# BINOMIAL SERIES EXPANSIONS

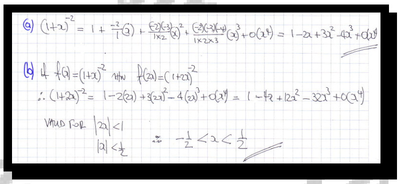
**Question 1 (**)**

The binomial expression $(1+x)^{-2}$ is to be expanded as an infinite convergent series, in ascending powers of $x$.

**a)** Determine the expansion of $(1+x)^{-2}$, up and including the term in $x^3$.

**b)** Use part **(a)** to find the expansion of $(1+2x)^{-2}$, up and including the term in $x^3$, stating the range of values of $x$ for which this expansion is valid.

$\boxed{1-2x+3x^2-4x^3+O\left(x^4\right)}$, $\boxed{1-4x+12x^2-32x^3+O\left(x^4\right)}$, $\boxed{-\frac{1}{2}<x<\frac{1}{2}}$



**Question 2** (**+)

The binomial expression $(1-x)^{-1}$ is to be expanded as an infinite convergent series, in ascending powers of $x$.

**a)** Determine the expansion of $(1-x)^{-1}$, up and including the term in $x^3$.

**b)** Use the expansion of part **(a)** to find the expansion of $\dfrac{1}{3-2x}$, up and including the term in $x^3$.

**c)** State the values of $x$ for which the expansion of $\dfrac{1}{3-2x}$ is valid.

$$\boxed{1+x+x^2+x^3+O\left(x^4\right)}, \quad \boxed{\frac{1}{3}+\frac{2}{9}x+\frac{4}{27}x^2+\frac{8}{81}x^3+O\left(x^4\right)}, \quad \boxed{-\frac{3}{2}<x<\frac{3}{2}}$$

(a) $(1-x)^{-1} = 1+(-1)(-x)+\frac{(-1)(-2)}{2!}(-x)^2+\frac{(-1)(-2)(-3)}{3!}(-x)^3+O(x^4)$

$= 1+x+x^2+x^3+O(x^4)$

(b) $\frac{1}{3-2x} = (3-2x)^{-1} = 3^{-1}\left[1-\frac{2}{3}x\right]^{-1} = \frac{1}{3}\left(1-\frac{2}{3}x\right)^{-1}$

REPLACE $x$ WITH $\frac{2}{3}x$ IN (a).

$= \frac{1}{3}\left[1+\left(\frac{2}{3}x\right)+\left(\frac{2}{3}x\right)^2+\left(\frac{2}{3}x\right)^3+O(x^4)\right]$

$= \frac{1}{3}\left[1+\frac{2}{3}x+\frac{4}{9}x^2+\frac{8}{27}x^3+O(x^4)\right]$

$= \frac{1}{3}+\frac{2}{9}x+\frac{4}{27}x^2+\frac{8}{81}x^3+O(x^4)$

(c) VALID FOR $\left|\frac{2}{3}x\right|<1$ $\quad \therefore -\frac{3}{2}<x<\frac{3}{2}$

**Question 3** (**+)

The binomial expression $(1+x)^{\frac{1}{2}}$ is to be expanded as an infinite convergent series, in ascending powers of $x$.

**a)** Determine the expansion of $(1+x)^{\frac{1}{2}}$, up and including the term in $x^3$.

**b)** Use the expansion of part **(a)** to find the expansion of $\sqrt{4+2x}$, up and including the term in $x^3$.

**c)** State the range of values of $x$ for which the expansion of $\sqrt{4+2x}$ is valid.

$\boxed{\phantom{x}}$, $\boxed{1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3+O\left(x^4\right)}$, $\boxed{2+\frac{1}{2}x-\frac{1}{16}x^2+\frac{1}{64}x^3+O\left(x^4\right)}$, $\boxed{-2<x<2}$

(a) $(1+x)^{\frac{1}{2}} = 1 + \frac{\frac{1}{2}}{1}(x) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{1\times 2}(x)^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{1\times 2\times 3}(x)^3 + O(x^4)$

$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + O(x^4)$

(b) $\sqrt{4+2x} = (4+2x)^{\frac{1}{2}} = 4^{\frac{1}{2}}\left(1+\frac{1}{2}x\right)^{\frac{1}{2}} = 2\left(1+\frac{1}{2}x\right)^{\frac{1}{2}}$

REPLACE $x$ FOR $\frac{1}{2}x$ IN EXPANSION OF PART (a)

$= 2\left[1 + \frac{1}{2}\left(\frac{1}{2}x\right) - \frac{1}{8}\left(\frac{1}{2}x\right)^2 + \frac{1}{16}\left(\frac{1}{2}x\right)^3 + O(x^4)\right]$

$= 2\left[1 + \frac{1}{4}x - \frac{1}{32}x^2 + \frac{1}{128}x^3 + O(x^4)\right]$

$= 2 + \frac{1}{2}x - \frac{1}{16}x^2 + \frac{1}{64}x^3 + O(x^4)$

(c) VALID FOR $\left|\frac{1}{2}x\right| < 1$, IE $|x| < 2$, IE $-2 < x < 2$

**Question 4** (**+)

The binomial expression $(1+x)^{\frac{1}{3}}$ is to be expanded as an infinite convergent series, in ascending powers of $x$.

**a)** Determine the expansion of $(1+x)^{\frac{1}{3}}$, up and including the term in $x^3$.

**b)** Use the expansion of part **(a)** to find the expansion of $(1-3x)^{\frac{1}{3}}$, up and including the term in $x^3$.

**c)** Use the expansion of part **(a)** to find the expansion of $(27-27x)^{\frac{1}{3}}$, up and including the term in $x^3$.

$$\boxed{1+\frac{1}{3}x-\frac{1}{9}x^2+\frac{5}{81}x^3+O\left(x^4\right)}, \quad \boxed{1-x-x^2-\frac{5}{3}x^3+O\left(x^4\right)},$$
$$\boxed{3-x-\frac{1}{3}x^2-\frac{5}{27}x^3+O\left(x^4\right)}$$

(a) $(1+x)^{\frac{1}{3}} = 1+\frac{\frac{1}{3}}{1}(x)+\frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{1\times 2}(x)^2+\frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{1\times 2\times 3}(x)^3+O(x^4)$

$= 1+\frac{1}{3}x-\frac{1}{9}x^2+\frac{5}{81}x^3+O(x^4)$

(b) $(1-3x)^{\frac{1}{3}} =$ REPLACE $x$ WITH $-3x$ ....

$= 1+\frac{1}{3}(-3x)-\frac{1}{9}(-3x)^2+\frac{5}{81}(-3x)^3+O(x^4)$

$= 1-x-x^2-\frac{5}{3}x^3+O(x^4)$

(c) $(27-27x)^{\frac{1}{3}} = 27^{\frac{1}{3}}(1-x)^{\frac{1}{3}} = 3(1-x)^{\frac{1}{3}}$

$= \dots$ REPLACE $x$ WITH $-x$

$= 3\left[1+\frac{1}{3}(-x)-\frac{1}{9}(-x)^2+\frac{5}{81}(-x)^3+O(x^4)\right]$

$= 3\left[1-\frac{1}{3}x-\frac{1}{9}x^2-\frac{5}{81}x^3+O(x^4)\right]$

$= 3-x-\frac{1}{3}x^2-\frac{5}{27}x^3+O(x^4)$

**Question 5** (**+)

$$f(x) = \frac{5x+3}{(1-x)(1+3x)}, \ |x| < \frac{1}{3}.$$

**a)** Express $f(x)$ into partial fractions.

**b)** Hence find the series expansion of $f(x)$, up and including the term in $x^3$.

$\boxed{\quad}$, $\boxed{f(x) = \frac{2}{1-x} + \frac{1}{1+3x}}$, $\boxed{f(x) = 3 - x + 11x^2 - 25x^3 + O(x^4)}$

(a) $f(x) = \frac{5x+3}{(1-x)(1+3x)} = \frac{A}{1-x} + \frac{B}{1+3x}$

$\boxed{5x+3 \equiv A(1+3x) + B(1-x)}$

If $x=1 \Rightarrow 8 = 4A \Rightarrow A=2$

If $x=-\frac{1}{3} \Rightarrow \frac{4}{3} = \frac{4}{3}B \Rightarrow B=1$

$f(x) = \frac{2}{1-x} + \frac{1}{1+3x}$

(b) $f(x) = 2(1-x)^{-1} + (1+3x)^{-1}$

- $2(1-x)^{-1} = 2\left[1 + \frac{-1}{1}(-x) + \frac{(-1)(-2)}{1\times2}(-x)^2 + \frac{(-1)(-2)(-3)}{1\times2\times3}(-x)^3 + O(x^4)\right]$
$= 2\left[1 + x + x^2 + x^3 + O(x^4)\right]$
$= 2 + 2x + 2x^2 + 2x^3 + O(x^4)$
- $(1+3x)^{-1} = 1 + (-3x) + (-3x)^2 + (-3x)^3 + O(x^4)$
$= 1 - 3x + 9x^2 - 27x^3 + O(x^4)$

REPLACED $x$ BY $-3x$ IN PREVIOUS EXPANSION OR FIND IT FROM SCRATCH

ADD $f(x) = 3 - x + 11x^2 - 25x^3 + O(x^4)$

**Question 6** (**+)

$$f(x) = \frac{2x}{(1+2x)^3}, \ x \neq -\frac{1}{2}.$$

**a)** Find the first 4 terms in the series expansion of $f(x)$.

**b)** State the range of values of $x$ for which the expansion of $f(x)$ is valid.

$\boxed{\quad}$, $\boxed{f(x) = 2x - 12x^2 + 48x^3 - 160x^4 + O(x^5)}$, $\boxed{-\frac{1}{2} < x < \frac{1}{2}}$

(a) $f(x) = \frac{2x}{(1+2x)^3} = 2x(1+2x)^{-3}$

$= 2x\left[1 + \frac{-3}{1}(2x)^1 + \frac{(-3)(-4)}{1\times2}(2x)^2 + \frac{(-3)(-4)(-5)}{1\times2\times3}(2x)^3 + O(x^4)\right]$

$= 2x\left[1 - 6x + 24x^2 - 80x^3 + O(x^4)\right]$

$= 2x - 12x^2 + 48x^3 - 160x^3 + O(x^5)$

(b) VALID FOR $|2x| < 1$

$|x| < \frac{1}{2}$ ie $-\frac{1}{2} < x < \frac{1}{2}$

**Question 7 (**+)**

$$f(x) = \frac{8x}{\sqrt{4-x}}.$$

Show that if $x$ is small, then

$$f(x) \approx 4x + \frac{1}{2}x^2 + \frac{3}{32}x^3.$$

$\boxed{\phantom{xx}}$, $\boxed{\text{proof}}$

$$\frac{8x}{\sqrt{4-x}} = 8x(4-x)^{-\frac{1}{2}} = 8x \times 4^{-\frac{1}{2}}\left(1-\tfrac{1}{4}x\right)^{-\frac{1}{2}} = 4x\left(1-\tfrac{1}{4}x\right)^{-\frac{1}{2}}$$

$$= 4x\left[1 + \frac{-\frac{1}{2}}{1}\left(-\tfrac{1}{4}x\right)^1 + \frac{-\frac{1}{2}\left(-\frac{3}{2}\right)}{1\times 2}\left(-\tfrac{1}{4}x\right)^2 + \frac{-\frac{1}{2}\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{1\times 2\times 3}\left(-\tfrac{1}{4}x\right)^3 + \dots\right]$$

$$= 4x\left[1 + \frac{1}{8}x + \frac{3}{128}x^2 + \frac{5}{1024}x^3 + \dots\right]$$

$$\approx 4x + \frac{1}{2}x^2 + \frac{3}{32}x^3$$

**Question 8 (**+)**

$$f(x) = 2\sqrt{1+4x} + \frac{4}{1+x}.$$

**a)** By combining the first 4 terms in the expansions of $(1+x)^{-1}$ and $(1+4x)^{\frac{1}{2}}$ show that

$$f(x) \approx 6 + 4x^3.$$

**b)** State range of values of $x$ for which the expansion of $f(x)$ is valid.

$\boxed{\phantom{xx}}$, $\boxed{-\frac{1}{4} < x < \frac{1}{4}}$

(a) $f(x) = 2(1+4x)^{\frac{1}{2}} + 4(1+x)^{-1}$

- $2(1+4x)^{\frac{1}{2}} = 2\left[1 + \frac{\frac{1}{2}}{1}(4x)^1 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{1\times 2}(4x)^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{1\times 2\times 3}(4x)^3 + O(x^4)\right]$
  $= 2\left[1 + 2x - 2x^2 + 4x^3 + O(x^4)\right]$
  $= 2 + 4x - 4x^2 + 8x^3 + O(x^4)$
- $4(1+x)^{-1} = 4\left[1 + \frac{-1}{1}(x)^1 + \frac{(-1)(-2)}{1\times 2}(x)^2 + \frac{(-1)(-2)(-3)}{1\times 2\times 3}(x)^3 + O(x^4)\right]$
  $= 4\left[1 - x + x^2 - x^3 + O(x^4)\right]$
  $= 4 - 4x + 4x^2 - 4x^3$

$\therefore f(x) = \begin{array}{l} 2 + \cancel{4x} - \cancel{4x^2} + 8x^3 + O(x^4) \\ 4 - \cancel{4x} + \cancel{4x^2} - 4x^3 + O(x^4) \end{array}$ $\quad \therefore f(x) \approx 6 + 4x^3$

(b) VALID FOR $|4x| < 1$, $|x| < \frac{1}{4}$ ie $-\frac{1}{4} < x < \frac{1}{4}$

**Question 9 (**+)**

$$f(x) = \frac{(1+2x)^2}{1-2x}, \quad x \neq \frac{1}{2}.$$

**a)** Find the first 4 terms in the series expansion of $f(x)$.

**b)** State the range of values of $x$ for which the expansion of $f(x)$ is valid.

$\boxed{f(x) = 1 + 6x + 16x^2 + 32x^3 + O(x^4)}$, $\boxed{-\frac{1}{2} < x < \frac{1}{2}}$

(a) $f(x) = \frac{(1+2x)^2}{1-2x} = (1+2x)^2(1-2x)^{-1}$

$= (1+4x+4x^2)\left(1 + \frac{(-1)}{1}(-2x) + \frac{(-1)(-2)}{1\times 2}(-2x)^2 + \frac{(-1)(-2)(-3)}{1\times 2\times 3}(-2x)^3 + \dots\right)$

$= (1+4x+4x^2)(1 + 2x + 4x^2 + 8x^3 + \dots)$

$= 1 + 2x + 4x^2 + 8x^3 + \dots$

$\quad 4x + 8x^2 + 16x^3 + \dots$

$\quad + 4x^2 + 8x^3 + \dots$

$= 1 + 6x + 16x^2 + 32x^3 + \dots$

(b) VALID FOR $|2x| < 1$

$|x| < \frac{1}{2}$

$\therefore -\frac{1}{2} < x < \frac{1}{2}$

**Question 10 (***)**

$$f(x) = (1+3x)\left(1 - \tfrac{2}{3}x\right)^{-2}.$$

**a)** Show that if $x$ is numerically small

$$f(x) \approx 1 + \frac{13}{3}x + \frac{16}{3}x^2 + \frac{140}{27}x^3.$$

**b)** State the range of values of $x$ for which the expansion of $f(x)$ is valid.

$\boxed{-\frac{3}{2} < x < \frac{3}{2}}$

(a) $(1+3x)\left(1-\frac{2}{3}x\right)^{-2} = (1+3x)\left[1 + \frac{-2}{1}\left(-\frac{2}{3}x\right)^1 + \frac{-2(-3)}{1\times 2}\left(-\frac{2}{3}x\right)^2 + \frac{-2(-3)(-4)}{1\times 2\times 3}\left(-\frac{2}{3}x\right)^3 + \dots\right]$

$f(x) = (1+3x)\left[1 + \frac{4}{3}x + \frac{4}{3}x^2 + \frac{32}{27}x^3 + \dots\right]$

$f(x) = 1 + \frac{4}{3}x + \frac{4}{3}x^2 + \frac{32}{27}x^3 + \dots$

$\quad 3x + 4x^2 + 4x^3 + \dots$

$f(x) = 1 + \frac{13}{3}x + \frac{16}{3}x^2 + \frac{140}{27}x^3 + \dots$

(b) VALID FOR $\left|\frac{2}{3}x\right| < 1$

$|x| < \frac{3}{2}$

$-\frac{3}{2} < x < \frac{3}{2}$

**Question 11** (***)

$$y = \sqrt{4-12x}\,, \quad -\frac{1}{3} < x < \frac{1}{3}.$$

**a)** Find the binomial expansion of $y$ in ascending powers of $x$ up and including the term in $x^3$, writing all coefficients in their simplest form.

**b)** Hence find the coefficient of $x^2$ in the expansion of

$$(12x-4)(4-12x)^{\frac{1}{2}}\,.$$

$\boxed{\text{C4M}}$, $\boxed{y = 2 - 3x - \frac{9}{4}x^2 - \frac{27}{8}x^3 + O\left(x^4\right)}$, $\boxed{-27}$

(a) $y = \sqrt{4-12x} = (4-12x)^{\frac{1}{2}} = 4^{\frac{1}{2}}(1-3x)^{\frac{1}{2}} = 2(1-3x)^{\frac{1}{2}}$

$= 2\left[1 + \frac{1}{2}(-3x) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{1\times 2}(-3x)^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{1\times 2\times 3}(-3x)^3 + O(x^4)\right]$

$= 2\left[1 - \frac{3}{2}x - \frac{9}{8}x^2 - \frac{27}{16}x^3 + O(x^4)\right]$

$= 2 - 3x - \frac{9}{4}x^2 - \frac{27}{8}x^3 + O(x^4)$

(b) $(12x-4)(4-12x)^{\frac{1}{2}}$

$(12x-4)\left(2 - 3x - \frac{9}{4}x^2 - \frac{27}{8}x^3 + O(x^4)\right)$

$-36x^2$ $\quad 9x^2$

$\therefore -27x^2$

$\therefore -27$

**Question 12** (***)

The binomial $(1+x)^{-\frac{1}{2}}$ is to be expanded as an infinite convergent series, in ascending powers of $x$.

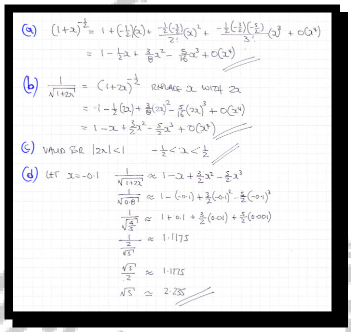
**a)** Find the series expansion of $(1+x)^{-\frac{1}{2}}$ up and including the term in $x^3$.

**b)** Use the expansion of part **(a)** to find the expansion of $\dfrac{1}{\sqrt{1+2x}}$, up and including the term in $x^3$.

**c)** State the range of values of $x$ for which the expansion of $\dfrac{1}{\sqrt{1+2x}}$ is valid.

**d)** Use the expansion of $\dfrac{1}{\sqrt{1+2x}}$ with $x=-0.1$ to show that $\sqrt{5}\approx 2.235$.

$\boxed{1-\frac{1}{2}x+\frac{3}{8}x^2-\frac{5}{16}x^3+O\left(x^4\right)}$, $\boxed{1-x+\frac{3}{2}x^2-\frac{5}{2}x^3+O\left(x^4\right)}$, $\boxed{-\frac{1}{2}<x<\frac{1}{2}}$


(a) $(1+x)^{-\frac{1}{2}} = 1+\left(-\frac{1}{2}\right)(x)+\frac{-\frac{1}{2}\left(-\frac{3}{2}\right)}{2!}(x)^2+\frac{-\frac{1}{2}\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}(x)^3+O(x^4)$

$= 1-\frac{1}{2}x+\frac{3}{8}x^2-\frac{5}{16}x^3+O(x^4)$

(b) $\frac{1}{\sqrt{1+2x}} = (1+2x)^{-\frac{1}{2}}$ REPLACE $x$ WITH $2x$

$= 1-\frac{1}{2}(2x)+\frac{3}{8}(2x)^2-\frac{5}{16}(2x)^3+O(x^4)$

$= 1-x+\frac{3}{2}x^2-\frac{5}{2}x^3+O(x^4)$

(c) VALID FOR $|2x|<1$ $\quad -\frac{1}{2}<x<\frac{1}{2}$

(d) LET $x=-0.1$ $\quad \frac{1}{\sqrt{1+2x}} \approx 1-x+\frac{3}{2}x^2-\frac{5}{2}x^3$

$\frac{1}{\sqrt{0.8}} \approx 1-(-0.1)+\frac{3}{2}(-0.1)^2-\frac{5}{2}(-0.1)^3$

$\frac{1}{\sqrt{\frac{4}{5}}} \approx 1+0.1+\frac{3}{2}(0.01)+\frac{5}{2}(0.001)$

$\frac{1}{\frac{2}{\sqrt{5}}} \approx 1.1175$

$\frac{\sqrt{5}}{2} \approx 1.1175$

$\sqrt{5} \simeq 2.235$

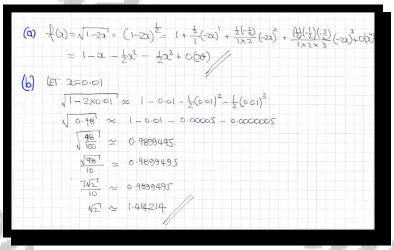
**Question 13** (***)

$$f(x) = \sqrt{1-2x}, \quad |x| < \frac{1}{2}.$$

**a)** Expand $f(x)$ as an infinite series, up and including the term in $x^3$.

**b)** By substituting $x = 0.01$ in the expansion, show that $\sqrt{2} \approx 1.414214$.

$$\boxed{f(x) = 1 - x - \frac{1}{2}x^2 - \frac{1}{2}x^3 + O\left(x^4\right)}$$



(a) $f(x) = \sqrt{1-2x} = (1-2x)^{\frac{1}{2}} = 1 + \frac{\frac{1}{2}}{1}(-2x)^1 + \frac{\frac{1}{2}(-\frac{1}{2})}{1\times 2}(-2x)^2 + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{1\times 2\times 3}(-2x)^3 + O(x^4)$

$= 1 - x - \frac{1}{2}x^2 - \frac{1}{2}x^3 + O(x^4)$

(b) LET $x = 0.01$

$\sqrt{1 - 2\times 0.01} \approx 1 - 0.01 - \frac{1}{2}(0.01)^2 - \frac{1}{2}(0.01)^3$

$\sqrt{0.98} \approx 1 - 0.01 - 0.00005 - 0.0000005$

$\sqrt{\frac{98}{100}} \simeq 0.9899495$

$\frac{\sqrt{98}}{10} \simeq 0.9899495$

$\frac{7\sqrt{2}}{10} \simeq 0.9899495$

$\sqrt{2} \simeq 1.414214$

**Question 14** **(***)**

$$f(x) \equiv \frac{18-19x}{(1-x)(2-3x)}, \; x \in \mathbb{R}, \; |x| < \frac{2}{3}.$$

**a)** Express $f(x)$ in partial fractions.

**b)** Hence, or otherwise, show that if $x$ is numerically small

$$f(x) \approx 9 + 13x + 19x^2 + 28x^3.$$

$\boxed{\phantom{x}}$, $\boxed{f(x) \equiv \frac{1}{1-x} + \frac{16}{2-3x}}$

(a) $f(x) = \frac{18-19x}{(1-x)(2-3x)} \equiv \frac{A}{1-x} + \frac{B}{2-3x}$

$\boxed{18-19x \equiv A(2-3x) + B(1-x)}$

- If $x=1$, $-1 = -A \Rightarrow A = 1$
- If $x=0$, $18 = 2A + B \Rightarrow B = 16$

$\therefore f(x) = \frac{1}{1-x} + \frac{16}{2-3x}$

(b) $\frac{1}{1-x} = (1-x)^{-1} = 1 + \frac{-1}{1}(-x) + \frac{-1(-2)}{1\times2}(-x)^2 + \frac{-1(-2)(-3)}{1\times2\times3}(-x)^3 + O(x^4)$

$= 1 + x + x^2 + x^3 + O(x^4)$

$\frac{16}{2-3x} = 16(2-3x)^{-1} = 16\times2^{-1}(1-\frac{3}{2}x) = 8(1-\frac{3}{2}x)^{-1}$

USE PREVIOUS EXPANSION $x \longmapsto \frac{3}{2}x$

$= 8\left[1 + \left(\frac{3}{2}x\right) + \left(\frac{3}{2}x\right)^2 + \left(\frac{3}{2}x\right)^3 + O(x^4)\right]$

$= 8 + 12x + 18x^2 + 27x^3 + O(x^4)$ ADD THE EXPANSIONS

$\therefore f(x) = 9 + 13x + 19x^2 + 28x^3 + O(x^4)$

**Question 15** (***)

$$f(x) \equiv \frac{2-x}{\sqrt{1+x}}, \quad |x|<1.$$

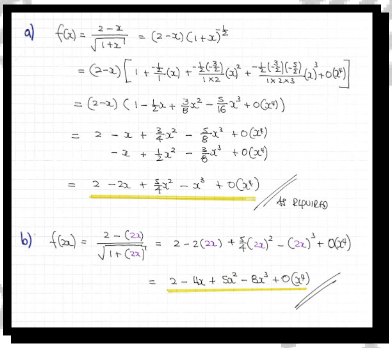
**a)** Show that the first four terms in the binomial expansion of $f(x)$ are

$$2-2x+\frac{5}{4}x^2-x^3.$$

**b)** Use the answer of part **(a)** to find the first four terms in the expansion of

$$g(x)=\frac{2-2x}{\sqrt{1+2x}}.$$

$\boxed{\phantom{M}}$ , $\boxed{g(x)=2-4x+5x^2-8x^3}$



a) $f(x) = \frac{2-x}{\sqrt{1+x}} = (2-x)(1+x)^{-\frac{1}{2}}$

$= (2-x)\left[1 + \frac{-\frac{1}{2}}{1}(x) + \frac{-\frac{1}{2}(-\frac{3}{2})}{1\times 2}(x)^2 + \frac{-\frac{1}{2}(-\frac{3}{2})(-\frac{5}{2})}{1\times 2\times 3}(x)^3 + O(x^4)\right]$

$= (2-x)\left(1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + O(x^4)\right)$

$= 2 - x + \frac{3}{4}x^2 - \frac{5}{8}x^3 + O(x^4)$

$\quad - x + \frac{1}{2}x^2 - \frac{3}{8}x^3 + O(x^4)$

$= 2 - 2x + \frac{5}{4}x^2 - x^3 + O(x^4)$ // AS REQUIRED

b) $f(2x) = \frac{2-(2x)}{\sqrt{1+(2x)}} = 2 - 2(2x) + \frac{5}{4}(2x)^2 - (2x)^3 + O(x^4)$

$= 2 - 4x + 5x^2 - 8x^3 + O(x^4)$ //

**Question 16** (***)

$$f(x)=\sqrt{1+\frac{1}{8}x}\ ,\ |x|<8\,.$$

**a)** Expand $f(x)$ as an infinite series, up and including the term in $x^2$.

**b)** By substituting $x=1$ in the expansion, show that

$$\sqrt{2}\approx\frac{256}{181}\quad\text{or}\quad\sqrt{2}\approx\frac{181}{128}.$$

$$\boxed{\phantom{xxxx}}\ ,\ \boxed{f(x)=1+\frac{1}{16}x-\frac{1}{512}x^2+O\left(x^3\right)}$$

a) EXPAND USING THE STANDARD FORMULA

$f(x)=\sqrt{1+\frac{1}{8}x}=\left(1+\frac{1}{8}x\right)^{\frac{1}{2}}$

$f(x)=1+\frac{\frac{1}{2}}{1}\left(\frac{1}{8}x\right)+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{1\times2}\left(\frac{1}{8}x\right)^2+O(x^3)$

$f(x)=1+\frac{1}{16}x-\frac{1}{512}x^2+O(x^3)$

b) LET $x=1$ IN THE ABOVE EXPANSION

$\sqrt{1+\frac{1}{8}\times1}\approx1+\frac{1}{16}\times1-\frac{1}{512}\times1^2$

$\sqrt{\frac{9}{8}}\approx\frac{543}{512}$

NOW WE HAVE TWO POSSIBLE APPROXIMATIONS

$\frac{3}{2\sqrt{2}}\approx\frac{543}{512}$

$\div3$

$\frac{1}{2\sqrt{2}}\approx\frac{181}{512}$

$\times2$

$\frac{1}{\sqrt{2}}\approx\frac{181}{256}$

$\sqrt{2}\approx\frac{256}{181}$

RATIONALISE LHS

$\frac{\sqrt{2}}{2}\approx\frac{181}{256}$

$\times2$

$\sqrt{2}\approx\frac{181}{128}$

**Question 17** (***)

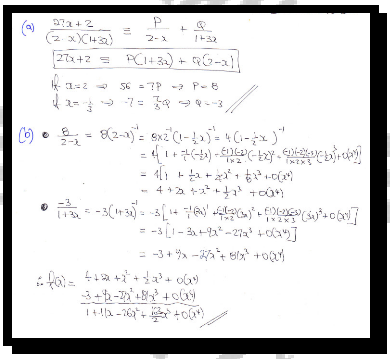
$$\frac{27x+2}{(2-x)(1+3x)} \equiv \frac{P}{2-x}+\frac{Q}{1+3x}.$$

