

Created by T. Madas

# TRIGONOMETRIC EQUATIONS

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**Question 1**

Solve each of the following trigonometric equations.

- a)  $\sec \theta = 4, \quad 0 \leq \theta < 360^\circ$
- b)  $3 \cot 2x - 1 = 4, \quad 0 \leq x < 180^\circ$
- c)  $2 \operatorname{cosec} 2y = 10, \quad 0 \leq y < 2\pi$
- d)  $8 \tan \varphi = \cot^2 \varphi, \quad 0 \leq \varphi < 2\pi$

$$\theta \approx 75.5^\circ, 284.5^\circ, \quad x \approx 15.5^\circ, 105.5^\circ, \quad y \approx 0.10^\circ, 1.47^\circ, 3.24^\circ, 4.61^\circ,$$

$$\varphi \approx 0.46^\circ, 3.61^\circ$$

Handwritten solutions for the trigonometric equations:

5. a)  $\sec \theta = 4$   
 $\Rightarrow \cos \theta = \frac{1}{4}$   
 $\arccos(\frac{1}{4}) = 75.51^\circ$   
 $\theta = 75.5^\circ \pm 324^\circ \quad \text{or } 1/33 \dots$   
 $\theta = 75.5^\circ$   
 $\theta_2 = 284.5^\circ$

b)  $3 \cot 2x - 1 = 4$   
 $\Rightarrow 3 \cot 2x = 5$   
 $\Rightarrow \cot 2x = \frac{5}{3}$   
 $\arccot(\frac{5}{3}) = 30.96^\circ \quad \text{or } 1/33 \dots$   
 $\Rightarrow 2x = 30.96^\circ \pm 180^\circ$   
 $\Rightarrow x = 15.48^\circ \pm 90^\circ$   
 $\therefore x_1 = 15.5^\circ$   
 $x_2 = 105.5^\circ$

c)  $2 \operatorname{cosec} 2y = 10$   
 $\Rightarrow \operatorname{cosec} 2y = 5$   
 $\Rightarrow \sin 2y = \frac{1}{5}$   
 $\arcsin(\frac{1}{5}) = 11.5^\circ$   
 $(2y = 11.5^\circ \pm 168.5^\circ)$   
 $2y = 239.0^\circ \pm 247^\circ \quad \text{or } 1/33 \dots$   
 $y = 11.75^\circ \pm 123.5^\circ$   
 $y = 0.10^\circ, 3.24^\circ, 1.47^\circ, 4.61^\circ$

d)  $8 \tan \varphi = \cot^2 \varphi$   
 $\Rightarrow 8 \tan^3 \varphi = 1$   
 $\Rightarrow \tan^3 \varphi = \frac{1}{8}$   
 $\Rightarrow \tan \varphi = \frac{1}{2}$   
 $\arctan(\frac{1}{2}) = 0.46^\circ$   
 $\therefore \varphi = 0.46^\circ \pm 180^\circ \quad \text{or } 1/33 \dots$   
 $\varphi_1 = 0.46^\circ$   
 $\varphi_2 = 3.61^\circ$

**Question 2**

Solve each of the following trigonometric equations.

a)  $2 \sec \theta = 3, \quad 0 \leq \theta < 360^\circ$

b)  $\cot 3x = \frac{1}{4}, \quad -90^\circ \leq x < 90^\circ$

c)  $5 - \operatorname{cosec} 2y = -1, \quad 0 \leq y < 2\pi$

d)  $27 \sin^2 \phi + 8 \operatorname{cosec} \phi = 0, \quad 0 \leq \phi < 2\pi$

$\theta \approx 48.2^\circ, 311.8^\circ, \quad x \approx -34.7^\circ, 25.3^\circ, 85.3^\circ, \quad y \approx 0.0837^\circ, 1.49^\circ, 3.23^\circ, 4.63^\circ,$

$\phi \approx 3.87^\circ, 5.55^\circ$

Handwritten solutions for the four trigonometric equations:

**a)**  $2 \sec \theta = 3$   
 $\Rightarrow \sec \theta = \frac{3}{2}$   
 $\Rightarrow \cos \theta = \frac{2}{3}$   
 $\arccos\left(\frac{2}{3}\right) = 48.2^\circ$   
 $\theta = 48.2^\circ \pm 360^\circ n$   
 $\theta = 311.8^\circ \pm 360^\circ n$   
 $\theta_1 = 48.2^\circ$   
 $\theta_2 = 311.8^\circ$

**b)**  $\cot 3x = \frac{1}{4}$   
 $\Rightarrow \tan 3x = 4$   
 $\arctan 4 = 75.96^\circ$   
 $3x = 75.96^\circ \pm 180^\circ n$   
 $x = 25.3^\circ \pm 60^\circ n$   
 $x_1 = 25.3^\circ$   
 $x_2 = 85.3^\circ$   
 $x_3 = -34.7^\circ$

**c)**  $5 - \operatorname{cosec} 2y = -1$   
 $\Rightarrow \operatorname{cosec} 2y = 6$   
 $\Rightarrow \sin 2y = \frac{1}{6}$   
 $\arcsin\left(\frac{1}{6}\right) = 9.59^\circ$   
 $2y = 9.59^\circ \pm 360^\circ n$   
 $2y = 297.01^\circ \pm 360^\circ n$   
 $y = 4.79^\circ \pm 180^\circ n$   
 $y_1 = 4.79^\circ$   
 $y_2 = 195.21^\circ$

**d)**  $27 \sin^2 \phi + 8 \operatorname{cosec} \phi = 0$   
 $\Rightarrow 27 \sin^2 \phi + \frac{8}{\sin \phi} = 0$   
 $\Rightarrow 27 \sin^3 \phi + 8 = 0$   
 $\Rightarrow \sin^3 \phi = -\frac{8}{27}$   
 $\Rightarrow \sin \phi = -\frac{2}{3}$   
 $\arcsin\left(-\frac{2}{3}\right) = -38.9^\circ$   
 $\phi = -38.9^\circ \pm 360^\circ n$   
 $\phi = 221.1^\circ \pm 360^\circ n$   
 $\phi_1 = 221.1^\circ$   
 $\phi_2 = 321.1^\circ$

**Question 3**

Solve each of the following trigonometric equations.

a)  $3\sec 2\theta = 7, \quad 0 \leq \theta < 180^\circ$

b)  $2\cot(x - 30^\circ) = 3, \quad 0 \leq x < 360^\circ$

c)  $5 - 2\operatorname{cosec} 3y = 9, \quad 0 \leq y < \pi$

d)  $27\cos \phi = \sec^2 \phi, \quad 0 \leq \phi < 2\pi$

$\theta \approx 32.3^\circ, 147.7^\circ, \quad x \approx 63.7^\circ, 243.7^\circ, \quad y = \frac{7\pi}{18}, \frac{11\pi}{18}, \quad \phi \approx 1.23^\circ, 5.05^\circ$

Handwritten solutions for the four trigonometric equations:

a)  $3\sec 2\theta = 7$   
 $\Rightarrow \sec 2\theta = \frac{7}{3}$   
 $\Rightarrow \cos 2\theta = \frac{3}{7}$   
 $\Rightarrow \arccos\left(\frac{3}{7}\right) = 64.46^\circ$   
 $(2\theta = 64.42 \pm 360^\circ)$   
 $(2\theta = 295.58 \pm 360^\circ)$   
 $(\theta = 32.21 \pm 180^\circ)$   
 $(\theta = 197.69 \pm 180^\circ)$   
 $\therefore \theta = 32.3^\circ, 197.7^\circ$

b)  $2\cot(x - 30^\circ) = 3$   
 $\Rightarrow \cot(x - 30^\circ) = \frac{3}{2}$   
 $\Rightarrow \tan(x - 30^\circ) = \frac{2}{3}$   
 $\Rightarrow \arctan\left(\frac{2}{3}\right) = 33.7^\circ$   
 $(x - 30 = 33.7 \pm 180^\circ)$   
 $(x = 63.7 \pm 180^\circ)$   
 $x = 63.7^\circ, 243.7^\circ$

c)  $5 - 2\operatorname{cosec} 3y = 9$   
 $\Rightarrow -2\operatorname{cosec} 3y = 4$   
 $\Rightarrow \operatorname{cosec} 3y = -2$   
 $\Rightarrow \sin 3y = -\frac{1}{2}$   
 $(3y = \frac{7\pi}{6} \pm 2\pi)$   
 $(3y = \frac{11\pi}{6} \pm 2\pi)$   
 $\therefore y = \frac{7\pi}{18}, \frac{11\pi}{18}$

d)  $27\cos \phi = \sec^2 \phi$   
 $\Rightarrow 27(\cos \phi) = \frac{1}{\cos^2 \phi}$   
 $\Rightarrow 27\cos^3 \phi = 1$   
 $\Rightarrow \cos^3 \phi = \frac{1}{27}$   
 $\Rightarrow \cos \phi = \frac{1}{3}$   
 $\arccos\left(\frac{1}{3}\right) = 1.23^\circ$   
 $(\phi = 1.23 \pm 2\pi)$   
 $(\phi = 5.05 \pm 2\pi)$   
 $\phi = 1.23^\circ, 5.05^\circ$

**Question 4**

Solve each of the following trigonometric equations.

a)  $2\sec\theta - 1 = 9, \quad 0 \leq \theta < 360^\circ$

b)  $2 + 3\cot(x - 20^\circ) = 8, \quad 0 \leq x < 360^\circ$

c)  $14 - 3\operatorname{cosec} 2y = 5, \quad 0 \leq y < \pi$

d)  $4\sin^3\varphi + \frac{1}{8}\operatorname{cosec}^2\varphi = 0, \quad 0 \leq \varphi < 2\pi$

$\theta \approx 78.5^\circ, 281.5^\circ, \quad x \approx 46.6^\circ, 226.6^\circ, \quad y \approx 0.170\pi, 1.40\pi, \quad \varphi = \frac{7\pi}{6}, \frac{11\pi}{6}$

Handwritten solutions for the four trigonometric equations:

(a)  $2\sec\theta - 1 = 9$   
 $\Rightarrow 2\sec\theta = 10$   
 $\Rightarrow \sec\theta = 5$   
 $\Rightarrow \operatorname{cosec}\theta = \frac{1}{5}$   
 $\bullet \operatorname{arccsc}\left(\frac{1}{5}\right) = 78.5^\circ$   
 $\theta_1 = 78.5^\circ \pm 360^\circ$   
 $\theta_2 = 281.5^\circ \pm 360^\circ$   
 $\theta_1 = 78.5^\circ$   
 $\theta_2 = 281.5^\circ$

(b)  $2 + 3\cot(x - 20^\circ) = 8$   
 $\Rightarrow 3\cot(x - 20^\circ) = 6$   
 $\Rightarrow \cot(x - 20^\circ) = 2$   
 $\Rightarrow \tan(x - 20^\circ) = \frac{1}{2}$   
 $\bullet \operatorname{arctan}\left(\frac{1}{2}\right) = 26.6^\circ$   
 $(x - 20^\circ) = 26.6^\circ \pm 180^\circ$   
 $(x = 46.6^\circ \pm 180^\circ)$   
 $\therefore x = 46.6^\circ, 226.6^\circ$

(c)  $14 - 3\operatorname{cosec} 2y = 5$   
 $\Rightarrow 9 = 3\operatorname{cosec} 2y$   
 $\Rightarrow 3 = \operatorname{cosec} 2y$   
 $\Rightarrow \sin 2y = \frac{1}{3}$   
 $\operatorname{arcsin}\left(\frac{1}{3}\right) = 0.345$   
 $(2y = 0.345 \pm 2\pi)$   
 $(2y = 2\pi \pm 0.345)$   
 $(y = 0.170 \pm \pi)$   
 $(y = 1.40 \pm \pi)$   
 $\therefore y_1 = 0.170\pi$   
 $y_2 = 1.40\pi$

(d)  $4\sin^3\varphi + \frac{1}{8}\operatorname{cosec}^2\varphi = 0$   
 $\Rightarrow 4\sin^3\varphi + \frac{1}{8\sin^2\varphi} = 0$   
 $\Rightarrow 32\sin^5\varphi + 1 = 0$   
 $\Rightarrow \sin^5\varphi = -\frac{1}{32}$   
 $\Rightarrow \sin\varphi = -\frac{1}{2}$   
 $\bullet \operatorname{arcsin}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$   
 $\Rightarrow \left(\frac{\varphi}{6} < \frac{\pi}{6} \pm 2\pi\right)$   
 $\varphi = \frac{11\pi}{6} \pm 2\pi$   
 $\varphi_1 = \frac{11\pi}{6}$   
 $\varphi_2 = \frac{7\pi}{6}$

Question 5

Solve each of the following trigonometric equations.

- a)  $2\sec\theta - 1 = 2\sec\theta\sin^2\theta$ ,  $0 \leq \theta < 180^\circ$ ,  $\theta \neq 90^\circ$
- b)  $\cos x \cot x + \sin x + 2\cot x = 0$ ,  $0 < x < 360^\circ$ ,  $x \neq 180^\circ$
- c)  $(\operatorname{cosec} y - \sin y)\sec^2 y = 2$ ,  $0 \leq y < \pi$ ,  $y \neq \frac{\pi}{2}$
- d)  $\operatorname{cosec}\phi - \sin\phi + 2\cos^2\phi \cot\phi = 0$ ,  $0 < \phi < 2\pi$ ,  $\phi \neq \pi$

$$\theta = 60^\circ, \quad x = 120^\circ, 240^\circ, \quad y = \frac{\pi}{6}, \frac{5\pi}{6}, \quad \phi = \frac{\pi}{2}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{3\pi}{2}$$

The image shows handwritten solutions for the four trigonometric equations. The solutions are written on grid paper and include the following steps:

- Question a:**  $2\sec\theta - 1 = 2\sec\theta\sin^2\theta$ . The solution involves dividing by  $\sec\theta$  (noting  $\sec\theta \neq 0$ ), leading to  $2 - \cos\theta = 2\sin^2\theta$ . This is then transformed into a quadratic in  $\cos\theta$ :  $2 - \cos\theta = 2(1 - \cos^2\theta)$ , which simplifies to  $2\cos^2\theta - \cos\theta - 2 = 0$ . Solving this quadratic gives  $\cos\theta = \frac{1}{2}$ , leading to  $\theta = 60^\circ$  (since  $0 \leq \theta < 180^\circ$ ).
- Question b:**  $\cos x \cot x + \sin x + 2\cot x = 0$ . The solution involves multiplying through by  $\sin x$  to get  $\cos x + \sin^2 x + 2\cos x = 0$ . This simplifies to  $3\cos x + \sin^2 x = 0$ . Using the identity  $\sin^2 x = 1 - \cos^2 x$ , it becomes  $-\cos^2 x + 3\cos x + 1 = 0$ . Solving this quadratic in  $\cos x$  gives  $\cos x = \frac{1}{2}$  or  $\cos x = -1$ . The solutions are  $x = 120^\circ, 240^\circ$  and  $x = 180^\circ$  (which is excluded).
- Question c:**  $(\operatorname{cosec} y - \sin y)\sec^2 y = 2$ . The solution involves multiplying through by  $\sin^2 y$  to get  $(1 - \sin^2 y)\sec^2 y = 2\sin^2 y$ . This simplifies to  $\cos^2 y \sec^2 y = 2\sin^2 y$ , which is  $1 = 2\sin^2 y$ . Thus,  $\sin y = \frac{1}{\sqrt{2}}$ , leading to  $y = \frac{\pi}{4}, \frac{3\pi}{4}$ .
- Question d:**  $\operatorname{cosec}\phi - \sin\phi + 2\cos^2\phi \cot\phi = 0$ . The solution involves multiplying through by  $\sin\phi$  to get  $1 - \sin^2\phi + 2\cos^2\phi \cos\phi = 0$ . This simplifies to  $\cos^2\phi + 2\cos^3\phi = 0$ . Factoring out  $\cos^2\phi$  gives  $\cos^2\phi(1 + 2\cos\phi) = 0$ . The solutions are  $\cos\phi = 0$  ( $\phi = \frac{\pi}{2}, \frac{3\pi}{2}$ ) and  $\cos\phi = -\frac{1}{2}$  ( $\phi = \frac{2\pi}{3}, \frac{4\pi}{3}$ ).

