x^2+1)^{\frac{1}{2}}+x\right]^{-1}$

$\Rightarrow f(-x)=-\ln\left[(x^2+1)^{\frac{1}{2}}+x\right]$

$\Rightarrow f(-x)=-f(x)$

∴ ODD FUNCTION

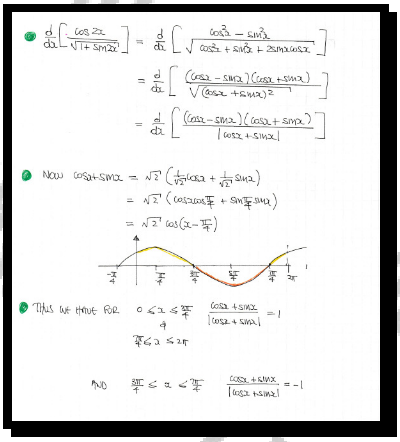
Question 226 (*****)

Show clearly that

$$\frac{d}{dx}\left(\frac{\cos 2x}{\sqrt{1+\sin 2x}}\right)=\begin{cases}-\sin x-\cos x & 0\le x\le \alpha\pi\\ \sin x+\cos x & \alpha\pi\le x\le \beta\pi\\ -\sin x-\cos x & \beta\pi\le x\le 2\pi\end{cases}$$

where α and β are constants to be found.

$\boxed{}$, $\boxed{\text{proof}}$



● $\frac{d}{dx}\left[\frac{\cos 2x}{\sqrt{1+\sin 2x}}\right]=\frac{d}{dx}\left[\frac{\cos^2 x-\sin^2 x}{\sqrt{\cos^2 x+\sin^2 x+2\sin x\cos x}}\right]$

$=\frac{d}{dx}\left[\frac{(\cos x-\sin x)(\cos x+\sin x)}{\sqrt{(\cos x+\sin x)^2}}\right]$

$=\frac{d}{dx}\left[\frac{(\cos x-\sin x)(\cos x+\sin x)}{|\cos x+\sin x|}\right]$

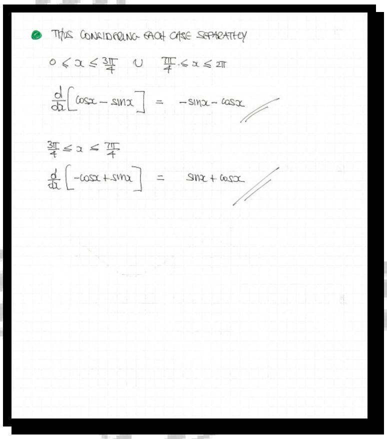
● NOW $\cos x+\sin x=\sqrt{2}\left(\frac{1}{\sqrt{2}}\cos x+\frac{1}{\sqrt{2}}\sin x\right)$

$=\sqrt{2}\left(\cos x\cos\frac{\pi}{4}+\sin\frac{\pi}{4}\sin x\right)$

$=\sqrt{2}\cos\left(x-\frac{\pi}{4}\right)$

● THUS WE HAVE FOR $0\le x\le\frac{3\pi}{4}$ & $\frac{7\pi}{4}\le x\le 2\pi$ $\quad\frac{\cos x+\sin x}{|\cos x+\sin x|}=1$

AND $\frac{3\pi}{4}\le x\le\frac{7\pi}{4}$ $\quad\frac{\cos x+\sin x}{|\cos x+\sin x|}=-1$



● THUS CONSIDERING EACH CASE SEPARATELY

$0\le x\le\frac{3\pi}{4}\ \cup\ \frac{7\pi}{4}\le x\le 2\pi$

$\frac{d}{dx}\left[\cos x-\sin x\right]=-\sin x-\cos x$ //

$\frac{3\pi}{4}\le x\le\frac{7\pi}{4}$

$\frac{d}{dx}\left[-\cos x+\sin x\right]=\sin x+\cos x$ //

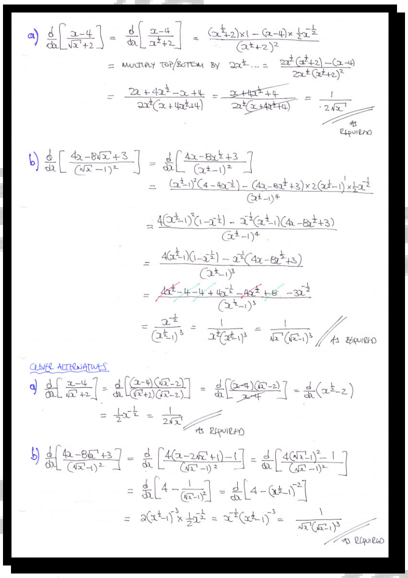
Question 227 (*****)

Show clearly that …

i. … $\frac{d}{dx}\left(\frac{x-4}{\sqrt{x}+2}\right)=\frac{1}{2\sqrt{x}}$

ii. … $\frac{d}{dx}\left(\frac{4x-8\sqrt{x}+3}{\left(\sqrt{x}-1\right)^2}\right)=\frac{1}{\sqrt{x}\left(\sqrt{x}-1\right)^3}$.

$\boxed{}$, $\boxed{\text{proof}}$



a) $\frac{d}{dx}\left[\frac{x-4}{\sqrt{x}+2}\right]=\frac{d}{dx}\left[\frac{x-4}{x^{\frac{1}{2}}+2}\right]=\frac{(x^{\frac{1}{2}}+2)\times1-(x-4)\times\frac{1}{2}x^{-\frac{1}{2}}}{(x^{\frac{1}{2}}+2)^2}$

$=$ MULTIPLY TOP/BOTTOM BY $2x^{\frac{1}{2}}\ldots=\frac{2x^{\frac{1}{2}}(x^{\frac{1}{2}}+2)-(x-4)}{2x^{\frac{1}{2}}(x^{\frac{1}{2}}+2)^2}$

$=\frac{2x+4x^{\frac{1}{2}}-x+4}{2x^{\frac{1}{2}}(x+4x^{\frac{1}{2}}+4)}=\frac{\cancel{x+4x^{\frac{1}{2}}+4}}{2x^{\frac{1}{2}}\cancel{(x+4x^{\frac{1}{2}}+4)}}=\frac{1}{2\sqrt{x}}$ // AS REQUIRED

b) $\frac{d}{dx}\left[\frac{4x-8\sqrt{x}+3}{(\sqrt{x}-1)^2}\right]=\frac{d}{dx}\left[\frac{4x-8x^{\frac{1}{2}}+3}{(x^{\frac{1}{2}}-1)^2}\right]$

$=\frac{(x^{\frac{1}{2}}-1)^2(4-4x^{-\frac{1}{2}})-(4x-8x^{\frac{1}{2}}+3)\times2(x^{\frac{1}{2}}-1)^1\times\frac{1}{2}x^{-\frac{1}{2}}}{(x^{\frac{1}{2}}-1)^4}$

$=\frac{4(x^{\frac{1}{2}}-1)^2(1-x^{-\frac{1}{2}})-x^{-\frac{1}{2}}(x^{\frac{1}{2}}-1)(4x-8x^{\frac{1}{2}}+3)}{(x^{\frac{1}{2}}-1)^4}$

$=\frac{4(x^{\frac{1}{2}}-1)(1-x^{-\frac{1}{2}})-x^{-\frac{1}{2}}(4x-8x^{\frac{1}{2}}+3)}{(x^{\frac{1}{2}}-1)^3}$

$=\frac{\cancel{4x^{\frac{1}{2}}}\cancel{-4}\cancel{-4}+4x^{-\frac{1}{2}}\cancel{-4x^{\frac{1}{2}}}\cancel{+8}-3x^{-\frac{1}{2}}}{(x^{\frac{1}{2}}-1)^3}$

$=\frac{x^{-\frac{1}{2}}}{(x^{\frac{1}{2}}-1)^3}=\frac{1}{x^{\frac{1}{2}}(x^{\frac{1}{2}}-1)^3}=\frac{1}{\sqrt{x}(\sqrt{x}-1)^3}$ // AS REQUIRED

CLEVER ALTERNATIVES

a) $\frac{d}{dx}\left[\frac{x-4}{\sqrt{x}+2}\right]=\frac{d}{dx}\left[\frac{(x-4)(\sqrt{x}-2)}{(\sqrt{x}+2)(\sqrt{x}-2)}\right]=\frac{d}{dx}\left[\frac{\cancel{(x-4)}(\sqrt{x}-2)}{\cancel{x-4}}\right]=\frac{d}{dx}(x^{\frac{1}{2}}-2)$

$=\frac{1}{2}x^{-\frac{1}{2}}=\frac{1}{2\sqrt{x}}$ // AS REQUIRED

b) $\frac{d}{dx}\left[\frac{4x-8\sqrt{x}+3}{(\sqrt{x}-1)^2}\right]=\frac{d}{dx}\left[\frac{4(x-2\sqrt{x}+1)-1}{(\sqrt{x}-1)^2}\right]=\frac{d}{dx}\left[\frac{4(\sqrt{x}-1)^2-1}{(\sqrt{x}-1)^2}\right]$

$=\frac{d}{dx}\left[4-\frac{1}{(\sqrt{x}-1)^2}\right]=\frac{d}{dx}\left[4-(x^{\frac{1}{2}}-1)^{-2}\right]$

$=2(x^{\frac{1}{2}}-1)^{-3}\times\frac{1}{2}x^{-\frac{1}{2}}=x^{-\frac{1}{2}}(x^{\frac{1}{2}}-1)^{-3}=\frac{1}{\sqrt{x}(\sqrt{x}-1)^3}$ // AS REQUIRED

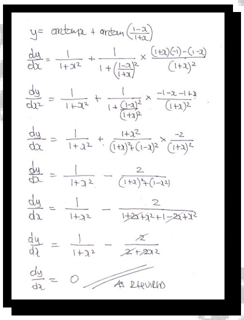
Question 228 **(*****)**

$$y = \arctan x + \arctan\left(\frac{1-x}{1+x}\right),\ \ x \in \mathbb{R}.$$

Without simplifying the above expression, use differentiation to show that

$$\frac{dy}{dx} = 0 \text{, for all values of } x .$$

$\boxed{\text{proof}}$



$$y = \arctan x + \arctan\left(\frac{1-x}{1+x}\right)$$

$$\frac{dy}{dx} = \frac{1}{1+x^2} + \frac{1}{1+\left(\frac{1-x}{1+x}\right)^2} \times \frac{(1+x)(-1)-(1-x)}{(1+x)^2}$$

$$\frac{dy}{dx^2} = \frac{1}{1+x^2} + \frac{1}{1+\frac{(1-x)^2}{(1+x)^2}} \times \frac{-1-x-1+x}{(1+x)^2}$$

$$\frac{dy}{dx} = \frac{1}{1+x^2} + \frac{1+x^2}{(1+x)^2+(1-x)^2} \times \frac{-2}{(1+x)^2}$$

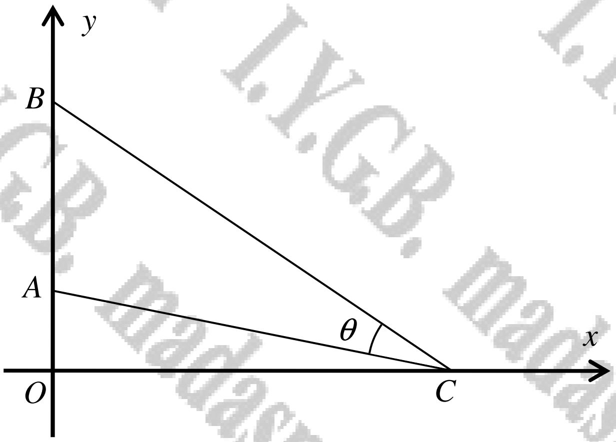
$$\frac{dy}{dx} = \frac{1}{1+x^2} - \frac{2}{(1+x)^2+(1-x^2)}$$

$$\frac{dy}{dx} = \frac{1}{1+x^2} - \frac{2}{1+\cancel{2x}+x^2+1-\cancel{2x}+x^2}$$

$$\frac{dy}{dx} = \frac{1}{1+x^2} - \frac{\cancel{2}}{\cancel{2}+2x^2}$$

$$\frac{dy}{dx} = 0$$ AS REQUIRED

Question 229 (*****)

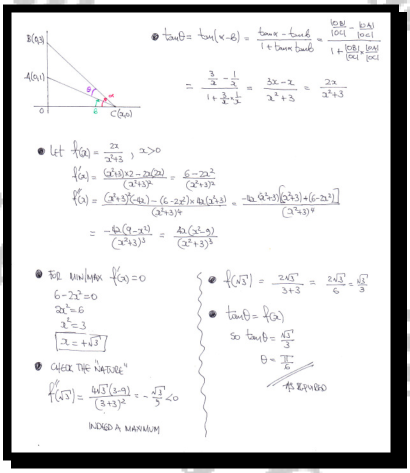


The figure above shows the triangle ABC, where $\measuredangle ACB = \theta$.

The points A and B have respective coordinates $(0,1)$ and $(0,3)$, while the **variable** point $C(x,0)$ lies on the positive x axis.

Show that as C varies, the maximum value of θ is $\frac{\pi}{6}$.

☐, proof



- $\tan\theta = \tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta} = \frac{\frac{|OB|}{|OC|} - \frac{|OA|}{|OC|}}{1+\frac{|OB|}{|OC|}\times\frac{|OA|}{|OC|}}$

$$= \frac{\frac{3}{x}-\frac{1}{x}}{1+\frac{3}{x}\times\frac{1}{x}} = \frac{3x-x}{x^2+3} = \frac{2x}{x^2+3}$$

- Let $f(x) = \frac{2x}{x^2+3}$, $x>0$

$$f'(x) = \frac{(x^2+3)\times2 - 2x(2x)}{(x^2+3)^2} = \frac{6-2x^2}{(x^2+3)^2}$$

$$f''(x) = \frac{(x^2+3)^2(-4x) - (6-2x^2)\times4x(x^2+3)}{(x^2+3)^4} = \frac{-4x(x^2+3)\left[(x^2+3)+(6-2x^2)\right]}{(x^2+3)^4}$$

$$= \frac{-4x(9-x^2)}{(x^2+3)^3} = \frac{4x(x^2-9)}{(x^2+3)^3}$$

- FOR MIN/MAX $f'(x)=0$

$6-2x^2=0$

$2x^2=6$

$x^2=3$

$\boxed{x=+\sqrt{3}}$

- CHECK THE "NATURE"

$f''(\sqrt{3}) = \frac{4\sqrt{3}(3-9)}{(3+3)^2} = -\frac{\sqrt{3}}{9} < 0$

INDEED A MAXIMUM

- $f(\sqrt{3}) = \frac{2\sqrt{3}}{3+3} = \frac{2\sqrt{3}}{6} = \frac{\sqrt{3}}{3}$

- $\tan\theta = f(x)$

so $\tan\theta = \frac{\sqrt{3}}{3}$

$\theta = \frac{\pi}{6}$

AS REQUIRED

Question 230 (*****)

A curve C has equation

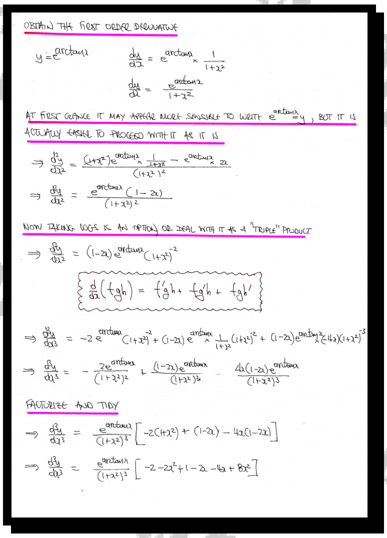
$$y = e^{\arctan x}, \ x \in \mathbb{R}.$$

a) Show, with detailed workings, that

$$\frac{d^3y}{dx^3} = \frac{\left(6x^2 - 6x - 1\right)e^{\arctan x}}{\left(1+x^2\right)^3}.$$

b) Deduce that C has a point of inflection, stating its coordinates.

$\boxed{}$, $\boxed{\left(\frac{1}{2}, e^{\arctan\frac{1}{2}}\right)}$



OBTAIN THE FIRST ORDER DERIVATIVE

$y = e^{\arctan x}$ $\qquad \frac{dy}{dx} = e^{\arctan x} \times \frac{1}{1+x^2}$

$$\frac{dy}{dx} = \frac{e^{\arctan x}}{1+x^2}$$

AT FIRST GLANCE IT MAY APPEAR MORE SENSIBLE TO WRITE $e^{\arctan x} = y$, BUT IT IS ACTUALLY EASIER TO PROCEED WITH IT AS IT IS

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{\cancel{(1+x^2)}e^{\arctan x} \times \frac{1}{\cancel{1+x^2}} - e^{\arctan x} \times 2x}{(1+x^2)^2}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{e^{\arctan x}(1-2x)}{(1+x^2)^2}$$

NOW TAKING LOGS IS AN OPTION OR DEAL WITH IT AS A "TRIPLE" PRODUCT

$$\Rightarrow \frac{d^2y}{dx^2} = (1-2x)e^{\arctan x}(1+x^2)^{-2}$$

$$\boxed{\frac{d}{dx}(fgh) = f'gh + fg'h + fgh'}$$

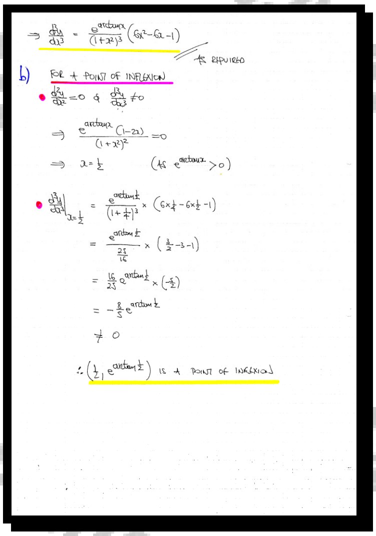
$$\Rightarrow \frac{d^3y}{dx^3} = -2e^{\arctan x}(1+x^2)^{-2} + (1-2x)e^{\arctan x} \times \frac{1}{1+x^2}(1+x^2)^{-2} + (1-2x)e^{\arctan x} \times (-4x)(1+x^2)^{-3}$$

$$\Rightarrow \frac{d^3y}{dx^3} = -\frac{2e^{\arctan x}}{(1+x^2)^2} + \frac{(1-2x)e^{\arctan x}}{(1+x^2)^3} - \frac{4x(1-2x)e^{\arctan x}}{(1+x^2)^3}$$

FACTORIZE AND TIDY

$$\Rightarrow \frac{d^3y}{dx^3} = \frac{e^{\arctan x}}{(1+x^2)^3}\left[-2(1+x^2) + (1-2x) - 4x(1-2x)\right]$$

$$\Rightarrow \frac{d^3y}{dx^3} = \frac{e^{\arctan x}}{(1+x^2)^3}\left[-2 - 2x^2 + 1 - 2x - 4x + 8x^2\right]$$



$$\Rightarrow \frac{d^3y}{dx^3} = \frac{e^{\arctan x}}{(1+x^2)^3}(6x^2 - 6x - 1)$$

AS REQUIRED

b) FOR A POINT OF INFLEXION

- $\frac{d^2y}{dx^2} = 0$ & $\frac{d^3y}{dx^3} \neq 0$

$$\Rightarrow \frac{e^{\arctan x}(1-2x)}{(1+x^2)^2} = 0$$

$$\Rightarrow x = \frac{1}{2} \qquad (\text{AS } e^{\arctan x} > 0)$$

- $\left.\frac{d^3y}{dx^3}\right|_{x=\frac{1}{2}} = \frac{e^{\arctan\frac{1}{2}}}{(1+\frac{1}{4})^3} \times \left(6 \times \frac{1}{4} - 6 \times \frac{1}{2} - 1\right)$

$$= \frac{e^{\arctan\frac{1}{2}}}{\frac{25}{16}} \times \left(\frac{3}{2} - 3 - 1\right)$$

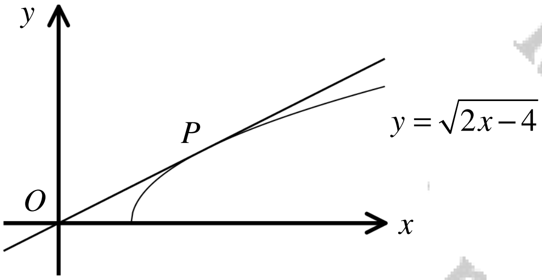
$$= \frac{16}{25}e^{\arctan\frac{1}{2}} \times \left(-\frac{5}{2}\right)$$

$$= -\frac{8}{5}e^{\arctan\frac{1}{2}}$$

$$\neq 0$$

$\therefore \left(\frac{1}{2}, e^{\arctan\frac{1}{2}}\right)$ IS A POINT OF INFLEXION

Question 231 (*****)



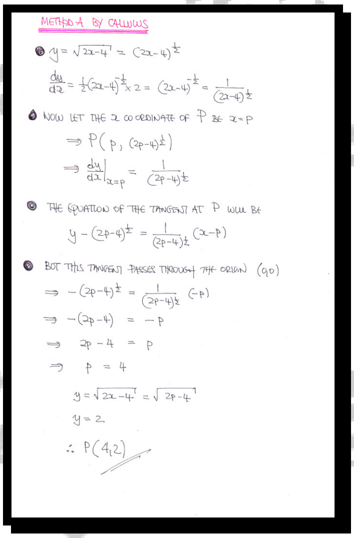
The figure above shows the graph of the curve C with equation

$$y=\sqrt{2x-4}\,,\ x\geq 2\,.$$

The point P lies on C, so that the tangent to the curve at the point P passes through the origin O.

Use a calculus method to find the coordinates of P.

$\boxed{}$, $\boxed{P(4,2)}$



METHOD A BY CALCULUS

- $y=\sqrt{2x-4}=(2x-4)^{\frac{1}{2}}$

  $\frac{dy}{dx}=\frac{1}{2}(2x-4)^{-\frac{1}{2}}\times 2=(2x-4)^{-\frac{1}{2}}=\frac{1}{(2x-4)^{\frac{1}{2}}}$

- NOW LET THE x COORDINATE OF P BE $x=p$

  $\Rightarrow P\left(p, (2p-4)^{\frac{1}{2}}\right)$

  $\Rightarrow \left.\frac{dy}{dx}\right|_{x=p}=\frac{1}{(2p-4)^{\frac{1}{2}}}$

- THE EQUATION OF THE TANGENT AT P WILL BE

  $y-(2p-4)^{\frac{1}{2}}=\frac{1}{(2p-4)^{\frac{1}{2}}}(x-p)$

- BUT THIS TANGENT PASSES THROUGH THE ORIGIN (0,0)

  $\Rightarrow -(2p-4)^{\frac{1}{2}}=\frac{1}{(2p-4)^{\frac{1}{2}}}(-p)$

  $\Rightarrow -(2p-4)=-p$

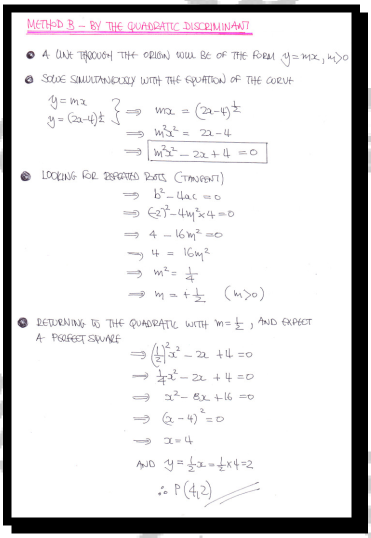
  $\Rightarrow 2p-4=p$

  $\Rightarrow p=4$

  $y=\sqrt{2x-4}=\sqrt{2p-4}$

  $y=2$

  $\therefore P(4,2)$



METHOD B – BY THE QUADRATIC DISCRIMINANT

- A LINE THROUGH THE ORIGIN WILL BE OF THE FORM $y=mx$, $m>0$
- SOLVE SIMULTANEOUSLY WITH THE EQUATION OF THE CURVE

  $\left.\begin{array}{l} y=mx \\ y=(2x-4)^{\frac{1}{2}} \end{array}\right\} \Rightarrow mx=(2x-4)^{\frac{1}{2}}$

  $\Rightarrow m^2x^2=2x-4$

  $\Rightarrow \boxed{m^2x^2-2x+4=0}$

- LOOKING FOR REPEATED ROOTS (TANGENT)

  $\Rightarrow b^2-4ac=0$

  $\Rightarrow (-2)^2-4m^2\times 4=0$

  $\Rightarrow 4-16m^2=0$

  $\Rightarrow 4=16m^2$

  $\Rightarrow m^2=\frac{1}{4}$

  $\Rightarrow m=+\frac{1}{2}\quad (m>0)$

- RETURNING TO THE QUADRATIC WITH $m=\frac{1}{2}$, AND EXPECT A PERFECT SQUARE

  $\Rightarrow \left(\frac{1}{2}\right)^2x^2-2x+4=0$

  $\Rightarrow \frac{1}{4}x^2-2x+4=0$

  $\Rightarrow x^2-8x+16=0$

  $\Rightarrow (x-4)^2=0$

  $\Rightarrow x=4$

  AND $y=\frac{1}{2}x=\frac{1}{2}\times 4=2$

  $\therefore P(4,2)$

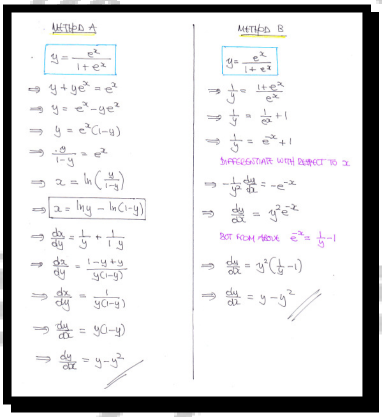
Question 232 (*****)

Given that

$$y = \frac{e^x}{1+e^x},$$

show that $\frac{dy}{dx} = f(y)$, where $f(y)$ is a simplified function to be determined.

$\boxed{}$, $\boxed{f(y) = y - y^2}$



METHOD A

$\boxed{y = \frac{e^x}{1+e^x}}$

$\Rightarrow y + ye^x = e^x$

$\Rightarrow y = e^x - ye^x$

$\Rightarrow y = e^x(1-y)$

$\Rightarrow \frac{y}{1-y} = e^x$

$\Rightarrow x = \ln\left(\frac{y}{1-y}\right)$

$\Rightarrow \boxed{x = \ln y - \ln(1-y)}$

$\Rightarrow \frac{dx}{dy} = \frac{1}{y} + \frac{1}{1-y}$

$\Rightarrow \frac{dx}{dy} = \frac{1-y+y}{y(1-y)}$

$\Rightarrow \frac{dx}{dy} = \frac{1}{y(1-y)}$

$\Rightarrow \frac{dy}{dx} = y(1-y)$

$\Rightarrow \frac{dy}{dx} = y - y^2$

METHOD B

$\boxed{y = \frac{e^x}{1+e^x}}$

$\Rightarrow \frac{1}{y} = \frac{1+e^x}{e^x}$

$\Rightarrow \frac{1}{y} = \frac{1}{e^x} + 1$

$\Rightarrow \frac{1}{y} = e^{-x} + 1$

DIFFERENTIATE WITH RESPECT TO x

$\Rightarrow -\frac{1}{y^2}\frac{dy}{dx} = -e^{-x}$

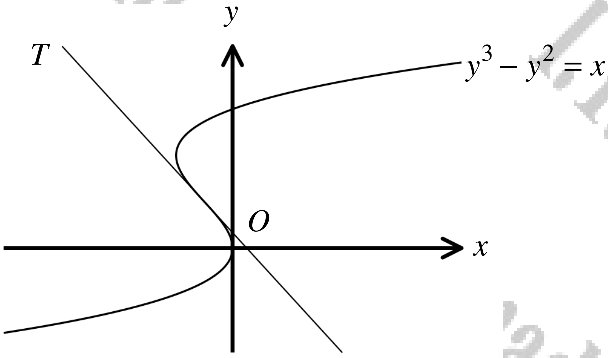
$\Rightarrow \frac{dy}{dx} = y^2 e^{-x}$

BUT FROM ABOVE $e^{-x} = \frac{1}{y} - 1$

$\Rightarrow \frac{dy}{dx} = y^2\left(\frac{1}{y} - 1\right)$

$\Rightarrow \frac{dy}{dx} = y - y^2$

Question 233 (*****)



The figure above shows the graph of a curve with equation

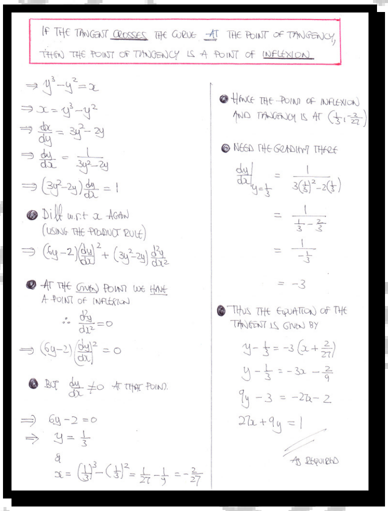
$$y^3 - y^2 = x, \ x \in \mathbb{R}, \ y \in \mathbb{R}.$$

There exists a tangent to the curve T, so that this tangent **crosses** the curve at the point of tangency.

Show that an equation of T is

$$27x + 9y = 1.$$

☐, proof



IF THE TANGENT CROSSES THE CURVE AT THE POINT OF TANGENCY, THEN THE POINT OF TANGENCY IS A POINT OF INFLEXION

$\Rightarrow y^3 - y^2 = x$

$\Rightarrow x = y^3 - y^2$

$\Rightarrow \frac{dx}{dy} = 3y^2 - 2y$

$\Rightarrow \frac{dy}{dx} = \frac{1}{3y^2 - 2y}$

$\Rightarrow (3y^2 - 2y)\frac{dy}{dx} = 1$

- Diff w.r.t x AGAIN (USING THE PRODUCT RULE)

$\Rightarrow (6y - 2)\left(\frac{dy}{dx}\right)^2 + (3y^2 - 2y)\frac{d^2y}{dx^2}$

- AT THE GIVEN POINT WE HAVE A POINT OF INFLEXION

$\therefore \frac{d^2y}{dx^2} = 0$

$\Rightarrow (6y - 2)\left(\frac{dy}{dx}\right)^2 = 0$

- BUT $\frac{dy}{dx} \neq 0$ AT THAT POINT.

$\Rightarrow 6y - 2 = 0$

$\Rightarrow y = \frac{1}{3}$

&

$x = \left(\frac{1}{3}\right)^3 - \left(\frac{1}{3}\right)^2 = \frac{1}{27} - \frac{1}{9} = -\frac{2}{27}$

- HENCE THE POINT OF INFLEXION AND TANGENCY IS AT $\left(\frac{1}{3}, -\frac{2}{27}\right)$

- NEED THE GRADIENT THERE

$\left.\frac{dy}{dx}\right|_{y=\frac{1}{3}} = \frac{1}{3\left(\frac{1}{3}\right)^2 - 2\left(\frac{1}{3}\right)}$

$= \frac{1}{\frac{1}{3} - \frac{2}{3}}$

$= \frac{1}{-\frac{1}{3}}$

$= -3$

- THUS THE EQUATION OF THE TANGENT IS GIVEN BY

$y - \frac{1}{3} = -3\left(x + \frac{2}{27}\right)$

$y - \frac{1}{3} = -3x - \frac{2}{9}$

$9y - 3 = -27x - 2$

$27x + 9y = 1$

AS REQUIRED

Question 234 **(*****)**

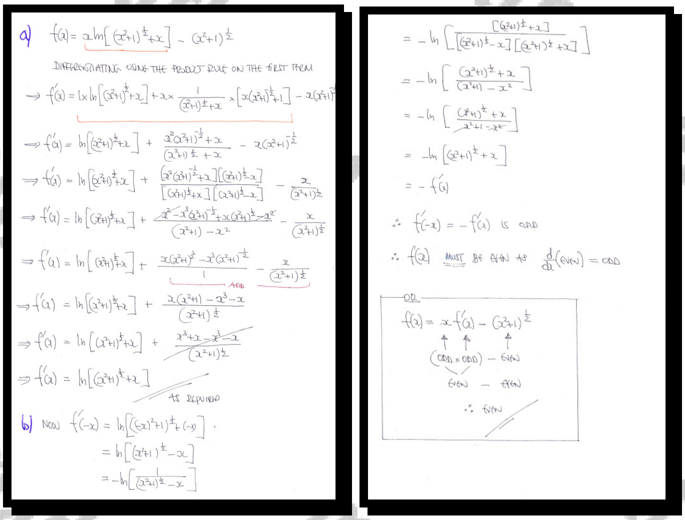
$$f(x) = x\ln\left[\left(x^2+1\right)^{\frac{1}{2}} + x\right] - \left(x^2+1\right)^{\frac{1}{2}},\ x \in \mathbb{R}.$$

Show clearly that …

a) … $f'(x) = \ln\left[\left(x^2+1\right)^{\frac{1}{2}} + x\right]$.

b) … $f(x)$ is an even function.

