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TRIGONOMETRY

THE DOUBLE ANGLE IDENTITIES

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Question 1

Prove the validity of each of the following trigonometric identities.

a) $\sec \theta \operatorname{cosec} \theta \equiv 2 \operatorname{cosec} 2\theta$

b) $\tan \theta + \cot \theta \equiv 2 \operatorname{cosec} 2\theta$

c) $\frac{1 - \cos 2x}{\sin 2x} \equiv \tan x$

d) $\frac{\cos 2\theta}{\cos \theta - \sin \theta} \equiv \cos \theta + \sin \theta$

e) $\frac{\cos 2x}{\sin x} + \frac{\sin 2x}{\cos x} \equiv \operatorname{cosec} x$

(a) $\text{LHS} = \sec \theta \operatorname{cosec} \theta = \frac{1}{\cos \theta} \times \frac{1}{\sin \theta} = \frac{1}{\sin \theta \cos \theta} = \frac{2}{2 \sin \theta \cos \theta} = \frac{2}{\sin 2\theta}$
 $= 2 \operatorname{cosec} 2\theta = \text{RHS}$

(b) $\text{LHS} = \tan \theta + \cot \theta = \frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta} = \frac{\sin^2 \theta + \cos^2 \theta}{\cos \theta \sin \theta} = \frac{1}{\cos \theta \sin \theta}$
 $= \frac{2}{2 \cos \theta \sin \theta} = \frac{2}{\sin 2\theta} = 2 \operatorname{cosec} 2\theta = \text{RHS}$

(c) $\text{LHS} = \frac{1 - \cos 2x}{\sin 2x} = \frac{1 - (1 - 2\cos^2 x)}{2 \sin x \cos x} = \frac{2 \cos^2 x}{2 \sin x \cos x} = \frac{\cos x}{\sin x} = \cot x = \text{RHS}$

(d) $\text{LHS} = \frac{\cos 2\theta}{\cos \theta - \sin \theta} = \frac{\cos^2 \theta - \sin^2 \theta}{\cos \theta - \sin \theta} = \frac{(\cos \theta - \sin \theta)(\cos \theta + \sin \theta)}{\cos \theta - \sin \theta} = \cos \theta + \sin \theta = \text{RHS}$

(e) $\text{LHS} = \frac{\cos 2x}{\sin x} + \frac{\sin 2x}{\cos x} = \frac{1 - 2\sin^2 x}{\sin x} + \frac{2 \sin x \cos x}{\cos x}$
 $= \frac{1}{\sin x} - \frac{2 \sin^2 x}{\sin x} + 2 \cos x = \operatorname{cosec} x - 2 \sin x + 2 \cos x = \text{RHS}$

OR
 $\text{LHS} = \frac{\cos 2x}{\sin x} + \frac{\sin 2x}{\cos x} = \frac{\cos 2x \cos x + \sin 2x \sin x}{\sin x \cos x} = \frac{\cos(2x - x)}{\sin x \cos x}$
 $= \frac{\cos x}{\sin x \cos x} = \operatorname{cosec} x = \text{RHS}$

Question 2

Prove the validity of each of the following trigonometric identities.

a) $\cot 2x + \operatorname{cosec} 2x \equiv \cot x$

b) $\cos 2x + \tan x \sin 2x \equiv 1$

c) $\frac{\sin x}{1 - \cos x} \equiv \cot \frac{1}{2}x$

d) $\sin 2\theta \equiv \frac{2 \tan \theta}{1 + \tan^2 \theta}$

e) $\frac{1}{\cos \theta - \sin \theta} - \frac{1}{\cos \theta + \sin \theta} \equiv 2 \sin \theta \sec 2\theta$

Handwritten solutions for the five trigonometric identities:

(a) $\text{LHS} = \cot 2x + \operatorname{cosec} 2x = \frac{\cos 2x}{\sin 2x} + \frac{1}{\sin 2x} = \frac{\cos 2x + 1}{\sin 2x}$
 $= \frac{(\cos^2 x - \sin^2 x) + 1}{2 \sin x \cos x} = \frac{2 \cos^2 x}{2 \sin x \cos x} = \frac{\cos x}{\sin x} = \cot x = \text{RHS}$

(b) $\text{LHS} = \cos 2x + \tan x \sin 2x = (1 - 2\sin^2 x) + \frac{\sin x}{\cos x} (2 \sin x \cos x)$
 $= 1 - 2\sin^2 x + 2\sin^2 x = 1 = \text{RHS}$

(c) $\text{LHS} = \frac{\sin x}{1 - \cos x} = \frac{2 \sin(\frac{x}{2}) \cos(\frac{x}{2})}{1 - (1 - 2\sin^2(\frac{x}{2}))}$
 $= \frac{2 \sin(\frac{x}{2}) \cos(\frac{x}{2})}{2 \sin^2(\frac{x}{2})} = \frac{\cos(\frac{x}{2})}{\sin(\frac{x}{2})} = \cot(\frac{x}{2}) = \text{RHS}$
* $\sin 2x = 2 \sin x \cos x$
 * $\sin^2(\theta) = 1 - \cos^2(\theta)$
 * $\cos(2x) = 1 - 2\sin^2(x)$
 * $\cos(2x) = 1 - 2\sin^2(x)$

(d) $\text{RHS} = \frac{2 \tan \theta}{1 + \tan^2 \theta} = \frac{2 \frac{\sin \theta}{\cos \theta}}{1 + \frac{\sin^2 \theta}{\cos^2 \theta}} = \frac{2 \sin \theta \cos \theta}{\cos^2 \theta + \sin^2 \theta} = \frac{2 \sin \theta \cos \theta}{1} = 2 \sin \theta \cos \theta = \sin 2\theta = \text{LHS}$

(e) $\text{LHS} = \frac{1}{\cos \theta - \sin \theta} - \frac{1}{\cos \theta + \sin \theta} = \frac{(\cos \theta + \sin \theta) - (\cos \theta - \sin \theta)}{(\cos \theta - \sin \theta)(\cos \theta + \sin \theta)}$
 $= \frac{2 \sin \theta}{\cos^2 \theta - \sin^2 \theta} = \frac{2 \sin \theta}{\cos 2\theta} = 2 \sin \theta \sec 2\theta = \text{RHS}$

Question 3

Prove the validity of each of the following trigonometric identities.

$$\text{a) } \frac{\tan 2\theta - \sin 2\theta}{\tan \theta} \equiv 2 \sin^2 \theta$$

$$\text{b) } \frac{\sec^2 \theta}{1 - \tan^2 \theta} \equiv \sec 2\theta$$

$$\text{c) } \cos 2\theta \equiv \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta}$$

$$\text{d) } \sqrt{2 + 2 \cos 2\theta} \equiv 2 \cos \theta$$

$$\text{e) } \tan 2\theta \sec \theta \equiv 2 \sin \theta \sec 2\theta$$

Handwritten solutions for questions a-e:

(a) $LHS = \frac{\tan 2\theta - \sin 2\theta}{\tan \theta} = \frac{\frac{\sin 2\theta}{\cos 2\theta} - \sin 2\theta}{\frac{\sin \theta}{\cos \theta}} = \frac{\sin 2\theta(1 - \cos 2\theta)}{\sin \theta \cos \theta}$
 $= \frac{2 \sin \theta \cos \theta (1 - (1 - 2 \sin^2 \theta))}{\sin \theta \cos \theta} = \frac{2 \sin \theta \cos \theta (2 \sin^2 \theta)}{\sin \theta \cos \theta} = 4 \sin^2 \theta$ (Note: The handwritten solution shows a different path, resulting in $2 \sin^2 \theta$ after simplification.)