Question 6

Solve each of the following trigonometric equations.

a)  $\sec \theta + \cos \theta = \frac{5}{2}, \quad 0 \leq \theta < 360^\circ, \quad \theta \neq 90^\circ, 270^\circ$

b)  $\sec x - \cos x = 8(\operatorname{cosec} x - \sin x), \quad 0 \leq x < 360^\circ, \quad x \neq 90^\circ$

c)  $2 \cot y - 3 \operatorname{cosec} y = 2 \sec y \operatorname{cosec} y, \quad 0 < y < 2\pi, \quad y \neq \frac{k\pi}{2}, k \in \mathbb{Z}$

d)  $(1 + \sec \phi)(1 - \cos \phi) = \tan \phi, \quad 0 \leq \phi < 2\pi, \quad \phi \neq \frac{\pi}{2}, \frac{3\pi}{2}$

e)  $\frac{\operatorname{cosec}^2 \psi \tan^2 \psi}{\cos \psi} + 8 = 0, \quad 0 \leq \psi < 360^\circ, \quad \psi \neq 90^\circ, 270^\circ$

$\theta = 60^\circ, 300^\circ, \quad x = 63.4^\circ, 243.6^\circ, \quad y = \frac{2\pi}{3}, \frac{4\pi}{3}, \quad \phi = 0, \pi, \quad \psi = 120^\circ, 240^\circ$

The image shows handwritten solutions for the five trigonometric equations. The solutions are written on grid paper and include the following steps:

- Question a:**  $\sec \theta + \cos \theta = \frac{5}{2}$ . Multiply by  $\cos \theta$  to get  $1 + \cos^2 \theta = \frac{5}{2} \cos \theta$ . Let  $u = \cos \theta$ , then  $2u^2 - 5u + 2 = 0$ . Factor to  $(2u-1)(u-2) = 0$ . Solutions:  $\cos \theta = \frac{1}{2}$  or  $\cos \theta = 2$  (invalid).  $\theta = 60^\circ, 300^\circ$ .
- Question b:**  $\sec x - \cos x = 8(\operatorname{cosec} x - \sin x)$ . Multiply by  $\sin x$  to get  $1 - \cos^2 x = 8(1 - \sin^2 x)$ . Let  $u = \sin x$ , then  $1 - (1-u^2) = 8(1-u^2)$ . Simplify to  $u^2 = 2$ . Solutions:  $\sin x = \pm \sqrt{2}$  (invalid) or  $\sin x = \pm \frac{1}{\sqrt{2}}$ .  $x = 45^\circ, 135^\circ, 225^\circ, 315^\circ$ .
- Question c:**  $2 \cot y - 3 \operatorname{cosec} y = 2 \sec y \operatorname{cosec} y$ . Multiply by  $\sin y$  to get  $2 \cos y - 3 = 2 \sin y$ . Let  $u = \tan y$ , then  $2 - 3 \sin y = 2 \sin y$ .  $5 \sin y = 2$ .  $\sin y = \frac{2}{5}$ . Solutions:  $y = \arcsin(\frac{2}{5}), \pi - \arcsin(\frac{2}{5})$ .
- Question d:**  $(1 + \sec \phi)(1 - \cos \phi) = \tan \phi$ . Simplify to  $1 - \cos^2 \phi + \sec \phi - \cos \phi = \frac{\sin \phi}{\cos \phi}$ . Let  $u = \cos \phi$ , then  $1 - u^2 + \frac{1}{u} - u = \frac{\sqrt{1-u^2}}{u}$ . Multiply by  $u$  to get  $u - u^3 + 1 - u^2 = \sqrt{1-u^2}$ . Square both sides to get a polynomial equation. Solutions:  $\phi = 0, \pi$ .
- Question e:**  $\frac{\operatorname{cosec}^2 \psi \tan^2 \psi}{\cos \psi} + 8 = 0$ . Multiply by  $\cos \psi$  to get  $\frac{\tan^2 \psi}{\cos \psi} + 8 = 0$ . Let  $u = \tan \psi$ , then  $\frac{u^2}{\sqrt{1-u^2}} + 8 = 0$ . Square both sides to get  $u^4 + 16u^2 + 8 = 0$ . Let  $v = u^2$ , then  $v^2 + 16v + 8 = 0$ . Solutions:  $v = -8 \pm \sqrt{64-8} = -8 \pm \sqrt{56}$ .  $\psi = 120^\circ, 240^\circ$ .

**Question 7 (hard questions)**

Solve each of the following trigonometric equations.

a)  $2 \sin \theta + 3 \sec \theta = 6 + \tan \theta, \quad 0 \leq \theta < 2\pi, \quad \theta \neq \frac{\pi}{2}, \frac{3\pi}{2}$

b)  $\sin^2 x \tan x + \cos^2 x \cot x + 2 \sin x \cos x = 2, \quad 0 < x < 360^\circ, \quad x \neq 90^\circ, 180^\circ, 270^\circ$

you may use in this part the fact that  $2 \sin x \cos x \equiv \sin 2x$

c)  $\sin y(1 + \tan y) + \cos y(1 + \cot y) = 0, \quad 0 < y < 360^\circ, \quad y \neq 90^\circ, 180^\circ, 270^\circ$

d)  $\frac{4}{2 \sec \phi - 2 \sin \phi + 1} = \cot \phi, \quad 0 < \phi < 2\pi, \quad \phi \neq \pi$

e)  $\frac{\cot \psi}{\operatorname{cosec} \psi - 1} - \frac{\cos \psi}{1 + \sin \psi} = 2, \quad 0 < \psi < 2\pi, \quad \psi \neq \frac{\pi}{2}, \pi, \frac{3\pi}{2}$

f)  $\frac{\cot \beta}{\operatorname{cosec} \beta - 1} + \frac{\operatorname{cosec} \beta - 1}{\cot \beta} = 4, \quad 0 \leq \beta < 360^\circ, \quad \beta \neq 90^\circ, 180^\circ, 270^\circ$

$\phi = \frac{\pi}{3}, \frac{5\pi}{3}, \quad x = 45^\circ, 225^\circ, \quad y = 135^\circ, 315^\circ, \quad \psi = \frac{\pi}{6}, \frac{5\pi}{6}, \quad \beta = 60^\circ, 300^\circ$

Handwritten solution for question a):

$$2 \sin \theta + 3 \sec \theta = 6 + \tan \theta$$

$$\Rightarrow 2 \sin \theta + \frac{3}{\cos \theta} = 6 + \frac{\sin \theta}{\cos \theta}$$

$$\Rightarrow 2 \sin \theta \cos \theta + 3 = 6 \cos \theta + \sin \theta$$

$$\Rightarrow 2 \sin \theta \cos \theta - \sin \theta = 6 \cos \theta - 3$$

$$\Rightarrow \sin \theta (2 \cos \theta - 1) = 3(2 \cos \theta - 1)$$

$$\Rightarrow \sin \theta (2 \cos \theta - 1) - 3(2 \cos \theta - 1) = 0$$

$$\Rightarrow (2 \cos \theta - 1)(\sin \theta - 3) = 0$$

Since  $\sin \theta \leq 1$ ,  $\cos \theta = \frac{1}{2}$   
 $\theta = \frac{\pi}{3} \pm 2\pi n$  or  $\theta = \frac{5\pi}{3} \pm 2\pi n$  for  $n \in \mathbb{Z}$

Handwritten solution for question b):

$$\frac{4}{2 \sec \phi - 2 \sin \phi + 1} = \cot \phi$$

$$\Rightarrow 4 = \frac{\cos \phi}{\sin \phi} (2 \sec \phi - 2 \sin \phi + 1)$$

$$\Rightarrow 4 = \frac{\cos \phi}{\sin \phi} \left( \frac{2}{\cos \phi} - 2 \sin \phi + 1 \right)$$

$$\Rightarrow 4 = \frac{2 - 2 \sin^2 \phi + \cos \phi}{\sin \phi}$$

$$\Rightarrow 4 \sin \phi = 2 - 2 \sin^2 \phi + \cos \phi$$

$$\Rightarrow 2 \sin^2 \phi - \cos \phi + 4 \sin \phi - 2 = 0$$

$$\Rightarrow 2 \sin^2 \phi - (2 \cos \phi) + 4 \sin \phi - 2 = 0$$

$$\Rightarrow (2 \cos \phi - 2) + 4 \sin \phi - 2 = 0$$

$$\Rightarrow 2(\cos \phi - 1) + 4 \sin \phi - 2 = 0$$

$$\Rightarrow 2 \cos \phi - 2 + 4 \sin \phi - 2 = 0$$

$$\Rightarrow 2 \cos \phi + 4 \sin \phi - 4 = 0$$

$$\Rightarrow \cos \phi + 2 \sin \phi - 2 = 0$$

$$\Rightarrow \cos \phi = 2 - 2 \sin \phi$$

$$\Rightarrow \cos^2 \phi = (2 - 2 \sin \phi)^2$$

$$\Rightarrow 1 - \sin^2 \phi = 4 - 8 \sin \phi + 4 \sin^2 \phi$$

$$\Rightarrow 3 \sin^2 \phi - 8 \sin \phi + 3 = 0$$

$$\Rightarrow \sin \phi = \frac{8 \pm \sqrt{64 - 36}}{6} = \frac{8 \pm 2\sqrt{10}}{6} = \frac{4 \pm \sqrt{10}}{3}$$

Handwritten solution for question c):

$$\frac{\cot \beta}{\operatorname{cosec} \beta - 1} + \frac{\operatorname{cosec} \beta - 1}{\cot \beta} = 4$$

$$\Rightarrow \frac{\cos \beta}{\sin \beta} \frac{1}{\frac{1}{\sin \beta} - 1} + \frac{\frac{1}{\sin \beta} - 1}{\frac{\cos \beta}{\sin \beta}} = 4$$

$$\Rightarrow \frac{\cos \beta}{\sin \beta} \frac{\sin \beta}{1 - \sin \beta} + \frac{1 - \sin \beta}{\cos \beta} = 4$$

$$\Rightarrow \frac{\cos \beta}{1 - \sin \beta} + \frac{1 - \sin \beta}{\cos \beta} = 4$$

$$\Rightarrow \frac{\cos^2 \beta + (1 - \sin \beta)^2}{(1 - \sin \beta) \cos \beta} = 4$$

$$\Rightarrow \frac{\cos^2 \beta + 1 - 2 \sin \beta + \sin^2 \beta}{(1 - \sin \beta) \cos \beta} = 4$$

$$\Rightarrow \frac{2 - 2 \sin \beta}{(1 - \sin \beta) \cos \beta} = 4$$

$$\Rightarrow \frac{2(1 - \sin \beta)}{(1 - \sin \beta) \cos \beta} = 4$$

$$\Rightarrow \frac{2}{\cos \beta} = 4$$

$$\Rightarrow \cos \beta = \frac{1}{2}$$

$$\Rightarrow \beta = 60^\circ, 300^\circ$$



**Question 8**

Solve each of the following equations.

a)  $2 \tan^2 \theta = 11 \sec \theta - 7, \quad 0 \leq \theta < 360^\circ$

b)  $4 \cot^2 x - 9 \operatorname{cosec} x + 6 = 0, \quad 0 \leq x < 360^\circ$

c)  $\sec^2 y + \tan y = 3, \quad 0 \leq y < 360^\circ$

d)  $2 \operatorname{cosec}^2 \phi + \cot^2 \phi = 11, \quad 0 \leq \phi < 360^\circ$

$\theta = 78.5^\circ, 281.5^\circ, \quad x = 30^\circ, 150^\circ, \quad y = 45^\circ, 225^\circ \quad y \approx 116.6^\circ, 296.6^\circ,$

$\phi = 30^\circ, 150^\circ, 210^\circ, 330^\circ$

The image shows handwritten solutions for the four parts of Question 8. The solutions are as follows:

- (a)**  $2 \tan^2 \theta = 11 \sec \theta - 7$   
 $\Rightarrow 2(\sec^2 \theta - 1) = 11 \sec \theta - 7$   
 $\Rightarrow 2 \sec^2 \theta - 11 \sec \theta + 5 = 0$   
 $\Rightarrow (2 \sec \theta - 1)(\sec \theta - 5) = 0$   
 $\Rightarrow \sec \theta = \frac{1}{2}$  or  $\sec \theta = 5$   
 $\cos \theta = \frac{1}{2}$  or  $\cos \theta = \frac{1}{5}$   
 $\theta = 78.5^\circ, 281.5^\circ$
- (b)**  $4 \cot^2 x - 9 \operatorname{cosec} x + 6 = 0$   
 $\Rightarrow 4(\frac{1}{\tan^2 x}) - 9 \operatorname{cosec} x + 6 = 0$   
 $\Rightarrow 4(\frac{1}{\sin^2 x}) - 9 \operatorname{cosec} x + 6 = 0$   
 $\Rightarrow 4 \operatorname{cosec}^2 x - 9 \operatorname{cosec} x + 6 = 0$   
 $\Rightarrow (4 \operatorname{cosec} x - 2)(\operatorname{cosec} x - 3) = 0$   
 $\Rightarrow \operatorname{cosec} x = \frac{1}{2}$  or  $\operatorname{cosec} x = 3$   
 $\sin x = 2$  or  $\sin x = \frac{1}{3}$   
 $x = 30^\circ, 150^\circ$
- (c)**  $\sec^2 y + \tan y = 3$   
 $\Rightarrow (1 + \tan^2 y) + \tan y = 3$   
 $\Rightarrow \tan^2 y + \tan y - 2 = 0$   
 $\Rightarrow (\tan y - 1)(\tan y + 2) = 0$   
 $\tan y = 1$  or  $\tan y = -2$   
 $y = 45^\circ, 225^\circ$   
 $y = 116.6^\circ, 296.6^\circ$
- (d)**  $2 \operatorname{cosec}^2 \phi + \cot^2 \phi = 11$   
 $\Rightarrow 2(1 + \cot^2 \phi) + \cot^2 \phi = 11$   
 $\Rightarrow 2 + 2 \cot^2 \phi + \cot^2 \phi = 11$   
 $\Rightarrow 3 \cot^2 \phi = 9$   
 $\cot^2 \phi = 3$   
 $\cot \phi = \pm \sqrt{3}$   
 $\tan \phi = \pm \frac{1}{\sqrt{3}}$   
 $\phi = 30^\circ, 150^\circ, 210^\circ, 330^\circ$

**Question 9**

Solve each of the following equations.

