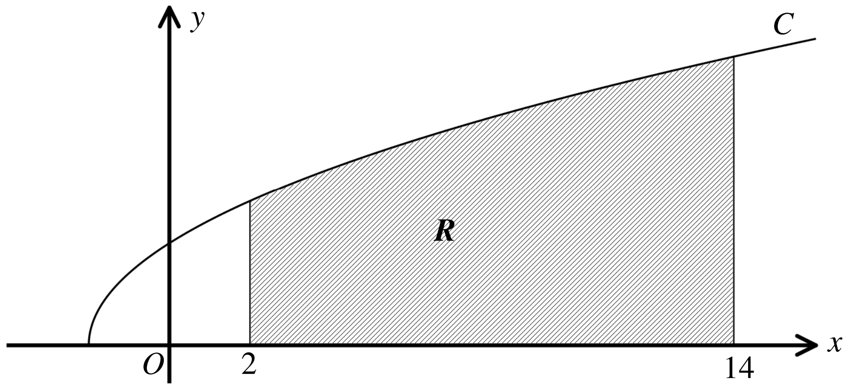
INTEGRATION IN PARAMETRIC

68 EXAM QUESTIONS

7 BASIC QUESTIONS

Question 1 **(**)**



The figure above shows the curve C, given parametrically by

$$x = t^2 - 2, \quad y = 6t \ , \ \ t \geq 0 .$$

The finite region R is bounded by C, the x axis and the straight lines with equations $x = 2$ and $x = 14$.

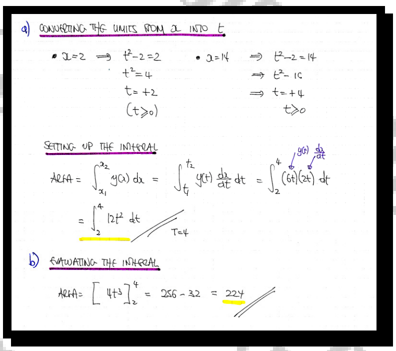
a) Show that the area of R is given by

$$\int_2^T 12t^2 \ dt ,$$

stating the value of T.

b) Hence find the area of R.

$\boxed{}$, $\boxed{T = 4}$, $\boxed{\text{area} = 224}$



a) CONVERTING THE UNITS FROM x INTO t

- $x = 2 \implies t^2 - 2 = 2$, $t^2 = 4$, $t = +2$ $(t \geq 0)$
- $x = 14 \implies t^2 - 2 = 14 \implies t^2 = 16 \implies t = +4$, $t \geq 0$

SETTING UP THE INTEGRAL

$$\text{AREA} = \int_{x_1}^{x_2} y(x)\, dx = \int_{t_1}^{t_2} y(t) \frac{dx}{dt}\, dt = \int_2^4 \underbrace{(6t)}_{y(t)}\underbrace{(2t)}_{\frac{dx}{dt}}\, dt$$

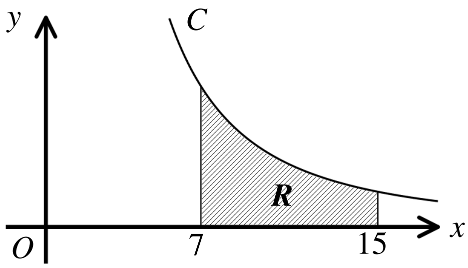
$$= \int_2^4 12t^2 \, dt$$

$T = 4$

b) EVALUATING THE INTEGRAL

$$\text{AREA} = \left[4t^3 \right]_2^4 = 256 - 32 = 224$$

Question 2 (**+)



The figure above shows the curve C, given parametrically by

$$x = 4t - 1, \quad y = \frac{16}{t^2}, \quad t > 0.$$

The finite region R is bounded by C, the x axis and the straight lines with equations $x = 7$ and $x = 15$.

a) Show that the area of R is given by

$$\int_{t_1}^{t_2} \frac{64}{t^2}\, dt,$$

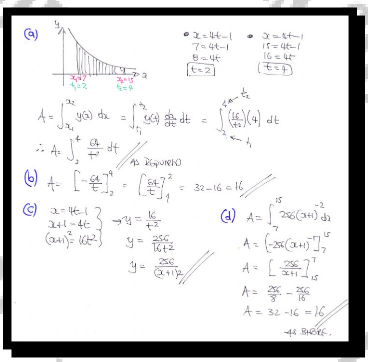
stating the values of t_1 and t_2.

b) Hence find the area of R.