(b) $LHS = \frac{\sec^2 \theta}{1 - \tan^2 \theta} = \frac{\frac{1}{\cos^2 \theta}}{1 - \frac{\sin^2 \theta}{\cos^2 \theta}} = \frac{1}{\cos^2 \theta - \sin^2 \theta} = \frac{1}{\cos 2\theta} = \sec 2\theta = RHS$

(c) $LHS = \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} = \frac{1 - \frac{\sin^2 \theta}{\cos^2 \theta}}{1 + \frac{\sin^2 \theta}{\cos^2 \theta}} = \frac{\cos^2 \theta - \sin^2 \theta}{\cos^2 \theta + \sin^2 \theta} = \frac{\cos 2\theta}{1} = \cos 2\theta = RHS$

(d) $LHS = \sqrt{2 + 2 \cos 2\theta} = \sqrt{2(1 + \cos 2\theta)} = \sqrt{2(1 + (2 \cos^2 \theta - 1))} = \sqrt{4 \cos^2 \theta} = 2 \cos \theta = RHS$

(e) $LHS = \tan 2\theta \sec \theta = \frac{\sin 2\theta}{\cos 2\theta} \times \frac{1}{\cos \theta} = \frac{2 \sin \theta \cos \theta}{\cos 2\theta \cos \theta} = \frac{2 \sin \theta}{\cos 2\theta} = 2 \sin \theta \sec 2\theta = RHS$

Question 4

Prove the validity of each of the following trigonometric identities.

a) $\frac{1 + \tan^2 x}{1 - \tan^2 x} \equiv \sec 2x$

b) $\cot x - \tan x \equiv 2 \cot 2x$

c) $\operatorname{cosec} 2\theta - \cot 2\theta \equiv \tan \theta$

d) $(3 \sin \theta + 5 \cos \theta)^2 \equiv 17 + 8 \cos 2\theta + 15 \sin 2\theta$

e) $2 \cot 2\theta + \tan \theta \equiv \cot \theta$

a) $\text{LHS} = \frac{1 + \tan^2 x}{1 - \tan^2 x} = \frac{1 + \frac{\sin^2 x}{\cos^2 x}}{1 - \frac{\sin^2 x}{\cos^2 x}} = \frac{\frac{\cos^2 x + \sin^2 x}{\cos^2 x}}{\frac{\cos^2 x - \sin^2 x}{\cos^2 x}} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x - \sin^2 x} = \frac{1}{\cos 2x} = \sec 2x = \text{RHS}$

b) $\text{LHS} = \cot x - \tan x = \frac{\cos x}{\sin x} - \frac{\sin x}{\cos x} = \frac{\cos^2 x - \sin^2 x}{\sin x \cos x} = \frac{\cos 2x}{\frac{1}{2} \sin 2x} = \frac{2 \cos 2x}{\sin 2x} = 2 \cot 2x = \text{RHS}$

c) $\text{LHS} = \operatorname{cosec} 2\theta - \cot 2\theta = \frac{1}{\sin 2\theta} - \frac{\cos 2\theta}{\sin 2\theta} = \frac{1 - \cos 2\theta}{\sin 2\theta} = \frac{1 - (1 - 2\sin^2 \theta)}{2 \sin \theta \cos \theta} = \frac{2 \sin^2 \theta}{2 \sin \theta \cos \theta} = \frac{\sin \theta}{\cos \theta} = \tan \theta = \text{RHS}$

d) $\text{LHS} = (3 \sin \theta + 5 \cos \theta)^2 = 9 \sin^2 \theta + 30 \sin \theta \cos \theta + 25 \cos^2 \theta$
 $= 9(\sin^2 \theta + \cos^2 \theta) + 15(2 \sin \theta \cos \theta) + 16 \cos^2 \theta$
 $= 9 + 15 \sin 2\theta + 16(\frac{1}{2} + \frac{1}{2} \cos 2\theta)$
 $= 9 + 15 \sin 2\theta + 8 + 8 \cos 2\theta$
 $= 17 + 15 \sin 2\theta + 8 \cos 2\theta = \text{RHS}$

e) $\text{LHS} = 2 \cot 2\theta + \tan \theta = \frac{2 \cos 2\theta}{\sin 2\theta} + \tan \theta = \frac{2 \cos 2\theta}{2 \sin \theta \cos \theta} + \tan \theta = \frac{\cos 2\theta}{\sin \theta \cos \theta} + \tan \theta$
 $= \frac{\cos^2 \theta - \sin^2 \theta}{\sin \theta \cos \theta} + \tan \theta = \cot \theta - \frac{\sin \theta}{\cos \theta} + \tan \theta = \cot \theta = \text{RHS}$

Question 5

Prove the validity of each of the following trigonometric identities.

a) $\frac{2 \tan x}{\tan x + \sin x} \equiv \sec^2\left(\frac{x}{2}\right)$

b) $\cot 2x \equiv \frac{\cot^2 x - 1}{2 \cot x}$

c) $\operatorname{cosec} \theta - \cot \theta \equiv \tan \frac{1}{2} \theta$

d) $2 - 2 \tan x - \frac{2 \tan x}{\tan 2x} \equiv (1 - \tan x)^2$

e) $\frac{\sin 2x + \sin x}{\cos 2x + \cos x + 1} \equiv \tan x$

Handwritten solutions for the five trigonometric identities:

(a) $LHS = \frac{2 \tan x}{\tan x + \sin x} = \frac{2 \left(\frac{\sin x}{\cos x}\right)}{\frac{\sin x}{\cos x} + \sin x} = \dots$ Multiply top & bottom by $\cos x$
 $= \frac{2 \sin x}{\sin x + \sin x \cos x} = \frac{2}{1 + \cos x} = \frac{2}{1 + (\cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2})}$
 $= \frac{2}{2 \cos^2 \frac{\theta}{2}} = \sec^2 \frac{\theta}{2} = RHS$

(b) $LHS = \cot 2x = \frac{\cos 2x}{\sin 2x} = \frac{\cos^2 x - \sin^2 x}{2 \sin x \cos x} = \frac{\cos^2 x - \sin^2 x}{2 \sin x \cos x}$
 $= \frac{\cos^2 x - \sin^2 x}{2 \sin x \cos x} = RHS$

(c) $LHS = \operatorname{cosec} \theta - \cot \theta = \frac{1}{\sin \theta} - \frac{\cos \theta}{\sin \theta} = \frac{1 - \cos \theta}{\sin \theta} = \frac{1 - (2 \cos^2 \frac{\theta}{2} - 1)}{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}}$
 $= \frac{2 - 2 \cos^2 \frac{\theta}{2}}{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}} = \frac{2 \sin^2 \frac{\theta}{2}}{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}} = \tan \frac{\theta}{2} = RHS$

(d) $LHS = 2 - 2 \tan x - \frac{2 \tan x}{\tan 2x} = 2 - 2 \tan x - \frac{2 \tan x}{\frac{2 \tan x}{1 - \tan^2 x}}$
 $= 2 - 2 \tan x - \frac{2 \tan x (1 - \tan^2 x)}{2 \tan x} = 2 - 2 \tan x - 1 + \tan^2 x$
 $= \tan^2 x - 2 \tan x + 1 = (\tan x - 1)^2 = RHS$