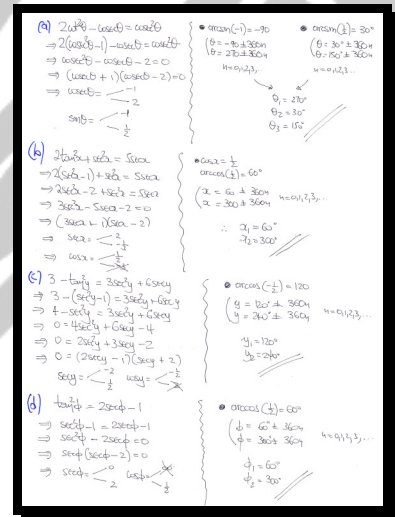
a)  $2 \cot^2 \theta - \operatorname{cosec} \theta = \operatorname{cosec}^2 \theta, \quad 0 \leq \theta < 360^\circ$

b)  $2 \tan^2 x + \sec^2 x = 5 \sec x, \quad 0 \leq x < 360^\circ$

c)  $3 - \tan^2 y = 3 \sec^2 y + 6 \sec y, \quad 0 \leq y < 360^\circ$

d)  $\tan^2 \phi = 2 \sec \phi - 1, \quad 0 \leq \phi < 360^\circ$

$\theta = 30^\circ, 150^\circ, 270^\circ, \quad x = 60^\circ, 300^\circ, \quad y = 120^\circ, 240^\circ, \quad \phi = 60^\circ, 300^\circ$



**Question 10**

Solve each of the following equations.

a)  $2 \cot^2 \theta + 6 = 9 \operatorname{cosec} \theta$ ,  $0 \leq \theta < 360^\circ$

b)  $5 \tan^2 x + 16 \sec x + 8 = 0$ ,  $0 \leq x < 360^\circ$

c)  $\operatorname{cosec} y + 5 \operatorname{cosec}^2 y = 6 \cot^2 y$ ,  $0 \leq y < 360^\circ$

d)  $2 \tan^2 \phi = 15 \sec \phi - 9$ ,  $0 \leq \phi < 360^\circ$

$\theta \approx 14.5^\circ, 165.5^\circ$ ,  $x \approx 109.5^\circ, 250.5^\circ$ ,  $y \approx 19.5^\circ, 160.5^\circ$   $y = 210^\circ, 330^\circ$ ,  
 $\phi \approx 81.8^\circ, 278.2^\circ$

(a)  $2 \cot^2 \theta + 6 = 9 \operatorname{cosec} \theta$   
 $\Rightarrow 2(\operatorname{cosec}^2 \theta - 1) + 6 = 9 \operatorname{cosec} \theta$   
 $\Rightarrow 2 \operatorname{cosec}^2 \theta - 2 + 6 = 9 \operatorname{cosec} \theta$   
 $\Rightarrow 2 \operatorname{cosec}^2 \theta - 9 \operatorname{cosec} \theta + 4 = 0$   
 $\Rightarrow (\operatorname{cosec} \theta - 4)(2 \operatorname{cosec} \theta - 1) = 0$   
 $\Rightarrow \operatorname{cosec} \theta = \frac{4}{1}$   
 $\Rightarrow \sin \theta = \frac{1}{4}$   
 $\Rightarrow \theta = \arcsin\left(\frac{1}{4}\right) = 14.48^\circ$   
 $\Rightarrow \theta = 14.48^\circ \pm 360^\circ$   
 $\Rightarrow \theta = 185.52^\circ \pm 360^\circ$   
 $\Rightarrow \theta = 14.5^\circ, 165.5^\circ$

(b)  $5 \tan^2 x + 16 \sec x + 8 = 0$   
 $\Rightarrow 5(\sec^2 x - 1) + 16 \sec x + 8 = 0$   
 $\Rightarrow 5 \sec^2 x - 5 + 16 \sec x + 8 = 0$   
 $\Rightarrow 5 \sec^2 x + 16 \sec x + 3 = 0$   
 $\Rightarrow (5 \sec x + 7)(\sec x + \frac{3}{5}) = 0$   
 $\Rightarrow \sec x = -\frac{7}{5}$   
 $\Rightarrow \cos x = -\frac{5}{7}$   
 $\Rightarrow x = \arccos\left(-\frac{5}{7}\right) = 109.47^\circ$   
 $\Rightarrow x = 109.47^\circ \pm 360^\circ$   
 $\Rightarrow x = 250.53^\circ \pm 360^\circ$   
 $\Rightarrow x = 109.5^\circ, 250.5^\circ$

(c)  $\operatorname{cosec} y + 5 \operatorname{cosec}^2 y = 6 \cot^2 y$   
 $\Rightarrow \operatorname{cosec} y + 5 \operatorname{cosec}^2 y = 6(\operatorname{cosec}^2 y - 1)$   
 $\Rightarrow \operatorname{cosec} y + 5 \operatorname{cosec}^2 y = 6 \operatorname{cosec}^2 y - 6$   
 $\Rightarrow 0 = \operatorname{cosec}^2 y - \operatorname{cosec} y - 6$   
 $\Rightarrow (\operatorname{cosec} y + 2)(\operatorname{cosec} y - 3) = 0$   
 $\Rightarrow \operatorname{cosec} y = \frac{3}{1}$   
 $\Rightarrow \sin y = \frac{1}{3}$   
 $\Rightarrow y = \arcsin\left(\frac{1}{3}\right) = 19.47^\circ$   
 $\Rightarrow y = 19.47^\circ \pm 360^\circ$   
 $\Rightarrow y = 160.53^\circ \pm 360^\circ$   
 $\Rightarrow y = 19.5^\circ, 160.5^\circ$

(d)  $2 \tan^2 \phi = 15 \sec \phi - 9$   
 $\Rightarrow 2(\sec^2 \phi - 1) = 15 \sec \phi - 9$   
 $\Rightarrow 2 \sec^2 \phi - 2 = 15 \sec \phi - 9$   
 $\Rightarrow 2 \sec^2 \phi - 15 \sec \phi + 7 = 0$   
 $\Rightarrow (2 \sec \phi - 7)(\sec \phi - 1) = 0$   
 $\Rightarrow \sec \phi = \frac{7}{2}$   
 $\Rightarrow \cos \phi = \frac{2}{7}$   
 $\Rightarrow \phi = \arccos\left(\frac{2}{7}\right) = 81.8^\circ$   
 $\Rightarrow \phi = 81.8^\circ \pm 360^\circ$   
 $\Rightarrow \phi = 278.2^\circ \pm 360^\circ$   
 $\Rightarrow \phi = 81.8^\circ, 278.2^\circ$

(d)  $2 \tan^2 \phi = 15 \sec \phi - 9$   
 $\Rightarrow 2(\sec^2 \phi - 1) = 15 \sec \phi - 9$   
 $\Rightarrow 2 \sec^2 \phi - 2 = 15 \sec \phi - 9$   
 $\Rightarrow 2 \sec^2 \phi - 15 \sec \phi + 7 = 0$   
 $\Rightarrow (2 \sec \phi - 7)(\sec \phi - 1) = 0$   
 $\Rightarrow \sec \phi = \frac{7}{2}$   
 $\Rightarrow \cos \phi = \frac{2}{7}$   
 $\therefore \sec \phi = \frac{7}{2}$   
 $\operatorname{arccos}\left(\frac{2}{7}\right) = 81.8^\circ$   
 $\phi = 81.8^\circ \pm 360^\circ$   
 $\phi = 278.2^\circ \pm 360^\circ$   
 $\phi = 81.8^\circ, 278.2^\circ$

**Question 11**

Solve each of the following equations.

a)  $4 \cot^2 \theta = 1 + \operatorname{cosec} \theta$ ,  $0 \leq \theta < 360^\circ$

b)  $4 \tan^2 x = 19 \sec x + 1$ ,  $0 \leq x < 360^\circ$

c)  $4 - \cot^2 y = 8 \operatorname{cosec} y + 3 \operatorname{cosec}^2 y$ ,  $0 \leq y < 360^\circ$

d)  $\sec^2 \phi = 2 \tan \phi$ ,  $0 \leq \phi < 360^\circ$

$\theta \approx 53.1^\circ, 126.9^\circ$   $\theta = 270^\circ$ ,  $x \approx 78.5^\circ, 281.5^\circ$ ,  $y \approx 203.6^\circ, 336.4^\circ$ ,

$\phi = 45^\circ, 225^\circ$

Handwritten solution for Question 11:

(a)  $4 \cot^2 \theta = 1 + \operatorname{cosec} \theta$   
 $\Rightarrow 4(\operatorname{cosec}^2 \theta - 1) = 1 + \operatorname{cosec} \theta$   
 $\Rightarrow 4 \operatorname{cosec}^2 \theta - 4 = 1 + \operatorname{cosec} \theta$   
 $\Rightarrow 4 \operatorname{cosec}^2 \theta - \operatorname{cosec} \theta - 5 = 0$   
 $\Rightarrow (4 \operatorname{cosec} \theta - 5)(\operatorname{cosec} \theta + 1) = 0$   
 $\Rightarrow \operatorname{cosec} \theta = \frac{5}{4}$  or  $\operatorname{cosec} \theta = -1$   
 $\Rightarrow \sin \theta = \frac{4}{5}$  or  $\sin \theta = -1$   
 $\bullet \operatorname{arcsin}(\frac{4}{5}) = 53.1^\circ$   $\bullet \operatorname{arcsin}(-1) = -90^\circ$   
 $(\theta = 53.1^\circ \pm 360^\circ)$   $(\theta = -90^\circ \pm 360^\circ)$   
 $(\theta = 126.9^\circ \pm 360^\circ)$   $(\theta = 270^\circ \pm 360^\circ)$   
 $\theta_1 = 53.1^\circ$   
 $\theta_2 = 126.9^\circ$   
 $\theta_3 = 270^\circ$

(b)  $4 \tan^2 x = 19 \sec x + 1$   
 $\Rightarrow 4(\sec^2 x - 1) = 19 \sec x + 1$   
 $\Rightarrow 4 \sec^2 x - 4 = 19 \sec x + 1$   
 $\Rightarrow 4 \sec^2 x - 19 \sec x - 5 = 0$   
 $\Rightarrow (4 \sec x + 1)(\sec x - 5) = 0$   
 $\Rightarrow \sec x = -\frac{1}{4}$  or  $\sec x = 5$   
 $\Rightarrow \cos x = -\frac{1}{4}$  or  $\cos x = \frac{1}{5}$   
 $\bullet \operatorname{arccos}(\frac{1}{5}) = 78.5^\circ$   
 $(x = 78.5^\circ \pm 360^\circ)$   
 $(x = 281.5^\circ \pm 360^\circ)$   
 $x_1 = 78.5^\circ$   
 $x_2 = 281.5^\circ$

(c)  $4 - \cot^2 y = 8 \operatorname{cosec} y + 3 \operatorname{cosec}^2 y$   
 $\Rightarrow 4 - (\operatorname{cosec}^2 y - 1) = 8 \operatorname{cosec} y + 3 \operatorname{cosec}^2 y$   
 $\Rightarrow 4 - \operatorname{cosec}^2 y + 1 = 8 \operatorname{cosec} y + 3 \operatorname{cosec}^2 y$   
 $\Rightarrow 0 = 4 \operatorname{cosec}^2 y + 8 \operatorname{cosec} y - 5$   
 $\Rightarrow 0 = (2 \operatorname{cosec} y - 1)(2 \operatorname{cosec} y + 5)$   
 $\operatorname{cosec} y = \frac{1}{2}$  or  $\operatorname{cosec} y = -\frac{5}{2}$   
 $\sin y = \frac{2}{1}$  or  $\sin y = -\frac{2}{5}$   
 $\bullet \operatorname{arcsin}(\frac{2}{1}) = 90^\circ$   $\bullet \operatorname{arcsin}(-\frac{2}{5}) = -23.7^\circ$   
 $(y = 90^\circ \pm 360^\circ)$   $(y = -23.7^\circ \pm 360^\circ)$   
 $(y = 236.3^\circ \pm 360^\circ)$   $(y = 336.3^\circ \pm 360^\circ)$   
 $y_1 = 90^\circ$   
 $y_2 = 236.3^\circ$   
 $y_3 = 336.3^\circ$

(d)  $\sec^2 \phi = 2 \tan \phi$   
 $\Rightarrow 1 + \tan^2 \phi = 2 \tan \phi$   
 $\Rightarrow \tan^2 \phi - 2 \tan \phi + 1 = 0$   
 $\Rightarrow (\tan \phi - 1)^2 = 0$   
 $\tan \phi = 1$   
 $\bullet \operatorname{arctan}(1) = 45^\circ$   
 $(\phi = 45^\circ \pm 180^\circ)$   $(\phi = 225^\circ \pm 360^\circ)$   
 $\phi_1 = 45^\circ$   
 $\phi_2 = 225^\circ$

Handwritten solution for Question 11 (d):

(d)  $\sec^2 \phi = 2 \tan \phi$   
 $\Rightarrow 1 + \tan^2 \phi = 2 \tan \phi$   
 $\Rightarrow \tan^2 \phi - 2 \tan \phi + 1 = 0$   
 $\Rightarrow (\tan \phi - 1)^2 = 0$   
 $\tan \phi = 1$   
 $\bullet \operatorname{arctan}(1) = 45^\circ$   
 $(\phi = 45^\circ \pm 180^\circ)$   $(\phi = 225^\circ \pm 360^\circ)$   
 $\phi_1 = 45^\circ$   
 $\phi_2 = 225^\circ$

**Question 12**

Solve each of the following equations.

a)  $2 \tan^2 \theta + 4 \tan \theta + 5 = \sec^2 \theta$ ,  $0 \leq \theta < 360^\circ$

b)  $2 \sec^2 x + 2 \tan^2 x = 1 + 4 \sec x$ ,  $0 \leq x < 360^\circ$

c)  $6 \cot^2 y + 3 \operatorname{cosec}^2 y = 2 + 6 \cot y$ ,  $0 \leq y < 2\pi$

d)  $4 \operatorname{cosec}^2 \phi + \cot^2 \phi = 1 - 9 \operatorname{cosec} \phi$ ,  $0 \leq \phi < 2\pi$

$\theta \approx 116.6^\circ, 296.6^\circ$ ,  $x \approx 48.2^\circ, 311.8^\circ$ ,  $y \approx 1.25^c, 4.39^c$ ,  $\phi = \frac{7\pi}{6}, \frac{11\pi}{6}$

Handwritten solution for Question 12:

(a)  $2 \tan^2 \theta + 4 \tan \theta + 5 = \sec^2 \theta$   
 $\Rightarrow 2 \tan^2 \theta + 4 \tan \theta + 5 = 1 + \tan^2 \theta$   
 $\Rightarrow \tan^2 \theta + 4 \tan \theta + 4 = 0$   
 $\Rightarrow (\tan \theta + 2)^2 = 0$   
 $\Rightarrow \tan \theta = -2$   
 $\bullet \arctan(-2) = -63.4^\circ$   
 $\theta = -63.4^\circ + 360^\circ = 296.6^\circ$   
 $\theta_1 = 116.6^\circ$   
 $\theta_2 = 296.6^\circ$

(b)  $2 \sec^2 x + 2 \tan^2 x = 1 + 4 \sec x$   
 $\Rightarrow 2 \sec^2 x + 2(\sec^2 x - 1) = 1 + 4 \sec x$   
 $\Rightarrow 2 \sec^2 x + 2 \sec^2 x - 2 = 1 + 4 \sec x$   
 $4 \sec^2 x - 4 \sec x - 3 = 0$   
 $\Rightarrow (2 \sec x + 1)(2 \sec x - 3) = 0$   
 $\Rightarrow \sec x = -\frac{1}{2}$   
 $\bullet \arccos(-\frac{1}{2}) = 120^\circ$   
 $x = 120^\circ + 360^\circ = 480^\circ$   
 $x = 240^\circ + 360^\circ = 600^\circ$   
 $x = 480^\circ, 600^\circ$

(c)  $6 \cot^2 y + 3 \operatorname{cosec}^2 y = 2 + 6 \cot y$   
 $\Rightarrow 6 \cot^2 y + 3(1 + \cot^2 y) = 2 + 6 \cot y$   
 $\Rightarrow 6 \cot^2 y + 3 + 3 \cot^2 y = 2 + 6 \cot y$   
 $\Rightarrow 9 \cot^2 y - 6 \cot y + 1 = 0$   
 $\Rightarrow (3 \cot y - 1)^2 = 0$   
 $\Rightarrow \cot y = \frac{1}{3}$   
 $\bullet \arctan(\frac{1}{3}) = 18.4^\circ$   
 $y = 1.25^c, 4.39^c$

(d)  $4 \operatorname{cosec}^2 \phi + \cot^2 \phi = 1 - 9 \operatorname{cosec} \phi$   
 $\Rightarrow 4 \operatorname{cosec}^2 \phi + (\operatorname{cosec}^2 \phi - 1) = 1 - 9 \operatorname{cosec} \phi$   
 $\Rightarrow 5 \operatorname{cosec}^2 \phi + \operatorname{cosec}^2 \phi - 2 = 0$   
 $\Rightarrow (5 \operatorname{cosec} \phi - 1)(\operatorname{cosec} \phi + 2) = 0$   
 $\Rightarrow \operatorname{cosec} \phi = \frac{1}{5}$   
 $\bullet \arcsin(\frac{1}{5}) = 11.3^\circ$   
 $\phi = \frac{7\pi}{6}, \frac{11\pi}{6}$

Question 13

Solve each of the following equations.

a)  $10\sec^2 \theta = 11\tan \theta + 16, \quad 0 \leq \theta < 360^\circ$

b)  $\cot^2 x = 7 - 2\operatorname{cosec} x, \quad 0 \leq x < 360^\circ$

c)  $\sec y = 13 - \frac{\tan^2 y + 16}{\sec y}, \quad 0 \leq y < 360^\circ$

d)  $(\operatorname{cosec} \phi + 1)^2 + 2(\cot \phi - 1)^2 = 9 - 4\cot \phi, \quad 0 \leq \phi < 360^\circ$

$\theta \approx 56.3^\circ, 158.2^\circ, 236.3^\circ, 338.2^\circ, \quad x = 30^\circ, 150^\circ, x \approx 194.5^\circ, 344.5^\circ,$

$y \approx 48.2^\circ, 78.5^\circ, 281.5^\circ, 311.8^\circ, \quad \phi \approx 48.6^\circ, 131.4^\circ, \phi = 210^\circ, 330^\circ$

$$\textcircled{a} \quad 10\sec^2 \theta = 11\tan \theta + 16$$

$$\Rightarrow 10(1 + \tan^2 \theta) = 11\tan \theta + 16$$

$$\Rightarrow 10 + 10\tan^2 \theta = 11\tan \theta + 16$$

$$\Rightarrow 10\tan^2 \theta - 11\tan \theta - 6 = 0$$

$$\Rightarrow (5\tan \theta + 2)(2\tan \theta - 3) = 0$$

$$\Rightarrow \tan \theta = \begin{cases} -\frac{2}{5} \\ \frac{3}{2} \end{cases}$$