**a)** Find the value of each of the constants $P$ and $Q$.

**b)** Hence show that if $x$ is sufficiently small

$$\frac{27x+2}{(2-x)(1+3x)} \approx 1+11x-26x^2+\frac{163}{2}x^3.$$

$\boxed{C4}$, $\boxed{P=8}$, $\boxed{Q=-3}$



(a) $\frac{27x+2}{(2-x)(1+3x)} \equiv \frac{P}{2-x}+\frac{Q}{1+3x}$

$\boxed{27x+2 \equiv P(1+3x)+Q(2-x)}$

If $x=2 \Rightarrow 56=7P \Rightarrow P=8$

If $x=-\frac{1}{3} \Rightarrow -7=\frac{7}{3}Q \Rightarrow Q=-3$

(b) • $\frac{8}{2-x}=8(2-x)^{-1}=8\times2^{-1}(1-\frac{1}{2}x)^{-1}=4(1-\frac{1}{2}x)^{-1}$

$=4\left[1+\frac{-1}{1}(-\frac{1}{2}x)+\frac{(-1)(-2)}{1\times2}(-\frac{1}{2}x)^2+\frac{(-1)(-2)(-3)}{1\times2\times3}(-\frac{1}{2}x)^3+O(x^4)\right]$

$=4\left[1+\frac{1}{2}x+\frac{1}{4}x^2+\frac{1}{8}x^3+O(x^4)\right]$

$=4+2x+x^2+\frac{1}{2}x^3+O(x^4)$

• $\frac{-3}{1+3x}=-3(1+3x)^{-1}=-3\left[1+\frac{-1}{1}(3x)^1+\frac{(-1)(-2)}{1\times2}(3x)^2+\frac{(-1)(-2)(-3)}{1\times2\times3}(3x)^3+O(x^4)\right]$

$=-3\left[1-3x+9x^2-27x^3+O(x^4)\right]$

$=-3+9x-27x^2+81x^3+O(x^4)$

$\therefore f(x)=4+2x+x^2+\frac{1}{2}x^3+O(x^4)$

$-3+9x-27x^2+81x^3+O(x^4)$

$1+11x-26x^2+\frac{163}{2}x^3+O(x^4)$

**Question 18** (***)

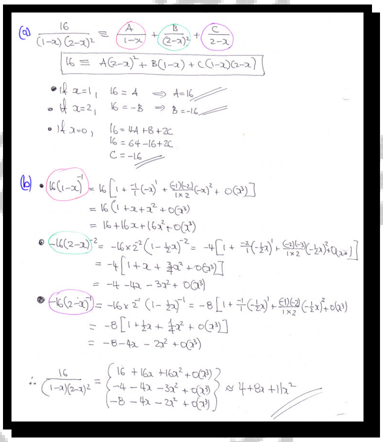
$$\frac{16}{(1-x)(2-x)^2} \equiv \frac{A}{1-x} + \frac{B}{(2-x)^2} + \frac{C}{2-x}.$$

**a)** Find the value of each of the constants $A$, $B$ and $C$.

**b)** Hence show that if $x$ is sufficiently small

$$\frac{16}{(1-x)(2-x)^2} \approx 4 + 8x + 11x^2 .$$

$\boxed{\phantom{x}}$, $\boxed{A = 16}$, $\boxed{B = -16}$, $\boxed{C = -16}$



(a) $\frac{16}{(1-x)(2-x)^2} \equiv \frac{A}{1-x} + \frac{B}{(2-x)^2} + \frac{C}{2-x}$

$\boxed{16 \equiv A(2-x)^2 + B(1-x) + C(1-x)(2-x)}$

- If $x=1$, $16 = A \Rightarrow A = 16$
- If $x=2$, $16 = -B \Rightarrow B = -16$
- If $x=0$, $16 = 4A + B + 2C$
  $16 = 64 - 16 + 2C$
  $C = -16$

(b)
- $16(1-x)^{-1} = 16\left[1 + \frac{-1}{1}(-x)^1 + \frac{(-1)(-2)}{1\times 2}(-x)^2 + O(x^3)\right]$
  $= 16(1 + x + x^2 + O(x^3))$
  $= 16 + 16x + 16x^2 + O(x^3)$
- $-16(2-x)^{-2} = -16 \times 2^{-2}\left(1 - \frac{1}{2}x\right)^{-2} = -4\left[1 + \frac{-2}{1}\left(-\frac{1}{2}x\right)^1 + \frac{(-2)(-3)}{1\times 2}\left(-\frac{1}{2}x\right)^2 + O(x^3)\right]$
  $= -4\left[1 + x + \frac{3}{4}x^2 + O(x^3)\right]$
  $= -4 - 4x - 3x^2 + O(x^3)$
- $-16(2-x)^{-1} = -16 \times 2^{-1}\left(1 - \frac{1}{2}x\right)^{-1} = -8\left[1 + \frac{-1}{1}\left(-\frac{1}{2}x\right)^1 + \frac{(-1)(-2)}{1\times 2}\left(-\frac{1}{2}x\right)^2 + O(x^3)\right]$
  $= -8\left[1 + \frac{1}{2}x + \frac{1}{4}x^2 + O(x^3)\right]$
  $= -8 - 4x - 2x^2 + O(x^3)$

$\therefore \frac{16}{(1-x)(2-x)^2} = \left\{\begin{array}{l} 16 + 16x + 16x^2 + O(x^3) \\ -4 - 4x - 3x^2 + O(x^3) \\ -8 - 4x - 2x^2 + O(x^3) \end{array}\right\} \approx 4 + 8x + 11x^2$

**Question 19** (***)

$$f(x)=\frac{15}{\sqrt{1-x}}, \ |x|<1.$$

**a)** Expand $f(x)$ as an infinite series, up and including the term in $x^3$.

**b)** By substituting $x=0.1$ in the expansion of $f(x)$, show that

$$\sqrt{10} \approx 3.162$$

$$\boxed{f(x)=15+\frac{15}{2}x+\frac{45}{8}x^2+\frac{75}{16}x^3+O\left(x^4\right)}$$

(a) $\frac{15}{\sqrt{1-x}} = 15(1-x)^{-\frac{1}{2}} = 15\left[1+\frac{-\frac{1}{2}}{1}(-x)^1+\frac{-\frac{1}{2}\left(-\frac{3}{2}\right)}{1\times 2}(-x)^2+\frac{-\frac{1}{2}\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{1\times 2\times 3}(-x)^3+\dots\right]$

$= 15\left[1+\frac{1}{2}x+\frac{3}{8}x^2+\frac{5}{16}x^3+\dots\right]$

$= 15+\frac{15}{2}x+\frac{45}{8}x^2+\frac{75}{16}x^3+\dots$

(b) $\frac{15}{\sqrt{1-x}} = 15+\frac{15}{2}x+\frac{45}{8}x^2+\frac{75}{16}x^3+\dots$

LET $x=0.1$

$\frac{15}{\sqrt{1-0.1}} = 15+\frac{15}{2}(0.1)+\frac{45}{8}(0.1)^2+\frac{75}{16}(0.1)^3+\dots$

$\frac{15}{\sqrt{\frac{9}{10}}} = 15+\frac{15}{20}+\frac{45}{800}+\frac{75}{16000}+\dots$

$\frac{15}{\frac{3}{\sqrt{10}}} = 15.8109375$

$5\sqrt{10} = 15.8109375$

$\sqrt{10} \simeq 3.162$ // (3 d.p)

**Question 20** (***)

In the convergent binomial expansion of

$$(1+bx)^n, \quad |bx|<1$$

the coefficient of $x$ is $-6$ and the coefficient of $x^2$ is $27$.

**a)** Show that $b=3$ and find the value of $n$.

**b)** Find the coefficient of $x^3$.

**c)** State the range of values of $x$ for which the above expansion is valid.

$\boxed{\phantom{proof}}$, $\boxed{n=-2}$, $\boxed{\left[x^3\right]=-108}$, $\boxed{-\frac{1}{3}<x<\frac{1}{3}}$

a) EXPAND $(1+bx)^n$ IN GENERAL FORM UP TO $x^3$

$$(1+bx)^n = 1 + \frac{n}{1}(bx)^1 + \frac{n(n-1)}{1\times 2}(bx)^2 + \frac{n(n-1)(n-2)}{1\times 2\times 3}(bx)^3 + \dots$$

$$(1+bx)^n = 1 + \underbrace{nb}_{-6}x + \underbrace{\tfrac{1}{2}n(n-1)b^2}_{27}x^2 + \underbrace{\tfrac{1}{6}n(n-1)(n-2)b^3}_{\text{REQUIRED IN (b)}}x^3 + \dots$$

SOLVING SIMULTANEOUSLY

$$\left\{\begin{array}{l} nb=-6 \\ \frac{1}{2}n(n-1)b^2=27 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} nb=-6 \\ n(n-1)b^2=54 \end{array}\right\} \Rightarrow$$

$$\Rightarrow \left\{\begin{array}{l} n^2b^2=36 \\ n^2(n-1)b^2=54n \end{array}\right\} \Rightarrow$$

$$\Rightarrow 36(n-1)=54n$$

$$\Rightarrow 36n-36=54n$$

$$\Rightarrow -36=18n$$

$$n=-2$$

$$\& \ b=3 \quad (nb=-6)$$

b) $[x^3]: \ \frac{1}{6}n(n-1)(n-2)b^3 = \frac{1}{6}(-2)(-3)(-4)\times 3^3 = -108$

c) VALID FOR $|bx|<1$

$|3x|<1$

$|x|<\frac{1}{3}$

$\therefore -\frac{1}{3}<x<\frac{1}{3}$

**Question 21** (***)

$$f(x) = \frac{20}{\sqrt{4+2x}}, \quad |x| < 2.$$

**a)** Expand $f(x)$ as an infinite series, up and including the term in $x^3$.

**b)** By substituting $x = \frac{1}{12}$ in the above expansion, show that

$$\sqrt{6} \approx 2.45.$$

$\boxed{C4O}$, $\boxed{f(x) = 10 - \frac{5}{2}x + \frac{15}{16}x^2 - \frac{25}{64}x^3 + O(x^4)}$

(a) $f(x) = \frac{20}{\sqrt{4+2x}} = 20(4+2x)^{-\frac{1}{2}} = 20 \times 4^{-\frac{1}{2}}(1+\frac{1}{2}x)^{-\frac{1}{2}} = 10(1+\frac{1}{2}x)^{-\frac{1}{2}}$

$f(x) = 10\left[1 + \frac{-\frac{1}{2}}{1}(\frac{1}{2}x)^1 + \frac{(-\frac{1}{2})(-\frac{3}{2})}{1\times 2}(\frac{1}{2}x)^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{1\times 2\times 3}(\frac{1}{2}x)^3 + O(x^4)\right]$

$f(x) = 10\left[1 - \frac{1}{4}x + \frac{3}{32}x^2 - \frac{5}{128}x^3 + O(x^4)\right]$

$f(x) = 10 - \frac{5}{2}x + \frac{15}{16}x^2 - \frac{25}{64}x^3 + O(x^4)$

(b) $\frac{20}{\sqrt{4+2x}} \approx 10 - \frac{5}{2}x + \frac{15}{16}x^2 - \frac{25}{64}x^3 + O(x^4)$

• LET $x = \frac{1}{12}$

$\Rightarrow \frac{20}{\sqrt{4+2\times\frac{1}{12}}} \simeq 10 - \frac{5}{2}\left(\frac{1}{12}\right) + \frac{15}{16}\left(\frac{1}{12}\right)^2 - \frac{25}{64}\left(\frac{1}{12}\right)^3$

$\Rightarrow 4\sqrt{6} \approx 10 - \frac{5}{24} + \frac{5}{768} - \frac{25}{110592}$

$\Rightarrow 4\sqrt{6} \simeq 9.79795...$

$\Rightarrow \sqrt{6} \approx 2.45$

AS REQUIRED

**Question 22** (***)

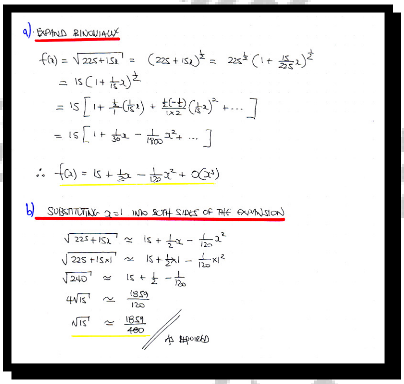
$$f(x)=\sqrt{225+15x}\,,\ |x|<15\,.$$

**a)** Expand $f(x)$ as an infinite series, up and including the term in $x^2$.

**b)** By substituting $x=1$ in the expansion of $f(x)$, show that

$$\sqrt{15}\approx\frac{1859}{480}\,.$$

$\boxed{\phantom{xxxx}}$ , $\boxed{f(x)=15+\frac{1}{2}x-\frac{1}{120}x^2+O(x^3)}$



a) EXPAND BINOMIALLY

$f(x)=\sqrt{225+15x}=(225+15x)^{\frac{1}{2}}=225^{\frac{1}{2}}\left(1+\frac{15}{225}x\right)^{\frac{1}{2}}$

$=15\left(1+\frac{1}{15}x\right)^{\frac{1}{2}}$

$=15\left[1+\frac{\frac{1}{2}}{1}\left(\frac{1}{15}x\right)+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{1\times2}\left(\frac{1}{15}x\right)^2+\dots\right]$

$=15\left[1+\frac{1}{30}x-\frac{1}{1800}x^2+\dots\right]$

$\therefore\ f(x)=15+\frac{1}{2}x-\frac{1}{120}x^2+O(x^3)$

b) SUBSTITUTING $x=1$ INTO BOTH SIDES OF THE EXPANSION

$\sqrt{225+15x}\simeq15+\frac{1}{2}x-\frac{1}{120}x^2$

$\sqrt{225+15\times1}\simeq15+\frac{1}{2}\times1-\frac{1}{120}\times1^2$

$\sqrt{240}\approx15+\frac{1}{2}-\frac{1}{120}$

$4\sqrt{15}\simeq\frac{1859}{120}$

$\sqrt{15}\simeq\frac{1859}{480}$

AS REQUIRED

**Question 23** (***)

$$f(x) = \frac{4x+1}{(1-2x)(1+x)}, \quad |x| < \frac{1}{2}.$$

**a)** Find the first four terms in the series expansion of $(1+x)^{-1}$.

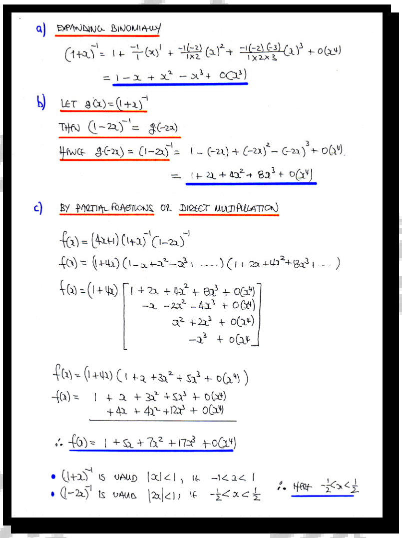
**b)** Hence, find the first four terms in the series expansion of $(1-2x)^{-1}$.

**c)** Hence show that

$$f(x) \approx 1 + 5x + 7x^2 + 17x^3,$$

stating the range of values of $x$ for which the above approximation is valid.

$\boxed{\quad}$, $\boxed{(1+x)^{-1} = 1 - x + x^2 - x^3 + O(x^4)}$, $\boxed{(1-2x)^{-1} = 1 + 2x + 4x^2 + 8x^3 + O(x^4)}$, $\boxed{-\frac{1}{2} < x < \frac{1}{2}}$



a) EXPANDING BINOMIALLY

$(1+x)^{-1} = 1 + \frac{-1}{1}(x)^1 + \frac{-1(-2)}{1\times 2}(x)^2 + \frac{-1(-2)(-3)}{1\times 2\times 3}(x)^3 + O(x^4)$

$= 1 - x + x^2 - x^3 + O(x^3)$

b) LET $g(x) = (1+x)^{-1}$

THEN $(1-2x)^{-1} = g(-2x)$

HENCE $g(-2x) = (1-2x)^{-1} = 1 - (-2x) + (-2x)^2 - (-2x)^3 + O(x^4)$

$= 1 + 2x + 4x^2 + 8x^3 + O(x^4)$

c) BY PARTIAL FRACTIONS OR DIRECT MULTIPLICATION

$f(x) = (4x+1)(1+x)^{-1}(1-2x)^{-1}$

$f(x) = (1+4x)(1 - x + x^2 - x^3 + \dots)(1 + 2x + 4x^2 + 8x^3 + \dots)$

$$f(x) = (1+4x)\left[\begin{array}{r} 1 + 2x + 4x^2 + 8x^3 + O(x^4) \\ -x - 2x^2 - 4x^3 + O(x^4) \\ x^2 + 2x^3 + O(x^4) \\ -x^3 + O(x^4) \end{array}\right]$$

$f(x) = (1+4x)(1 + x + 3x^2 + 5x^3 + O(x^4))$

$f(x) = 1 + x + 3x^2 + 5x^3 + O(x^4)$
$\quad + 4x + 4x^2 + 12x^3 + O(x^4)$

$\therefore f(x) = 1 + 5x + 7x^2 + 17x^3 + O(x^4)$

- $(1+x)^{-1}$ IS VALID $|x| < 1$, IE $-1 < x < 1$
- $(1-2x)^{-1}$ IS VALID $|2x| < 1$, IE $-\frac{1}{2} < x < \frac{1}{2}$

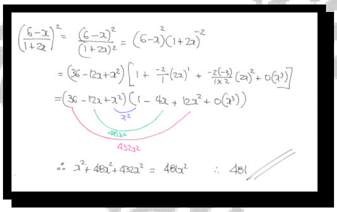
$\therefore$ HERE $-\frac{1}{2} < x < \frac{1}{2}$

**Question 24** (***)

$$f(x)=\left(\frac{6-x}{1+2x}\right)^2, \quad |x|<\frac{1}{2}.$$

Determine the value of the coefficient of $x^2$ in the binomial expansion of $f(x)$.

$\boxed{C4}$, $\boxed{481}$



$$\left(\frac{6-x}{1+2x}\right)^2=\frac{(6-x)^2}{(1+2x)^2}=(6-x)^2(1+2x)^{-2}$$

$$=\left(36-12x+x^2\right)\left[1+\frac{-2}{1}(2x)^1+\frac{-2(-3)}{1\times 2}(2x)^2+O(x^3)\right]$$

$$=\left(36-12x+x^2\right)\left(1-4x+12x^2+O(x^3)\right)$$

$x^2$ $48x^2$ $432x^2$

$$\therefore\ x^2+48x^2+432x^2=481x^2 \qquad \therefore 481$$

**Question 25** (***)

$$f(x) = (1-x)^{\frac{1}{3}}, \ -1 < x < 1.$$

**a)** Find the binomial expansion of $f(x)$ in ascending powers of $x$ up and including the term in $x^2$.

$$g(x) = (8-3x)^{\frac{1}{3}}, \ -\frac{8}{3} < x < \frac{8}{3}.$$

**b)** Use the result of part **(a)** to find the binomial expansion of $g(x)$ in ascending powers of $x$ up and including the term in $x^2$.

**c)** Hence, show that

$$\sqrt[3]{7} \approx \frac{551}{288}.$$

$\boxed{\quad}$, $\boxed{f(x) = 1 - \frac{1}{3}x - \frac{1}{9}x^2 + O(x^3)}$, $\boxed{g(x) = 2 - \frac{1}{4}x - \frac{1}{32}x^2 + O(x^3)}$

a) $f(x) = (1-x)^{\frac{1}{3}}$

$\Rightarrow f(x) = 1 + \frac{\frac{1}{3}}{1}(-x)^1 + \frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{1 \times 2}(-x)^2 + O(x^3)$

$\Rightarrow f(x) = 1 - \frac{1}{3}x - \frac{1}{9}x^2 + O(x^3)$

b) USING PART (a)

$\Rightarrow g(x) = (8-3x)^{\frac{1}{3}} = 8^{\frac{1}{3}}\left(1 - \frac{3}{8}x\right)^{\frac{1}{3}} = 2\left(1 - \frac{3}{8}x\right)^{\frac{1}{3}}$

$\Rightarrow g(x) = 2f\left(\frac{3}{8}x\right)$

$\Rightarrow g(x) = 2\left[1 - \frac{1}{3}\left(\frac{3}{8}x\right)^1 - \frac{1}{9}\left(\frac{3}{8}x\right)^2 + O(x^3)\right]$

$\Rightarrow g(x) = 2\left[1 - \frac{1}{8}x - \frac{1}{64}x^2 + O(x^3)\right]$

$\Rightarrow g(x) = 2 - \frac{1}{4}x - \frac{1}{32}x^2 + O(x^3)$

c) LET $x = \frac{1}{3}$ IN BOTH SIDES OF THE EXPANSION OF $g(x)$

$\Rightarrow (8-3x)^{\frac{1}{3}} \approx 2 - \frac{1}{4}x - \frac{1}{32}x^2$

$\Rightarrow \left(8 - 3 \times \frac{1}{3}\right)^{\frac{1}{3}} \approx 2 - \frac{1}{4}\left(\frac{1}{3}\right) - \frac{1}{32}\left(\frac{1}{3}\right)^2$

$\Rightarrow 7^{\frac{1}{3}} \approx 2 - \frac{1}{12} - \frac{1}{288}$

$\Rightarrow \sqrt[3]{7} \approx \frac{551}{288}$

AS REQUIRED

**Question 26** (***)

In the series expansion of

$$(1+ax)^n, \ |ax|<1,$$

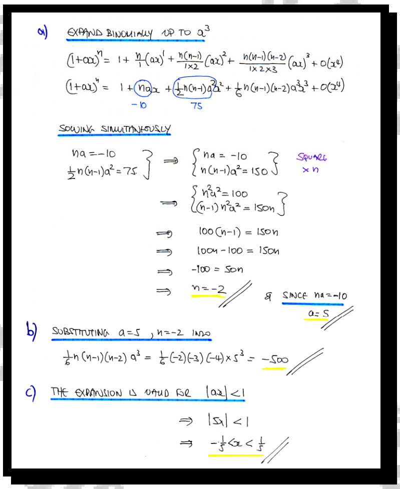
the coefficient of $x$ is $-10$ and the coefficient of $x^2$ is $75$.

**a)** Show that $n=-2$ and find the value of $a$.

**b)** Find the coefficient of $x^3$.

**c)** State the range of values of $x$ for which the above expansion is valid.

$\boxed{\text{SHOWN}}$, $\boxed{a=5}$, $\boxed{[x^3]:-500}$, $\boxed{-\frac{1}{5}<x<\frac{1}{5}}$



a) EXPAND BINOMIALLY UP TO $x^3$

$(1+ax)^n = 1 + \frac{n}{1}(ax)^1 + \frac{n(n-1)}{1\times 2}(ax)^2 + \frac{n(n-1)(n-2)}{1\times 2\times 3}(ax)^3 + O(x^4)$

$(1+ax)^n = 1 + \underbrace{na}_{-10}x + \underbrace{\tfrac{1}{2}n(n-1)a^2}_{75}x^2 + \tfrac{1}{6}n(n-1)(n-2)a^3x^3 + O(x^4)$

SOLVING SIMULTANEOUSLY

$\left.\begin{array}{l} na=-10 \\ \frac{1}{2}n(n-1)a^2=75 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} na=-10 \\ n(n-1)a^2=150 \end{array}\right\}$ SQUARE $\times n$

$\Rightarrow \left\{\begin{array}{l} n^2a^2=100 \\ (n-1)n^2a^2=150n \end{array}\right\}$

$\Rightarrow 100(n-1)=150n$

$\Rightarrow 100n-100=150n$

$\Rightarrow -100=50n$

$\Rightarrow n=-2$ // & SINCE $na=-10$

$a=5$ //

b) SUBSTITUTING $a=5$, $n=-2$ INTO

$\frac{1}{6}n(n-1)(n-2)a^3 = \frac{1}{6}(-2)(-3)(-4)\times 5^3 = -500$ //

c) THE EXPANSION IS VALID FOR $|ax|<1$

$\Rightarrow |5x|<1$

$\Rightarrow -\frac{1}{5}<x<\frac{1}{5}$ //

**Question 27** (***)

$$f(x) = \sqrt{1-x}, \ -1 < x < 1,$$

**a)** Expand $f(x)$ in ascending powers of $x$, up and including the term in $x^2$.

**b)** Use the expansion of part **(a)** to show that if $y$ is numerically small

$$\sqrt{1-4y+y^2} \approx 1 - 2y - \frac{3}{2}y^2.$$

$\boxed{\phantom{xxxx}}$, $\boxed{f(x) = 1 - \frac{1}{2}x - \frac{1}{8}x^2 + O(x^3)}$

a) EXPANDING BINOMIALLY UP TO $x^2$

$\Rightarrow f(x) = \sqrt{1-x} = (1-x)^{\frac{1}{2}} = 1 + \frac{\frac{1}{2}}{1}(-x)^1 + \frac{\frac{1}{2}(-\frac{1}{2})}{1\times 2}(-x)^2 + \dots$

$\Rightarrow f(x) = \sqrt{1-x} = 1 - \frac{1}{2}x - \frac{1}{8}x^2 + O(x^3)$

b) LET $x = (4y - y^2)$ — "CAREFUL WITH THE MINUS ON $y^2$"

$\Rightarrow -\frac{1}{2}x = -\frac{1}{2}(4y-y^2) = -2y + \frac{1}{2}y^2$

$\Rightarrow -\frac{1}{8}x^2 = -\frac{1}{8}(4y-y^2)^2 = -\frac{1}{8}(16y^2 - 8y^3 + y^4) = -2y^2 + y^3 - \frac{1}{8}y^4$

HENCE WE HAVE

$\Rightarrow \sqrt{1-4y+y^2} = \sqrt{1-(4y-y^2)}$

$= 1 - 2y + \frac{1}{2}y^2 - 2y^2 + y^3 - \frac{1}{8}y^4 + \dots$

$= 1 - 2y - \frac{3}{2}y^2 + O(y^3)$

**Question 28** (***)

$$f(x) = \frac{4+x}{(1+3x)^2}, \quad |x| < \frac{1}{3}.$$

**a)** Find the series expansion of $(1+3x)^{-1}$, up and including the term in $x^3$.

**b)** By differentiating both sides of the expansion found in part **(a)**, show that

$$(1+3x)^{-2} = 1 - 6x + 27x^2 + \ldots$$

**c)** Hence find the first three terms in the series expansion of $f(x)$.

$\boxed{\phantom{xx}}$, $\boxed{(1+3x)^{-1} = 1 - 3x + 9x^2 - 27x^3 + O(x^4)}$, $\boxed{f(x) = 4 - 23x + 102x^2 + O(x^3)}$

a) PROCEED AS FOLLOWS

$$\Rightarrow (1+3x)^{-1} = 1 + \frac{-1}{1}(3x)^1 + \frac{-1(-2)}{1\times 2}(3x)^2 + \frac{-1(-2)(-3)}{1\times 2\times 3}(3x)^3 + O(x^4)$$

$$\Rightarrow (1+3x)^{-1} = 1 - 3x + 9x^2 - 27x^3 + O(x^4)$$

b) DIFFERENTIATING AS SUGGESTED

$$\Rightarrow \frac{d}{dx}\left[(1+3x)^{-1}\right] = \frac{d}{dx}\left[1 - 3x + 9x^2 - 27x^3 + O(x^4)\right]$$

$$\Rightarrow -3(1+3x)^{-2} = 0 - 3 + 18x - 81x^2 + O(x^3)$$

$$\Rightarrow (1+3x)^{-2} = \frac{-3}{-3} + \frac{18x}{-3} - \frac{81x^2}{-3} + O(x^3)$$

$$\Rightarrow (1+3x)^{-2} = 1 - 6x + 27x^2 + O(x^3)$$

c) USING PART (b)

$$f(x) = \frac{4+x}{(1+3x)^2} = (4+x)(1+3x)^{-2}$$

$$= (4+x)\left[1 - 6x + 27x^2 + O(x^3)\right]$$

$$= 4 - 24x + 108x^2 + O(x^3)$$

$$\phantom{= 4 - 2} x - 6x^2 + O(x^3)$$

$$= 4 - 23x + 102x^2 + O(x^3)$$

**Question 29** (***)

$$f(x) = (1-2x)^{-\frac{1}{2}}.$$

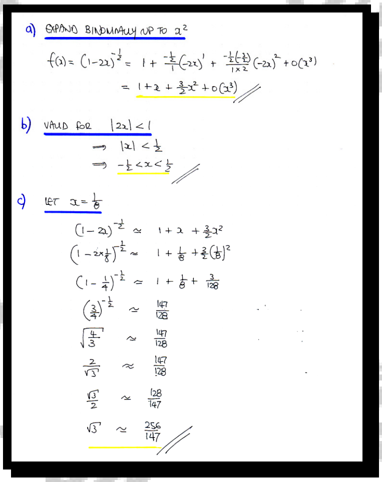
**a)** Expand $f(x)$ up and including the term in $x^2$.