(e) $LHS = \frac{\sin 2x + \sin x}{\cos 2x + \cos x + 1} = \frac{2 \sin x \cos x + \sin x}{\cos^2 x - \sin^2 x + \cos x + 1} = \frac{\sin x (2 \cos x + 1)}{\cos^2 x + \cos x - \sin^2 x + 1}$
 $= \frac{\sin x (2 \cos x + 1)}{\cos x (2 \cos x + 1)} = \tan x = RHS$

Question 7

Prove the validity of each of the following trigonometric identities.

a) $4\operatorname{cosec}^2 2\theta - \operatorname{cosec}^2 \theta \equiv \sec^2 \theta$

b) $\frac{\tan 2\theta + \sin 2\theta}{\tan 2\theta} \equiv 2\cos^2 \theta$

c) $(\cos x + \sin x)(\operatorname{cosec} x - \sec x) \equiv 2\cot 2x$

d) $\frac{2\sec^2 \theta - \cos 2\theta - 1}{2\tan \theta + \sin 2\theta} \equiv \tan \theta$

(a) LHS = $4\operatorname{cosec}^2 2\theta - \operatorname{cosec}^2 \theta = \frac{4}{\sin^2 2\theta} - \frac{1}{\sin^2 \theta} = \frac{4}{(2\sin\theta\cos\theta)^2} - \frac{1}{\sin^2 \theta} = \frac{4}{4\sin^2\theta\cos^2\theta} - \frac{1}{\sin^2\theta} = \frac{1}{\sin^2\theta\cos^2\theta} - \frac{1}{\sin^2\theta}$
 $= \frac{1 - \cos^2\theta}{\sin^2\theta\cos^2\theta} = \frac{\sin^2\theta}{\sin^2\theta\cos^2\theta} = \frac{1}{\cos^2\theta} = \sec^2\theta = \text{RHS}$

(b) LHS = $\frac{\tan 2\theta + \sin 2\theta}{\tan 2\theta} = \frac{\frac{\sin 2\theta}{\cos 2\theta} + \sin 2\theta}{\frac{\sin 2\theta}{\cos 2\theta}} = \frac{\frac{\sin 2\theta + \sin 2\theta \cos 2\theta}{\cos 2\theta}}{\frac{\sin 2\theta}{\cos 2\theta}} = \frac{\sin 2\theta + \sin 2\theta \cos 2\theta}{\sin 2\theta} = 1 + \cos 2\theta$
 $= 1 + (2\cos^2\theta - 1) = 2\cos^2\theta = \text{RHS}$

(c) LHS = $(\cos x + \sin x)(\operatorname{cosec} x - \sec x) = \cos x \operatorname{cosec} x - \cos x \sec x + \sin x \operatorname{cosec} x - \sin x \sec x$
 $= \frac{\cos x}{\sin x} - \frac{1}{\cos x} + \frac{\sin x}{\sin x} - \frac{1}{\cos x} = \cot x - \frac{1}{\cos x} + 1 - \frac{1}{\cos x}$
 $= \frac{1}{\tan x} - \frac{1}{\cos x} = \frac{1 - \tan x}{\tan x} = 2 \left(\frac{1 - \tan x}{2\tan x} \right)$
 $= 2 \times \frac{1}{2\tan 2x} = 2\cot 2x = \text{RHS}$

(d) LHS = $\frac{2\sec^2 \theta - \cos 2\theta - 1}{2\tan \theta + \sin 2\theta} = \frac{2\sec^2 \theta - (2\cos^2 \theta - 1) - 1}{2\tan \theta + 2\sin\theta\cos\theta} = \frac{2\sec^2 \theta - 2\cos^2 \theta}{2\tan \theta + 2\sin\theta\cos\theta} = \frac{\frac{2}{\cos^2 \theta} - 2\cos^2 \theta}{2\left(\frac{\sin \theta}{\cos \theta} + \sin\theta\cos\theta\right)}$
 $= \frac{1 - \cos^4 \theta}{\sin\theta\cos\theta} = \frac{(1 - \cos^2 \theta)(1 + \cos^2 \theta)}{\sin\theta\cos\theta(1 + \cos^2 \theta)} = \frac{1 - \cos^2 \theta}{\sin\theta\cos\theta} = \frac{\sin^2 \theta}{\sin\theta\cos\theta} = \frac{\sin \theta}{\cos \theta} = \tan \theta = \text{RHS}$

Question 8

Prove the validity of each of the following trigonometric identities.

a) $4\operatorname{cosec}^2 2\theta - \sec^2 \theta \equiv \operatorname{cosec}^2 \theta$

b) $2\cos^4 \theta + \frac{1}{2}\sin^2 2\theta - 1 \equiv \cos 2\theta$

c) $\frac{\cos 2x}{\sqrt{1 + \sin 2x}} \equiv \cos x - \sin x$

d) $\frac{\sqrt{2 - 2\cos x}}{\sin x} \equiv \sec \frac{x}{2}$

Handwritten solutions for the trigonometric identities:

(a) LHS = $4 \operatorname{cosec}^2 2\theta - \sec^2 \theta = \frac{4}{(\sin 2\theta)^2} - \frac{1}{\cos^2 \theta}$
 $= \frac{4}{(2\sin\theta\cos\theta)^2} - \frac{1}{\cos^2 \theta} = \frac{4}{4\sin^2\theta\cos^2\theta} - \frac{1}{\cos^2\theta} = \frac{1}{\sin^2\theta\cos^2\theta} - \frac{1}{\cos^2\theta}$
 $= \frac{1 - \sin^2\theta}{\sin^2\theta\cos^2\theta} = \frac{\cos^2\theta}{\sin^2\theta\cos^2\theta} = \frac{1}{\sin^2\theta} = \operatorname{cosec}^2\theta = \text{RHS}$

(b) LHS = $2\cos^4\theta + \frac{1}{2}\sin^2 2\theta - 1$
 $= \frac{1}{2}(4\cos^4\theta + \sin^2 2\theta) - 1$
 $= \frac{1}{2}(4\cos^4\theta + 4\cos^2\theta\sin^2\theta) - 1$
 $= \frac{1}{2}(4\cos^2\theta(\cos^2\theta + \sin^2\theta)) - 1$
 $= \frac{1}{2}(4\cos^2\theta) - 1 = 2\cos^2\theta - 1 = \cos 2\theta = \text{RHS}$

(c) LHS = $\frac{\cos 2x}{\sqrt{1 + \sin 2x}} = \frac{\cos^2 x - \sin^2 x}{\sqrt{1 + 2\sin x \cos x}} = \frac{(\cos x - \sin x)(\cos x + \sin x)}{\sqrt{(\cos x + \sin x)^2}} = \frac{(\cos x - \sin x)(\cos x + \sin x)}{\cos x + \sin x} = \cos x - \sin x = \text{RHS}$

(d) LHS = $\frac{\sqrt{2 - 2\cos x}}{\sin x} = \frac{\sqrt{2(1 - \cos x)}}{2\sin \frac{x}{2}} = \frac{\sqrt{2} \cdot \sqrt{1 - \cos x}}{2\sin \frac{x}{2}}$
 $= \frac{\sqrt{2} \cdot \sqrt{1 - 2\cos^2 \frac{x}{2} + 1}}{2\sin \frac{x}{2}} = \frac{\sqrt{2} \cdot \sqrt{2 - 2\cos^2 \frac{x}{2}}}{2\sin \frac{x}{2}} = \frac{2\sin \frac{x}{2}}{2\sin \frac{x}{2}} = 1 = \sec \frac{x}{2} = \text{RHS}$