$\therefore \theta = 56.3^\circ, 158.2^\circ, 236.3^\circ, 338.2^\circ$

$$\textcircled{b} \quad \cot^2 x = 7 - 2\operatorname{cosec} x$$

$$\Rightarrow (\operatorname{cosec} x - 1)^2 = 7 - 2\operatorname{cosec} x$$

$$\Rightarrow \operatorname{cosec}^2 x + 2\operatorname{cosec} x - 8 = 0$$

$$\Rightarrow (\operatorname{cosec} x - 2)(\operatorname{cosec} x + 4) = 0$$

$$\Rightarrow \operatorname{cosec} x = \begin{cases} 2 \\ -4 \end{cases}$$

$$\Rightarrow \sin x = \begin{cases} \frac{1}{2} \\ -\frac{1}{4} \end{cases}$$

$\therefore \sin x = \frac{1}{2} \Rightarrow x = 30^\circ, 150^\circ, 210^\circ, 330^\circ$

$\therefore \sin x = -\frac{1}{4} \Rightarrow x = 194.5^\circ, 344.5^\circ$

$$\textcircled{c} \quad \sec y = 13 - \frac{\tan^2 y + 16}{\sec y}$$

$$\Rightarrow \sec^2 y = 13\sec y - \tan^2 y - 16$$

$$\Rightarrow \sec^2 y = 13\sec y - (\sec^2 y - 1) - 16$$

$$\Rightarrow \sec^2 y = 13\sec y - \sec^2 y + 1 - 16$$

$$\Rightarrow 2\sec^2 y - 13\sec y + 15 = 0$$

$$\Rightarrow (2\sec y - 3)(\sec y - 5) = 0$$

$$\Rightarrow \sec y = \begin{cases} \frac{3}{2} \\ 5 \end{cases}$$

$\therefore \sec y = \frac{3}{2} \Rightarrow y = 48.2^\circ, 311.8^\circ$

$\therefore \sec y = 5 \Rightarrow y = 78.5^\circ, 281.5^\circ$

$$\textcircled{d} \quad (\operatorname{cosec} \phi + 1)^2 + 2(\cot \phi - 1)^2 = 9 - 4\cot \phi$$

$$\Rightarrow \operatorname{cosec}^2 \phi + 2\operatorname{cosec} \phi + 1 + 2(\cot^2 \phi - 2\cot \phi + 1) = 9 - 4\cot \phi$$

$$\Rightarrow \operatorname{cosec}^2 \phi + 2\operatorname{cosec} \phi + 1 + 2\cot^2 \phi - 4\cot \phi + 2 = 9 - 4\cot \phi$$

$$\Rightarrow \operatorname{cosec}^2 \phi + 2\cot^2 \phi + 2\operatorname{cosec} \phi - 6 = 0$$

$$\Rightarrow 3\operatorname{cosec}^2 \phi + 2\operatorname{cosec} \phi - 6 = 0$$

$$\Rightarrow (3\operatorname{cosec} \phi - 4)(\operatorname{cosec} \phi + 2) = 0$$

$$\Rightarrow \operatorname{cosec} \phi = \begin{cases} \frac{4}{3} \\ -2 \end{cases}$$

$$\Rightarrow \sin \phi = \begin{cases} \frac{3}{4} \\ -\frac{1}{2} \end{cases}$$

$\therefore \sin \phi = \frac{3}{4} \Rightarrow \phi = 48.6^\circ, 131.4^\circ$

$\therefore \sin \phi = -\frac{1}{2} \Rightarrow \phi = 210^\circ, 330^\circ$

**Question 14**

Solve each of the following equations.

a)  $3 \tan^2 \theta = 8 \sec \theta, \quad 0 \leq \theta < 2\pi$

b)  $\operatorname{cosec}^2 x = 2 \cot x + 9, \quad 0 \leq x < 2\pi$

c)  $\operatorname{cosec}^2 y + 7(1 + \operatorname{cosec} y) + \cot^2 y = 0, \quad 0 \leq y < 2\pi$

d)  $6 \tan \phi = \frac{2 - 3 \sec^2 \phi}{\tan \phi - 1}, \quad 0 \leq \phi < 2\pi$

$\theta \approx 1.23^\circ, 5.05^\circ, \quad x \approx 0.245^\circ, 2.68^\circ, 3.39^\circ, 5.82^\circ, \quad y \approx 3.67^\circ, 3.87^\circ, 5.55^\circ, 5.76^\circ,$

$\phi \approx 0.322^\circ, 3.46^\circ$

a)  $3 \tan^2 \theta = 8 \sec \theta$   
 $\Rightarrow 3(\sec^2 \theta - 1) = 8 \sec \theta$   
 $\Rightarrow 3 \sec^2 \theta - 3 = 8 \sec \theta$   
 $\Rightarrow 3 \sec^2 \theta - 8 \sec \theta - 3 = 0$   
 $\Rightarrow (3 \sec \theta + 1)(\sec \theta - 3) = 0$   
 $\Rightarrow \sec \theta = \frac{3}{-1}$   
 $\Rightarrow \cos \theta = -\frac{1}{3}$

$\arccos\left(-\frac{1}{3}\right) = 1.231^\circ$   
 $(\theta = 1.231) \pm 2\pi$   
 $(\theta = 5.052) \pm 2\pi$   
 $\theta_1 = 1.23^\circ$   
 $\theta_2 = 5.05^\circ$

b)  $\operatorname{cosec}^2 x = 2 \cot x + 9$   
 $\Rightarrow 1 + \cot^2 x = 2 \cot x + 9$   
 $\Rightarrow \cot^2 x - 2 \cot x - 8 = 0$   
 $\Rightarrow (\cot x + 2)(\cot x - 4) = 0$   
 $\Rightarrow \cot x = \frac{-2}{-4}$   
 $\Rightarrow \tan x = \frac{1}{2}$

$\arctan\left(\frac{1}{2}\right) = 0.464^\circ$   
 $\arctan\left(\frac{1}{2}\right) = 0.245^\circ$   
 $\therefore x = 0.245^\circ \pm \pi$   
 $\therefore x = 0.245^\circ, 3.39^\circ, 2.68^\circ, 5.82^\circ$

c)  $\operatorname{cosec}^2 y + 7(1 + \operatorname{cosec} y) + \cot^2 y = 0$   
 $\Rightarrow \operatorname{cosec}^2 y + 7 + 7 \operatorname{cosec} y + \cot^2 y = 0$   
 $\Rightarrow \operatorname{cosec}^2 y + 7 + 7 \operatorname{cosec} y + \operatorname{cosec}^2 y - 1 = 0$   
 $\Rightarrow 2 \operatorname{cosec}^2 y + 7 \operatorname{cosec} y + 6 = 0$   
 $\Rightarrow (2 \operatorname{cosec} y + 3)(\operatorname{cosec} y + 2) = 0$   
 $\operatorname{cosec} y = \frac{-2}{2}$   
 $\operatorname{cosec} y = -1$   
 $\operatorname{cosec}\left(-\frac{\pi}{2}\right) = -1$   
 $\operatorname{cosec}(y) = -0.75^\circ$

$(y = -\frac{\pi}{2}) \pm 2\pi$   
 $(y = \frac{3\pi}{2}) \pm 2\pi$   
 $\therefore y_1 = 5.76^\circ$   
 $y_2 = 3.67^\circ$   
 $y_3 = 5.55^\circ$   
 $y_4 = 3.87^\circ$

d)  $6 \tan \phi = \frac{2 - 3 \sec^2 \phi}{\tan \phi - 1}$   
 $\Rightarrow 6 \tan^2 \phi - 6 \tan \phi = 2 - 3 \sec^2 \phi$   
 $\Rightarrow 6 \tan^2 \phi - 6 \tan \phi = 2 - 3(1 + \tan^2 \phi)$   
 $\Rightarrow 6 \tan^2 \phi - 6 \tan \phi = 2 - 3 - 3 \tan^2 \phi$   
 $\Rightarrow 9 \tan^2 \phi - 6 \tan \phi + 1 = 0$   
 $\Rightarrow (3 \tan \phi - 1)(3 \tan \phi - 1) = 0$   
 $\tan \phi = \frac{1}{3}$

$\arctan\left(\frac{1}{3}\right) = 0.332^\circ$   
 $\phi = 0.332^\circ \pm \pi$   
 $\therefore \phi_1 = 0.322^\circ$   
 $\phi_2 = 3.46^\circ$

Question 15

Solve each of the following equations.

a)  $5 \tan^2 \theta - 12 \sec \theta + 9 = 0, \quad 0 \leq \theta < 360^\circ$

b)  $4 \cot^2 x - 11 \operatorname{cosec} x + 1 = 0, \quad 0 \leq x < 360^\circ$

c)  $\frac{5 + \tan^2 y}{\sec y} = 9 - \sec y, \quad 0 \leq y < 2\pi$

d)  $\frac{\sec^2 \varphi - 2}{\tan \varphi} = \frac{\tan \varphi - 1}{2}, \quad 0 \leq \varphi < 2\pi$

$\theta = 60^\circ, 300^\circ, \quad x \approx 19.5^\circ, 160.5^\circ, \quad y \approx 1.32^\circ, 4.97^\circ,$

$\varphi \approx 0.785^\circ, 2.03^\circ, 3.93^\circ, 5.18^\circ$

Handwritten solutions for Question 15:

a)  $5 \tan^2 \theta - 12 \sec \theta + 9 = 0$   
 $\Rightarrow 5(\sec^2 \theta - 1) - 12 \sec \theta + 9 = 0$   
 $\Rightarrow 5 \sec^2 \theta - 5 - 12 \sec \theta + 9 = 0$   
 $\Rightarrow 5 \sec^2 \theta - 12 \sec \theta + 4 = 0$   
 $\Rightarrow (5 \sec \theta - 2)(\sec \theta - 2) = 0$   
 $\Rightarrow \sec \theta = \frac{2}{5} \quad \text{or} \quad \sec \theta = 2$   
 $\therefore \operatorname{arccos}(\frac{2}{5}) = 60^\circ$   
 $\Rightarrow \theta = 60^\circ \pm 360^\circ \quad \text{or} \quad \theta = 300^\circ \pm 360^\circ$   
 $\therefore \theta = 60^\circ$   
 $\theta_2 = 300^\circ$

b)  $4 \cot^2 x - 11 \operatorname{cosec} x + 1 = 0$   
 $\Rightarrow 4(\operatorname{cosec}^2 x - 1) - 11 \operatorname{cosec} x + 1 = 0$   
 $\Rightarrow 4 \operatorname{cosec}^2 x - 4 - 11 \operatorname{cosec} x + 1 = 0$   
 $\Rightarrow 4 \operatorname{cosec}^2 x - 11 \operatorname{cosec} x - 3 = 0$   
 $\Rightarrow (4 \operatorname{cosec} x + 1)(\operatorname{cosec} x - 3) = 0$   
 $\Rightarrow \operatorname{cosec} x = \frac{1}{4} \quad \text{or} \quad \operatorname{cosec} x = 3$   
 $\Rightarrow \sin x = \frac{4}{1} \quad \text{or} \quad \sin x = \frac{1}{3}$   
 $\therefore \operatorname{arcsin}(\frac{1}{3}) = 19.47^\circ$   
 $\Rightarrow x = 19.5^\circ \pm 360^\circ \quad \text{or} \quad x = 160.5^\circ \pm 360^\circ$   
 $\therefore x = 19.5^\circ$   
 $x_2 = 160.5^\circ$

c)  $\frac{5 + \tan^2 y}{\sec y} = 9 - \sec y$   
 $\Rightarrow 5 + \tan^2 y = 9 \sec y - \sec^2 y$   
 $\Rightarrow 5 + (\sec^2 y - 1) = 9 \sec y - \sec^2 y$   
 $\Rightarrow 4 + \sec^2 y = 9 \sec y - \sec^2 y$   
 $\Rightarrow 2 \sec^2 y - 9 \sec y + 4 = 0$   
 $\Rightarrow (2 \sec y - 1)(\sec y - 4) = 0$   
 $\Rightarrow \sec y = \frac{1}{2} \quad \text{or} \quad \sec y = 4$   
 $\therefore \operatorname{arccos}(\frac{1}{2}) = 1.107^\circ$   
 $\Rightarrow y = 1.32^\circ \pm 2\pi \quad \text{or} \quad y = 4.97^\circ \pm 2\pi$   
 $\therefore y = 1.32^\circ, 4.97^\circ$

d)  $\frac{\sec^2 \varphi - 2}{\tan \varphi} = \frac{\tan \varphi - 1}{2}$   
 $\Rightarrow 2(\sec^2 \varphi - 2) = \tan^2 \varphi - \tan \varphi$   
 $\Rightarrow 2(\tan^2 \varphi + 1 - 2) = \tan^2 \varphi - \tan \varphi$   
 $\Rightarrow 2 \tan^2 \varphi = \tan^2 \varphi - \tan \varphi$   
 $\Rightarrow \tan^2 \varphi + \tan \varphi - 2 = 0$   
 $\Rightarrow (\tan \varphi - 1)(\tan \varphi + 2) = 0$   
 $\therefore \tan \varphi = 1 \quad \text{or} \quad \tan \varphi = -2$   
 $\operatorname{arctan}(1) = \frac{\pi}{4} = 0.785$   
 $\operatorname{arctan}(-2) = -1.107$   
 $\therefore \varphi = 0.785^\circ \pm \pi$   
 $\varphi = -1.107^\circ \pm \pi$   
 $\therefore \varphi = 0.785^\circ, 3.93^\circ, 2.03^\circ, 5.18^\circ$



Question 16

Solve each of the following equations.

a)  $\sec \theta = \frac{1 - \tan^2 \theta}{4 \sec \theta - 9}, \quad 0 \leq \theta < 2\pi$

b)  $\frac{\sec^2 x + 8}{4 - \tan x} = 3 \tan x, \quad 0 \leq x < 2\pi$

c)  $\frac{1 - 2 \operatorname{cosec}^2 y}{2 \cot y} - 2 = \cot y, \quad 0 \leq y < 2\pi$

d)  $\frac{2 \cot^2 \varphi + 5}{\operatorname{cosec} \varphi} + 2 \operatorname{cosec} \varphi = 13, \quad 0 \leq \varphi < 2\pi$

$\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \quad x \approx 0.983^\circ, 4.12^\circ, \quad y \approx 2.03^\circ, 5.18^\circ, \quad \varphi \approx 0.340^\circ, 2.80^\circ$

Handwritten solutions for Question 16:

a)  $\sec \theta = \frac{1 - \tan^2 \theta}{4 \sec \theta - 9}$   
 $\Rightarrow 4 \sec^2 \theta - 9 \sec \theta = 1 - \tan^2 \theta$   
 $\Rightarrow 4 \sec^2 \theta - 9 \sec \theta = 1 - (\sec^2 \theta - 1)$   
 $\Rightarrow 5 \sec^2 \theta - 9 \sec \theta - 2 = 0$   
 $\Rightarrow (5 \sec \theta + 1)(\sec \theta - 2) = 0$   
 $\Rightarrow \sec \theta = -\frac{1}{5} \text{ or } \sec \theta = 2$   
 $\tan \theta = \frac{3}{4} \text{ or } \frac{4}{3}$   
 $\theta = \frac{\pi}{3} \text{ or } \frac{5\pi}{3}$

b)  $\frac{\sec^2 x + 8}{4 - \tan x} = 3 \tan x$   
 $\Rightarrow \sec^2 x + 8 = 3 \tan x (4 - \tan x)$   
 $\Rightarrow \sec^2 x + 8 = 12 \tan x - 3 \tan^2 x$   
 $\Rightarrow 4 \tan^2 x - 12 \tan x + 9 = 0$   
 $\Rightarrow (2 \tan x - 3)^2 = 0$   
 $\tan x = \frac{3}{2}$   
 $\arctan(\frac{3}{2}) = 0.983^\circ$   
 $\therefore x = 0.983^\circ$   
 $x_2 = 4.12^\circ$

c)  $\frac{1 - 2 \operatorname{cosec}^2 y}{2 \cot y} - 2 = \cot y$   
 $\Rightarrow \frac{1 - 2 \operatorname{cosec}^2 y}{2 \cot y} = 2 + \cot y$   
 $\Rightarrow 1 - 2 \operatorname{cosec}^2 y = 4 \cot y + 2 \cot^2 y$   
 $\Rightarrow 1 - 2(1 + \cot^2 y) = 4 \cot y + 2 \cot^2 y$   
 $\Rightarrow 1 - 2 - 2 \cot^2 y = 4 \cot y + 2 \cot^2 y$   
 $\Rightarrow 0 = 4 \cot y + 4 \cot^2 y + 1$   
 $\Rightarrow 0 = (2 \cot y + 1)^2$   
 $2 \cot y = -\frac{1}{2}$   
 $\cot y = -\frac{1}{4}$   
 $\arctan(-2) = -1.107$   
 $y = 2.03^\circ$   
 $y_2 = 5.18^\circ$

d)  $\frac{2 \cot^2 \varphi + 5}{\operatorname{cosec} \varphi} + 2 \operatorname{cosec} \varphi = 13$   
 $\Rightarrow \frac{2 \cot^2 \varphi + 5}{\operatorname{cosec} \varphi} + 2 \operatorname{cosec} \varphi = 13$   
 $\Rightarrow 2 \cot^2 \varphi + 5 + 2 \operatorname{cosec}^2 \varphi = 13 \operatorname{cosec} \varphi$   
 $\Rightarrow 2 \cot^2 \varphi + 5 + 2 \operatorname{cosec}^2 \varphi - 13 \operatorname{cosec} \varphi = 0$   
 $\Rightarrow 4 \operatorname{cosec}^2 \varphi - 13 \operatorname{cosec} \varphi + 3 = 0$   
 $\Rightarrow (4 \operatorname{cosec} \varphi - 1)(\operatorname{cosec} \varphi - 3) = 0$   
 $\operatorname{cosec} \varphi = \frac{1}{4} \text{ or } \operatorname{cosec} \varphi = 3$   
 $\varphi = 0.340^\circ$   
 $\varphi_2 = 2.80^\circ$