**b)** State the values of $x$ for which the expansion is valid.

**c)** By substituting $x = \frac{1}{8}$ in the expansion of part **(a)** show that

$$\sqrt{3} \approx \frac{256}{147}.$$

$\boxed{\phantom{XXX}}$, $\boxed{1 + x + \frac{3}{2}x^2 + O(x^3)}$, $\boxed{-\frac{1}{2} < x < \frac{1}{2}}$


a) EXPAND BINOMIALLY UP TO $x^2$

$f(x) = (1-2x)^{-\frac{1}{2}} = 1 + \frac{-\frac{1}{2}}{1}(-2x)^1 + \frac{-\frac{1}{2}(-\frac{3}{2})}{1\times 2}(-2x)^2 + O(x^3)$

$= 1 + x + \frac{3}{2}x^2 + O(x^3)$

b) VALID FOR $|2x| < 1$

$\Rightarrow |x| < \frac{1}{2}$

$\Rightarrow -\frac{1}{2} < x < \frac{1}{2}$

c) LET $x = \frac{1}{8}$

$(1-2x)^{-\frac{1}{2}} \simeq 1 + x + \frac{3}{2}x^2$

$(1 - 2\times\frac{1}{8})^{-\frac{1}{2}} \simeq 1 + \frac{1}{8} + \frac{3}{2}(\frac{1}{8})^2$

$(1 - \frac{1}{4})^{-\frac{1}{2}} \simeq 1 + \frac{1}{8} + \frac{3}{128}$

$(\frac{3}{4})^{-\frac{1}{2}} \simeq \frac{147}{128}$

$\sqrt{\frac{4}{3}} \simeq \frac{147}{128}$

$\frac{2}{\sqrt{3}} \approx \frac{147}{128}$

$\frac{\sqrt{3}}{2} \simeq \frac{128}{147}$

$\sqrt{3} \simeq \frac{256}{147}$

**Question 30** (***+)

$$f(x) = (1+ax)^n, \ a \in \mathbb{R}, \ n \in \mathbb{R}.$$

It is given that the series expansion of $f(x)$ is

$$1 + 2x + \frac{1}{2}x^2 + bx^3 + O\left(x^4\right).$$

**a)** Show that $a = \frac{3}{2}$ and find the value of $n$.

**b)** Find the value of $b$.

**c)** State the range of values of $x$ for which the above expansion is valid.

$\boxed{\phantom{xx}}$, $\boxed{n = \frac{4}{3}}$, $\boxed{b = -\frac{1}{6}}$, $\boxed{-\frac{2}{3} < x < \frac{2}{3}}$

(a) $(1+ax)^n = 1 + \frac{n}{1}(ax) + \frac{n(n-1)}{1\times 2}(ax)^2 + \frac{n(n-1)(n-2)}{1\times 2\times 3}(ax)^3 + O(x^4)$

$= 1 + \underbrace{\boxed{na}}_{2}x + \underbrace{\boxed{\tfrac{1}{2}n(n-1)a^2}}_{\frac{1}{2}}x^2 + \underbrace{\boxed{\tfrac{1}{6}n(n-1)(n-2)a^3}}_{b}x^3 + O(x^4)$

- $an = 2 \Rightarrow a = \frac{2}{n}$
- $\frac{1}{2}n(n-1)a^2 = \frac{1}{2} \Rightarrow n(n-1)a^2 = 1$

$\Rightarrow n(n-1)\left(\frac{2}{n}\right)^2 = 1$

$\Rightarrow \cancel{n}(n-1)\frac{4}{n^{\cancel{2}}} = 1$

$\Rightarrow \frac{4(n-1)}{n} = 1$

$\Rightarrow 4n - 4 = n$

$\Rightarrow 3n = 4$ $\quad \therefore n = \frac{4}{3}$

hence $a = \frac{2}{4/3}$ $\quad \therefore a = \frac{3}{2}$

(b) $b = \frac{1}{6} \times \frac{4}{3} \times \frac{1}{3} \times \left(-\frac{2}{3}\right) \times \left(\frac{3}{2}\right)^2$

$b = -\frac{1}{6}$

(c) $|ax| < 1 \Rightarrow \left|\frac{3}{2}x\right| < 1$

$|x| < \frac{2}{3}$ $\quad \therefore -\frac{2}{3} < x < \frac{2}{3}$

**Question 31** (***+)

In the series expansion of

$$(1+ax)^n, \ |ax|<1, \ \ a,n \in \mathbb{R},$$

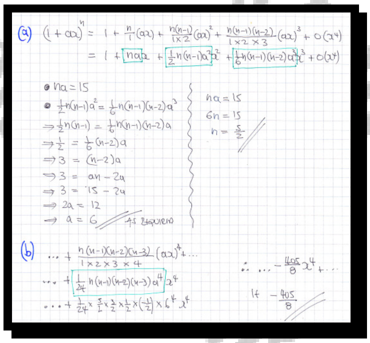
the coefficient of $x$ is $15$ and the coefficients of $x^2$ and $x^3$ are equal.

**a)** Given that $n$ is not a positive integer, show that $a=6$.

**b)** Find the value of $n$.

**c)** Find the coefficient of $x^4$.

$\boxed{\phantom{a=6}}$, $\boxed{n=\frac{5}{2}}$, $\boxed{\left[x^4\right]=-\frac{405}{8}}$



**Question 32** (***+)

$$f(x)=\frac{8x^2+17x}{(1-x)(3+2x)^2}, \quad |x|<1.$$

**a)** Express $f(x)$ into partial fractions.

**b)** Hence show that

$$f(x)\approx\frac{1}{27}x(7x+51).$$

$$\boxed{f(x)=\frac{1}{(1-x)}-\frac{3}{(3+2x)^2}-\frac{2}{(3+2x)}}$$

(a) $f(x)=\frac{8x^2+17x}{(1-x)(3+2x)^3}=\frac{A}{1-x}+\frac{B}{(3+2x)^2}+\frac{C}{3+2x}$

$$\boxed{8x^2+17x \equiv A(3+2x)^2+B(1-x)+C(3+2x)(1-x)}$$

- If $x=1 \Rightarrow 25=25A \Rightarrow \boxed{A=1}$
- If $x=-\frac{3}{2} \Rightarrow -\frac{15}{2}=\frac{5}{2}B \Rightarrow \boxed{B=-3}$
- If $x=0 \Rightarrow 0=9A+B+3C$

$0=9-3+3C$

$\boxed{C=-2}$

$\therefore f(x)=\frac{1}{1-x}-\frac{3}{(3+2x)^2}-\frac{2}{3+2x}$ //

(b) $f(x)=(1-x)^{-1}-3(3+2x)^{-2}-2(3+2x)^{-1}$

- $(1-x)^{-1}=1+\frac{-1}{1}(-x)^1+\frac{(-1)(-2)}{1\times 2}(-x)^2+O(x^3)$

$=1+x+x^2+O(x^3)$

- $-3(3+2x)^{-2}=-3\times 3^{-2}\left(1+\frac{2}{3}x\right)^{-2}=-\frac{1}{3}\left(1+\frac{2}{3}x\right)^{-2}$

$=-\frac{1}{3}\left[1+\frac{-2}{1}\left(\frac{2}{3}x\right)^1+\frac{(-2)(-3)}{1\times 2}\left(\frac{2}{3}x\right)^2+O(x^3)\right]$

$=-\frac{1}{3}\left[1-\frac{4}{3}x+\frac{4}{3}x^2+O(x^3)\right]$

$=-\frac{1}{3}+\frac{4}{9}x-\frac{4}{9}x^2+O(x^3)$

- $-2(3+2x)^{-1}=-2\times 3^{-1}\left(1+\frac{2}{3}x\right)^{-1}=-\frac{2}{3}\left(1+\frac{2}{3}x\right)^{-1}$

$=-\frac{2}{3}\left[1+\frac{(-1)}{1}\left(\frac{2}{3}x\right)^1+\frac{(-1)(-2)}{1\times 2}\left(\frac{2}{3}x\right)^2+O(x^3)\right]$

$=-\frac{2}{3}\left[1-\frac{2}{3}x+\frac{4}{9}x^2+O(x^3)\right]$

$=-\frac{2}{3}+\frac{4}{9}x-\frac{8}{27}x^2+O(x^3)$

$\therefore f(x)= 1+x+x^2+O(x^3)$

$-\frac{1}{3}+\frac{4}{9}x-\frac{4}{9}x^2+O(x^3)$

$-\frac{2}{3}+\frac{4}{9}x-\frac{8}{27}x^2+O(x^3)$

$\frac{17}{9}x+\frac{7}{27}x^2$ //

AS REQUIRED

**Question 33** (***+)

The algebraic expression $\sqrt[3]{1-3x}$ is to be expanded as an infinite convergent series, in ascending powers of $x$.

**a)** Find the first 4 terms in the series expansion of $\sqrt[3]{1-3x}$.

**b)** State the range of values of $x$ for which the expansion is valid.

**c)** By substituting a suitable value for $x$ in the expansion, show that

$$\sqrt[3]{997} \approx 9.989989983.$$

$\boxed{\text{SYN-C}}$, $\boxed{1-x-x^2-\frac{5}{3}x^3+O\left(x^4\right)}$, $\boxed{-\frac{1}{3}<x<\frac{1}{3}}$

a) EXPANDING UP TO $x^3$

$$\sqrt[3]{1-3x} = (1-3x)^{\frac{1}{3}} = 1 + \frac{\frac{1}{3}}{1}(-3x)^1 + \frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{1\times 2}(-3x)^2 + \frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{1\times 2\times 3}(-3x)^3 + O(x^4)$$

$$= 1 - x - x^2 - \frac{5}{3}x^3 + O(x^4)$$

b) VALIDITY IS FOR $|3x|<1$

$$|x| < \frac{1}{3}$$

$$-\frac{1}{3} < x < \frac{1}{3}$$

c) LET $x = 0.001$

$$\Rightarrow \sqrt[3]{1-3x} \approx 1 - x - x^2 - \frac{5}{3}x^3$$

$$\Rightarrow \sqrt[3]{1-3(0.001)} \approx 1 - (0.001) - (0.001)^2 - \frac{5}{3}(0.001)^3$$

$$\Rightarrow \sqrt[3]{0.997} \approx 0.9989989983\ldots$$

$$\Rightarrow \sqrt[3]{\frac{997}{1000}} \approx 0.9989989983\ldots$$

$$\Rightarrow \frac{\sqrt[3]{997}}{\sqrt[3]{1000}} \approx 0.9989989983\ldots$$

$$\Rightarrow \frac{\sqrt[3]{997}}{10} \approx 0.9989989983\ldots$$

$$\Rightarrow \sqrt[3]{997} \approx 9.989989983\ldots$$

AS REQUIRED

**Question 34 (***+)**

The binomial expression $(1+12x)^{\frac{3}{4}}$ is to be expanded as an infinite convergent series, in ascending powers of $x$.

**a)** Find the first 4 terms in the expansion of $(1+12x)^{\frac{3}{4}}$.

**b)** State the range of values of $x$ for which the expansion is valid.

**c)** By substituting a suitable value for $x$ in the expansion show that

$$\left(\frac{53}{50}\right)^{\frac{3}{4}} \approx 1.04467 .$$

$\boxed{\phantom{x}}$, $\boxed{1+9x-\frac{27}{2}x^2+\frac{135}{2}x^3+O\left(x^4\right)}$, $\boxed{-\frac{1}{12}<x<\frac{1}{12}}$

(a) $(1+12x)^{\frac{3}{4}} = 1+\frac{3}{4}(12x)+\frac{\frac{3}{4}\left(-\frac{1}{4}\right)}{2!}(12x)^2+\frac{\frac{3}{4}\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)}{3!}(12x)^3+O(x^4)$

$= 1+9x-\frac{27}{2}x^2+\frac{135}{2}x^3+O(x^4)$

(b) VALID FOR $|12x|<1$ $\quad -\frac{1}{12}<x<\frac{1}{12}$

(c) $\frac{53}{50} = 1+12x$

$\frac{3}{50} = 12x$

$x = 0.005$

HENCE $(1+12x)^{3/4} \approx 1+9x-\frac{27}{2}x^2+\frac{135}{2}x^3$

$(1+12(0.005))^{\frac{3}{4}} \approx 1+9(0.005)-\frac{27}{2}(0.005)^2+\frac{135}{2}(0.005)^3$

$\left(\frac{53}{50}\right)^{\frac{3}{4}} \approx 1.044670938\ldots$

$\left(\frac{53}{50}\right)^{\frac{3}{4}} \approx 1.04467$

**Question 35 (***+)**

$$f(x)=\sqrt{1+8x}\,,\ |x|<\frac{1}{8}$$

**a)** Expand $f(x)$ up and including the term in $x^3$.

**b)** By considering $\sqrt{1.08}$ and the series obtained in part (a), show that

$$\sqrt{3}\approx 1.73205\,.$$

$$\boxed{f(x)=1+4x-8x^2+32x^3+O\left(x^4\right)}$$

(a) $f(x)=\sqrt{1+8x}=(1+8x)^{\frac{1}{2}}=1+\frac{\frac{1}{2}}{1}(8x)^1+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{1\times 2}(8x)^2+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{1\times 2\times 3}(8x)^3+\dots$

$f(x)=1+4x-8x^2+32x^3+O(x^4)$

(b) LET $x=0.01$

$\sqrt{1+8\times 0.01}\approx 1+4(0.01)-8(0.01)^2+32(0.01)^3$

$\sqrt{1.08}\approx 1.039232$

$\sqrt{\frac{108}{100}}\approx 1.039232$

$\frac{\sqrt{108}}{10}\approx 1.039232$

$\sqrt{108}\approx 10.39232$

$6\sqrt{3}\approx 10.39232$

$\sqrt{3}\approx 1.73205$

**Question 36** (***+)

$$f(x) = \frac{1}{\sqrt{1+4x}}, \quad -\tfrac{1}{4} < x < \tfrac{1}{4}.$$

**a)** Find the binomial series expansion of $f(x)$ up and including the term in $x^3$.

**b)** Hence determine the coefficient of $x^3$ in the binomial expansion of $f\left(x+x^2\right)$.

$\boxed{\text{MP2-G}}$, $\boxed{f(x) = 1 - 2x + 6x^2 - 20x^3 + O\left(x^4\right)}$, $\boxed{\left[x^3\right] = -8}$

a) $f(x) = \frac{1}{\sqrt{1+4x}} = (1+4x)^{-\frac{1}{2}} = 1 + \frac{-\frac{1}{2}}{1}(4x) + \frac{-\frac{1}{2}\left(-\frac{3}{2}\right)}{1\times 2}(4x)^2 + \frac{-\frac{1}{2}\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{1\times 2\times 3}(4x)^3 + \dots$

$f(x) = 1 - 2x + 6x^2 - 20x^3 + O(x^4)$

b) Now $f(x+x^2)$

$\therefore f(x+x^2) = 1 - 2(x+x^2) + 6(x+x^2)^2 - 20(x+x^2)^3 + O(x^4)$

$= \dots\dots\dots + 6[\dots + 2x^3 + \dots] - 20[x^3 + O(x^4)] + O(x^4)$

$= \dots\dots + 12x^3 - 20x^3 + O(x^4)$

$\therefore [x^3] = -8$

**Question 37** (***+)

$$f(x) \equiv \frac{1+x}{(1-2x)\left(1+2x^2\right)} \equiv \frac{A}{1-2x} + \frac{Bx+C}{1+2x^2}, \quad |x| < \frac{1}{2}$$

**a)** Find the value of each of the constants $A$, $B$ and $C$.

**b)** Find the binomial expansion of $f(x)$, up and including the term in $x^3$.

$\boxed{A=1,\ B=1,\ C=0}$, $\boxed{f(x) = 1+3x+4x^2+6x^3+O\left(x^4\right)}$

(a) $\dfrac{1+x}{(1-2x)(1+2x^2)} \equiv \dfrac{A}{1-2x} + \dfrac{Bx+C}{1+2x^2}$

$\boxed{1+x \equiv A(1+2x^2) + (Bx+C)(1-2x)}$

If $x=\frac{1}{2}$ $\quad \frac{3}{2} = A \times \frac{3}{2} \Rightarrow \boxed{A=1}$

If $x=0$ $\quad 1 = A + C \Rightarrow \boxed{C=0}$

If $x=1$ $\quad 2 = 3A + B(-1) \Rightarrow \boxed{B=1}$

(b) $f(x) = \dfrac{1}{1-2x} + \dfrac{x}{1+2x^2} = (1-2x)^{-1} + x(1+2x^2)^{-1}$

• $(1-2x)^{-1} = 1 + \frac{-1}{1}(-2x)^1 + \frac{(-1)(-2)}{1\times 2}(-2x)^2 + \frac{(-1)(-2)(-3)}{1\times 2\times 3}(-2x)^3 + O(x^4)$

$= 1 + 2x + 4x^2 + 8x^3 + O(x^4)$

• $x(1+2x^2)^{-1} = x\left[1 + \frac{-1}{1}(2x^2)^1 + O(x^4)\right]$

$= x\left[1 - 2x^2 + O(x^4)\right]$

$= x - 2x^3 + O(x^5)$

$\therefore f(x) = 1 + 2x + 4x^2 + 8x^3 + O(x^4)$

$\phantom{\therefore f(x) = 1 +}\ x \phantom{+ 4x^2} - 2x^3 + O(x^5)$

$1 + 3x + 4x^2 + 6x^3 + O(x^4)$

**Question 38** (***+)

$$f(x) = \frac{3x-1}{(1-2x)^2}, \quad |x| < \frac{1}{2}$$

Show that if $x$ is small, then

$$f(x) \approx -1 - x + 4x^3.$$

$\boxed{\phantom{xx}}$, $\boxed{\text{proof}}$

PROCEED AS FOLLOWS

$\Rightarrow f(x) = \dfrac{3x-1}{(1-2x)^2} = (-1+3x)(1-2x)^{-2}$

$\Rightarrow f(x) = (-1+3x)\left[1 + \frac{-2}{1}(-2x)^1 + \frac{-2(-3)}{1\times 2}(-2x)^2 + \frac{-2(-3)(-4)}{1\times 2\times 3}(-2x)^3 + O(x^4)\right]$

$\Rightarrow f(x) = (-1+3x)\left[1 + 4x + 12x^2 + 32x^3 + O(x^4)\right]$

$\Rightarrow f(x) = -1 \div 4x - \cancel{12x^2} - 32x^3 + O(x^4)$

$\phantom{\Rightarrow f(x) = -1} + 3x + \cancel{12x^2} + 36x^3 + O(x^4)$

$\Rightarrow f(x) = -1 + x + 4x^3 + O(x^4)$

AS REQUIRED

**Question 39** (***+)

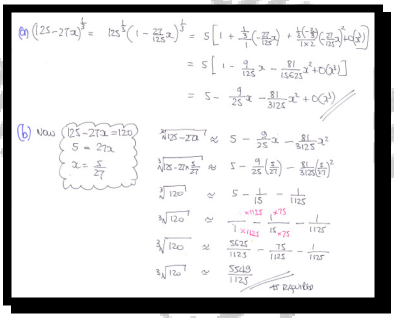
$$(125-27x)^{\frac{1}{3}}, \quad |x|<\frac{125}{27}$$

**a)** Find the first three terms in the series expansion of $f(x)$.

**b)** Use first three terms in the series expansion of $f(x)$ to show that

$$\sqrt[3]{120} \approx \frac{5549}{1125}.$$

$\boxed{\phantom{xx}}$, $\boxed{f(x)=5-\frac{9}{25}x-\frac{81}{3125}x^2+O\left(x^3\right)}$



(a) $(125-27x)^{\frac{1}{3}} = 125^{\frac{1}{3}}\left(1-\frac{27}{125}x\right)^{\frac{1}{3}} = 5\left[1+\frac{\frac{1}{3}}{1}\left(-\frac{27}{125}x\right)+\frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{1\times 2}\left(-\frac{27}{125}x\right)^2+O(x^3)\right]$

$= 5\left[1-\frac{9}{125}x-\frac{81}{15625}x^2+O(x^3)\right]$

$= 5-\frac{9}{25}x-\frac{81}{3125}x^2+O(x^3)$

(b) Now $125-27x=120$, $5=27x$, $x=\frac{5}{27}$

$\sqrt[3]{125-27x} \approx 5-\frac{9}{25}x-\frac{81}{3125}x^2$

$\sqrt[3]{125-27\times\frac{5}{27}} \approx 5-\frac{9}{25}\left(\frac{5}{27}\right)-\frac{81}{3125}\left(\frac{5}{27}\right)^2$

$\sqrt[3]{120} \approx 5-\frac{1}{15}-\frac{1}{1125}$

$\sqrt[3]{120} \approx \frac{5^{\times 1125}}{1_{\times 1125}}-\frac{1^{\times 75}}{15_{\times 75}}-\frac{1}{1125}$

$\sqrt[3]{120} \approx \frac{5625}{1125}-\frac{75}{1125}-\frac{1}{1125}$

$\sqrt[3]{120} \approx \frac{5549}{1125}$ as required

**Question 40 (***+)**

$$f(x) = \frac{(1-x)^2}{\sqrt{1+2x}}, \quad |x| < \frac{1}{2}$$

Show that if $x$ is small, then

$$f(x) \approx 1 - 3x + \frac{9}{2}x^2 - \frac{13}{2}x^3 .$$

$\boxed{\phantom{xxxx}}$, $\boxed{\text{proof}}$

PROCEED AS FOLLOWS

$$\Rightarrow f(x) = \frac{(1-x)^2}{\sqrt{1+2x}} = (1-x)^2(1+2x)^{-\frac{1}{2}}$$

$$\Rightarrow f(x) = (1-2x+x^2)\left[1 + \frac{-\frac{1}{2}}{1}(2x)^1 + \frac{-\frac{1}{2}\left(-\frac{3}{2}\right)}{1\times 2}(2x)^2 + \frac{-\frac{1}{2}\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{1\times 2\times 3}(2x)^3 + O(x^4)\right]$$

$$\Rightarrow f(x) = (1-2x+x^2)\left[1 - x + \frac{3}{2}x^2 - \frac{5}{2}x^3 + O(x^4)\right]$$

MULTIPLYING FULLY

$$\Rightarrow f(x) = \begin{array}{l} 1 - x + \frac{3}{2}x^2 - \frac{5}{2}x^3 + O(x^4) \\ \quad - 2x + 2x^2 - 3x^3 + O(x^4) \\ \qquad\quad x^2 - x^3 + O(x^4) \end{array}$$

$$\Rightarrow f(x) = 1 - 3x + \frac{9}{2}x^2 - \frac{13}{2}x^3 + O(x^4)$$

// AS REQUIRED

**Question 41** (***+)

The algebraic expression $\dfrac{1}{\sqrt[3]{1+x}}$ is to be expanded as an infinite convergent series, in ascending powers of $x$.

**a)** Expand $\dfrac{1}{\sqrt[3]{1+x}}$ up and including the term in $x^3$.

**b)** Use the expansion of part **(a)** to find the expansion of $\left(1+\dfrac{3}{4}x\right)^{-\frac{1}{3}}$ up and including the term in $x^3$.

**c)** Hence find the expansion of $\sqrt[3]{\dfrac{256}{4+3x}}$ up and including the term in $x^3$.

$\boxed{\phantom{xxxx}}$, $\boxed{1-\frac{1}{3}x+\frac{2}{9}x^2-\frac{14}{81}x^3+O\left(x^4\right)}$, $\boxed{1-\frac{1}{4}x+\frac{1}{8}x^2-\frac{7}{96}x^3+O\left(x^4\right)}$, $\boxed{4-x+\frac{1}{2}x^2-\frac{7}{24}x^3+O\left(x^4\right)}$

a) EXPANDING BINOMIALLY UP TO $x^3$

$$\frac{1}{\sqrt[3]{1+x}}=(1+x)^{-\frac{1}{3}}=1+\frac{-\frac{1}{3}}{1}(x)^1+\frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)}{1\times 2}(x)^2+\frac{-\frac{1}{3}\left(-\frac{4}{3}\right)\left(-\frac{7}{3}\right)}{1\times 2\times 3}(x)^3+O(x^4)$$

$$\frac{1}{\sqrt[3]{1+x}}=1-\frac{1}{3}x+\frac{2}{9}x^2-\frac{14}{81}x^3+O(x^4)$$

b) USING PART (a)

IF $f(x)=(1+x)^{-\frac{1}{3}}$, THEN $f\left(\frac{3}{4}x\right)=\left(1+\frac{3}{4}x\right)^{-\frac{1}{3}}$

$$\therefore\ \left(1+\frac{3}{4}x\right)^{-\frac{1}{3}}=1-\frac{1}{3}\left(\frac{3}{4}x\right)+\frac{2}{9}\left(\frac{3}{4}x\right)^2-\frac{14}{81}\left(\frac{3}{4}x\right)^3+O(x^4)$$

$$\left(1+\frac{3}{4}x\right)^{-\frac{1}{3}}=1-\frac{1}{4}x+\frac{1}{8}x^2-\frac{7}{96}x^3+O(x^4)$$

c) MANIPULATE AS FOLLOWS

$$\sqrt[3]{\frac{256}{4+3x}}=\sqrt[3]{\frac{64}{1+\frac{3}{4}x}}=\frac{\sqrt[3]{64}}{\sqrt[3]{1+\frac{3}{4}x}}=4\left(1+\frac{3}{4}x\right)^{-\frac{1}{3}}$$

DIVIDE "TOP & BOTTOM" BY 4

$$=4\left[1-\frac{1}{4}x+\frac{1}{8}x^2-\frac{7}{96}x^3+O(x^4)\right]$$

$$=4-x+\frac{1}{2}x^2-\frac{7}{24}x^3+O(x^4)$$

**Question 42** (***+)

$$f(x) \equiv \frac{1}{(2-3x)^3}, \quad |x| < \frac{2}{3}$$

**a)** Find the series expansion of $f(x)$, up and including the term in $x^2$.

It is given that

$$\frac{2+px}{(2-3x)^3} \equiv \frac{1}{4} + \frac{1}{8}x + qx^2 + \dots$$

where $p$ and $q$ are non zero constants.

**b)** Determine the value of $p$ and the value of $q$.

$\boxed{\phantom{xx}}$, $\boxed{f(x) = \frac{1}{8} + \frac{9}{16}x + \frac{27}{16}x^2 + O(x^3)}$, $\boxed{p = -8,\ q = -\frac{9}{8}}$

(a) $f(x) = \frac{1}{(2-3x)^3} = (2-3x)^{-3} = 2^{-3}\left(1 - \frac{3}{2}x\right)^{-3} = \frac{1}{8}\left(1 - \frac{3}{2}x\right)^{-3}$

$= \frac{1}{8}\left[1 + \frac{(-3)}{1}\left(-\frac{3}{2}x\right) + \frac{(-3)(-4)}{1 \times 2}\left(-\frac{3}{2}x\right)^2 + O(x^3)\right]$

$= \frac{1}{8}\left[1 + \frac{9}{2}x + \frac{27}{2}x^2 + O(x^3)\right]$

$= \frac{1}{8} + \frac{9}{16}x + \frac{27}{16}x^2 + O(x^3)$

(b) $\frac{2+px}{(2-3x)^3} = (2+px)(2-3x)^{-3} = (2+px)\left(\frac{1}{8} + \frac{9}{16}x + \frac{27}{16}x^2 + O(x^3)\right)$

$= \frac{1}{4} + \frac{9}{8}x + \frac{27}{8}x^2 + O(x^3)$

$\quad\ \frac{1}{8}px + \frac{9}{16}px^2 + O(x^3)$

$= \frac{1}{4} + \underbrace{\left(\frac{9}{8} + \frac{1}{8}p\right)}_{\frac{1}{8}}x + \underbrace{\left(\frac{27}{8} + \frac{9}{16}p\right)}_{q}x^2 + O(x^3)$

$\therefore \frac{9}{8} + \frac{1}{8}p = \frac{1}{8}$ & $\frac{27}{8} + \frac{9}{16}(-8) = q$

$9 + p = 1$ $\quad q = -\frac{9}{8}$

$p = -8$

**Question 43** (***+)

$$f(x) \equiv \left(\tfrac{1}{4} - x\right)^{-\frac{3}{2}}, \ |x| < \tfrac{1}{4}$$

**a)** Find the series expansion of $f(x)$, up and including the term in $x^3$.

**b)** Use the result of part **(a)** to obtain the series expansion of

$$\sqrt{\tfrac{1}{4} - x}, \ |x| < \tfrac{1}{4},$$

up and including the term in $x^3$.