$\boxed{\text{proof}}$



a) $f(x) = \underbrace{x\ln\left[(x^2+1)^{\frac{1}{2}}+x\right]} - (x^2+1)^{\frac{1}{2}}$

DIFFERENTIATING USING THE PRODUCT RULE ON THE FIRST TERM

$\Rightarrow f'(x) = \underbrace{1\times\ln\left[(x^2+1)^{\frac{1}{2}}+x\right] + x\times\frac{1}{(x^2+1)^{\frac{1}{2}}+x}\times\left[x(x^2+1)^{-\frac{1}{2}}+1\right]} - x(x^2+1)^{-\frac{1}{2}}$

$\Rightarrow f'(x) = \ln\left[(x^2+1)^{\frac{1}{2}}+x\right] + \frac{x^2(x^2+1)^{-\frac{1}{2}}+x}{(x^2+1)^{\frac{1}{2}}+x} - x(x^2+1)^{-\frac{1}{2}}$

$\Rightarrow f'(x) = \ln\left[(x^2+1)^{\frac{1}{2}}+x\right] + \frac{\left[x^2(x^2+1)^{-\frac{1}{2}}+x\right]\left[(x^2+1)^{\frac{1}{2}}-x\right]}{\left[(x^2+1)^{\frac{1}{2}}+x\right]\left[(x^2+1)^{\frac{1}{2}}-x\right]} - \frac{x}{(x^2+1)^{\frac{1}{2}}}$

$\Rightarrow f'(x) = \ln\left[(x^2+1)^{\frac{1}{2}}+x\right] + \frac{\cancel{x^2} - x^3(x^2+1)^{-\frac{1}{2}} + x(x^2+1)^{\frac{1}{2}} \cancel{- x^2}}{(x^2+1)-x^2} - \frac{x}{(x^2+1)^{\frac{1}{2}}}$

$\Rightarrow f'(x) = \ln\left[(x^2+1)^{\frac{1}{2}}+x\right] + \underbrace{\frac{x(x^2+1)^{\frac{1}{2}} - x^3(x^2+1)^{-\frac{1}{2}}}{1}}_{\text{ADD}} - \underbrace{\frac{x}{(x^2+1)^{\frac{1}{2}}}}$

$\Rightarrow f'(x) = \ln\left[(x^2+1)^{\frac{1}{2}}+x\right] + \frac{x(x^2+1) - x^3 - x}{(x^2+1)^{\frac{1}{2}}}$

$\Rightarrow f'(x) = \ln\left[(x^2+1)^{\frac{1}{2}}+x\right] + \cancel{\frac{x^3+x-x^3-x}{(x^2+1)^{\frac{1}{2}}}}$

$\Rightarrow f'(x) = \ln\left[(x^2+1)^{\frac{1}{2}}+x\right]$

AS REQUIRED

b) NOW $f'(-x) = \ln\left[((-x)^2+1)^{\frac{1}{2}}+(-x)\right]$.

$= \ln\left[(x^2+1)^{\frac{1}{2}}-x\right]$

$= -\ln\left[\frac{1}{(x^2+1)^{\frac{1}{2}}-x}\right]$

$= -\ln\left[\frac{\left[(x^2+1)^{\frac{1}{2}}+x\right]}{\left[(x^2+1)^{\frac{1}{2}}-x\right]\left[(x^2+1)^{\frac{1}{2}}+x\right]}\right]$

$= -\ln\left[\frac{(x^2+1)^{\frac{1}{2}}+x}{(x^2+1)-x^2}\right]$

$= -\ln\left[\frac{(x^2+1)^{\frac{1}{2}}+x}{\cancel{x^2+1-x^2}}\right]$

$= -\ln\left[(x^2+1)^{\frac{1}{2}}+x\right]$

$= -f'(x)$

$\therefore$ $f'(-x) = -f'(x)$ IS ODD

$\therefore$ $f(x)$ MUST BE EVEN AS $\frac{d}{dx}(\text{EVEN}) = \text{ODD}$

OR

$f(x) = x f'(x) - (x^2+1)^{\frac{1}{2}}$

(ODD × ODD) − EVEN

EVEN − EVEN

$\therefore$ EVEN

Question 235 (*****)

A curve is defined in its largest real domain by the equation

$$y = \arccos\left[\frac{a\cos x + b}{a + b\cos x}\right],$$

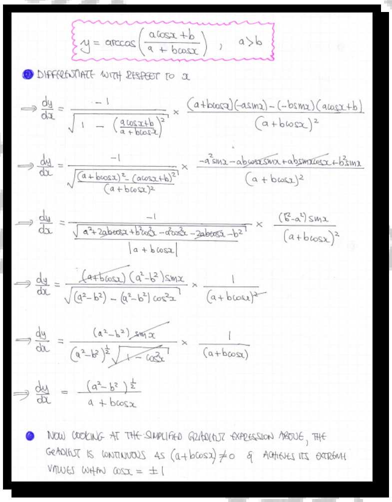
where a and b are constants such $a > b > 0$.

Show that y increases with x at a rate which lies between

$$\sqrt{\frac{a-b}{a+b}} \quad \text{and} \quad \sqrt{\frac{a+b}{a-b}}.$$

You may assume that $\frac{d}{dx}[\arccos x] = -\frac{1}{\sqrt{1-x^2}}$.

$\boxed{}$, $\boxed{\text{proof}}$



$$\boxed{y = \arccos\left(\frac{a\cos x + b}{a + b\cos x}\right), \quad a > b}$$

● DIFFERENTIATE WITH RESPECT TO x

$$\Longrightarrow \frac{dy}{dx} = \frac{-1}{\sqrt{1 - \left(\frac{a\cos x + b}{a + b\cos x}\right)^2}} \times \frac{(a+b\cos x)(-a\sin x) - (-b\sin x)(a\cos x + b)}{(a+b\cos x)^2}$$

$$\Longrightarrow \frac{dy}{dx} = \frac{-1}{\sqrt{\frac{(a+b\cos x)^2 - (a\cos x + b)^2}{(a+b\cos x)^2}}} \times \frac{-a^2\sin x - \cancel{ab\cos x\sin x} + \cancel{ab\sin x\cos x} + b^2\sin x}{(a+b\cos x)^2}$$

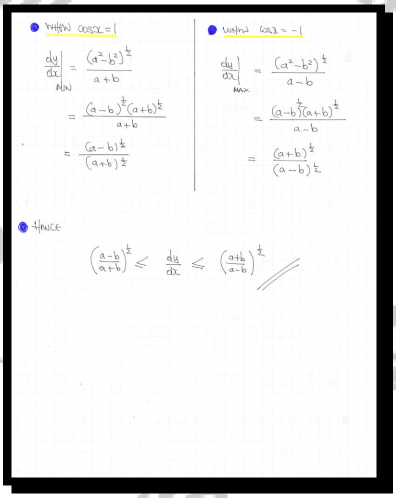
$$\Longrightarrow \frac{dy}{dx} = \frac{-1}{\frac{\sqrt{a^2 + \cancel{2ab\cos x} + b^2\cos^2 x - a^2\cos^2 x - \cancel{2ab\cos x} - b^2}}{|a+b\cos x|}} \times \frac{(b^2-a^2)\sin x}{(a+b\cos x)^2}$$

$$\Longrightarrow \frac{dy}{dx} = \frac{\cancel{(a+b\cos x)}(a^2-b^2)\sin x}{\sqrt{(a^2-b^2) - (a^2-b^2)\cos^2 x}} \times \frac{1}{(a+b\cos x)^{\cancel{2}}}$$

$$\Longrightarrow \frac{dy}{dx} = \frac{(a^2-b^2)\cancel{\sin x}}{(a^2-b^2)^{\frac{1}{2}}\cancel{\sqrt{1-\cos^2 x}}} \times \frac{1}{(a+b\cos x)}$$

$$\Longrightarrow \frac{dy}{dx} = \frac{(a^2-b^2)^{\frac{1}{2}}}{a+b\cos x}$$

● NOW LOOKING AT THE SIMPLIFIED GRADIENT EXPRESSION ABOVE, THE GRADIENT IS CONTINUOUS AS $(a+b\cos x) \neq 0$ & ACHIEVES ITS EXTREMA VALUES WHEN $\cos x = \pm 1$



● WHEN $\cos x = 1$

$$\left.\frac{dy}{dx}\right|_{\text{MIN}} = \frac{(a^2-b^2)^{\frac{1}{2}}}{a+b} = \frac{(a-b)^{\frac{1}{2}}(a+b)^{\frac{1}{2}}}{a+b} = \frac{(a-b)^{\frac{1}{2}}}{(a+b)^{\frac{1}{2}}}$$

● WHEN $\cos x = -1$

$$\left.\frac{dy}{dx}\right|_{\text{MAX}} = \frac{(a^2-b^2)^{\frac{1}{2}}}{a-b} = \frac{(a-b)^{\frac{1}{2}}(a+b)^{\frac{1}{2}}}{a-b} = \frac{(a+b)^{\frac{1}{2}}}{(a-b)^{\frac{1}{2}}}$$

● HENCE

$$\left(\frac{a-b}{a+b}\right)^{\frac{1}{2}} \leqslant \frac{dy}{dx} \leqslant \left(\frac{a+b}{a-b}\right)^{\frac{1}{2}} \quad //$$

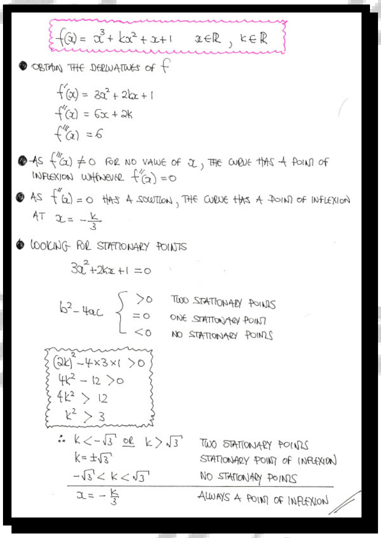
Question 236 **(*****)**

The function f is defined, in terms of the real constant k, by

$$f(x) \equiv x^3 + kx^2 + x + 1, \ x \in \mathbb{R}.$$

Investigate the number of turning points of f for different values of k, distinguishing further which ones are stationary

$\boxed{}$, $\boxed{\text{investigation}}$



$$f(x) = x^3 + kx^2 + x + 1 \quad x \in \mathbb{R}, \ k \in \mathbb{R}$$

- OBTAIN THE DERIVATIVES OF f

$$f'(x) = 3x^2 + 2kx + 1$$
$$f''(x) = 6x + 2k$$
$$f'''(x) = 6$$

- AS $f'''(x) \neq 0$ FOR NO VALUE OF x, THE CURVE HAS A POINT OF INFLEXION WHENEVER $f''(x) = 0$
- AS $f''(x) = 0$ HAS A SOLUTION, THE CURVE HAS A POINT OF INFLEXION AT $x = -\frac{k}{3}$
- LOOKING FOR STATIONARY POINTS

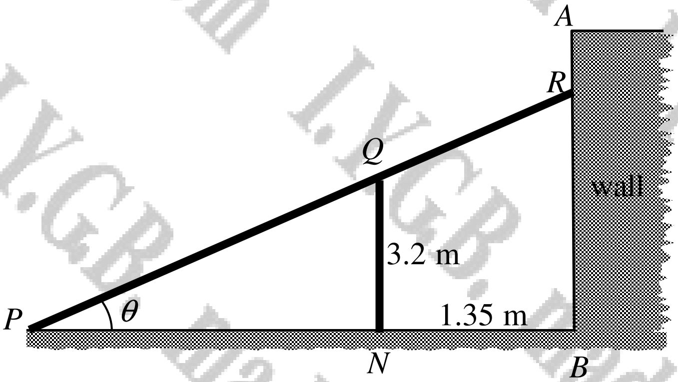
$$3x^2 + 2kx + 1 = 0$$

$$b^2 - 4ac \begin{cases} > 0 & \text{TWO STATIONARY POINTS} \\ = 0 & \text{ONE STATIONARY POINT} \\ < 0 & \text{NO STATIONARY POINTS} \end{cases}$$

$$(2k)^2 - 4 \times 3 \times 1 > 0$$
$$4k^2 - 12 > 0$$
$$4k^2 > 12$$
$$k^2 > 3$$

$\therefore$ $k < -\sqrt{3}$ OR $k > \sqrt{3}$	TWO STATIONARY POINTS
$k = \pm\sqrt{3}$	STATIONARY POINT OF INFLEXION
$-\sqrt{3} < k < \sqrt{3}$	NO STATIONARY POINTS
$x = -\frac{k}{3}$	ALWAYS A POINT OF INFLEXION

Question 237 (*****)



The figure above shows the wall AB of a certain structure, which is supported by a straight rigid beam PR, where P is on level ground and R is at some point on the wall.

In order to increase the rigidity of the support, the beam is rested on a steady pole NQ, of height 3.2 metres.

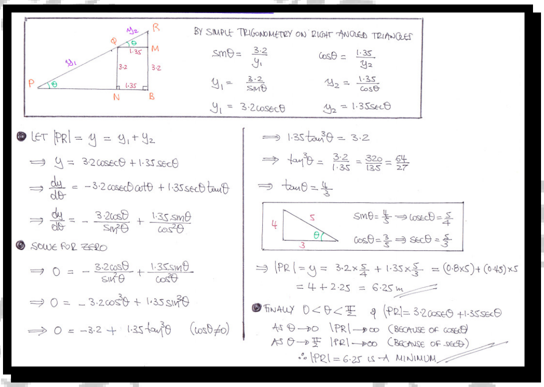
The pole is placed at a distance of 1.35 metres from the bottom of the wall B.

The beam PR is forming an acute angle θ with the horizontal ground PNB.

The angle θ is chosen so that the length of the beam PR, is least.

Determine the least value for the length of the beam PR, assuming that R lies on the wall, fully justifying that this is indeed the minimum value.

$\boxed{}$, $\boxed{6.25}$



BY SIMPLE TRIGONOMETRY ON RIGHT ANGLED TRIANGLES

$\sin\theta = \frac{3.2}{y_1}$ $\qquad$ $\cos\theta = \frac{1.35}{y_2}$

$y_1 = \frac{3.2}{\sin\theta}$ $\qquad$ $y_2 = \frac{1.35}{\cos\theta}$

$y_1 = 3.2\operatorname{cosec}\theta$ $\qquad$ $y_2 = 1.35\sec\theta$

- LET $|PR| = y = y_1 + y_2$

$\Rightarrow y = 3.2\operatorname{cosec}\theta + 1.35\sec\theta$

$\Rightarrow \frac{dy}{d\theta} = -3.2\operatorname{cosec}\theta\cot\theta + 1.35\sec\theta\tan\theta$

$\Rightarrow \frac{dy}{d\theta} = -\frac{3.2\cos\theta}{\sin^2\theta} + \frac{1.35\sin\theta}{\cos^2\theta}$

- SOLVE FOR ZERO

$\Rightarrow 0 = -\frac{3.2\cos\theta}{\sin^2\theta} + \frac{1.35\sin\theta}{\cos^2\theta}$

$\Rightarrow 0 = -3.2\cos^3\theta + 1.35\sin^3\theta$

$\Rightarrow 0 = -3.2 + 1.35\tan^3\theta \quad (\cos\theta \neq 0)$

$\Rightarrow 1.35\tan^3\theta = 3.2$

$\Rightarrow \tan^3\theta = \frac{3.2}{1.35} = \frac{320}{135} = \frac{64}{27}$

$\Rightarrow \tan\theta = \frac{4}{3}$

$\sin\theta = \frac{4}{5} \Rightarrow \operatorname{cosec}\theta = \frac{5}{4}$

$\cos\theta = \frac{3}{5} \Rightarrow \sec\theta = \frac{5}{3}$

$\Rightarrow |PR| = y = 3.2 \times \frac{5}{4} + 1.35 \times \frac{5}{3} = (0.8\times5) + (0.45)\times5$

$= 4 + 2.25 = 6.25\text{ m}$

- FINALLY $0 < \theta < \frac{\pi}{2}$ & $|PR| = 3.2\operatorname{cosec}\theta + 1.35\sec\theta$

AS $\theta \to 0$ $|PR| \to \infty$ (BECAUSE OF $\operatorname{cosec}\theta$)

AS $\theta \to \frac{\pi}{2}$ $|PR| \to \infty$ (BECAUSE OF $\sec\theta$)

$\therefore |PR| = 6.25$ IS A MINIMUM

Question 238 (*****)

The variables x and y are such so that

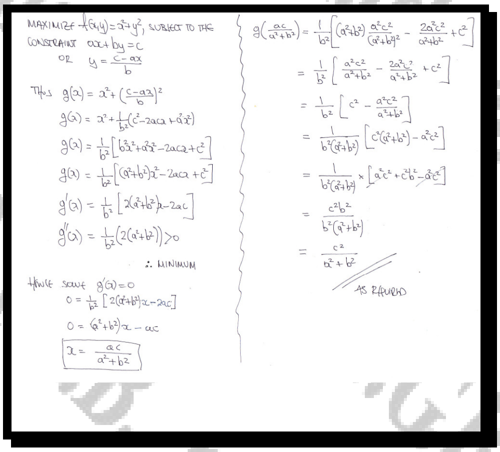
$$ax+by=c,$$

where a, b and c are non zero constants.

Show that the minimum value of x^2+y^2 is

$$\frac{c^2}{a^2+b^2}.$$

$\boxed{}$, $\boxed{\text{proof}}$



MAXIMIZE $f(x,y)=x^2+y^2$, SUBJECT TO THE CONSTRAINT $ax+by=c$

OR $y=\frac{c-ax}{b}$

THUS $g(x)=x^2+\left(\frac{c-ax}{b}\right)^2$

$$g(x)=x^2+\frac{1}{b^2}(c^2-2acx+a^2x^2)$$
$$g(x)=\frac{1}{b^2}\left[b^2x^2+a^2x^2-2acx+c^2\right]$$
$$g(x)=\frac{1}{b^2}\left[(a^2+b^2)x^2-2acx+c^2\right]$$
$$g'(x)=\frac{1}{b^2}\left[2(a^2+b^2)x-2ac\right]$$
$$g''(x)=\frac{1}{b^2}\left(2(a^2+b^2)\right)>0$$

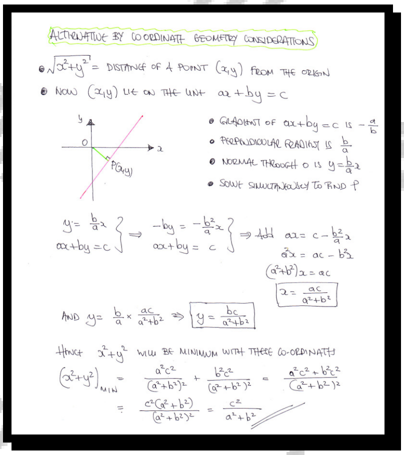
∴ MINIMUM

HENCE SOLVE $g'(x)=0$

$$0=\frac{1}{b^2}\left[2(a^2+b^2)x-2ac\right]$$
$$0=(a^2+b^2)x-ac$$
$$\boxed{x=\frac{ac}{a^2+b^2}}$$

$$g\left(\frac{ac}{a^2+b^2}\right)=\frac{1}{b^2}\left[(a^2+b^2)\frac{a^2c^2}{(a^2+b^2)^2}-\frac{2a^2c^2}{a^2+b^2}+c^2\right]$$
$$=\frac{1}{b^2}\left[\frac{a^2c^2}{a^2+b^2}-\frac{2a^2c^2}{a^2+b^2}+c^2\right]$$
$$=\frac{1}{b^2}\left[c^2-\frac{a^2c^2}{a^2+b^2}\right]$$
$$=\frac{1}{b^2(a^2+b^2)}\left[c^2(a^2+b^2)-a^2c^2\right]$$
$$=\frac{1}{b^2(a^2+b^2)}\times\left[\cancel{a^2c^2}+c^2b^2-\cancel{a^2c^2}\right]$$
$$=\frac{c^2b^2}{b^2(a^2+b^2)}$$
$$=\frac{c^2}{a^2+b^2}$$

AS REQUIRED



ALTERNATIVE BY COORDINATE GEOMETRY CONSIDERATIONS

- $\sqrt{x^2+y^2}$ = DISTANCE OF A POINT (x,y) FROM THE ORIGIN
- NOW (x,y) LIE ON THE LINE $ax+by=c$
- GRADIENT OF $ax+by=c$ IS $-\frac{a}{b}$
- PERPENDICULAR GRADIENT IS $\frac{b}{a}$
- NORMAL THROUGH O IS $y=\frac{b}{a}x$
- SOLVE SIMULTANEOUSLY TO FIND P

$$\left.\begin{array}{l} y=\frac{b}{a}x \\ ax+by=c \end{array}\right\} \Rightarrow \left.\begin{array}{l} -by=-\frac{b^2}{a}x \\ ax+by=c \end{array}\right\} \Rightarrow \text{ADD} \quad ax=c-\frac{b^2}{a}x$$
$$a^2x=ac-b^2x$$
$$(a^2+b^2)x=ac$$
$$\boxed{x=\frac{ac}{a^2+b^2}}$$

AND $y=\frac{b}{a}\times\frac{ac}{a^2+b^2} \Rightarrow \boxed{y=\frac{bc}{a^2+b^2}}$

HENCE x^2+y^2 WILL BE MINIMUM WITH THESE CO-ORDINATES

$$\left(x^2+y^2\right)_{\text{MIN}}=\frac{a^2c^2}{(a^2+b^2)^2}+\frac{b^2c^2}{(a^2+b^2)^2}=\frac{a^2c^2+b^2c^2}{(a^2+b^2)^2}$$
$$=\frac{c^2(a^2+b^2)}{(a^2+b^2)^2}=\frac{c^2}{a^2+b^2}$$

Question 239 (*****)

The point $A(a,0)$ lies on the circle with Cartesian equation

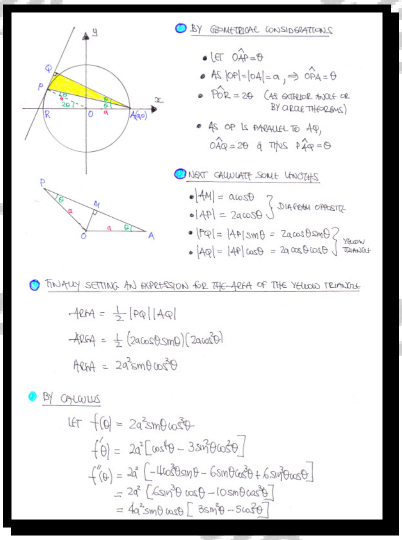
$$x^2+y^2=a^2.$$

The point P is also on the same circle , and the point Q lies on the tangent to the circle through P, so that AQP is a right angle.

Use a calculus method to show that for all possible positions of P, the largest area of the triangle AQP is

$$\frac{3\sqrt{3}}{8}a^2.$$

SP-S, proof



BY GEOMETRICAL CONSIDERATIONS

- LET $O\hat{A}P=\theta$
- AS $|OP|=|OA|=a$, $\Rightarrow O\hat{P}A=\theta$
- $P\hat{O}R=2\theta$ (AS EXTERIOR ANGLE OR BY CIRCLE THEOREMS)
- AS OP IS PARALLEL TO AQ, $O\hat{A}Q=2\theta$ & THUS $P\hat{A}Q=\theta$

NEXT CALCULATE SOME LENGTHS

- $|AM|=a\cos\theta$
- $|AP|=2a\cos\theta$

$\}$ DIAGRAM OPPOSITE

- $|PQ|=|AP|\sin\theta=2a\cos\theta\sin\theta$
- $|AQ|=|AP|\cos\theta=2a\cos\theta\cos\theta$

$\}$ YELLOW TRIANGLE

FINALLY SETTING AN EXPRESSION FOR THE AREA OF THE YELLOW TRIANGLE

$$\text{AREA}=\frac{1}{2}|PQ||AQ|$$

$$\text{AREA}=\frac{1}{2}(2a\cos\theta\sin\theta)(2a\cos^2\theta)$$

$$\text{AREA}=2a^2\sin\theta\cos^3\theta$$

BY CALCULUS

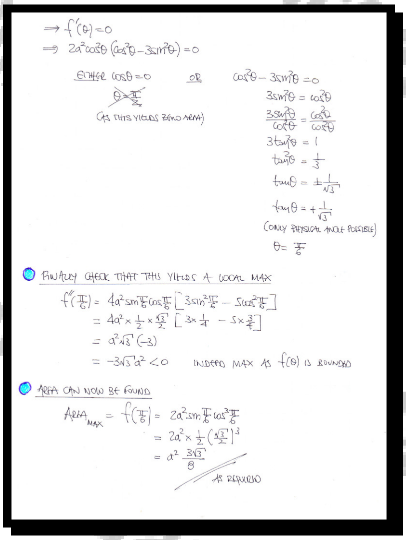
$$\text{LET } f(\theta)=2a^2\sin\theta\cos^3\theta$$

$$f'(\theta)=2a^2\left[\cos^4\theta-3\sin^2\theta\cos^2\theta\right]$$

$$f''(\theta)=2a^2\left[-4\cos^3\theta\sin\theta-6\sin\theta\cos^3\theta+6\sin^3\theta\cos\theta\right]$$

$$=2a^2\left[6\sin^3\theta\cos\theta-10\sin\theta\cos^3\theta\right]$$

$$=4a^2\sin\theta\cos\theta\left[3\sin^2\theta-5\cos^2\theta\right]$$



$$\Rightarrow f'(\theta)=0$$

$$\Rightarrow 2a^2\cos^2\theta(\cos^2\theta-3\sin^2\theta)=0$$

EITHER $\cos\theta=0$ OR $\cos^2\theta-3\sin^2\theta=0$

$\cancel{\theta=\frac{\pi}{2}}$

(AS THIS YIELDS ZERO AREA)

$$3\sin^2\theta=\cos^2\theta$$

$$\frac{3\sin^2\theta}{\cos^2\theta}=\frac{\cos^2\theta}{\cos^2\theta}$$

$$3\tan^2\theta=1$$

$$\tan^2\theta=\frac{1}{3}$$

$$\tan\theta=\pm\frac{1}{\sqrt{3}}$$

$$\tan\theta=+\frac{1}{\sqrt{3}}$$

(ONLY PHYSICAL ANGLE POSSIBLE)

$$\theta=\frac{\pi}{6}$$

FINALLY CHECK THAT THIS YIELDS A LOCAL MAX

$$f''\left(\tfrac{\pi}{6}\right)=4a^2\sin\tfrac{\pi}{6}\cos\tfrac{\pi}{6}\left[3\sin^2\tfrac{\pi}{6}-5\cos^2\tfrac{\pi}{6}\right]$$

$$=4a^2\times\frac{1}{2}\times\frac{\sqrt{3}}{2}\left[3\times\frac{1}{4}-5\times\frac{3}{4}\right]$$

$$=a^2\sqrt{3}(-3)$$

$$=-3\sqrt{3}a^2<0$$

INDEED MAX AS $f(\theta)$ IS BOUNDED

AREA CAN NOW BE FOUND

$$\text{AREA}_{\text{MAX}}=f\left(\tfrac{\pi}{6}\right)=2a^2\sin\tfrac{\pi}{6}\cos^3\tfrac{\pi}{6}$$

$$=2a^2\times\frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)^3$$

$$=a^2\,\frac{3\sqrt{3}}{8}$$

AS REQUIRED

Question 240 (*****)

A curve has equation

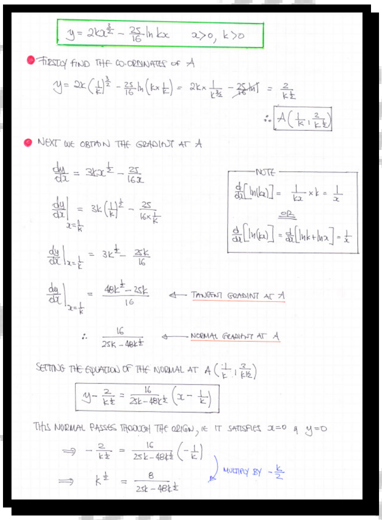
$$y = 2kx^{\frac{3}{2}} - \frac{25}{16}\ln kx\,, \quad x \in \mathbb{R}\,, \quad x > 0\,.$$

where k is a positive constant.

The point A lies on the curve, where $x = \frac{1}{k}$.

Given that the normal to the curve at A passes through the origin O, find an equation of the normal to the curve at A.

[illegible] , $\boxed{y = 4x}$



$\boxed{y = 2kx^{\frac{3}{2}} - \frac{25}{16}\ln kx \qquad x>0,\ k>0}$

● FIRSTLY FIND THE CO-ORDINATES OF A

$y = 2k\left(\frac{1}{k}\right)^{\frac{3}{2}} - \frac{25}{16}\ln\left(k \times \frac{1}{k}\right) = 2k \times \frac{1}{k^{3/2}} - \cancel{\frac{25}{16}\ln 1} = \frac{2}{k^{\frac{1}{2}}}$

$\therefore \boxed{A\left(\frac{1}{k}, \frac{2}{k^{\frac{1}{2}}}\right)}$

● NEXT WE OBTAIN THE GRADIENT AT A

$\frac{dy}{dx} = 3kx^{\frac{1}{2}} - \frac{25}{16x}$

$\left.\frac{dy}{dx}\right|_{x=\frac{1}{k}} = 3k\left(\frac{1}{k}\right)^{\frac{1}{2}} - \frac{25}{16 \times \frac{1}{k}}$

$\left.\frac{dy}{dx}\right|_{x=\frac{1}{k}} = 3k^{\frac{1}{2}} - \frac{25k}{16}$

$\left.\frac{dy}{dx}\right|_{x=\frac{1}{k}} = \frac{48k^{\frac{1}{2}} - 25k}{16}$ ← TANGENT GRADIENT AT A

$\therefore \frac{16}{25k - 48k^{\frac{1}{2}}}$ ← NORMAL GRADIENT AT A

NOTE

$\frac{d}{dx}[\ln(kx)] = \frac{1}{kx} \times k = \frac{1}{x}$

OR

$\frac{d}{dx}[\ln(kx)] = \frac{d}{dx}[\ln k + \ln x] = \frac{1}{x}$

SETTING THE EQUATION OF THE NORMAL AT $A\left(\frac{1}{k}, \frac{2}{k^{1/2}}\right)$

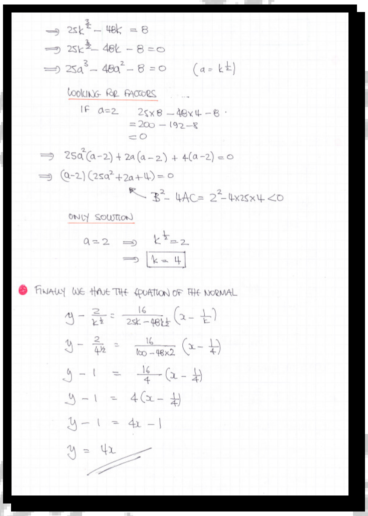
$\boxed{y - \frac{2}{k^{\frac{1}{2}}} = \frac{16}{25k - 48k^{\frac{1}{2}}}\left(x - \frac{1}{k}\right)}$

THIS NORMAL PASSES THROUGH THE ORIGIN, IE IT SATISFIES $x=0$ & $y=0$

$\Rightarrow -\frac{2}{k^{\frac{1}{2}}} = \frac{16}{25k - 48k^{\frac{1}{2}}}\left(-\frac{1}{k}\right)$

MULTIPLY BY $-\frac{k}{2}$

$\Rightarrow k^{\frac{1}{2}} = \frac{8}{25k - 48k^{\frac{1}{2}}}$



$\Rightarrow 25k^{\frac{3}{2}} - 48k = 8$

$\Rightarrow 25k^{\frac{3}{2}} - 48k - 8 = 0$

$\Rightarrow 25a^3 - 48a^2 - 8 = 0 \qquad (a = k^{\frac{1}{2}})$

LOOKING FOR FACTORS ……

IF $a=2$ $\quad 25 \times 8 - 48 \times 4 - 8$

$= 200 - 192 - 8$

$= 0$

$\Rightarrow 25a^2(a-2) + 2a(a-2) + 4(a-2) = 0$

$\Rightarrow (a-2)(25a^2 + 2a + 4) = 0$

$B^2 - 4AC = 2^2 - 4 \times 25 \times 4 < 0$

ONLY SOLUTION

$a = 2 \Rightarrow k^{\frac{1}{2}} = 2$

$\Rightarrow \boxed{k = 4}$

● FINALLY WE HAVE THE EQUATION OF THE NORMAL

$y - \frac{2}{k^{\frac{1}{2}}} = \frac{16}{25k - 48k^{\frac{1}{2}}}\left(x - \frac{1}{k}\right)$

$y - \frac{2}{4^{1/2}} = \frac{16}{100 - 48 \times 2}\left(x - \frac{1}{4}\right)$

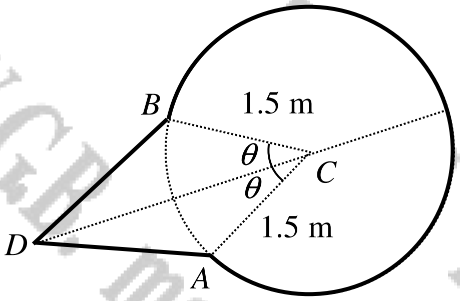
$y - 1 = \frac{16}{4}\left(x - \frac{1}{4}\right)$

$y - 1 = 4\left(x - \frac{1}{4}\right)$

$y - 1 = 4x - 1$

$y = 4x$

Question 241 (*****)



The figure above shows a circle with centre at C and radius 1.5 metres.

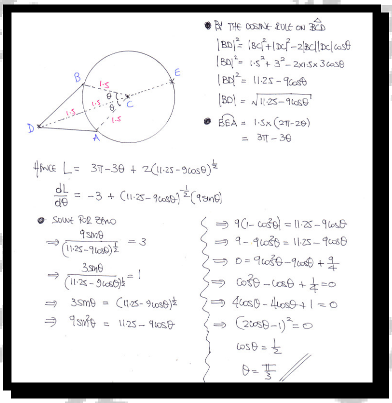
The points A and B lie on the circle so that $\measuredangle BCA = 2\theta$, $0 < \theta < \pi$.