Question 17

Solve each of the following trigonometric equations.

a)  $\sec \theta = \frac{1 - \tan^2 \theta}{4 \sec \theta - 9}, \quad 0 \leq \theta < 2\pi$

b)  $\frac{\sec^2 x + 8}{4 - \tan x} = 3 \tan x, \quad 0 \leq x < 2\pi$

c)  $\frac{1 - 2 \operatorname{cosec}^2 y}{2 \cot y} - 2 = \cot y, \quad 0 \leq y < 2\pi$

d)  $\frac{2 \cot^2 \phi + 5}{\operatorname{cosec} \phi} + 2 \operatorname{cosec} \phi = 13, \quad 0 \leq \phi < 2\pi$

$\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \quad x = 0.983^\circ, 4.12^\circ, \quad y = 2.03^\circ, 5.18^\circ, \quad \phi = 0.340^\circ, 2.80^\circ$

Handwritten solutions for Question 17:

a)  $\sec \theta = \frac{1 - \tan^2 \theta}{4 \sec \theta - 9}$   
 $\Rightarrow 4 \sec^2 \theta - 9 \sec \theta = 1 - \tan^2 \theta$   
 $\Rightarrow 4 \sec^2 \theta - 9 \sec \theta = 1 - (\sec^2 \theta - 1)$   
 $\Rightarrow 4 \sec^2 \theta - 9 \sec \theta = 2 - \sec^2 \theta$   
 $\Rightarrow (4 \sec^2 \theta + \sec^2 \theta) - 9 \sec \theta - 2 = 0$   
 $\Rightarrow 5 \sec^2 \theta - 9 \sec \theta - 2 = 0$   
 $\Rightarrow (5 \sec \theta + 1)(\sec \theta - 2) = 0$   
 $\Rightarrow \sec \theta = -\frac{1}{5} \text{ or } \sec \theta = 2$   
 $\Rightarrow \cos \theta = -\frac{1}{5} \text{ or } \cos \theta = \frac{1}{2}$   
 $\Rightarrow \theta = \frac{\pi}{3}, \frac{5\pi}{3} \text{ or } \theta = \arccos(-\frac{1}{5})$   
 $\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \arccos(-\frac{1}{5}), 2\pi - \arccos(-\frac{1}{5})$

b)  $\frac{\sec^2 x + 8}{4 - \tan x} = 3 \tan x$   
 $\Rightarrow (\sec^2 x + 8) = 3 \tan x (4 - \tan x)$   
 $\Rightarrow (1 + \tan^2 x) + 8 = 12 \tan x - 3 \tan^2 x$   
 $\Rightarrow 4 \tan^2 x - 12 \tan x + 9 = 0$   
 $\Rightarrow (2 \tan x - 3)^2 = 0$   
 $\Rightarrow \tan x = \frac{3}{2}$   
 $\Rightarrow \arctan(\frac{3}{2}) = 0.983^\circ$   
 $\Rightarrow 2 = 0.983^\circ + 2\pi \text{ or } \pi + 0.983^\circ$   
 $\therefore x = 0.983^\circ, 4.12^\circ$

c)  $\frac{1 - 2 \operatorname{cosec}^2 y}{2 \cot y} - 2 = \cot y$   
 $\Rightarrow \frac{1 - 2 \operatorname{cosec}^2 y}{2 \cot y} = 2 + \cot y$   
 $\Rightarrow 1 - 2 \operatorname{cosec}^2 y = 4 \cot y + 2 \cot^2 y$   
 $\Rightarrow 1 - 2(1 + \cot^2 y) = 4 \cot y + 2 \cot^2 y$   
 $\Rightarrow 1 - 2 - 2 \cot^2 y = 4 \cot y + 2 \cot^2 y$   
 $\Rightarrow 0 = 4 \cot y + 4 \cot^2 y + 1$   
 $\Rightarrow 0 = (2 \cot y + 1)^2$   
 $\Rightarrow 2 \cot y + 1 = 0$   
 $\Rightarrow \cot y = -\frac{1}{2}$   
 $\Rightarrow \tan y = -2$   
 $\Rightarrow \arctan(-2) = -1.107$   
 $\Rightarrow y = -1.107 \text{ or } 2\pi - 1.107$   
 $y = 2.03^\circ, 5.18^\circ$

d)  $\frac{2 \cot^2 \phi + 5}{\operatorname{cosec} \phi} + 2 \operatorname{cosec} \phi = 13$   
 $\Rightarrow \frac{2 \cot^2 \phi + 5}{\operatorname{cosec} \phi} + 2 \operatorname{cosec} \phi = 13$   
 $\Rightarrow 2 \cot^2 \phi + 5 + 2 \operatorname{cosec}^2 \phi = 13 \operatorname{cosec} \phi$   
 $\Rightarrow 2 \cot^2 \phi + 5 + 2 \operatorname{cosec}^2 \phi - 13 \operatorname{cosec} \phi = 0$   
 $\Rightarrow 4 \operatorname{cosec}^2 \phi - 13 \operatorname{cosec} \phi + 3 = 0$   
 $\Rightarrow (4 \operatorname{cosec} \phi - 1)(\operatorname{cosec} \phi - 3) = 0$   
 $\Rightarrow \operatorname{cosec} \phi = \frac{1}{4} \text{ or } \operatorname{cosec} \phi = 3$   
 $\Rightarrow \sin \phi = 4 \text{ or } \sin \phi = \frac{1}{3}$   
 $\Rightarrow \sin \phi = \frac{1}{3}$   
 $\Rightarrow \phi = 0.340^\circ \text{ or } 2\pi - 0.340^\circ$   
 $\phi = 0.340^\circ, 2.80^\circ$

**Question 17**

Solve each of the following trigonometric equations.

- a)  $\cos(\theta + 30^\circ) = \sin \theta, \quad 0 \leq \theta < 360^\circ$
- b)  $3 \cos(x + 30^\circ) = \sin(x - 60^\circ), \quad 0 \leq x < 360^\circ$
- c)  $\sin(y - 30^\circ) = \sin(y + 45^\circ), \quad 0 \leq y < 360^\circ$
- d)  $\sin(\phi + 30^\circ) = \cos(\phi - 45^\circ), \quad 0 \leq \phi < 360^\circ$
- e)  $\cos(\alpha - 60^\circ) = \cos(\alpha - 45^\circ), \quad 0 \leq \alpha < 360^\circ$

$\theta = 30^\circ, 210^\circ, \quad x = 60^\circ, 240^\circ, \quad y = 82.5^\circ, 262.5^\circ, \quad \phi = 52.5^\circ, 232.5^\circ,$   
 $\alpha = 52.5^\circ, 232.5^\circ$

The image shows handwritten solutions for the five trigonometric equations. The solutions are as follows:

- (a)**  $\cos(\theta + 30^\circ) = \sin \theta$   
 $\Rightarrow \cos \theta \cos 30^\circ - \sin \theta \sin 30^\circ = \sin \theta$   
 $\Rightarrow \frac{\sqrt{3}}{2} \cos \theta - \frac{1}{2} \sin \theta = \sin \theta$   
 $\Rightarrow \sqrt{3} \cos \theta = 2 \sin \theta$   
 $\Rightarrow \sqrt{3} = 2 \tan \theta$   
 $\Rightarrow \tan \theta = \frac{\sqrt{3}}{2}$   
 $\Rightarrow \theta = 30^\circ, 210^\circ$
- (b)**  $3 \cos(x + 30^\circ) = \sin(x - 60^\circ)$   
 $\Rightarrow 3(\cos x \cos 30^\circ - \sin x \sin 30^\circ) = \sin x \cos 60^\circ - \cos x \sin 60^\circ$   
 $\Rightarrow \frac{3\sqrt{3}}{2} \cos x - \frac{3}{2} \sin x = \frac{1}{2} \sin x - \frac{\sqrt{3}}{2} \cos x$   
 $\Rightarrow 2\sqrt{3} \cos x - 2 \sin x = \sin x - \sqrt{3} \cos x$   
 $\Rightarrow 3\sqrt{3} \cos x = 3 \sin x$   
 $\Rightarrow \sqrt{3} \cos x = \sin x$   
 $\Rightarrow \tan x = \sqrt{3}$   
 $\Rightarrow x = 60^\circ, 240^\circ$
- (c)**  $\sin(y - 30^\circ) = \sin(y + 45^\circ)$   
 $\Rightarrow \sin y \cos 30^\circ - \cos y \sin 30^\circ = \sin y \cos 45^\circ + \cos y \sin 45^\circ$   
 $\Rightarrow \frac{\sqrt{3}}{2} \sin y - \frac{1}{2} \cos y = \frac{\sqrt{2}}{2} \sin y + \frac{\sqrt{2}}{2} \cos y$   
 $\Rightarrow \sqrt{3} \sin y - \cos y = \sqrt{2} \sin y + \sqrt{2} \cos y$   
 $\Rightarrow (\sqrt{3} - \sqrt{2}) \sin y = (\sqrt{2} + 1) \cos y$   
 $\Rightarrow \tan y = \frac{\sqrt{2} + 1}{\sqrt{3} - \sqrt{2}}$   
 $\Rightarrow y = 82.5^\circ, 262.5^\circ$
- (d)**  $\sin(\phi + 30^\circ) = \cos(\phi - 45^\circ)$   
 $\Rightarrow \sin \phi \cos 30^\circ + \cos \phi \sin 30^\circ = \cos \phi \cos 45^\circ + \sin \phi \sin 45^\circ$   
 $\Rightarrow \frac{\sqrt{3}}{2} \sin \phi + \frac{1}{2} \cos \phi = \frac{\sqrt{2}}{2} \cos \phi + \frac{\sqrt{2}}{2} \sin \phi$   
 $\Rightarrow \sqrt{3} \sin \phi + \cos \phi = \sqrt{2} \cos \phi + \sqrt{2} \sin \phi$   
 $\Rightarrow (\sqrt{3} - \sqrt{2}) \sin \phi = (\sqrt{2} - 1) \cos \phi$   
 $\Rightarrow \tan \phi = \frac{\sqrt{2} - 1}{\sqrt{3} - \sqrt{2}}$   
 $\Rightarrow \phi = 52.5^\circ, 232.5^\circ$
- (e)**  $\cos(\alpha - 60^\circ) = \cos(\alpha - 45^\circ)$   
 $\Rightarrow \cos \alpha \cos 60^\circ + \sin \alpha \sin 60^\circ = \cos \alpha \cos 45^\circ + \sin \alpha \sin 45^\circ$   
 $\Rightarrow \frac{1}{2} \cos \alpha + \frac{\sqrt{3}}{2} \sin \alpha = \frac{\sqrt{2}}{2} \cos \alpha + \frac{\sqrt{2}}{2} \sin \alpha$   
 $\Rightarrow \cos \alpha + \sqrt{3} \sin \alpha = \sqrt{2} \cos \alpha + \sqrt{2} \sin \alpha$   
 $\Rightarrow (1 - \sqrt{2}) \cos \alpha = (\sqrt{2} - \sqrt{3}) \sin \alpha$   
 $\Rightarrow \tan \alpha = \frac{\sqrt{2} - \sqrt{3}}{\sqrt{2} - 1}$   
 $\Rightarrow \alpha = 52.5^\circ, 232.5^\circ$

**Question 18**

Solve each of the following trigonometric equations.

- a)  $\sin(\theta - 45^\circ) = \sin \theta, \quad 0 \leq \theta < 360^\circ$
- b)  $\cos(x - 30^\circ) = \sin(x + 30^\circ), \quad 0 \leq x < 360^\circ$
- c)  $\cos(y - 30^\circ) = \sin(y + 45^\circ), \quad 0 \leq y < 360^\circ$
- d)  $\sin(\phi - 30^\circ) = \cos(\phi - 45^\circ), \quad 0 \leq \phi < 360^\circ$
- e)  $\cos(\alpha - 60^\circ) = \cos(\alpha + 60^\circ), \quad 0 \leq \alpha < 360^\circ$

$\theta = 112.5^\circ, 292.5^\circ, \quad x = 45^\circ, 225^\circ, \quad y = 37.5^\circ, 217.5^\circ, \quad \phi = 82.5^\circ, 262.5^\circ,$

$\alpha = 0^\circ, 180^\circ$

**(a)**  $\sin(\theta - 45^\circ) = \sin \theta$   
 $\Rightarrow \sin \theta \cos 45^\circ - \cos \theta \sin 45^\circ = \sin \theta$   
 $\Rightarrow \sin \theta \frac{\sqrt{2}}{2} - \cos \theta \frac{\sqrt{2}}{2} = \sin \theta$   
 $\Rightarrow \sqrt{2} \sin \theta - \sqrt{2} \cos \theta = 2 \sin \theta$   
 $\Rightarrow \frac{\sqrt{2} \sin \theta}{\sqrt{2}} - \frac{\sqrt{2} \cos \theta}{\sqrt{2}} = \frac{2 \sin \theta}{\sqrt{2}}$   
 $\Rightarrow \sqrt{2} \sin \theta - \sqrt{2} \cos \theta = \sqrt{2} \sin \theta$   
 $\Rightarrow \sqrt{2} - \sqrt{2} \cos \theta = \sqrt{2}$   
 $\Rightarrow \cos \theta = 0$   
 $\Rightarrow \theta = 90^\circ \text{ or } 270^\circ$   
 $\therefore \theta_1 = 90^\circ$   
 $\theta_2 = 270^\circ$

**(b)**  $\cos(x - 30^\circ) = \sin(x + 30^\circ)$   
 $\Rightarrow \cos x \cos 30^\circ + \sin x \sin 30^\circ = \sin x \cos 30^\circ + \cos x \sin 30^\circ$   
 $\Rightarrow \frac{\sqrt{3}}{2} \cos x + \frac{1}{2} \sin x = \frac{\sqrt{3}}{2} \sin x + \frac{1}{2} \cos x$   
 $\Rightarrow \sqrt{3} \cos x + \sin x = \sqrt{3} \sin x + \cos x$   
 $\Rightarrow \frac{\sqrt{3} \cos x}{\sqrt{3}} + \frac{\sin x}{\sqrt{3}} = \frac{\sqrt{3} \sin x}{\sqrt{3}} + \frac{\cos x}{\sqrt{3}}$   
 $\Rightarrow \cos x + \frac{1}{\sqrt{3}} \sin x = \sin x + \frac{1}{\sqrt{3}} \cos x$   
 $\Rightarrow \cos x - \frac{1}{\sqrt{3}} \cos x = \sin x - \frac{1}{\sqrt{3}} \sin x$   
 $\Rightarrow \cos x (1 - \frac{1}{\sqrt{3}}) = \sin x (1 - \frac{1}{\sqrt{3}})$   
 $\Rightarrow \tan x = 1$   
 $\Rightarrow x = 45^\circ$   
 $\therefore x_1 = 45^\circ$   
 $x_2 = 225^\circ$