*No credit will be given for obtaining a direct expansion in this part.*

$\boxed{\phantom{xxx}}$, $\boxed{f(x) = 8 + 48x + 240x^2 + 1120x^3 + O(x^4)}$,

$\boxed{\sqrt{\tfrac{1}{4} - x} = \tfrac{1}{2} - x - x^2 - 2x^3 + O(x^4)}$

a) CREATE A "ONE" AND "EXPAND

$$\left(\tfrac{1}{4}-x\right)^{-\frac{3}{2}} = \left(\tfrac{1}{4}\right)^{-\frac{3}{2}}(1-4x)^{-\frac{3}{2}} = 4^{\frac{3}{2}}(1-4x)^{-\frac{3}{2}}$$

$$= 8\left[1-4x\right]^{-\frac{3}{2}}$$

$$= 8\left[1 + \frac{-\frac{3}{2}}{1!}(-4x)^1 + \frac{-\frac{3}{2}\left(-\frac{5}{2}\right)}{2!}(-4x)^2 + \frac{\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)}{3!}(-4x)^3 + \dots\right]$$

$$= 8\left[1 + 6x + 30x^2 + 140x^3 + \dots\right]$$

$$= \underline{8 + 48x + 240x^2 + 1120x^3 + \dots}$$

b) PROCEED AS FOLLOWS

$$\sqrt{\tfrac{1}{4}-x} = \left(\tfrac{1}{4}-x\right)^{\frac{1}{2}} = \left(\tfrac{1}{4}-x\right)^2\left(\tfrac{1}{4}-x\right)^{-\frac{3}{2}}$$

$$= \left(\tfrac{1}{16} - \tfrac{1}{2}x + x^2\right)\left(8 + 48x + 240x^2 + 1120x^3 + \dots\right)$$

$$\begin{aligned} = \tfrac{1}{2} + 3x &+ 15x^2 + 70x^3 + \dots \\ - 4x &- 24x^2 - 120x^3 + \dots \\ & \ \ \ 8x^2 + 48x^3 + \dots \end{aligned}$$

$$= \underline{\tfrac{1}{2} - x - x^2 - 2x^3 + \dots}$$

**Question 44** (***+)

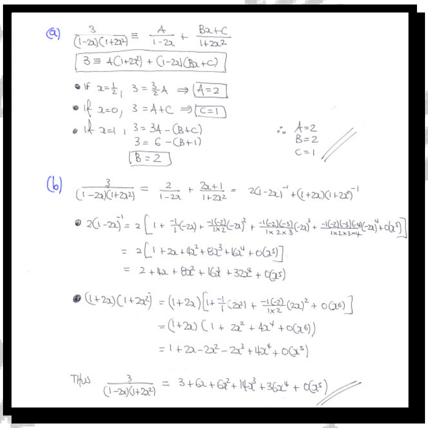
$$\frac{3}{(1-2x)\left(1+2x^2\right)} \equiv \frac{A}{1-2x} + \frac{Bx+C}{1+2x^2}, \ |x| < \frac{1}{2}.$$

**a)** Find the value of each of the constants $A$, $B$ and $C$.

**b)** Hence or otherwise find the first five terms in the binomial expansion of

$$\frac{3}{(1-2x)\left(1+2x^2\right)}.$$

$\boxed{A=2}$, $\boxed{B=2}$, $\boxed{C=1}$, $\boxed{3+6x+6x^2+14x^3+36x^4+O\left(x^5\right)}$



(a) $\frac{3}{(1-2x)(1+2x^2)} \equiv \frac{A}{1-2x} + \frac{Bx+C}{1+2x^2}$

$\boxed{3 \equiv A(1+2x^2) + (1-2x)(Bx+C)}$

- If $x=\frac{1}{2}$, $3 = \frac{3}{2}A \Rightarrow \boxed{A=2}$
- If $x=0$, $3 = A+C \Rightarrow \boxed{C=1}$
- If $x=1$, $3 = 3A-(B+C)$
  $3 = 6-(B+1)$
  $\boxed{B=2}$

$\therefore A=2, B=2, C=1$

(b) $\frac{3}{(1-2x)(1+2x^2)} = \frac{2}{1-2x} + \frac{2x+1}{1+2x^2} = 2(1-2x)^{-1} + (1+2x)(1+2x^2)^{-1}$

- $2(1-2x)^{-1} = 2\left[1 + \frac{-1}{1}(-2x) + \frac{-1(-2)}{1\times2}(-2x)^2 + \frac{-1(-2)(-3)}{1\times2\times3}(-2x)^3 + \frac{-1(-2)(-3)(-4)}{1\times2\times3\times4}(-2x)^4 + O(x^5)\right]$
  $= 2\left[1+2x+4x^2+8x^3+16x^4+O(x^5)\right]$
  $= 2+4x+8x^2+16x^3+32x^4+O(x^5)$
- $(1+2x)(1+2x^2)^{-1} = (1+2x)\left[1+\frac{-1}{1}(2x^2)+\frac{-1(-2)}{1\times2}(2x)^2+O(x^6)\right]$
  $= (1+2x)(1-2x^2+4x^4+O(x^6))$
  $= 1+2x-2x^2-4x^3+4x^4+O(x^5)$

Thus $\frac{3}{(1-2x)(1+2x^2)} = 3+6x+6x^2+14x^3+36x^4+O(x^5)$

**Question 45** (***+)

The function $f(x)$ is defined in terms of the non zero constant $n$, by

$$f(x) = (3+2x)^n, \ -\frac{3}{2} < x < \frac{3}{2}.$$

**a)** Given that $n$ is not a positive integer, find in terms of $n$ the ratio of the coefficient of $x^3$ to the coefficient of $x^2$ in binomial expansion of $f(x)$.

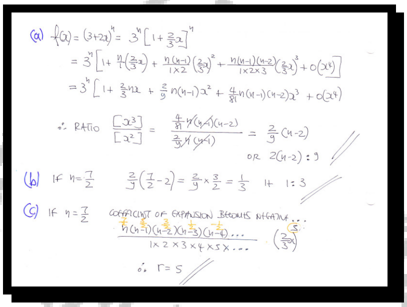
It is now given that $n = \frac{7}{2}$.

**b)** Evaluate the ratio found in part (**a**).

The coefficient of $x^r$ in the binomial expansion of $f(x)$ is negative.

**c)** Find the smallest value of $r$.

$\boxed{4.}$, $\boxed{[x^3]:[x^2] = 2(n-2):9}$, $\boxed{[x^3]:[x^2] = 1:3}$, $\boxed{r=5}$



**Question 46** (****)

$$f(x) = \sqrt{\frac{4-x}{4+x}}, \quad |x| < 4.$$

**a)** Expand $f(x)$ as an infinite convergent series, up and including the term in $x^2$.

**b)** By substituting $x = 0.5$ in the expansion of part **(a)**, show that

$$\sqrt{7} \approx \frac{339}{128}.$$

$\boxed{\phantom{xxxx}}$, $\boxed{f(x) = 1 - \frac{1}{4}x + \frac{1}{32}x^2 + O(x^3)}$

PROCEED AS FOLLOWS

$$\Rightarrow f(x) = \sqrt{\frac{4-x}{4+x}} = \frac{\sqrt{4-x}}{\sqrt{4+x}} = (4-x)^{\frac{1}{2}}(4+x)^{-\frac{1}{2}}$$

$$= 4^{\frac{1}{2}}\left(1-\tfrac{1}{4}x\right)^{\frac{1}{2}} \times 4^{-\frac{1}{2}}\left(1+\tfrac{1}{4}x\right)^{-\frac{1}{2}}$$

$$= \left(1-\tfrac{1}{4}x\right)^{\frac{1}{2}}\left(1+\tfrac{1}{4}x\right)^{-\frac{1}{2}}$$

EXPAND BINOMIALLY WE OBTAIN

$$\Rightarrow f(x) = \left[1 + \frac{\frac{1}{2}}{1}\left(-\tfrac{1}{4}x\right)^1 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{1\times 2}\left(-\tfrac{1}{4}x\right)^2 + O(x^3)\right]\left[1 + \frac{-\frac{1}{2}}{1}\left(\tfrac{1}{4}x\right)^1 + \frac{-\frac{1}{2}\left(-\frac{3}{2}\right)}{1\times 2}\left(\tfrac{1}{4}x\right)^2 + O(x^3)\right]$$

$$\Rightarrow f(x) = \left[1 - \tfrac{1}{8}x - \tfrac{1}{128}x^2 + O(x^3)\right]\left[1 - \tfrac{1}{8}x + \tfrac{3}{128}x^2 + O(x^3)\right]$$

MULTIPLYING OUT YIELDS

$$f(x) = 1 - \tfrac{1}{8}x + \tfrac{3}{128}x^2 + O(x^3)$$
$$\qquad - \tfrac{1}{8}x + \tfrac{1}{64}x^2 + O(x^3)$$
$$\qquad\qquad - \tfrac{1}{128}x^2 + O(x^3)$$

$$f(x) = 1 - \tfrac{1}{4}x + \tfrac{1}{32}x^2 + O(x^3)$$

ALTERNATIVE TO PART (a)

$$f(x) = \sqrt{\frac{4-x}{4+x}} = \frac{\sqrt{4-x}}{\sqrt{4+x}} = \frac{\sqrt{4-x}\sqrt{4-x}}{\sqrt{4+x}\sqrt{4-x}} = \frac{4-x}{\sqrt{16-x^2}}$$

$$= (4-x)(16-x^2)^{-\frac{1}{2}} = (4-x)\times 16^{-\frac{1}{2}}\left(1-\tfrac{1}{16}x^2\right)^{-\frac{1}{2}}$$

$$= \tfrac{1}{4}(4-x)\left(1-\tfrac{1}{16}x^2\right)^{-\frac{1}{2}}$$

EXPANDING UP TO $x^2$

$$= \tfrac{1}{4}(4-x)\left[1 + \frac{-\frac{1}{2}}{1}\left(-\tfrac{1}{16}x^2\right) + O(x^4)\right]$$

$$= \left(1-\tfrac{1}{4}x\right)\left[1 + \tfrac{1}{32}x^2 + O(x^4)\right]$$

$$= 1 + \tfrac{1}{32}x^2 - \tfrac{1}{4}x + O(x^3)$$

$$= 1 - \tfrac{1}{4}x + \tfrac{1}{32}x^2 + O(x^3)$$ AS BEFORE

b) USING THE EXPANSION OF PART (a)

$$\Rightarrow \sqrt{\frac{4-x}{4+x}} \approx 1 - \tfrac{1}{4}x + \tfrac{1}{32}x^2$$

$$\Rightarrow \sqrt{\frac{4-0.5}{4+0.5}} \approx 1 - \tfrac{1}{4}\left(\tfrac{1}{2}\right) + \tfrac{1}{32}\left(\tfrac{1}{2}\right)^2$$

$$\Rightarrow \sqrt{\frac{7/2}{9/2}} \approx 1 - \frac{1}{8} + \frac{1}{128}$$

$$\Rightarrow \sqrt{\frac{7}{9}} \approx \frac{113}{128}$$

$$\Rightarrow \frac{\sqrt{7}}{3} \approx \frac{113}{128}$$

$$\Rightarrow \sqrt{7} \approx \frac{339}{128}$$ AS REQUIRED

**Question 47** (****)

$$f(x) \equiv (1-8x)^{\frac{1}{4}}, \ |x| < \frac{1}{8}.$$

**a)** Find the first four terms in the binomial series expansion of $f(x)$.

The term of lowest degree in the series expansion of

$$(1+ax)\left(1+bx^2\right)^5 - f(x),$$

is the term in $x^3$.

**b)** Determine the value of each of the constants $a$ and $b$, and hence state the coefficient of $x^3$.

$\boxed{\quad}$, $\boxed{1-2x-6x^2-28x^3+O\left(x^4\right)}$, $\boxed{a=-2}$, $\boxed{b=-\frac{6}{5}}$, $\boxed{\left[x^3\right]=40}$

a) EXPAND BINOMIALLY UP TO $x^3$

$$(1-8x)^{\frac{1}{4}} = 1 + \frac{\frac{1}{4}}{1}(-8x) + \frac{\frac{1}{4}\left(-\frac{3}{4}\right)}{1\times 2}(-8x)^2 + \frac{\frac{1}{4}\left(-\frac{3}{4}\right)\left(-\frac{7}{4}\right)}{1\times 2\times 3}(-8x)^3 + O(x^4)$$

$$(1-8x)^{\frac{1}{4}} = 1 - 2x - 6x^2 - 28x^3 + O(x^4)$$

b) PROCEED AS FOLLOWS.

$$(1+ax)(1+bx^2)^5 - (1-8x)^{\frac{1}{4}}$$

$$= (1+ax)\left[1 + \frac{5}{1}(bx^2)^1 + O(x^4)\right] - \left[1-2x-6x^2-28x^3+O(x^4)\right]$$

$$= (1+ax)\left[1+5bx^2+O(x^4)\right] - \left[1-2x-6x^2-28x^3+O(x^4)\right]$$

$$= 1 + ax + 5bx^2 + 5abx^3 + O(x^4) - 1 + 2x + 6x^2 + 28x^3 + O(x^4)$$

$$= \underbrace{(a+2)}_{\text{ZERO}}x + \underbrace{(5b+6)}_{\text{ZERO}}x^2 + (5ab+28)x^3 + O(x^4)$$

$\therefore a = -2$

$b = -\frac{6}{5}$

$\therefore$ COEFFICIENT OF $x^3$ IS $5ab + 28$

$= 5(-2)\left(-\frac{6}{5}\right) + 28$

$= 40$

**Question 48** (****)

$$\frac{2-3x^2}{(2x+1)\left(x^2+1\right)} \equiv \frac{A}{2x+1}+\frac{B}{x^2+1}+\frac{Cx}{x^2+1}.$$

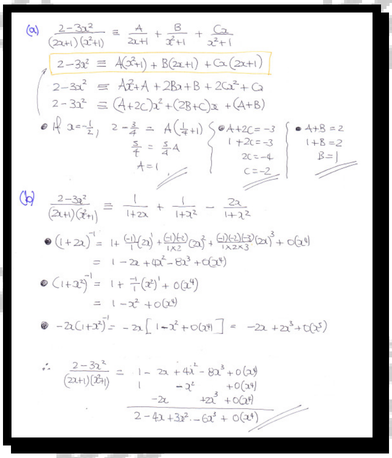
**a)** Find the value of each of the constants $A$, $B$ and $C$ in the above identity.

**b)** Hence, or otherwise, determine the series expansion of

$$\frac{2-3x^2}{(2x+1)\left(x^2+1\right)},$$

up and including the term in $x^3$.

$\boxed{A=1}$, $\boxed{B=1}$, $\boxed{C=-2}$, $\boxed{2-4x+3x^2-6x^3+O\left(x^4\right)}$



(a) $\frac{2-3x^2}{(2x+1)(x^2+1)} \equiv \frac{A}{2x+1}+\frac{B}{x^2+1}+\frac{Cx}{x^2+1}$

$\boxed{2-3x^2 \equiv A(x^2+1)+B(2x+1)+Cx(2x+1)}$

$2-3x^2 \equiv Ax^2+A+2Bx+B+2Cx^2+Cx$

$2-3x^2 \equiv (A+2C)x^2+(2B+C)x+(A+B)$

• If $x=-\frac{1}{2}$, $2-\frac{3}{4} = A\left(\frac{1}{4}+1\right)$, $\frac{5}{4}=\frac{5}{4}A$, $A=1$

• $A+2C=-3$, $1+2C=-3$, $2C=-4$, $C=-2$

• $A+B=2$, $1+B=2$, $B=1$

(b) $\frac{2-3x^2}{(2x+1)(x^2+1)} \equiv \frac{1}{1+2x}+\frac{1}{1+x^2}-\frac{2x}{1+x^2}$

• $(1+2x)^{-1} = 1+\frac{(-1)}{1}(2x)^1+\frac{(-1)(-2)}{1\times2}(2x)^2+\frac{(-1)(-2)(-3)}{1\times2\times3}(2x)^3+O(x^4)$
$= 1-2x+4x^2-8x^3+O(x^4)$

• $(1+x^2)^{-1} = 1+\frac{-1}{1}(x^2)^1+O(x^4)$
$= 1-x^2+O(x^4)$

• $-2x(1+x^2)^{-1} = -2x\left[1-x^2+O(x^4)\right] = -2x+2x^3+O(x^5)$

$\therefore \frac{2-3x^2}{(2x+1)(x^2+1)} = 1-2x+4x^2-8x^3+O(x^4)$
$\quad 1 - x^2 + O(x^4)$
$\quad -2x + 2x^3 + O(x^4)$
$= 2-4x+3x^2-6x^3+O(x^4)$

**Question 49** (****)

$$f(x) = \sqrt{1-x}, \quad -1 < x < 1.$$

**a)** Expand $f(x)$ up and including the term in $x^3$.

**b)** Show clearly that

$$8 \times \sqrt{1 - \frac{1}{64}} = 3\sqrt{7}.$$

**c)** By using the **first two** terms of the expansion obtained in part **(a)** and the result shown in part **(b)**, show further that

$$\sqrt{7} \approx \frac{127}{48}.$$

$\boxed{\phantom{xx}}$, $\boxed{1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 + O(x^4)}$

a) EXPANDING BINOMIALLY UP TO $x^3$

$\Rightarrow \sqrt{1-x} = (1-x)^{\frac{1}{2}} = 1 + \frac{\frac{1}{2}}{1}(-x) + \frac{\frac{1}{2}(-\frac{1}{2})}{1\times 2}(-x)^2 + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{1\times 2\times 3}(-x)^3 + O(x^4)$

$\Rightarrow \sqrt{1-x} = 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 + O(x^4)$

b) PROCEED AS FOLLOWS

$8 \times \sqrt{1 - \frac{1}{64}} = 8\sqrt{\frac{63}{64}} = \frac{8\sqrt{63}}{\sqrt{64}} = \frac{8\sqrt{9}\sqrt{7}}{\sqrt{64}}$

$= \frac{8 \times 3\sqrt{7}}{8} = 3\sqrt{7}$ AS REQUIRED

c) COMBINING RESULTS

$\sqrt{1-x} \approx 1 - \frac{1}{2}x$ (FOR SMALL $x$, FIRST 2 TERMS)

LET $x = \frac{1}{64}$

$\Rightarrow \sqrt{1 - \frac{1}{64}} \approx 1 - \frac{1}{2} \times \frac{1}{64}$

$\Rightarrow 8\sqrt{1 - \frac{1}{64}} \approx 8\left[1 - \frac{1}{128}\right]$

$\Rightarrow 3\sqrt{7} \simeq 8 - \frac{1}{16}$

$\Rightarrow 3\sqrt{7} \simeq \frac{127}{16}$

$\Rightarrow \sqrt{7} \simeq \frac{127}{48}$ AS REQUIRED

**Question 50** (****)

The algebraic expression $\frac{1+x}{1+3x}$ is to be expanded as an infinite convergent series, in ascending powers of $x$.

**a)** Find the first 4 terms in the binomial expansion of

$$\frac{1+x}{1+3x}.$$

**b)** State the range of values of $x$ for which the expansion is valid.

**c)** By substituting a suitable value for $x$ in the above expansion show that

$$\frac{101}{103} \approx 0.980582.$$

$\boxed{1-2x+6x^2-18x^3+O\left(x^4\right)}$, $\boxed{-\frac{1}{3}<x<\frac{1}{3}}$

(a) $\frac{1+x}{1+3x} = (1+x)(1+3x)^{-1}$

$= (1+x)\left[1 + \frac{-1}{1}(3x)^1 + \frac{(-1)(-2)}{1\times 2}(3x)^2 + \frac{(-1)(-2)(-3)}{1\times 2\times 3}(3x)^3 + O(x^4)\right]$

$= (1+x)\left[1 - 3x + 9x^2 - 27x^3 + O(x^4)\right]$

$= 1 - 3x + 9x^2 - 27x^3 + O(x^4)$

$\quad\;\; x - 3x^2 + 9x^3 + O(x^4)$

$1 - 2x + 6x^2 - 18x^3 + O(x^4)$

(b) VALID FOR $|3x|<1$

$|x|<\frac{1}{3}$ $\quad\therefore\; -\frac{1}{3}<x<\frac{1}{3}$

(c) $\frac{1+x}{1+3x} \approx 1 - 2x + 6x^2 - 18x^3$

• LET $x = 0.01$

$\frac{1+0.01}{1+0.03} \simeq 1 - 2(0.01) + 6(0.01)^2 - 18(0.01)^3$

$\frac{1.01}{1.03} \approx 1 - 0.02 + 0.0006 - 0.000018$

$\frac{101}{103} \approx 0.980582$

**Question 51** (****)

The algebraic expression $\sqrt{9-x}$ is to be expanded as an infinite convergent series, in ascending powers of $x$.

**a)** Find the first 4 terms in the series expansion of $\sqrt{9-x}$.

**b)** State the range of values of $x$ for which the expansion is valid.

**c)** By substituting a suitable value for $x$ in the expansion show that

$$\sqrt{850} \approx 29.1548.$$

$$\boxed{3-\frac{1}{6}x-\frac{1}{216}x^2-\frac{1}{3888}x^3+O\left(x^4\right)}, \quad \boxed{-9<x<9}$$

(a) $\sqrt{9-x} = (9-x)^{\frac{1}{2}} = 9^{\frac{1}{2}}\left(1-\frac{1}{9}x\right)^{\frac{1}{2}} = 3\left(1-\frac{1}{9}x\right)^{\frac{1}{2}}$

$= 3\left[1+\frac{\frac{1}{2}}{1}\left(-\frac{1}{9}x\right)+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{1\times 2}\left(-\frac{1}{9}x\right)^2+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{1\times 2\times 3}\left(-\frac{1}{9}x\right)^3+O(x^4)\right]$

$= 3\left[1-\frac{1}{18}x-\frac{1}{648}x^2-\frac{1}{11664}x^3+O(x^4)\right]$

$= 3-\frac{1}{6}x-\frac{1}{216}x^2-\frac{1}{3888}x^3+O(x^4)$

(b) VALID $\left|\frac{1}{9}x\right|<1$ $\therefore -9<x<9$

$|x|<9$

(c) LET $x=0.5$

$\sqrt{9-x} \approx 3-\frac{1}{6}x-\frac{1}{216}x^2-\frac{1}{3888}x^3$

$\sqrt{9-0.5} \approx 3-\frac{1}{6}(0.5)-\frac{1}{216}(0.5)^2-\frac{1}{3888}(0.5)^3$

$\sqrt{8.5} \approx 2.915477109\ldots$

$\sqrt{\frac{850}{100}} \approx 2.915477109\ldots$

$\frac{\sqrt{850}}{10} \approx 2.915477109$

$\sqrt{850} \approx 29.1548$ AS REQUIRED

**Question 52** (****)

$$f(x)=\frac{5x^2-52x+4}{(1+2x)(2-x)^2},\ |x|<\frac{1}{2}.$$

Show that if $x$ is numerically small

$$f(x)\approx 1-14x+17x^2-42x^3.$$

proof

$$f(x)=\frac{5x^2-52x+4}{(2-x)^2(1+2x)}$$

SPLIT INTO PARTIAL FRACTION

$$\frac{5x^2-52x+4}{(2-x)^2(1+2x)}\equiv\frac{A}{(2-x)^2}+\frac{B}{2-x}+\frac{C}{1+2x}$$

$$5x^2-52x+4\equiv A(1+2x)+B(2-x)(1+2x)+C(2-x)^2$$

- IF $x=2 \Rightarrow 20-104+4=5A$
  $5A=-80$
  $\boxed{A=-16}$
- IF $x=-\frac{1}{2}\Rightarrow \frac{5}{4}+26+4=\frac{25}{4}C$
  $\frac{125}{4}=\frac{25}{4}C$
  $\boxed{C=5}$
- IF $x=0 \Rightarrow 4=A+2B+4C$
  $4=-16+2B+20$
  $\boxed{B=0}$

$$\therefore f(x)=\frac{5}{1+2x}-\frac{16}{(2-x)^2}$$

- $\frac{5}{1+2x}=5(1+2x)^{-1}$
  $=5\left[1+\frac{-1}{1}(2x)^1+\frac{-1(-2)}{1\times2}(2x)^2+\frac{-1(-2)(-3)}{1\times2\times3}(2x)^3+O(x^4)\right]$
  $=5\left[1-2x+4x^2-8x^3+O(x^4)\right]$
  $=5-10x+20x^2-40x^3+O(x^4)$
- $-\frac{16}{(2-x)^2}=-16(2-x)^{-2}=-16\times2^{-2}\left(1-\frac{1}{2}x\right)^{-2}=-4\left(1-\frac{1}{2}x\right)^{-2}$
  $=-4\left[1+\frac{-2}{1}\left(-\frac{1}{2}x\right)^1+\frac{-2(-3)}{1\times2}\left(-\frac{1}{2}x\right)^2+\frac{-2(-3)(-4)}{1\times2\times3}\left(-\frac{1}{2}x\right)^3+O(x^4)\right]$
  $=-4\left[1+x+\frac{3}{4}x^2+\frac{1}{2}x^3+O(x^4)\right]$
  $=-4-4x-3x^2-2x^3+O(x^4)$
- $f(x)=5-10x+20x^2-40x^3+O(x^4)$
  $\quad -4-4x-3x^2-2x^3+O(x^4)$
  $\quad 1-14x+17x^2-42x^3+O(x^4)$

ALTERNATIVE

- $(1+2x)^{-1}=1+\frac{-1}{1}(2x)^1+\frac{-1(-2)}{1\times2}(2x)^2+\frac{-1(-2)(-3)}{1\times2\times3}(2x)^3+O(x^4)$
  $=1-2x+4x^2-8x^3+O(x^4)$
- $(2-x)^{-2}=2^{-2}\left(1-\frac{1}{2}x\right)^{-2}=\frac{1}{4}\left[1+\frac{-2}{1}\left(-\frac{1}{2}x\right)^1+\frac{-2(-3)}{1\times2}\left(-\frac{1}{2}x\right)^2+\frac{-2(-3)(-4)}{1\times2\times3}\left(-\frac{1}{2}x\right)^3+O(x^4)\right]$
  $=\frac{1}{4}\left[1+x+\frac{3}{4}x^2+\frac{1}{2}x^3+O(x^4)\right]$
  $=\frac{1}{4}+\frac{1}{4}x+\frac{3}{16}x^2+\frac{1}{8}x^3+O(x^4)$

$\therefore f(x)=(4-52x+5x^2)\left[1-2x+4x^2-8x^3+O(x^4)\right]\left[\frac{1}{4}+\frac{1}{4}x+\frac{3}{16}x^2+\frac{1}{8}x^3+O(x^4)\right]$

$\Rightarrow f(x)=(4-52x+5x^2)\left[\begin{array}{l}\frac{1}{4}+\frac{1}{4}x+\frac{3}{16}x^2+\frac{1}{8}x^3+O(x^4)\\ \quad -\frac{1}{2}x-\frac{1}{2}x^2-\frac{3}{8}x^3+O(x^4)\\ \quad\quad x^2+x^3+O(x^4)\\ \quad\quad\quad -2x^3+O(x^4)\end{array}\right]$

$\Rightarrow f(x)=(4-52x+5x^2)\left[\frac{1}{4}-\frac{1}{4}x+\frac{11}{16}x^2-\frac{5}{4}x^3+O(x^4)\right]$

$\Rightarrow f(x)=1-x+\frac{11}{4}x^2-5x^3+O(x^4)$
$\quad -13x+13x^2-\frac{143}{4}x^3+O(x^4)$
$\quad \frac{5}{4}x^2-\frac{5}{4}x^3+O(x^4)$

$\Rightarrow f(x)=1-14x+17x^2-42x^3+O(x^4)$

**Question 53** (****)

$$\frac{2x^2-3}{(3-2x)(1-x)^2} \equiv \frac{A}{3-2x} + \frac{B}{1-x} + \frac{C}{(1-x)^2}.$$

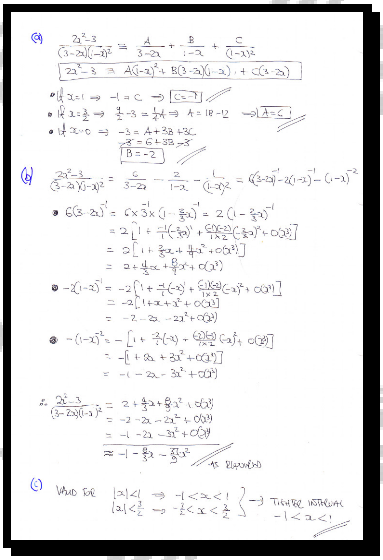
**a)** Find the value of each of the constants $A$, $B$ and $C$.