The point D lies outside the circle so that the line segments BD and AD are equal in length and the length of DC is 3 metres. The point E lies on the circle so that DCE is a straight line segment of length 4.5 metres.

The **total length** of the line segment BD, the line segment AD and the circular arc $\overset{\frown}{AEB}$ denoted by L.

Given that θ varies, show that L has a stationary value when $\theta = \dfrac{\pi}{3}$ and determine further the value L and the nature of this stationary value.

$\boxed{}$, $\boxed{\text{inflection at } \left(\dfrac{\pi}{3}, 2\pi + 3\sqrt{3}\right)}$



- BY THE COSINE RULE ON $\hat{BCD}$

$|BD|^2 = |BC|^2 + |DC|^2 - 2|BC||DC|\cos\theta$

$|BD|^2 = 1.5^2 + 3^2 - 2\times1.5\times3\cos\theta$

$|BD|^2 = 11.25 - 9\cos\theta$

$|BD| = \sqrt{11.25 - 9\cos\theta}$

- $\overset{\frown}{BEA} = 1.5\times(2\pi - 2\theta)$
  $= 3\pi - 3\theta$

HENCE $L = 3\pi - 3\theta + 2(11.25 - 9\cos\theta)^{\frac{1}{2}}$

$\frac{dL}{d\theta} = -3 + (11.25 - 9\cos\theta)^{-\frac{1}{2}}(9\sin\theta)$

- SOLVE FOR ZERO

$\Rightarrow \frac{9\sin\theta}{(11.25 - 9\cos\theta)^{\frac{1}{2}}} = 3$

$\Rightarrow \frac{3\sin\theta}{(11.25 - 9\cos\theta)^{\frac{1}{2}}} = 1$

$\Rightarrow 3\sin\theta = (11.25 - 9\cos\theta)^{\frac{1}{2}}$

$\Rightarrow 9\sin^2\theta = 11.25 - 9\cos\theta$

$\Rightarrow 9(1 - \cos^2\theta) = 11.25 - 9\cos\theta$

$\Rightarrow 9 - 9\cos^2\theta = 11.25 - 9\cos\theta$

$\Rightarrow 0 = 9\cos^2\theta - 9\cos\theta + \frac{9}{4}$

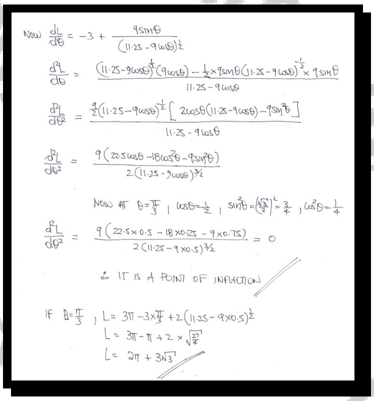
$\Rightarrow \cos^2\theta - \cos\theta + \frac{1}{4} = 0$

$\Rightarrow 4\cos\theta - 4\cos\theta + 1 = 0$

$\Rightarrow (2\cos\theta - 1)^2 = 0$

$\cos\theta = \frac{1}{2}$

$\theta = \frac{\pi}{3}$



NOW $\frac{dL}{d\theta} = -3 + \frac{9\sin\theta}{(11.25 - 9\cos\theta)^{\frac{1}{2}}}$

$\frac{d^2L}{d\theta} = \frac{(11.25 - 9\cos\theta)^{\frac{1}{2}}(9\cos\theta) - \frac{1}{2}\times9\sin\theta(11.25 - 9\cos\theta)^{-\frac{1}{2}}\times9\sin\theta}{11.25 - 9\cos\theta}$

$\frac{d^2L}{d\theta^2} = \frac{\frac{9}{2}(11.25 - 9\cos\theta)^{-\frac{1}{2}}\left[2\cos\theta(11.25 - 9\cos\theta) - 9\sin^2\theta\right]}{11.25 - 9\cos\theta}$

$\frac{d^2L}{d\theta^2} = \frac{9(22.5\cos\theta - 18\cos^2\theta - 9\sin^2\theta)}{2(11.25 - 9\cos\theta)^{3/2}}$

NOW AT $\theta = \frac{\pi}{3}$, $\cos\theta = \frac{1}{2}$, $\sin^2\theta = \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{4}$, $\cos^2\theta = \frac{1}{4}$

$\frac{d^2L}{d\theta^2} = \frac{9(22.5\times0.5 - 18\times0.25 - 9\times0.75)}{2(11.25 - 9\times0.5)^{3/2}} = 0$

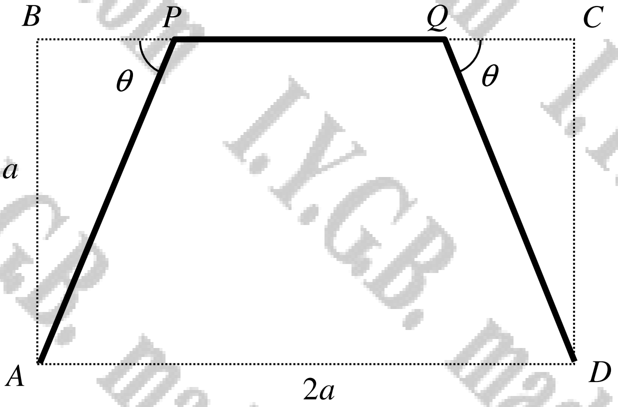
$\therefore$ IT IS A POINT OF INFLECTION

IF $\theta = \frac{\pi}{3}$, $L = 3\pi - 3\times\frac{\pi}{3} + 2(11.25 - 9\times0.5)^{\frac{1}{2}}$

$L = 3\pi - \pi + 2\times\sqrt{\frac{27}{4}}$

$L = 2\pi + 3\sqrt{3}$

Question 242 (*****)

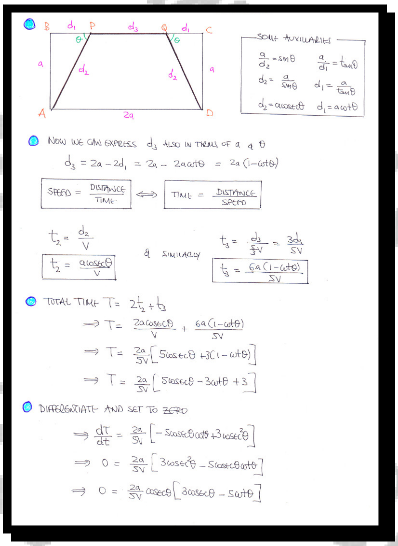


The figure above shows a network $APQD$ inside a rectangle $ABCD$, where $|AB| = a$ and $|BC| = 2a$. The endpoints of the network A and D are fixed. The points P and Q are variable so that they always lie on BC with $|AP| = |QD|$. The angles BPA and CQD are both equal to θ. A particle travels with constant speed v on the sections AP and QD, and with constant speed $\frac{5}{3}v$ on the section PQ.

Let T be the total time for the journey $APQD$.

Given that the positions of the points P and Q can be varied as appropriate, show that the minimum value of T is $\dfrac{14a}{5v}$, fully justifying that this is the minimum value.

$\boxed{}$, $\boxed{\text{proof}}$



SOME AUXILIARIES

$\frac{a}{d_2} = \sin\theta \qquad \frac{a}{d_1} = \tan\theta$

$d_2 = \frac{a}{\sin\theta} \qquad d_1 = \frac{a}{\tan\theta}$

$d_2 = a\operatorname{cosec}\theta \qquad d_1 = a\cot\theta$

NOW WE CAN EXPRESS d_3 ALSO IN TERMS OF a & θ

$d_3 = 2a - 2d_1 = 2a - 2a\cot\theta = 2a(1-\cot\theta)$

$\boxed{\text{SPEED} = \frac{\text{DISTANCE}}{\text{TIME}}} \Longleftrightarrow \boxed{\text{TIME} = \frac{\text{DISTANCE}}{\text{SPEED}}}$

$t_2 = \frac{d_2}{v}$ & SIMILARLY $t_3 = \frac{d_3}{\frac{5}{3}v} = \frac{3d_3}{5v}$

$\boxed{t_2 = \frac{a\operatorname{cosec}\theta}{v}}$ $\boxed{t_3 = \frac{6a(1-\cot\theta)}{5v}}$

TOTAL TIME $T = 2t_2 + t_3$

$\Rightarrow T = \frac{2a\operatorname{cosec}\theta}{v} + \frac{6a(1-\cot\theta)}{5v}$

$\Rightarrow T = \frac{2a}{5v}\left[5\operatorname{cosec}\theta + 3(1-\cot\theta)\right]$

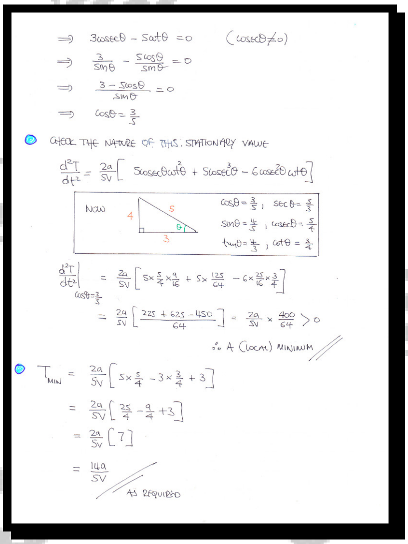
$\Rightarrow T = \frac{2a}{5v}\left[5\operatorname{cosec}\theta - 3\cot\theta + 3\right]$

DIFFERENTIATE AND SET TO ZERO

$\Rightarrow \frac{dT}{dt} = \frac{2a}{5v}\left[-5\operatorname{cosec}\theta\cot\theta + 3\operatorname{cosec}^2\theta\right]$

$\Rightarrow 0 = \frac{2a}{5v}\left[3\operatorname{cosec}^2\theta - 5\operatorname{cosec}\theta\cot\theta\right]$

$\Rightarrow 0 = \frac{2a}{5v}\operatorname{cosec}\theta\left[3\operatorname{cosec}\theta - 5\cot\theta\right]$



$\Rightarrow 3\operatorname{cosec}\theta - 5\cot\theta = 0 \qquad (\operatorname{cosec}\theta \neq 0)$

$\Rightarrow \frac{3}{\sin\theta} - \frac{5\cos\theta}{\sin\theta} = 0$

$\Rightarrow \frac{3 - 5\cos\theta}{\sin\theta} = 0$

$\Rightarrow \cos\theta = \frac{3}{5}$

CHECK THE NATURE OF THIS STATIONARY VALUE

$\frac{d^2T}{dt^2} = \frac{2a}{5v}\left[5\operatorname{cosec}\theta\cot^2\theta + 5\operatorname{cosec}^3\theta - 6\operatorname{cosec}^2\theta\cot\theta\right]$

NOW

$\cos\theta = \frac{3}{5}, \ \sec\theta = \frac{5}{3}$

$\sin\theta = \frac{4}{5}, \ \operatorname{cosec}\theta = \frac{5}{4}$

$\tan\theta = \frac{4}{3}, \ \cot\theta = \frac{3}{4}$

$\left.\frac{d^2T}{dt^2}\right|_{\cos\theta = \frac{3}{5}} = \frac{2a}{5v}\left[5\times\frac{5}{4}\times\frac{9}{16} + 5\times\frac{125}{64} - 6\times\frac{25}{16}\times\frac{3}{4}\right]$

$= \frac{2a}{5v}\left[\frac{225+625-450}{64}\right] = \frac{2a}{5v}\times\frac{400}{64} > 0$

$\therefore$ A (LOCAL) MINIMUM

$T_{\text{MIN}} = \frac{2a}{5v}\left[5\times\frac{5}{4} - 3\times\frac{3}{4} + 3\right]$

$= \frac{2a}{5v}\left[\frac{25}{4} - \frac{9}{4} + 3\right]$

$= \frac{2a}{5v}\left[7\right]$

$= \frac{14a}{5v}$

AS REQUIRED

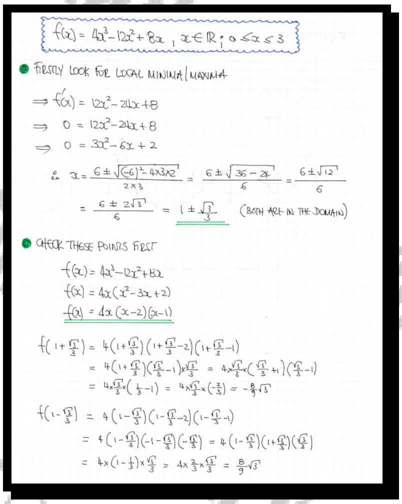
Question 243 (*****)

The function f is defined as

$$f(x) \equiv 4x^3 - 12x^2 + 8x, \quad x \in \mathbb{R}, \ 0 \le x \le 3.$$

Find the range of f, and hence sketch its graph, showing clearly the coordinates of any relevant points.

$\boxed{\quad}$, $\boxed{-\frac{8}{9}\sqrt{3} \le f(x) \le 24}$



$\boxed{f(x) = 4x^3 - 12x^2 + 8x, \ x \in \mathbb{R}; \ 0 \le x \le 3}$

- FIRSTLY LOOK FOR LOCAL MINIMA/MAXIMA

$\Rightarrow f'(x) = 12x^2 - 24x + 8$

$\Rightarrow 0 = 12x^2 - 24x + 8$

$\Rightarrow 0 = 3x^2 - 6x + 2$

$\therefore x = \frac{6 \pm \sqrt{(-6)^2 - 4 \times 3 \times 2}}{2 \times 3} = \frac{6 \pm \sqrt{36 - 24}}{6} = \frac{6 \pm \sqrt{12}}{6}$

$= \frac{6 \pm 2\sqrt{3}}{6} = \underline{\underline{1 \pm \frac{\sqrt{3}}{3}}}$ (BOTH ARE IN THE DOMAIN)

- CHECK THESE POINTS FIRST

$f(x) = 4x^3 - 12x^2 + 8x$

$f(x) = 4x(x^2 - 3x + 2)$

$\underline{\underline{f(x) = 4x(x-2)(x-1)}}$

$f(1+\frac{\sqrt{3}}{3}) = 4(1+\frac{\sqrt{3}}{3})(1+\frac{\sqrt{3}}{3}-2)(1+\frac{\sqrt{3}}{3}-1)$

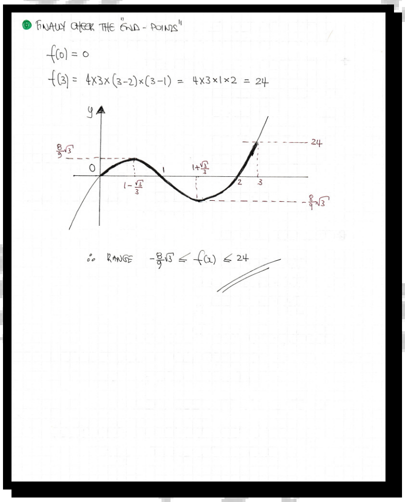
$= 4(1+\frac{\sqrt{3}}{3})(\frac{\sqrt{3}}{3}-1) \times \frac{\sqrt{3}}{3} = 4 \times \frac{\sqrt{3}}{3} \times (\frac{\sqrt{3}}{3}+1)(\frac{\sqrt{3}}{3}-1)$

$= 4 \times \frac{\sqrt{3}}{3} \times (\frac{1}{3}-1) = 4 \times \frac{\sqrt{3}}{3} \times (-\frac{2}{3}) = -\frac{8}{9}\sqrt{3}$

$f(1-\frac{\sqrt{3}}{3}) = 4(1-\frac{\sqrt{3}}{3})(1-\frac{\sqrt{3}}{3}-2)(1-\frac{\sqrt{3}}{3}-1)$

$= 4(1-\frac{\sqrt{3}}{3})(-1-\frac{\sqrt{3}}{3})(-\frac{\sqrt{3}}{3}) = 4(1-\frac{\sqrt{3}}{3})(1+\frac{\sqrt{3}}{3})(\frac{\sqrt{3}}{3})$

$= 4 \times (1-\frac{1}{3}) \times \frac{\sqrt{3}}{3} = 4 \times \frac{2}{3} \times \frac{\sqrt{3}}{3} = \frac{8}{9}\sqrt{3}$



- FINALLY CHECK THE "END - POINTS"

$f(0) = 0$

$f(3) = 4 \times 3 \times (3-2) \times (3-1) = 4 \times 3 \times 1 \times 2 = 24$

$\therefore$ RANGE $-\frac{8}{9}\sqrt{3} \le f(x) \le 24$

Question 244 (*****)

$$y=\arccos x\,,\ -1\le x\le 1\,.$$

a) Show that

$$\frac{dy}{dx}=-\frac{1}{\sqrt{1-x^2}}\,.$$

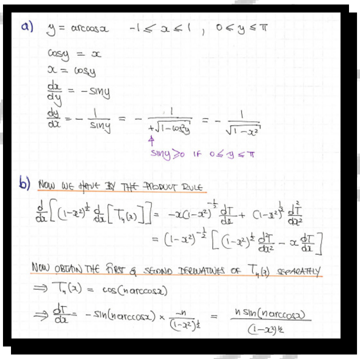
The Chebyshev polynomials of the first kind $T_n(x)$ is a family of functions defined as

$$T_n(x)=\cos(n\arccos x)\,,\ -1\le x\le 1\,,\ n\in\mathbb{N}\,.$$

b) Show further that

$$\frac{d}{dx}\left[\left(1-x^2\right)^{\frac{1}{2}}\frac{d}{dx}\left[T_n(x)\right]\right]=\frac{-n^2\,T_n(x)}{\sqrt{1-x^2}}\,.$$

$\boxed{}$, $\boxed{\text{proof}}$



a) $y=\arccos x \quad -1\le x\le 1\,,\ 0\le y\le\pi$

$\cos y = x$

$x=\cos y$

$\frac{dx}{dy}=-\sin y$

$\frac{dy}{dx}=-\frac{1}{\sin y}=-\frac{1}{+\sqrt{1-\cos^2 y}}=-\frac{1}{\sqrt{1-x^2}}$

$\uparrow$ $\sin y\ge 0$ IF $0\le y\le\pi$

b) NOW WE HAVE BY THE PRODUCT RULE

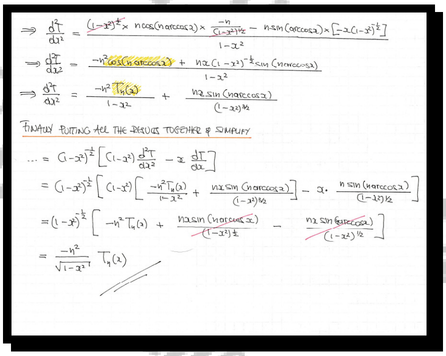
$\frac{d}{dx}\left[(1-x^2)^{\frac12}\frac{d}{dx}\left[T_n(x)\right]\right]=-x(1-x^2)^{-\frac12}\frac{dT}{dx}+(1-x^2)^{\frac12}\frac{d^2T}{dx^2}$

$=(1-x^2)^{-\frac12}\left[(1-x^2)^{\frac12}\frac{d^2T}{dx^2}-x\frac{dT}{dx}\right]$

NOW OBTAIN THE FIRST & SECOND DERIVATIVES OF $T_n(x)$ SEPARATELY

$\Rightarrow T_n(x)=\cos(n\arccos x)$

$\Rightarrow \frac{dT}{dx}=-\sin(n\arccos x)\times\frac{-n}{(1-x^2)^{\frac12}}=\frac{n\sin(n\arccos x)}{(1-x^2)^{1/2}}$



$\Rightarrow \frac{d^2T}{dx^2}=\frac{\cancel{(1-x^2)^{\frac12}}\times n\cos(n\arccos x)\times\frac{-n}{\cancel{(1-x^2)^{1/2}}}-n\sin(\arccos x)\times\left[-x(1-x^2)^{-\frac12}\right]}{1-x^2}$

$\Rightarrow \frac{d^2T}{dx^2}=\frac{-n^2\cos(n\arccos x)+nx(1-x^2)^{-\frac12}\sin(n\arccos x)}{1-x^2}$

$\Rightarrow \frac{d^2T}{dx^2}=\frac{-n^2T_n(x)}{1-x^2}+\frac{nx\sin(n\arccos x)}{(1-x^2)^{3/2}}$

FINALLY PUTTING ALL THE RESULTS TOGETHER & SIMPLIFY

$\dots=(1-x^2)^{-\frac12}\left[(1-x^2)\frac{d^2T}{dx^2}-x\frac{dT}{dx}\right]$

$=(1-x^2)^{-\frac12}\left[(1-x^2)\left[\frac{-n^2T_n(x)}{1-x^2}+\frac{nx\sin(n\arccos x)}{(1-x^2)^{3/2}}\right]-x\cdot\frac{n\sin(n\arccos x)}{(1-x^2)^{1/2}}\right]$

$=(1-x^2)^{-\frac12}\left[-n^2T_n(x)+\cancel{\frac{nx\sin(n\arccos x)}{(1-x^2)^{\frac12}}}-\cancel{\frac{nx\sin(n\arccos x)}{(1-x^2)^{1/2}}}\right]$

$=\frac{-n^2}{\sqrt{1-x^2}}T_n(x)$

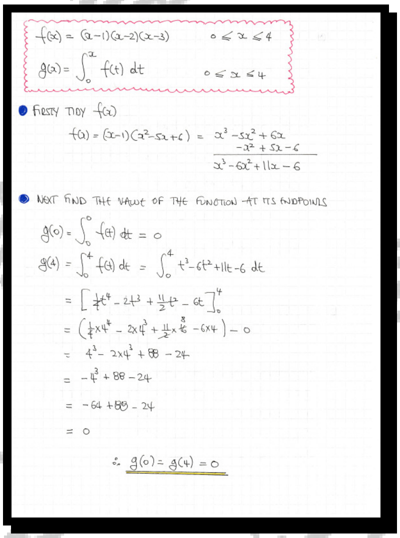
Question 245 (*****)

The real functions f and g have a common domain $0 \le x \le 4$, and defined as

$$f(x) \equiv (x-1)(x-2)(x-3) \qquad \text{and} \qquad g(x) \equiv \int_0^x f(t)\ dt .$$

Use a detailed algebraic method to determine the range of g.

$\boxed{}$, $\boxed{-\frac{9}{4} \le g(x) \le 0}$



$$\boxed{\begin{array}{ll} f(x) = (x-1)(x-2)(x-3) & 0 \leqslant x \leqslant 4 \\ g(x) = \int_0^x f(t)\ dt & 0 \leqslant x \leqslant 4 \end{array}}$$

- FIRSTLY TIDY $f(x)$

$$f(x) = (x-1)(x^2-5x+6) = \begin{array}{r} x^3 - 5x^2 + 6x \\ -x^2 + 5x - 6 \\ \hline x^3 - 6x^2 + 11x - 6 \end{array}$$

- NEXT FIND THE VALUE OF THE FUNCTION AT ITS ENDPOINTS

$g(0) = \int_0^0 f(t)\ dt = 0$

$g(4) = \int_0^4 f(t)\ dt = \int_0^4 t^3 - 6t^2 + 11t - 6\ dt$

$= \left[\frac{1}{4}t^4 - 2t^3 + \frac{11}{2}t^2 - 6t\right]_0^4$

$= \left(\frac{1}{\cancel{4}} \times 4^{\cancel{4}} - 2\times 4^3 + \frac{11}{\cancel{2}} \times \overset{8}{\cancel{16}} - 6\times 4\right) - 0$

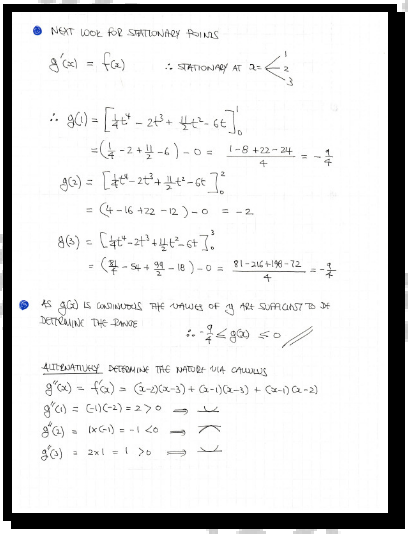
$= 4^3 - 2\times 4^3 + 88 - 24$

$= -4^3 + 88 - 24$

$= -64 + 88 - 24$

$= 0$

$\therefore \underline{g(0) = g(4) = 0}$



- NEXT LOOK FOR STATIONARY POINTS

$g'(x) = f(x) \qquad \therefore$ STATIONARY AT $x = \left\langle \begin{array}{l} 1 \\ 2 \\ 3 \end{array}\right.$

$\therefore g(1) = \left[\frac{1}{4}t^4 - 2t^3 + \frac{11}{2}t^2 - 6t\right]_0^1$

$= \left(\frac{1}{4} - 2 + \frac{11}{2} - 6\right) - 0 = \frac{1 - 8 + 22 - 24}{4} = -\frac{9}{4}$

$g(2) = \left[\frac{1}{4}t^4 - 2t^3 + \frac{11}{2}t^2 - 6t\right]_0^2$

$= (4 - 16 + 22 - 12) - 0 = -2$

$g(3) = \left[\frac{1}{4}t^4 - 2t^3 + \frac{11}{2}t^2 - 6t\right]_0^3$

$= \left(\frac{81}{4} - 54 + \frac{99}{2} - 18\right) - 0 = \frac{81 - 216 + 198 - 72}{4} = -\frac{9}{4}$

- AS $g(x)$ IS CONTINUOUS THE VALUES OF g ARE SUFFICIENT TO DETERMINE THE RANGE

$\therefore -\frac{9}{4} \leqslant g(x) \leqslant 0$ //

<u>ALTERNATIVELY</u> DETERMINE THE NATURE VIA CALCULUS

$g''(x) = f'(x) = (x-2)(x-3) + (x-1)(x-3) + (x-1)(x-2)$

$g''(1) = (-1)(-2) = 2 > 0 \implies \smile$

$g''(2) = 1\times(-1) = -1 < 0 \implies \frown$

$g''(3) = 2\times 1 = 1 > 0 \implies \smile$

Question 246 (*****)

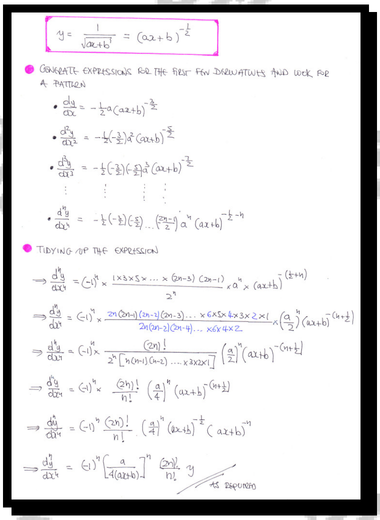
$$y = \frac{1}{\sqrt{ax+b}}, \ x \geq 0,$$

where a and b are positive constants.

Show, by a detailed method, that

$$\frac{d^n y}{dx^n} = \frac{(-1)^n y(2n)!}{n!}\left(\frac{a}{4(ax+b)}\right)^n.$$

$\boxed{}$, $\boxed{\text{proof}}$



$$\boxed{y = \frac{1}{\sqrt{ax+b}} = (ax+b)^{-\frac{1}{2}}}$$

- GENERATE EXPRESSIONS FOR THE FIRST FEW DERIVATIVES AND LOOK FOR A PATTERN

  - $\frac{dy}{dx} = -\frac{1}{2}a(ax+b)^{-\frac{3}{2}}$
  - $\frac{d^2y}{dx^2} = -\frac{1}{2}\left(-\frac{3}{2}\right)a^2(ax+b)^{-\frac{5}{2}}$
  - $\frac{d^3y}{dx^3} = -\frac{1}{2}\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)a^3(ax+b)^{-\frac{7}{2}}$

    $\vdots \qquad \vdots \qquad \vdots \quad \vdots$
  - $\frac{d^ny}{dx^n} = -\frac{1}{2}\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\dots\left(-\frac{2n-1}{2}\right)a^n(ax+b)^{-\frac{1}{2}-n}$

- TIDYING UP THE EXPRESSION

$$\Rightarrow \frac{d^ny}{dx^n} = (-1)^n \times \frac{1\times3\times5\times\dots\times(2n-3)(2n-1)}{2^n} \times a^n \times (ax+b)^{-\left(\frac{1}{2}+n\right)}$$

$$\Rightarrow \frac{d^ny}{dx^n} = (-1)^n \times \frac{2n(2n-1)(2n-2)(2n-3)\dots\times6\times5\times4\times3\times2\times1}{2n(2n-2)(2n-4)\dots\times6\times4\times2} \times \left(\frac{a}{2}\right)^n (ax+b)^{-\left(n+\frac{1}{2}\right)}$$

$$\Rightarrow \frac{d^ny}{dx^n} = (-1)^n \times \frac{(2n)!}{2^n\left[n(n-1)(n-2)\dots\times3\times2\times1\right]}\left(\frac{a}{2}\right)^n (ax+b)^{-\left(n+\frac{1}{2}\right)}$$

$$\Rightarrow \frac{d^ny}{dx^n} = (-1)^n \times \frac{(2n)!}{n!}\left(\frac{a}{4}\right)^n (ax+b)^{-\left(n+\frac{1}{2}\right)}$$

$$\Rightarrow \frac{d^ny}{dx^n} = (-1)^n \frac{(2n)!}{n!}\left(\frac{a}{4}\right)^n (ax+b)^{-\frac{1}{2}}(ax+b)^{-n}$$

$$\Rightarrow \frac{d^ny}{dx^n} = (-1)^n\left[\frac{a}{4(ax+b)}\right]^n \frac{(2n)!}{n!}y$$

AS REQUIRED

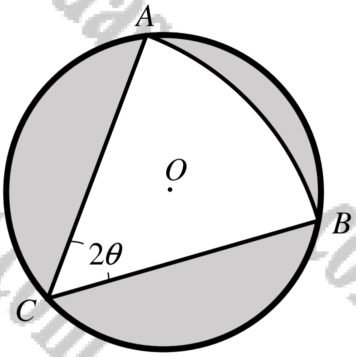
Question 247 (*****)

$$2x\tan x = 1,\ x \neq \frac{1}{2}n\pi,\ n \in \mathbb{N}$$

a) Show that the above equation has a solution in the interval $(0.6, 0.7)$.

b) Use the Newton Raphson method to find the solution of this equation, correct to 5 decimal places.

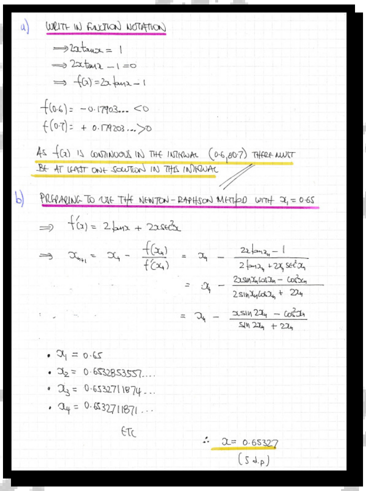
The figure below shows a circle, centre at O. The points A, B and C lie on the circumference of this circle. A circular sector ABC, subtending an angle of 2θ at C, is inscribed in this circle.



c) Determine the greatest proportion of the area of the circle, which can be covered by this sector.