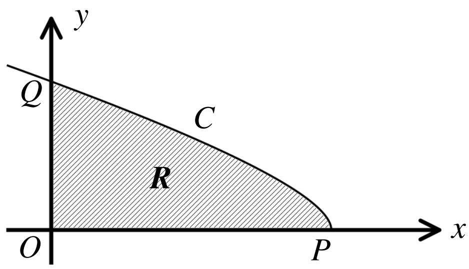
c) Find a Cartesian equation of C, in the form $y = f(x)$.

d) Use the Cartesian equation of C to verify the result of part (b).

$\boxed{t_1 = 2}$, $\boxed{t_2 = 4}$, $\boxed{\text{area} = 16}$, $\boxed{y = \frac{256}{(x+1)^2}}$



Question 3 (+)**



The figure above shows the curve C, given parametrically by

$$x = 4 - t^2, \quad y = t(t+3), \quad t \geq 0.$$

The curve meets the coordinate axes at the points P and Q.

a) Find the coordinates of P and Q.

The finite region R is bounded by C and the coordinate axes.

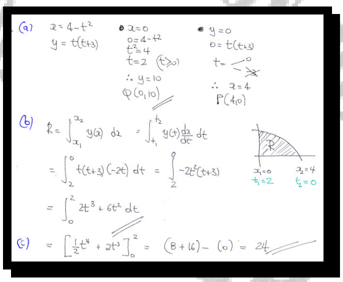
b) Show that the area of R is given by

$$\int_{t_1}^{t_2} 2t^3 + 6t^2 \; dt,$$

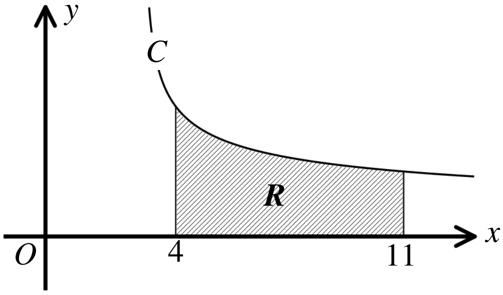
stating the values of t_1 and t_2.

c) Hence find the area of R.

$\boxed{P(4,0) \;\&\; Q(0,10)}$, $\boxed{t_1 = 0}$, $\boxed{t_2 = 2}$, $\boxed{\text{area} = 24}$



Question 4 (**+)



The figure above shows the curve C, given parametrically by

$$x = t^3 + 3 \quad , \quad y = \frac{2}{3t} \, , \quad t > 0 \, .$$

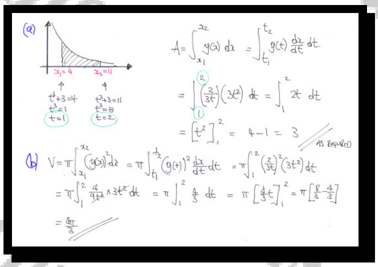
The finite region R is bounded by C, the x axis and the straight lines with equations $x = 4$ and $x = 11$.

a) Show that the area of R is 3 square units.

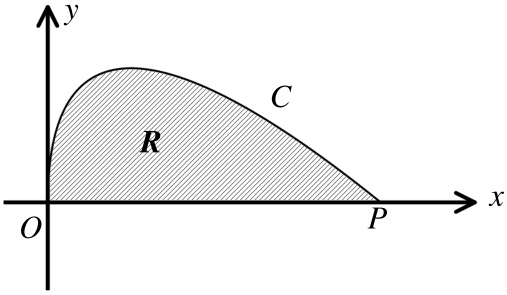
The region R is revolved in the x axis by 2π radians to form a solid of revolution S.

b) Find the volume of S.

$$\boxed{V = \frac{4\pi}{3}}$$



Question 5 (***)



The figure above shows the curve C, given parametrically by

$$x = 6t^2, \quad y = t - t^2, \quad t \geq 0.$$

The curve meets the x axis at the origin O and at the point P.

a) Show that the x coordinate of P is 6.

The finite region R, bounded by C and the x axis, is revolved in the x axis by 2π radians to form a solid of revolution, whose volume is denoted by V.