**b)** Hence show that for small $x$

$$\frac{2x^2-3}{(3-2x)(1-x)^2} \approx -1 - \frac{8}{3}x - \frac{37}{9}x^2.$$

**c)** State the range of values of $x$ for which the approximation in part **(b)** is valid.

$\boxed{A=6,\ B=-2,\ C=-1}$, $\boxed{-1<x<1}$


(a) $\frac{2x^2-3}{(3-2x)(1-x)^2} \equiv \frac{A}{3-2x} + \frac{B}{1-x} + \frac{C}{(1-x)^2}$

$\boxed{2x^2-3 \equiv A(1-x)^2 + B(3-2x)(1-x) + C(3-2x)}$

- If $x=1 \Rightarrow -1 = C \Rightarrow \boxed{C=-1}$
- If $x=\frac{3}{2} \Rightarrow \frac{9}{2}-3 = \frac{1}{4}A \Rightarrow A = 18-12 \Rightarrow \boxed{A=6}$
- If $x=0 \Rightarrow -3 = A+3B+3C$

$\cancel{-3} = 6+3B\cancel{-3}$

$\boxed{B=-2}$

(b) $\frac{2x^2-3}{(3-2x)(1-x)^2} = \frac{6}{3-2x} - \frac{2}{1-x} - \frac{1}{(1-x)^2} = 6(3-2x)^{-1} - 2(1-x)^{-1} - (1-x)^{-2}$

- $6(3-2x)^{-1} = 6\times3^{-1}\times(1-\frac{2}{3}x)^{-1} = 2(1-\frac{2}{3}x)^{-1}$

$= 2\left[1 + \frac{-1}{1}(-\frac{2}{3}x)^1 + \frac{(-1)(-2)}{1\times2}(-\frac{2}{3}x)^2 + O(x^3)\right]$

$= 2\left[1 + \frac{2}{3}x + \frac{4}{9}x^2 + O(x^3)\right]$

$= 2 + \frac{4}{3}x + \frac{8}{9}x^2 + O(x^3)$

- $-2(1-x)^{-1} = -2\left[1 + \frac{-1}{1}(-x)^1 + \frac{(-1)(-2)}{1\times2}(-x)^2 + O(x^3)\right]$

$= -2\left[1+x+x^2+O(x^3)\right]$

$= -2-2x-2x^2+O(x^3)$

- $-(1-x)^{-2} = -\left[1 + \frac{-2}{1}(-x) + \frac{(-2)(-3)}{1\times2}(-x)^2 + O(x^3)\right]$

$= -\left[1+2x+3x^2+O(x^3)\right]$

$= -1-2x-3x^2+O(x^3)$

$\therefore \frac{2x^2-3}{(3-2x)(1-x)^2} = 2+\frac{4}{3}x+\frac{8}{9}x^2+O(x^3)$

$= -2-2x-2x^2+O(x^3)$

$= -1-2x-3x^2+O(x^3)$

$\approx -1-\frac{8}{3}x-\frac{37}{9}x^2$ AS REQUIRED

(c) VALID FOR $|x|<1 \Rightarrow -1<x<1$, $|x|<\frac{3}{2} \Rightarrow -\frac{3}{2}<x<\frac{3}{2}$ $\Rightarrow$ TIGHTER INTERVAL $-1<x<1$

**Question 54** (****)

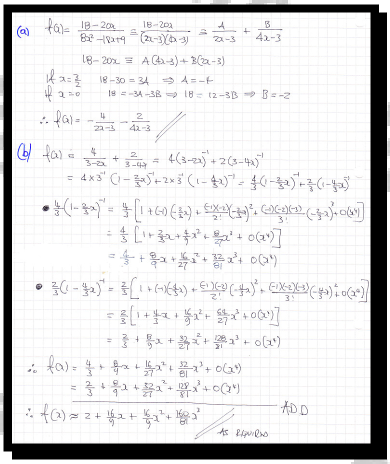
$$f(x) = \frac{18-20x}{8x^2-18x+9}, \quad -\frac{3}{4} < x < \frac{3}{4}.$$

**a)** Express $f(x)$ into partial fractions.

**b)** Hence show that

$$f(x) \approx 2 + \frac{16}{9}x + \frac{16}{9}x^2 + \frac{160}{81}x^3.$$

$$\boxed{f(x) = \frac{2}{3-4x} + \frac{4}{3-2x}}$$



(a) $f(x) = \frac{18-20x}{8x^2-18x+9} \equiv \frac{18-20x}{(2x-3)(4x-3)} \equiv \frac{A}{2x-3} + \frac{B}{4x-3}$

$18-20x \equiv A(4x-3) + B(2x-3)$

If $x = \frac{3}{2}$ $\quad 18-30 = 3A \Rightarrow A = -4$

If $x = 0$ $\quad 18 = -3A-3B \Rightarrow 18 = 12-3B \Rightarrow B = -2$

$\therefore f(x) = -\frac{4}{2x-3} - \frac{2}{4x-3}$

(b) $f(x) = \frac{4}{3-2x} + \frac{2}{3-4x} = 4(3-2x)^{-1} + 2(3-4x)^{-1}$

$= 4 \times 3^{-1}\left(1-\frac{2}{3}x\right)^{-1} + 2 \times 3^{-1}\left(1-\frac{4}{3}x\right)^{-1} = \frac{4}{3}\left(1-\frac{2}{3}x\right)^{-1} + \frac{2}{3}\left(1-\frac{4}{3}x\right)^{-1}$

- $\frac{4}{3}\left(1-\frac{2}{3}x\right)^{-1} = \frac{4}{3}\left[1 + (-1)\left(-\frac{2}{3}x\right) + \frac{(-1)(-2)}{2!}\left(-\frac{2}{3}x\right)^2 + \frac{(-1)(-2)(-3)}{3!}\left(-\frac{2}{3}x\right)^3 + O(x^4)\right]$
  $= \frac{4}{3}\left[1 + \frac{2}{3}x + \frac{4}{9}x^2 + \frac{8}{27}x^3 + O(x^4)\right]$
  $= \frac{4}{3} + \frac{8}{9}x + \frac{16}{27}x^2 + \frac{32}{81}x^3 + O(x^4)$

- $\frac{2}{3}\left(1-\frac{4}{3}x\right)^{-1} = \frac{2}{3}\left[1 + (-1)\left(-\frac{4}{3}x\right) + \frac{(-1)(-2)}{2!}\left(-\frac{4}{3}x\right)^2 + \frac{(-1)(-2)(-3)}{3!}\left(-\frac{4}{3}x\right)^3 + O(x^4)\right]$
  $= \frac{2}{3}\left[1 + \frac{4}{3}x + \frac{16}{9}x^2 + \frac{64}{27}x^3 + O(x^4)\right]$
  $= \frac{2}{3} + \frac{8}{9}x + \frac{32}{27}x^2 + \frac{128}{81}x^3 + O(x^4)$

$\therefore f(x) = \frac{4}{3} + \frac{8}{9}x + \frac{16}{27}x^2 + \frac{32}{81}x^3 + O(x^4)$

$\quad = \frac{2}{3} + \frac{8}{9}x + \frac{32}{27}x^2 + \frac{128}{81}x^3 + O(x^4)$ ADD

$\therefore f(x) \approx 2 + \frac{16}{9}x + \frac{16}{9}x^2 + \frac{160}{81}x^3$

AS REQUIRED

**Question (****)**

$$f(x) = \frac{12}{\sqrt{1-2x}}, \quad x \in \mathbb{R}, \quad x \le \frac{1}{2}.$$

Use a quadratic approximation for $f(x)$ to solve the equation

$$f(x) = 16 - 67x - 2x^2.$$

$\boxed{\phantom{xxxx}}$, $\boxed{x \approx \dfrac{1}{20}}$

EXPAND BINOMIALLY UP TO $x^2$

$$\frac{1}{\sqrt{1-2x}} = (1-2x)^{-\frac{1}{2}} = 1 + \frac{-\frac{1}{2}}{1}(-2x) + \frac{-\frac{1}{2}\left(-\frac{3}{2}\right)}{1\times 2}(-2x)^2 + O(x^3)$$

$$= 1 + x + \frac{3}{2}x^2 + O(x^3)$$

NOW THE GIVEN EQUATION

$$\frac{12}{\sqrt{1-2x}} = 16 - 67x - 2x^2$$

$$12\left(1 + x + \frac{3}{2}x^2\right) \simeq 16 - 67x - 2x^2$$

$$12 + 12x + 18x^2 \simeq 16 - 67x - 2x^2$$

$$20x^2 + 79x - 4 \simeq 0$$

FACTORIZE OR QUADRATIC FORMULA

$$(20x + 1)(x - 4) = 0$$

$$x = \begin{cases} \frac{1}{20} \\ -4 \end{cases}$$

NOW THE VALIDITY OF THIS EXPANSION IS $-\frac{1}{2} < x < \frac{1}{2}$

$\therefore x = \frac{1}{20}$ (AS WELL AS $x=4$ CANNOT GO INTO THE RADICAL)

**Question 55** (****)

$$f(x) = \sqrt{1-x}\,,\ -1 < x < 1.$$

**a)** Expand $f(x)$ up and including the term in $x^3$.

**b)** Show clearly that

$$9\sqrt{1-\frac{1}{81}} = 4\sqrt{5}\,.$$

**c)** By using the **first two** terms of the expansion obtained in part **(a)** and the result shown in part **(b)**, show further that

$$\sqrt{5} \approx \frac{161}{72}\,.$$

$$\boxed{1-\frac{1}{2}x-\frac{1}{8}x^2-\frac{1}{16}x^3+O\left(x^4\right)}$$

(a) $f(x) = \sqrt{1-x} = (1-x)^{\frac{1}{2}}$

$= 1 + \frac{\frac{1}{2}}{1}(-x)^1 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{1\times 2}(-x)^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{1\times 2\times 3}(-x)^3 + O(x^4)$

$= 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 + O(x^4)$

(b) $9\sqrt{1-\frac{1}{81}} = 9\sqrt{\frac{81}{81}-\frac{1}{81}} = 9\sqrt{\frac{80}{81}} = 9\frac{\sqrt{80}}{\sqrt{81}} = \cancel{9}\times\frac{\sqrt{80}}{\cancel{9}}$

$= \sqrt{16\times 5} = 4\sqrt{5}$

(c) $\sqrt{1-x} \approx 1 - \frac{1}{2}x$ (FOR VERY SMALL $x$)

LET $x = \frac{1}{81}$

$\Rightarrow \sqrt{1-\frac{1}{81}} \approx 1 - \frac{1}{2}\left(\frac{1}{81}\right)$

$\Rightarrow 9\sqrt{1-\frac{1}{81}} \approx 9\left[1 - \frac{1}{162}\right]$

$\Rightarrow 4\sqrt{5} \approx 9 - \frac{1}{18}$

$\Rightarrow 4\sqrt{5} \approx \frac{161}{18}$

$\Rightarrow \sqrt{5} \approx \frac{161}{72}$ AS REQUIRED

**Question 56** (****)

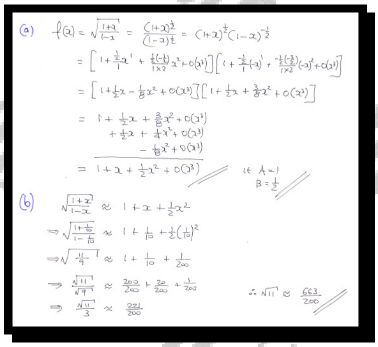
$$f(x)=\sqrt{\frac{1+x}{1-x}}\approx 1+Ax+Bx^2 \text{, for small } x.$$

**a)** Show that $B=\frac{1}{2}$ and find the value of $A$.

**b)** By using $x=\frac{1}{10}$ in the above expansion, show clearly that

$$\sqrt{11}\approx\frac{663}{200}.$$

$\boxed{A=1}$



(a) $f(x)=\sqrt{\frac{1+x}{1-x}}=\frac{(1+x)^{\frac{1}{2}}}{(1-x)^{\frac{1}{2}}}=(1+x)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}}$

$=\left[1+\frac{\frac{1}{2}}{1}x^1+\frac{\frac{1}{2}(-\frac{1}{2})}{1\times 2}x^2+O(x^3)\right]\left[1+\frac{-\frac{1}{2}}{1}(-x)^1+\frac{-\frac{1}{2}(-\frac{3}{2})}{1\times 2}(-x)^2+O(x^3)\right]$

$=\left[1+\frac{1}{2}x-\frac{1}{8}x^2+O(x^3)\right]\left[1+\frac{1}{2}x+\frac{3}{8}x^2+O(x^3)\right]$

$=1+\frac{1}{2}x+\frac{3}{8}x^2+O(x^3)$
$\quad+\frac{1}{2}x+\frac{1}{4}x^2+O(x^3)$
$\quad-\frac{1}{8}x^2+O(x^3)$

$=1+x+\frac{1}{2}x^2+O(x^3)$ ie $A=1$, $B=\frac{1}{2}$

(b) $\sqrt{\frac{1+x}{1-x}}\approx 1+x+\frac{1}{2}x^2$

$\Rightarrow\sqrt{\frac{1+\frac{1}{10}}{1-\frac{1}{10}}}\approx 1+\frac{1}{10}+\frac{1}{2}\left(\frac{1}{10}\right)^2$

$\Rightarrow\sqrt{\frac{11}{9}}\approx 1+\frac{1}{10}+\frac{1}{200}$

$\Rightarrow\frac{\sqrt{11}}{\sqrt{9}}\approx\frac{200}{200}+\frac{20}{200}+\frac{1}{200}$

$\Rightarrow\frac{\sqrt{11}}{3}\approx\frac{221}{200}$ $\quad\therefore\sqrt{11}\approx\frac{663}{200}$

**Question 57** (****)

$$f(x) = \sqrt{1-x}, \quad -1 < x < 1.$$

**a)** Expand $f(x)$ up and including the term in $x^3$.

**b)** Show carefully that

$$17\sqrt{1-\frac{1}{289}} = 12\sqrt{2}.$$

**c)** By using the **first two** terms of the expansion obtained in part (**a**) and the result shown in part (**b**), show further that

$$\sqrt{2} \approx \frac{577}{408}.$$

$$\boxed{1-\frac{1}{2}x-\frac{1}{8}x^2-\frac{1}{16}x^3+O\left(x^4\right)}$$

(a) $\sqrt{1-x} = (1-x)^{\frac{1}{2}} = 1 + \frac{\frac{1}{2}}{1}(-x) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{1\times2}(-x)^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{1\times2\times3}(-x)^3 + O(x^4)$

$= 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 + O(x^4)$

(b) $17\sqrt{1-\frac{1}{289}} = 17\sqrt{\frac{289}{289}-\frac{1}{289}} = 17\sqrt{\frac{288}{289}} = 17 \times \frac{\sqrt{288}}{\sqrt{289}}$

$= \cancel{17} \times \frac{\sqrt{144\times2}}{\cancel{17}} = 12\sqrt{2}$ AS REQUIRED

(c) IF $x$ IS NUMERICALLY SMALL

$\Rightarrow \sqrt{1-x} \approx 1 - \frac{1}{2}x$

$\Rightarrow \sqrt{1-\frac{1}{289}} \approx 1 - \frac{1}{2}\left(\frac{1}{289}\right)$

$\Rightarrow 17\sqrt{1-\frac{1}{289}} \approx 17\left[1-\frac{1}{578}\right]$

$\Rightarrow 12\sqrt{2} \approx \frac{577}{34}$

$\Rightarrow \sqrt{2} \approx \frac{577}{408}$

AS REQUIRED

**Question 58** (****)

$$f(x) \equiv \frac{4x(9x-10)}{(2-x)(2-3x)^2}, \; x \in \mathbb{R}, \; |x| < \frac{2}{3}, \; x \neq 0.$$

**a)** Find the values of the constants $A$, $B$ and $C$ given that

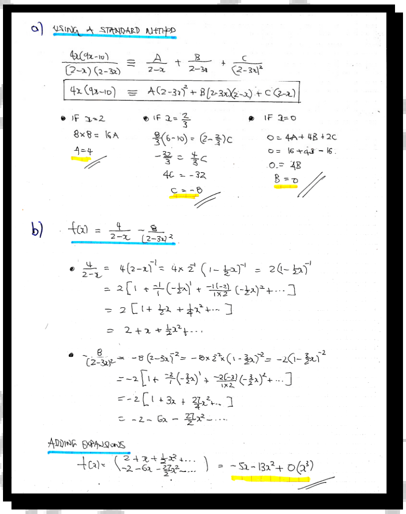
$$f(x) \equiv \frac{A}{2-x} + \frac{B}{2-3x} + \frac{C}{(2-3x)^2}.$$

**b)** Hence, or otherwise, find the binomial series expansion of $f(x)$, up and including the term in $x^2$.

The equation $f(x) = -0.63$ is known to have a positive solution which is further known to be numerically small.

**c)** Use part **(b)** to find this solution.

$\boxed{A=4}$, $\boxed{B=0}$, $\boxed{C=-8}$, $\boxed{f(x) = -5x - 13x^2 + O(x^3)}$, $\boxed{x = 0.1}$



a) USING A STANDARD METHOD

$$\frac{4x(9x-10)}{(2-x)(2-3x)} \equiv \frac{A}{2-x} + \frac{B}{2-3x} + \frac{C}{(2-3x)^2}$$

$$\boxed{4x(9x-10) \equiv A(2-3x)^2 + B(2-3x)(2-x) + C(2-x)}$$

- IF $x=2$: $8 \times 8 = 16A$, $A=4$
- IF $x=\frac{2}{3}$: $\frac{8}{3}(6-10) = (2-\frac{2}{3})C$, $-\frac{32}{3} = \frac{4}{3}C$, $4C = -32$, $C=-8$
- IF $x=0$: $0 = 4A + 4B + 2C$, $0 = 16 + 4B - 16$, $0 = 4B$, $B=0$

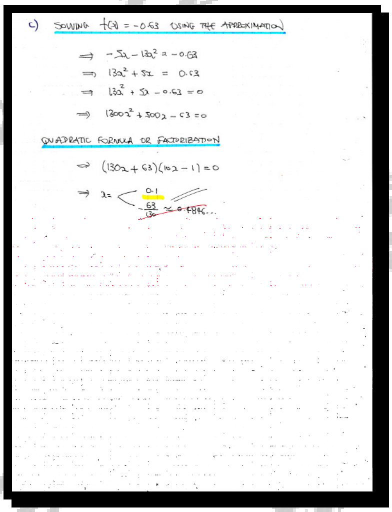
b) $f(x) = \frac{4}{2-x} - \frac{8}{(2-3x)^2}$

- $\frac{4}{2-x} = 4(2-x)^{-1} = 4 \times 2^{-1}(1-\frac{1}{2}x)^{-1} = 2(1-\frac{1}{2}x)^{-1}$
$= 2\left[1 + \frac{-1}{1}(-\frac{1}{2}x)^1 + \frac{-1(-2)}{1 \times 2}(-\frac{1}{2}x)^2 + \dots\right]$
$= 2\left[1 + \frac{1}{2}x + \frac{1}{4}x^2 + \dots\right]$
$= 2 + x + \frac{1}{2}x^2 + \dots$

- $-\frac{8}{(2-3x)^2} = -8(2-3x)^{-2} = -8 \times 2^{-2} \times (1-\frac{3}{2}x)^{-2} = -2(1-\frac{3}{2}x)^{-2}$
$= -2\left[1 + \frac{-2}{1}(-\frac{3}{2}x)^1 + \frac{-2(-3)}{1 \times 2}(-\frac{3}{2}x)^2 + \dots\right]$
$= -2\left[1 + 3x + \frac{27}{4}x^2 + \dots\right]$
$= -2 - 6x - \frac{27}{2}x^2 - \dots$

ADDING EXPANSIONS

$$f(x) = \begin{pmatrix} 2 + x + \frac{1}{2}x^2 + \dots \\ -2 - 6x - \frac{27}{2}x^2 - \dots \end{pmatrix} = -5x - 13x^2 + O(x^3)$$



c) SOLVING $f(x) = -0.63$ USING THE APPROXIMATION

$\Rightarrow -5x - 13x^2 = -0.63$

$\Rightarrow 13x^2 + 5x = 0.63$

$\Rightarrow 13x^2 + 5x - 0.63 = 0$

$\Rightarrow 1300x^2 + 500x - 63 = 0$

QUADRATIC FORMULA OR FACTORIZATION

$\Rightarrow (130x + 63)(10x - 1) = 0$

$\Rightarrow x = \begin{cases} 0.1 \\ \cancel{-\frac{63}{130} \approx -0.4846\dots} \end{cases}$

**Question 59** (****)

$$f(x) = (1+kx)^{-3}, \ |kx| < 1,$$

where $k$ is a non zero constant.

**a)** Expand $f(x)$, in terms of $k$, as an infinite convergent series up and including the term in $x^3$.

$$g(x) = \frac{6-x}{(1+kx)^3}, \ |kx| < 1.$$

The coefficient of $x^2$ in the expansion of $g(x)$ is $3$.

**b)** Find the possible values of $k$.

$\boxed{\phantom{x}}$, $\boxed{1 - 3kx + 6k^2x^2 - 10k^3x^3 + O(x^4)}$, $\boxed{k = -\frac{1}{3}, \frac{1}{4}}$

(a) $(1+kx)^{-3} = 1 + \frac{-3}{1}(kx)^1 + \frac{-3(-4)}{1\times 2}(kx)^2 + \frac{(-3)(-4)(-5)}{1\times 2\times 3}(kx)^3 + O(x^4)$

$= 1 - 3kx + 6k^2x^2 - 10k^3x^3 + O(x^4)$

(b) $\frac{6-x}{(1+kx)^3} = (6-x)(1+kx)^{-3}$

$= (6-x)(1 - 3kx + 6k^2x^2 + O(x^3))$

$3kx^2$

$36k^2x^2$

Hence $3kx^2 + 36k^2x^2 = 3x^2$

$3k + 36k^2 = 3$

$12k^2 + k - 1 = 0$

$(3k+1)(4k-1) = 0$

$\therefore k = \frac{1}{4}, -\frac{1}{3}$

**Question 60** (****)

$$f(x) = \frac{1}{\sqrt{1-x}} - \sqrt{1+x}, \; |x| < 1.$$

**a)** Show clearly that

$$f(x) = \frac{1}{2}x^2 + \frac{1}{4}x^3 + O\left(x^4\right).$$

**b)** Hence show that $f(x)$ has a minimum at the origin.

$\boxed{\text{proof}}$

(a) $\frac{1}{\sqrt{1-x}} = (1-x)^{-\frac{1}{2}} = 1 + \frac{-\frac{1}{2}}{1}(-x)^1 + \frac{-\frac{1}{2}\left(-\frac{3}{2}\right)}{1\times2}(-x)^2 + \frac{-\frac{1}{2}\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{1\times2\times3}(-x)^3 + O(x^4)$

$= 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + O(x^4)$

$-\sqrt{1+x} = -(1+x)^{\frac{1}{2}} = -\left[1 + \frac{\frac{1}{2}}{1}(x)^1 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{1\times2}(x)^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{1\times2\times3}(x)^3 + O(x^4)\right]$

$= -\left[1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + O(x^4)\right]$

$= -1 - \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{16}x^3 + O(x^4)$

$\therefore f(x) = \cancel{1} + \cancel{\frac{1}{2}x} + \frac{3}{8}x^2 + \frac{5}{16}x^3 + O(x^4)$

$\cancel{-1} - \cancel{\frac{1}{2}x} + \frac{1}{8}x^2 - \frac{1}{16}x^3 + O(x^4)$

$f(x) = \frac{1}{2}x^2 + \frac{1}{4}x^3 + O(x^4)$

(b) $f'(x) = x + \frac{3}{4}x^2 + O(x^3)$ $\quad f'(0) = 0$ & $f''(0) = 1 > 0$

$f''(x) = 1 + \frac{3}{2}x + O(x^2)$

$\therefore$ INDEED A MINIMUM AT THE ORIGIN

**Question 61** (****)

$$f(x) \equiv \frac{16x^2+3x-2}{x^2(3x-2)}, \; x \in \mathbb{R}, \; |x| < \frac{2}{3}, \; x \neq 0.$$

**a)** Determine the value of each of the constants $A$, $B$ and $C$ given that

$$f(x) \equiv \frac{A}{x^2} + \frac{B}{x} + \frac{C}{(3x-2)}.$$

**b)** Find the binomial series expansion of $\dfrac{1}{3x-2}$, up and including the term in $x^3$.

**c)** Hence, or otherwise, show that if $x$ is numerically small

$$\frac{16x^2+3x-2}{(3x-2)} \approx 1 - 8x^2 - 12x^3 - 18x^4 - 27x^5.$$

$\boxed{\phantom{xx}}$, $\boxed{A=1}$, $\boxed{B=0}$, $\boxed{C=16}$, $\boxed{-\frac{1}{2} - \frac{3}{4}x - \frac{9}{8}x^2 - \frac{27}{16}x^3 + O(x^4)}$

a) $f(x) \equiv \dfrac{16x^2+3x-2}{x^2(3x-2)} \equiv \dfrac{A}{x^2} + \dfrac{B}{x} + \dfrac{C}{3x-2}$

$16x^2+3x-2 \equiv A(3x-2) + Bx(3x-2) + Cx^2$

- IF $x=0$: $-2 = -2A$, $A=1$
- IF $x=\frac{2}{3}$: $\frac{64}{9} + \cancel{2} - \cancel{2} = C \times \frac{4}{9}$, $C=16$
- IF $x=1$: $17 = A+B+C$, $17 = 1+B+16$, $B=0$

b) $\dfrac{1}{3x-2} = -\dfrac{1}{2-3x} = -(2-3x)^{-1} = -(2)^{-1}\left[1-\frac{3}{2}x\right]^{-1}$

$= -\frac{1}{2}\left(1-\frac{3}{2}x\right)^{-1}$

$= -\frac{1}{2}\left[1 + \frac{-1}{1}\left(-\frac{3}{2}x\right)^1 + \frac{-1(-2)}{1\times2}\left(-\frac{3}{2}x\right)^2 + \frac{(-1)(-2)(-3)}{1\times2\times3}\left(-\frac{3}{2}x\right)^3 + \dots\right]$

$= -\frac{1}{2}\left[1 + \frac{3}{2}x + \frac{9}{4}x^2 + \frac{27}{8}x^3 + \dots\right]$

$= -\frac{1}{2} - \frac{3}{4}x - \frac{9}{8}x^2 - \frac{27}{16}x^3 - \dots$

c) METHOD A (ONLY UP TO $x^3$ IS DIRECTLY AVAILABLE)

$\dfrac{16x^2+3x-2}{3x-2} = (-2+3x+16x^2)\left[-\frac{1}{2} - \frac{3}{4}x - \frac{9}{8}x^2 - \frac{27}{16}x^3 + \dots\right]$

$= 1 + \cancel{\frac{3}{2}x} + \cancel{\frac{9}{4}x^2} + \cancel{\frac{27}{8}x^3} + \dots$

$\quad \cancel{-\frac{3}{2}x} \cancel{-\frac{9}{4}x^2} \cancel{-\frac{27}{8}x^3} + \dots$

$\quad -8x^2 - 12x^3 + \dots$

$= 1 - 8x^2 - 12x^3 + \dots$

METHOD B (USING PREVIOUS PARTS)

$\dfrac{16x^2+3x-2}{x^2(3x-2)} = \dfrac{1}{x^2} + \dfrac{16}{3x-2}$

$\dfrac{1}{x^2}\left(\dfrac{16x^2+3x-2}{3x-2}\right) = \dfrac{1}{x^2} + 16\left(\dfrac{1}{3x-2}\right)$

$\dfrac{16x^2+3x-2}{3x-2} = 1 + 16x^2\left(\dfrac{1}{3x-2}\right)$

$\dfrac{16x^2+3x-2}{3x-2} = 1 + 16x^2\left[-\frac{1}{2} - \frac{3}{4}x - \frac{9}{8}x^2 - \frac{27}{16}x^3 + O(x^4)\right]$

$\dfrac{16x^2+3x-2}{3x-2} = 1 - 8x^2 - 12x^3 - 18x^4 - 27x^5 + O(x^6)$

**Question 62** (****)

$$f(x) = \sqrt{1-x}, \quad -1 < x < 1.$$

**a)** Expand $f(x)$ up and including the term in $x^3$.

**b)** Show clearly that

$$7\sqrt{1-\frac{1}{49}} = 4\sqrt{3}.$$

**c)** By using the **first two** terms of the expansion obtained in part **(a)** and the result obtained in part **(b)**, show further that

$$\sqrt{3} \approx \frac{97}{56}.$$

$\boxed{\phantom{x}}$, $\boxed{1-\frac{1}{2}x-\frac{1}{8}x^2-\frac{1}{16}x^3+O\left(x^4\right)}$

a) $f(x) = \sqrt{1-x} = (1-x)^{\frac{1}{2}}$

$= 1 + \frac{\frac{1}{2}}{1}(-x)^1 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{1\times 2}(-x)^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{1\times 2\times 3}(-x)^3 + \dots$

$= 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 + O(x^4)$

b) BY DIRECT PROOF

$7\sqrt{1-\frac{1}{49}} = 7\sqrt{\frac{48}{49}} = \cancel{7}\frac{\sqrt{48}}{\cancel{\sqrt{49}}} = \sqrt{48}$

$= \sqrt{16\times 3} = 4\sqrt{3}$

c) FROM PART (a)

$\Rightarrow \sqrt{1-x} \approx 1 - \frac{1}{2}x$

$\Rightarrow 7\sqrt{1-x} \approx 7\left(1-\frac{1}{2}x\right)$

$\Rightarrow 7\sqrt{1-x} \approx 7 - \frac{7}{2}x$

LET $x = \frac{1}{49}$ (AS IN PART b)

$\Rightarrow 7\sqrt{1-\frac{1}{49}} \approx 7 - \frac{7}{2}\left(\frac{1}{49}\right)$

$\Rightarrow 4\sqrt{3} \approx 7 - \frac{7}{98}$

$\Rightarrow 4\sqrt{3} \approx \frac{97}{14}$

$\Rightarrow \sqrt{3} \approx \frac{97}{56}$

AS REQUIRED

**Question 63** (****+)

$$f(x) = \sqrt{\frac{1+ax}{4-x}}, \quad -1 < x < 1.$$

The value of the constant $a$ is such so that the coefficient of $x^2$ in the convergent binomial expansion of $f(x)$ is $\frac{1}{64}$.

Find the value of $a$.

$\boxed{\phantom{a=1}}$, $\boxed{a = \frac{1}{4}}$

$$\sqrt{\frac{1+ax}{4-x}} = \frac{(1+ax)^{\frac{1}{2}}}{(4-x)^{\frac{1}{2}}} = (1+ax)^{\frac{1}{2}}(4-x)^{-\frac{1}{2}} = (1+ax)^{\frac{1}{2}} \times 4^{-\frac{1}{2}}\left(1-\tfrac{1}{4}x\right)^{-\frac{1}{2}}$$

$$= \tfrac{1}{2}(1+ax)^{\frac{1}{2}}\left(1-\tfrac{1}{4}x\right)^{-\frac{1}{2}}$$

$$= \tfrac{1}{2}\left[1 + \frac{\frac{1}{2}}{1}(ax)^1 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{1\times 2}(ax)^2 + O(x^3)\right]\left[1 + \frac{-\frac{1}{2}}{1}\left(-\tfrac{1}{4}x\right)^1 + \frac{-\frac{1}{2}\left(-\frac{3}{2}\right)}{1\times 2}\left(-\tfrac{1}{4}x\right)^2 + O(x^3)\right]$$

$$= \tfrac{1}{2}\left[1 + \tfrac{1}{2}ax - \tfrac{1}{8}a^2x^2 + O(x^3)\right]\left[1 + \tfrac{1}{8}x + \tfrac{3}{128}x^2 + O(x^3)\right]$$

$$= \ldots$$

$-\frac{1}{8}a^2$

$\frac{1}{16}a$

$\frac{3}{128}$

Hence $\frac{1}{2}\left[-\frac{1}{8}a^2 + \frac{1}{16}a + \frac{3}{128}\right] = \frac{1}{64}$

$$-\tfrac{1}{8}a^2 + \tfrac{1}{16}a + \tfrac{3}{128} = \tfrac{1}{32} \quad (\times 128)$$

$$-16a^2 + 8a + 3 = 4$$

$$16a^2 - 8a + 1 = 0$$

$$(4a-1)^2 = 0$$

$$a = \tfrac{1}{4}$$

**Question 64** (****+)

$$f(x) \equiv \frac{1}{\sqrt{1-ax}} - \sqrt{1+bx},$$

where $a$ and $b$ are constants so that $a > b > 0$.