You may give the answer as a percentage, correct to two decimal places

$\boxed{}$, $\boxed{x \approx 0.65327}$, $\boxed{\approx 52.45}$



a) WRITE IN FUNCTION NOTATION

$\Rightarrow 2x\tan x = 1$

$\Rightarrow 2x\tan x - 1 = 0$

$\Rightarrow f(x) = 2x\tan x - 1$

$f(0.6) = -0.17903\ldots < 0$

$f(0.7) = +0.179203\ldots > 0$

AS $f(x)$ IS CONTINUOUS IN THE INTERVAL $(0.6, 0.7)$ THERE MUST BE AT LEAST ONE SOLUTION IN THIS INTERVAL

b) PREPARING TO USE THE NEWTON-RAPHSON METHOD WITH $x_1 = 0.65$

$\Rightarrow f'(x) = 2\tan x + 2x\sec^2 x$

$\Rightarrow x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{2x\tan x_n - 1}{2\tan x_n + 2x_n\sec^2 x_n}$

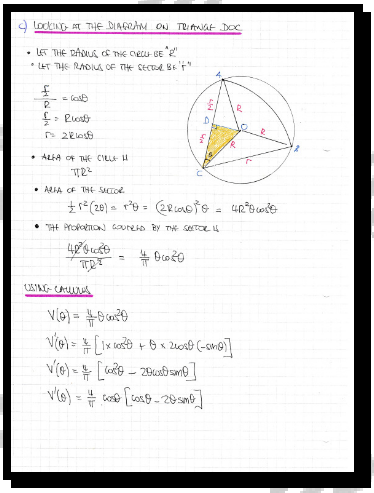
$= x_n - \frac{2x\sin x_n\cos x_n - \cos^2 x_n}{2\sin x_n\cos x_n + 2x_n}$

$= x_n - \frac{x\sin 2x_n - \cos^2 x_n}{\sin 2x_n + 2x_n}$

- $x_1 = 0.65$
- $x_2 = 0.6532853557\ldots$
- $x_3 = 0.6532711874\ldots$
- $x_4 = 0.6532711871\ldots$

ETC

$\therefore x = 0.65327$ (5 d.p)



c) LOOKING AT THE DIAGRAM ON TRIANGLE DOC

- LET THE RADIUS OF THE CIRCLE BE "R"
- LET THE RADIUS OF THE SECTOR BE "r"

$\frac{\frac{r}{2}}{R} = \cos\theta$

$\frac{r}{2} = R\cos\theta$

$r = 2R\cos\theta$

- AREA OF THE CIRCLE IS πR^2
- AREA OF THE SECTOR $\frac{1}{2}r^2(2\theta) = r^2\theta = (2R\cos\theta)^2\theta = 4R^2\theta\cos^2\theta$
- THE PROPORTION COVERED BY THE SECTOR IS $\frac{4\cancel{R^2}\theta\cos^2\theta}{\pi\cancel{R^2}} = \frac{4}{\pi}\theta\cos^2\theta$

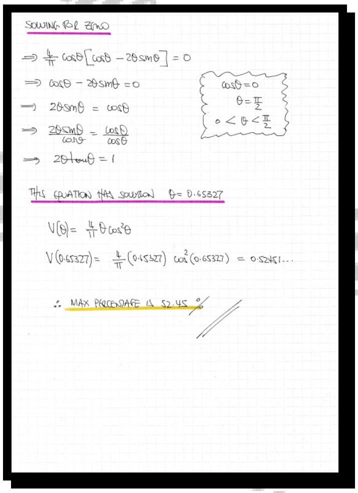
USING CALCULUS

$V(\theta) = \frac{4}{\pi}\theta\cos^2\theta$

$V'(\theta) = \frac{4}{\pi}\left[1 \times \cos^2\theta + \theta \times 2\cos\theta(-\sin\theta)\right]$

$V'(\theta) = \frac{4}{\pi}\left[\cos^2\theta - 2\theta\cos\theta\sin\theta\right]$

$V'(\theta) = \frac{4}{\pi}\cos\theta\left[\cos\theta - 2\theta\sin\theta\right]$



SOLVING FOR ZERO

$\Rightarrow \frac{4}{\pi}\cos\theta\left[\cos\theta - 2\theta\sin\theta\right] = 0$

$\Rightarrow \cos\theta - 2\theta\sin\theta = 0$

$\Rightarrow 2\theta\sin\theta = \cos\theta$

$\Rightarrow \frac{2\theta\sin\theta}{\cos\theta} = \frac{\cos\theta}{\cos\theta}$

$\Rightarrow 2\theta\tan\theta = 1$

$\cos\theta = 0$

$\theta = \frac{\pi}{2}$

$0 < \theta < \frac{\pi}{2}$

THIS EQUATION HAS SOLUTION $\theta = 0.65327$

$V(\theta) = \frac{4}{\pi}\theta\cos^2\theta$

$V(0.65327) = \frac{4}{\pi}(0.65327)\cos^2(0.65327) = 0.52451\ldots$

$\therefore$ MAX PERCENTAGE IS 52.45 %

Question 248 (*****)

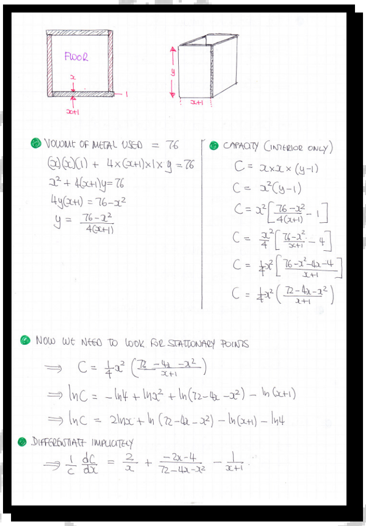
A lump of metal, of volume 76 cubic units is **moulded** into the shape of a cuboidal box, with a square base, rectangular sided and no lid.

All the faces of the box are 1 unit thick.

All the metal is moulded in the construction of this box, and the construction it has maximum capacity.

If the internal width of the box is x, find the value of x which maximises the capacity of he box, and hence determine this maximum capacity.

$\boxed{\quad}$, $\boxed{x=4}$, $\boxed{C_{\text{max}}=38\frac{2}{5}}$



VOLUME OF METAL USED = 76

$(x)(x)(1) + 4\times(x+1)\times1\times y = 76$

$x^2 + 4(x+1)y = 76$

$4y(x+1) = 76 - x^2$

$y = \frac{76-x^2}{4(x+1)}$

CAPACITY (INTERIOR ONLY)

$C = x \times x \times (y-1)$

$C = x^2(y-1)$

$C = x^2\left[\frac{76-x^2}{4(x+1)} - 1\right]$

$C = \frac{x^2}{4}\left[\frac{76-x^2}{x+1} - 4\right]$

$C = \frac{1}{4}x^2\left[\frac{76-x^2-4x-4}{x+1}\right]$

$C = \frac{1}{4}x^2\left(\frac{72-4x-x^2}{x+1}\right)$

NOW WE NEED TO LOOK FOR STATIONARY POINTS

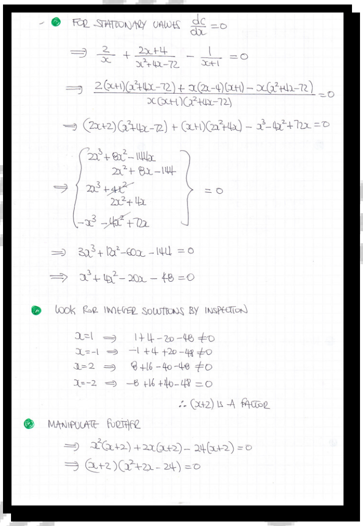
$\Rightarrow C = \frac{1}{4}x^2\left(\frac{72-4x-x^2}{x+1}\right)$

$\Rightarrow \ln C = -\ln 4 + \ln x^2 + \ln(72-4x-x^2) - \ln(x+1)$

$\Rightarrow \ln C = 2\ln x + \ln(72-4x-x^2) - \ln(x+1) - \ln 4$

DIFFERENTIATE IMPLICITLY

$\Rightarrow \frac{1}{C}\frac{dC}{dx} = \frac{2}{x} + \frac{-2x-4}{72-4x-x^2} - \frac{1}{x+1}$



FOR STATIONARY VALUES $\frac{dC}{dx} = 0$

$\Rightarrow \frac{2}{x} + \frac{2x+4}{x^2+4x-72} - \frac{1}{x+1} = 0$

$\Rightarrow \frac{2(x+1)(x^2+4x-72) + x(2x-4)(x+1) - x(x^2+4x-72)}{x(x+1)(x^2+4x-72)} = 0$

$\Rightarrow (2x+2)(x^2+4x-72) + (x+1)(2x^2+4x) - x^3 - 4x^2 + 72x = 0$

$$\Rightarrow \left\{\begin{array}{l} 2x^3 + 8x^2 - 144x \\ \quad 2x^2 + 8x - 144 \\ 2x^3 + \cancel{4x^2} \\ \quad 2x^2 + 4x \\ -x^3 - \cancel{4x^2} + 72x \end{array}\right\} = 0$$

$\Rightarrow 3x^3 + 12x^2 - 60x - 144 = 0$

$\Rightarrow x^3 + 4x^2 - 20x - 48 = 0$

LOOK FOR INTEGER SOLUTIONS BY INSPECTION

$x=1 \Rightarrow 1+4-20-48 \neq 0$

$x=-1 \Rightarrow -1+4+20-48 \neq 0$

$x=2 \Rightarrow 8+16-40-48 \neq 0$

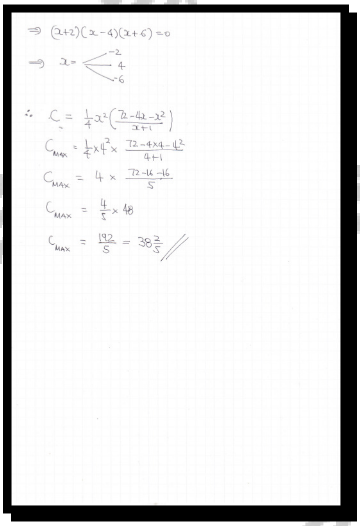
$x=-2 \Rightarrow -8+16+40-48 = 0$

$\therefore (x+2)$ IS A FACTOR

MANIPULATE FURTHER

$\Rightarrow x^2(x+2) + 2x(x+2) - 24(x+2) = 0$

$\Rightarrow (x+2)(x^2+2x-24) = 0$



$\Rightarrow (x+2)(x-4)(x+6) = 0$

$\Rightarrow x = \begin{cases} -2 \\ 4 \\ -6 \end{cases}$

$\therefore C = \frac{1}{4}x^2\left(\frac{72-4x-x^2}{x+1}\right)$

$C_{\text{MAX}} = \frac{1}{4}\times4^2\times\frac{72-4\times4-4^2}{4+1}$

$C_{\text{MAX}} = 4 \times \frac{72-16-16}{5}$

$C_{\text{MAX}} = \frac{4}{5}\times 48$

$C_{\text{MAX}} = \frac{192}{5} = 38\frac{2}{5}$ //

Question 249 (*****)

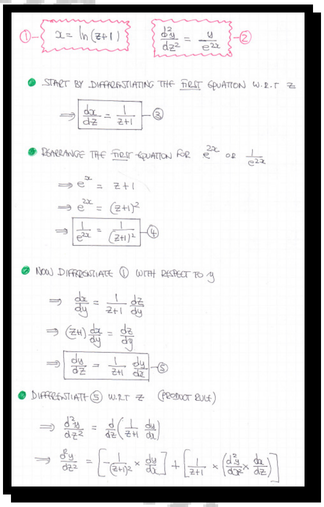
The variables x, y and z satisfy the following relationships.

$$x = \ln(z+1) \qquad \text{and} \qquad \frac{d^2y}{dz^2} = \frac{y}{e^{2x}}.$$

Show that

$$\frac{d^2y}{dx^2} = \frac{dy}{dx} + y.$$

$\boxed{}$, $\boxed{\text{proof}}$



$$① - \boxed{x = \ln(z+1)} \qquad \boxed{\frac{d^2y}{dz^2} = \frac{y}{e^{2x}}} - ②$$

- START BY DIFFERENTIATING THE FIRST EQUATION W.R.T z

$$\Rightarrow \boxed{\frac{dx}{dz} = \frac{1}{z+1}} - ③$$

- REARRANGE THE FIRST EQUATION FOR e^{2x} OR $\frac{1}{e^{2x}}$

$$\Rightarrow e^{x} = z+1$$

$$\Rightarrow e^{2x} = (z+1)^2$$

$$\Rightarrow \boxed{\frac{1}{e^{2x}} = \frac{1}{(z+1)^2}} - ④$$

- NOW DIFFERENTIATE ① WITH RESPECT TO y

$$\Rightarrow \frac{dx}{dy} = \frac{1}{z+1}\frac{dz}{dy}$$

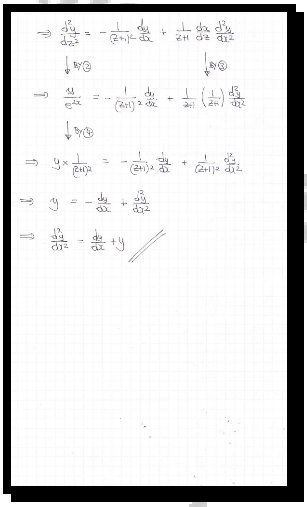
$$\Rightarrow (z+1)\frac{dx}{dy} = \frac{dz}{dy}$$

$$\Rightarrow \boxed{\frac{dy}{dz} = \frac{1}{z+1}\frac{dy}{dx}} - ⑤$$

- DIFFERENTIATE ⑤ W.R.T z (PRODUCT RULE)

$$\Rightarrow \frac{d^2y}{dz^2} = \frac{d}{dz}\left(\frac{1}{z+1}\frac{dy}{dx}\right)$$

$$\Rightarrow \frac{d^2y}{dz^2} = \left[-\frac{1}{(z+1)^2} \times \frac{dy}{dx}\right] + \left[\frac{1}{z+1} \times \left(\frac{d^2y}{dx^2} \times \frac{dx}{dz}\right)\right]$$



$$\Rightarrow \frac{d^2y}{dz^2} = -\frac{1}{(z+1)^2}\frac{dy}{dx} + \frac{1}{z+1}\frac{dx}{dz}\frac{d^2y}{dx^2}$$

$\downarrow$ BY ② $\qquad \downarrow$ BY ③

$$\Rightarrow \frac{y}{e^{2x}} = -\frac{1}{(z+1)^2}\frac{dy}{dx} + \frac{1}{z+1}\left(\frac{1}{z+1}\right)\frac{d^2y}{dx^2}$$

$\downarrow$ BY ④

$$\Rightarrow y \times \frac{1}{(z+1)^2} = -\frac{1}{(z+1)^2}\frac{dy}{dx} + \frac{1}{(z+1)^2}\frac{d^2y}{dx^2}$$

$$\Rightarrow y = -\frac{dy}{dx} + \frac{d^2y}{dx^2}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{dy}{dx} + y \quad //$$

Question 250 (*****)

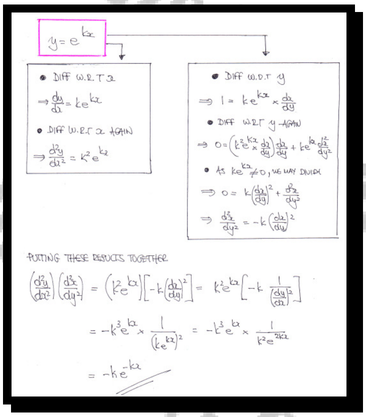
$$y = \mathrm{e}^{kx}, \ k \neq 0 .$$

Find a simplified expression for

$$\left[\frac{d^2 y}{dx^2}\right]\left[\frac{d^2 x}{dy^2}\right],$$

giving the answer in terms of k and e^{kx}.

$\boxed{}$, $\boxed{\text{proof}}$



$\boxed{y = e^{kx}}$

- DIFF W.R.T x

$\Rightarrow \frac{dy}{dx} = ke^{kx}$

- DIFF W.R.T x AGAIN

$\Rightarrow \frac{d^2y}{dx^2} = k^2e^{kx}$

- DIFF W.R.T y

$\Rightarrow 1 = ke^{kx} \times \frac{dx}{dy}$

- DIFF W.R.T y AGAIN

$\Rightarrow 0 = \left(k^2e^{kx} \times \frac{dx}{dy}\right)\frac{dx}{dy} + ke^{kx}\frac{d^2x}{dy^2}$

- AS $ke^{kx} \neq 0$, WE MAY DIVIDE

$\Rightarrow 0 = k\left(\frac{dx}{dy}\right)^2 + \frac{d^2x}{dy^2}$

$\Rightarrow \frac{d^2x}{dy^2} = -k\left(\frac{dx}{dy}\right)^2$

PUTTING THESE RESULTS TOGETHER

$$\left(\frac{d^2y}{dx^2}\right)\left(\frac{d^2x}{dy^2}\right) = \left(k^2e^{kx}\right)\left[-k\left(\frac{dx}{dy}\right)^2\right] = k^2e^{kx}\left[-k\frac{1}{\left(\frac{dy}{dx}\right)^2}\right]$$

$$= -k^3e^{kx} \times \frac{1}{\left(ke^{kx}\right)^2} = -k^3e^{kx} \times \frac{1}{k^2e^{2kx}}$$

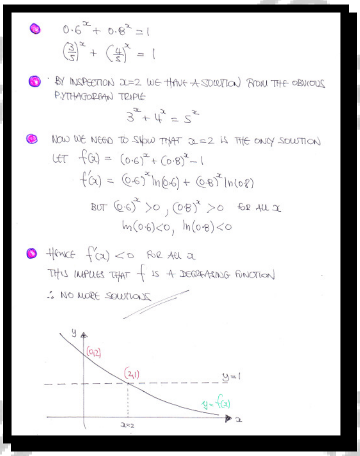
$$= -ke^{-kx}$$

Question 251 (*****)

$$0.6^x + 0.8^x = 1, \ x \in \mathbb{R}.$$

Find the only solution of the above equation, fully justifying the fact that it is the only solution.

$\boxed{\text{SPY}}$, $\boxed{x=2}$



- $0.6^x + 0.8^x = 1$

  $\left(\frac{3}{5}\right)^x + \left(\frac{4}{5}\right)^x = 1$

- BY INSPECTION $x=2$ WE HAVE A SOLUTION FROM THE OBVIOUS PYTHAGOREAN TRIPLE

  $$3^x + 4^x = 5^x$$

- NOW WE NEED TO SHOW THAT $x=2$ IS THE ONLY SOLUTION

  LET $f(x) = (0.6)^x + (0.8)^x - 1$

  $f'(x) = (0.6)^x \ln(0.6) + (0.8)^x \ln(0.8)$

  BUT $(0.6)^x > 0$, $(0.8)^x > 0$ FOR ALL x

  $\ln(0.6) < 0$, $\ln(0.8) < 0$

- HENCE $f'(x) < 0$ FOR ALL x

  THIS IMPLIES THAT f IS A DECREASING FUNCTION

  $\therefore$ NO MORE SOLUTIONS

Question 252 **(*****)**

From a thin sheet of metal, a circular sector of area A is removed.

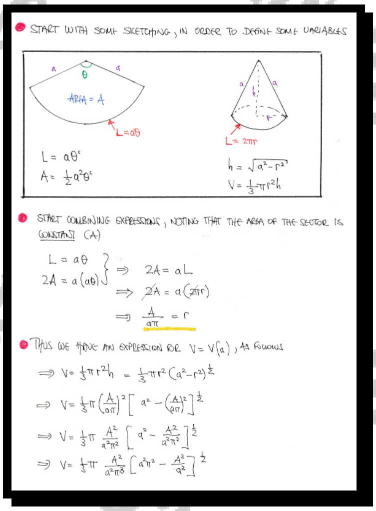
The circular sector is folded without any overlapping into the curved surface of a right circular cone of volume V.

The measurements of the circular sector are such so that V is maximum.

Find the angle subtended by the circular sector at its centre and show further that the maximum value of V is

$$\frac{1}{9}\sqrt[4]{\frac{12A^6}{\pi^2}}.$$

SP-I , $\boxed{\theta = \frac{2\pi}{\sqrt{3}}}$



- START WITH SOME SKETCHING, IN ORDER TO DEFINE SOME VARIABLES

$L = a\theta^c$

$A = \frac{1}{2}a^2\theta^c$

$h = \sqrt{a^2 - r^2}$

$V = \frac{1}{3}\pi r^2 h$

- START COMBINING EXPRESSIONS, NOTING THAT THE AREA OF THE SECTOR IS CONSTANT (A)

$$\left.\begin{array}{l} L = a\theta \\ 2A = a(a\theta) \end{array}\right\} \Rightarrow 2A = aL$$

$$\Rightarrow \cancel{2}A = a(\cancel{2}\pi r)$$

$$\Rightarrow \frac{A}{a\pi} = r$$

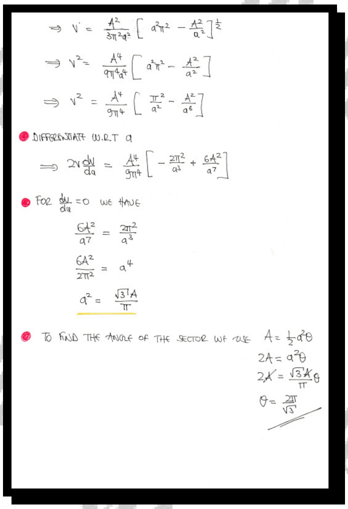
- THUS WE HAVE AN EXPRESSION FOR $V = V(a)$, AS FOLLOWS

$$\Rightarrow V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2 (a^2 - r^2)^{\frac{1}{2}}$$

$$\Rightarrow V = \frac{1}{3}\pi\left(\frac{A}{a\pi}\right)^2\left[a^2 - \left(\frac{A}{a\pi}\right)^2\right]^{\frac{1}{2}}$$

$$\Rightarrow V = \frac{1}{3}\pi\,\frac{A^2}{a^2\pi^2}\left[a^2 - \frac{A^2}{a^2\pi^2}\right]^{\frac{1}{2}}$$

$$\Rightarrow V = \frac{1}{3}\pi\,\frac{A^2}{a^2\pi^3}\left[a^2\pi^2 - \frac{A^2}{a^2}\right]^{\frac{1}{2}}$$



$$\Rightarrow V = \frac{A^2}{3\pi^2 a^2}\left[a^2\pi^2 - \frac{A^2}{a^2}\right]^{\frac{1}{2}}$$

$$\Rightarrow V^2 = \frac{A^4}{9\pi^4 a^4}\left[a^2\pi^2 - \frac{A^2}{a^2}\right]$$

$$\Rightarrow V^2 = \frac{A^4}{9\pi^4}\left[\frac{\pi^2}{a^2} - \frac{A^2}{a^6}\right]$$

- DIFFERENTIATE W.R.T a

$$\Rightarrow 2V\frac{dV}{da} = \frac{A^4}{9\pi^4}\left[-\frac{2\pi^2}{a^3} + \frac{6A^2}{a^7}\right]$$

- FOR $\frac{dV}{da} = 0$ WE HAVE

$$\frac{6A^2}{a^7} = \frac{2\pi^2}{a^3}$$

$$\frac{6A^2}{2\pi^2} = a^4$$

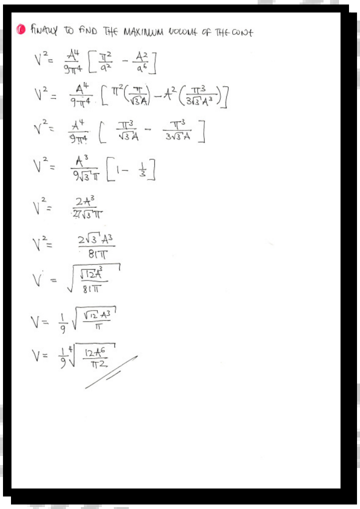
$$a^2 = \frac{\sqrt{3}A}{\pi}$$

- TO FIND THE ANGLE OF THE SECTOR WE USE $A = \frac{1}{2}a^2\theta$

$$2A = a^2\theta$$

$$2\cancel{A} = \frac{\sqrt{3}\cancel{A}}{\pi}\theta$$

$$\theta = \frac{2\pi}{\sqrt{3}}$$



- FINALLY TO FIND THE MAXIMUM VOLUME OF THE CONE

$$V^2 = \frac{A^4}{9\pi^4}\left[\frac{\pi^2}{a^2} - \frac{A^2}{a^6}\right]$$

$$V^2 = \frac{A^4}{9\pi^4}\left[\pi^2\left(\frac{\pi}{\sqrt{3}A}\right) - A^2\left(\frac{\pi^3}{3\sqrt{3}A^3}\right)\right]$$

$$V^2 = \frac{A^4}{9\pi^4}\left[\frac{\pi^3}{\sqrt{3}A} - \frac{\pi^3}{3\sqrt{3}A}\right]$$

$$V^2 = \frac{A^3}{9\sqrt{3}\pi}\left[1 - \frac{1}{3}\right]$$

$$V^2 = \frac{2A^3}{27\sqrt{3}\pi}$$

$$V^2 = \frac{2\sqrt{3}A^3}{81\pi}$$

$$V = \sqrt{\frac{\sqrt{12}A^3}{81\pi}}$$

$$V = \frac{1}{9}\sqrt{\frac{\sqrt{12}A^3}{\pi}}$$

$$V = \frac{1}{9}\sqrt[4]{\frac{12A^6}{\pi^2}}$$

Question 253 (*****)

The variable point P lies on the positive x axis and the variable point Q lies on the curve with equation

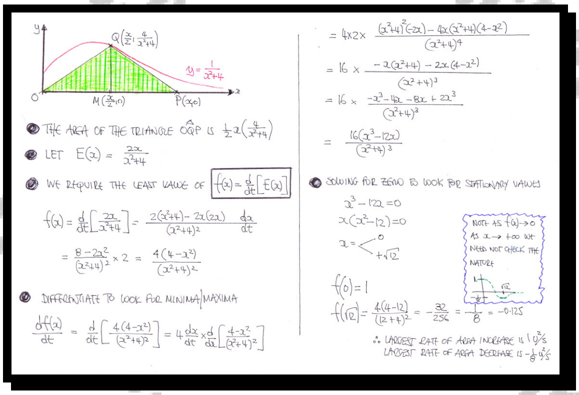
$$y = \frac{1}{x^2+4}, \ x \in \mathbb{R}, \ x \geq 0 .$$

The x coordinate Q is always half the x coordinate of P.

The point P starts at the origin O and begins to move in the positive x direction at constant rate.

Determine the largest rate of area increase and the largest rate of area decrease of the triangle OPQ, as P is moving away from O.

$\boxed{}$, $\boxed{\text{largest rate of area increase } = 1}$, $\boxed{\text{largest rate of area decrease } = 0.125}$



● THE AREA OF THE TRIANGLE $O\hat{Q}P$ IS $\frac{1}{2}x\left(\frac{4}{x^2+4}\right)$

● LET $E(x) = \frac{2x}{x^2+4}$

● WE REQUIRE THE LEAST VALUE OF $\boxed{f(x) = \frac{d}{dt}\left[E(x)\right]}$

$$f(x) = \frac{d}{dt}\left[\frac{2x}{x^2+4}\right] = \frac{2(x^2+4) - 2x(2x)}{(x^2+4)^2}\frac{dx}{dt}$$

$$= \frac{8-2x^2}{(x^2+4)^2} \times 2 = \frac{4(4-x^2)}{(x^2+4)^2}$$

● DIFFERENTIATE TO LOOK FOR MINIMA/MAXIMA

$$\frac{df(x)}{dt} = \frac{d}{dt}\left[\frac{4(4-x^2)}{(x^2+4)^2}\right] = 4\frac{dx}{dt} \times \frac{d}{dx}\left[\frac{4-x^2}{(x^2+4)^2}\right]$$

$$= 4 \times 2 \times \frac{(x^2+4)^2(-2x) - 4x(x^2+4)(4-x^2)}{(x^2+4)^4}$$

$$= 16 \times \frac{-x(x^2+4) - 2x(4-x^2)}{(x^2+4)^3}$$

$$= 16 \times \frac{-x^3-4x-8x+2x^3}{(x^2+4)^3}$$

$$= \frac{16(x^3-12x)}{(x^2+4)^3}$$

● SOLVING FOR ZERO TO LOOK FOR STATIONARY VALUES

$$x^3 - 12x = 0$$

$$x(x^2-12) = 0$$

$$x = \begin{cases} 0 \\ +\sqrt{12} \end{cases}$$

NOTE AS $f(x) \to 0$ AS $x \to +\infty$ WE NEED NOT CHECK THE NATURE

$$f(0) = 1$$

$$f(\sqrt{12}) = \frac{4(4-12)}{(12+4)^2} = -\frac{32}{256} = -\frac{1}{8} = -0.125$$

∴ LARGEST RATE OF AREA INCREASE IS 1 u^2/s
LARGEST RATE OF AREA DECREASE IS $-\frac{1}{8}$ u^2/s

Question 254 (*****)

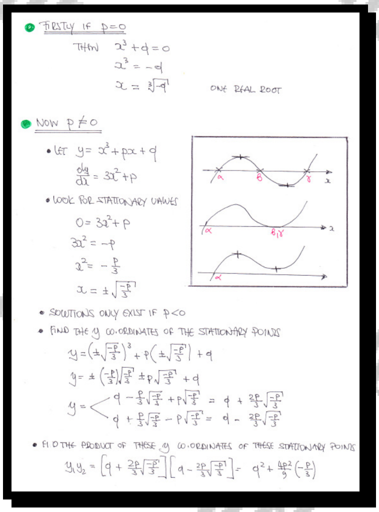
$$x^3 + px + q = 0\,,\ x \in \mathbb{R}$$

The cubic equation above is given in terms of the real constants, p and q.

Use differentiation to determine the conditions that p and q must satisfy so that the above equation has …

- … one real root.
- … three real roots, of which one is repeated.
- … three distinct real roots.

$\boxed{\quad}$, $\boxed{p \geq 0 \ \cup \ \left[p < 0 \cap q^2 - \dfrac{4p^3}{27} > 0\right]}$, $\boxed{p < 0 \cap q^2 - \dfrac{4p^3}{27} = 0}$, $\boxed{p < 0 \cap q^2 - \dfrac{4p^3}{27} < 0}$



● FIRSTLY IF $p=0$

THEN $x^3+q=0$

$x^3=-q$

$x=\sqrt[3]{-q}$ ONE REAL ROOT

● NOW $p \neq 0$

- LET $y=x^3+px+q$

$\frac{dy}{dx}=3x^2+p$

- LOOK FOR STATIONARY VALUES

$0=3x^2+p$

$3x^2=-p$

$x^2=-\frac{p}{3}$

$x=\pm\sqrt{\frac{-p}{3}}$

- SOLUTIONS ONLY EXIST IF $p<0$
- FIND THE y CO-ORDINATES OF THE STATIONARY POINTS

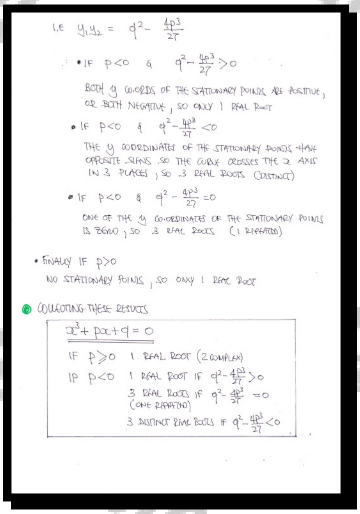
$y=\left(\pm\sqrt{\frac{-p}{3}}\right)^3+p\left(\pm\sqrt{\frac{-p}{3}}\right)+q$

$y=\pm\left(\frac{-p}{3}\right)\sqrt{\frac{-p}{3}}\pm p\sqrt{\frac{-p}{3}}+q$

$y=\begin{cases} q-\frac{p}{3}\sqrt{\frac{-p}{3}}+p\sqrt{\frac{-p}{3}}=q+\frac{2p}{3}\sqrt{\frac{-p}{3}} \\ q+\frac{p}{3}\sqrt{\frac{-p}{3}}-p\sqrt{\frac{-p}{3}}=q-\frac{2p}{3}\sqrt{\frac{-p}{3}} \end{cases}$

- FIND THE PRODUCT OF THESE y CO-ORDINATES OF THESE STATIONARY POINTS

$y_1y_2=\left[q+\frac{2p}{3}\sqrt{\frac{-p}{3}}\right]\left[q-\frac{2p}{3}\sqrt{\frac{-p}{3}}\right]=q^2+\frac{4p^2}{9}\left(-\frac{p}{3}\right)$



I.E $y_1y_2=q^2-\frac{4p^3}{27}$

- IF $p<0$ & $q^2-\frac{4p^3}{27}>0$

BOTH y CO-ORDS OF THE STATIONARY POINTS ARE POSITIVE, OR BOTH NEGATIVE, SO ONLY 1 REAL ROOT

- IF $p<0$ & $q^2-\frac{4p^3}{27}<0$

THE y COORDINATES OF THE STATIONARY POINTS HAVE OPPOSITE SIGNS SO THE CURVE CROSSES THE x AXIS IN 3 PLACES, SO 3 REAL ROOTS (DISTINCT)

- IF $p<0$ & $q^2-\frac{4p^3}{27}=0$

ONE OF THE y CO-ORDINATES OF THE STATIONARY POINTS IS ZERO, SO 3 REAL ROOTS (1 REPEATED)

- FINALLY IF $p>0$

NO STATIONARY POINTS, SO ONLY 1 REAL ROOT

● COLLECTING THESE RESULTS

$x^3+px+q=0$

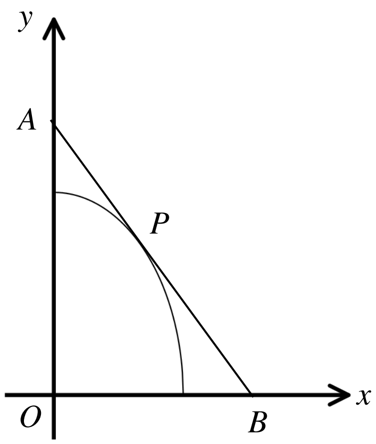
IF $p \geqslant 0$ 1 REAL ROOT (2 COMPLEX)

IF $p<0$ 1 REAL ROOT IF $q^2-\frac{4p^3}{27}>0$

3 REAL ROOTS IF $q^2-\frac{4p^3}{27}=0$ (ONE REPEATED)

3 DISTINCT REAL ROOTS IF $q^2-\frac{4p^3}{27}<0$

Question 255 (*****)



The figure above shows the curve C with equation

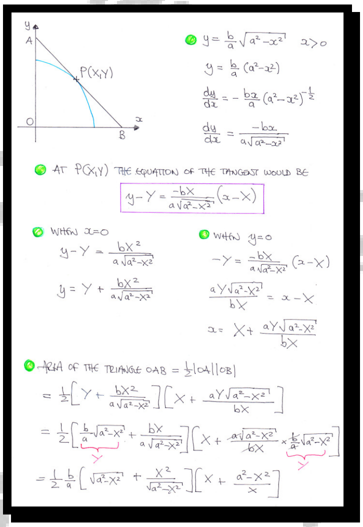
$$y = \frac{b}{a}\sqrt{a^2 - x^2}\ ,\ x \geq 0\ ,$$

where a and b are constants such that $b > a > 0$.

The point P lies on C and the tangent to C at P meets the coordinate axes at the points A and B, as shown in the figure.

Show with full justification that the minimum area of the triangle AOB, where O is the origin, is ab.