Question 9

Prove the validity of each of the following trigonometric identities.

a) $8\cos^4\left(\frac{1}{2}\theta\right) \equiv \cos 2\theta + 4\cos\theta + 3$

b) $\sqrt{1 + \sin 2\theta} \equiv \sin\theta + \cos\theta$

c) $\sin^4\theta + \cos^4\theta \equiv \frac{1}{2}(2 - \sin^2 2\theta)$

d) $\sin^4\theta + \cos^4\theta \equiv \frac{1}{4}(3 + \cos 4\theta)$

Handwritten solutions for Question 9:

a) $8\cos^4\left(\frac{1}{2}\theta\right) = 8\cos^2\left(\frac{1}{2}\theta\right)^2 = 2[2\cos^2\left(\frac{1}{2}\theta\right)]^2 = 2[1 + \cos\theta]^2 = 2(1 + 2\cos\theta + \cos^2\theta) = 2 + 4\cos\theta + 2\cos^2\theta = 2 + 4\cos\theta + (\cos 2\theta + 1) = 3 + 4\cos\theta + \cos 2\theta = \text{RHS}$

b) $\text{LHS} = \sqrt{1 + \sin 2\theta} = \sqrt{1 + 2\sin\theta\cos\theta} = \sqrt{\cos^2\theta + \sin^2\theta + 2\sin\theta\cos\theta} = \sqrt{(\cos\theta + \sin\theta)^2} = \cos\theta + \sin\theta = \text{RHS}$

c) $\text{LHS} = \sin^4\theta + \cos^4\theta = \frac{1}{2}[4\sin^4\theta + 4\cos^4\theta] = \frac{1}{2}[(1 - \cos 2\theta)^2 + (1 + \cos 2\theta)^2] = \frac{1}{2}[1 - 2\cos 2\theta + \cos^2 2\theta + 1 + 2\cos 2\theta + \cos^2 2\theta] = \frac{1}{2}[2 + 2\cos^2 2\theta] = \frac{1}{2}[2 + (1 + \cos 4\theta)] = \frac{1}{2}[3 + \cos 4\theta] = \text{RHS}$

d) $\text{LHS} = \sin^4\theta + \cos^4\theta = \dots$ identical as in part (c) ... $= \frac{1}{2}[2 + 2\cos^2 2\theta] = \frac{1}{2}[2 + (1 + \cos 4\theta)] = \frac{1}{2}[3 + \cos 4\theta] = \text{RHS}$

Handwritten notes in boxes:

- For (a): $\cos 2\theta = 2\cos^2\theta - 1$, $2\cos^2\theta = 1 + \cos 2\theta$, $2\cos^2\left(\frac{1}{2}\theta\right) = 1 + \cos\theta$
- For (c): $\cos 2\theta = 2\cos^2\theta - 1$, $1 + \cos 2\theta = 2\cos^2\theta$, $1 + \cos 2\theta = 2\cos^2\theta$, $2\sin^2\theta = 1 - \cos 2\theta$, $4\sin^2\theta = (1 - \cos 2\theta)^2$
- For (d): $\cos 2\theta = 2\cos^2\theta - 1$, $\cos 4\theta = 2\cos^2 2\theta - 1$, $2\cos^2 2\theta = 1 + \cos 4\theta$

Question 10

Solve each of the following trigonometric equations.

a) $\sin 2\theta = \tan \theta, \quad 0 \leq \theta \leq 180^\circ$

b) $2 \sin 2x = \cos x, \quad 0 \leq x < 180^\circ$

c) $\sin 2y + \sin y = 0, \quad 0 \leq y < 360^\circ$

d) $4 \sin \phi \cos \phi = 1, \quad 0 \leq \phi < \pi$

$\theta = 0^\circ, 45^\circ, 135^\circ, 180^\circ, \quad x = 90^\circ, x \approx 14.5^\circ, 165.5^\circ, \quad y = 0^\circ, 120^\circ, 180^\circ, 240^\circ,$

$\phi = \frac{\pi}{12}, \frac{5\pi}{12}$

Handwritten solutions for the four trigonometric equations:

(a) $\sin 2\theta = \tan \theta$
 $\Rightarrow 2\sin\theta \cos\theta = \frac{\sin\theta}{\cos\theta}$
 $\Rightarrow 2\sin\theta \cos^2\theta = \sin\theta$
 $\Rightarrow 2\sin\theta \cos^2\theta - \sin\theta = 0$
 $\Rightarrow \sin\theta(2\cos^2\theta - 1) = 0$
 $\Rightarrow \sin\theta \cos 2\theta = 0$
 • $\sin\theta = 0$
 $\theta = 0^\circ, 180^\circ$
 • $\cos 2\theta = 0$
 $\cos 2\theta = 0 \Rightarrow 2\theta = 90^\circ, 270^\circ$
 $\theta = 45^\circ, 135^\circ$
 $\therefore \theta = 0^\circ, 45^\circ, 135^\circ, 180^\circ$

(b) $2 \sin 2x = \cos x$
 $\Rightarrow 2(2\sin x \cos x) = \cos x$
 $\Rightarrow 4\sin x \cos x - \cos x = 0$
 $\Rightarrow \cos x(4\sin x - 1) = 0$
 • $\cos x = 0$
 $\cos x = 0 \Rightarrow x = 90^\circ, 270^\circ$
 • $4\sin x - 1 = 0$
 $4\sin x = 1 \Rightarrow \sin x = \frac{1}{4}$
 $x = \arcsin(\frac{1}{4}) \approx 14.5^\circ$
 $x = 180^\circ - 14.5^\circ = 165.5^\circ$
 $\therefore x = 90^\circ, 14.5^\circ, 165.5^\circ, 270^\circ$

(c) $\sin 2y + \sin y = 0$
 $\Rightarrow 2\sin y \cos y + \sin y = 0$
 $\Rightarrow \sin y(2\cos y + 1) = 0$
 • $\sin y = 0$
 $\sin y = 0 \Rightarrow y = 0^\circ, 180^\circ, 360^\circ$
 • $2\cos y + 1 = 0$
 $2\cos y = -1 \Rightarrow \cos y = -\frac{1}{2}$
 $y = 120^\circ, 240^\circ$
 $\therefore y = 0^\circ, 120^\circ, 180^\circ, 240^\circ, 360^\circ$

(d) $4 \sin \phi \cos \phi = 1$
 $\Rightarrow 2(2\sin\phi \cos\phi) = 1$
 $\Rightarrow 2\sin 2\phi = 1$
 $\Rightarrow \sin 2\phi = \frac{1}{2}$
 • $\sin 2\phi = \frac{1}{2}$
 $2\phi = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$
 $\phi = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}$
 $\therefore \phi = \frac{\pi}{12}, \frac{5\pi}{12}$

Question 11

Solve each of the following trigonometric equations.

a) $2 \sin 2\theta = \cot \theta, \quad 0 \leq \theta \leq \pi$

b) $3 \sin 2x = 2 \cos x, \quad 0 \leq x < 180^\circ$

c) $\sin 4y = \sin 2y, \quad 0 \leq y < 180^\circ$

d) $\sin \varphi + \frac{1}{4} \sec \varphi = 0, \quad 0 \leq \varphi < \pi$

$$\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$$

$$x = 90^\circ, x \approx 19.5^\circ, 160.5^\circ$$

$$y = 0^\circ, 30^\circ, 90^\circ, 150^\circ$$

$$\varphi = \frac{7\pi}{12}, \frac{11\pi}{12}$$

Handwritten solutions for the four trigonometric equations:

(a) $2 \sin 2\theta = \cot \theta$
 $\Rightarrow 2(2 \sin \theta \cos \theta) = \frac{\cos \theta}{\sin \theta}$
 $\Rightarrow 4 \sin^2 \theta \cos \theta = \cos \theta$
 $\Rightarrow 4 \sin^2 \theta \cos \theta - \cos \theta = 0$
 $\Rightarrow \cos \theta (4 \sin^2 \theta - 1) = 0$
 Solutions: $\theta = \frac{\pi}{2}, \frac{\pi}{3}, \frac{5\pi}{6}$

(b) $3 \sin 2x = 2 \cos x$
 $\Rightarrow 3(2 \sin x \cos x) = 2 \cos x$
 $\Rightarrow 6 \sin x \cos x = 2 \cos x$
 $\Rightarrow 6 \sin x \cos x - 2 \cos x = 0$
 $\Rightarrow 2 \cos x (3 \sin x - 1) = 0$
 Solutions: $x = 90^\circ, x \approx 19.5^\circ, 160.5^\circ$

(c) $\sin 4y = \sin 2y$
 $\Rightarrow \sin(2 \cdot 2y) = \sin 2y$
 $\Rightarrow 2 \sin 2y \cos 2y = \sin 2y$
 $\Rightarrow \sin 2y (2 \cos 2y - 1) = 0$
 Solutions: $y = 0^\circ, 30^\circ, 90^\circ, 150^\circ$

(d) $\sin \varphi + \frac{1}{4} \sec \varphi = 0$
 $\Rightarrow 4 \sin \varphi + \sec \varphi = 0$
 $\Rightarrow 4 \sin \varphi + \frac{1}{\cos \varphi} = 0$
 $\Rightarrow 4 \sin \varphi \cos \varphi + 1 = 0$
 $\Rightarrow 2(2 \sin \varphi \cos \varphi) + 1 = 0$
 $\Rightarrow 2 \sin 2\varphi + 1 = 0$
 $\Rightarrow \sin 2\varphi = -\frac{1}{2}$
 Solutions: $\varphi = \frac{7\pi}{12}, \frac{11\pi}{12}$

Question 12

Solve each of the following trigonometric equations.

a) $\cos \theta - \sin 2\theta = 0, \quad 0 \leq \theta \leq 360^\circ$

b) $\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} = 4, \quad 0 \leq x < 360^\circ$

c) $2 \cos y = 2 \tan y \sin y + \sec y, \quad 0 \leq y < 2\pi$

d) $2 \cos \phi + \operatorname{cosec} \phi = 0, \quad 0 \leq \phi < 2\pi$

$$\theta = 30^\circ, 90^\circ, 150^\circ, 270^\circ, \quad x = 15^\circ, 75^\circ, 195^\circ, 255^\circ, \quad y = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6},$$

$$\phi = \frac{3\pi}{4}, \frac{7\pi}{4}$$

Handwritten solutions for the trigonometric equations in Question 12:

a) $\cos \theta - \sin 2\theta = 0$
 $\cos \theta - 2 \sin \theta \cos \theta = 0$
 $\cos \theta (1 - 2 \sin \theta) = 0$
 $\cos \theta = 0$ or $\sin \theta = \frac{1}{2}$
 $\cos \theta = 0 \Rightarrow \theta = 90^\circ, 270^\circ$
 $\sin \theta = \frac{1}{2} \Rightarrow \theta = 30^\circ, 150^\circ$
 $\therefore \theta = 30^\circ, 90^\circ, 150^\circ, 270^\circ$

b) $\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} = 4$
 $\frac{\sin^2 x + \cos^2 x}{\sin x \cos x} = 4$
 $\frac{1}{\sin x \cos x} = 4$
 $\frac{1}{2 \sin x \cos x} = 4$
 $\frac{1}{\sin 2x} = 4$
 $\sin 2x = \frac{1}{4}$
 $2x = \arcsin \left(\frac{1}{4} \right)$
 $x = \frac{1}{2} \arcsin \left(\frac{1}{4} \right)$
 $x = 15^\circ, 75^\circ, 195^\circ, 255^\circ$

c) $2 \cos y = 2 \tan y \sin y + \sec y$
 $2 \cos y = 2 \frac{\sin y}{\cos y} \sin y + \frac{1}{\cos y}$
 $2 \cos y = \frac{2 \sin^2 y + 1}{\cos y}$
 $2 \cos^2 y = 2 \sin^2 y + 1$
 $2 \cos^2 y - 2 \sin^2 y = 1$
 $2 \cos 2y = 1$
 $\cos 2y = \frac{1}{2}$
 $2y = 60^\circ, 300^\circ$
 $y = 30^\circ, 150^\circ$

d) $2 \cos \phi + \operatorname{cosec} \phi = 0$
 $2 \cos \phi + \frac{1}{\sin \phi} = 0$
 $2 \cos^2 \phi + 1 = 0$
 $2 \cos^2 \phi = -1$
 $\cos^2 \phi = -\frac{1}{2}$
 $\cos \phi = \pm \frac{\sqrt{2}}{2}$
 $\phi = 135^\circ, 225^\circ$

Question 13

Solve each of the following trigonometric equations.

a) $2 \cos 2\theta = 1 + \cos \theta, \quad 0 \leq \theta < 360^\circ$

b) $\cos 2x + 3 \sin x = 2, \quad 0 \leq x < 360^\circ$

c) $\cos 2y + \sin y = 0, \quad 0 \leq y < 360^\circ$

d) $2(1 - \cos 2\phi) = \tan \phi, \quad 0 \leq \phi < 180^\circ$

$\theta = 0^\circ, \theta \approx 138.6^\circ, 221.4^\circ, \quad x = 30^\circ, 90^\circ, 150^\circ, \quad y = 90^\circ, 210^\circ, 330^\circ,$

$\phi = 0^\circ, 15^\circ, 75^\circ$

Handwritten solutions for the four trigonometric equations:

(a) $2 \cos 2\theta = 1 + \cos \theta$
 $\Rightarrow 2(2\cos^2\theta - 1) = 1 + \cos \theta$
 $\Rightarrow 4\cos^2\theta - 2 = 1 + \cos \theta$
 $\Rightarrow 4\cos^2\theta - \cos \theta - 3 = 0$
 $\Rightarrow (\cos \theta - 1)(4\cos \theta + 3) = 0$
 $\Rightarrow \cos \theta = \frac{1}{4}$
 $\Rightarrow \theta = 0^\circ, 138.6^\circ, 221.4^\circ$

(b) $\cos 2x + 3 \sin x = 2$
 $\Rightarrow 1 - 2\sin^2 x + 3 \sin x = 2$
 $\Rightarrow -2\sin^2 x + 3 \sin x - 1 = 0$
 $\Rightarrow 2\sin^2 x - 3 \sin x + 1 = 0$
 $\Rightarrow (2\sin x - 1)(\sin x - 1) = 0$
 $\Rightarrow \sin x = \frac{1}{2}, 1$
 $\Rightarrow x = 30^\circ, 150^\circ, 90^\circ$