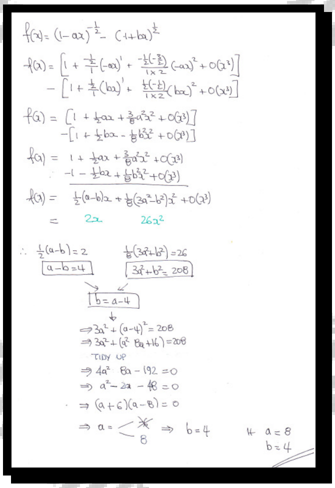
The function $f$ is defined in a suitable domain of $x$, and furthermore the values of $x$ are small enough so that $f(x)$ has a binomial series expansion.

Given that

$$f(x) \approx 2x + 26x^2,$$

determine the value of $a$ and the value of $b$.

$\boxed{\phantom{a=8}}$, $\boxed{a=8}$, $\boxed{b=4}$



$$f(x) = (1-ax)^{-\frac{1}{2}} - (1+bx)^{\frac{1}{2}}$$

$$f(x) = \left[1 + \frac{-\frac{1}{2}}{1}(-ax)^1 + \frac{-\frac{1}{2}\left(-\frac{3}{2}\right)}{1\times 2}(-ax)^2 + O(x^3)\right] - \left[1 + \frac{\frac{1}{2}}{1}(bx)^1 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{1\times 2}(bx)^2 + O(x^3)\right]$$

$$f(x) = \left[1 + \frac{1}{2}ax + \frac{3}{8}a^2x^2 + O(x^3)\right] - \left[1 + \frac{1}{2}bx - \frac{1}{8}b^2x^2 + O(x^3)\right]$$

$$f(x) = 1 + \frac{1}{2}ax + \frac{3}{8}a^2x^2 + O(x^3) - 1 - \frac{1}{2}bx + \frac{1}{8}b^2x^2 + O(x^3)$$

$$f(x) = \frac{1}{2}(a-b)x + \frac{1}{8}(3a^2-b^2)x^2 + O(x^3)$$

$$= 2x \qquad 26x^2$$

$$\therefore \ \frac{1}{2}(a-b) = 2 \qquad \frac{1}{8}(3a^2+b^2) = 26$$

$$\boxed{a-b=4} \qquad \boxed{3a^2+b^2=208}$$

$$\boxed{b=a-4}$$

$$\Rightarrow 3a^2 + (a-4)^2 = 208$$

$$\Rightarrow 3a^2 + (a^2 - 8a + 16) = 208$$

TIDY UP

$$\Rightarrow 4a^2 - 8a - 192 = 0$$

$$\Rightarrow a^2 - 2a - 48 = 0$$

$$\Rightarrow (a+6)(a-8) = 0$$

$$\Rightarrow a = \cancel{-6},\ 8 \Rightarrow b = 4$$

$$\therefore \ a = 8,\ b = 4$$

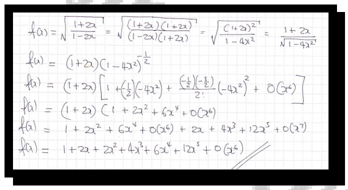
**Question 65** (****+)

$$f(x)=\sqrt{\frac{1+2x}{1-2x}}, \quad |x|<\frac{1}{2}.$$

By writing $f(x)$ in the form $f(x)=\dfrac{1+ax}{\sqrt{1+bx^2}}$, show that

$$f(x)=1+2x+2x^2+4x^3+6x^4+12x^5+O\left(x^6\right).$$

$\boxed{\text{proof}}$



$$f(x)=\sqrt{\frac{1+2x}{1-2x}}=\sqrt{\frac{(1+2x)(1+2x)}{(1-2x)(1+2x)}}=\sqrt{\frac{(1+2x)^2}{1-4x^2}}=\frac{1+2x}{\sqrt{1-4x^2}}$$

$$f(x)=(1+2x)(1-4x^2)^{-\frac{1}{2}}$$

$$f(x)=(1+2x)\left[1+\left(-\tfrac{1}{2}\right)(-4x^2)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(-4x^2)^2+O(x^6)\right]$$

$$f(x)=(1+2x)(1+2x^2+6x^4+O(x^6))$$

$$f(x)=1+2x^2+6x^4+O(x^6)+2x+4x^3+12x^5+O(x^7)$$

$$f(x)=1+2x+2x^2+4x^3+6x^4+12x^5+O(x^6)$$

**Question 66** (****+)

$$f(x) = \left(\tfrac{1}{2} - x\right)^{-3}, \quad |x| < \tfrac{1}{2}.$$

**a)** Expand $f(x)$, up and including the term in $x^3$.

$$g(x) = \frac{a + bx}{\left(\tfrac{1}{2} - x\right)^3}.$$

The coefficients of $x^2$ and $x^3$ in the expansion of $g(x)$ are $42$ and $136$ respectively.

**b)** Show that $a = \frac{1}{4}$ and find the value of $b$.

$\boxed{\phantom{MM}}$, $\boxed{f(x) = 8 + 48x + 192x^2 + 640x^3 + O\left(x^4\right)}$, $\boxed{b = -\frac{1}{8}}$

a) EXPAND BINOMIALLY UP TO $x^3$

$\Rightarrow f(x) = \left(\frac{1}{2} - x\right)^{-3} = \left(\frac{1}{2}\right)^{-3}\left[1 - 2x\right]^{-3} = 8(1 - 2x)^{-3}$

$\Rightarrow f(x) = 8\left[1 + \frac{-3}{1}(-2x)^1 + \frac{-3(-4)}{1 \times 2}(-2x)^2 + \frac{-3(-4)(-5)}{1 \times 2 \times 3}(-2x)^3 + O(x^4)\right]$

$\Rightarrow f(x) = 8\left[1 + 6x + 24x^2 + 80x^3 + O(x^4)\right]$

$\Rightarrow f(x) = 8 + 48x + 192x^2 + 640x^3 + O(x^4)$ //

b) PROCEED AS FOLLOWS

$\Rightarrow g(x) = \frac{a + bx}{\left(\frac{1}{2} - x\right)^3} = (a + bx)\left(\frac{1}{2} - x\right)^{-3}$

$\Rightarrow g(x) = (a + bx)\left[8 + 48x + 192x^2 + 640x^3 + O(x^4)\right]$

$\Rightarrow g(x) = 8a + 48ax + 192ax^2 + 640ax^3 + O(x^4)$
$\qquad\qquad 8bx + 48bx^2 + 192bx^3 + O(x^4)$

$\Rightarrow g(x) = 8a + (48a + 8b)x + \underbrace{(192a + 48b)}_{42}x^2 + \underbrace{(640a + 192b)}_{136}x^3 + O(x^4)$

FINALLY WE HAVE

$\left.\begin{array}{l} 192a + 48b = 42 \\ 640a + 192b = 136 \end{array}\right\} \Rightarrow \left.\begin{array}{l} 32a + 8b = 7 \\ 80a + 24b = 17 \end{array}\right\} \begin{array}{l} \times(-3) \\ \times 1 \end{array}$

$\left.\begin{array}{l} -96a - 24b = -21 \\ 80a + 24b = 17 \end{array}\right\}$ ADDING $-16a = -4$

$a = \frac{1}{4}$ //

AND $32a + 8b = 7$

$8 + 8b = 7$

$b = -\frac{1}{8}$ //

**Question 67** (****+)

In the convergent expansion of

$$(1+kx)^n, \quad |kx|<1,$$

where $k$ and $n$ are non zero constants, the coefficient of $x^2$ is 12 and the coefficient of $x^3$ is $32$.

Given the coefficient of $x$ is negative determine the values of $k$ and $n$.

$\boxed{\text{MP2-V}}$, $\boxed{n=\frac{6}{5}}$, $\boxed{k=-10}$

a) EXPAND IN TERMS OF $k$ & $n$, UP TO $x^3$

$$(1+kx)^n = 1+\frac{n}{1}(kx)^1+\frac{n(n-1)}{1\times 2}(kx)^2+\frac{n(n-1)(n-2)}{1\times 2\times 3}(kx)^3+O(x^4)$$

$$(1+kx)^n = 1+\underbrace{nk}_{\text{NEGATIVE}}x+\underbrace{\tfrac{1}{2}n(n-1)k^2}_{12}x^2+\underbrace{\tfrac{1}{6}n(n-1)(n-2)k^3}_{32}x^3+O(x^4)$$

FORMING & SOLVING EQUATIONS

$$\left.\begin{array}{l}\frac{1}{2}n(n-1)k^2=12\\ \frac{1}{6}n(n-1)(n-2)k^3=32\end{array}\right\}\Rightarrow \begin{array}{l}n(n-1)k^2=24\\ n(n-1)(n-2)k^3=192\end{array}$$

DIVIDING THE EQUATIONS YIELDS

$$\Rightarrow \frac{n(n-1)(n-2)k^3}{n(n-1)k^2}=\frac{192}{24} \qquad n\neq 0,\ k\neq 0,\ n\neq 1$$

$$\Rightarrow k(n-2)=8$$

$$\Rightarrow k=\frac{8}{n-2}$$

SUBSTITUTE INTO $n(n-1)k^2=24$

$$\Rightarrow n(n-1)\left(\frac{8}{n-2}\right)^2=24$$

$$\Rightarrow \frac{64n(n-1)}{(n-2)^2}=24$$

$$\Rightarrow 64n^2-64n=24(n-2)^2$$

$$\Rightarrow 64n^2-64n=24(n^2-4n+4)$$

$$\Rightarrow 64n^2-64n=24n^2-96n+96$$

$$\Rightarrow 40n^2+32n-96=0$$

$$\Rightarrow 5n^2+4n-12=0$$

$$\Rightarrow (5n-6)(n+2)$$

$$\Rightarrow n = \begin{cases}6/5\\ -2\end{cases}$$

& $k=\dfrac{8}{n-2}=\begin{cases}\dfrac{8}{\frac{6}{5}-2}=-10\\[2mm] \dfrac{8}{-4}=-2\end{cases}$

EITHER $n=\frac{6}{5}$, $k=-10$ $\quad nk<0$, GIVEN

OR ~~$n=-2$, $k=2$~~

**Question 68** (****+)

$$\frac{A+Bx}{(2-x)^3} \equiv \frac{1}{4} + Cx^2 + Dx^3 + \dots,$$

where $A$, $B$, $C$ and $D$ are constants, and $|x|<2$

Determine the value of $A$, $B$, $C$ and $D$.

$\boxed{\phantom{A=2}}$, $\boxed{A=2}$, $\boxed{B=-3}$, $\boxed{C=-\frac{3}{16}}$, $\boxed{D=-\frac{1}{4}}$

PROCEED AS FOLLOWS

$$\frac{A+Bx}{(2-x)^3} = (A+Bx)(2-x)^{-3} = (A+Bx)\times 2^{-3}\left(1-\tfrac{1}{2}x\right)^{-3}$$

$$= \tfrac{1}{8}(A+Bx)\left[1 + \tfrac{-3}{1}\left(-\tfrac{1}{2}x\right)^1 + \tfrac{-3(-4)}{1\times 2}\left(-\tfrac{1}{2}x\right)^2 + \tfrac{-3(-4)(-5)}{1\times 2\times 3}\left(-\tfrac{1}{2}x\right)^3 + O(x^4)\right]$$

$$= \tfrac{1}{8}(A+Bx)\left[1 + \tfrac{3}{2}x + \tfrac{3}{2}x^2 + \tfrac{5}{4}x^3 + O(x^4)\right]$$

$$= (A+Bx)\left[\tfrac{1}{8} + \tfrac{3}{16}x + \tfrac{3}{16}x^2 + \tfrac{5}{32}x^3 + O(x^4)\right]$$

$$= \tfrac{1}{8}A + \tfrac{3}{16}Ax + \tfrac{3}{16}Ax^2 + \tfrac{5}{32}Ax^3 + O(x^4)$$
$$+ \tfrac{1}{8}Bx + \tfrac{3}{16}Bx^2 + \tfrac{3}{16}Bx^3 + O(x^4)$$

$$\underbrace{\tfrac{1}{8}A}_{\frac{1}{4}} + \underbrace{\left(\tfrac{3}{16}A + \tfrac{1}{8}B\right)}_{0}x + \underbrace{\left(\tfrac{3}{16}A + \tfrac{3}{16}B\right)}_{C}x^2 + \underbrace{\left(\tfrac{5}{32}A + \tfrac{3}{16}B\right)}_{D}x^3 + O(x^4)$$

COMPARING COEFFICIENTS

- $\frac{1}{8}A = \frac{1}{4}$
  $A = 2$
- $\frac{3}{16}A + \frac{1}{8}B = 0$
  $3A + 2B = 0$
  $6 + 2B = 0$
  $B = -3$
- $C = \frac{3}{16}A + \frac{3}{16}B$
  $C = \frac{3}{16}(A+B)$
  $C = \frac{3}{16}(-1)$
  $C = -\frac{3}{16}$
- $D = \frac{5}{32}A + \frac{3}{16}B$
  $D = \frac{5}{16} - \frac{9}{16}$
  $D = -\frac{1}{4}$

**Question 69 (****+)**

$$f(x) \equiv \sqrt[3]{1+12x}\,.$$

It is given that the equation

$$f(x)+(6x-5)^2 = 24-15x$$

has a solution $\alpha$, which is numerically small.

Use a quadratic approximation for $f(x)$ to find an approximate value for $\alpha$.

$\boxed{\text{MP}}$, $\boxed{\alpha \approx \frac{1}{20}}$

EXPAND $f(x)$ UP TO $x^2$

$$f(x) = (1+12x)^{\frac{1}{3}} = 1 + \frac{\frac{1}{3}}{1}(12x) + \frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{1\times 2}(12x)^2 + O(x^3)$$
$$= 1 + 4x - 16x^2 + O(x^3)$$

NOW LOOKING AT THE EQUATION

$$f(x) + (6x-5)^2 = 24-15x$$
$$\left[1+4x-16x^2+O(x^3)\right] + \left[36x^2-60x+25\right] = 24-15x$$

FOR $|x| \ll 1$ WE HAVE

$$1+4x-16x^2+36x^2-60x+25 \approx 24-15x$$
$$20x^2-41x+2=0$$

BY QUADRATIC FORMULA OF FACTORIZING

$$(x-2)(20x-1)=0$$

$$x = \begin{cases} \frac{1}{20} \\ \cancel{2} \end{cases} \qquad |x| < \frac{1}{12}$$

**Question 70** (****+)

The function $f$ is defined as

$$f(x) \equiv \frac{ax+b}{(1-x)(1+2x)}, \quad x \in \mathbb{R}, \ |x| < \frac{1}{2},$$

where $a$ and $b$ are constants.

**a)** Find the values of the constants $P$ and $Q$ in terms of $a$ and $b$, given that

$$f(x) \equiv \frac{P}{(1-x)} + \frac{Q}{(1+2x)}.$$

The binomial series expansion of $f(x)$, up and including the term in $x^3$ is

$$f(x) = 1 + 13x + Ax^2 + Bx^3 + \ldots,$$

where $A$ and $B$ are constants.

**b)** Determine the value of the constants …

**i.** … $a$ and $b$.

**ii.** … $A$ and $B$.

$\boxed{\phantom{xxxx}}$, $\boxed{P = \frac{a+b}{3}}$, $\boxed{P = \frac{2b-a}{3}}$, $\boxed{a = 14}$, $\boxed{b = 1}$, $\boxed{A = -11}$, $\boxed{B = 37}$

a) BY COVER UP OR STANDARD TECHNIQUES

$$\frac{ax+b}{(1-x)(1-2x)} \equiv \frac{P}{1-x} + \frac{Q}{1+2x}$$

$$ax + b \equiv P(1+2x) + Q(1-x)$$

- IF $x=1$: $a+b = 3P$, $P = \frac{a+b}{3}$
- IF $x = -\frac{1}{2}$: $-\frac{1}{2}a + b = \frac{3}{2}Q$, $-a+2b = 3Q$, $Q = \frac{2b-a}{3}$

b) COMPARING $f(x)$ IN TERMS OF $a$ & $b$ IN THE EXPANSION GIVEN

$$f(x) = 1 + 13x + Ax^2 + Bx^3 + \ldots$$

$$\Rightarrow \frac{\frac{a+b}{3}}{1-x} + \frac{\frac{2b-a}{3}}{1+2x} \equiv 1 + 13x + Ax^2 + Bx^3 + \ldots \quad \times 3$$

$$\Rightarrow (a+b)(1-x)^{-1} + (2b-a)(1+2x)^{-1} \equiv 3 + 39x + 3Ax^2 + 3Bx^3 + \ldots$$

$$\Rightarrow (a+b)(1+x+x^2+x^3) + (2b-a)(1-2x+4x^2-8x^3) \equiv 3 + 39x + 3Ax^2 + 3Bx^3$$

HERE WE USED STANDARD EXPANSIONS

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots$$

$$\frac{1}{1+2x} = 1 - (2x) + (2x)^2 - (2x)^3 + \ldots = 1 - 2x + 4x^2 - 8x^3$$

COMPARING CONSTANTS

$(a+b) + (2b-a) = 3$

$3b = 3$

$b = 1$

COMPARING $x^1$

$(a+b) - 2(2b-a) = 39$

$3a - 3b = 39$

$a - b = 13$

$\therefore a = 14$

FINALLY COMPARING $[x^2]$ & $[x^3]$

$(a+b) + 4(2b-a) = 3A$

$9b - 3a = 3A$

$A = 3b - a$

$A = 3 - 14$

$A = -11$

$(a+b) - 8(2b-a) = 3B$

$9a - 15b = 3B$

$3a - 5b = B$

$B = 3 \times 14 - 5 \times 1$

$B = 42 - 5$

$B = 37$

**Question 71** (****+)

The function $f$ is defined as

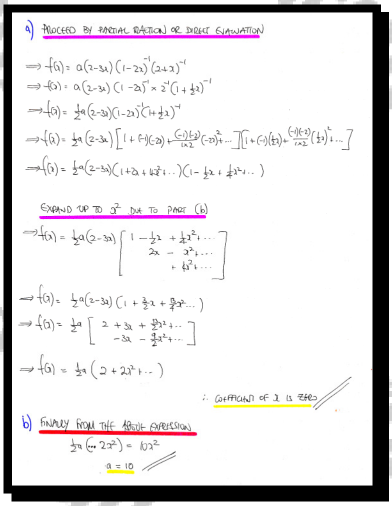
$$f(x) \equiv \frac{a(2-3x)}{(1-2x)(2+x)}, \quad x \in \mathbb{R}, \; |x| < \frac{1}{2}, \; x \neq 0.$$

where $a$ is a non zero constant.

**a)** Show that for all values of the constant $a$, the coefficient of $x$ in the binomial series expansion of $f(x)$, is zero.

**b)** Find the value of $a$, given that the coefficient of $x^2$ in the binomial series expansion of $f(x)$, is $10$.

$\boxed{\phantom{proof}}$, $\boxed{a=10}$



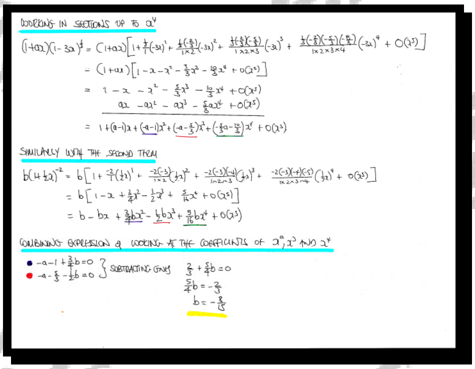
**Question 72** (****+)

$$f(x) = (1+ax)(1-3x)^{\frac{1}{3}} + \frac{b}{\left(1+\frac{1}{2}x\right)^2}, \quad |3x|<1, \ |ax|<1.$$

In the binomial expansion of $f(x)$ the coefficients of $x^2$ and $x^3$ are both zero.

Show clearly that the coefficient of $x^4$ is $-\frac{7}{6}$.

$\boxed{\phantom{xxxx}}$, $\boxed{\text{proof}}$



WORKING IN SECTIONS UP TO $x^4$

$$(1+ax)(1-3x)^{\frac{1}{3}} = (1+ax)\left[1+\frac{\frac{1}{3}}{1}(-3x)^1+\frac{\frac{1}{3}(-\frac{2}{3})}{1\times2}(-3x)^2+\frac{\frac{1}{3}(-\frac{2}{3})(-\frac{5}{3})}{1\times2\times3}(-3x)^3+\frac{\frac{1}{3}(-\frac{2}{3})(-\frac{5}{3})(-\frac{8}{3})}{1\times2\times3\times4}(-3x)^4+O(x^5)\right]$$
$$= (1+ax)\left[1-x-x^2-\frac{5}{3}x^3-\frac{10}{3}x^4+O(x^5)\right]$$
$$= 1-x-x^2-\frac{5}{3}x^3-\frac{10}{3}x^4+O(x^5)$$
$$\quad ax-ax^2-ax^3-\frac{5}{3}ax^4+O(x^5)$$
$$= 1+(a-1)x+(-a-1)x^2+\left(-a-\frac{5}{3}\right)x^3+\left(-\frac{5}{3}a-\frac{10}{3}\right)x^4+O(x^5)$$

SIMILARLY WITH THE SECOND TERM

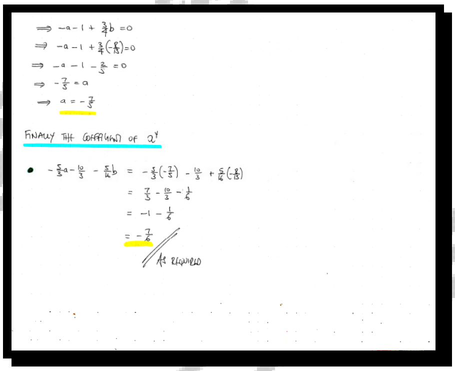
$$b\left(1+\frac{1}{2}x\right)^{-2} = b\left[1+\frac{-2}{1}\left(\frac{1}{2}x\right)^1+\frac{-2(-3)}{1\times2}\left(\frac{1}{2}x\right)^2+\frac{-2(-3)(-4)}{1\times2\times3}\left(\frac{1}{2}x\right)^3+\frac{-2(-3)(-4)(-5)}{1\times2\times3\times4}\left(\frac{1}{2}x\right)^4+O(x^5)\right]$$
$$= b\left[1-x+\frac{3}{4}x^2-\frac{1}{2}x^3+\frac{5}{16}x^4+O(x^5)\right]$$
$$= b-bx+\frac{3}{4}bx^2-\frac{1}{2}bx^3+\frac{5}{16}bx^4+O(x^5)$$

COMBINING EXPRESSION & LOOKING AT THE COEFFICIENTS OF $x^2$, $x^3$ AND $x^4$

- $-a-1+\frac{3}{4}b=0$
- $-a-\frac{5}{3}-\frac{1}{2}b=0$

SUBTRACTING GIVES $\frac{2}{3}+\frac{5}{4}b=0$

$$\frac{5}{4}b=-\frac{2}{3}$$
$$b=-\frac{8}{15}$$



$$\Rightarrow -a-1+\frac{3}{4}b=0$$
$$\Rightarrow -a-1+\frac{3}{4}\left(-\frac{8}{15}\right)=0$$
$$\Rightarrow -a-1-\frac{2}{5}=0$$
$$\Rightarrow -\frac{7}{5}=a$$
$$\Rightarrow a=-\frac{7}{5}$$

FINALLY THE COEFFICIENT OF $x^4$

- $-\frac{5}{3}a-\frac{10}{3}-\frac{5}{16}b = -\frac{5}{3}\left(-\frac{7}{5}\right)-\frac{10}{3}+\frac{5}{16}\left(-\frac{8}{15}\right)$

$$= \frac{7}{3}-\frac{10}{3}-\frac{1}{6}$$
$$= -1-\frac{1}{6}$$
$$= -\frac{7}{6}$$

AS REQUIRED

**Question 73** **(****+)**

If $x$ is sufficiently small find the series expansion of

$$\frac{10x^2-x-6}{(2+3x)\left(1-2x^2\right)},$$

up and including the term in $x^3$.

$$\boxed{\frac{10x^2-x-6}{(2+3x)\left(1-2x^2\right)}=-3+4x-7x^2+\frac{19}{2}x^3+O\left(x^4\right)}$$

PROCEED BY PARTIAL FRACTIONS

$$\Rightarrow \frac{10x^2-x-6}{(2+3x)(1-2x^2)} \equiv \frac{A}{2+3x}+\frac{Bx+C}{1-2x^2}$$

$$\Rightarrow 10x^2-x-6 \equiv A(1-2x^2)+(2+3x)(Bx+C)$$

$$\Rightarrow 10x^2-x-6 \equiv (3B-2A)x^2+(2B+3C)x+(A+2C)$$

- $3B-2A=10$
- $2B+3C=-1$, $\boxed{6B+9C=-3}$
- $-A+2C=-6$, $\boxed{A=-6-2C}$

THUS WE HAVE

$$3B-2(-6-2C)=10$$
$$3B+12+4C=10$$
$$3B+4C=-2$$
$$\boxed{-6B-8C=4}$$

$C=1$, $A=-8$, $B=-2$

HENCE WE NOW HAVE

$$\frac{10x^2-x-6}{(2+3x)(1-2x^2)}=\frac{-8}{2+3x}+\frac{-2x+1}{1-2x^2}$$

$$=(1-2x)(1-2x^2)^{-1}-8(2+3x)^{-1}$$

$$=(1-2x)(1-2x^2)^{-1}-8\times 2^{-1}\left(1+\tfrac{3}{2}x\right)^{-1}$$

$$=(1-2x)(1-2x^2)^{-1}-4\left(1+\tfrac{3}{2}x\right)^{-1}$$

USING $(1-x)^{-1}=1+x+x^2+x^3+O(x^4)$
USING $(1+x)^{-1}=1-x+x^2+x^3+O(x^4)$

$$\frac{10x^2-x-6}{(2+3x)(1-2x^2)}=\left\{\begin{array}{l}(1-2x)\left[1+2x^2+O(x^4)\right]\\ -4\left(1-\frac{3}{2}x+\frac{9}{4}x^2-\frac{27}{8}x^3+O(x^4)\right)\end{array}\right\}$$

$$=\left\{\begin{array}{l}1-2x+2x^2-4x^3+O(x^4)\\ -4+6x-9x^2+\frac{27}{2}x^3+O(x^4)\end{array}\right\}$$

$$=-3+4x-7x^2+\frac{19}{2}x^3+O(x^4)$$

**Question 74** (****+)

In the convergent expansion of

$$\left(1+\tfrac{4}{7}nx\right)^{n},\ n \in \mathbb{R},\ n \notin \mathbb{N},\ n \neq 0,$$

the coefficients of $x^2$ and $x^3$ are non zero and equal.

**a)** Determine the possible values of $n$.

**b)** State with justification which value, values or indeed if any of the values of $n$ produces a valid expansion for $x=1$.

$\boxed{\phantom{x}}$, $\boxed{n=-\tfrac{3}{2},\ \tfrac{7}{2}}$, $\boxed{\text{only } n=-\tfrac{3}{2} \text{ produces a valid expansion for } x=1}$

(a) $\left(1+\frac{4n}{7}x\right)^{n} = 1 + \frac{n}{1}\left(\frac{4n}{7}x\right)^{1} + \frac{n(n-1)}{1\times 2}\left(\frac{4n}{7}x\right)^{2} + \frac{n(n-1)(n-2)}{1\times 2\times 3}\left(\frac{4n}{7}x\right)^{3} + O(x^4)$

$= 1 + \frac{4}{7}n^2x + \frac{8}{49}n^3(n-1)x^2 + \frac{32}{1029}n^4(n-1)(n-2)x^3 + O(x^4)$

$(7\times7\times7)\times 3$

Now $\frac{8}{49}n^3(n-1) = \frac{32}{1029}n^4(n-1)(n-2)$

$\frac{8}{49} = \frac{32}{1029}n(n-2)$ (As $n \neq 0, n \neq 1$)

$\frac{1}{49} = \frac{4}{1029}n(n-2)$ ($1029 = 7\times7\times3\times7$)

$21 = 4n(n-2)$

$21 = 4n^2 - 8n$

$0 = 4n^2 - 8n - 21$

$0 = (2n+3)(2n-7)$

$n = \left\langle \begin{array}{l} -\frac{3}{2} \\ \frac{7}{2} \end{array}\right.$

(b) • IF $n = -\frac{3}{2}$

$\left(1+\frac{4}{7}nx\right)^{n} = \left(1-\frac{6}{7}x\right)^{-\frac{3}{2}}$

VALID FOR $\left|\frac{6}{7}x\right| < 1$

$|x| < \frac{7}{6}$

$-\frac{7}{6} < x < \frac{7}{6}$

$x=1$ IS OK

• IF $n = \frac{7}{2}$

$\left(1+\frac{4}{7}nx\right)^{n} = (1+2x)^{\frac{7}{2}}$

VALID FOR $|2x| < 1$

$|x| < \frac{1}{2}$

$-\frac{1}{2} < x < \frac{1}{2}$

$x=1$ IS NOT OK

**Question 75** (*****)

$$f(x) \equiv \frac{1}{(1-5x)^2}, \quad |x| < \frac{1}{5}.$$

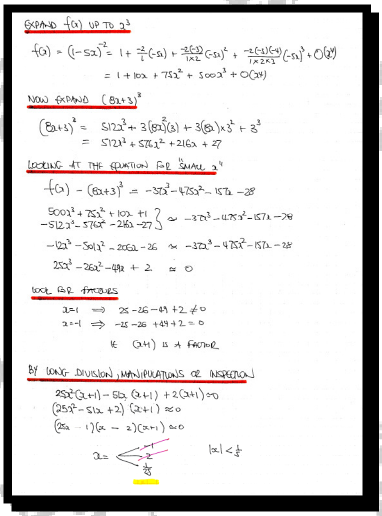
It is given that the equation

$$f(x) - (8x+3)^3 = -37x^3 - 475x^2 - 157x + 27$$

has a solution $\alpha$, which is numerically small.