☐ , $\boxed{\text{proof}}$



$y = \frac{b}{a}\sqrt{a^2 - x^2} \quad x > 0$

$y = \frac{b}{a}(a^2 - x^2)$

$\frac{dy}{dx} = -\frac{bx}{a}(a^2 - x^2)^{-\frac{1}{2}}$

$\frac{dy}{dx} = \frac{-bx}{a\sqrt{a^2 - x^2}}$

AT P(X,Y) THE EQUATION OF THE TANGENT WOULD BE

$\boxed{y - Y = \frac{-bX}{a\sqrt{a^2 - X^2}}(x - X)}$

WHEN $x = 0$

$y - Y = \frac{bX^2}{a\sqrt{a^2 - X^2}}$

$y = Y + \frac{bX^2}{a\sqrt{a^2 - X^2}}$

WHEN $y = 0$

$-Y = \frac{-bX}{a\sqrt{a^2 - X^2}}(x - X)$

$\frac{aY\sqrt{a^2 - X^2}}{bX} = x - X$

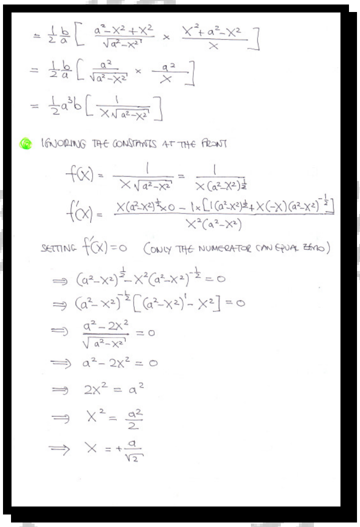
$x = X + \frac{aY\sqrt{a^2 - X^2}}{bX}$

AREA OF THE TRIANGLE OAB $= \frac{1}{2}|OA||OB|$

$= \frac{1}{2}\left[Y + \frac{bX^2}{a\sqrt{a^2 - X^2}}\right]\left[X + \frac{aY\sqrt{a^2 - X^2}}{bX}\right]$

$= \frac{1}{2}\left[\underbrace{\frac{b}{a}\sqrt{a^2 - X^2}}_{Y} + \frac{bX}{a\sqrt{a^2 - X^2}}\right]\left[X + \frac{\cancel{a}\sqrt{a^2 - X^2}}{\cancel{b}X} \times \underbrace{\frac{\cancel{b}}{\cancel{a}}\sqrt{a^2 - X^2}}_{Y}\right]$

$= \frac{1}{2}\frac{b}{a}\left[\sqrt{a^2 - X^2} + \frac{X^2}{\sqrt{a^2 - X^2}}\right]\left[X + \frac{a^2 - X^2}{X}\right]$



$= \frac{1}{2}\frac{b}{a}\left[\frac{a^2 - X^2 + X^2}{\sqrt{a^2 - X^2}} \times \frac{X^2 + a^2 - X^2}{X}\right]$

$= \frac{1}{2}\frac{b}{a}\left[\frac{a^2}{\sqrt{a^2 - X^2}} \times \frac{a^2}{X}\right]$

$= \frac{1}{2}a^3b\left[\frac{1}{X\sqrt{a^2 - X^2}}\right]$

IGNORING THE CONSTANTS AT THE FRONT

$f(X) = \frac{1}{X\sqrt{a^2 - X^2}} = \frac{1}{X(a^2 - X^2)^{\frac{1}{2}}}$

$f'(X) = \frac{X(a^2 - X^2)^{\frac{1}{2}} \times 0 - 1 \times \left[1(a^2 - X^2)^{\frac{1}{2}} + X(-X)(a^2 - X^2)^{-\frac{1}{2}}\right]}{X^2(a^2 - X^2)}$

SETTING $f'(X) = 0$ (ONLY THE NUMERATOR CAN EQUAL ZERO)

$\Rightarrow (a^2 - X^2)^{\frac{1}{2}} - X^2(a^2 - X^2)^{-\frac{1}{2}} = 0$

$\Rightarrow (a^2 - X^2)^{-\frac{1}{2}}\left[(a^2 - X^2)^1 - X^2\right] = 0$

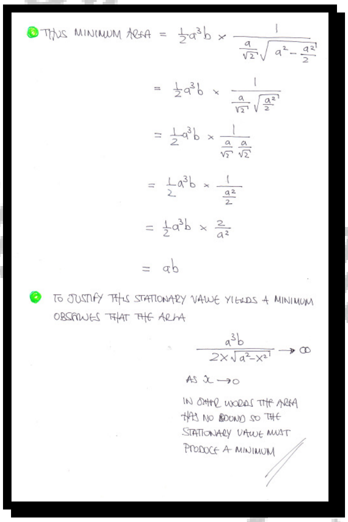
$\Rightarrow \frac{a^2 - 2X^2}{\sqrt{a^2 - X^2}} = 0$

$\Rightarrow a^2 - 2X^2 = 0$

$\Rightarrow 2X^2 = a^2$

$\Rightarrow X^2 = \frac{a^2}{2}$

$\Rightarrow X = +\frac{a}{\sqrt{2}}$



THUS MINIMUM AREA $= \frac{1}{2}a^3b \times \frac{1}{\frac{a}{\sqrt{2}}\sqrt{a^2 - \frac{a^2}{2}}}$

$= \frac{1}{2}a^3b \times \frac{1}{\frac{a}{\sqrt{2}}\sqrt{\frac{a^2}{2}}}$

$= \frac{1}{2}a^3b \times \frac{1}{\frac{a}{\sqrt{2}}\frac{a}{\sqrt{2}}}$

$= \frac{1}{2}a^3b \times \frac{1}{\frac{a^2}{2}}$

$= \frac{1}{2}a^3b \times \frac{2}{a^2}$

$= ab$

TO JUSTIFY THIS STATIONARY VALUE YIELDS A MINIMUM OBSERVES THAT THE AREA

$\frac{a^3b}{2X\sqrt{a^2 - X^2}} \rightarrow \infty$

AS $x \rightarrow 0$

IN OTHER WORDS THE AREA
HAS NO BOUND SO THE
STATIONARY VALUE MUST
PRODUCE A MINIMUM

Question 256 **(*****)**

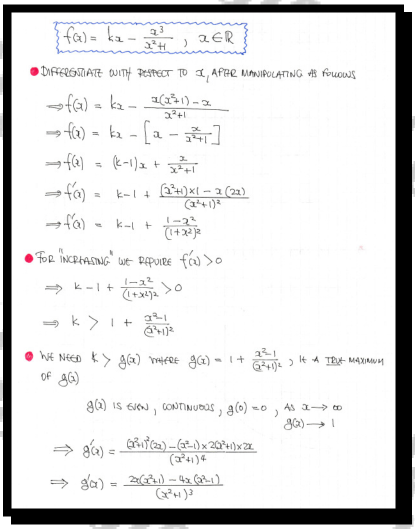
The function f is defined as

$$f(x) \equiv kx - \frac{x^3}{x^2+1}, \ x \in \mathbb{R},$$

where k is a positive constant.

Given that f is increasing for $x \in \mathbb{R}$, show that $k > \frac{9}{8}$ and hence sketch the graph of f, showing clearly the behaviour of f at $\pm\sqrt{2}$.

$\boxed{}$, $\boxed{\text{graph}}$



$$\boxed{f(x) = kx - \frac{x^3}{x^2+1}, \ x \in \mathbb{R}}$$

- DIFFERENTIATE WITH RESPECT TO x, AFTER MANIPULATING AS FOLLOWS

$$\Rightarrow f(x) = kx - \frac{x(x^2+1)-x}{x^2+1}$$

$$\Rightarrow f(x) = kx - \left[x - \frac{x}{x^2+1}\right]$$

$$\Rightarrow f(x) = (k-1)x + \frac{x}{x^2+1}$$

$$\Rightarrow f'(x) = k-1 + \frac{(x^2+1)\times 1 - x(2x)}{(x^2+1)^2}$$

$$\Rightarrow f'(x) = k-1 + \frac{1-x^2}{(1+x^2)^2}$$

- FOR "INCREASING" WE REQUIRE $f'(x) > 0$

$$\Rightarrow k-1+\frac{1-x^2}{(1+x^2)^2} > 0$$

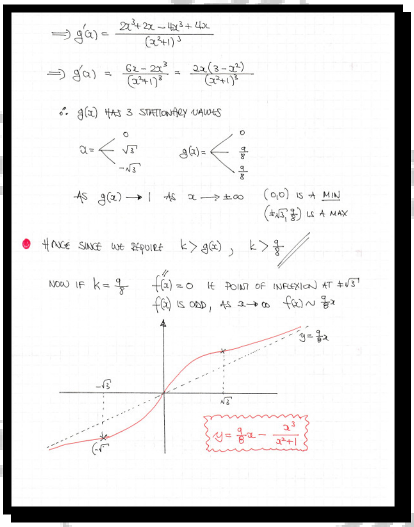
$$\Rightarrow k > 1 + \frac{x^2-1}{(x^2+1)^2}$$

- WE NEED $k > g(x)$ WHERE $g(x) = 1 + \frac{x^2-1}{(x^2+1)^2}$, IE A TRUE MAXIMUM OF $g(x)$

$g(x)$ IS EVEN, CONTINUOUS, $g(0) = 0$, AS $x \to \infty$ $g(x) \to 1$

$$\Rightarrow g'(x) = \frac{(x^2+1)^2(2x) - (x^2-1)\times 2(x^2+1)\times 2x}{(x^2+1)^4}$$

$$\Rightarrow g'(x) = \frac{2x(x^2+1) - 4x(x^2-1)}{(x^2+1)^3}$$



$$\Rightarrow g'(x) = \frac{2x^3+2x-4x^3+4x}{(x^2+1)^3}$$

$$\Rightarrow g'(x) = \frac{6x-2x^3}{(x^2+1)^3} = \frac{2x(3-x^2)}{(x^2+1)^3}$$

$\therefore$ $g(x)$ HAS 3 STATIONARY VALUES

$$x = \begin{cases} 0 \\ \sqrt{3} \\ -\sqrt{3} \end{cases} \qquad g(x) = \begin{cases} 0 \\ \frac{9}{8} \\ \frac{9}{8} \end{cases}$$

AS $g(x) \to 1$ AS $x \to \pm\infty$ $(0,0)$ IS A MIN
$(\pm\sqrt{3}, \frac{9}{8})$ IS A MAX

- HENCE SINCE WE REQUIRE $k > g(x)$, $k > \frac{9}{8}$ //

NOW IF $k = \frac{9}{8}$ $f''(x) = 0$ IE POINT OF INFLEXION AT $\pm\sqrt{3}$
$f(x)$ IS ODD, AS $x \to \infty$ $f(x) \sim \frac{9}{8}x$

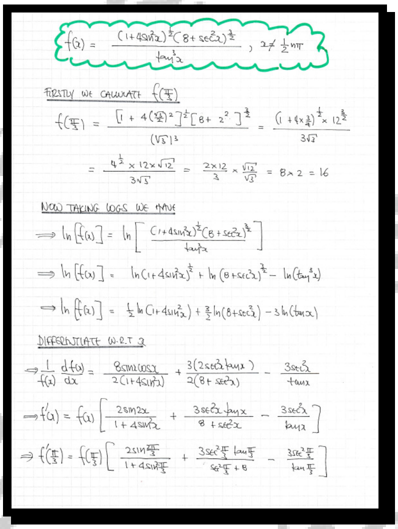
Question 257 (*****)

The function f is defined as

$$f(x) \equiv \frac{\left(1+4\sin^2 x\right)^{\frac{1}{2}}\left(8+\sec^2 x\right)^{\frac{3}{2}}}{\tan^3 x}, \quad x \in \mathbb{R}, \quad x \neq \frac{1}{2}n\pi .$$

Find in exact simplified form the value of $f'\left(\frac{1}{3}\pi\right)$.

$\boxed{f'\left(\frac{1}{3}\pi\right) = -44\sqrt{3}}$



$$f(x) = \frac{(1+4\sin^2 x)^{\frac{1}{2}}(8+\sec^2 x)^{\frac{3}{2}}}{\tan^3 x}, \quad x \neq \frac{1}{2}n\pi$$

FIRSTLY WE CALCULATE $f\left(\frac{\pi}{3}\right)$

$$f\left(\frac{\pi}{3}\right) = \frac{\left[1+4\left(\frac{\sqrt{3}}{2}\right)^2\right]^{\frac{1}{2}}\left[8+2^2\right]^{\frac{3}{2}}}{(\sqrt{3})^3} = \frac{\left(1+4\times\frac{3}{4}\right)^{\frac{1}{2}}\times 12^{\frac{3}{2}}}{3\sqrt{3}}$$

$$= \frac{4^{\frac{1}{2}}\times 12\times\sqrt{12}}{3\sqrt{3}} = \frac{2\times 12}{3}\times\frac{\sqrt{12}}{\sqrt{3}} = 8\times 2 = 16$$

NOW TAKING LOGS WE HAVE

$$\Longrightarrow \ln[f(x)] = \ln\left[\frac{(1+4\sin^2 x)^{\frac{1}{2}}(8+\sec^2 x)^{\frac{3}{2}}}{\tan^3 x}\right]$$

$$\Longrightarrow \ln[f(x)] = \ln(1+4\sin^2 x)^{\frac{1}{2}} + \ln(8+\sec^2 x)^{\frac{3}{2}} - \ln(\tan^3 x)$$

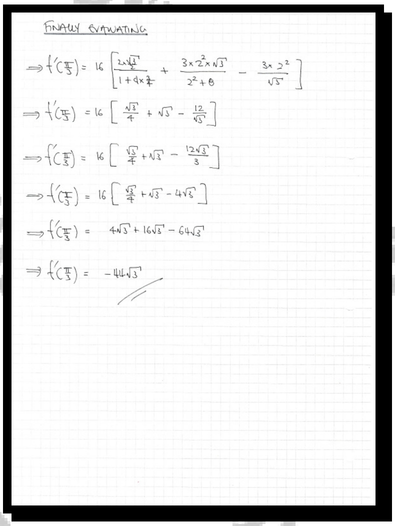
$$\Longrightarrow \ln[f(x)] = \frac{1}{2}\ln(1+4\sin^2 x) + \frac{3}{2}\ln(8+\sec^2 x) - 3\ln(\tan x)$$

DIFFERENTIATE W.R.T x

$$\Longrightarrow \frac{1}{f(x)}\frac{df(x)}{dx} = \frac{8\sin x\cos x}{2(1+4\sin^2 x)} + \frac{3(2\sec^2 x\tan x)}{2(8+\sec^2 x)} - \frac{3\sec^2 x}{\tan x}$$

$$\Longrightarrow f'(x) = f(x)\left[\frac{2\sin 2x}{1+4\sin^2 x} + \frac{3\sec^2 x\tan x}{8+\sec^2 x} - \frac{3\sec^2 x}{\tan x}\right]$$

$$\Longrightarrow f'\left(\frac{\pi}{3}\right) = f\left(\frac{\pi}{3}\right)\left[\frac{2\sin\frac{2\pi}{3}}{1+4\sin^2\frac{\pi}{3}} + \frac{3\sec^2\frac{\pi}{3}\tan\frac{\pi}{3}}{\sec^2\frac{\pi}{3}+8} - \frac{3\sec^2\frac{\pi}{3}}{\tan\frac{\pi}{3}}\right]$$



FINALLY EVALUATING

$$\Longrightarrow f'\left(\frac{\pi}{3}\right) = 16\left[\frac{2\times\frac{\sqrt{3}}{2}}{1+4\times\frac{3}{4}} + \frac{3\times 2^2\times\sqrt{3}}{2^2+8} - \frac{3\times 2^2}{\sqrt{3}}\right]$$

$$\Longrightarrow f'\left(\frac{\pi}{3}\right) = 16\left[\frac{\sqrt{3}}{4} + \sqrt{3} - \frac{12}{\sqrt{3}}\right]$$

$$\Longrightarrow f'\left(\frac{\pi}{3}\right) = 16\left[\frac{\sqrt{3}}{4} + \sqrt{3} - \frac{12\sqrt{3}}{3}\right]$$

$$\Longrightarrow f'\left(\frac{\pi}{3}\right) = 16\left[\frac{\sqrt{3}}{4} + \sqrt{3} - 4\sqrt{3}\right]$$

$$\Longrightarrow f'\left(\frac{\pi}{3}\right) = 4\sqrt{3} + 16\sqrt{3} - 64\sqrt{3}$$

$$\Longrightarrow f'\left(\frac{\pi}{3}\right) = -44\sqrt{3}$$

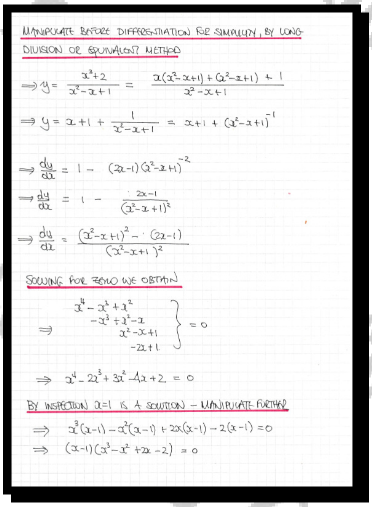
Question 258 (*****)

The curve C has equation

$$y = \frac{x^3+2}{x^2-x+1}.$$

Find the coordinates of the stationary point of C and determine its nature.

$\boxed{}$, $\boxed{\text{point of inflexion at } (1,3)}$



MANIPULATE BEFORE DIFFERENTIATION FOR SIMPLICITY, BY LONG DIVISION OR EQUIVALENT METHOD

$$\Rightarrow y = \frac{x^3+2}{x^2-x+1} = \frac{x(x^2-x+1)+(x^2-x+1)+1}{x^2-x+1}$$

$$\Rightarrow y = x+1+\frac{1}{x^2-x+1} = x+1+(x^2-x+1)^{-1}$$

$$\Rightarrow \frac{dy}{dx} = 1-(2x-1)(x^2-x+1)^{-2}$$

$$\Rightarrow \frac{dy}{dx} = 1-\frac{2x-1}{(x^2-x+1)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(x^2-x+1)^2-(2x-1)}{(x^2-x+1)^2}$$

SOLVING FOR ZERO WE OBTAIN

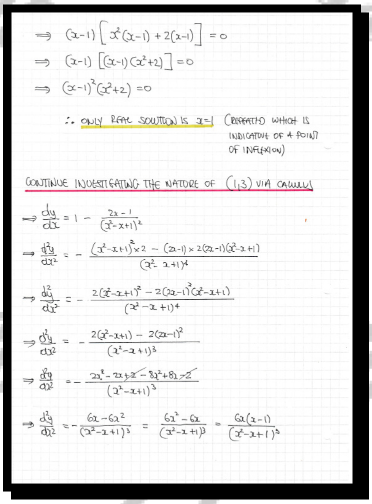
$$\Rightarrow \left.\begin{array}{r} x^4-x^3+x^2 \\ -x^3+x^2-x \\ x^2-x+1 \\ -2x+1 \end{array}\right\} = 0$$

$$\Rightarrow x^4-2x^3+3x^2-4x+2 = 0$$

BY INSPECTION $x=1$ IS A SOLUTION – MANIPULATE FURTHER

$$\Rightarrow x^3(x-1)-x^2(x-1)+2x(x-1)-2(x-1)=0$$

$$\Rightarrow (x-1)(x^3-x^2+2x-2)=0$$



$$\Rightarrow (x-1)\left[x^2(x-1)+2(x-1)\right]=0$$

$$\Rightarrow (x-1)\left[(x-1)(x^2+2)\right]=0$$

$$\Rightarrow (x-1)^2(x^2+2)=0$$

$\therefore$ ONLY REAL SOLUTION IS $x=1$ (REPEATED WHICH IS INDICATIVE OF A POINT OF INFLEXION)

CONTINUE INVESTIGATING THE NATURE OF $(1,3)$ VIA CALCULUS

$$\Rightarrow \frac{dy}{dx} = 1-\frac{2x-1}{(x^2-x+1)^2}$$

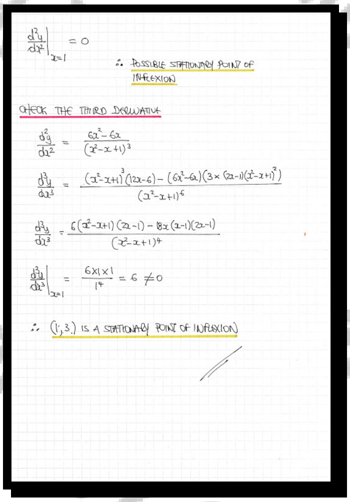
$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{(x^2-x+1)^2\times 2-(2x-1)\times 2(2x-1)(x^2-x+1)}{(x^2-x+1)^4}$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{2(x^2-x+1)^2-2(2x-1)^2(x^2-x+1)}{(x^2-x+1)^4}$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{2(x^2-x+1)-2(2x-1)^2}{(x^2-x+1)^3}$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{2x^2-2x\cancel{+2}-8x^2+8x\cancel{-2}}{(x^2-x+1)^3}$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{6x-6x^2}{(x^2-x+1)^3} = \frac{6x^2-6x}{(x^2-x+1)^3} = \frac{6x(x-1)}{(x^2-x+1)^3}$$



$$\left.\frac{d^2y}{dx^2}\right|_{x=1} = 0$$

$\therefore$ POSSIBLE STATIONARY POINT OF INFLEXION

CHECK THE THIRD DERIVATIVE

$$\frac{d^2y}{dx^2} = \frac{6x^2-6x}{(x^2-x+1)^3}$$

$$\frac{d^3y}{dx^3} = \frac{(x^2-x+1)^3(12x-6)-(6x^2-6x)(3\times(2x-1)(x^2-x+1)^2)}{(x^2-x+1)^6}$$

$$\frac{d^3y}{dx^3} = \frac{6(x^2-x+1)(2x-1)-18x(x-1)(2x-1)}{(x^2-x+1)^4}$$

$$\left.\frac{d^3y}{dx^3}\right|_{x=1} = \frac{6\times1\times1}{1^4} = 6 \neq 0$$

$\therefore$ $(1,3)$ IS A STATIONARY POINT OF INFLEXION

Question 259 (*****)

A curve C is defined in the largest real domain by the equation

$$y = \log_x 2 .$$

a) Sketch a detailed graph of C.

The point P, where $x = 2$ lies on C.

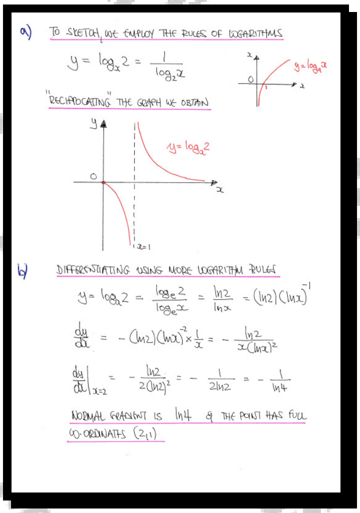
The normal to C at P meets C again at the point Q.

b) Show that the x coordinate of Q is a solution of the equation

$$[1 + x\ln 4 - \ln 16]\ln x = \ln 2 .$$

c) Use an iterative formula of the form $x_{n+1} = e^{f(x_n)}$, with a suitable starting value, to find the coordinates of Q, correct to 3 decimal places.

$\boxed{\quad}$, $\boxed{Q(0.518,-1.054)}$



a) TO SKETCH, WE EMPLOY THE RULES OF LOGARITHMS

$$y = \log_x 2 = \frac{1}{\log_2 x}$$

"RECIPROCATING" THE GRAPH WE OBTAIN

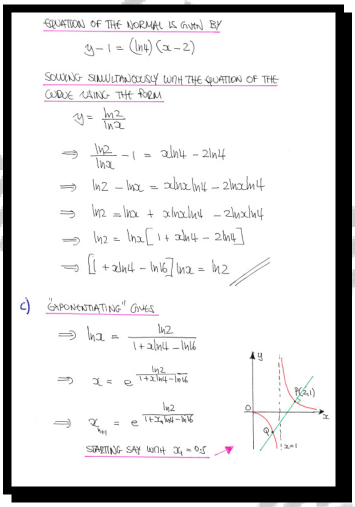
b) DIFFERENTIATING USING MORE LOGARITHM RULES

$$y = \log_x 2 = \frac{\log_e 2}{\log_e x} = \frac{\ln 2}{\ln x} = (\ln 2)(\ln x)^{-1}$$

$$\frac{dy}{dx} = -(\ln 2)(\ln x)^{-2} \times \frac{1}{x} = -\frac{\ln 2}{x(\ln x)^2}$$

$$\left.\frac{dy}{dx}\right|_{x=2} = -\frac{\ln 2}{2(\ln 2)^2} = -\frac{1}{2\ln 2} = -\frac{1}{\ln 4}$$

NORMAL GRADIENT IS $\ln 4$ & THE POINT HAS FULL CO-ORDINATES $(2,1)$



EQUATION OF THE NORMAL IS GIVEN BY

$$y - 1 = (\ln 4)(x - 2)$$

SOLVING SIMULTANEOUSLY WITH THE EQUATION OF THE CURVE USING THE FORM

$$y = \frac{\ln 2}{\ln x}$$

$$\Rightarrow \frac{\ln 2}{\ln x} - 1 = x\ln 4 - 2\ln 4$$

$$\Rightarrow \ln 2 - \ln x = x\ln x\ln 4 - 2\ln x\ln 4$$

$$\Rightarrow \ln 2 = \ln x + x\ln x\ln 4 - 2\ln x\ln 4$$

$$\Rightarrow \ln 2 = \ln x[1 + x\ln 4 - 2\ln 4]$$

$$\Rightarrow [1 + x\ln 4 - \ln 16]\ln x = \ln 2$$

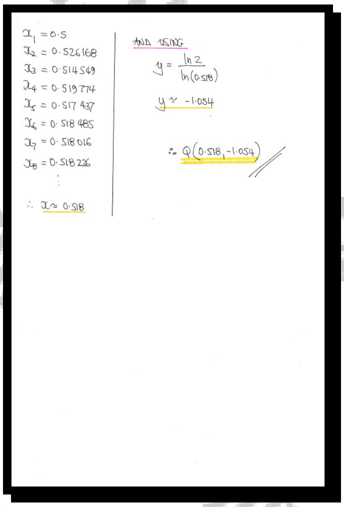
c) "EXPONENTIATING" GIVES

$$\Rightarrow \ln x = \frac{\ln 2}{1 + x\ln 4 - \ln 16}$$

$$\Rightarrow x = e^{\frac{\ln 2}{1 + x\ln 4 - \ln 16}}$$

$$\Rightarrow x_{n+1} = e^{\frac{\ln 2}{1 + x_n\ln 4 - \ln 16}}$$

STARTING SAY WITH $x_1 = 0.5$



$x_1 = 0.5$
$x_2 = 0.526168$
$x_3 = 0.514549$
$x_4 = 0.519774$
$x_5 = 0.517437$
$x_6 = 0.518485$
$x_7 = 0.518016$
$x_8 = 0.518226$
$\vdots$

$\therefore x \simeq 0.518$

AND USING

$$y = \frac{\ln 2}{\ln(0.518)}$$

$y \simeq -1.054$

$\therefore Q(0.518, -1.054)$

Question 260 (*****)

A sphere of radius r, whose centre is at O, is fixed on a horizontal plane.

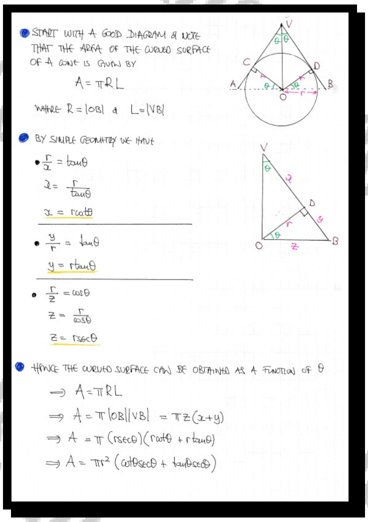
A thin right circular conical shell, **without** a base is placed over the sphere.

The axis of the conical shell is vertical and passes through O. The circumference of the missing base of the conical shell is at the same horizontal level as O.

Show that the minimum value of the outer surface area of the conical shell is

$$\frac{3\sqrt{3}\pi r^2}{2}.$$

☐, proof



● START WITH A GOOD DIAGRAM & NOTE THAT THE AREA OF THE CURVED SURFACE OF A CONE IS GIVEN BY

$A = \pi R L$

WHERE $R = |OB|$ & $L = |VB|$

● BY SIMPLE GEOMETRY WE HAVE

- $\frac{r}{x} = \tan\theta$

  $x = \frac{r}{\tan\theta}$

  $\underline{x = r\cot\theta}$

- $\frac{y}{r} = \tan\theta$

  $\underline{y = r\tan\theta}$

- $\frac{r}{z} = \cos\theta$

  $z = \frac{r}{\cos\theta}$

  $\underline{z = r\sec\theta}$

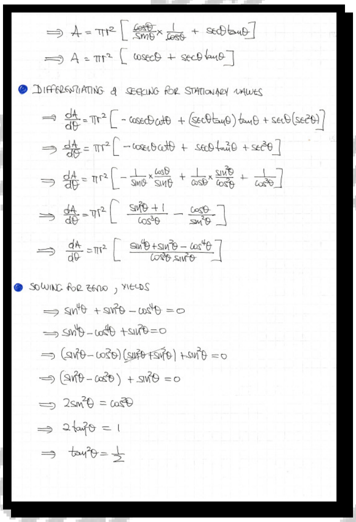
● HENCE THE CURVED SURFACE CAN BE OBTAINED AS A FUNCTION OF θ

$\Rightarrow A = \pi R L$

$\Rightarrow A = \pi |OB||VB| = \pi z(x+y)$

$\Rightarrow A = \pi(r\sec\theta)(r\cot\theta + r\tan\theta)$

$\Rightarrow A = \pi r^2(\cot\theta\sec\theta + \tan\theta\sec\theta)$



$\Rightarrow A = \pi r^2\left[\frac{\cos\theta}{\sin\theta}\times\frac{1}{\cos\theta} + \sec\theta\tan\theta\right]$

$\Rightarrow A = \pi r^2\left[\operatorname{cosec}\theta + \sec\theta\tan\theta\right]$

● DIFFERENTIATING & SEEKING FOR STATIONARY VALUES

$\Rightarrow \frac{dA}{d\theta} = \pi r^2\left[-\operatorname{cosec}\theta\cot\theta + (\sec\theta\tan\theta)\tan\theta + \sec\theta(\sec^2\theta)\right]$

$\Rightarrow \frac{dA}{d\theta} = \pi r^2\left[-\operatorname{cosec}\theta\cot\theta + \sec\theta\tan^2\theta + \sec^3\theta\right]$

$\Rightarrow \frac{dA}{d\theta} = \pi r^2\left[-\frac{1}{\sin\theta}\times\frac{\cos\theta}{\sin\theta} + \frac{1}{\cos\theta}\times\frac{\sin^2\theta}{\cos^2\theta} + \frac{1}{\cos^3\theta}\right]$

$\Rightarrow \frac{dA}{d\theta} = \pi r^2\left[\frac{\sin^2\theta + 1}{\cos^3\theta} - \frac{\cos\theta}{\sin^2\theta}\right]$

$\Rightarrow \frac{dA}{d\theta} = \pi r^2\left[\frac{\sin^4\theta + \sin^2\theta - \cos^4\theta}{\cos^3\theta\sin^2\theta}\right]$

● SOLVING FOR ZERO, YIELDS

$\Rightarrow \sin^4\theta + \sin^2\theta - \cos^4\theta = 0$

$\Rightarrow \sin^4\theta - \cos^4\theta + \sin^2\theta = 0$

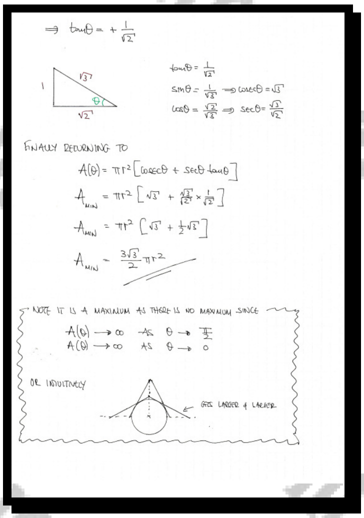
$\Rightarrow (\sin^2\theta - \cos^2\theta)\cancel{(\sin^2\theta + \sin^2\theta)} + \sin^2\theta = 0$

$\Rightarrow (\sin^2\theta - \cos^2\theta) + \sin^2\theta = 0$

$\Rightarrow 2\sin^2\theta = \cos^2\theta$

$\Rightarrow 2\tan^2\theta = 1$

$\Rightarrow \tan^2\theta = \frac{1}{2}$



$\Rightarrow \tan\theta = +\frac{1}{\sqrt{2}}$

$\tan\theta = \frac{1}{\sqrt{2}}$

$\sin\theta = \frac{1}{\sqrt{3}} \Rightarrow \operatorname{cosec}\theta = \sqrt{3}$

$\cos\theta = \frac{\sqrt{2}}{\sqrt{3}} \Rightarrow \sec\theta = \frac{\sqrt{3}}{\sqrt{2}}$

FINALLY RETURNING TO

$A(\theta) = \pi r^2\left[\operatorname{cosec}\theta + \sec\theta\tan\theta\right]$

$A_{MIN} = \pi r^2\left[\sqrt{3} + \frac{\sqrt{3}}{\sqrt{2}}\times\frac{1}{\sqrt{2}}\right]$

$A_{MIN} = \pi r^2\left[\sqrt{3} + \frac{1}{2}\sqrt{3}\right]$

$A_{MIN} = \frac{3\sqrt{3}}{2}\pi r^2$

NOTE IT IS A MAXIMUM AS THERE IS NO MAXIMUM SINCE

$A(\theta) \to \infty$ AS $\theta \to \frac{\pi}{2}$

$A(\theta) \to \infty$ AS $\theta \to 0$

OR INTUITIVELY

Question 261 (*****)

The function $y = f(x)$ satisfies the following relationship.