**(c)**  $\cos(y - 30^\circ) = \sin(y + 45^\circ)$   
 $\Rightarrow \cos y \cos 30^\circ + \sin y \sin 30^\circ = \sin y \cos 45^\circ + \cos y \sin 45^\circ$   
 $\Rightarrow \frac{\sqrt{3}}{2} \cos y + \frac{1}{2} \sin y = \frac{\sqrt{2}}{2} \sin y + \frac{\sqrt{2}}{2} \cos y$   
 $\Rightarrow \sqrt{3} \cos y + \sin y = \sqrt{2} \sin y + \sqrt{2} \cos y$   
 $\Rightarrow \frac{\sqrt{3} \cos y}{\sqrt{2}} + \frac{\sin y}{\sqrt{2}} = \frac{\sqrt{2} \sin y}{\sqrt{2}} + \frac{\sqrt{2} \cos y}{\sqrt{2}}$   
 $\Rightarrow \frac{\sqrt{3}}{\sqrt{2}} \cos y + \frac{1}{\sqrt{2}} \sin y = \sin y + \cos y$   
 $\Rightarrow \frac{\sqrt{3}}{\sqrt{2}} \cos y - \cos y = \sin y - \frac{1}{\sqrt{2}} \sin y$   
 $\Rightarrow \cos y (\frac{\sqrt{3}-\sqrt{2}}{\sqrt{2}}) = \sin y (\frac{\sqrt{2}-1}{\sqrt{2}})$   
 $\Rightarrow \tan y = \frac{\sqrt{3}-\sqrt{2}}{\sqrt{2}-1}$   
 $\Rightarrow y = 37.5^\circ$   
 $\therefore y_1 = 37.5^\circ$   
 $y_2 = 217.5^\circ$

**(d)**  $\sin(\phi - 30^\circ) = \cos(\phi - 45^\circ)$   
 $\Rightarrow \sin \phi \cos 30^\circ - \cos \phi \sin 30^\circ = \cos \phi \cos 45^\circ + \sin \phi \sin 45^\circ$   
 $\Rightarrow \frac{\sqrt{3}}{2} \sin \phi - \frac{1}{2} \cos \phi = \frac{\sqrt{2}}{2} \cos \phi + \frac{\sqrt{2}}{2} \sin \phi$   
 $\Rightarrow \sqrt{3} \sin \phi - \cos \phi = \sqrt{2} \cos \phi + \sqrt{2} \sin \phi$   
 $\Rightarrow \frac{\sqrt{3} \sin \phi}{\sqrt{2}} - \frac{\cos \phi}{\sqrt{2}} = \frac{\sqrt{2} \cos \phi}{\sqrt{2}} + \frac{\sqrt{2} \sin \phi}{\sqrt{2}}$   
 $\Rightarrow \frac{\sqrt{3}}{\sqrt{2}} \sin \phi - \frac{\cos \phi}{\sqrt{2}} = \cos \phi + \sin \phi$   
 $\Rightarrow \frac{\sqrt{3}}{\sqrt{2}} \sin \phi - \sin \phi = \cos \phi + \frac{\cos \phi}{\sqrt{2}}$   
 $\Rightarrow \sin \phi (\frac{\sqrt{3}-\sqrt{2}}{\sqrt{2}}) = \cos \phi (1 + \frac{1}{\sqrt{2}})$   
 $\Rightarrow \tan \phi = \frac{\sqrt{3}-\sqrt{2}}{\sqrt{2} + 1}$   
 $\Rightarrow \phi = 82.5^\circ$   
 $\therefore \phi_1 = 82.5^\circ$   
 $\phi_2 = 262.5^\circ$

**(e)**  $\cos(\alpha - 60^\circ) = \cos(\alpha + 60^\circ)$   
 $\Rightarrow \cos \alpha \cos 60^\circ + \sin \alpha \sin 60^\circ = \cos \alpha \cos 60^\circ - \sin \alpha \sin 60^\circ$   
 $\Rightarrow \frac{1}{2} \cos \alpha + \frac{\sqrt{3}}{2} \sin \alpha = \frac{1}{2} \cos \alpha - \frac{\sqrt{3}}{2} \sin \alpha$   
 $\Rightarrow \frac{\sqrt{3}}{2} \sin \alpha + \frac{\sqrt{3}}{2} \sin \alpha = 0$   
 $\Rightarrow \sqrt{3} \sin \alpha = 0$   
 $\Rightarrow \sin \alpha = 0$   
 $\Rightarrow \alpha = 0^\circ$   
 $\therefore \alpha_1 = 0^\circ$   
 $\alpha_2 = 180^\circ$

**Question 19**

Solve each of the following trigonometric equations.

a)  $\sin\left(\theta + \frac{\pi}{4}\right) = \sin \theta, \quad 0 \leq \theta < 2\pi$

b)  $\cos\left(x + \frac{\pi}{6}\right) = \cos\left(x + \frac{2\pi}{3}\right), \quad 0 \leq \theta < 2\pi$

c)  $\sin\left(\frac{\pi}{3} - y\right) = \cos\left(y + \frac{5\pi}{6}\right), \quad 0 \leq y < 2\pi \text{ (very hard)}$

d)  $2\cos\left(\varphi + \frac{\pi}{2}\right) + \sin\left(\varphi + \frac{\pi}{3}\right) = 0, \quad 0 \leq \varphi < 2\pi$

e)  $\sqrt{2}\cos\left(\alpha + \frac{\pi}{4}\right) = \sin\left(\alpha + \frac{\pi}{6}\right), \quad 0 \leq \alpha < 2\pi$

$\theta = \frac{3\pi}{8}, \frac{11\pi}{8}$	$x = \frac{7\pi}{12}, \frac{19\pi}{12}$	$y = \frac{\pi}{2}, \frac{3\pi}{2}$	$\varphi = \frac{\pi}{6}, \frac{7\pi}{6}$	$\alpha = \frac{\pi}{12}, \frac{13\pi}{12}$
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(a)  $\sin\left(\theta + \frac{\pi}{4}\right) = \sin \theta$   
 $\Rightarrow \sin \theta \cos \frac{\pi}{4} + \cos \theta \sin \frac{\pi}{4} = \sin \theta$   
 $\Rightarrow \sin \theta \frac{\sqrt{2}}{2} + \cos \theta \frac{\sqrt{2}}{2} = \sin \theta$   
 $\Rightarrow \sqrt{2} \cos \theta + \sqrt{2} \sin \theta = 2 \sin \theta$   
 $\Rightarrow \sqrt{2} \cos \theta = \sqrt{2} \sin \theta$   
 $\Rightarrow \tan \theta = \frac{\sqrt{2}}{\sqrt{2}} = 1$   
 $\Rightarrow \theta = \frac{\pi}{4} \pm n\pi$   
 $\theta_1 = \frac{\pi}{4}$   
 $\theta_2 = \frac{5\pi}{4}$

(b)  $\cos\left(x + \frac{\pi}{6}\right) = \cos\left(x + \frac{2\pi}{3}\right)$   
 $\Rightarrow \cos x \cos \frac{\pi}{6} - \sin x \sin \frac{\pi}{6} = \cos x \cos \frac{2\pi}{3} - \sin x \sin \frac{2\pi}{3}$   
 $\Rightarrow \cos x \frac{\sqrt{3}}{2} - \sin x \frac{1}{2} = \cos x \left(-\frac{1}{2}\right) - \sin x \frac{\sqrt{3}}{2}$   
 $\Rightarrow \sqrt{3} \cos x - \sin x = -\cos x - \sqrt{3} \sin x$   
 $\Rightarrow \sqrt{3} \cos x + \cos x = -\sqrt{3} \sin x - \sin x$   
 $\Rightarrow \cos x (\sqrt{3} + 1) = -\sin x (\sqrt{3} + 1)$   
 $\Rightarrow \tan x = \frac{\cos x (\sqrt{3} + 1)}{-\sin x (\sqrt{3} + 1)} = -1$   
 $\Rightarrow x = \frac{3\pi}{4} \pm n\pi$   
 $x_1 = \frac{3\pi}{4}$   
 $x_2 = \frac{7\pi}{4}$

(c)  $\sin\left(\frac{\pi}{3} - y\right) = \cos\left(y + \frac{5\pi}{6}\right)$   
 $\Rightarrow \sin \frac{\pi}{3} \cos y - \cos \frac{\pi}{3} \sin y = \cos y \cos \frac{5\pi}{6} + \sin y \sin \frac{5\pi}{6}$   
 $\Rightarrow \frac{\sqrt{3}}{2} \cos y - \frac{1}{2} \sin y = \cos y \left(-\frac{\sqrt{3}}{2}\right) + \sin y \left(\frac{1}{2}\right)$   
 $\Rightarrow \sqrt{3} \cos y - \sin y = -\sqrt{3} \cos y + \sin y$   
 $\Rightarrow 2\sqrt{3} \cos y = 2 \sin y$   
 $\Rightarrow \tan y = \sqrt{3}$   
 $\Rightarrow y = \frac{\pi}{3} \pm n\pi$   
 $y_1 = \frac{\pi}{3}$   
 $y_2 = \frac{4\pi}{3}$

(d)  $2\cos\left(\varphi + \frac{\pi}{2}\right) + \sin\left(\varphi + \frac{\pi}{3}\right) = 0$   
 $\Rightarrow 2\cos \varphi \cos \frac{\pi}{2} - 2\sin \varphi \sin \frac{\pi}{2} + \sin \varphi \cos \frac{\pi}{3} + \cos \varphi \sin \frac{\pi}{3} = 0$   
 $\Rightarrow -2\sin \varphi + \frac{1}{2} \sin \varphi + \frac{\sqrt{3}}{2} \cos \varphi = 0$   
 $\Rightarrow \frac{3}{2} \sin \varphi = \frac{\sqrt{3}}{2} \cos \varphi$   
 $\Rightarrow \sqrt{3} \sin \varphi = \cos \varphi$   
 $\Rightarrow \tan \varphi = \frac{\cos \varphi}{\sqrt{3} \sin \varphi} = \frac{1}{\sqrt{3}}$   
 $\Rightarrow \varphi = \frac{\pi}{6} \pm n\pi$   
 $\varphi_1 = \frac{\pi}{6}$   
 $\varphi_2 = \frac{7\pi}{6}$

(e)  $\sqrt{2}\cos\left(\alpha + \frac{\pi}{4}\right) = \sin\left(\alpha + \frac{\pi}{6}\right)$   
 $\Rightarrow \sqrt{2} \cos \alpha \cos \frac{\pi}{4} - \sqrt{2} \sin \alpha \sin \frac{\pi}{4} = \sin \alpha \cos \frac{\pi}{6} + \cos \alpha \sin \frac{\pi}{6}$   
 $\Rightarrow \sqrt{2} \cos \alpha \frac{\sqrt{2}}{2} - \sqrt{2} \sin \alpha \frac{\sqrt{2}}{2} = \sin \alpha \frac{\sqrt{3}}{2} + \cos \alpha \frac{1}{2}$   
 $\Rightarrow 2 \cos \alpha - 2 \sin \alpha = \frac{\sqrt{3}}{2} \sin \alpha + \frac{1}{2} \cos \alpha$   
 $\Rightarrow \frac{3}{2} \cos \alpha - \frac{5}{2} \sin \alpha = 0$   
 $\Rightarrow 3 \cos \alpha = 5 \sin \alpha$   
 $\Rightarrow \tan \alpha = \frac{3}{5}$   
 $\Rightarrow \alpha = \arctan\left(\frac{3}{5}\right) \approx 0.5907 \text{ rad}$   
 $\Rightarrow \alpha_1 = \frac{\pi}{12}$   
 $\alpha_2 = \frac{13\pi}{12}$

**Question 20**

Solve each of the following trigonometric equations.

a)  $\sin(\theta - 20^\circ) = \sin(\theta + 60^\circ), \quad 0 \leq \theta < 360^\circ$

b)  $\cos(x - 35^\circ) = \cos(x - 55^\circ), \quad 0 \leq x < 360^\circ$

c)  $\sin(y - 48^\circ) = \cos(y + 12^\circ), \quad 0 \leq y < 360^\circ$

d)  $\sin(\phi + 72^\circ) = \cos(\phi - 38^\circ), \quad 0 \leq \phi < 360^\circ$

e)  $\cos(\alpha - 36^\circ) = \cos(\alpha - 72^\circ), \quad 0 \leq \alpha < 360^\circ$

$\theta = 70^\circ, 250^\circ, \quad x = 45^\circ, 225^\circ, \quad y = 63^\circ, 243^\circ, \quad \phi = 28^\circ, 208^\circ, \quad \alpha = 54^\circ, 234^\circ$

Handwritten solution for Question 20a:

$$\sin(\theta - 20^\circ) = \sin(\theta + 60^\circ)$$

$$\Rightarrow \sin\theta \cos 20^\circ - \cos\theta \sin 20^\circ = \sin\theta \cos 60^\circ + \cos\theta \sin 60^\circ$$

$$\Rightarrow \sin\theta(\cos 20^\circ - \cos 60^\circ) = \cos\theta(\sin 20^\circ + \sin 60^\circ)$$

$$\Rightarrow \frac{\sin\theta}{\cos\theta} \frac{(\cos 20^\circ - \cos 60^\circ)}{(\sin 20^\circ + \sin 60^\circ)} = 1$$

$$\Rightarrow \tan\theta = \frac{\cos 20^\circ - \cos 60^\circ}{\sin 20^\circ + \sin 60^\circ} = \frac{0.9397 - 0.5}{0.3420 + 0.5774} = \frac{0.4397}{0.9194} \approx 0.4782$$

$$\Rightarrow \theta = \arctan(0.4782) \approx 27.5^\circ$$

General solutions:

$$\theta = 27.5^\circ + 180^\circ n \quad n = 0, 1, 2, 3$$

$$\therefore \theta_1 = 27.5^\circ$$

$$\theta_2 = 207.5^\circ$$


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Handwritten solution for Question 20b:

$$\cos(x - 35^\circ) = \cos(x - 55^\circ)$$

$$\Rightarrow \cos x \cos 35^\circ + \sin x \sin 35^\circ = \cos x \cos 55^\circ + \sin x \sin 55^\circ$$

$$\Rightarrow \cos x(\cos 35^\circ - \cos 55^\circ) = \sin x(\sin 55^\circ - \sin 35^\circ)$$

$$\Rightarrow \frac{\cos x}{\sin x} \frac{(\cos 35^\circ - \cos 55^\circ)}{(\sin 55^\circ - \sin 35^\circ)} = 1$$

$$\Rightarrow \cot x = \frac{\cos 35^\circ - \cos 55^\circ}{\sin 55^\circ - \sin 35^\circ} = \frac{0.8192 - 0.5638}{0.8192 - 0.5736} = \frac{0.2554}{0.2456} \approx 1.0399$$

$$\Rightarrow \tan x = 1$$

General solutions:

$$x = 45^\circ + 180^\circ n \quad n = 0, 1, 2, 3$$

$$\therefore x_1 = 45^\circ$$

$$x_2 = 225^\circ$$


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Handwritten solution for Question 20c:

$$\sin(y - 48^\circ) = \cos(y + 12^\circ)$$

$$\Rightarrow \sin y \cos 48^\circ - \cos y \sin 48^\circ = \cos y \cos 12^\circ + \sin y \sin 12^\circ$$

$$\Rightarrow \sin y(\cos 48^\circ + \sin 12^\circ) = \cos y(\cos 12^\circ + \sin 48^\circ)$$

$$\Rightarrow \frac{\sin y}{\cos y} \frac{(\cos 48^\circ + \sin 12^\circ)}{(\cos 12^\circ + \sin 48^\circ)} = 1$$

$$\Rightarrow \tan y = \frac{\cos 48^\circ + \sin 12^\circ}{\cos 12^\circ + \sin 48^\circ} = \frac{0.6691 + 0.2079}{0.9781 + 0.7431} = \frac{0.8770}{1.7212} \approx 0.5096$$

$$\Rightarrow y = \arctan(0.5096) \approx 27.1^\circ$$

General solutions:

$$y = 27.1^\circ + 180^\circ n \quad n = 0, 1, 2, 3$$

$$\therefore y_1 = 27.1^\circ$$

$$y_2 = 207.1^\circ$$

Handwritten solution for Question 20d:

$$\sin(\phi + 72^\circ) = \cos(\phi - 38^\circ)$$

$$\Rightarrow \sin\phi \cos 72^\circ + \cos\phi \sin 72^\circ = \cos\phi \cos 38^\circ + \sin\phi \sin 38^\circ$$

$$\Rightarrow \sin\phi(\cos 72^\circ - \sin 38^\circ) = \cos\phi(\cos 38^\circ - \sin 72^\circ)$$

$$\Rightarrow \frac{\sin\phi}{\cos\phi} \frac{(\cos 72^\circ - \sin 38^\circ)}{(\cos 38^\circ - \sin 72^\circ)} = 1$$

$$\Rightarrow \tan\phi = \frac{\cos 72^\circ - \sin 38^\circ}{\cos 38^\circ - \sin 72^\circ} = \frac{0.3090 - 0.6157}{0.7880 - 0.9516} = \frac{-0.3067}{-0.1636} \approx 1.8753$$

$$\Rightarrow \phi = \arctan(1.8753) \approx 61.5^\circ$$

General solutions:

$$\phi = 61.5^\circ + 180^\circ n \quad n = 0, 1, 2, 3$$

$$\therefore \phi_1 = 61.5^\circ$$

$$\phi_2 = 241.5^\circ$$


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Handwritten solution for Question 20e:

$$\cos(\alpha - 36^\circ) = \cos(\alpha - 72^\circ)$$

$$\Rightarrow \cos\alpha \cos 36^\circ + \sin\alpha \sin 36^\circ = \cos\alpha \cos 72^\circ + \sin\alpha \sin 72^\circ$$

$$\Rightarrow \cos\alpha(\cos 36^\circ - \cos 72^\circ) = \sin\alpha(\sin 72^\circ - \sin 36^\circ)$$

$$\Rightarrow \frac{\cos\alpha}{\sin\alpha} \frac{(\cos 36^\circ - \cos 72^\circ)}{(\sin 72^\circ - \sin 36^\circ)} = 1$$

$$\Rightarrow \cot\alpha = \frac{\cos 36^\circ - \cos 72^\circ}{\sin 72^\circ - \sin 36^\circ} = \frac{0.8090 - 0.3090}{0.9516 - 0.5938} = \frac{0.5000}{0.3578} \approx 1.3974$$

$$\Rightarrow \tan\alpha = 1.3974$$

General solutions:

$$\alpha = 45.5^\circ + 180^\circ n \quad n = 0, 1, 2, 3$$

$$\therefore \alpha_1 = 45.5^\circ$$

$$\alpha_2 = 225.5^\circ$$

**Question 21**

Solve each of the following trigonometric equations.