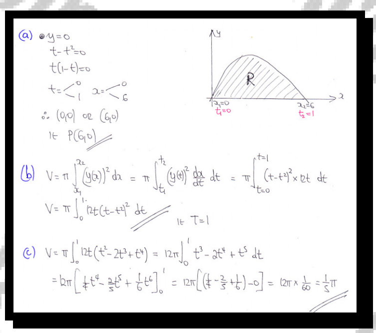
b) Show clearly that

$$V = \pi \int_0^T 12t\left(t - t^2\right)^2 \, dt,$$

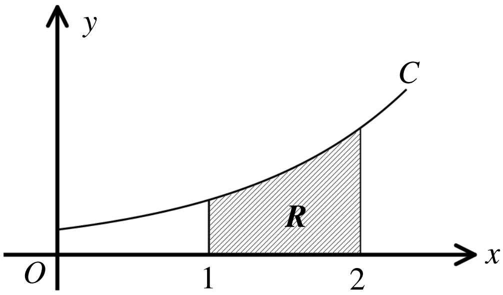
stating the value of T.

c) Hence find the value of V.

$\boxed{}$, $\boxed{T = 1}$, $\boxed{V = \frac{\pi}{5}}$



Question 6 (***)



The figure above shows the curve C, given parametrically by

$$x = \ln t, \quad y = t + \sqrt{t} \;, \quad 1 \le t \le 10 .$$

The finite region R is bounded by C, the straight lines with equations $x = 1$ and $x = 2$, and the x axis.

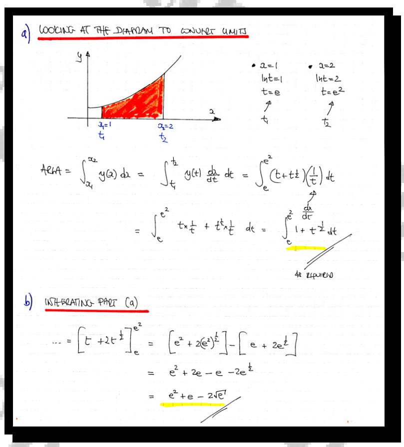
a) Show that the area of R is given by

$$\int_{T_1}^{T_2} 1 + t^{-\frac{1}{2}} \; dt ,$$

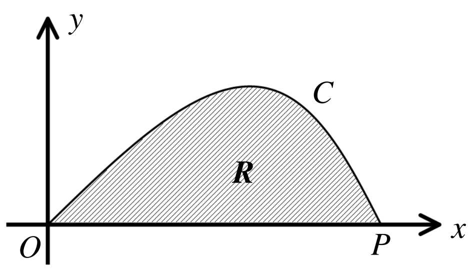
stating the values of T_1 and T_2.

b) Hence find an exact value for the area of R.

$\boxed{}$, $\boxed{T_1 = \mathrm{e},\; T_2 = \mathrm{e}^2}$, $\boxed{\mathrm{e}^2 + \mathrm{e} - 2\mathrm{e}^{\frac{1}{2}}}$,



Question 7 (***)



The figure above shows the curve C, given parametrically by

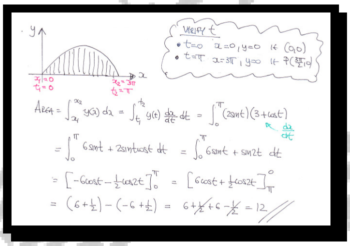
$$x = 3t + \sin t, \quad y = 2\sin t \ , \quad 0 \le t \le \pi \ .$$

The curve meets the coordinate axes at the point P and at the origin O.

The finite region R is bounded by C and the x axis.

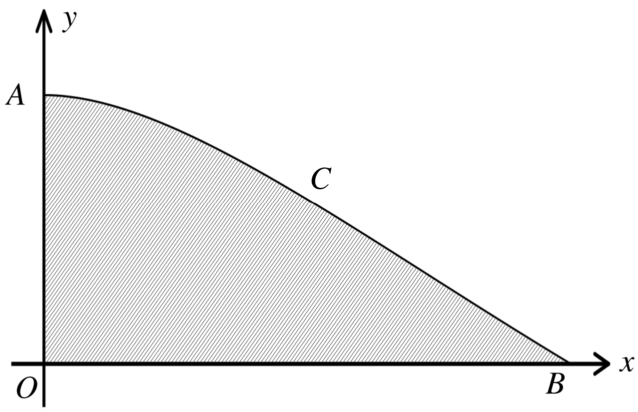
Determine the area of R.