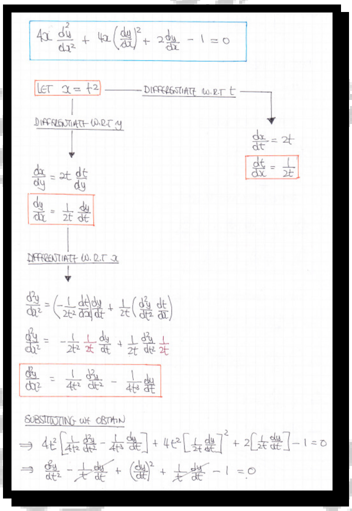
$$4x\frac{d^2y}{dx^2} + 4x\left(\frac{dy}{dx}\right)^2 + 2\frac{dy}{dx} - 1 = 0.$$

It is further given that $x = t^2$ and $y = \ln v$.

Show that

$$\frac{d^2v}{dt^2} = v.$$

SP W, proof



$$\boxed{4x\frac{d^2y}{dx^2} + 4x\left(\frac{dy}{dx}\right)^2 + 2\frac{dy}{dx} - 1 = 0}$$

$\boxed{\text{LET } x = t^2}$ — DIFFERENTIATE W.R.T t

$\frac{dx}{dt} = 2t$

$\boxed{\frac{dt}{dx} = \frac{1}{2t}}$

DIFFERENTIATE W.R.T y

$\frac{dx}{dy} = 2t\frac{dt}{dy}$

$\boxed{\frac{dy}{dx} = \frac{1}{2t}\frac{dy}{dt}}$

DIFFERENTIATE W.R.T x

$\frac{d^2y}{dx^2} = \left(-\frac{1}{2t^2}\frac{dt}{dx}\right)\frac{dy}{dt} + \frac{1}{2t}\left(\frac{d^2y}{dt^2}\frac{dt}{dx}\right)$

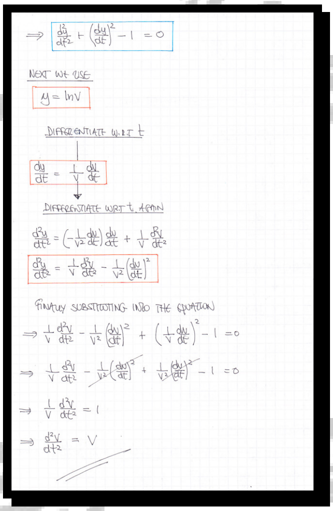
$\frac{d^2y}{dx^2} = -\frac{1}{2t^2}\frac{1}{2t}\frac{dy}{dt} + \frac{1}{2t}\frac{d^2y}{dt^2}\frac{1}{2t}$

$\boxed{\frac{d^2y}{dx^2} = \frac{1}{4t^2}\frac{d^2y}{dt^2} - \frac{1}{4t^3}\frac{dy}{dt}}$

SUBSTITUTING WE OBTAIN

$\Rightarrow 4t^2\left[\frac{1}{4t^2}\frac{d^2y}{dt^2} - \frac{1}{4t^3}\frac{dy}{dt}\right] + 4t^2\left[\frac{1}{2t}\frac{dy}{dt}\right]^2 + 2\left[\frac{1}{2t}\frac{dy}{dt}\right] - 1 = 0$

$\Rightarrow \frac{d^2y}{dt^2} - \cancel{\frac{1}{t}\frac{dy}{dt}} + \left(\frac{dy}{dt}\right)^2 + \cancel{\frac{1}{t}\frac{dy}{dt}} - 1 = 0$



$\Rightarrow \boxed{\frac{d^2y}{dt^2} + \left(\frac{dy}{dt}\right)^2 - 1 = 0}$

NEXT WE USE

$\boxed{y = \ln v}$

DIFFERENTIATE W.R.T t

$\boxed{\frac{dy}{dt} = \frac{1}{v}\frac{dv}{dt}}$

DIFFERENTIATE W.R.T t, AGAIN

$\frac{d^2y}{dt^2} = \left(-\frac{1}{v^2}\frac{dv}{dt}\right)\frac{dv}{dt} + \frac{1}{v}\frac{d^2v}{dt^2}$

$\boxed{\frac{d^2y}{dt^2} = \frac{1}{v}\frac{d^2v}{dt^2} - \frac{1}{v^2}\left(\frac{dv}{dt}\right)^2}$

FINALLY SUBSTITUTING INTO THE EQUATION

$\Rightarrow \frac{1}{v}\frac{d^2v}{dt^2} - \frac{1}{v^2}\left(\frac{dv}{dt}\right)^2 + \left(\frac{1}{v}\frac{dv}{dt}\right)^2 - 1 = 0$

$\Rightarrow \frac{1}{v}\frac{d^2v}{dt^2} - \cancel{\frac{1}{v^2}\left(\frac{dv}{dt}\right)^2} + \cancel{\frac{1}{v^2}\left(\frac{dv}{dt}\right)^2} - 1 = 0$

$\Rightarrow \frac{1}{v}\frac{d^2v}{dt^2} = 1$

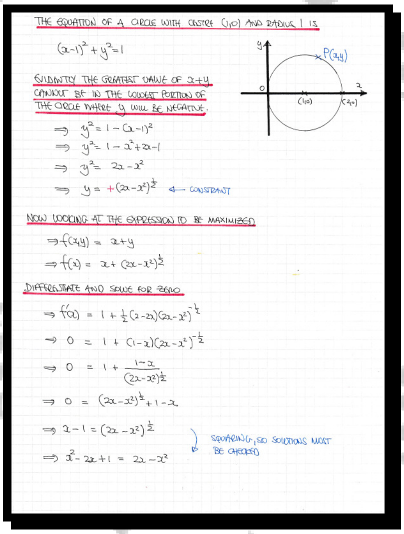
$\Rightarrow \frac{d^2v}{dt^2} = v$

Question 262 (*****)

The point $P(x,y)$ lies on a circle with centre at $(1,0)$ and radius 1.

Find, in exact surd form, the greatest value of $x+y$, for all the possible positions of the point P.

$\boxed{\text{MP2-S}}$, $\boxed{1+\sqrt{2}}$



THE EQUATION OF A CIRCLE WITH CENTRE (1,0) AND RADIUS 1 IS

$(x-1)^2+y^2=1$

EVIDENTLY THE GREATEST VALUE OF $x+y$ CANNOT BE IN THE LOWEST PORTION OF THE CIRCLE WHERE y WILL BE NEGATIVE.

$\Rightarrow y^2=1-(x-1)^2$

$\Rightarrow y^2=1-x^2+2x-1$

$\Rightarrow y^2=2x-x^2$

$\Rightarrow y=+(2x-x^2)^{\frac{1}{2}}$ ← CONSTRAINT

NOW LOOKING AT THE EXPRESSION TO BE MAXIMIZED

$\Rightarrow f(x,y)=x+y$

$\Rightarrow f(x)=x+(2x-x^2)^{\frac{1}{2}}$

DIFFERENTIATE AND SOLVE FOR ZERO

$\Rightarrow f'(x)=1+\frac{1}{2}(2-2x)(2x-x^2)^{-\frac{1}{2}}$

$\Rightarrow 0=1+(1-x)(2x-x^2)^{-\frac{1}{2}}$

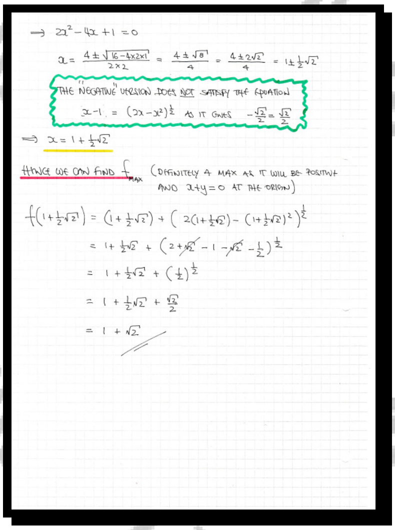
$\Rightarrow 0=1+\frac{1-x}{(2x-x^2)^{\frac{1}{2}}}$

$\Rightarrow 0=(2x-x^2)^{\frac{1}{2}}+1-x$

$\Rightarrow x-1=(2x-x^2)^{\frac{1}{2}}$

SQUARING, SO SOLUTIONS MUST BE CHECKED

$\Rightarrow x^2-2x+1=2x-x^2$



$\Rightarrow 2x^2-4x+1=0$

$x=\frac{4\pm\sqrt{16-4\times2\times1}}{2\times2}=\frac{4\pm\sqrt{8}}{4}=\frac{4\pm2\sqrt{2}}{4}=1\pm\frac{1}{2}\sqrt{2}$

"NEGATIVE" VERSION DOES NOT SATISFY THE EQUATION

$x-1=(2x-x^2)^{\frac{1}{2}}$ AS IT GIVES $-\frac{\sqrt{2}}{2}=\frac{\sqrt{2}}{2}$

$\Rightarrow x=1+\frac{1}{2}\sqrt{2}$

HENCE WE CAN FIND f_{MAX} (DEFINITELY A MAX AS IT WILL BE POSITIVE AND $x+y=0$ AT THE ORIGIN)

$f\left(1+\frac{1}{2}\sqrt{2}\right)=\left(1+\frac{1}{2}\sqrt{2}\right)+\left(2\left(1+\frac{1}{2}\sqrt{2}\right)-\left(1+\frac{1}{2}\sqrt{2}\right)^2\right)^{\frac{1}{2}}$

$=1+\frac{1}{2}\sqrt{2}+\left(2+\cancel{\sqrt{2}}-1-\cancel{\sqrt{2}}-\frac{1}{2}\right)^{\frac{1}{2}}$

$=1+\frac{1}{2}\sqrt{2}+\left(\frac{1}{2}\right)^{\frac{1}{2}}$

$=1+\frac{1}{2}\sqrt{2}+\frac{\sqrt{2}}{2}$

$=1+\sqrt{2}$ //

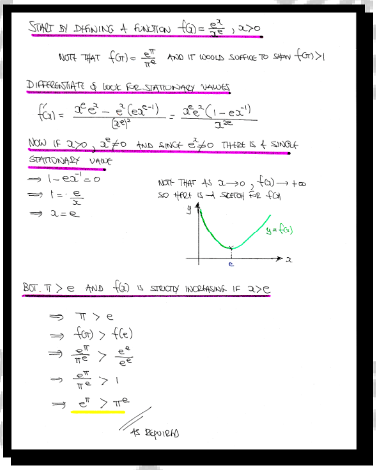
Question 263 (*****)

Assuming that $\pi \approx 3.14$ and $\mathrm{e} \approx 2.7$, show **without** any calculating aid that

$$\mathrm{e}^{\pi} > \pi^{\mathrm{e}} .$$

You must show a detailed method in this question.

☐, $\boxed{\text{proof}}$



START BY DEFINING A FUNCTION $f(x) = \frac{e^x}{x^e}$, $x>0$

NOTE THAT $f(\pi) = \frac{e^\pi}{\pi^e}$ AND IT WOULD SUFFICE TO SHOW $f(\pi)>1$

DIFFERENTIATE & LOOK FOR STATIONARY VALUES

$$f'(x) = \frac{x^e e^x - e^x(ex^{e-1})}{(x^e)^2} = \frac{x^e e^x(1-ex^{-1})}{x^{2e}}$$

NOW IF $x>0$, $x^e \neq 0$ AND SINCE $e^x \neq 0$ THERE IS A SINGLE STATIONARY VALUE

$\Rightarrow 1 - ex^{-1} = 0$

$\Rightarrow 1 = \frac{e}{x}$

$\Rightarrow x = e$

NOTE THAT AS $x \to 0$, $f(x) \to +\infty$ SO HERE IS A SKETCH FOR $f(x)$

BUT $\pi > e$ AND $f(x)$ IS STRICTLY INCREASING IF $x>e$

$\Rightarrow \pi > e$

$\Rightarrow f(\pi) > f(e)$

$\Rightarrow \frac{e^\pi}{\pi^e} > \frac{e^e}{e^e}$

$\Rightarrow \frac{e^\pi}{\pi^e} > 1$

$\Rightarrow e^\pi > \pi^e$

// AS REQUIRED

Question 264 (*****)

A general curve C has equation

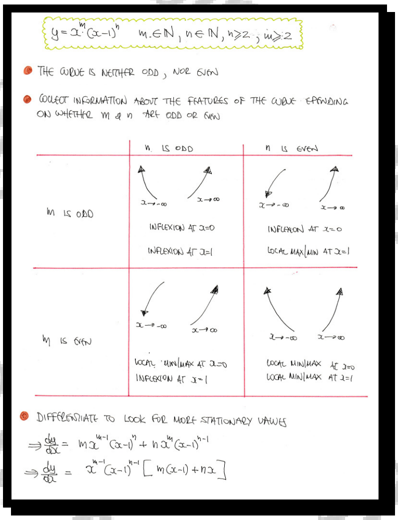
$$y = x^m (x-1)^n ,$$

where $x \in \mathbb{R}$, $m \in \mathbb{N}$, $m \geq 2$, $n \in \mathbb{N}$, $n \geq 2$.

Sketch in four separate of axes, the 4 separate shapes which C can take, $m \geq 2$.

The sketches must contain the coordinates of any stationary points.

☐ , graph



$\boxed{y = x^m (x-1)^n \quad m \in \mathbb{N},\ n \in \mathbb{N},\ n \geqslant 2,\ m \geqslant 2}$

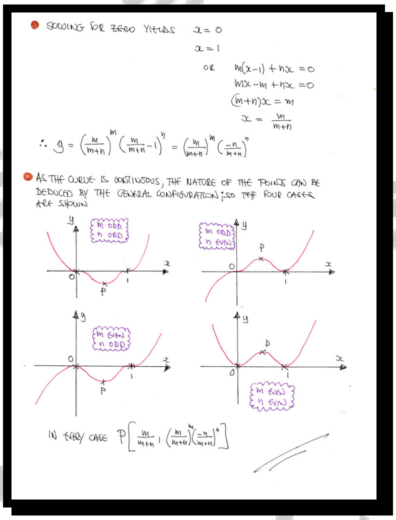
- THE CURVE IS NEITHER ODD, NOR EVEN
- COLLECT INFORMATION ABOUT THE FEATURES OF THE CURVE DEPENDING ON WHETHER m & n ARE ODD OR EVEN

	n IS ODD	n IS EVEN
m IS ODD	$x \to -\infty$ $x \to \infty$ INFLEXION AT $x=0$ INFLEXION AT $x=1$	$x \to -\infty$ $x \to \infty$ INFLEXION AT $x=0$ LOCAL MAX/MIN AT $x=1$
m IS EVEN	$x \to -\infty$ $x \to \infty$ LOCAL MIN/MAX AT $x=0$ INFLEXION AT $x=1$	$x \to -\infty$ $x \to \infty$ LOCAL MIN/MAX AT $x=0$ LOCAL MIN/MAX AT $x=1$

- DIFFERENTIATE TO LOOK FOR MORE STATIONARY VALUES

$$\Rightarrow \frac{dy}{dx} = m x^{m-1}(x-1)^n + n x^m (x-1)^{n-1}$$

$$\Rightarrow \frac{dy}{dx} = x^{m-1}(x-1)^{n-1}\left[m(x-1) + nx \right]$$



- SOLVING FOR ZERO YIELDS $x = 0$

$x = 1$

OR $m(x-1) + nx = 0$

$mx - m + nx = 0$

$(m+n)x = m$

$x = \frac{m}{m+n}$

$$\therefore y = \left(\frac{m}{m+n}\right)^m \left(\frac{m}{m+n} - 1\right)^n = \left(\frac{m}{m+n}\right)^m \left(\frac{-n}{m+n}\right)^n$$

- AS THE CURVE IS CONTINOOUS, THE NATURE OF THE POINTS CAN BE DEDUCED BY THE GENERAL CONFIGURATION; SO THE FOUR CASES ARE SHOWN

IN EVERY CASE $P\left[\frac{m}{m+n}, \left(\frac{m}{m+n}\right)^m \left(\frac{-n}{m+n}\right)^n\right]$

Question 265 (*****)

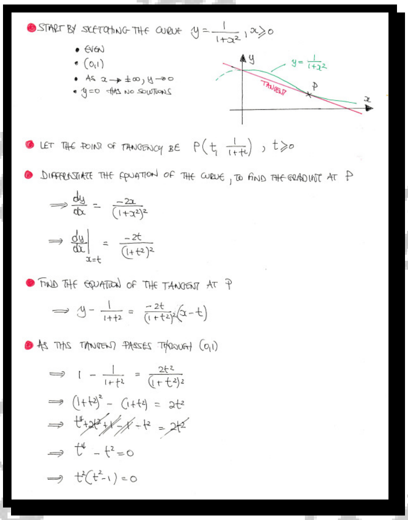
The point P lies on the curve C with equation

$$y = \frac{1}{1+x^2}, \quad x \in \mathbb{R}, \quad x \geq 0.$$

The straight line L is the tangent to the C at P.

Determine an equation for L, given further that L meets C at the point $(0,1)$.

$\boxed{}$, $\boxed{2y + x = 2}$



● START BY SKETCHING THE CURVE $y = \frac{1}{1+x^2}$, $x \geq 0$

- EVEN
- $(0,1)$
- AS $x \to \pm\infty$, $y \to 0$
- $y = 0$ HAS NO SOLUTIONS

● LET THE POINT OF TANGENCY BE $P\left(t, \frac{1}{1+t^2}\right)$, $t \geq 0$

● DIFFERENTIATE THE EQUATION OF THE CURVE, TO FIND THE GRADIENT AT P

$$\Rightarrow \frac{dy}{dx} = \frac{-2x}{(1+x^2)^2}$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{x=t} = \frac{-2t}{(1+t^2)^2}$$

● FIND THE EQUATION OF THE TANGENT AT P

$$\Rightarrow y - \frac{1}{1+t^2} = \frac{-2t}{(1+t^2)^2}(x-t)$$

● AS THIS TANGENT PASSES THROUGH $(0,1)$

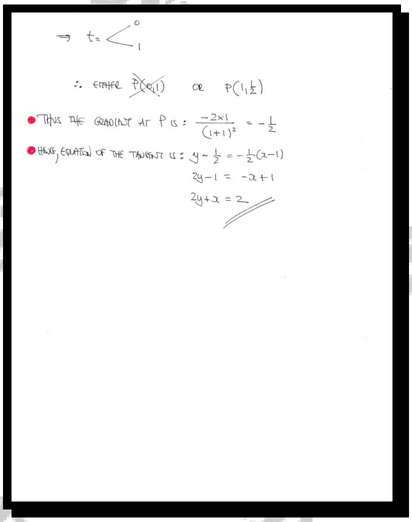
$$\Rightarrow 1 - \frac{1}{1+t^2} = \frac{2t^2}{(1+t^2)^2}$$

$$\Rightarrow (1+t^2)^2 - (1+t^2) = 2t^2$$

$$\Rightarrow t^4 + \cancel{2t^2} + \cancel{1} - \cancel{1} - t^2 = \cancel{2t^2}$$

$$\Rightarrow t^4 - t^2 = 0$$

$$\Rightarrow t^2(t^2-1) = 0$$



$$\Rightarrow t = \begin{cases} 0 \\ 1 \end{cases}$$

$\therefore$ EITHER ~~$P(0,1)$~~ OR $P\left(1, \frac{1}{2}\right)$

● THUS THE GRADIENT AT P IS: $\frac{-2 \times 1}{(1+1)^2} = -\frac{1}{2}$

● HENCE, EQUATION OF THE TANGENT IS: $y - \frac{1}{2} = -\frac{1}{2}(x-1)$

$$2y - 1 = -x + 1$$

$$2y + x = 2$$

Question 266 (*****)

The point P has rational coordinates and lies on the curve C with equation

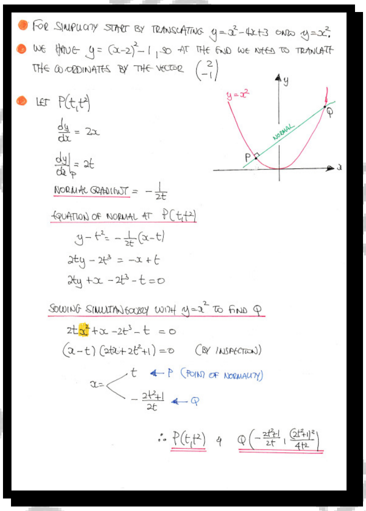
$$y = x^2 - 4x + 3, \quad x \in \mathbb{R}.$$

The straight line L is the normal to the C at P.

L meets the curve again at the point Q.

Given that $|PQ| = \sqrt{8}$, determine the possible coordinates of P and Q.

$$\boxed{}, \boxed{P\left(\frac{3}{2},-\frac{3}{4}\right) \cap Q\left(\frac{7}{2},\frac{5}{4}\right) \quad \cup \quad P\left(\frac{5}{2},-\frac{3}{4}\right) \cap Q\left(\frac{1}{2},\frac{5}{4}\right)}$$



- FOR SIMPLICITY START BY TRANSLATING $y=x^2-4x+3$ ONTO $y=x^2$.
- WE HAVE $y=(x-2)^2-1$, SO AT THE END WE NEED TO TRANSLATE THE COORDINATES BY THE VECTOR $\begin{pmatrix}2\\-1\end{pmatrix}$
- LET $P(t,t^2)$

$$\frac{dy}{dx} = 2x$$

$$\left.\frac{dy}{dx}\right|_P = 2t$$

NORMAL GRADIENT $= -\frac{1}{2t}$

EQUATION OF NORMAL AT $P(t,t^2)$

$$y-t^2 = -\frac{1}{2t}(x-t)$$

$$2ty-2t^3 = -x+t$$

$$2ty+x-2t^3-t=0$$

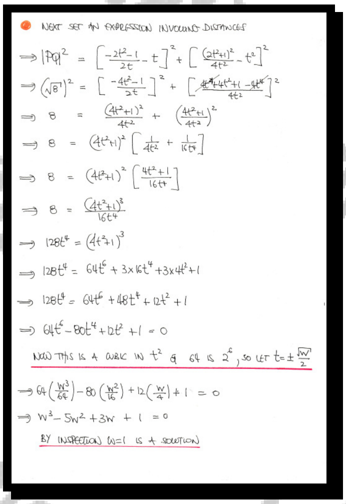
SOLVING SIMULTANEOUSLY WITH $y=x^2$ TO FIND Q

$$2tx^2+x-2t^3-t=0$$

$$(x-t)(2tx+2t^2+1)=0 \quad \text{(BY INSPECTION)}$$

$$x = \begin{cases} t & \leftarrow P \text{ (POINT OF NORMALITY)} \\ -\frac{2t^2+1}{2t} & \leftarrow Q \end{cases}$$

$$\therefore \underline{\underline{P(t,t^2)}} \ \& \ \underline{\underline{Q\left(-\frac{2t^2+1}{2t}, \frac{(2t^2+1)^2}{4t^2}\right)}}$$



- NEXT SET AN EXPRESSION INVOLVING DISTANCES

$$\Rightarrow |PQ|^2 = \left[\frac{-2t^2-1}{2t}-t\right]^2 + \left[\frac{(2t^2+1)^2}{4t^2}-t^2\right]^2$$

$$\Rightarrow (\sqrt{8})^2 = \left[\frac{-4t^2-1}{2t}\right]^2 + \left[\frac{\cancel{4t^4}+4t^2+1\cancel{-4t^4}}{4t^2}\right]^2$$

$$\Rightarrow 8 = \frac{(4t^2+1)^2}{4t^2} + \left(\frac{4t^2+1}{4t^2}\right)^2$$

$$\Rightarrow 8 = (4t^2+1)^2\left[\frac{1}{4t^2}+\frac{1}{16t^4}\right]$$

$$\Rightarrow 8 = (4t^2+1)^2\left[\frac{4t^2+1}{16t^4}\right]$$

$$\Rightarrow 8 = \frac{(4t^2+1)^3}{16t^4}$$

$$\Rightarrow 128t^4 = (4t^2+1)^3$$

$$\Rightarrow 128t^4 = 64t^6+3\times16t^4+3\times4t^2+1$$

$$\Rightarrow 128t^4 = 64t^6+48t^4+12t^2+1$$

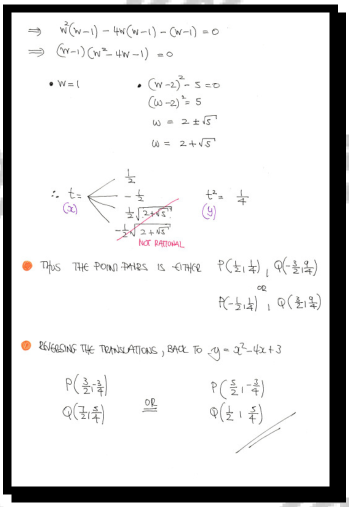
$$\Rightarrow 64t^6-80t^4+12t^2+1=0$$

NOW THIS IS A CUBIC IN t^2 & 64 IS 2^6, SO LET $t=\pm\frac{\sqrt{w}}{2}$

$$\Rightarrow 64\left(\frac{w^3}{64}\right)-80\left(\frac{w^2}{16}\right)+12\left(\frac{w}{4}\right)+1=0$$

$$\Rightarrow w^3-5w^2+3w+1=0$$

BY INSPECTION $w=1$ IS A SOLUTION



$$\Rightarrow w^2(w-1)-4w(w-1)-(w-1)=0$$

$$\Rightarrow (w-1)(w^2-4w-1)=0$$

- $w=1$
- $(w-2)^2-5=0$
  $(w-2)^2=5$
  $w=2\pm\sqrt{5}$
  $w=2+\sqrt{5}$

$$\therefore \underset{(x)}{t} = \begin{cases} \frac{1}{2} \\ -\frac{1}{2} \\ \cancel{\frac{1}{2}\sqrt{2+\sqrt{5}}} \\ \cancel{-\frac{1}{2}\sqrt{2+\sqrt{5}}} \end{cases} \qquad \underset{(y)}{t^2} = \frac{1}{4}$$

NOT RATIONAL

- THUS THE POINT PAIRS IS EITHER $P\left(\frac{1}{2},\frac{1}{4}\right)$, $Q\left(-\frac{3}{2},\frac{9}{4}\right)$
  OR
  $P\left(-\frac{1}{2},\frac{1}{4}\right)$, $Q\left(\frac{3}{2},\frac{9}{4}\right)$
- REVERSING THE TRANSLATIONS, BACK TO $y=x^2-4x+3$

$P\left(\frac{3}{2},-\frac{3}{4}\right)$ $\qquad$ $P\left(\frac{5}{2},-\frac{3}{4}\right)$
$Q\left(\frac{7}{2},\frac{5}{4}\right)$ $\quad$ OR $\quad$ $Q\left(\frac{1}{2},\frac{5}{4}\right)$

Question 267 (*****)

Leibniz rule states that the n^{th} derivative of the product of the functions $f(x)$ and $g(x)$ satisfies

$$\left[f(x)\,g(x)\right]^n = \sum_{r=0}^{n}\left[\binom{n}{r}\left[\left[f(x)\right]^{(r)}\left[f(x)\right]^{(n-r)}\right]\right],$$

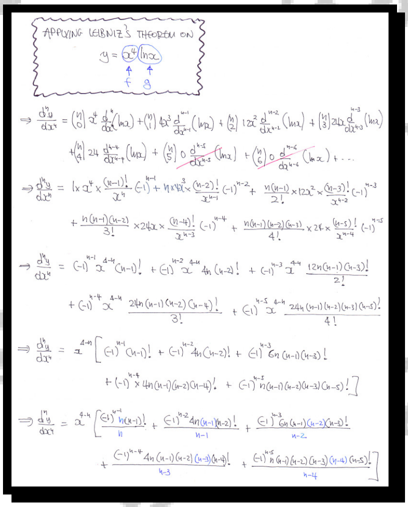
where $f^0(x) = f(x)$, $f^1(x) = f'(x)$, $f^2(x) = f''(x)$, … , $f^k(x) = \dfrac{d^k}{dx^k}\left[f(x)\right]$.

Show, by a detailed method, that

$$\frac{d^n}{dx^n}\left[x^4\ln x\right] = n!\,x^{4-n}\sum_{r=0}^{4}\left[\binom{4}{r}f(n,r)\right],$$

where $f(n,r)$ is a function to be found.

$$\boxed{\frac{d^n}{dx^n}\left[x^4\ln x\right] = n!\,x^{4-n}\sum_{r=0}^{4}\left[\binom{4}{r}\frac{(-1)^{n-r-1}}{n-r}\right]}$$



APPLYING LEIBNIZ'S THEOREM ON

$$y = \underset{\substack{\uparrow\\ f}}{x^4}\,\underset{\substack{\uparrow\\ g}}{\ln x}$$

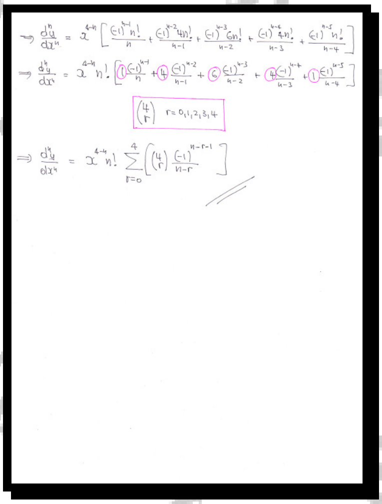
$$\Rightarrow \frac{d^n y}{dx^n} = \binom{n}{0}x^4\frac{d^n}{dx^n}(\ln x) + \binom{n}{1}4x^3\frac{d^{n-1}}{dx^{n-1}}(\ln x) + \binom{n}{2}12x^2\frac{d^{n-2}}{dx^{n-2}}(\ln x) + \binom{n}{3}24x\frac{d^{n-3}}{dx^{n-3}}(\ln x)$$
$$+\binom{n}{4}24\frac{d^{n-4}}{dx^{n-4}}(\ln x) + \binom{n}{5}0\cancel{\frac{d^{n-5}}{dx^{n-5}}(\ln x)} + \binom{n}{6}0\cancel{\frac{d^{n-6}}{dx^{n-6}}}(\ln x) + \dots$$

$$\Rightarrow \frac{d^n y}{dx^n} = 1\times x^4\times\frac{(n-1)!}{x^n}(-1)^{n-1} + n\times 4x^3\times\frac{(n-2)!}{x^{n-1}}(-1)^{n-2} + \frac{n(n-1)}{2!}\times 12x^2\times\frac{(n-3)!}{x^{n-2}}(-1)^{n-3}$$
$$+\frac{n(n-1)(n-2)}{3!}\times 24x\times\frac{(n-4)!}{x^{n-3}}(-1)^{n-4} + \frac{n(n-1)(n-2)(n-3)}{4!}\times 24\times\frac{(n-5)!}{x^{n-4}}(-1)^{n-5}$$

$$\Rightarrow \frac{d^n y}{dx^n} = (-1)^{n-1}x^{4-n}(n-1)! + (-1)^{n-2}x^{4-n}4n(n-2)! + (-1)^{n-3}x^{4-n}\frac{12n(n-1)(n-3)!}{2!}$$
$$+(-1)^{n-4}x^{4-n}\frac{24n(n-1)(n-2)(n-4)!}{3!} + (-1)^{n-5}x^{4-n}\frac{24n(n-1)(n-2)(n-3)(n-5)!}{4!}$$

$$\Rightarrow \frac{d^n y}{dx^n} = x^{4-n}\Big[(-1)^{n-1}(n-1)! + (-1)^{n-2}4n(n-2)! + (-1)^{n-3}6n(n-1)(n-3)!$$
$$+(-1)^{n-4}\times 4n(n-1)(n-2)(n-4)! + (-1)^{n-5}n(n-1)(n-2)(n-3)(n-5)!\Big]$$

$$\Rightarrow \frac{d^n y}{dx^n} = x^{4-n}\Big[\frac{(-1)^{n-1}n(n-1)!}{n} + \frac{(-1)^{n-2}4n(n-1)(n-2)!}{n-1} + \frac{(-1)^{n-3}6n(n-1)(n-2)(n-3)!}{n-2}$$
$$+\frac{(-1)^{n-4}4n(n-1)(n-2)(n-3)(n-4)!}{n-3} + \frac{(-1)^{n-5}n(n-1)(n-2)(n-3)(n-4)(n-5)!}{n-4}\Big]$$



$$\Rightarrow \frac{d^n y}{dx^n} = x^{4-n}\left[\frac{(-1)^{n-1}n!}{n} + \frac{(-1)^{n-2}4n!}{n-1} + \frac{(-1)^{n-3}6n!}{n-2} + \frac{(-1)^{n-4}4n!}{n-3} + \frac{(-1)^{n-5}n!}{n-4}\right]$$

$$\Rightarrow \frac{d^n y}{dx^n} = x^{4-n}n!\left[1\frac{(-1)^{n-1}}{n} + 4\frac{(-1)^{n-2}}{n-1} + 6\frac{(-1)^{n-3}}{n-2} + 4\frac{(-1)^{n-4}}{n-3} + 1\frac{(-1)^{n-5}}{n-4}\right]$$

$$\boxed{\binom{4}{r}\quad r=0,1,2,3,4}$$

$$\Rightarrow \frac{d^n y}{dx^n} = x^{4-n}n!\sum_{r=0}^{4}\left[\binom{4}{r}\frac{(-1)^{n-r-1}}{n-r}\right]$$

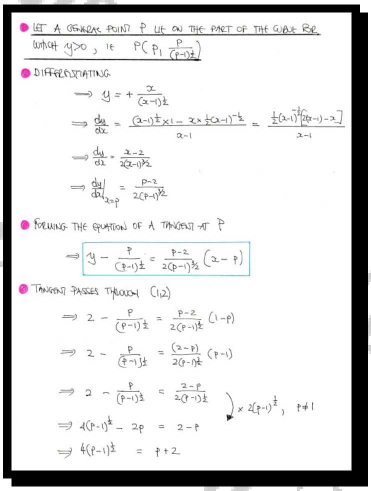
Question 268 (*****)

A curve C has equation

$$y^2 = \frac{x^2}{x-1}, \quad x \in \mathbb{R}, \quad x > 1.$$

Show that there exist exactly two tangents to C which pass through the point $(1,2)$, and find their equations.