(c) $\cos 2y + \sin y = 0$
 $\Rightarrow 1 - 2\sin^2 y + \sin y = 0$
 $\Rightarrow 2\sin^2 y - \sin y - 1 = 0$
 $\Rightarrow (2\sin y + 1)(\sin y - 1) = 0$
 $\Rightarrow \sin y = -\frac{1}{2}, 1$
 $\Rightarrow y = 90^\circ, 210^\circ, 330^\circ$

(d) $2(1 - \cos 2\phi) = \tan \phi$
 $\Rightarrow 2[1 - (1 - 2\sin^2 \phi)] = \frac{\sin \phi}{\cos \phi}$
 $\Rightarrow 2[1 - 1 + 2\sin^2 \phi] = \frac{\sin \phi}{\cos \phi}$
 $\Rightarrow 4\sin^2 \phi = \frac{\sin \phi}{\cos \phi}$
 $\Rightarrow 4\sin \phi \cos \phi = \sin \phi$
 $\Rightarrow 4\sin \phi \cos \phi - \sin \phi = 0$
 $\Rightarrow \sin \phi (4\cos \phi - 1) = 0$
 $\Rightarrow \sin \phi = 0, \cos \phi = \frac{1}{4}$
 $\Rightarrow \phi = 0^\circ, 15^\circ, 75^\circ$

Question 14

Solve each of the following trigonometric equations.

a) $\cos 2\theta - 7 \sin \theta - 4 = 0, \quad 0 \leq \theta < 360^\circ$

b) $3 \cos 2x = \sin x + 2, \quad 0 \leq x < 360^\circ$

c) $3 \cos 2y = 7 \cos y, \quad 0 \leq y < 360^\circ$

d) $\cos 2\phi = \sin \phi, \quad 0 \leq \phi < 360^\circ$

$\theta = 210^\circ, 330^\circ, \quad x \approx 19.5^\circ, 160.5^\circ \quad x = 210^\circ, 330^\circ, \quad y \approx 109.5^\circ, 250.5^\circ,$

$\phi = 30^\circ, 150^\circ, 270^\circ$

Handwritten solutions for the four trigonometric equations:

(a) $\cos 2\theta - 7\sin\theta - 4 = 0$
 $\Rightarrow (1 - 2\sin^2\theta) - 7\sin\theta - 4 = 0$
 $\Rightarrow -2\sin^2\theta - 7\sin\theta - 3 = 0$
 $\Rightarrow 2\sin^2\theta + 7\sin\theta + 3 = 0$
 $\Rightarrow (2\sin\theta + 1)(\sin\theta + 3) = 0$
 $\Rightarrow \sin\theta = -\frac{1}{2}$
 $\Rightarrow \arcsin(-\frac{1}{2}) = -30^\circ$
 $\theta = 330^\circ$
 $\theta = 210^\circ \pm 360^\circ n, n \in \mathbb{Z}$

(b) $3\cos 2x = \sin x + 2$
 $\Rightarrow 3(1 - 2\sin^2 x) = \sin x + 2$
 $\Rightarrow 3 - 6\sin^2 x = \sin x + 2$
 $\Rightarrow -6\sin^2 x - \sin x + 1 = 0$
 $\Rightarrow (3\sin x - 1)(2\sin x + 1) = 0$
 $\Rightarrow \sin x = \frac{1}{3}$
 $\Rightarrow x = 19.5^\circ$
 $x = 160.5^\circ$
 $x = 210^\circ$
 $x = 330^\circ$

(c) $3\cos 2y = 7\cos y$
 $\Rightarrow 3(2\cos^2 y - 1) = 7\cos y$
 $\Rightarrow 6\cos^2 y - 3 = 7\cos y$
 $\Rightarrow (2\cos y - 3)(2\cos y + 1) = 0$
 $\Rightarrow \cos y = \frac{3}{2}$ (no solution)
 $\Rightarrow \cos y = -\frac{1}{2}$
 $\Rightarrow \arcsin(-\frac{1}{2}) = 109.5^\circ$
 $y = 250.5^\circ$

(d) $\cos 2\phi = \sin \phi$
 $\Rightarrow 1 - 2\sin^2 \phi = \sin \phi$
 $\Rightarrow -2\sin^2 \phi - \sin \phi + 1 = 0$
 $\Rightarrow 2\sin^2 \phi + \sin \phi - 1 = 0$
 $\Rightarrow (2\sin \phi - 1)(\sin \phi + 1) = 0$
 $\Rightarrow \sin \phi = \frac{1}{2}$
 $\phi = 30^\circ$
 $\phi = 150^\circ$
 $\phi = 270^\circ$

Question 15

Solve each of the following trigonometric equations.

- a) $3 \cos 2\theta - 5 \sin \theta = 4, \quad 0 \leq \theta < 360^\circ$
- b) $3 \cos 2x = 1 - \sin x, \quad 0 \leq x < 360^\circ$
- c) $\cos 2y - 7 \cos y + 4 = 0, \quad 0 \leq y < 360^\circ$
- d) $\cos 2\phi + 6 \cos \phi + 5 = 0, \quad 0 \leq \phi < 360^\circ$

$$\theta = 210^\circ, 330^\circ \quad \theta \approx 199.5^\circ, 340.5^\circ, \quad x \approx 41.8^\circ, 138.2^\circ \quad x = 210^\circ, 330^\circ,$$

$$y = 60^\circ, 300^\circ, \quad \phi = 180^\circ$$

Handwritten solutions for the four trigonometric equations:

- a)** $3 \cos 2\theta - 5 \sin \theta = 4$
 $3(1 - 2\sin^2 \theta) - 5 \sin \theta = 4$
 $3 - 6\sin^2 \theta - 5 \sin \theta = 4$
 $0 = 6\sin^2 \theta + 5 \sin \theta + 1$
 $(3 \sin \theta + 1)(2 \sin \theta + 1) = 0$
 $\sin \theta = -\frac{1}{3}$
 $\arcsin(-\frac{1}{3}) = 19.47^\circ$
 $\theta = 199.5^\circ, 340.5^\circ$
 $\sin \theta = -\frac{1}{2}$
 $\arcsin(-\frac{1}{2}) = 30^\circ$
 $\theta = 210^\circ, 330^\circ$
 $\therefore \theta = 340.5^\circ, 199.5^\circ, 330^\circ, 210^\circ$
- b)** $3 \cos 2x = 1 - \sin x$
 $3(1 - 2\sin^2 x) = 1 - \sin x$
 $3 - 6\sin^2 x = 1 - \sin x$
 $0 = 6\sin^2 x - \sin x - 2$
 $0 = (3 \sin x - 2)(2 \sin x + 1)$
 $\sin x = \frac{2}{3}$
 $\arcsin(\frac{2}{3}) = 41.8^\circ$
 $\sin x = -\frac{1}{2}$
 $\arcsin(-\frac{1}{2}) = 30^\circ$
 $x = 210^\circ, 330^\circ$
 $\therefore x = 41.8^\circ, 138.2^\circ, 330^\circ, 210^\circ$
- c)** $\cos 2y - 7 \cos y + 4 = 0$
 $(2 \cos^2 y - 1) - 7 \cos y + 4 = 0$
 $2 \cos^2 y - 7 \cos y + 3 = 0$
 $(2 \cos y - 3)(\cos y - 1) = 0$
 $\cos y = \frac{3}{2}$
 $\cos y = 1$
 $y = 0^\circ, 360^\circ$
 $\therefore y = 60^\circ, 300^\circ$
- d)** $\cos 2\phi + 6 \cos \phi + 5 = 0$
 $(2 \cos^2 \phi - 1) + 6 \cos \phi + 5 = 0$
 $2 \cos^2 \phi + 6 \cos \phi + 4 = 0$
 $\cos^2 \phi + 3 \cos \phi + 2 = 0$
 $(\cos \phi + 1)(\cos \phi + 2) = 0$
 $\cos \phi = -1$
 $\arccos(-1) = 180^\circ$
 $\phi = 180^\circ$

Question 16

Solve each of the following trigonometric equations.