$\boxed{\text{K}}$, $\boxed{\text{area} = 12}$



30 BASIC QUESTIONS

Question 1 (***+)



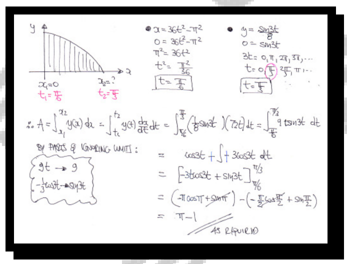
The figure above shows the curve C, with parametric equations

$$x = 36t^2 - \pi^2, \quad y = \frac{\sin 3t}{8}, \quad \frac{\pi}{6} \le t \le \frac{\pi}{3}.$$

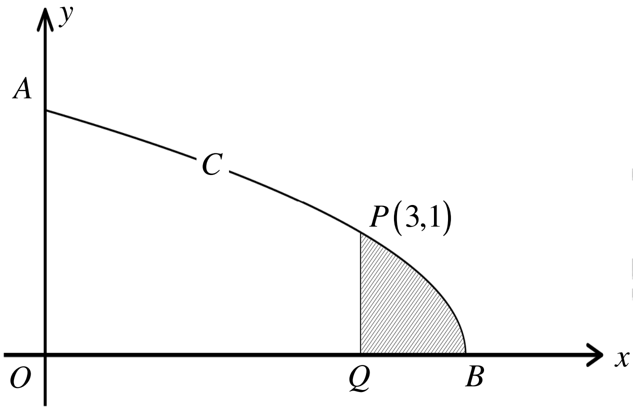
The curve meets the coordinate axes at the points A and B.

By setting up and evaluating a suitable integral in parametric, show that the area bounded by C and the coordinate axes is $(\pi - 1)$ square units.

[ ], proof



Question 2 (***+)



The figure above shows the curve C, with parametric equations

$$x = 4\sin^2 t \ , \ y = 2\cos t \, , \ 0 \le t \le \frac{\pi}{2} .$$

The curve meets the coordinate axes at the points A and B.

The point $P(3,1)$ lies on C.

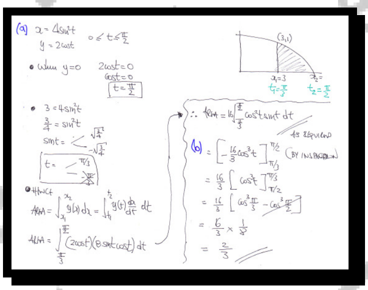
The point Q lies on the x axis so that PQ is parallel to the y axis.

a) Show that the area of the shaded region bounded by C, the line PQ and the x axis is given by the integral

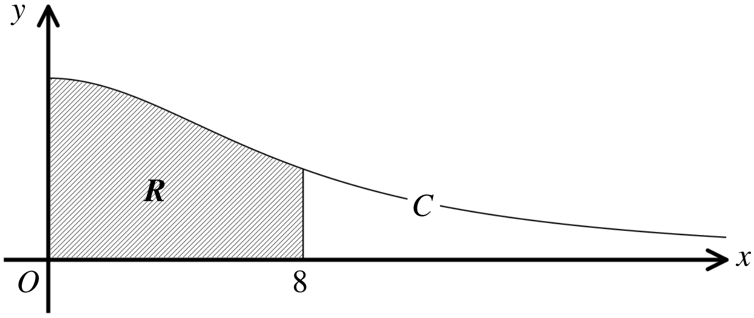
$$16\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cos^2 t \sin t \ dt .$$

b) Evaluate the above integral to find the area of the shaded region.

$\boxed{}$, $\boxed{\text{area} = \frac{2}{3}}$



Question 3 (***+)



The figure above shows the curve C with parametric equations

$$x = 8\tan t\,,\ \ y = \cos^2 t\,,\ \ 0 \le t < \frac{\pi}{2}.$$

The finite region R is bounded by C, the coordinate axes and the straight line with equation $x = 8$.

The region R is revolved in the x axis by 2π radians to form a solid of revolution S.

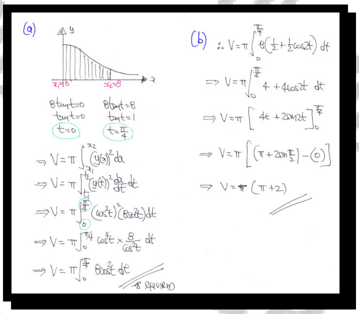
a) Show the volume of S is given by the integral

$$8\pi \int_{t_1}^{t_2} \cos^2 t \ dt\,,$$

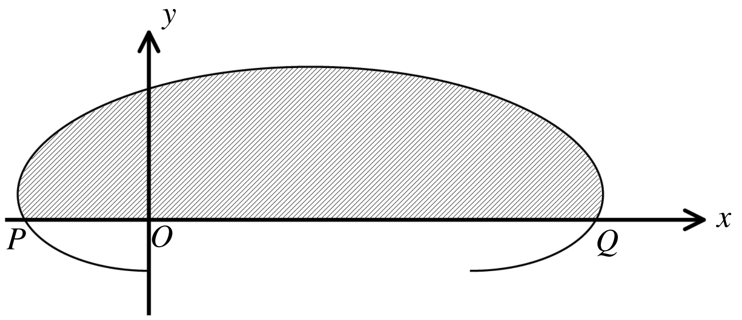
for some appropriate limits t_1 and t_2.

b) Hence find an exact value for the volume of S.