a)  $\sin 2\theta = \tan \theta, \quad 0 \leq \theta \leq 180^\circ$

b)  $2 \sin 2x = \cos x, \quad 0 \leq x < 180^\circ$

c)  $\sin 2y + \sin y = 0, \quad 0 \leq y < 360^\circ$

d)  $4 \sin \phi \cos \phi = 1, \quad 0 \leq \phi < \pi$

$\theta = 0^\circ, 45^\circ, 135^\circ, 180^\circ, \quad x = 90^\circ, x \approx 14.5^\circ, 165.5^\circ, \quad y = 0^\circ, 120^\circ, 180^\circ, 240^\circ,$

$\phi = \frac{\pi}{12}, \frac{5\pi}{12}$

Handwritten solution for Question 21:

(a)  $\sin 2\theta = \tan \theta$   
 $\Rightarrow 2\sin\theta\cos\theta = \frac{\sin\theta}{\cos\theta}$   
 $\Rightarrow 2\sin\theta\cos^2\theta = \sin\theta$   
 $\Rightarrow 2\sin\theta\cos^2\theta - \sin\theta = 0$   
 $\Rightarrow \sin\theta(2\cos^2\theta - 1) = 0$   
 $\Rightarrow \sin\theta\cos 2\theta = 0$   
 •  $\sin\theta = 0$   
 $\theta = 0^\circ + 360^\circ n$   
 $\theta = 180^\circ + 360^\circ n$   
 •  $\cos 2\theta = 0$   
 $\cos 2\theta = 0 \Rightarrow 2\theta = 90^\circ + 180^\circ n$   
 $\theta = 45^\circ + 90^\circ n$   
 $\theta = 135^\circ + 180^\circ n$   
 $\therefore \theta = 0, 45^\circ, 135^\circ, 180^\circ$

(b)  $2 \sin 2x = \cos x$   
 $\Rightarrow 2(2\sin x \cos x) = \cos x$   
 $\Rightarrow 4\sin x \cos x - \cos x = 0$   
 $\Rightarrow \cos x(4\sin x - 1) = 0$   
 •  $\cos x = 0$   
 $\cos x = 0 \Rightarrow x = 90^\circ + 180^\circ n$   
 $x = 270^\circ + 360^\circ n$   
 $\therefore x = 90^\circ, 270^\circ$   
 •  $4\sin x - 1 = 0$   
 $4\sin x = 1 \Rightarrow \sin x = \frac{1}{4}$   
 $x = \arcsin\left(\frac{1}{4}\right) \approx 14.5^\circ$   
 $x = 180^\circ - \arcsin\left(\frac{1}{4}\right) \approx 165.5^\circ$   
 $\therefore x = 90^\circ, 14.5^\circ, 165.5^\circ, 270^\circ$

(c)  $\sin 2y + \sin y = 0$   
 $\Rightarrow 2\sin y \cos y + \sin y = 0$   
 $\Rightarrow \sin y(2\cos y + 1) = 0$   
 •  $\sin y = 0$   
 $\sin y = 0 \Rightarrow y = 0^\circ + 360^\circ n$   
 $y = 180^\circ + 360^\circ n$   
 $\therefore y = 0, 180, 360, 540$   
 •  $2\cos y + 1 = 0$   
 $2\cos y = -1 \Rightarrow \cos y = -\frac{1}{2}$   
 $\cos y = -\frac{1}{2} \Rightarrow y = 120^\circ + 360^\circ n$   
 $y = 240^\circ + 360^\circ n$   
 $\therefore y = 0, 120, 240, 360$

(d)  $4 \sin \phi \cos \phi = 1$   
 $\Rightarrow 2(2\sin\phi\cos\phi) = 1$   
 $\Rightarrow 2\sin 2\phi = 1$   
 $\Rightarrow \sin 2\phi = \frac{1}{2}$   
 •  $\sin 2\phi = \frac{1}{2}$   
 $2\phi = \frac{\pi}{6} + 2\pi n$   
 $2\phi = \frac{5\pi}{6} + 2\pi n$   
 $\phi = \frac{\pi}{12} + \pi n$   
 $\phi = \frac{5\pi}{12} + \pi n$   
 $\therefore \phi = \frac{\pi}{12}, \frac{5\pi}{12}$

**Question 22**

Solve each of the following trigonometric equations.

a)  $2 \sin 2\theta = \cot \theta, \quad 0 \leq \theta \leq \pi$

b)  $3 \sin 2x = 2 \cos x, \quad 0 \leq x < 180^\circ$

c)  $\sin 4y = \sin 2y, \quad 0 \leq y < 180^\circ$

d)  $\sin \phi + \frac{1}{4} \sec \phi = 0, \quad 0 \leq \phi < \pi$

$\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \quad x = 90^\circ, x \approx 19.5^\circ, 160.5^\circ, \quad y = 0^\circ, 30^\circ, 90^\circ, 150^\circ, \quad \phi = \frac{7\pi}{12}, \frac{11\pi}{12}$

Handwritten solutions for the four trigonometric equations:

(a)  $2 \sin 2\theta = \cot \theta$   
 $\Rightarrow 2(2 \sin \theta \cos \theta) = \frac{\cos \theta}{\sin \theta}$   
 $\Rightarrow 4 \sin \theta \cos \theta = \frac{\cos \theta}{\sin \theta}$   
 $\Rightarrow 4 \sin^2 \theta \cos \theta - \cos \theta = 0$   
 $\Rightarrow \cos \theta (4 \sin^2 \theta - 1) = 0$

(b)  $3 \sin 2x = 2 \cos x$   
 $\Rightarrow 3(2 \sin x \cos x) = 2 \cos x$   
 $\Rightarrow 6 \sin x \cos x = 2 \cos x$   
 $\Rightarrow 6 \sin x \cos x - 2 \cos x = 0$   
 $\Rightarrow 2 \cos x (3 \sin x - 1) = 0$

(c)  $\sin 4y = \sin 2y$   
 $\Rightarrow \sin(2 \cdot 2y) = \sin 2y$   
 $\Rightarrow 2 \sin 2y \cos 2y - \sin 2y = 0$   
 $\Rightarrow \sin 2y (2 \cos 2y - 1) = 0$

(d)  $\sin \phi + \frac{1}{4} \sec \phi = 0$   
 $\Rightarrow 4 \sin \phi + \sec \phi = 0$   
 $\Rightarrow 4 \sin \phi + \frac{1}{\cos \phi} = 0$   
 $\Rightarrow 4 \sin \phi \cos \phi + 1 = 0$   
 $\Rightarrow 2(2 \sin \phi \cos \phi) + 1 = 0$   
 $\Rightarrow 2 \sin 2\phi + 1 = 0$



Question 23

Solve each of the following trigonometric equations.

a)  $\cos \theta - \sin 2\theta = 0, \quad 0 \leq \theta \leq 360^\circ$

b)  $\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} = 4, \quad 0 \leq x < 360^\circ$

c)  $2 \cos y = 2 \tan y \sin y + \sec y, \quad 0 \leq y < 2\pi$

d)  $2 \cos \phi + \operatorname{cosec} \phi = 0, \quad 0 \leq \phi < 2\pi$

$\theta = 30^\circ, 90^\circ, 150^\circ, 270^\circ, \quad x = 15^\circ, 75^\circ, 195^\circ, 255^\circ, \quad y = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6},$

$\phi = \frac{3\pi}{4}, \frac{7\pi}{4}$

Handwritten solutions for the trigonometric equations in Question 23:

a)  $\cos \theta - \sin 2\theta = 0$   
 $\cos \theta - 2 \sin \theta \cos \theta = 0$   
 $\cos \theta (1 - 2 \sin \theta) = 0$   
 $\cos \theta = 0 \quad \text{or} \quad \sin \theta = \frac{1}{2}$   
 $\cos \theta = 0 \Rightarrow \theta = 90^\circ, 270^\circ$   
 $\sin \theta = \frac{1}{2} \Rightarrow \theta = 30^\circ, 150^\circ$   
 $\therefore \theta = 30^\circ, 90^\circ, 150^\circ, 270^\circ$

b)  $\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} = 4$   
 $\frac{\sin^2 x + \cos^2 x}{\sin x \cos x} = 4$   
 $\frac{1}{\sin x \cos x} = 4$   
 $\frac{1}{\frac{1}{2} \sin 2x} = 4$   
 $\frac{2}{\sin 2x} = 4$   
 $\sin 2x = \frac{1}{2}$   
 $2x = 30^\circ, 150^\circ$   
 $x = 15^\circ, 75^\circ$   
 $\therefore x = 15^\circ, 75^\circ, 195^\circ, 255^\circ$

c)  $2 \cos y = 2 \tan y \sin y + \sec y$   
 $2 \cos y = \frac{2 \sin y}{\cos y} \sin y + \frac{1}{\cos y}$   
 $2 \cos^2 y = 2 \sin^2 y + 1$   
 $2 \cos^2 y - 2 \sin^2 y = 1$   
 $2(\cos^2 y - \sin^2 y) = 1$   
 $2 \cos 2y = 1$   
 $\cos 2y = \frac{1}{2}$   
 $2y = 60^\circ, 300^\circ$   
 $y = 30^\circ, 150^\circ$   
 $\therefore y = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

d)  $2 \cos \phi + \operatorname{cosec} \phi = 0$   
 $2 \cos \phi + \frac{1}{\sin \phi} = 0$   
 $2 \cos^2 \phi + 1 = 0$   
 $2 \cos^2 \phi = -1$   
 $\cos^2 \phi = -\frac{1}{2}$   
 $\cos \phi = \pm \frac{\sqrt{2}}{2}$   
 $\phi = \frac{3\pi}{4}, \frac{7\pi}{4}$

**Question 24**

Solve each of the following trigonometric equations.

a)  $2 \cos 2\theta = 1 + \cos \theta, \quad 0 \leq \theta < 360^\circ$

b)  $\cos 2x + 3 \sin x = 2, \quad 0 \leq x < 360^\circ$

c)  $\cos 2y + \sin y = 0, \quad 0 \leq y < 360^\circ$

d)  $2(1 - \cos 2\varphi) = \tan \varphi, \quad 0 \leq \varphi < 180^\circ$

$\theta = 0^\circ, \theta \approx 138.6^\circ, 221.4^\circ, \quad x = 30^\circ, 90^\circ, 150^\circ, \quad y = 90^\circ, 210^\circ, 330^\circ,$

$\varphi = 0^\circ, 15^\circ, 75^\circ$

Handwritten solutions for Question 24:

(a)  $2 \cos 2\theta = 1 + \cos \theta$   
 $\Rightarrow 2(2\cos^2\theta - 1) = 1 + \cos \theta$   
 $\Rightarrow 4\cos^2\theta - 2 = 1 + \cos \theta$   
 $\Rightarrow 4\cos^2\theta - \cos \theta - 3 = 0$   
 $\Rightarrow (4\cos\theta - 3)(\cos\theta + 1) = 0$   
 $\Rightarrow \cos\theta = \frac{3}{4}$  or  $\cos\theta = -1$   
 $\Rightarrow \theta = 0, 138.6, 221.4$

(b)  $\cos 2x + 3 \sin x = 2$   
 $\Rightarrow 1 - 2\sin^2 x + 3 \sin x = 2$   
 $\Rightarrow -2\sin^2 x + 3 \sin x - 1 = 0$   
 $\Rightarrow 2\sin^2 x - 3 \sin x + 1 = 0$   
 $\Rightarrow (2\sin x - 1)(\sin x - 1) = 0$   
 $\Rightarrow \sin x = \frac{1}{2}$  or  $\sin x = 1$   
 $\Rightarrow x = 30, 150, 90$

(c)  $\cos 2y + \sin y = 0$   
 $\Rightarrow 1 - 2\sin^2 y + \sin y = 0$   
 $\Rightarrow 2\sin^2 y - \sin y - 1 = 0$   
 $\Rightarrow (2\sin y + 1)(\sin y - 1) = 0$   
 $\Rightarrow \sin y = -\frac{1}{2}$  or  $\sin y = 1$   
 $\Rightarrow y = 90, 210, 330$

(d)  $2(1 - \cos 2\varphi) = \tan \varphi$   
 $\Rightarrow 2[1 - (1 - 2\sin^2 \varphi)] = \frac{\sin \varphi}{\cos \varphi}$   
 $\Rightarrow 2[1 - 1 + 2\sin^2 \varphi] = \frac{\sin \varphi}{\cos \varphi}$   
 $\Rightarrow 4\sin^2 \varphi = \frac{\sin \varphi}{\cos \varphi}$   
 $\Rightarrow 4\sin^2 \varphi \cos \varphi = \sin \varphi$   
 $\Rightarrow 4\sin \varphi \cos^2 \varphi - \sin \varphi = 0$   
 $\Rightarrow \sin \varphi (4\cos^2 \varphi - 1) = 0$   
 $\Rightarrow \sin \varphi = 0$  or  $\cos^2 \varphi = \frac{1}{4}$   
 $\Rightarrow \varphi = 0, 15, 75$

**Question 25**

Solve each of the following trigonometric equations.

a)  $\cos 2\theta = 1 + \sin \theta, \quad 0 \leq \theta < 360^\circ$

b)  $\cos 2x + 3\cos x = 1, \quad 0 \leq x < 2\pi$

c)  $3\cos 2y = 1 - \sin y, \quad 0 \leq y < 360^\circ$

d)  $2\cos \phi + 1 = \sin\left(\frac{1}{2}\phi\right), \quad 0 \leq \phi < 360^\circ$

$\theta = 0^\circ, 180^\circ, 210^\circ, 330^\circ, \quad x = \frac{\pi}{3}, \frac{5\pi}{3}, \quad y \approx 41.8^\circ, 138.2^\circ \quad y = 210^\circ, 330^\circ,$

$\phi = 97.2^\circ, 262.8^\circ$

Handwritten solutions for the four trigonometric equations:

**(a)**  $\cos 2\theta = 1 + \sin \theta$   
 $\rightarrow 1 - 2\sin^2 \theta = 1 + \sin \theta$   
 $\rightarrow 0 = 2\sin^2 \theta + \sin \theta$   
 $\rightarrow 0 = \sin \theta (2\sin \theta + 1)$   
 $\rightarrow \sin \theta = 0 \text{ or } \sin \theta = -\frac{1}{2}$   
 Solutions:  $\theta = 0^\circ, 180^\circ, 210^\circ, 330^\circ$