$\boxed{\text{MP2}}$, $\boxed{y = 2}$, $\boxed{27y = 4x + 50}$



● LET A GENERAL POINT P LIE ON THE PART OF THE CURVE FOR WHICH $y>0$, IE $P\left(p, \frac{p}{(p-1)^{\frac{1}{2}}}\right)$

● DIFFERENTIATING

$$\Longrightarrow y = +\frac{x}{(x-1)^{\frac{1}{2}}}$$

$$\Longrightarrow \frac{dy}{dx} = \frac{(x-1)^{\frac{1}{2}} \times 1 - x \times \frac{1}{2}(x-1)^{-\frac{1}{2}}}{x-1} = \frac{\frac{1}{2}(x-1)^{-\frac{1}{2}}[2(x-1)-x]}{x-1}$$

$$\Longrightarrow \frac{dy}{dx} = \frac{x-2}{2(x-1)^{3/2}}$$

$$\Longrightarrow \left.\frac{dy}{dx}\right|_{x=p} = \frac{p-2}{2(p-1)^{3/2}}$$

● FORMING THE EQUATION OF A TANGENT AT P

$$\Longrightarrow \boxed{y - \frac{p}{(p-1)^{\frac{1}{2}}} = \frac{p-2}{2(p-1)^{3/2}}(x-p)}$$

● TANGENT PASSES THROUGH (1,2)

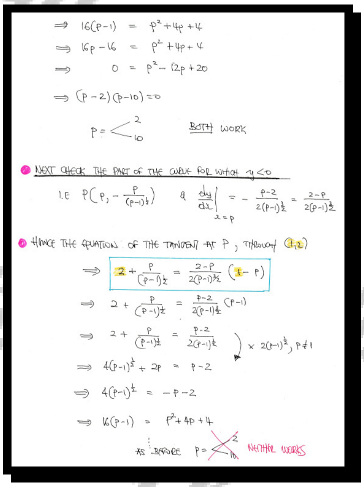
$$\Longrightarrow 2 - \frac{p}{(p-1)^{\frac{1}{2}}} = \frac{p-2}{2(p-1)^{\frac{3}{2}}}(1-p)$$

$$\Longrightarrow 2 - \frac{p}{(p-1)^{\frac{1}{2}}} = \frac{(2-p)}{2(p-1)^{\frac{3}{2}}}(p-1)$$

$$\Longrightarrow 2 - \frac{p}{(p-1)^{\frac{1}{2}}} = \frac{2-p}{2(p-1)^{\frac{1}{2}}} \quad \Big) \times 2(p-1)^{\frac{1}{2}}, \ p \neq 1$$

$$\Longrightarrow 4(p-1)^{\frac{1}{2}} - 2p = 2-p$$

$$\Longrightarrow 4(p-1)^{\frac{1}{2}} = p+2$$



$$\Longrightarrow 16(p-1) = p^2+4p+4$$

$$\Longrightarrow 16p-16 = p^2+4p+4$$

$$\Longrightarrow 0 = p^2-12p+20$$

$$\Longrightarrow (p-2)(p-10) = 0$$

$$p = \begin{cases} 2 \\ 10 \end{cases} \qquad \underline{\text{BOTH}} \text{ WORK}$$

● NEXT CHECK THE PART OF THE CURVE FOR WHICH $y<0$

$$\text{I.E } P\left(p, -\frac{p}{(p-1)^{\frac{1}{2}}}\right) \quad \& \quad \left.\frac{dy}{dx}\right|_{x=p} = -\frac{p-2}{2(p-1)^{\frac{3}{2}}} = \frac{2-p}{2(p-1)^{\frac{3}{2}}}$$

● HENCE THE EQUATION OF THE TANGENT AT P, THROUGH (1,2)

$$\Longrightarrow \boxed{2 + \frac{p}{(p-1)^{\frac{1}{2}}} = \frac{2-p}{2(p-1)^{3/2}}(1-p)}$$

$$\Longrightarrow 2 + \frac{p}{(p-1)^{\frac{1}{2}}} = \frac{p-2}{2(p-1)^{\frac{3}{2}}}(p-1)$$

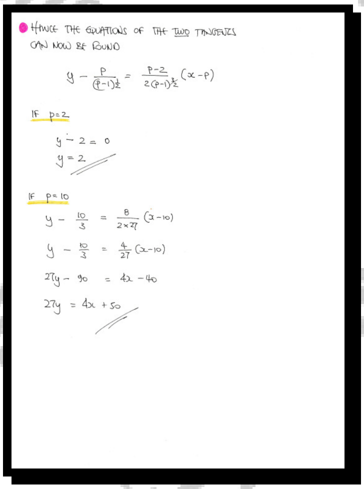
$$\Longrightarrow 2 + \frac{p}{(p-1)^{\frac{1}{2}}} = \frac{p-2}{2(p-1)^{\frac{1}{2}}} \quad \Big) \times 2(p-1)^{\frac{1}{2}}, \ p \neq 1$$

$$\Longrightarrow 4(p-1)^{\frac{1}{2}} + 2p = p-2$$

$$\Longrightarrow 4(p-1)^{\frac{1}{2}} = -p-2$$

$$\Longrightarrow 16(p-1) = p^2+4p+4$$

$$\text{AS BEFORE } \cancel{p = \begin{cases} 2 \\ 10 \end{cases}} \quad \text{NEITHER WORKS}$$



● HENCE THE EQUATIONS OF THE TWO TANGENTS CAN NOW BE FOUND

$$y - \frac{p}{(p-1)^{\frac{1}{2}}} = \frac{p-2}{2(p-1)^{3/2}}(x-p)$$

IF $p=2$

$$y - 2 = 0$$

$$y = 2$$

IF $p=10$

$$y - \frac{10}{3} = \frac{8}{2 \times 27}(x-10)$$

$$y - \frac{10}{3} = \frac{4}{27}(x-10)$$

$$27y - 90 = 4x - 40$$

$$27y = 4x + 50$$

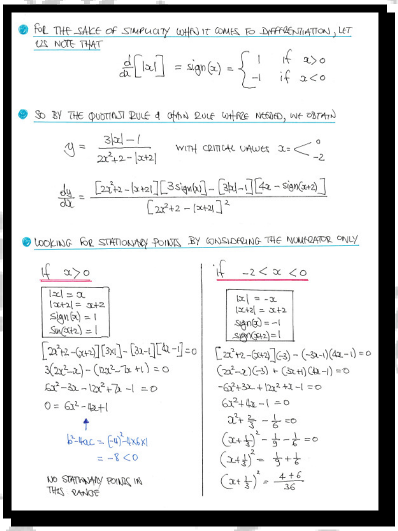
Question 269 (*****)

A curve C has equation

$$y=\frac{3|x|-1}{2x^2+2-|x+2|}, \quad x\in\mathbb{R}, \quad x\neq 0, \quad x\neq \frac{1}{2}.$$

Find, in exact simplified surd form, the y coordinate of the stationary point of C.

$\boxed{}$, $\boxed{y=7-2\sqrt{10}}$



FOR THE SAKE OF SIMPLICITY WHEN IT COMES TO DIFFERENTIATION, LET US NOTE THAT

$$\frac{d}{dx}\left[|x|\right]=\text{sign}(x)=\begin{cases}1 & \text{if } x>0\\ -1 & \text{if } x<0\end{cases}$$

SO BY THE QUOTIENT RULE & CHAIN RULE WHERE NEEDED, WE OBTAIN

$$y=\frac{3|x|-1}{2x^2+2-|x+2|} \quad \text{WITH CRITICAL VALUES } x=\left\langle\begin{matrix}0\\-2\end{matrix}\right.$$

$$\frac{dy}{dx}=\frac{\left[2x^2+2-|x+2|\right]\left[3\,\text{sign}(x)\right]-\left[3|x|-1\right]\left[4x-\text{sign}(x+2)\right]}{\left[2x^2+2-|x+2|\right]^2}$$

LOOKING FOR STATIONARY POINTS BY CONSIDERING THE NUMERATOR ONLY

If $x>0$

$$\boxed{\begin{matrix}|x|=x\\|x+2|=x+2\\\text{sign}(x)=1\\\text{Sin}(x+2)=1\end{matrix}}$$

$\left[2x^2+2-(x+2)\right]\left[3\times1\right]-\left[3x-1\right]\left[4x-1\right]=0$

$3(2x^2-x)-(12x^2-7x+1)=0$

$6x^2-3x-12x^2+7x-1=0$

$0=6x^2-4x+1$

$\uparrow$

$b^2-4ac=(-4)^2-4\times6\times1$

$=-8<0$

NO STATIONARY POINTS IN THIS RANGE

if $-2<x<0$

$$\boxed{\begin{matrix}|x|=-x\\|x+2|=x+2\\\text{sign}(x)=-1\\\text{sign}(x+2)=1\end{matrix}}$$

$\left[2x^2+2-(x+2)\right](-3)-(-3x-1)(4x-1)=0$

$(2x^2-x)(-3)+(3x+1)(4x-1)=0$

$-6x^2+3x+12x^2+x-1=0$

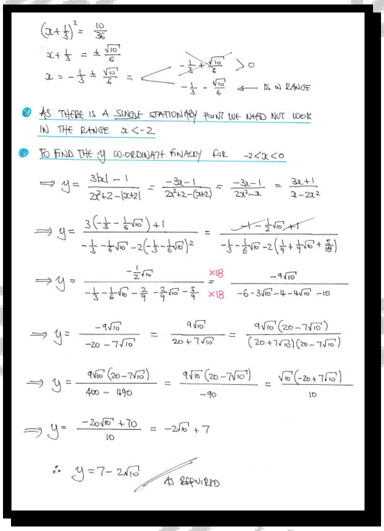
$6x^2+4x-1=0$

$x^2+\frac{2}{3}-\frac{1}{6}=0$

$\left(x+\frac{1}{3}\right)^2-\frac{1}{9}-\frac{1}{6}=0$

$\left(x+\frac{1}{3}\right)^2=\frac{1}{9}+\frac{1}{6}$

$\left(x+\frac{1}{3}\right)^2=\frac{4+6}{36}$



$\left(x+\frac{1}{3}\right)^2=\frac{10}{36}$

$x+\frac{1}{3}=\pm\frac{\sqrt{10}}{6}$

$x=-\frac{1}{3}\pm\frac{\sqrt{10}}{6}=\left\langle\begin{matrix}\cancel{-\frac{1}{3}+\frac{\sqrt{10}}{6}}>0\\-\frac{1}{3}-\frac{\sqrt{10}}{6} \longleftarrow \text{IS IN RANGE}\end{matrix}\right.$

AS THERE IS A SINGLE STATIONARY POINT WE NEED NOT LOOK IN THE RANGE $x<-2$

TO FIND THE y CO-ORDINATE FINALLY FOR $-2<x<0$

$\Rightarrow y=\frac{3|x|-1}{2x^2+2-|x+2|}=\frac{-3x-1}{2x^2+2-(x+2)}=\frac{-3x-1}{2x^2-x}=\frac{3x+1}{x-2x^2}$

$\Rightarrow y=\frac{3\left(-\frac{1}{3}-\frac{1}{6}\sqrt{10}\right)+1}{-\frac{1}{3}-\frac{1}{6}\sqrt{10}-2\left(-\frac{1}{3}-\frac{1}{6}\sqrt{10}\right)^2}=\frac{\cancel{-1}-\frac{1}{2}\sqrt{10}\cancel{+1}}{-\frac{1}{3}-\frac{1}{6}\sqrt{10}-2\left(\frac{1}{9}+\frac{1}{9}\sqrt{10}+\frac{5}{18}\right)}$

$\Rightarrow y=\frac{-\frac{1}{2}\sqrt{10}}{-\frac{1}{3}-\frac{1}{6}\sqrt{10}-\frac{2}{9}-\frac{2}{9}\sqrt{10}-\frac{5}{9}}\overset{\times18}{\underset{\times18}{=}}\frac{-9\sqrt{10}}{-6-3\sqrt{10}-4-4\sqrt{10}-10}$

$\Rightarrow y=\frac{-9\sqrt{10}}{-20-7\sqrt{10}}=\frac{9\sqrt{10}}{20+7\sqrt{10}}=\frac{9\sqrt{10}(20-7\sqrt{10})}{(20+7\sqrt{10})(20-7\sqrt{10})}$

$\Rightarrow y=\frac{9\sqrt{10}(20-7\sqrt{10})}{400-490}=\frac{9\sqrt{10}(20-7\sqrt{10})}{-90}=\frac{\sqrt{10}(-20+7\sqrt{10})}{10}$

$\Rightarrow y=\frac{-20\sqrt{10}+70}{10}=-2\sqrt{10}+7$

$\therefore y=7-2\sqrt{10}$ AS REQUIRED

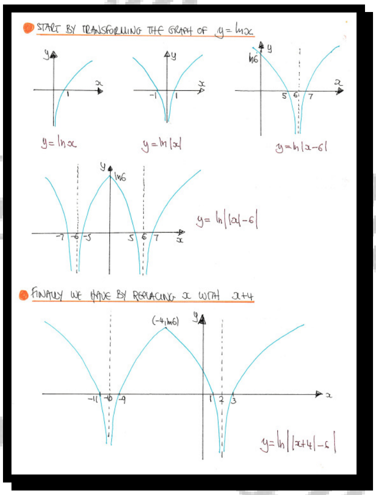
Question 270 (*****)

A curve, defined in the largest real domain, has equation

$$y = \ln \big| |x+4| - 6 \big| .$$

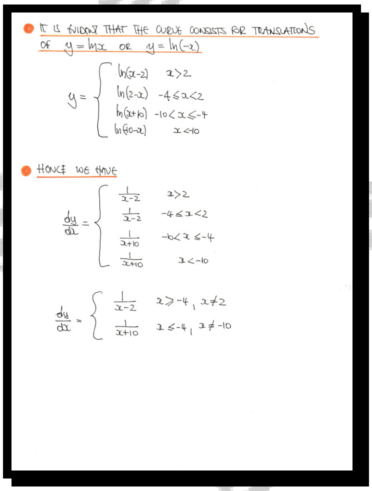
Determine, in its simplest form, an expression for $\frac{dy}{dx}$.

☐ , $\boxed{\frac{dy}{dx} = \begin{cases} \frac{1}{x-2}, & x \geq -4, \ x \neq 2 \\ \frac{1}{x+10}, & x \leq -4, \ x \neq -10 \end{cases}}$



● START BY TRANSFORMING THE GRAPH OF $y = \ln x$

● FINALLY WE HAVE BY REPLACING x WITH $x+4$



● IT IS EVIDENT THAT THE CURVE CONSISTS FOR TRANSLATIONS OF $y = \ln x$ OR $y = \ln(-x)$

$$y = \begin{cases} \ln(x-2) & x > 2 \\ \ln(2-x) & -4 \leqslant x < 2 \\ \ln(x+10) & -10 < x \leqslant -4 \\ \ln(-10-x) & x < -10 \end{cases}$$

● HENCE WE HAVE

$$\frac{dy}{dx} = \begin{cases} \frac{1}{x-2} & x > 2 \\ \frac{1}{x-2} & -4 \leqslant x < 2 \\ \frac{1}{x+10} & -10 < x \leqslant -4 \\ \frac{1}{x+10} & x < -10 \end{cases}$$

$$\frac{dy}{dx} = \begin{cases} \frac{1}{x-2} & x \geqslant -4, \ x \neq 2 \\ \frac{1}{x+10} & x \leqslant -4, \ x \neq -10 \end{cases}$$

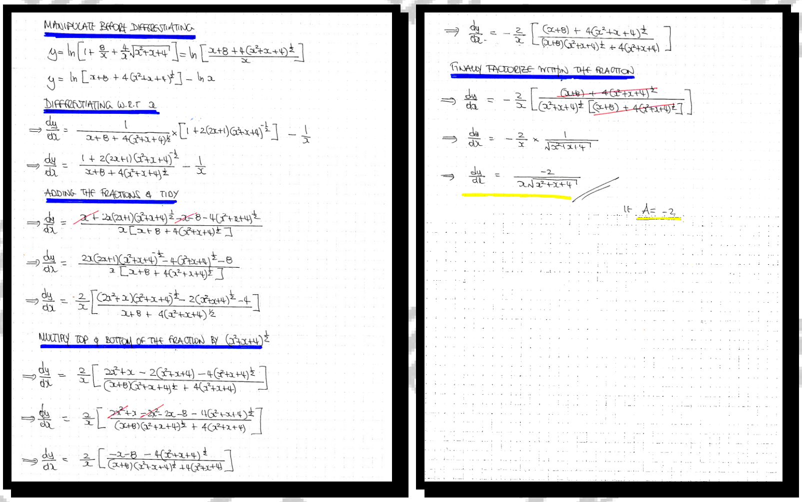
Question 271 (*****)

Show that

$$\frac{d}{dx}\left[\ln\left(1+\frac{8}{x}+\frac{4}{x}\sqrt{x^2+x+4}\right)\right] = \frac{A}{x\sqrt{x^2+x+4}},$$

where A is a non zero constant.

$\boxed{}$, $\boxed{A=-2}$



MANIPULATE BEFORE DIFFERENTIATING

$y = \ln\left[1+\frac{8}{x}+\frac{4}{x}\sqrt{x^2+x+4}\right] = \ln\left[\frac{x+8+4(x^2+x+4)^{\frac{1}{2}}}{x}\right]$

$y = \ln\left[x+8+4(x^2+x+4)^{\frac{1}{2}}\right] - \ln x$

DIFFERENTIATING W.R.T x

$\Rightarrow \frac{dy}{dx} = \frac{1}{x+8+4(x^2+x+4)^{\frac{1}{2}}} \times \left[1+2(2x+1)(x^2+x+4)^{-\frac{1}{2}}\right] - \frac{1}{x}$

$\Rightarrow \frac{dy}{dx} = \frac{1+2(2x+1)(x^2+x+4)^{-\frac{1}{2}}}{x+8+4(x^2+x+4)^{\frac{1}{2}}} - \frac{1}{x}$

ADDING THE FRACTIONS & TIDY

$\Rightarrow \frac{dy}{dx} = \frac{\cancel{x}+2x(2x+1)(x^2+x+4)^{\frac{1}{2}}\cancel{-x}-8-4(x^2+x+4)^{\frac{1}{2}}}{x\left[x+8+4(x^2+x+4)^{\frac{1}{2}}\right]}$

$\Rightarrow \frac{dy}{dx} = \frac{2x(2x+1)(x^2+x+4)^{-\frac{1}{2}}-4(x^2+x+4)^{\frac{1}{2}}-8}{x\left[x+8+4(x^2+x+4)^{\frac{1}{2}}\right]}$

$\Rightarrow \frac{dy}{dx} = \frac{2}{x}\left[\frac{(2x^2+x)(x^2+x+4)^{-\frac{1}{2}}-2(x^2+x+4)^{\frac{1}{2}}-4}{x+8+4(x^2+x+4)^{\frac{1}{2}}}\right]$

MULTIPLY TOP & BOTTOM OF THE FRACTION BY $(x^2+x+4)^{\frac{1}{2}}$

$\Rightarrow \frac{dy}{dx} = \frac{2}{x}\left[\frac{2x^2+x-2(x^2+x+4)-4(x^2+x+4)^{\frac{1}{2}}}{(x+8)(x^2+x+4)^{\frac{1}{2}}+4(x^2+x+4)}\right]$

$\Rightarrow \frac{dy}{dx} = \frac{2}{x}\left[\frac{\cancel{2x^2}+x\cancel{-2x^2}-2x-8-4(x^2+x+4)^{\frac{1}{2}}}{(x+8)(x^2+x+4)^{\frac{1}{2}}+4(x^2+x+4)}\right]$

$\Rightarrow \frac{dy}{dx} = \frac{2}{x}\left[\frac{-x-8-4(x^2+x+4)^{\frac{1}{2}}}{(x+8)(x^2+x+4)^{\frac{1}{2}}+4(x^2+x+4)}\right]$

$\Rightarrow \frac{dy}{dx} = -\frac{2}{x}\left[\frac{(x+8)+4(x^2+x+4)^{\frac{1}{2}}}{(x+8)(x^2+x+4)^{\frac{1}{2}}+4(x^2+x+4)}\right]$

FINALLY FACTORIZE WITHIN THE FRACTION

$\Rightarrow \frac{dy}{dx} = -\frac{2}{x}\left[\frac{\cancel{(x+8)+4(x^2+x+4)^{\frac{1}{2}}}}{(x^2+x+4)^{\frac{1}{2}}\left[\cancel{(x+8)+4(x^2+x+4)^{\frac{1}{2}}}\right]}\right]$

$\Rightarrow \frac{dy}{dx} = -\frac{2}{x} \times \frac{1}{\sqrt{x^2+x+4}}$

$\Rightarrow \frac{dy}{dx} = \frac{-2}{x\sqrt{x^2+x+4}}$

$\therefore A = -2$

Question 272 (*****)

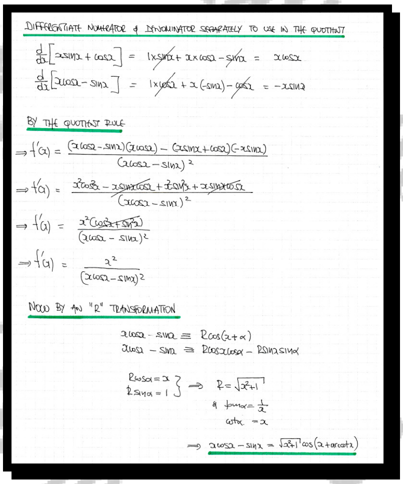
$$f(x) \equiv \frac{x\sin x + \cos x}{x\cos x - \sin x}, \quad x \in (0, \pi].$$

Show that

$$f'(x) = g(x)\sec^2\left[x + \operatorname{arccot} x\right],$$

where $g(x)$ is a rational function to be found.

$$\boxed{g(x) = \frac{x^2}{x^2+1}}$$



DIFFERENTIATE NUMERATOR & DENOMINATOR SEPARATELY TO USE IN THE QUOTIENT

$$\frac{d}{dx}\left[x\sin x + \cos x\right] = 1\times\cancel{\sin x} + x\times\cos x - \cancel{\sin x} = x\cos x$$

$$\frac{d}{dx}\left[x\cos x - \sin x\right] = 1\times\cancel{\cos x} + x(-\sin x) - \cancel{\cos x} = -x\sin x$$

BY THE QUOTIENT RULE

$$\Rightarrow f'(x) = \frac{(x\cos x - \sin x)(x\cos x) - (x\sin x + \cos x)(-x\sin x)}{(x\cos x - \sin x)^2}$$

$$\Rightarrow f'(x) = \frac{x^2\cos^2 x - \cancel{x\sin x\cos x} + x^2\sin^2 x + \cancel{x\sin x\cos x}}{(x\cos x - \sin x)^2}$$

$$\Rightarrow f'(x) = \frac{x^2(\cancel{\cos^2 x + \sin^2 x})}{(x\cos x - \sin x)^2}$$

$$\Rightarrow f'(x) = \frac{x^2}{(x\cos x - \sin x)^2}$$

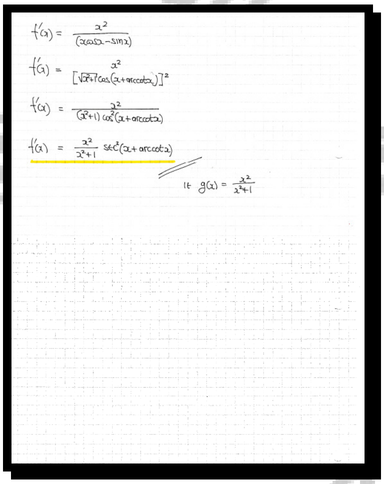
NOW BY AN "R" TRANSFORMATION

$$x\cos x - \sin x \equiv R\cos(x+\alpha)$$
$$x\cos x - \sin x \equiv R\cos x\cos\alpha - R\sin x\sin\alpha$$

$$\left.\begin{array}{l} R\cos\alpha = x \\ R\sin\alpha = 1 \end{array}\right\} \Rightarrow R = \sqrt{x^2+1}$$

$$\text{\& } \tan\alpha = \frac{1}{x}$$
$$\cot\alpha = x$$

$$\Rightarrow x\cos x - \sin x = \sqrt{x^2+1}\cos(x + \operatorname{arccot} x)$$



$$f'(x) = \frac{x^2}{(x\cos x - \sin x)}$$

$$f'(x) = \frac{x^2}{\left[\sqrt{x^2+1}\cos(x+\operatorname{arccot} x)\right]^2}$$

$$f'(x) = \frac{x^2}{(x^2+1)\cos^2(x+\operatorname{arccot} x)}$$

$$f'(x) = \frac{x^2}{x^2+1}\sec^2(x+\operatorname{arccot} x)$$

ie $g(x) = \frac{x^2}{x^2+1}$

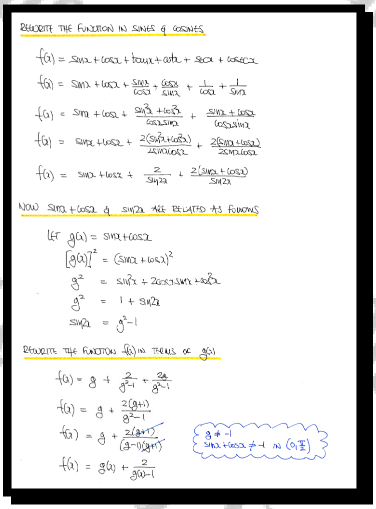
Question 273 (*****)

The function f is defined as

$$f(x) \equiv \sin x + \cos x + \tan x + \cot x + \sec x + \operatorname{cosec} x\,, \qquad x \in \left(0, \tfrac{1}{2}\pi\right).$$

Determine with full justification the range of f.

$\boxed{}$, $\boxed{f(x) \in \left[2+3\sqrt{2}, \infty\right)}$



REWRITE THE FUNCTION IN SINES & COSINES

$$f(x) = \sin x + \cos x + \tan x + \cot x + \sec x + \operatorname{cosec} x$$
$$f(x) = \sin x + \cos x + \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} + \frac{1}{\cos x} + \frac{1}{\sin x}$$
$$f(x) = \sin x + \cos x + \frac{\sin^2 x + \cos^2 x}{\cos x \sin x} + \frac{\sin x + \cos x}{\cos x \sin x}$$
$$f(x) = \sin x + \cos x + \frac{2(\sin^2 x + \cos^2 x)}{2\sin x \cos x} + \frac{2(\sin x + \cos x)}{2\sin x \cos x}$$
$$f(x) = \sin x + \cos x + \frac{2}{\sin 2x} + \frac{2(\sin x + \cos x)}{\sin 2x}$$

NOW $\sin x + \cos x$ & $\sin 2x$ ARE RELATED AS FOLLOWS

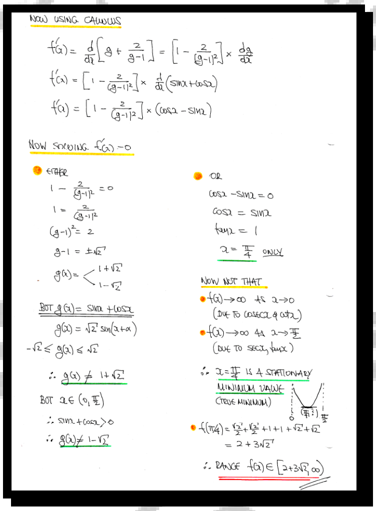
LET $g(x) = \sin x + \cos x$
$$[g(x)]^2 = (\sin x + \cos x)^2$$
$$g^2 = \sin^2 x + 2\cos x \sin x + \cos^2 x$$
$$g^2 = 1 + \sin 2x$$
$$\sin 2x = g^2 - 1$$

REWRITE THE FUNCTION $f(x)$ IN TERMS OF $g(x)$

$$f(x) = g + \frac{2}{g^2-1} + \frac{2g}{g^2-1}$$
$$f(x) = g + \frac{2(g+1)}{g^2-1}$$
$$f(x) = g + \frac{2\cancel{(g+1)}}{(g-1)\cancel{(g+1)}}$$

$g \neq -1$
$\sin x + \cos x \neq -1$ IN $\left(0, \frac{\pi}{2}\right)$

$$f(x) = g(x) + \frac{2}{g(x)-1}$$



NOW USING CALCULUS

$$f'(x) = \frac{d}{dx}\left[g + \frac{2}{g-1}\right] = \left[1 - \frac{2}{(g-1)^2}\right] \times \frac{dg}{dx}$$
$$f'(x) = \left[1 - \frac{2}{(g-1)^2}\right] \times \frac{d}{dx}(\sin x + \cos x)$$
$$f'(x) = \left[1 - \frac{2}{(g-1)^2}\right] \times (\cos x - \sin x)$$

NOW SOLVING $f'(x) = 0$

- EITHER

$$1 - \frac{2}{(g-1)^2} = 0$$
$$1 = \frac{2}{(g-1)^2}$$
$$(g-1)^2 = 2$$
$$g - 1 = \pm\sqrt{2}$$
$$g(x) = \begin{cases} 1+\sqrt{2} \\ 1-\sqrt{2} \end{cases}$$

BUT $g(x) = \sin x + \cos x$
$$g(x) = \sqrt{2}\sin(x+\alpha)$$
$$-\sqrt{2} \leqslant g(x) \leqslant \sqrt{2}$$
$$\therefore\ g(x) \neq 1+\sqrt{2}$$

BUT $x \in \left(0, \frac{\pi}{2}\right)$
$$\therefore\ \sin x + \cos x > 0$$
$$\therefore\ g(x) \neq 1-\sqrt{2}$$

- OR

$$\cos x - \sin x = 0$$
$$\cos x = \sin x$$
$$\tan x = 1$$
$$x = \frac{\pi}{4} \text{ ONLY}$$

NOW NOT THAT

- $f(x) \to \infty$ AS $x \to 0$ (DUE TO $\operatorname{cosec} x$ & $\cot x$)
- $f(x) \to \infty$ AS $x \to \frac{\pi}{2}$ (DUE TO $\sec x$, $\tan x$)

$\therefore\ x = \frac{\pi}{4}$ IS A STATIONARY MINIMUM VALUE (TRUE MINIMUM)

- $f(\pi/4) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + 1 + 1 + \sqrt{2} + \sqrt{2}$
$$= 2 + 3\sqrt{2}$$

$\therefore$ RANGE $f(x) \in \left[2+3\sqrt{2}, \infty\right)$

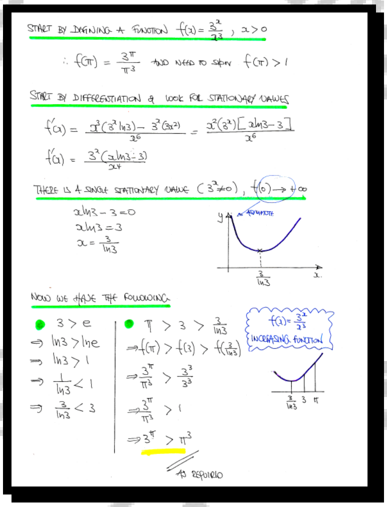
Question 274 (*****)

Assuming that $\pi \approx 3.14$, show **without** any calculating aid that

$$3^{\pi} > \pi^{3}.$$

You must show a detailed method in this question.

$\boxed{\text{[illegible]}}$, $\boxed{\text{proof}}$



START BY DEFINING A FUNCTION $f(x) = \frac{3^x}{x^3}$, $x > 0$

$\therefore f(\pi) = \frac{3^{\pi}}{\pi^3}$ AND NEED TO SHOW $f(\pi) > 1$

START BY DIFFERENTIATION & LOOK FOR STATIONARY VALUES

$$f'(x) = \frac{x^3(3^x \ln 3) - 3^x(3x^2)}{x^6} = \frac{x^2(3^x)[x\ln 3 - 3]}{x^6}$$

$$f'(x) = \frac{3^x(x\ln 3 - 3)}{x^4}$$

THERE IS A SINGLE STATIONARY VALUE $(3^x \neq 0)$, $f(0) \to +\infty$

$x\ln 3 - 3 = 0$

$x \ln 3 = 3$

$x = \frac{3}{\ln 3}$

NOW WE HAVE THE FOLLOWING

- $3 > e$
- $\Rightarrow \ln 3 > \ln e$
- $\Rightarrow \ln 3 > 1$
- $\Rightarrow \frac{1}{\ln 3} < 1$
- $\Rightarrow \frac{3}{\ln 3} < 3$

- $\pi > 3 > \frac{3}{\ln 3}$
- $\Rightarrow f(\pi) > f(3) > f\left(\frac{3}{\ln 3}\right)$
- $\Rightarrow \frac{3^{\pi}}{\pi^3} > \frac{3^3}{3^3}$
- $\Rightarrow \frac{3^{\pi}}{\pi^3} > 1$
- $\Rightarrow 3^{\pi} > \pi^3$

$f(x) = \frac{3^x}{x^3}$ INCREASING FUNCTION

// AS REQUIRED

Question 275 (*****)

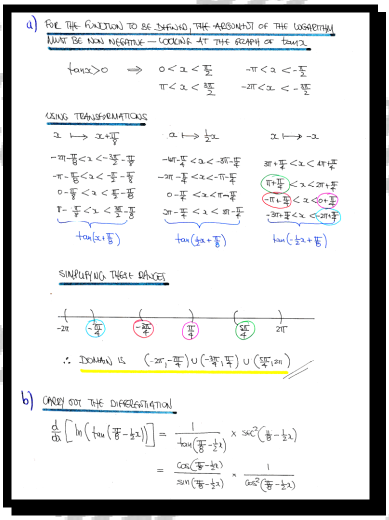
The function f is defined in the largest possible real domain, contained in the interval $(-2\pi, 2\pi)$, and its equation is

$$f(x) \equiv \ln\left[\tan\left(\tfrac{1}{8}\pi - \tfrac{1}{2}x\right)\right].$$

a) Find the domain of f.

b) Show that $f'(x) \equiv \dfrac{k}{\sqrt{1-\sin 2x}}$, for some constant k.