$\boxed{}$, $\boxed{\text{volume} = \pi(\pi + 2)}$



Question 4 (***+)



The figure above shows a curve known as a re-entrant cycloid, with parametric equations

$$x = \theta - 4\sin\theta, \quad y = 1 - 2\cos\theta, \quad 0 \le \theta \le 2\pi.$$

The curve crosses the x axis at the points P and Q.

a) Find the value of θ at the points P and Q.

b) Show that the area of the finite region bounded by the curve and the x axis, shown shaded in the figure above, is given by the integral

$$\int_{\theta_1}^{\theta_2} 1 - 6\cos\theta + 8\cos^2\theta \; d\theta,$$

where θ_1 and θ_2 must be stated.

c) Find an exact value for the above integral.

$\boxed{}$, $\boxed{\theta_P = \frac{1}{3}\pi, \quad \theta_Q = \frac{5}{3}\pi}$, $\boxed{\theta_1 = \frac{1}{3}\pi, \theta_2 = \frac{5}{3}\pi}$, $\boxed{\frac{20}{3}\pi + 4\sqrt{3}}$

a) SOLVING $y=0$

$\Rightarrow 0 = 1 - 2\cos\theta$

$\Rightarrow 2\cos\theta = 1$

$\Rightarrow \cos\theta = \frac{1}{2}$

$\theta = \frac{\pi}{3}, \ \frac{5\pi}{3} \leftarrow \left(2\pi - \frac{\pi}{3}\right)$

CHECK WHICH ONE IS WHICH

$\theta = \frac{\pi}{3} \Rightarrow x = \frac{\pi}{3} - 4\sin\frac{\pi}{3}$

$\Rightarrow x = -2.4160\ldots$

$\theta = \frac{5\pi}{3} \Rightarrow x = \frac{5\pi}{3} - 4\sin\frac{5\pi}{3}$

$\Rightarrow x = 8.700\ldots$

$\therefore$ AT P $\theta = \pi/3$

AT Q $\theta = 5\pi/3$

b) SETTING UP AN AREA INTEGRAL IN PARAMETRIC

$\Rightarrow \text{AREA} = \int_{x_1}^{x_2} y(x)\,dx = \int_{\theta_1}^{\theta_2} y(\theta)\frac{dx}{d\theta}\,d\theta$

$\Rightarrow \text{AREA} = \int_{\frac{\pi}{3}\,(\text{AT } P)}^{\frac{5\pi}{3}\,(\text{AT } Q)} \underbrace{(1-2\cos\theta)}_{y(\theta)}\underbrace{(1-4\cos\theta)}_{dx/d\theta}\,d\theta$

$\Rightarrow \text{AREA} = \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} 1 - 4\cos\theta - 2\cos\theta + 8\cos^2\theta\,d\theta$

$\Rightarrow \text{AREA} = \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} 1 - 6\cos\theta + 8\cos^2\theta\,d\theta$

AS REQUIRED

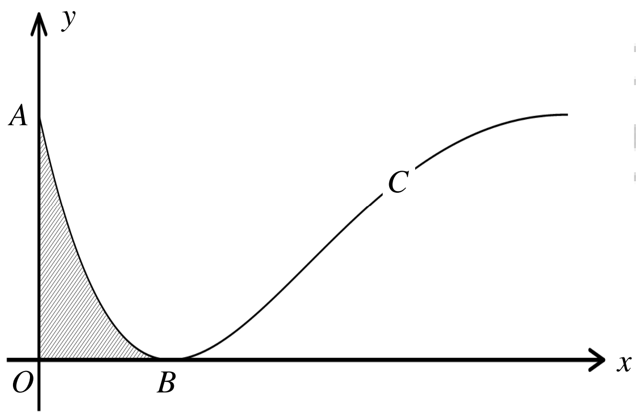
c) INTEGRATING USING $\cos 2\theta \equiv 2\cos^2\theta - 1 \Rightarrow \cos^2\theta \equiv \frac{1}{2} + \frac{1}{2}\cos 2\theta$