- a) $\cos 2\theta = 7 \cos \theta + 3, \quad 0 \leq \theta < 360^\circ$
- b) $2 \cos 2x = 4 \cos x - 3, \quad 0 \leq x < 360^\circ$
- c) $6 \cos 2y + 5 \cos y + 3 = 0, \quad 0 \leq y < 360^\circ$
- d) $5 \cos 2\phi + 22 \sin \phi = 9, \quad 0 \leq \phi < 360^\circ$

$\theta = 120^\circ, 240^\circ, \quad x = 60^\circ, 300^\circ, \quad y \approx 70.5^\circ, 138.6^\circ, 221.4^\circ, 289.5^\circ, \quad \phi \approx 11.5^\circ, 168.5^\circ$

The image shows handwritten solutions for the four trigonometric equations. The solutions are as follows:

- (a)** $\cos 2\theta = 7 \cos \theta + 3$
 $\Rightarrow 2 \cos^2 \theta - 1 = 7 \cos \theta + 3$
 $\Rightarrow 2 \cos^2 \theta - 7 \cos \theta - 4 = 0$
 $\Rightarrow (2 \cos \theta + 1)(\cos \theta - 4) = 0$
 $\Rightarrow \cos \theta = -\frac{1}{2}$
 $\theta = 120^\circ, 240^\circ$
- (b)** $2 \cos 2x = 4 \cos x - 3$
 $\Rightarrow 2(2 \cos^2 x - 1) = 4 \cos x - 3$
 $\Rightarrow 4 \cos^2 x - 2 = 4 \cos x - 3$
 $\Rightarrow 4 \cos^2 x - 4 \cos x + 1 = 0$
 $\Rightarrow (2 \cos x - 1)(2 \cos x - 1) = 0$
 $\cos x = \frac{1}{2}$
 $x = 60^\circ, 300^\circ$
- (c)** $6 \cos 2y + 5 \cos y + 3 = 0$
 $\Rightarrow 6(2 \cos^2 y - 1) + 5 \cos y + 3 = 0$
 $\Rightarrow 12 \cos^2 y - 6 + 5 \cos y + 3 = 0$
 $\Rightarrow 12 \cos^2 y + 5 \cos y - 3 = 0$
 $\Rightarrow (3 \cos y - 1)(4 \cos y + 3) = 0$
 $\cos y = \frac{1}{3}$
 $y \approx 70.5^\circ, 289.5^\circ$
- (d)** $5 \cos 2\phi + 22 \sin \phi = 9$
 $\Rightarrow 5(1 - 2 \sin^2 \phi) + 22 \sin \phi = 9$
 $\Rightarrow 5 - 10 \sin^2 \phi + 22 \sin \phi = 9$
 $\Rightarrow 0 = 10 \sin^2 \phi - 22 \sin \phi + 4$
 $\Rightarrow 0 = 5 \sin^2 \phi - 11 \sin \phi + 2$
 $\Rightarrow 0 = (5 \sin \phi - 1)(\sin \phi - 2)$
 $\sin \phi = \frac{1}{5}$
 $\phi \approx 11.5^\circ, 168.5^\circ$

Question 17

Solve each of the following trigonometric equations.

a) $\cos 2\theta + 9\sin \theta + 4 = 0, \quad 0 \leq \theta < 360^\circ$

b) $3\cos 2x = 9 - 14\cos x, \quad 0 \leq x < 360^\circ$

c) $2\cos 2y + 7\cos y = 0, \quad 0 \leq y < 360^\circ$

d) $2\cos 2\phi = 1 - 2\sin \phi, \quad 0 \leq \phi < 360^\circ$

$\theta = 210^\circ, 330^\circ, \quad x \approx 48.2^\circ, 311.8^\circ, \quad y \approx 75.5^\circ, 284.5^\circ, \quad \phi = 54^\circ, 126^\circ, 198^\circ, 342^\circ$

Handwritten solutions for the four trigonometric equations:

(a) $\cos 2\theta + 9\sin \theta + 4 = 0$
 $\Rightarrow (1 - 2\sin^2 \theta) + 9\sin \theta + 4 = 0$
 $\Rightarrow 0 = 2\sin^2 \theta - 9\sin \theta - 5$
 $\Rightarrow (2\sin \theta + 1)(\sin \theta - 5) = 0$
 $\Rightarrow \sin \theta = -\frac{1}{2}$

(b) $3\cos 2x = 9 - 14\cos x$
 $\Rightarrow 3(2\cos^2 x - 1) = 9 - 14\cos x$
 $\Rightarrow 6\cos^2 x - 3 = 9 - 14\cos x$
 $\Rightarrow 6\cos^2 x + 14\cos x - 12 = 0$
 $\Rightarrow 3\cos^2 x + 7\cos x - 6 = 0$
 $\Rightarrow (3\cos x - 2)(\cos x + 3) = 0$
 $\Rightarrow \cos x = \frac{2}{3}$

(c) $2\cos 2y + 7\cos y = 0$
 $2(2\cos^2 y - 1) + 7\cos y = 0$
 $4\cos^2 y - 2 + 7\cos y = 0$
 $4\cos^2 y + 7\cos y - 2 = 0$
 $(4\cos y - 1)(\cos y + 2) = 0$
 $\Rightarrow \cos y = \frac{1}{4}$

(d) $2\cos 2\phi = 1 - 2\sin \phi$
 $2(1 - 2\sin^2 \phi) = 1 - 2\sin \phi$
 $2 - 4\sin^2 \phi = 1 - 2\sin \phi$
 $0 = 4\sin^2 \phi - 2\sin \phi - 1$
 $\Rightarrow \sin \phi = \frac{2 \pm \sqrt{4+4}}{8}$
 $\Rightarrow \sin \phi = \frac{2 \pm 2\sqrt{2}}{8}$
 $\Rightarrow \sin \phi = \frac{1 \pm \sqrt{2}}{4}$

Question 18

Solve each of the following trigonometric equations.

a) $\cos 2\theta = 1 + \sin \theta, \quad 0 \leq \theta < 360^\circ$

b) $\cos 2x + 3\cos x = 1, \quad 0 \leq x < 2\pi$

c) $3\cos 2y = 1 - \sin y, \quad 0 \leq y < 360^\circ$

d) $2\cos \phi + 1 = \sin\left(\frac{1}{2}\phi\right), \quad 0 \leq \phi < 360^\circ$

$\theta = 0^\circ, 180^\circ, 210^\circ, 330^\circ, \quad x = \frac{\pi}{3}, \frac{5\pi}{3}, \quad y \approx 41.8^\circ, 138.2^\circ \quad y = 210^\circ, 330^\circ, \quad \phi = 97.2^\circ, 262.8^\circ$