SPX , $\boxed{\left(-2\pi, -\tfrac{7}{4}\pi\right) \cup \left(-\tfrac{3}{4}\pi, \tfrac{1}{4}\pi\right) \cup \left(\tfrac{5}{4}\pi, 2\pi\right)}$



a) FOR THE FUNCTION TO BE DEFINED, THE ARGUMENT OF THE LOGARITHM MUST BE NON NEGATIVE — LOOKING AT THE GRAPH OF $\tan x$

$\tan x > 0 \Rightarrow 0 < x < \frac{\pi}{2}$ $-\pi < x < -\frac{\pi}{2}$

$\pi < x < \frac{3\pi}{2}$ $-2\pi < x < -\frac{3\pi}{2}$

USING TRANSFORMATIONS

$x \longmapsto x + \frac{\pi}{8}$

$-2\pi - \frac{\pi}{8} < x < -\frac{3\pi}{2} - \frac{\pi}{8}$

$-\pi - \frac{\pi}{8} < x < -\frac{\pi}{2} - \frac{\pi}{8}$

$0 - \frac{\pi}{8} < x < \frac{\pi}{2} - \frac{\pi}{8}$

$\pi - \frac{\pi}{8} < x < \frac{3\pi}{2} - \frac{\pi}{8}$

$\tan\left(x + \frac{\pi}{8}\right)$

$x \longmapsto \frac{1}{2}x$

$-4\pi - \frac{\pi}{4} < x < -3\pi - \frac{\pi}{4}$

$-2\pi - \frac{\pi}{4} < x < -\pi - \frac{\pi}{4}$

$0 - \frac{\pi}{4} < x < \pi - \frac{\pi}{4}$

$2\pi - \frac{\pi}{4} < x < 3\pi - \frac{\pi}{4}$

$\tan\left(\frac{1}{2}x + \frac{\pi}{8}\right)$

$x \longmapsto -x$

$3\pi + \frac{\pi}{4} < x < 4\pi + \frac{\pi}{4}$

$\pi + \frac{\pi}{4} < x < 2\pi + \frac{\pi}{4}$

$-\pi + \frac{\pi}{4} < x < 0 + \frac{\pi}{4}$

$-3\pi + \frac{\pi}{4} < x < -2\pi + \frac{\pi}{4}$

$\tan\left(-\frac{1}{2}x + \frac{\pi}{8}\right)$

SIMPLIFYING THESE RANGES

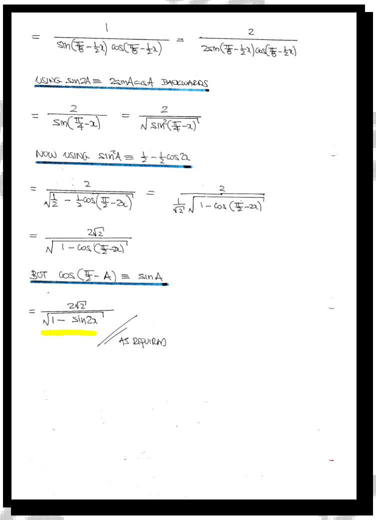
$-2\pi \quad -\frac{7\pi}{4} \quad -\frac{3\pi}{4} \quad \frac{\pi}{4} \quad \frac{5\pi}{4} \quad 2\pi$

$\therefore$ DOMAIN IS $\left(-2\pi, -\frac{7\pi}{4}\right) \cup \left(-\frac{3\pi}{4}, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, 2\pi\right)$

b) CARRY OUT THE DIFFERENTIATION

$$\frac{d}{dx}\left[\ln\left(\tan\left(\frac{\pi}{8} - \frac{1}{2}x\right)\right)\right] = \frac{1}{\tan\left(\frac{\pi}{8} - \frac{1}{2}x\right)} \times \sec^2\left(\frac{\pi}{8} - \frac{1}{2}x\right)$$

$$= \frac{\cos\left(\frac{\pi}{8} - \frac{1}{2}x\right)}{\sin\left(\frac{\pi}{8} - \frac{1}{2}x\right)} \times \frac{1}{\cos^2\left(\frac{\pi}{8} - \frac{1}{2}x\right)}$$



$$= \frac{1}{\sin\left(\frac{\pi}{8} - \frac{1}{2}x\right)\cos\left(\frac{\pi}{8} - \frac{1}{2}x\right)} = \frac{2}{2\sin\left(\frac{\pi}{8} - \frac{1}{2}x\right)\cos\left(\frac{\pi}{8} - \frac{1}{2}x\right)}$$

USING $\sin 2A \equiv 2\sin A\cos A$ BACKWARDS

$$= \frac{2}{\sin\left(\frac{\pi}{4} - x\right)} = \frac{2}{\sqrt{\sin^2\left(\frac{\pi}{4} - x\right)}}$$

NOW USING $\sin^2 A \equiv \frac{1}{2} - \frac{1}{2}\cos 2x$

$$= \frac{2}{\sqrt{\frac{1}{2} - \frac{1}{2}\cos\left(\frac{\pi}{2} - 2x\right)}} = \frac{2}{\frac{1}{\sqrt{2}}\sqrt{1 - \cos\left(\frac{\pi}{2} - 2x\right)}}$$

$$= \frac{2\sqrt{2}}{\sqrt{1 - \cos\left(\frac{\pi}{2} - 2x\right)}}$$

BUT $\cos\left(\frac{\pi}{2} - A\right) \equiv \sin A$

$$= \frac{2\sqrt{2}}{\sqrt{1 - \sin 2x}}$$

AS REQUIRED

Question 276 **(*****)**

A curve has equation

$$y = x^{x^{-1}}, \ x \in \mathbb{R}.$$

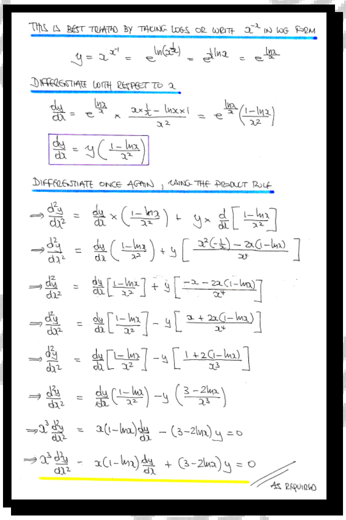
a) Show that y is a solution of the following differential equation.

$$x^3 \frac{d^2 y}{dx^2} - x(1 - \ln x)\frac{dy}{dx} + (3 - 2\ln x)\, y = 0.$$

b) Show further that

$$\left.\frac{d^2 y}{dx^2}\right|_{x=\mathrm{e}} = -\mathrm{e}^{\mathrm{e}^{-1}-3}.$$

$\boxed{\quad\quad}$, $\boxed{\text{proof}}$



THIS IS BEST TREATED BY TAKING LOGS OR WRITE x^{-1} IN LOG FORM

$$y = x^{x^{-1}} = e^{\ln(x^{\frac{1}{x}})} = e^{\frac{1}{x}\ln x} = e^{\frac{\ln x}{x}}$$

DIFFERENTIATE WITH RESPECT TO x

$$\frac{dy}{dx} = e^{\frac{\ln x}{x}} \times \frac{x \times \frac{1}{x} - \ln x \times 1}{x^2} = e^{\frac{\ln x}{x}}\left(\frac{1-\ln x}{x^2}\right)$$

$$\boxed{\frac{dy}{dx} = y\left(\frac{1-\ln x}{x^2}\right)}$$

DIFFERENTIATE ONCE AGAIN, USING THE PRODUCT RULE

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{dy}{dx} \times \left(\frac{1-\ln x}{x^2}\right) + y \times \frac{d}{dx}\left[\frac{1-\ln x}{x^2}\right]$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{dy}{dx}\left(\frac{1-\ln x}{x^2}\right) + y\left[\frac{x^2(-\frac{1}{x}) - 2x(1-\ln x)}{x^4}\right]$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{dy}{dx}\left[\frac{1-\ln x}{x^2}\right] + y\left[\frac{-x - 2x(1-\ln x)}{x^4}\right]$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{dy}{dx}\left[\frac{1-\ln x}{x^2}\right] - y\left[\frac{x + 2x(1-\ln x)}{x^4}\right]$$

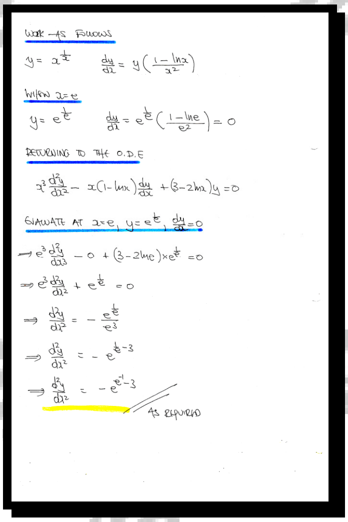
$$\Rightarrow \frac{d^2y}{dx^2} = \frac{dy}{dx}\left[\frac{1-\ln x}{x^2}\right] - y\left[\frac{1 + 2(1-\ln x)}{x^3}\right]$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{dy}{dx}\left(\frac{1-\ln x}{x^2}\right) - y\left(\frac{3-2\ln x}{x^3}\right)$$

$$\Rightarrow x^3\frac{d^2y}{dx^2} = x(1-\ln x)\frac{dy}{dx} - (3-2\ln x)y = 0$$

$$\Rightarrow x^3\frac{d^2y}{dx^2} - x(1-\ln x)\frac{dy}{dx} + (3-2\ln x)y = 0$$

AS REQUIRED



WORK AS FOLLOWS

$$y = x^{\frac{1}{x}} \qquad \frac{dy}{dx} = y\left(\frac{1-\ln x}{x^2}\right)$$

WHEN $x = e$

$$y = e^{\frac{1}{e}} \qquad \frac{dy}{dx} = e^{\frac{1}{e}}\left(\frac{1-\ln e}{e^2}\right) = 0$$

RETURNING TO THE O.D.E

$$x^3\frac{d^2y}{dx^2} - x(1-\ln x)\frac{dy}{dx} + (3-2\ln x)y = 0$$

EVALUATE AT $x = e$, $y = e^{\frac{1}{e}}$, $\frac{dy}{dx} = 0$

$$\Rightarrow e^3\frac{d^2y}{dx^3} - 0 + (3-2\ln e) \times e^{\frac{1}{e}} = 0$$

$$\Rightarrow e^3\frac{d^2y}{dx^2} + e^{\frac{1}{e}} = 0$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{e^{\frac{1}{e}}}{e^3}$$

$$\Rightarrow \frac{d^2y}{dx^2} = -e^{\frac{1}{e}-3}$$

$$\Rightarrow \frac{d^2y}{dx^2} = -e^{e^{-1}-3}$$

AS REQUIRED

Question 277 (*****)

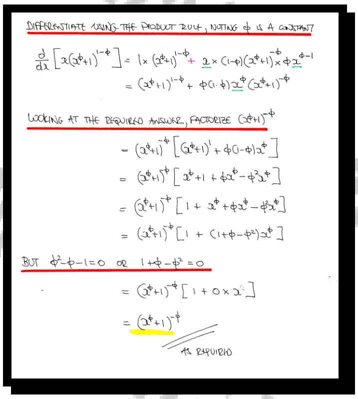
The positive solution of the quadratic equation $x^2 - x - 1 = 0$ is denoted by ϕ, and is commonly known as the golden section or golden number.

This implies that $\phi^2 - \phi - 1 = 0$, $\phi = \frac{1}{2}\left(1+\sqrt{5}\right) \approx 1.62$.

Show, with a detailed method, that

$$\frac{d}{dx}\left[x\left(x^{\phi}+1\right)^{1-\phi}\right] = \left(x^{\phi}+1\right)^{-\phi}.$$

$\boxed{}$, $\boxed{\text{proof}}$



DIFFERENTIATE USING THE PRODUCT RULE, NOTING ϕ IS A CONSTANT

$$\frac{d}{dx}\left[x(x^{\phi}+1)^{1-\phi}\right] = 1 \times (x^{\phi}+1)^{1-\phi} + x \times (1-\phi)(x^{\phi}+1)^{-\phi} \times \phi x^{\phi-1}$$
$$= (x^{\phi}+1)^{1-\phi} + \phi(1-\phi)x^{\phi}(x^{\phi}+1)^{-\phi}$$

LOOKING AT THE REQUIRED ANSWER, FACTORIZE $(x^{\phi}+1)^{-\phi}$

$$= (x^{\phi}+1)^{-\phi}\left[(x^{\phi}+1)^{1} + \phi(1-\phi)x^{\phi}\right]$$
$$= (x^{\phi}+1)^{-\phi}\left[x^{\phi}+1 + \phi x^{\phi} - \phi^2 x^{\phi}\right]$$
$$= (x^{\phi}+1)^{-\phi}\left[1 + x^{\phi} + \phi x^{\phi} - \phi^2 x^{\phi}\right]$$
$$= (x^{\phi}+1)^{-\phi}\left[1 + (1+\phi-\phi^2)x^{\phi}\right]$$

BUT $\phi^2 - \phi - 1 = 0$ OR $1 + \phi - \phi^2 = 0$

$$= (x^{\phi}+1)^{-\phi}\left[1 + 0 \times x^{\phi}\right]$$
$$= (x^{\phi}+1)^{-\phi}$$

AS REQUIRED

Question 278 (*****)

The curve C is defined in the greatest real domain by the equation

$$y = \frac{x}{(y-2)(y+1)(y-3)}.$$

a) Show that

$$\frac{dy}{dx} = \frac{1}{2(y-1)\left(ay^2+by+c\right)},$$

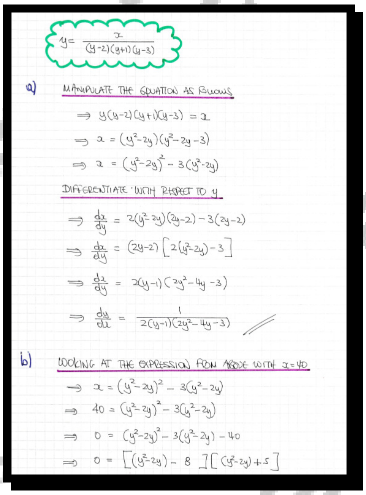
where a, b and c are integers to be found.

b) Determine the exact value of the gradient at the points on C, where $x = 40$.

c) Sketch the graph of C.

The sketch must include the coordinates of any points where C meets the coordinate axes, the coordinates of the points of infinite gradient. You must also find, with a full algebraic method, the line of symmetry of C.

$\boxed{}$, $\boxed{a=2,\quad b=-4,\quad c=-3}$, $\boxed{\pm\frac{1}{78}}$



$$y = \frac{x}{(y-2)(y+1)(y-3)}$$

a) MANIPULATE THE EQUATION AS FOLLOWS

$\Rightarrow y(y-2)(y+1)(y-3) = x$

$\Rightarrow x = (y^2-2y)(y^2-2y-3)$

$\Rightarrow x = (y^2-2y)^2 - 3(y^2-2y)$

DIFFERENTIATE WITH RESPECT TO y

$\Rightarrow \frac{dx}{dy} = 2(y^2-2y)(2y-2) - 3(2y-2)$

$\Rightarrow \frac{dx}{dy} = (2y-2)\left[2(y^2-2y)-3\right]$

$\Rightarrow \frac{dx}{dy} = 2(y-1)(2y^2-4y-3)$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2(y-1)(2y^2-4y-3)}$ //

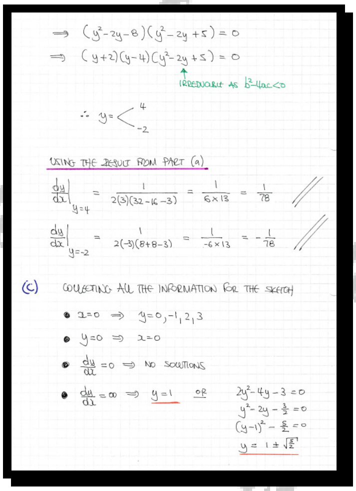
b) LOOKING AT THE EXPRESSION FROM ABOVE WITH $x=40$

$\Rightarrow x = (y^2-2y)^2 - 3(y^2-2y)$

$\Rightarrow 40 = (y^2-2y)^2 - 3(y^2-2y)$

$\Rightarrow 0 = (y^2-2y)^2 - 3(y^2-2y) - 40$

$\Rightarrow 0 = \left[(y^2-2y) - 8\right]\left[(y^2-2y)+5\right]$



$\Rightarrow (y^2-2y-8)(y^2-2y+5) = 0$

$\Rightarrow (y+2)(y-4)(y^2-2y+5) = 0$

↑ IRREDUCIBLE AS $b^2-4ac<0$

$\therefore y = \begin{cases} 4 \\ -2 \end{cases}$

USING THE RESULT FROM PART (a)

$\left.\frac{dy}{dx}\right|_{y=4} = \frac{1}{2(3)(32-16-3)} = \frac{1}{6\times13} = \frac{1}{78}$ //

$\left.\frac{dy}{dx}\right|_{y=-2} = \frac{1}{2(-3)(8+8-3)} = \frac{1}{-6\times13} = -\frac{1}{78}$ //

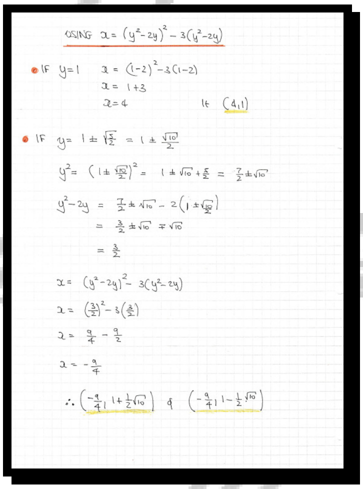
(c) COLLECTING ALL THE INFORMATION FOR THE SKETCH

- $x=0 \Rightarrow y=0,-1,2,3$
- $y=0 \Rightarrow x=0$
- $\frac{dy}{dx}=0 \Rightarrow$ NO SOLUTIONS
- $\frac{dy}{dx}=\infty \Rightarrow y=1$ OR $2y^2-4y-3=0$

$y^2-2y-\frac{3}{2}=0$

$(y-1)^2-\frac{5}{2}=0$

$y = 1 \pm \sqrt{\frac{5}{2}}$



USING $x = (y^2-2y)^2 - 3(y^2-2y)$

- IF $y=1$ $\quad x = (1-2)^2-3(1-2)$

$x = 1+3$

$x=4$ $\quad$ IE $(4,1)$

- IF $y = 1\pm\sqrt{\frac{5}{2}} = 1 \pm \frac{\sqrt{10}}{2}$

$y^2 = \left(1\pm\frac{\sqrt{10}}{2}\right)^2 = 1 \pm \sqrt{10} + \frac{5}{2} = \frac{7}{2} \pm \sqrt{10}$

$y^2-2y = \frac{7}{2}\pm\sqrt{10} - 2\left(1\pm\frac{\sqrt{10}}{2}\right)$

$= \frac{3}{2} \pm \sqrt{10} \mp \sqrt{10}$

$= \frac{3}{2}$

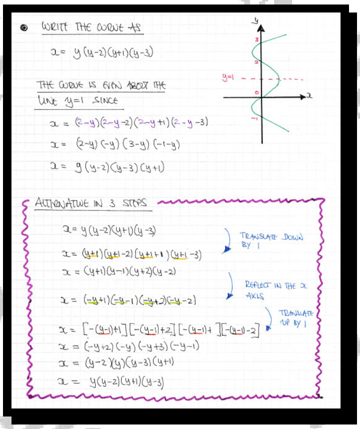
$x = (y^2-2y)^2 - 3(y^2-2y)$

$x = \left(\frac{3}{2}\right)^2 - 3\left(\frac{3}{2}\right)$

$x = \frac{9}{4} - \frac{9}{2}$

$x = -\frac{9}{4}$

$\therefore \left(-\frac{9}{4}, 1+\frac{1}{2}\sqrt{10}\right)$ & $\left(-\frac{9}{4}, 1-\frac{1}{2}\sqrt{10}\right)$



- WRITE THE CURVE AS

$x = y(y-2)(y+1)(y-3)$

THE CURVE IS EVEN ABOUT THE LINE $y=1$ SINCE

$x = (2-y)(2-y-2)(2-y+1)(2-y-3)$

$x = (2-y)(-y)(3-y)(-1-y)$

$x = y(y-2)(y-3)(y+1)$

ALTERNATIVE IN 3 STEPS

$x = y(y-2)(y+1)(y-3)$

TRANSLATE DOWN BY 1

$x = (y+1)(y+1-2)(y+1+1)(y+1-3)$

$x = (y+1)(y-1)(y+2)(y-2)$

REFLECT IN THE x AXIS

$x = (-y+1)(-y-1)(-y+2)(-y-2)$

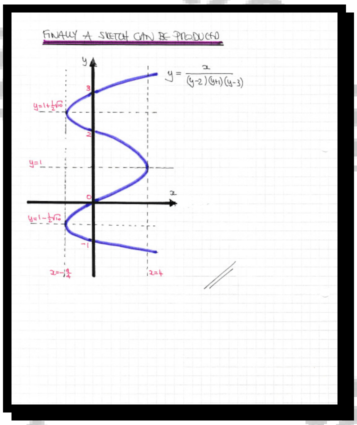
TRANSLATE UP BY 1

$x = \left[-(y-1)+1\right]\left[-(y-1)+2\right]\left[-(y-1)+\ \right]\left[-(y-1)-2\right]$

$x = (-y+2)(-y)(-y+3)(-y-1)$

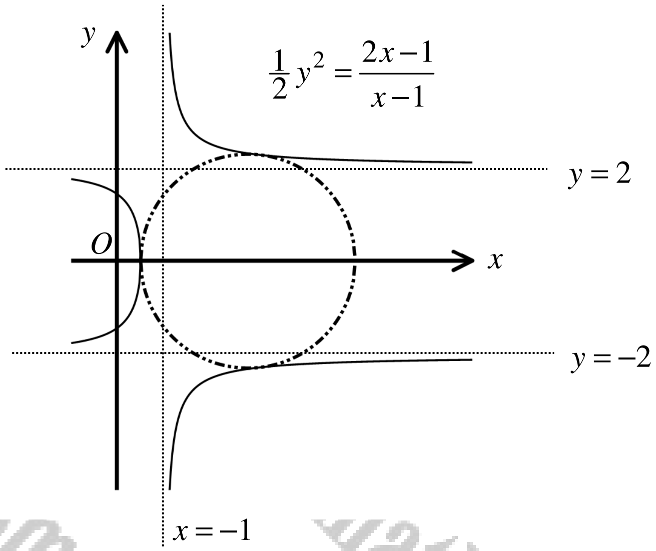
$x = (y-2)(y)(y-3)(y+1)$

$x = y(y-2)(y+1)(y-3)$



FINALLY A SKETCH CAN BE PRODUCED

Question 279 (*****)

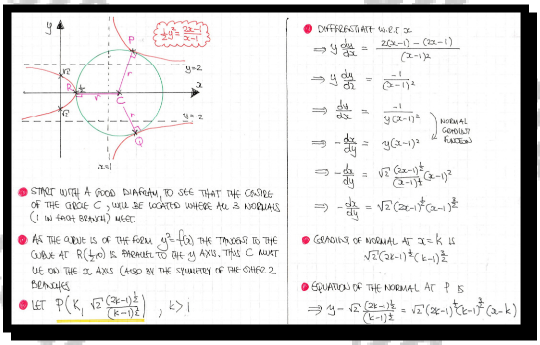


The figure above shows the curve with equation $\frac{1}{2}y^2 = \frac{2x-1}{x-1}$, whose three asymptotes are marked with dotted lines.

A circle centred at the point C and of radius r is drawn, so that it touches all three branches of the curve, as shown in the figure.

Determine the coordinates of C and the value of r.

$\boxed{}$, $\boxed{C\left(\frac{11}{4},0\right)}$, $\boxed{r = \frac{9}{4}}$



- START WITH A GOOD DIAGRAM, TO SEE THAT THE CENTRE OF THE CIRCLE C, WILL BE LOCATED WHERE ALL 3 NORMALS (1 IN EACH BRANCH) MEET.
- AS THE CURVE IS OF THE FORM $y^2 = f(x)$ THE TANGENT TO THE CURVE AT $R(\frac{1}{2},0)$ IS PARALLEL TO THE y AXIS. THUS C MUST LIE ON THE x AXIS (ALSO BY THE SYMMETRY OF THE OTHER 2 BRANCHES
- LET $P\left(k, \sqrt{2}\frac{(2k-1)^{\frac{1}{2}}}{(k-1)^{\frac{1}{2}}}\right)$, $k>1$

- DIFFERENTIATE W.R.T x

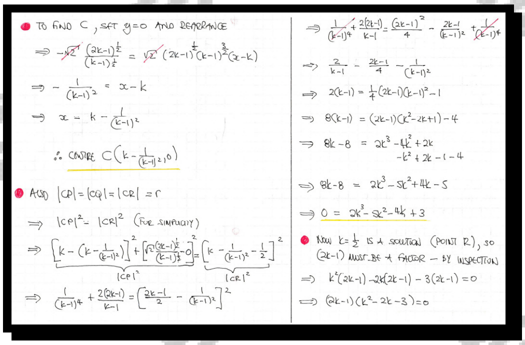
$$\Rightarrow y\frac{dy}{dx} = \frac{2(x-1)-(2x-1)}{(x-1)2}$$
$$\Rightarrow y\frac{dy}{dx} = \frac{-1}{(x-1)^2}$$
$$\Rightarrow \frac{dy}{dx} = \frac{-1}{y(x-1)^2}$$
$$\Rightarrow -\frac{dx}{dy} = y(x-1)^2$$

NORMAL GRADIENT FUNCTION

$$\Rightarrow -\frac{dx}{dy} = \sqrt{2}\frac{(2x-1)^{\frac{1}{2}}}{(x-1)^{\frac{1}{2}}}(x-1)^2$$
$$\Rightarrow -\frac{dx}{dy} = \sqrt{2}(2x-1)^{\frac{1}{2}}(x-1)^{\frac{3}{2}}$$

- GRADIENT OF NORMAL AT $x=k$ IS $\sqrt{2}(2k-1)^{\frac{1}{2}}(k-1)^{\frac{3}{2}}$
- EQUATION OF THE NORMAL AT P IS

$$\Rightarrow y - \sqrt{2}\frac{(2k-1)^{\frac{1}{2}}}{(k-1)^{\frac{1}{2}}} = \sqrt{2}(2k-1)^{\frac{1}{2}}(k-1)^{\frac{3}{2}}(x-k)$$



- TO FIND C, SET $y=0$ AND REARRANGE

$$\Rightarrow -\cancel{\sqrt{2}}\frac{(2k-1)^{\frac{1}{2}}}{(k-1)^{\frac{1}{2}}} = \cancel{\sqrt{2}}(2k-1)^{\frac{1}{2}}(k-1)^{\frac{3}{2}}(x-k)$$
$$\Rightarrow -\frac{1}{(k-1)^2} = x-k$$
$$\Rightarrow x = k - \frac{1}{(k-1)^2}$$

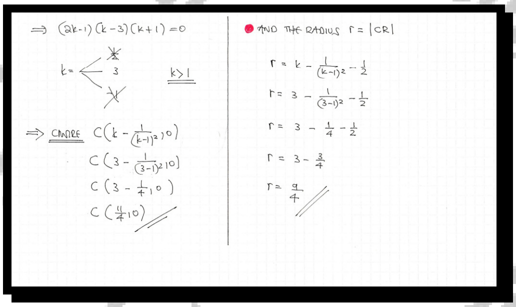
$\therefore$ CENTRE $C\left(k-\frac{1}{(k-1)^2},0\right)$

- ALSO $|CP| = |CQ| = |CR| = r$

$$\Rightarrow |CP|^2 = |CR|^2 \quad \text{(FOR SIMPLICITY)}$$
$$\Rightarrow \underbrace{\left[k-\left(k-\frac{1}{(k-1)^2}\right)\right]^2 + \left[\sqrt{2}\frac{(2k-1)^{\frac{1}{2}}}{(k-1)^{\frac{1}{2}}}-0\right]^2}_{|CP|^2} = \underbrace{\left[k-\frac{1}{(k-1)^2}-\frac{1}{2}\right]^2}_{|CR|^2}$$
$$\Rightarrow \frac{1}{(k-1)^4} + \frac{2(2k-1)}{k-1} = \left[\frac{2k-1}{2} - \frac{1}{(k-1)^2}\right]^2$$
$$\Rightarrow \cancel{\frac{1}{(k-1)^4}} + \frac{2(2k-1)}{k-1} = \frac{(2k-1)^2}{4} - \frac{2k-1}{(k-1)^2} + \cancel{\frac{1}{(k-1)^4}}$$
$$\Rightarrow \frac{2}{k-1} = \frac{2k-1}{4} - \frac{1}{(k-1)^2}$$
$$\Rightarrow 2(k-1) = \frac{1}{4}(2k-1)(k-1)^2 - 1$$
$$\Rightarrow 8(k-1) = (2k-1)(k^2-2k+1) - 4$$
$$\Rightarrow 8k-8 = 2k^3 - 4k^2 + 2k$$
$$-k^2 + 2k - 1 - 4$$
$$\Rightarrow 8k-8 = 2k^3 - 5k^2 + 4k - 5$$
$$\Rightarrow 0 = 2k^3 - 5k^2 - 4k + 3$$

- NOW $k=\frac{1}{2}$ IS A SOLUTION (POINT R), SO $(2k-1)$ MUST BE A FACTOR – BY INSPECTION

$$\Rightarrow k^2(2k-1) - 2k(2k-1) - 3(2k-1) = 0$$
$$\Rightarrow (2k-1)(k^2-2k-3) = 0$$



$$\Rightarrow (2k-1)(k-3)(k+1) = 0$$

$$k = \begin{cases} \cancel{\frac{1}{2}} \\ 3 \\ \cancel{-1} \end{cases} \qquad \underline{\underline{k>1}}$$

$$\Rightarrow \underline{\underline{\text{CENTRE}}} \; C\left(k-\frac{1}{(k-1)^2},0\right)$$
$$C\left(3-\frac{1}{(3-1)^2},0\right)$$
$$C\left(3-\frac{1}{4},0\right)$$
$$C\left(\frac{11}{4},0\right)$$

- AND THE RADIUS $r = |CR|$

$$r = k - \frac{1}{(k-1)^2} - \frac{1}{2}$$
$$r = 3 - \frac{1}{(3-1)^2} - \frac{1}{2}$$
$$r = 3 - \frac{1}{4} - \frac{1}{2}$$
$$r = 3 - \frac{3}{4}$$
$$r = \frac{9}{4}$$

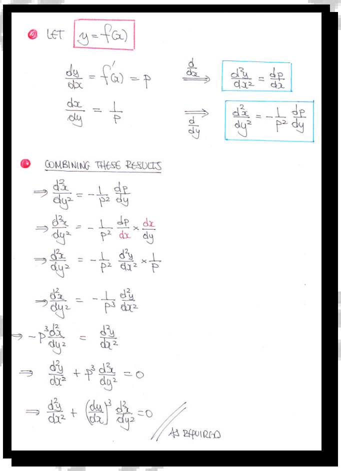
Question 280 (*****)

The function with equation $y = f(x)$ has smooth first and second derivatives.

Show that

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 \frac{d^2x}{dy^2} = 0 .$$

$\boxed{}$, $\boxed{\text{proof}}$



- LET $\boxed{y = f(x)}$

$$\frac{dy}{dx} = f'(x) = p \quad \xrightarrow{\frac{d}{dx}} \quad \boxed{\frac{d^2y}{dx^2} = \frac{dp}{dx}}$$

$$\frac{dx}{dy} = \frac{1}{p} \quad \xrightarrow[\frac{d}{dy}]{} \quad \boxed{\frac{d^2x}{dy^2} = -\frac{1}{p^2}\frac{dp}{dy}}$$

- COMBINING THESE RESULTS

$$\Rightarrow \frac{d^2x}{dy^2} = -\frac{1}{p^2}\frac{dp}{dy}$$

$$\Rightarrow \frac{d^2x}{dy^2} = -\frac{1}{p^2}\frac{dp}{dx} \times \frac{dx}{dy}$$

$$\Rightarrow \frac{d^2x}{dy^2} = -\frac{1}{p^2}\frac{d^2y}{dx^2} \times \frac{1}{p}$$

$$\Rightarrow \frac{d^2x}{dy^2} = -\frac{1}{p^3}\frac{d^2y}{dx^2}$$

$$\Rightarrow -p^3\frac{d^2x}{dy^2} = \frac{d^2y}{dx^2}$$

$$\Rightarrow \frac{d^2y}{dx^2} + p^3\frac{d^2x}{dy^2} = 0$$

$$\Rightarrow \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3\frac{d^2x}{dy^2} = 0$$

// AS REQUIRED