$\text{AREA} = \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} 1 - 6\cos\theta + 8\left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right)d\theta = \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} 5 - 6\cos\theta + 4\cos 2\theta\,d\theta$

$= \left[5\theta - 6\sin\theta + 2\sin 2\theta\right]_{\frac{\pi}{3}}^{\frac{5\pi}{3}} = \left(\frac{25}{3}\pi + 3\sqrt{3} - \sqrt{3}\right) - \left(\frac{5\pi}{3} - 3\sqrt{3} + \sqrt{3}\right)$

$= \frac{20\pi}{3} + 4\sqrt{3}$

Question 5 (***+)



The figure above shows the curve C, with parametric equations

$$x = t^2, \quad y = 1 + \cos t \;, \; 0 \le t \le 2\pi \,.$$

The curve meets the coordinate axes at the points A and B.

a) Show that the area of the shaded region bounded by C and the coordinate axes is given by the integral

$$\int_{t_1}^{t_2} 2t\left(1+\cos t\right)\, dt \,,$$

where t_1 and t_2 are constants to be stated.

b) Evaluate the above parametric integral to find an exact value for the area of the shaded region.

$\boxed{}$, $\boxed{t_1 = 0,\ t_2 = \pi}$, $\boxed{\text{area} = \pi^2 - 4}$

[solution overleaf]

[solution of the previous question]

a) OBTAIN THE LIMITS FIRST

- AT A, $x=0$

$t^2=0$

$t=0$

- AT B, $y=0$

$1+\cos t=0$

$\cos t=-1$

$t=\pi$ ⇐ ONLY SOLUTION IN $0 \leq t \leq 2\pi$

SETTING UP AN INTEGRAL

$$\text{AREA} = \int_{x_1}^{x_2} y(x)\,dx = \int_{t_1}^{t_2} y(t)\frac{dx}{dt}\,dt = \int_0^{\pi} (1+\cos t)(2t)\,dt$$

$$\text{AREA} = \int_0^{\pi} 2t(1+\cos t)\,dt$$

AS REQUIRED

b) INTEGRATION BY PARTS (IGNORING LIMITS)

$$\int 2t(1+\cos t)\,dt$$

$$= 2t(t+\sin t) - 2\int t+\sin t\,dt$$

$$= 2t^2 - 2t\sin t - 2\left(\tfrac{1}{2}t^2 - \cos t\right) + C$$

$$= t^2 - 2t\sin t + 2\cos t + C$$

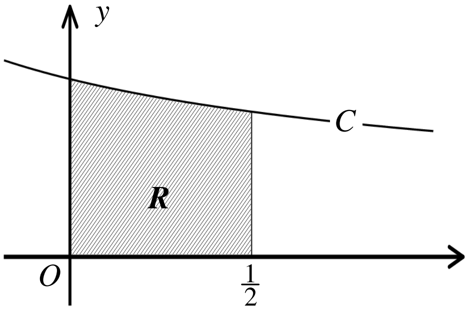
$2t$	2
$t+\sin t$	$1+\cos t$

FINALLY THE AREA

$$\text{AREA} = \left[t^2 - 2t\sin t + 2\cos t\right]_0^{\pi} = (\pi^2 - 0 - 2) - (0 - 0 + 2)$$

$$= \pi^2 - 4$$

Question 6 (***+)



The figure above shows part of the curve C, with parametric equations

$$x = \cos 2\theta, \quad y = \sec\theta \ , \ 0 < \theta < \frac{\pi}{2}.$$

The finite region R is bounded by C, the straight line with equation $x = \frac{1}{2}$ and the coordinate axes.

a) Show that the area of R is given by the integral

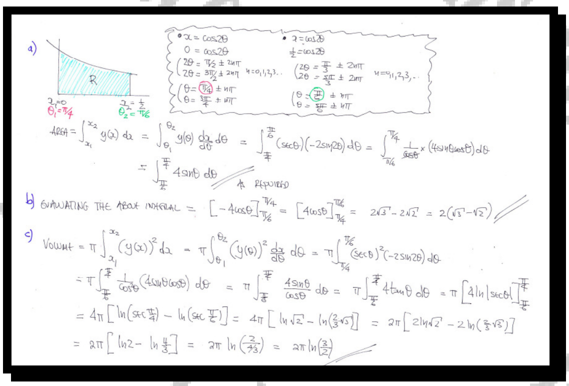
$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} 4\sin\theta \ d\theta .$$

b) Evaluate the above integral to find an exact value for R.