**(b)**  $\cos 2x + 3\cos x = 1$   
 $\rightarrow 2\cos^2 x - 1 + 3\cos x = 1$   
 $\rightarrow 2\cos^2 x + 3\cos x - 2 = 0$   
 $\rightarrow (2\cos x - 1)(\cos x + 2) = 0$   
 $\rightarrow \cos x = \frac{1}{2}$   
 $\rightarrow x = \frac{\pi}{3}, \frac{5\pi}{3}$

**(c)**  $3\cos 2y = 1 - \sin y$   
 $\rightarrow 3(1 - 2\sin^2 y) = 1 - \sin y$   
 $\rightarrow 3 - 6\sin^2 y = 1 - \sin y$   
 $\rightarrow 0 = 6\sin^2 y - \sin y - 2$   
 $\rightarrow (3\sin y - 2)(2\sin y + 1) = 0$   
 $\rightarrow \sin y = \frac{2}{3}$   
 $\rightarrow y \approx 41.8^\circ, 138.2^\circ$

**(d)**  $2\cos \phi + 1 = \sin\left(\frac{1}{2}\phi\right)$   
 $\rightarrow 2\left(1 - 2\sin^2\left(\frac{\phi}{2}\right)\right) + 1 = \sin\left(\frac{\phi}{2}\right)$   
 $\rightarrow 2 - 4\sin^2\left(\frac{\phi}{2}\right) + 1 = \sin\left(\frac{\phi}{2}\right)$   
 $\rightarrow 0 = 4\sin^2\left(\frac{\phi}{2}\right) + \sin\left(\frac{\phi}{2}\right) - 3$   
 $\rightarrow (4\sin\left(\frac{\phi}{2}\right) - 3)\left(\sin\left(\frac{\phi}{2}\right) + 1\right) = 0$   
 $\rightarrow \sin\left(\frac{\phi}{2}\right) = \frac{3}{4}$   
 $\rightarrow \frac{\phi}{2} = \arcsin\left(\frac{3}{4}\right) \approx 46.6^\circ$   
 $\rightarrow \phi = 97.2^\circ, 262.8^\circ$

Question 26

- a)  $\cos 2\theta - 7\sin \theta - 4 = 0, \quad 0 \leq \theta < 360^\circ$
- b)  $3\cos 2x = \sin x + 2, \quad 0 \leq x < 360^\circ$
- c)  $3\cos 2y = 7\cos y, \quad 0 \leq y < 360^\circ$
- d)  $\cos 2\phi = \sin \phi, \quad 0 \leq \phi < 360^\circ$

$\theta = 210^\circ, 330^\circ, \quad x \approx 19.5^\circ, 160.5^\circ \quad x = 210^\circ, 330^\circ, \quad y \approx 109.5^\circ, 250.5^\circ,$   
 $\phi = 30^\circ, 150^\circ, 270^\circ$

The image shows handwritten solutions for each part of Question 26:

- a)**  $\cos 2\theta - 7\sin \theta - 4 = 0$   
 $\Rightarrow (1 - 2\sin^2 \theta) - 7\sin \theta - 4 = 0$   
 $\Rightarrow -2\sin^2 \theta - 7\sin \theta - 4 = 0$   
 $\Rightarrow 2\sin^2 \theta + 7\sin \theta + 4 = 0$   
 $\Rightarrow (2\sin \theta + 4)(\sin \theta + 1) = 0$   
 $\Rightarrow \sin \theta = -2$  (no solution)  
 $\Rightarrow \sin \theta = -1$   
 $\Rightarrow \theta = 270^\circ$
- b)**  $3\cos 2x = \sin x + 2$   
 $\Rightarrow 3(1 - 2\sin^2 x) = \sin x + 2$   
 $\Rightarrow 3 - 6\sin^2 x = \sin x + 2$   
 $\Rightarrow -6\sin^2 x - \sin x + 1 = 0$   
 $\Rightarrow (3\sin x - 1)(2\sin x + 1) = 0$   
 $\Rightarrow \sin x = \frac{1}{3}$  or  $\sin x = -\frac{1}{2}$   
 $\Rightarrow x \approx 19.5^\circ, 160.5^\circ$  or  $x = 210^\circ, 330^\circ$
- c)**  $3\cos 2y = 7\cos y$   
 $\Rightarrow 3(2\cos^2 y - 1) = 7\cos y$   
 $\Rightarrow 6\cos^2 y - 3 = 7\cos y$   
 $\Rightarrow 6\cos^2 y - 7\cos y - 3 = 0$   
 $\Rightarrow (3\cos y + 1)(2\cos y - 3) = 0$   
 $\Rightarrow \cos y = -\frac{1}{3}$  or  $\cos y = \frac{3}{2}$  (no solution)  
 $\Rightarrow y \approx 109.5^\circ, 250.5^\circ$
- d)**  $\cos 2\phi = \sin \phi$   
 $\Rightarrow 1 - 2\sin^2 \phi = \sin \phi$   
 $\Rightarrow -2\sin^2 \phi - \sin \phi + 1 = 0$   
 $\Rightarrow 2\sin^2 \phi + \sin \phi - 1 = 0$   
 $\Rightarrow (2\sin \phi - 1)(\sin \phi + 1) = 0$   
 $\Rightarrow \sin \phi = \frac{1}{2}$  or  $\sin \phi = -1$   
 $\Rightarrow \phi = 30^\circ, 150^\circ$  or  $\phi = 270^\circ$

**Question 27**

Solve each of the following trigonometric equations.

- a)  $3 \cos 2\theta - 5 \sin \theta = 4, \quad 0 \leq \theta < 360^\circ$
- b)  $3 \cos 2x = 1 - \sin x, \quad 0 \leq x < 360^\circ$
- c)  $\cos 2y - 7 \cos y + 4 = 0, \quad 0 \leq y < 360^\circ$
- d)  $\cos 2\phi + 6 \cos \phi + 5 = 0, \quad 0 \leq \phi < 360^\circ$

$$\theta = 210^\circ, 330^\circ \quad \theta \approx 199.5^\circ, 340.5^\circ, \quad x \approx 41.8^\circ, 138.2^\circ \quad x = 210^\circ, 330^\circ,$$

$$y = 60^\circ, 300^\circ, \quad \phi = 180^\circ$$

Handwritten solutions for the four trigonometric equations:

**a)**  $3 \cos 2\theta - 5 \sin \theta = 4$   
 $3(1 - 2\sin^2 \theta) - 5 \sin \theta = 4$   
 $3 - 6\sin^2 \theta - 5 \sin \theta = 4$   
 $0 = 6\sin^2 \theta + 5 \sin \theta + 1$   
 $(3\sin \theta + 1)(2\sin \theta + 1) = 0$   
 $\sin \theta = -\frac{1}{3}$   
 $\arcsin(-\frac{1}{3}) = -19.47^\circ$   
 $\theta = -19.47^\circ + 360^\circ$   
 $\theta = 199.53^\circ$   
 $\theta = 210^\circ, 330^\circ$

**b)**  $3 \cos 2x = 1 - \sin x$   
 $3(1 - 2\sin^2 x) = 1 - \sin x$   
 $3 - 6\sin^2 x = 1 - \sin x$   
 $0 = 6\sin^2 x - \sin x - 2$   
 $0 = (2\sin x - 2)(3\sin x + 1)$   
 $\sin x = \frac{2}{3}$   
 $\arcsin(\frac{2}{3}) = 41.8^\circ$   
 $x = 41.8^\circ, 138.2^\circ$   
 $\sin x = -\frac{1}{2}$   
 $\arcsin(-\frac{1}{2}) = -30^\circ$   
 $x = -30^\circ + 360^\circ$   
 $x = 210^\circ, 330^\circ$

**c)**  $\cos 2y - 7 \cos y + 4 = 0$   
 $(2\cos^2 y - 1) - 7 \cos y + 4 = 0$   
 $2\cos^2 y - 7 \cos y + 3 = 0$   
 $(2\cos y - 1)(\cos y - 3) = 0$   
 $\cos y = \frac{1}{2}$   
 $y = 60^\circ, 300^\circ$

**d)**  $\cos 2\phi + 6 \cos \phi + 5 = 0$   
 $(2\cos^2 \phi - 1) + 6 \cos \phi + 5 = 0$   
 $2\cos^2 \phi + 6 \cos \phi + 4 = 0$   
 $\cos^2 \phi + 3 \cos \phi + 2 = 0$   
 $(\cos \phi + 1)(\cos \phi + 2) = 0$   
 $\cos \phi = -1$   
 $\phi = 180^\circ$

**Question 28**

Solve each of the following trigonometric equations.

- a)  $\cos 2\theta = 7 \cos \theta + 3, \quad 0 \leq \theta < 360^\circ$
- b)  $2 \cos 2x = 4 \cos x - 3, \quad 0 \leq x < 360^\circ$
- c)  $6 \cos 2y + 5 \cos y + 3 = 0, \quad 0 \leq y < 360^\circ$
- d)  $5 \cos 2\phi + 22 \sin \phi = 9, \quad 0 \leq \phi < 360^\circ$

$\theta = 120^\circ, 240^\circ, \quad x = 60^\circ, 300^\circ, \quad y \approx 70.5^\circ, 138.6^\circ, 221.4^\circ, 289.5^\circ,$

$\phi \approx 11.5^\circ, 168.5^\circ$

The handwritten solution shows the following steps:

- Part (a):**  $\cos 2\theta = 7 \cos \theta + 3 \Rightarrow 2 \cos^2 \theta - 1 = 7 \cos \theta + 3 \Rightarrow 2 \cos^2 \theta - 7 \cos \theta - 4 = 0 \Rightarrow (2 \cos \theta + 1)(\cos \theta - 4) = 0 \Rightarrow \cos \theta = -\frac{1}{2}$ . Solutions:  $\theta = 120^\circ, 240^\circ$ .
- Part (b):**  $2 \cos 2x = 4 \cos x - 3 \Rightarrow 2(2 \cos^2 x - 1) = 4 \cos x - 3 \Rightarrow 4 \cos^2 x - 2 = 4 \cos x - 3 \Rightarrow 4 \cos^2 x - 4 \cos x + 1 = 0 \Rightarrow (2 \cos x - 1)(2 \cos x - 1) = 0 \Rightarrow \cos x = \frac{1}{2}$ . Solutions:  $x = 60^\circ, 300^\circ$ .
- Part (c):**  $6 \cos 2y + 5 \cos y + 3 = 0 \Rightarrow 6(2 \cos^2 y - 1) + 5 \cos y + 3 = 0 \Rightarrow 12 \cos^2 y - 6 + 5 \cos y + 3 = 0 \Rightarrow 12 \cos^2 y + 5 \cos y - 3 = 0 \Rightarrow (3 \cos y - 1)(4 \cos y + 3) = 0 \Rightarrow \cos y = \frac{1}{3}$ . Solutions:  $y \approx 70.5^\circ, 289.5^\circ$ .
- Part (d):**  $5 \cos 2\phi + 22 \sin \phi = 9 \Rightarrow 5(1 - 2 \sin^2 \phi) + 22 \sin \phi = 9 \Rightarrow 5 - 10 \sin^2 \phi + 22 \sin \phi = 9 \Rightarrow 0 = 10 \sin^2 \phi - 22 \sin \phi + 4 \Rightarrow 0 = 5 \sin \phi - 11 \sin \phi + 2 \Rightarrow 0 = (5 \sin \phi - 1)(\sin \phi - 2)$ . Solutions:  $\phi \approx 11.5^\circ, 168.5^\circ$ .

**Question 29**

Solve each of the following trigonometric equations.

a)  $\cos 2\theta + 9 \sin \theta + 4 = 0, \quad 0 \leq \theta < 360^\circ$

b)  $3 \cos 2x = 9 - 14 \cos x, \quad 0 \leq x < 360^\circ$

c)  $2 \cos 2y + 7 \cos y = 0, \quad 0 \leq y < 360^\circ$

d)  $2 \cos 2\phi = 1 - 2 \sin \phi, \quad 0 \leq \phi < 360^\circ$

$\theta = 210^\circ, 330^\circ, \quad x \approx 48.2^\circ, 311.8^\circ, \quad y \approx 75.5^\circ, 284.5^\circ, \quad \phi = 54^\circ, 126^\circ, 198^\circ, 342^\circ$

The image shows handwritten solutions for the four trigonometric equations. The solutions are as follows:

**(a)**  $\cos 2\theta + 9 \sin \theta + 4 = 0$   
 $\Rightarrow (-2 \sin^2 \theta) + 9 \sin \theta + 4 = 0$   
 $\Rightarrow 0 = 2 \sin^2 \theta - 9 \sin \theta - 5$   
 $\Rightarrow (2 \sin \theta + 1)(\sin \theta - 5) = 0$   
 $\Rightarrow \sin \theta = -\frac{1}{2}$

$\bullet \sin \theta = -\frac{1}{2}, \text{ arcsin}(\frac{-1}{2}) = -30^\circ$   
 $(\theta = -30^\circ \pm 360^\circ, \theta = 210^\circ \pm 360^\circ, \dots)$   
 $\therefore \theta = 210^\circ$

**(b)**  $3 \cos 2x = 9 - 14 \cos x$   
 $\Rightarrow 3(2 \cos^2 x - 1) = 9 - 14 \cos x$   
 $\Rightarrow 6 \cos^2 x - 3 = 9 - 14 \cos x$   
 $\Rightarrow 6 \cos^2 x + 14 \cos x - 12 = 0$   
 $\Rightarrow 3 \cos^2 x + 7 \cos x - 6 = 0$   
 $\Rightarrow (3 \cos x - 2)(\cos x + 3) = 0$   
 $\cos x = \frac{2}{3}$

$\bullet \cos x = \frac{2}{3}, \text{ arcsin}(\frac{2}{3}) = 48.2^\circ$   
 $(x = 48.2^\circ \pm 360^\circ, x = 311.8^\circ \pm 360^\circ, \dots)$   
 $x_1 = 48.2^\circ$   
 $x_2 = 311.8^\circ$

**(c)**  $2 \cos 2y + 7 \cos y = 0$   
 $2(2 \cos^2 y - 1) + 7 \cos y = 0$   
 $4 \cos^2 y - 2 + 7 \cos y = 0$   
 $4 \cos^2 y + 7 \cos y - 2 = 0$   
 $(4 \cos y - 1)(\cos y + 2) = 0$   
 $\cos y = \frac{1}{4}$

$\bullet \text{arccos}(\frac{1}{4}) = 75.5^\circ$   
 $(y = 75.5^\circ \pm 360^\circ, y = 284.5^\circ \pm 360^\circ, \dots)$   
 $y_1 = 75.5^\circ$   
 $y_2 = 284.5^\circ$

**(d)**  $2 \cos 2\phi = 1 - 2 \sin \phi$   
 $2(1 - 2 \sin^2 \phi) = 1 - 2 \sin \phi$   
 $2 - 4 \sin^2 \phi = 1 - 2 \sin \phi$   
 $0 = 4 \sin^2 \phi - 2 \sin \phi - 1$   
 quadratic formula  
 $\sin \phi = \frac{2 \pm \sqrt{4 + 4}}{8}$   
 $\sin \phi = \frac{2 \pm 2\sqrt{2}}{8}$   
 $\sin \phi = \frac{1 \pm \sqrt{2}}{4}$

$\bullet \text{arcsin}(\frac{1 + \sqrt{2}}{4}) = 54^\circ$      $\bullet \text{arcsin}(\frac{1 - \sqrt{2}}{4}) = -18^\circ$   
 $(\phi = 54^\circ \pm 360^\circ, \phi = 126^\circ \pm 360^\circ, \dots)$   
 $(\phi = -18^\circ \pm 360^\circ, \phi = 198^\circ \pm 360^\circ, \dots)$   
 $(\dots)$   
 $\therefore \phi = 54^\circ, 126^\circ, 198^\circ, 342^\circ$

**Question 30**

Solve each of the following trigonometric equations.

a)  $\tan \theta(1 + \cos 2\theta) = 2\sin^2 2\theta, \quad 0 \leq \theta \leq 90^\circ$

b)  $4 \tan 2\varphi + 3 \cot \varphi \sec^2 \varphi = 0, \quad 0 \leq \varphi < 2\pi$  (hard)

$$\theta = 0^\circ, 15^\circ, 75^\circ, 90^\circ, \quad \varphi = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$$