Find an approximate value for $\alpha$.

MP2-I, $\boxed{\alpha \approx \frac{1}{25}}$



EXPAND $f(x)$ UP TO $x^3$

$f(x) = (1-5x)^{-2} = 1 + \frac{-2}{1}(-5x) + \frac{-2(-3)}{1\times 2}(-5x)^2 + \frac{-2(-3)(-4)}{1\times 2\times 3}(-5x)^3 + O(x^4)$

$= 1 + 10x + 75x^2 + 500x^3 + O(x^4)$

NOW EXPAND $(8x+3)^3$

$(8x+3)^3 = 512x^3 + 3(8x)^2(3) + 3(8x)\times 3^2 + 3^3$

$= 512x^3 + 576x^2 + 216x + 27$

LOOKING AT THE EQUATION FOR "SMALL $x$"

$f(x) - (8x+3)^3 = -37x^3 - 475x^2 - 157x - 28$

$\left.\begin{array}{l} 500x^3 + 75x^2 + 10x + 1 \\ -512x^3 - 576x^2 - 216x - 27 \end{array}\right\} \approx -37x^3 - 475x^2 - 157x - 28$

$-12x^3 - 501x^2 - 206x - 26 \approx -37x^3 - 475x^2 - 157x - 28$

$25x^3 - 26x^2 - 49x + 2 \approx 0$

LOOK FOR FACTORS

$x = 1 \Rightarrow 25 - 26 - 49 + 2 \neq 0$

$x = -1 \Rightarrow -25 - 26 + 49 + 2 = 0$

IE $(x+1)$ IS A FACTOR

BY LONG DIVISION, MANIPULATIONS OR INSPECTION

$25x^2(x+1) - 51x(x+1) + 2(x+1) \approx 0$

$(25x^2 - 51x + 2)(x+1) \approx 0$

$(25x - 1)(x - 2)(x+1) \approx 0$

$x = \cancel{-1}, \cancel{2}, \frac{1}{25}$ $\quad |x| < \frac{1}{5}$

**Question 76** (*****)

By considering the binomial expansion of

$$\frac{1}{(1-\cos\theta)^2},$$

sum each of the following series.

- $$\sum_{r=1}^{\infty}\left[\frac{r}{2^{r-1}}\right].$$

- $$\sum_{r=1}^{\infty}\left[\frac{r}{(-2)^{r-1}}\right].$$

$\boxed{\phantom{x}}$, $\boxed{\sum_{r=1}^{\infty}\left[\frac{r}{2^{r-1}}\right]=4}$, $\boxed{\sum_{r=1}^{\infty}\left[\frac{r}{(-2)^{r-1}}\right]=\frac{4}{9}}$

- START BY A SIMPLE SUBSTITUTION, $x=\cos\theta$

$$\frac{1}{(1-\cos\theta)^2}=\frac{1}{(1-x)^2}=(1-x)^{-2}$$

$$=1+\frac{-2}{1}(-x)^1+\frac{-2(-3)}{1\times2}(-x)^2+\frac{(-2)(-3)(-4)}{1\times2\times3}(-x)^3+O(x^4)$$

$$=1+2x+3x^2+4x^3+O(x^4) \qquad |x|<1$$

$$=1+2\cos\theta+3\cos^2\theta+4\cos^3\theta+\dots \qquad |\cos\theta|<1$$

$$=\sum_{r=1}^{\infty}r(\cos\theta)^{r-1}$$

- NOW $\sum_{r=1}^{\infty}\frac{r}{2^{r-1}}=\dots$ IS THE ABOVE EXPANSION WITH $\cos\theta=\frac{1}{2}$ (I.E $\theta=\frac{\pi}{3}$)

$$=\frac{1}{(1-\frac{1}{2})^2}=\frac{1}{(\frac{1}{2})^2}=\frac{1}{\frac{1}{4}}=4$$

- AND $\sum_{r=1}^{\infty}\frac{r}{(-2)^{r-1}}=\dots$ IS THE ABOVE EXPANSION WITH $\cos\theta=-\frac{1}{2}$ (I.E $\theta=\frac{2\pi}{3}$)

$$=\frac{1}{(1-(-\frac{1}{2}))^2}=\frac{1}{(\frac{3}{2})^2}=\frac{1}{\frac{9}{4}}=\frac{4}{9}$$

**Question 77** (*****)

$$f(x) \equiv \frac{1-x}{1+x+x^2+x^3}, \quad -1 < x < 1.$$

Show that $f(x)$ can be written in the form

$$f(x) = g(x)\sum_{r=0}^{\infty}\left(x^{4r}\right),$$

where $g(x)$ is a simplified function to be found.

$\boxed{\phantom{xx}}$, $\boxed{g(x) = (1-x)^2}$

$$f(x) = \frac{1-x}{1+x+x^2+x^3} = \frac{1-x}{(1+x)+x^2(1+x)} = \frac{1-x}{(1+x)(1+x^2)}$$

$$= \frac{(1-x)(1-x)}{(1-x)(1+x)(1+x^2)} = \frac{(1-x)^2}{(1-x^2)(1+x^2)}$$

$$= \frac{(1-x)^2}{1-x^4}$$

NOW USING STANDARD EXPANSIONS, OR THE SUM TO INFINITY OF A G.P.

$$\text{e.g.} \quad \frac{1}{1-x} = 1+x+x^2+x^3+\dots$$

$$\dots = (1-x)^2(1+x^4+x^8+x^{12}+\dots)$$

$$\dots = (1-x)^2\sum_{r=0}^{\infty} x^{4r}$$

LONGER ALTERNATIVE

$$f(x) = \frac{1-x}{1+x+x^2+x^4} = \dots = \frac{1-x}{(1+x)(1+x^2)} \quad \dots \text{ NOW PARTIAL FRACTIONS}$$

$$\frac{1-x}{(1+x)(1+x^2)} \equiv \frac{A}{1+x} + \frac{Bx+C}{1+x^2}$$

$$1-x \equiv A(1+x^2) + (1+x)(Bx+C)$$

IF $x=-1 \implies 2 = 2A \implies A=1$

IF $x=0 \implies 1 = A+C \implies C=0$

IF $x=1 \implies 0 = 2A+2B \implies B=-1$

THUS WE HAVE

$$\frac{1}{1+x} = 1-x+x^2-x^3+x^4-\dots$$

$$f(x) = \frac{1}{1+x} - \frac{x}{1+x^2}$$

$$f(x) = (1-x+x^2-x^3+\dots) - x(1-x^2+x^4-x^6+\dots)$$

$$f(x) = 1 - x + x^2 - \cancel{x^3} + x^4 - x^5 + x^6 - \cancel{x^7} + x^8 - x^9 + x^{10} - \dots$$
$$\quad\quad -x \quad\quad + \cancel{x^3} \quad\quad -x^5 \quad\quad + \cancel{x^7} \quad\quad -x^9$$

$$f(x) = (1-2x+x^2) + (x^4-2x^5+x^6) + (x^8-2x^9+x^{10}) + \dots$$

$$f(x) = (1-2x+x^2) + x^4(1-2x+x^2) + x^8(1-2x+x^2) + \dots$$

$$f(x) = (1-2x+x^2)\left[1+x^4+x^8+x^{12}+\dots\right]$$

$$f(x) = (1-x)^2\sum_{r=0}^{\infty} x^{4r}$$

AS BEFORE

**Question 78** (*****)

$$S = 1 + \frac{2}{4} + \frac{2\cdot 3}{4\cdot 8} + \frac{2\cdot 3\cdot 4}{4\cdot 8\cdot 12} + \frac{2\cdot 3\cdot 4\cdot 5}{4\cdot 8\cdot 12\cdot 16} + \dots$$

By considering a suitable binomial series, or other wise, find the sum to infinity of $S$.

V, $\square$, $\boxed{S_\infty = \frac{16}{9}}$

MANIPULATE THE SERIES STEP BY STEP

$$\Rightarrow S = 1 + \frac{2}{4} + \frac{2\times3}{4\times8} + \frac{2\times3\times4}{4\times8\times12} + \frac{2\times3\times4\times5}{4\times8\times12\times16} + \dots$$

$$\Rightarrow S = 1 + \frac{2}{4(1)} + \frac{2\times3}{4^2(1\times2)} + \frac{2\times3\times4}{4^3(1\times2\times3)} + \frac{2\times3\times4\times5}{4^4(1\times2\times3\times4)} + \dots$$

$$\Rightarrow S = 1 + \frac{2}{1!}\left(\frac{1}{4}\right) + \frac{2\times3}{2!}\left(\frac{1}{4}\right)^2 + \frac{2\times3\times4}{3!}\left(\frac{1}{4}\right)^3 + \frac{2\times3\times4\times5}{4!}\left(\frac{1}{4}\right)^4 + \dots$$

FINALLY WE NEED TO "TAKE CARE" OF THE SIGNS, IN ORDER TO FORM A CONVERGENT BINOMIAL EXPANSION

$$\Rightarrow S = 1 + \frac{-2}{1!}\left(-\frac{1}{4}\right)^1 + \frac{(-2)(-3)}{2!}\left(-\frac{1}{4}\right)^2 + \frac{(-2)(-3)(-4)}{3!}\left(-\frac{1}{4}\right)^3 + \frac{(-2)(-3)(-4)(-5)}{4!}\left(-\frac{1}{4}\right)^4 + \dots$$

$$\Rightarrow S = \left(1 - \frac{1}{4}\right)^{-2}$$

$$\Rightarrow S = \left(\frac{3}{4}\right)^{-2}$$

$$\Rightarrow S = \left(\frac{9}{16}\right)^{-1}$$

$$\Rightarrow S = \frac{16}{9}$$

**Question 79** (*****)

Without the use of any calculating aid and by showing full workings, show that

$$\left(\frac{6}{5}\right)^{\frac{6}{5}} \approx 1.24 .$$

proof

$$\left(\frac{6}{5}\right)^{\frac{6}{5}} = \left(1 + \frac{1}{5}\right)^{\frac{6}{5}}$$

- USE $(1+x)^{\frac{6}{5}}$ $\quad |x|<1$

$$= 1 + \frac{6}{5}x + O(x^2)$$

- HERE $x = \frac{1}{5}$

$$\therefore \left(\frac{6}{5}\right)^{\frac{6}{5}} \approx 1 + \frac{6}{5}\times\frac{1}{5}$$

$$= 1 + \frac{6}{25} = 1 + \frac{24}{100} = 1.24$$

**Question 80** (*****)

$$g(x) \equiv \sum_{r=0}^{\infty} f(x,r) \;-\; \frac{1-x}{\sqrt{1-x^2}\,\sqrt[3]{1-x^3}}, \quad -1 < x < 1 .$$

Given that the first term of the series expansion of $g(x)$ is $\frac{1}{5}x^5$, determine in exact simplified form a simplified expression of $f(x,r)$.

$\boxed{\phantom{xx}}$, $\boxed{f(x,r) = \dfrac{(-x)^r}{r!}}$

$$\sum_{r=0}^{\infty} f(x,r) - \frac{1-x}{(1-x^2)^{\frac{1}{2}}(1-x^3)^{\frac{1}{3}}} = \frac{1}{5}x^5 + O(x^6)$$

- START BY REWRITING THE FRACTION AS FOLLOWS

$$(1-x)(1-x^2)^{-\frac{1}{2}}(1-x^3)^{-\frac{1}{3}}$$

- OBTAIN EACH EXPANSION SEPARATELY

$$(1-x^2)^{-\frac{1}{2}} = 1 + \frac{-\frac{1}{2}}{1}(-x^2)^1 + \frac{-\frac{1}{2}(-\frac{3}{2})}{1\times 2}(-x^2)^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{1\times 2\times 3}(-x^2)^3 + O(x^8)$$

$$= 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6 + O(x^8)$$

$$(1-x^3)^{-\frac{1}{3}} = 1 + \frac{-\frac{1}{3}}{1}(-x^3)^1 + \frac{-\frac{1}{3}(-\frac{4}{3})}{1\times 2}(-x^3)^2 + O(x^9)$$

$$= 1 + \frac{1}{3}x^3 + \frac{2}{9}x^6 + O(x^9)$$

- COLLECTING ALL THE RESULTS

$$(1-x)\left[1+\frac{1}{3}x^3+\frac{2}{9}x^6+O(x^9)\right]\left[1+\frac{1}{2}x^2+\frac{3}{8}x^4+\frac{5}{16}x^6+O(x^8)\right]$$

$$= (1-x)\left[1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + O(x^6) + \frac{1}{3}x^3 + \frac{1}{6}x^5 + O(x^6)\right]$$

$$= (1-x)\left(1 + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{3}{8}x^4 + \frac{1}{6}x^5 + O(x^6)\right)$$

$$= 1 + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{3}{8}x^4 + \frac{1}{6}x^5 + O(x^6)$$
$$\quad - x - \frac{1}{2}x^3 - \frac{1}{3}x^4 - \frac{3}{8}x^5 + O(x^6)$$

$$= 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{5}{24}x^5 + O(x^6)$$

- THUS WE NOW HAVE

$$\sum_{r=0}^{5} f(x,r) - \left(1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{5}{24}x^5\right) = \frac{1}{5}x^5$$

$$\sum_{r=0}^{5} f(x,r) = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{120}x^5$$

$$\sum_{r=0}^{5} f(x,r) = \frac{x^0}{0!} - \frac{x^1}{1!} + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 - \frac{1}{5!}x^5$$

$$\therefore\ f(x,r) = \frac{(-x)^r}{r!}$$

**Question 81** (*****)

$$S = 1 - \frac{1}{4} + \frac{1\cdot 3}{4\cdot 8} - \frac{1\cdot 3\cdot 5}{4\cdot 8\cdot 12} + \frac{1\cdot 3\cdot 5\cdot 7}{4\cdot 8\cdot 12\cdot 16} - \ldots$$

Find the sum to infinity of $S$, by considering the binomial series expansion of $(1+x)^n$ for suitable values of $x$ and $n$.

$\boxed{\phantom{MP2}}$, $\boxed{S_\infty = \sqrt{\frac{2}{3}}}$

START BY CREATING FACTORIALS IN THE DENOMINATORS

$$\Rightarrow S = 1 - \frac{1}{4} + \frac{1\times 3}{4\times 8} - \frac{1\times 3\times 5}{4\times 8\times 12} + \frac{1\times 3\times 5\times 7}{4\times 8\times 12\times 16} + \ldots$$

$$\Rightarrow S = 1 - \frac{1}{4^1\times 1} + \frac{1\times 3}{4^2(1\times 2)} - \frac{1\times 3\times 5}{4^3(1\times 2\times 3)} + \frac{1\times 3\times 5\times 7}{4^4(1\times 2\times 3\times 4)} + \ldots$$

$$\Rightarrow S = 1 - \frac{2^1(\frac{1}{2})}{4^1\times 1} + \frac{2^2(\frac{1}{2})(\frac{3}{2})}{4^2(1\times 2)} - \frac{2^3(\frac{1}{2})(\frac{3}{2})(\frac{5}{2})}{4^3(1\times 2\times 3)} + \frac{2^4(\frac{1}{2})(\frac{3}{2})(\frac{5}{2})(\frac{7}{2})}{4^4(1\times 2\times 3\times 4)} + \ldots$$

CREATE WHAT LOOKS LIKE A BINOMIAL EXPANSION

$$\Rightarrow S = 1 - \frac{\frac{1}{2}}{1}\left(\tfrac{1}{2}\right)^1 + \frac{\frac{1}{2}(\frac{3}{2})}{1\times 2}\left(\tfrac{1}{2}\right)^2 - \frac{\frac{1}{2}(\frac{3}{2})(\frac{5}{2})}{1\times 2\times 3}\left(\tfrac{1}{2}\right)^3 + \frac{\frac{1}{2}(\frac{3}{2})(\frac{5}{2})(\frac{7}{2})}{1\times 2\times 3\times 4}\left(\tfrac{1}{2}\right)^4 + \ldots$$

FINALLY DEAL WITH THE MINUS SIGNS

$$\Rightarrow S = 1 + \frac{-\frac{1}{2}}{1}\left(\tfrac{1}{2}\right)^1 + \frac{-\frac{1}{2}(-\frac{3}{2})}{1\times 2}\left(\tfrac{1}{2}\right)^2 + \frac{-\frac{1}{2}(-\frac{3}{2})(-\frac{5}{2})}{1\times 2\times 3}\left(\tfrac{1}{2}\right)^3 + \frac{-\frac{1}{2}(-\frac{3}{2})(-\frac{5}{2})(-\frac{7}{2})}{1\times 2\times 3\times 4}\left(\tfrac{1}{2}\right)^4 + \ldots$$

$$\Rightarrow S = \left(1 + \tfrac{1}{2}\right)^{-\frac{1}{2}}$$ (BY CONSIDERING THE EXPANSION $(1+x)^{-\frac{1}{2}}$ WITH $x = \frac{1}{2}$)

$$\Rightarrow S = \left(\sqrt{\tfrac{3}{2}}\right)^{-1}$$

$$\therefore\ 1 - \frac{1}{4} + \frac{1\times 3}{4\times 8} - \frac{1\times 3\times 5}{4\times 8\times 12} + \frac{1\times 3\times 5\times 7}{4\times 8\times 12\times 16} = \sqrt{\tfrac{2}{3}}$$

**Question 82** (*****)

Without the use of any calculating aid and by showing full workings, show that

$$(0.9)^{0.9} \approx 0.91.$$

$\boxed{\phantom{MP2}}$, $\boxed{\text{proof}}$

MANIPULATING INTO A BINOMIAL

$$(0.9)^{0.9} = (1-x)^{\frac{9}{10}}$$ WITH $x = 0.1$

$$= 1 + \frac{\frac{9}{10}}{1}(-x) + O(x^2)$$

$$= 1 - \frac{9}{10}x + O(x^2)$$

NOW LET $x = \frac{1}{10}$

$$\therefore\ 0.9^{0.9} \simeq 1 - \frac{9}{10}\left(\frac{1}{10}\right)$$

$$\simeq 1 - \frac{9}{100}$$

$$\approx \frac{91}{100}$$

$$\simeq 0.91$$

**Question 83** (*****)

$$f(x) = \frac{1}{\sqrt{1-x}}, \quad -1 < x < 1.$$

**a)** By manipulating the general term of binomial expansion of $f(x)$ show that

$$f(x) = \sum_{r=0}^{\infty} \binom{2r}{r} \left(\tfrac{1}{4}x\right)^r .$$

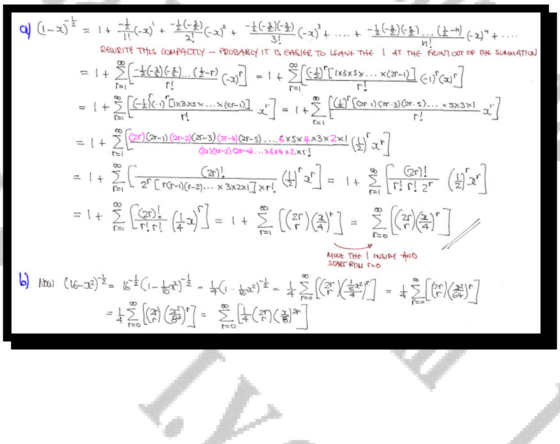
**b)** Find a similar expression for $\dfrac{1}{\sqrt{16-x^2}}$ and show further that

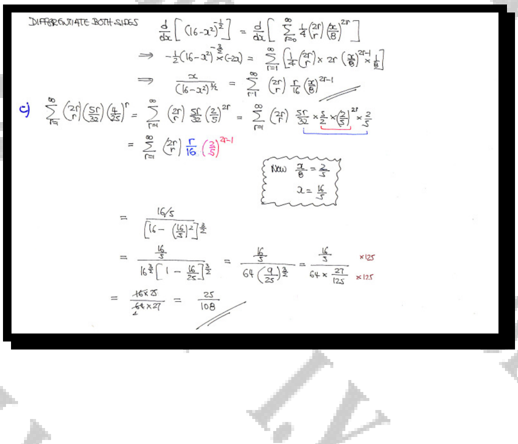
$$\frac{x}{\left(16-x^2\right)^{\frac{3}{2}}} = \sum_{r=1}^{\infty} \binom{2r}{r} \left(\tfrac{1}{16}r\right)\left(\tfrac{1}{8}x\right)^{2r-1} .$$

**c)** Determine the exact value of

$$\sum_{r=1}^{\infty} \binom{2r}{r} \left(\tfrac{5}{32}r\right)\left(\tfrac{4}{25}\right)^r .$$

$\boxed{\phantom{xx}}$, $\boxed{\dfrac{25}{108}}$





**Question 84** (*****)

$$f(x) \equiv \frac{2-3x}{(1-x)(1-2x)}, \quad -\tfrac{1}{2} < x < \tfrac{1}{2}.$$

Show that $f(x)$ can be written in the form

$$f(x) = \sum_{r=0}^{\infty}\left[x^r g(r)\right],$$

where $g(r)$ is a simplified function to be found.

$\boxed{\phantom{xx}}$, $\boxed{g(r) = 2^r + 1}$

- START BY REWRITING & SPLITTING INTO PARTIAL FRACTIONS BY INSPECTION

$$\Rightarrow f(x) = \frac{2-3x}{(1-x)(1-2x)} = (2-3x) \times \frac{1}{(1-x)(1-2x)}$$

$$\Rightarrow f(x) = (2-3x) \times \left[\frac{\frac{1}{-1}}{1-x} + \frac{\frac{1}{1/2}}{1-2x}\right]$$

$$\Rightarrow f(x) = (2-3x)\left[\frac{2}{1-2x} - \frac{1}{1-x}\right]$$

- NEXT WE USE STANDARD EXPANSIONS OR THE SUM TO INFINITY OF A G.P.

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \dots,$$

TO OBTAIN:

$$\Rightarrow f(x) = (2-3x)\left[\begin{array}{l} 2(1+2x+4x^2+8x^3+\dots) \\ -1-x-x^2-x^3-\dots \end{array}\right]$$

$$\Rightarrow f(x) = (2-3x)\left[\begin{array}{l} 2+4x+8x^2+16x^3+\dots \\ -1-x-x^2-x^3-\dots \end{array}\right]$$

$$\Rightarrow f(x) = (2-3x)(1+3x+7x^2+15x^3+\dots)$$

$$\Rightarrow f(x) = (2-3x)\sum_{r=0}^{\infty}(2^{r+1}-1)x^r$$

$$\Rightarrow f(x) = 2\sum_{r=0}^{\infty}(2^{r+1}-1)x^r - 3\sum_{r=0}^{\infty}(2^{r+1}-1)x^{r+1}$$

- ADJUST THE "FIRST" SUMMATION AS FOLLOWS

$$\Rightarrow f(x) = 2 + 2\sum_{r=1}^{\infty}(2^{r+1}-1)x^r - 3\sum_{r=0}^{\infty}(2^{r+1}-1)x^{r+1}$$

- NEXT ADJUST THE FIRST SUMMATION SO IT STARTS FROM r=0 AGAIN

$$f(x) = 2 + 2\sum_{r=0}^{\infty}(2^{r+2}-1)x^{r+1} - 3\sum_{r=0}^{\infty}(2^{r+1}-1)x^{r+1}$$

$$f(x) = 2 + \sum_{r=0}^{\infty}\left[2(2^{r+2}-1) - 3(2^{r+1}-1)\right]x^{r+1}$$

$$f(x) = 2 + \sum_{r=0}^{\infty}(4\times 2^{r+1} - 2 - 3\times 2^{r+1} + 3)x^{r+1}$$

$$f(x) = 2 + \sum_{r=0}^{\infty}(2^{r+1}+1)x^{r+1}$$

- ADJUST THE SUMMATION SO THAT IT STARTS FROM r=1

$$f(x) = 2 + \sum_{r=1}^{\infty}(2^r+1)x^r$$

$$f(x) = (2^0+1)x^0 + \sum_{r=1}^{\infty}(2^r+1)x^r$$

$$f(x) = \sum_{r=0}^{\infty}(2^r+1)x^r$$

ANOTHER APPROACH THOUGH NOT AS FORMAL IS AS FOLLOWS

$$f(x) = (2-3x)(1+3x+7x^2+15x^3+\dots) \leftarrow \text{FROM EARLIER}$$

$$f(x) = \begin{array}{l} 2+6x+14x^2+30x^3+\dots \\ \phantom{2}-3x-9x^2-21x^3-\dots \end{array}$$

$$f(x) = 2+3x+5x^2+9x^3+\dots$$

WHICH ONE MIGHT DEDUCE IS $\sum_{r=0}^{\infty}(2^r+1)x^r$

**Question 85** (*****)

Show by considering a suitable binomial expansion that

$$1+\frac{1}{24}+\frac{1\cdot 4}{24\cdot 48}+\frac{1\cdot 4\cdot 7}{24\cdot 48\cdot 72}+\frac{1\cdot 4\cdot 7\cdot 10}{24\cdot 48\cdot 72\cdot 96}-...=\frac{2}{\sqrt[3]{7}}.$$

V, , proof

$$S=1+\frac{1}{24}+\frac{1\times4}{24\times48}+\frac{1\times4\times7}{24\times48\times72}+\frac{1\times4\times7\times10}{24\times48\times72\times96}+\dots$$

$$S=1+\frac{1}{24(1)}+\frac{1\times4}{24^2(1\times2)}+\frac{1\times4\times7}{24^3(1\times2\times3)}+\frac{1\times4\times7\times10}{24^4(1\times2\times3\times4)}+\dots$$

$$S=1+\frac{3\left(\frac{1}{3}\right)}{24(1)}+\frac{3^2\left(\frac{1}{3}\times\frac{4}{3}\right)}{24^2(1\times2)}+\frac{3^3\left(\frac{1}{3}\times\frac{4}{3}\times\frac{7}{3}\right)}{24^3(1\times2\times3)}+\frac{3^4\left(\frac{1}{3}\times\frac{4}{3}\times\frac{7}{3}\times\frac{10}{3}\right)}{24^4(1\times2\times3\times4)}+\dots$$

NOW FOR BINOMIAL CONVERGENT SERIES THE POWERS MUST BE DECREASING
IE $n(n-1)(n-2)\dots$ ETC
THUS ADJUSTING THE SIGNS

$$S=1+\frac{\left(-\frac{1}{3}\right)}{1}\left(-\frac{1}{8}\right)+\frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)}{1\times2}\left(-\frac{1}{8}\right)^2+\frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\left(-\frac{7}{3}\right)}{1\times2\times3}\left(-\frac{1}{8}\right)^3+\frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\left(-\frac{7}{3}\right)\left(-\frac{10}{3}\right)}{1\times2\times3\times4}\left(-\frac{1}{8}\right)^4+\dots$$

THIS IS THE BINOMIAL SERIES EXPANSION OF

$\left(1-\frac{1}{8}x\right)^{-\frac{1}{3}}$ WITH $x=1$ (NOTE THIS SERIES CONVERGES FOR $-8<x<8$)

$$S=\left(1-\frac{1}{8}\right)^{-\frac{1}{3}}=\left(\frac{7}{8}\right)^{-\frac{1}{3}}=\left(\frac{8}{7}\right)^{\frac{1}{3}}=\frac{2}{\sqrt[3]{7}}$$

**Question 86** (*****)

$$S=\frac{3}{8}+\frac{3\times9}{8\times16}+\frac{3\times9\times15}{8\times16\times24}+\frac{3\times9\times15\times21}{8\times16\times24\times32}+\frac{3\times9\times15\times21\times27}{8\times16\times24\times32\times40}...$$

By considering a suitable binomial expansion, show that $S=1$.