The region R is rotated by 2π radians in the x axis to form a solid of revolution S.

c) Use parametric integration to find an exact value for the volume of S.

$\boxed{}$, $\boxed{\text{area} = 2\sqrt{3} - 2\sqrt{2}}$, $\boxed{\text{volume} = 2\pi \ln\left(\frac{3}{2}\right)}$



a)

$x_1 = 0$, $\theta_1 = \pi/4$; $x_2 = \frac{1}{2}$, $\theta_2 = \pi/6$

• $x = \cos 2\theta$
$0 = \cos 2\theta$
$2\theta = \pi/2 \pm 2n\pi$, $2\theta = 3\pi/2 \pm 2n\pi$, $n = 0,1,2,3,\ldots$
$\theta = \pi/4 \pm n\pi$, $\theta = \frac{3\pi}{4} \pm n\pi$

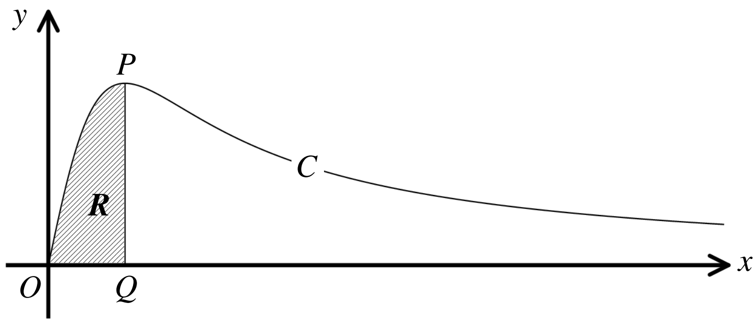
• $x = \cos 2\theta$
$\frac{1}{2} = \cos 2\theta$
$2\theta = \frac{\pi}{3} \pm 2n\pi$, $2\theta = \frac{5\pi}{3} \pm 2n\pi$, $n = 0,1,2,3,\ldots$
$\theta = \frac{\pi}{6} \pm n\pi$, $\theta = \frac{5\pi}{6} \pm n\pi$

$$\text{AREA} = \int_{x_1}^{x_2} y(x)\,dx = \int_{\theta_1}^{\theta_2} y(\theta)\frac{dx}{d\theta}\,d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{6}} (\sec\theta)(-2\sin 2\theta)\,d\theta = \int_{\pi/6}^{\pi/4} \frac{1}{\cos\theta} \times (4\sin\theta\cos\theta)\,d\theta$$
$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} 4\sin\theta\,d\theta$$
AS REQUIRED

b) EVALUATING THE ABOVE INTEGRAL $= \left[-4\cos\theta\right]_{\pi/6}^{\pi/4} = \left[4\cos\theta\right]_{\pi/4}^{\pi/6} = 2\sqrt{3} - 2\sqrt{2} = 2(\sqrt{3} - \sqrt{2})$

c)
$$\text{VOLUME} = \pi\int_{x_1}^{x_2} (y(x))^2\,dx = \pi\int_{\theta_1}^{\theta_2} (y(\theta))^2 \frac{dx}{d\theta}\,d\theta = \pi\int_{\pi/4}^{\pi/6} (\sec\theta)^2(-2\sin 2\theta)\,d\theta$$
$$= \pi\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1}{\cos^2\theta}(4\sin\theta\cos\theta)\,d\theta = \pi\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{4\sin\theta}{\cos\theta}\,d\theta = \pi\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} 4\tan\theta\,d\theta = \pi\left[4\ln|\sec\theta|\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$$
$$= 4\pi\left[\ln\left(\sec\tfrac{\pi}{4}\right) - \ln\left(\sec\tfrac{\pi}{6}\right)\right] = 4\pi\left[\ln\sqrt{2} - \ln\left(\tfrac{2}{3}\sqrt{3}\right)\right] = 2\pi\left[2\ln\sqrt{2} - 2\ln\left(\tfrac{2}{3}\sqrt{3}\right)\right]$$
$$= 2\pi\left[\ln 2 - \ln\tfrac{4}{3}\right] = 2\pi\ln\left(\frac{2}{4/3}\right) = 2\pi\ln\left(\frac{3}{2}\right)$$

Question 7 (***+)



The figure above shows the curve C with parametric equations

$$x = 6\tan\theta, \quad y = \sin 2\theta, \quad 0 \le \theta < \frac{\pi}{2}.$$

The curve has a single stationary point at P.

a) Find the coordinates of P.