Handwritten solutions for Question 18:

a) $\cos 2\theta = 1 + \sin \theta$
 $\rightarrow 1 - 2\sin^2 \theta = 1 + \sin \theta$
 $\rightarrow 0 = 2\sin^2 \theta + \sin \theta$
 $\rightarrow 0 = \sin \theta (2\sin \theta + 1)$
 $\rightarrow \sin \theta = 0$ or $\sin \theta = -\frac{1}{2}$
 Solutions: $\theta = 0^\circ, 180^\circ, 210^\circ, 330^\circ$

b) $\cos 2x + 3\cos x = 1$
 $\rightarrow 2\cos^2 x - 1 + 3\cos x = 1$
 $\rightarrow 2\cos^2 x + 3\cos x - 2 = 0$
 $\Rightarrow (2\cos x - 1)(\cos x + 2) = 0$
 $\rightarrow \cos x = \frac{1}{2}$ or $\cos x = -2$
 Solutions: $x = \frac{\pi}{3}, \frac{5\pi}{3}$

c) $3\cos 2y = 1 - \sin y$
 $\rightarrow 3(1 - 2\sin^2 y) = 1 - \sin y$
 $\rightarrow 3 - 6\sin^2 y = 1 - \sin y$
 $\rightarrow 0 = 6\sin^2 y - \sin y - 2$
 $\rightarrow (3\sin y - 2)(2\sin y + 1) = 0$
 $\rightarrow \sin y = \frac{2}{3}$ or $\sin y = -\frac{1}{2}$
 Solutions: $y \approx 41.8^\circ, 138.2^\circ, 210^\circ, 330^\circ$

d) $2\cos \phi + 1 = \sin\left(\frac{1}{2}\phi\right)$
 $\rightarrow 2\left(1 - 2\sin^2\left(\frac{\phi}{2}\right)\right) + 1 = \sin\left(\frac{\phi}{2}\right)$
 $\rightarrow 2 - 4\sin^2\left(\frac{\phi}{2}\right) + 1 = \sin\left(\frac{\phi}{2}\right)$
 $\rightarrow 0 = 4\sin^2\left(\frac{\phi}{2}\right) + \sin\left(\frac{\phi}{2}\right) - 3$
 $\rightarrow (4\sin\left(\frac{\phi}{2}\right) - 3)\left(\sin\left(\frac{\phi}{2}\right) + 1\right) = 0$
 $\rightarrow \sin\left(\frac{\phi}{2}\right) = \frac{3}{4}$ or $\sin\left(\frac{\phi}{2}\right) = -1$
 Solutions: $\phi = 97.2^\circ, 262.8^\circ$

Question 19

Solve each of the following trigonometric equations.

a) $\tan \theta(1 + \cos 2\theta) = 2\sin^2 2\theta, \quad 0 \leq \theta \leq 90^\circ$

b) $4 \tan 2\phi + 3 \cot \phi \sec^2 \phi = 0, \quad 0 \leq \phi < 2\pi$ (hard)

$$\theta = 0^\circ, 15^\circ, 75^\circ, 90^\circ, \quad \phi = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$$

(a) $\tan \theta(1 + \cos 2\theta) = 2\sin^2 2\theta$
 $\tan \theta(1 + (2\cos^2 \theta - 1)) = 2(\sin 2\theta)^2$
 $\tan \theta(2\cos^2 \theta) = 2(2\sin \theta \cos \theta)^2$
 $\frac{\sin \theta}{\cos \theta} \times 2\cos^2 \theta = 8\sin^2 \theta \cos^2 \theta$ (BACK TO THE ORIGINAL EXPRESSION) (SOMEHOW, REMOVAL OF RHS NOT NEEDED)
 $2\sin \theta \cos^2 \theta = 2\sin^3 \theta \cos^2 \theta$
 $\sin 2\theta = 2\sin^3 \theta$
 $0 = 2\sin^3 \theta - \sin 2\theta$
 $0 = \sin 2\theta(2\sin^2 \theta - 1)$

• $\sin 2\theta = 0$
 $2\theta = 0 \pm 360n$
 $2\theta = 180 \pm 360n$
 $\theta = 0 \pm 180n$
 $\theta = 90 \pm 180n$

• $\sin 2\theta = \frac{1}{2}$
 $2\theta = 30 \pm 360n$
 $2\theta = 150 \pm 360n$
 $\theta = 15 \pm 180n$
 $\theta = 75 \pm 180n$

$\therefore \theta = 0^\circ, 90^\circ, 15^\circ, 75^\circ$

(b) $4 \tan 2\phi + 3 \cot \phi \sec^2 \phi = 0$
 $4 \left(\frac{2 \tan \phi}{1 - \tan^2 \phi} \right) + \frac{3}{\tan \phi} (1 + \tan^2 \phi) = 0$
 $\frac{8 \tan \phi}{1 - \tan^2 \phi} + \frac{3(1 + \tan^2 \phi)}{\tan \phi} = 0$
 $8 \tan^2 \phi + 3(1 + \tan^2 \phi)(1 - \tan^2 \phi) = 0$
 $8T + 3(1+T)(1-T) = 0$
 where $T = \tan^2 \phi$
 $8T + 3(1 - T^2) = 0$
 $8T + 3 - 3T^2 = 0$
 $3T^2 - 8T - 3 = 0$
 $(3T + 1)(T - 3) = 0$

Then $T = \begin{cases} 3 \\ -\frac{1}{3} \end{cases}$
 $\tan^2 \phi = \begin{cases} 3 \\ -\frac{1}{3} \end{cases}$
 $\tan \phi = \begin{cases} \sqrt{3} \\ -\frac{1}{\sqrt{3}} \end{cases}$
 $\phi = \frac{\pi}{3} \pm n\pi$
 $\phi = \frac{5\pi}{3} \pm n\pi$
 $\therefore \phi = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

Question 20

Show clearly that each of the following trigonometric equations has no real roots, regardless of the solution interval.

a) $\cos 2\theta = 23 + 14\cos\theta$

b) $\cos 2x + \cos x + 2 = 0$

proof

(a) METHOD A (BY SOLUTION)
 $\cos 2\theta = 23 + 14\cos\theta$
 $\Rightarrow 2\cos^2\theta - 1 = 23 + 14\cos\theta$
 $\Rightarrow 2\cos^2\theta - 14\cos\theta - 24 = 0$
 $\Rightarrow \cos^2\theta - 7\cos\theta - 12 = 0$
 $\cos\theta = \frac{7 \pm \sqrt{49 - 4(-12)}}{2}$
 $\cos\theta = \frac{7 \pm \sqrt{121}}{2}$
 $\cos\theta = \frac{-1 \pm 11}{2} < -1$
 $\frac{9 \pm 24}{2} > 1$
 \therefore NO SOLUTIONS

METHOD B:
 $\cos 2\theta = 23 + 14\cos\theta$
 RANGE: $-1 \leq \cos\theta \leq 1$
 $-1 \leq \cos 2\theta \leq 1$
 $9 \leq 23 + 14\cos\theta \leq 37$
 "GRAPHS" DO NOT INTERSECT
 \therefore NO SOLUTIONS

(b) $\cos 2x + \cos x + 2 = 0$
 $\Rightarrow 2\cos^2 x - 1 + \cos x + 2 = 0$
 $\Rightarrow 2\cos^2 x + \cos x + 1 = 0$
 $b^2 - 4ac = 1^2 - 4(2)(1) = -7 < 0$
 NO SOLUTIONS