, proof

$$S=\frac{3}{8}+\frac{3\times9}{8\times16}+\frac{3\times9\times15}{8\times16\times24}+\frac{3\times9\times15\times21}{8\times16\times24\times32}+\frac{3\times9\times15\times21\times27}{8\times16\times24\times32\times40}+\dots$$

$$\Rightarrow S=\frac{3}{8}+\frac{3^2(1\times3)}{8^2(1\times2)}+\frac{3^3(1\times3\times5)}{8^3(1\times2\times3)}+\frac{3^4(1\times3\times5\times7)}{8^4(1\times2\times3\times4)}+\frac{3^5(1\times3\times5\times7\times9)}{8^5(1\times2\times3\times4\times5)}+\dots$$

$$\Rightarrow S=\frac{\frac{1}{2}\times3}{4^1(1)}+\frac{\left(\frac{1}{2}\times\frac{3}{2}\right)3^2}{4^2(1\times2)}+\frac{\left(\frac{1}{2}\times\frac{3}{2}\times\frac{5}{2}\right)3^3}{4^3(1\times2\times3)}+\frac{\left(\frac{1}{2}\times\frac{3}{2}\times\frac{5}{2}\times\frac{7}{2}\right)3^4}{4^4(1\times2\times3\times4)}+\frac{\left(\frac{1}{2}\times\frac{3}{2}\times\frac{5}{2}\times\frac{7}{2}\times\frac{9}{2}\right)3^5}{4^5(1\times2\times3\times4\times5)}+\dots$$

$$\Rightarrow S=\frac{\frac{1}{2}}{1!}\left(\frac{3}{4}\right)^1+\frac{\frac{1}{2}\times\frac{3}{2}}{2!}\left(\frac{3}{4}\right)^2+\frac{\left(\frac{1}{2}\times\frac{3}{2}\times\frac{5}{2}\right)}{3!}\left(\frac{3}{4}\right)^3+\frac{\frac{1}{2}\times\frac{3}{2}\times\frac{5}{2}\times\frac{7}{2}}{4!}\left(\frac{3}{4}\right)^4+\frac{\frac{1}{2}\times\frac{3}{2}\times\frac{5}{2}\times\frac{7}{2}\times\frac{9}{2}}{5!}\left(\frac{3}{4}\right)^5+\dots$$

THIS IS ALMOST A BINOMIAL APART FROM THE SIGNS ("BITS" OUGHT TO DECREASE BY 1 EVERY TIME)
BY INSPECTION INSERT "MINUSES", ENSURING EACH TERM IS STILL OVERALL A "PLUS"

$$\Rightarrow S=\frac{-\frac{1}{2}}{1!}\left(-\frac{3}{4}\right)^1+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(-\frac{3}{4}\right)^2+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}\left(-\frac{3}{4}\right)^3+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)}{4!}\left(-\frac{3}{4}\right)^4+\dots$$

ADDING 1 TO BOTH SIDES TO GET A SUITABLE BINOMIAL

$$\Rightarrow 1+S=1+\frac{-\frac{1}{2}}{1!}\left(-\frac{3}{4}\right)^1+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(-\frac{3}{4}\right)^2+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}\left(-\frac{3}{4}\right)^3+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)}{4!}\left(-\frac{3}{4}\right)^4+\dots$$

$$\Rightarrow 1+S=\left(1-\frac{3}{4}\right)^{-\frac{1}{2}}$$

$$\Rightarrow S=\left(\frac{1}{4}\right)^{-\frac{1}{2}}-1$$

$$\Rightarrow S=2-1=1$$

AS REQUIRED

**Question 87** (*****)

The function $f$ is defined in terms of the real constants, $a$, $b$ and $c$, by

$$f(x) = \left(a + bx + cx^2\right)(1-x)^{-3}, \quad x \in \mathbb{R}, \quad |x| < 1.$$

**a)** Show that

$$f(x) = a + (3a+b)x + \frac{1}{2}\sum_{n=2}^{\infty}\Big[\big[a(n+1)(n+2) + bn(n+1) + cn(n-1)\big]x^n\Big].$$

**b)** Use the expression of part **(a)** to deduce the value of

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n}.$$

$\boxed{\phantom{000}}$, $\boxed{6}$

a) STARTING WITH THE BINOMIAL EXPANSION OF $(1-x)^{-3}$

$$\Rightarrow (1-x)^{-3} = 1 + \frac{-3}{1}(-x)^1 + \frac{(-3)(-4)}{1\times2}(-x)^2 + \frac{(-3)(-4)(-5)}{1\times2\times3}(-x)^3 + \frac{(-3)(-4)(-5)(-6)}{1\times2\times3\times4}(-x)^4 + \dots$$

$$\Rightarrow (1-x)^{-3} = 1 + \frac{3}{1}x + \frac{3\times4}{1\times2}x^2 + \frac{3\times4\times5}{1\times2\times3}x^3 + \frac{3\times4\times5\times6}{1\times2\times3\times4}x^4 + \dots$$

$$\dots + \frac{3\times4\times5\times6\times\dots(n+2)}{1\times2\times3\times4\times\dots n}x^n$$

THUS THE COEFFICIENT OF $x^n$ IS

$$\frac{1}{2}\left(\frac{2\times3\times5\times\dots\times(n+2)}{n!}\right) = \frac{1}{2}\,\frac{(n+2)!}{n!} = \frac{1}{2}\,\frac{(n+2)(n+1)\times\cancel{n!}}{\cancel{n!}}$$

$$(1-x)^3 = \sum_{n=0}^{\infty}\tfrac{1}{2}(n+1)(n+2)x^n$$

THUS LOOKING NOW AT $f(x)$

$$\Rightarrow f(x) = (a+bx+cx^2)(1-x)^{-3} = (a+bx+cx^2)\sum_{n=0}\tfrac{1}{2}(n+1)(n+2)x^n$$

$$\Rightarrow f(x) = \tfrac{1}{2}a\sum_{n=0}^{\infty}(n+1)(n+2)x^n + \tfrac{1}{2}b\sum_{n=0}^{\infty}(n+1)(n+2)x^{n+1} + \tfrac{1}{2}c\sum_{n=0}^{\infty}(n+1)(n+2)x^{n+2}$$

$$\Rightarrow f(x) = \tfrac{1}{2}a\times2\times x^0 + \tfrac{1}{2}b\times1\times2\times x^1 + \tfrac{1}{2}a\times2\times3\times x^1 + \tfrac{1}{2}a\sum_{n=2}^{\infty}(n+1)(n+2)x^n + \tfrac{1}{2}b\sum_{n=1}^{\infty}(n+1)(n+2)x^{n+1} + \tfrac{1}{2}c\sum_{n=0}^{\infty}(n+1)(n+2)x^{n+2}$$

$$f(x) = (a+bx+cx^2)(1-x)^{-3} = a + (b+3a)x + \tfrac{1}{2}a\sum_{n=2}^{\infty}(n+1)(n+2)x^n + \tfrac{1}{2}b\sum_{n=1}^{\infty}(n+1)(n+2)x^{n+1} + \tfrac{1}{2}c\sum_{n=0}^{\infty}(n+1)(n+2)x^{n+2}$$

"ADJUST" THE SUMMATIONS SO THEY ALL START FROM $n=2$

$$\Rightarrow f(x) = a + (3a+b)x + \tfrac{1}{2}a\sum_{n=2}^{\infty}(n+1)(n+2)x^n + \tfrac{1}{2}b\sum_{n=2}^{\infty}n(n+1)x^n + \tfrac{1}{2}c\sum_{n=2}^{\infty}(n-1)(n)x^n$$

$$\Rightarrow f(x) = a + (3a+b)x + \tfrac{1}{2}\sum_{n=2}^{\infty}\big[a(n+1)(n+2) + bn(n+1) + cn(n-1)\big]x^n$$

b) NOW LOOKING AT THE COEFFICIENT OF $x^n$ (MULTIPLIED THE $\frac{1}{2}$ INTO THE SUM)

$$\begin{pmatrix}\tfrac{1}{2}an^2 + \tfrac{3}{2}an + a\\ \tfrac{1}{2}bn^2 + \tfrac{1}{2}bn\\ \tfrac{1}{2}cn^2 - \tfrac{1}{2}cn\end{pmatrix}x^n$$

LOOK FOR THIS TO REDUCE TO $n^2x^n$

$\therefore \boxed{a=0}$ $\quad \frac{1}{2}b - \frac{1}{2}c = 0$, $\boxed{b=c}$ $\quad \frac{1}{2}b + \frac{1}{2}c = 1$, $\frac{1}{2}b + \frac{1}{2}b = 1$, $\boxed{b=c=1}$

$$\Rightarrow f(x) = (a+bx+cx^2)(1-x)^{-3} = a + (3a+b)x + \tfrac{1}{2}\sum_{n=2}^{\infty}\big[a(n+1)(n+2) + bn(n+1) + cn(n-1)\big]x^n$$

LET $a=0$, $b=1$, $c=1$

$$\Rightarrow f(x) = (x+x^2)(1-x)^{-3} = x + \sum_{n=2}^{\infty}n^2x$$

$$\Rightarrow f\left(\tfrac{1}{2}\right) = \left(\tfrac{1}{2}+\tfrac{1}{4}\right)\left(1-\tfrac{1}{2}\right)^{-3} = \tfrac{1}{2} + \sum_{n=2}^{\infty}n^2\left(\tfrac{1}{2}\right)^n$$

$$\Rightarrow \tfrac{3}{4}\times\left(\tfrac{1}{2}\right)^{-3} = \tfrac{1}{2} + \sum_{n=2}^{\infty}\frac{n^2}{2^n}$$

$$\Rightarrow \tfrac{3}{4}\times8 = \tfrac{1}{2} + \sum_{n=2}^{\infty}\frac{n^2}{2^n}$$

$$\Rightarrow 6 = \sum_{n=1}^{\infty}\frac{n^2}{2^n} \quad \left(\text{AS } \frac{1^2}{2^1} = \frac{1}{2}\right)$$

$$\therefore \sum_{n=1}^{\infty}\frac{n^2}{2^n} = 6$$

**Question 88** (*****)

The first three terms of a series $S$ are

$$S = 7 + 9x + 8x^2 + \ldots$$

The $n^{\text{th}}$ term of $S$ is given by

$$A\left(\tfrac{3}{4}x\right)^n + B\left(\tfrac{1}{3}x\right)^n,$$

where $A$ and $B$ are non zero constants.

Given that the sum to infinity of $S$ is 19, determine the value of $x$.

$\boxed{\text{SP-F}}$, $\boxed{x = \frac{12}{19}}$

$S = 7 + 9x + 8x^2 + \ldots \qquad \left[A\left(\frac{3}{4}\right)^n + B\left(\frac{1}{3}\right)^n\right]x^n$

- IF $n=0$: $(A+B)x^0 = 7$
  IF $n=1$: $\left(\frac{3}{4}A + \frac{1}{3}B\right)x^1 = 9$

$$\left.\begin{aligned} A + B &= 7 \\ \tfrac{3}{4}A + \tfrac{1}{3}B &= 9 \end{aligned}\right\} \begin{aligned} &\times 9 \\ &\times 12 \end{aligned} \qquad \begin{aligned} 9A + 9B &= 63 \\ 9A + 4B &= 108 \\ \hline 5B &= -45 \\ B &= -9 \\ A &= 16 \end{aligned}$$

- NOW THE SERIES IS

$$S = \sum_{n=0}^{\infty} \left[16\left(\tfrac{3}{4}\right)^n - 9\left(\tfrac{1}{3}\right)^n\right]x^n$$

$$S = 16\sum_{n=0}^{\infty}\left[\left(\tfrac{3}{4}\right)^n x^n\right] - 9\sum_{n=0}^{\infty}\left[\left(\tfrac{1}{3}\right)^n x^n\right]$$

$$S = 16\sum_{n=0}^{\infty}\left(\tfrac{3}{4}x\right)^n - 9\sum_{n=0}^{\infty}\left(\tfrac{1}{3}x\right)^n$$

$$S = 16\left[1 + \tfrac{3}{4}x + \tfrac{9}{16}x^2 + \tfrac{27}{64}x^3 + \ldots\right] - 9\left[1 + \tfrac{1}{3}x + \tfrac{1}{9}x^2 + \tfrac{1}{27}x^3 + \ldots\right]$$

$$S = 16 \times \left(\frac{1}{1 - \frac{3}{4}x}\right) - 9\left(\frac{1}{1 - \frac{1}{3}x}\right)$$

NOW THE SUM TO INFINITY IS 19

$$\Rightarrow \frac{16}{1 - \frac{3}{4}x} - \frac{9}{1 - \frac{1}{3}x} = 19$$

$$\Rightarrow \frac{64}{4 - 3x} - \frac{27}{3 - x} = 19$$

$$\Rightarrow 64(3 - x) - 27(4 - 3x) = 19(3 - x)(4 - 3x)$$

$$\Rightarrow 192 - 64x - 108 + 81x = 19(3x - 4)(x - 3)$$

$$\Rightarrow 84 + 17x = 19(3x^2 - 13x + 12)$$

$$\Rightarrow 84 + 17x = 57x^2 - 19 \times 13x + 19 \times 12$$

$$\Rightarrow 0 = 57x^2 - 247x - 17x + 228 - 84$$

$$\Rightarrow 0 = 57x^2 - 264x + 144$$

$$\Rightarrow 0 = 19x^2 - 88x + 48$$

$$\Rightarrow 0 = (19x - 12)(x - 4)$$

$$\Rightarrow x = \begin{cases} 4 \\ \frac{12}{19} \end{cases}$$

BUT IN ORDER TO CONVERGE $\left|\frac{3}{4}x\right| < 1$ & $\left|\frac{1}{3}x\right| < 1$

$|x| < \frac{4}{3}$ & $|x| < 3$

$\therefore x = \frac{12}{19}$ // (AS 4 IS GREATER THAN $\frac{4}{3}$ OR EVEN 3)

**Question 89** (*****)

$$f(x) \equiv \frac{1-7x}{(1+x)(1-3x)}, \quad -\tfrac{1}{3} < x < \tfrac{1}{3}.$$

Show that $f(x)$ can be written in the form

$$f(x) = 1 - \sum_{r=1}^{\infty} \left[ x^r g(r) \right],$$

where $g(r)$ is a simplified function to be found.

$\boxed{\phantom{xx}}$ , $\boxed{g(r) = 3^r + 2\times(-1)^{r+1}}$

$$f(x) = \frac{1-7x}{(1+x)(1-3x)} \quad |x| < \tfrac{1}{3}$$

- BY PARTIAL FRACTIONS OR DIRECT EXPANSIONS USING STANDARD BINOMIAL EXPANSION, OR THE SUM TO INFINITY OF A GEOMETRIC SERIES.

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \dots$$
$$(1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \dots$$

- THUS WE HAVE

$$\Rightarrow f(x) = (1-7x)(1+x)^{-1}(1-3x)^{-1}$$
$$\Rightarrow f(x) = (1-7x)(1 - x + x^2 - x^3 + x^4 - \dots)(1 + 3x + 9x^2 + 27x^3 + 81x^4 + \dots)$$
$$\Rightarrow f(x) = (1-7x)\left[\begin{array}{l} 1 + 3x + 9x^2 + 27x^3 + 81x^4 + 243x^5 + 729x^6 + \dots \\ \quad -x - 3x^2 - 9x^3 - 27x^4 - 81x^5 - 243x^6 - \\ \qquad x^2 + 3x^3 + 9x^4 + 27x^5 + 81x^6 \\ \qquad -x^3 - 3x^4 - 9x^5 - 27x^6 \\ \qquad x^4 + 3x^5 + 9x^6 \\ \qquad -x^5 - 3x^6 \end{array}\right]$$

$$\Rightarrow f(x) = (1-7x)(1 + 2x + 7x^2 + 20x^3 + 61x^4 + 182x^5 + \dots)$$

$$\begin{aligned} \Rightarrow f(x) &= 1 + 2x + 7x^2 + 20x^3 + 61x^4 + 182x^5 \\ &\quad - 7x - 14x^2 - 49x^3 - 140x^4 - 427x^5 \\ &= 1 - 5x - 7x^2 - 29x^3 - 79x^4 - 245x^5 \end{aligned}$$

- BY INSPECTION

N/A $\quad 3^1+2 \quad 3^2-2 \quad 3^3+2 \quad 3^4-2 \quad 3^5+2$

- HENCE WE MAY WRITE

$$f(x) = 1 - \sum_{r=1}^{\infty}\left[\left[3^r + (-1)^{r+1}\times 2\right]x^r\right]$$

$$f(x) = 1 - \sum_{r=1}^{\infty}\left[\left(3^r + 2(-1)^{r+1}\right)x^r\right]$$

**Question 90** (*****)

The product operator $\prod$, is defined as

$$\prod_{i=1}^{k}\left[u_i\right] = u_1 \times u_2 \times u_3 \times u_4 \times \ldots \times u_{k-1} \times u_k .$$

Find the sum to infinity of the following expression

$$\sum_{k=1}^{\infty}\left[\prod_{r=1}^{k}\left(\frac{8r-7}{40r}\right)\right].$$

$\boxed{\text{SP-K}}$ , $\boxed{\sqrt[8]{\dfrac{5}{4}}-1}$

START BY WRITING A FEW TERMS EXPLICITLY & LOOK FOR A PATTERN

$$\sum_{k=1}^{\infty}\left[\prod_{r=1}^{k}\left(\frac{8r-7}{40r}\right)\right] = \prod_{r=1}^{1}\left(\frac{8r-7}{40r}\right) + \prod_{r=1}^{2}\left(\frac{8r-7}{40r}\right) + \prod_{r=1}^{3}\left(\frac{8r-7}{40r}\right) + \prod_{r=1}^{4}\left(\frac{8r-7}{40r}\right) + \ldots$$

$$= \frac{1}{40} + \frac{1}{40}\times\frac{9}{80} + \frac{1}{40}\times\frac{9}{80}\times\frac{17}{120} + \frac{1}{40}\times\frac{9}{80}+\frac{17}{120}\times\frac{25}{160} + \ldots$$

$$= \frac{1}{40} + \frac{1\times 9}{40\times 80} + \frac{1\times 9\times 17}{40\times 80\times 120} + \frac{1\times 9\times 17\times 25}{40\times 80\times 120\times 160}$$

$$= \frac{1}{40\times 1} + \frac{1\times 9}{40^2(1\times 2)} + \frac{1\times 9\times 17}{40^3(1\times 2\times 3)} + \frac{1\times 9\times 17\times 25}{40^4(1\times 2\times 3\times 4)}$$

THIS RESEMBLES A BINOMIAL EXPANSION DUE TO THE FACTORIALS AT THE DENOMINATOR
THE NEXT ITEM IS TO "CREATE" NUMERATORS OF THE FORM $n(n-1)(n-2)(n-3)\ldots$

BY INSPECTION THIS WILL OCCUR AS $-\frac{1}{8}, -\frac{9}{8}, \frac{-17}{8}, \frac{-25}{8}$

THUS TRY AND ADJUST THE SERIES

$$= \frac{1}{(-8)(-5)\,1!} + \frac{1\times 9}{(-8)^2(-5)^2\,2!} + \frac{1\times 9\times 17}{(-8)^3(-5)^3\,3!} + \frac{1\times 9\times 17\times 25}{(-8)^4(-5)^4\,4!}$$

$$\sum_{k=1}^{\infty}\left[\prod_{r=1}^{k}\left(\frac{8r-7}{40r}\right)\right] = \underbrace{\frac{-\frac{1}{8}}{1!}\left(-\frac{1}{5}\right) + \frac{\left(-\frac{1}{8}\right)\left(-\frac{9}{8}\right)}{2!}\left(-\frac{1}{5}\right)^2 + \frac{\left(-\frac{1}{8}\right)\left(-\frac{9}{8}\right)\left(-\frac{17}{8}\right)}{3!}\left(-\frac{1}{5}\right)^3 + \frac{\left(-\frac{1}{8}\right)\left(-\frac{9}{8}\right)\left(-\frac{17}{8}\right)\left(-\frac{25}{8}\right)}{4!}\left(-\frac{1}{5}\right)^4 + \ldots}_{\text{THIS IS A BINOMIAL EXPANSION WITH THE "1" MISSING AT THE FRONT}}$$

$$= \left(1-\frac{1}{5}\right)^{-\frac{1}{8}} - 1$$

$$= \left(\frac{4}{5}\right)^{-\frac{1}{8}} - 1$$

$$= \sqrt[8]{\frac{5}{4}} - 1$$

**Question 91** (*****)

It is given that for $x \in \mathbb{R}$, $-\frac{1}{k} < x < \frac{1}{k}$, $k > 0$,

$$f(x,k) \equiv \frac{k+1}{(1-x)(1+kx)}.$$

Given further that

$$f(x,k) \equiv \sum_{r=0}^{\infty} \left[ a_r x^r \right],$$

where $a_r$ are functions of $k$, show that

$$\sum_{r=0}^{\infty} \left[ {a_r}^2 x^r \right] = \frac{(1-kx)(1+k)^2}{(1-x)(1+kx)\left(1-k^2 x\right)}.$$

You may assume that $\sum_{r=0}^{\infty} \left[ {a_r}^2 x^r \right]$ converges.

proof

$$\boxed{f(x,k) = \frac{k+1}{(1-x)(1+kx)} \qquad -\frac{1}{k} < x < \frac{1}{k}, \quad k>0}$$

- REWRITE & EXPAND BINOMIALLY

$$\Rightarrow f(x,k) = (k+1)(1-x)^{-1}(1+kx)^{-1} = (1+k)\left[1+x+x^2+x^3+x^4+O(x^5)\right]\left[1-kx+k^2x^2-k^3x^3+k^4x^4+O(x^5)\right]$$

$$\boxed{\begin{aligned}(1-x)^{-1} &= 1+x+x^2+x^3+x^4+\dots \\ (1+x)^{-1} &= 1-x+x^2-x^3+x^4+\dots\end{aligned}}$$

- TIDYING UP THE EXPRESSION

$$\Rightarrow f(x,k) = (1+k)\left[\begin{array}{l} 1 - kx + k^2x^2 - k^3x^3 + k^4x^4 - \dots \\ \quad\; x - kx^2 + k^2x^3 - k^3x^4 + \dots \\ \qquad\quad x^2 - kx^3 + k^2x^4 - \dots \\ \qquad\qquad x^3 - kx^4 + \dots \\ \qquad\qquad\quad x^4 - \dots \end{array}\right]$$

$$\Rightarrow f(x,k) = (1+k)\left[1 + (1-k)x + (1-k+k^2)x^2 + (1-k+k^2-k^3)x^3 + (1-k+k^2-k^3+k^4)x^4 + \dots\right]$$

- USING THE IDENTITY $\boxed{a^n - b^n \equiv \left(a^{n-1} - a^{n-2}b^1 + a^{n-3}b^2 - a^{n-4}b^3 + \dots\ b^n\right)(a-b)}$

$$\Rightarrow f(x,k) = \overset{a_0}{(1+k)} + \overset{a_1}{(1-k^2)}x + \overset{a_2}{(1+k^3)}x^2 + \overset{a_3}{(1-k^4)}x^3 + \overset{a_4}{(1-k^5)}x^4 + O(x^5)$$

$$\Rightarrow f(x,k) = \sum_{r=0}^{\infty}\left[\left[1+k^{r+1}(-1)^r\right]x^r\right]$$

$$\Rightarrow f(x,k) = \sum_{r=0}^{\infty}\left[\left[1+k(-k)^r\right]x^r\right]$$

- NEXT CONSIDER THE REQUIRED SERIES

$$\Rightarrow g(x,k) = \sum_{r=0}^{\infty}\left[\left[1+k(-k)^r\right]^2 x^r\right] = \sum_{r=0}^{\infty}\left[\left[1+2k(-k)^r+k^2(-k)^{2r}\right]x^r\right]$$

$$\Rightarrow g(x,k) = \sum_{r=0}^{\infty}\left[x^r\right] + 2k\sum_{r=0}^{\infty}\left[(-k)^r x^r\right] + k^2\sum_{r=0}^{\infty}\left[(k^2)^r x^r\right]$$

$$\Rightarrow g(x,k) = \sum_{r=0}^{\infty}(x^r) + 2k\sum_{r=0}^{\infty}(-kx)^r + k^2\sum(k^2x)^r$$

- NEXT RECALLING THE STANDARD EXPANSIONS WE USED EARLIER

$$(1-x)^{-1} = 1+x+x^2+x^3+\dots = \sum_{r=0}^{\infty}x^r$$

$$(1+x)^{-1} = 1-x+x^2-x^3+\dots = \sum_{r=0}^{\infty}(-x)^r$$

$$\Rightarrow g(x,k) = \frac{1}{1-x} + \frac{2k}{1+kx} + \frac{k^2}{1-k^2x}$$

$$\Rightarrow g(x,k) = \frac{(1+kx)(1-k^2x) + 2k(1-x)(1-k^2x) + k^2(1-x)(1+kx)}{(1-x)(1+kx)(1-k^2x)}$$

$$\Rightarrow g(x,k) = \frac{\left\{\begin{array}{l} 1 + kx - k^2x - \cancel{k^3x^2} \\ 2k - 2k^3x + \cancel{2k^3x^2} \\ \quad -2kx \\ k^2 + k^3x \\ \quad -k^3x - \cancel{k^3x^2} \end{array}\right\}}{(1-x)(1+kx)(1-k^2x)}$$

$$\Rightarrow g(x,k) = \frac{(k^2+2k+1) - (k^3+2k^2+k)x}{(1-x)(1+kx)(1-k^2x)}$$

$$\Rightarrow g(x,k) = \frac{(k^2+2k+1) - (k^2+2k+1)kx}{(1-x)(1+kx)(1-k^2x)}$$

$$\Rightarrow g(x,k) = \frac{(k^2+2k+1)(1-kx)}{(1-x)(1+kx)(1-k^2x)}$$

$$\Rightarrow g(x,k) = \frac{(1+k)^2(1-kx)}{(1-x)(1+kx)(1-k^2x)}$$

**Question 92** (*****)

Consider the following infinite series, $S$.

$$S=\frac{5}{18}-\frac{5\times8}{18\times24}+\frac{5\times8\times11}{18\times24\times30}-\frac{5\times8\times11\times14}{18\times24\times30\times36}+\frac{5\times8\times11\times14\times17}{18\times24\times30\times36\times42}-\ldots$$

Given that $S$ converges, show that

$$S=9A-41,$$

where $A$ is an exact simplified surd.

$\boxed{A=\sqrt[3]{96}}$

PROCEED AS FOLLOWS

$$S=\frac{5}{18}-\frac{5\times8}{18\times24}+\frac{5\times8\times11}{18\times24\times30}-\frac{5\times8\times11\times14}{18\times24\times30\times36}+\cdots$$

$$S=\frac{5}{6^1\times3}-\frac{5\times8}{6^2(3\times4)}+\frac{5\times8\times11}{6^3(3\times4\times5)}-\frac{5\times8\times11\times14}{6^4(3\times4\times5\times6)}+\cdots$$

$$S=\frac{\frac{5}{3}}{3}\left(\frac{1}{2}\right)-\frac{\frac{5}{3}\left(\frac{8}{3}\right)}{4\times3}\left(\frac{1}{2}\right)^2+\frac{\frac{5}{3}\left(\frac{8}{3}\right)\left(\frac{11}{3}\right)}{5\times4\times3}\left(\frac{1}{2}\right)^3-\frac{\frac{5}{3}\left(\frac{8}{3}\right)\left(\frac{11}{3}\right)\left(\frac{14}{3}\right)}{6\times5\times4\times3}\left(\frac{1}{2}\right)^4+\cdots$$

$$S\times\frac{1}{2}\times\left(\frac{1}{2}\right)^2=\frac{\frac{5}{3}}{3\times2}\left(\frac{1}{2}\right)^3-\frac{\frac{5}{3}\left(\frac{8}{3}\right)}{4\times3\times2}\left(\frac{1}{2}\right)^4+\frac{\frac{5}{3}\left(\frac{8}{3}\right)\left(\frac{11}{3}\right)}{5\times4\times3\times2}-\frac{\frac{5}{3}\left(\frac{8}{3}\right)\left(\frac{11}{3}\right)\left(\frac{14}{3}\right)}{6\times5\times4\times3\times2}$$

$$S\times\frac{1}{8}\times\frac{2}{3}\times\frac{1}{3}=\frac{\frac{1}{3}\left(\frac{2}{3}\right)\left(\frac{5}{3}\right)}{3!}\left(\frac{1}{2}\right)^3-\frac{\frac{1}{3}\left(\frac{2}{3}\right)\left(\frac{5}{3}\right)\left(\frac{8}{3}\right)}{4!}\left(\frac{1}{2}\right)^4+\frac{\frac{1}{3}\left(\frac{2}{3}\right)\left(\frac{5}{3}\right)\left(\frac{8}{3}\right)\left(\frac{11}{3}\right)}{5!}\left(\frac{1}{2}\right)^5-\frac{\frac{1}{3}\left(\frac{2}{3}\right)\left(\frac{5}{3}\right)\left(\frac{8}{3}\right)\left(\frac{11}{3}\right)\left(\frac{14}{3}\right)}{6!}\left(\frac{1}{2}\right)^6,\ldots$$

$$\frac{1}{36}S=\frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3!}\left(\frac{1}{2}\right)^3+\frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\left(-\frac{8}{3}\right)}{4!}\left(\frac{1}{2}\right)^4+\frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\left(-\frac{8}{3}\right)\left(-\frac{11}{3}\right)}{5!}\left(\frac{1}{2}\right)^5+\frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\left(-\frac{8}{3}\right)\left(-\frac{11}{3}\right)\left(-\frac{14}{3}\right)}{6!}\left(\frac{1}{2}\right)^6+\cdots$$

ADD THE MISSING TERMS TO PRODUCE A BINOMIAL EXPANSION

$$\frac{1}{36}S=\underbrace{\left[1+\frac{\frac{1}{3}}{1!}\left(\frac{1}{2}\right)^1+\frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{2!}\left(\frac{1}{2}\right)^2\right]+\left[\frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3!}\left(\frac{1}{2}\right)^3+\frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\left(-\frac{8}{3}\right)}{4!}\left(\frac{1}{2}\right)^4+\cdots\right]}_{\text{PERFECT BINOMIAL EXPANSION}}-\left[1+\frac{\frac{1}{3}}{1!}\left(\frac{1}{2}\right)^1+\frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{2!}\left(\frac{1}{2}\right)^2\right]$$

$$\Rightarrow \frac{1}{36}S=\left(1+\frac{1}{2}\right)^{\frac{1}{3}}-\left(1+\frac{1}{6}-\frac{1}{36}\right)$$

$$\Rightarrow S=36\left(\frac{3}{2}\right)^{\frac{1}{3}}-(36+6-1)$$

$$\Rightarrow S=36\sqrt[3]{\frac{3}{2}}-41$$

$$\Rightarrow S=9\times4\times\sqrt{\frac{3}{2}}-41$$

$$\Rightarrow S=9\times\sqrt[3]{64}\times\sqrt[3]{\frac{3}{2}}-41$$

$$\Rightarrow S=9\sqrt[3]{96}-41$$