The point Q lies on the x axis so that PQ is parallel to the y axis. The finite region R is bounded by C, the x axis and the straight line segment PQ. The region R is revolved in the x axis by 2π radians to form a solid of revolution S.

b) Show the volume of S is given by the integral

$$\pi \int_0^{\frac{\pi}{4}} 24\sin^2\theta \; d\theta.$$

c) Hence find an exact value for the volume of S.

$\boxed{P(6,1)}$, $\boxed{\text{volume} = 3\pi(\pi - 2)}$

(a) $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2\cos 2\theta}{6\sec^2\theta}$

FOR T.P, $\frac{dy}{dx} = 0$

$2\cos 2\theta = 0$

$2\theta = \frac{\pi}{2} \pm 2n\pi$, $2\theta = \frac{3\pi}{2} \pm 2n\pi$ $\quad n = 0,1,2,3,\ldots$

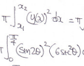
$\theta = \frac{\pi}{4} \pm n\pi$, $\theta = \frac{3\pi}{4} \pm n\pi$

$\therefore \theta = \frac{\pi}{4} \Rightarrow x = 6\tan\frac{\pi}{4} = 6$

$\Rightarrow y = \sin\frac{\pi}{2} = 1$

$\therefore P(6,1)$

(b)



(BY INSPECTION)

$V = \pi\int_{x_1}^{x_2} (y(x))^2\,dx = \pi\int_{\theta_1}^{\theta_2} (y(\theta))^2 \frac{dx}{d\theta}\,d\theta$

$V = \pi\int_0^{\frac{\pi}{4}} (\sin 2\theta)^2 (6\sec^2\theta)\,d\theta$

$V = \pi\int_0^{\frac{\pi}{4}} (2\sin\theta\cos\theta)^2 \left(\frac{6}{\cos^2\theta}\right) d\theta$

$V = \pi\int_0^{\frac{\pi}{4}} 4\sin^2\theta\cancel{\cos^2\theta} \times \frac{6}{\cancel{\cos^2\theta}}\,d\theta$

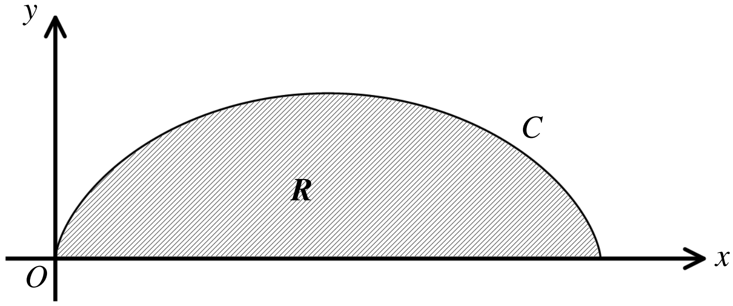
$V = \pi\int_0^{\frac{\pi}{4}} 24\sin^2\theta\,d\theta$ AS REQUIRED

(c) $V = \pi\int_0^{\frac{\pi}{4}} 24\left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right) d\theta = \pi\int_0^{\frac{\pi}{4}} 12 - 12\cos 2\theta\,d\theta$

$= \pi\left[12\theta - 6\sin 2\theta\right]_0^{\frac{\pi}{4}} = \pi\left[(3\pi - 6) - (0 - 0)\right] = \pi(3\pi - 6)$

$= 3\pi(\pi - 2)$

Question 8 (***+)



The figure above shows a cycloid C, whose parametric equations are

$$x = \theta - \sin\theta, \quad y = 1 - \cos\theta \ , \quad 0 \le \theta \le 2\pi .$$

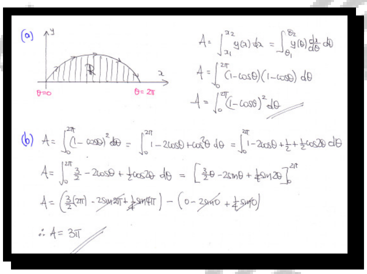
The finite region R is bounded by C and the x axis.

a) Show, with full justification, that the area of R is given by

$$\int_0^{2\pi} (1-\cos\theta)^2 \ d\theta .$$

b) Hence find the area of R.

$\boxed{}$, $\boxed{\text{area} = 3\pi}$



(a) $A = \int_{x_1}^{x_2} y(x)\,dx = \int_{\theta_1}^{\theta_2} y(\theta)\frac{dx}{d\theta}\,d\theta$

$A = \int_0^{2\pi} (1-\cos\theta)(1-\cos\theta)\,d\theta$

$A = \int_0^{2\pi} (1-\cos\theta)^2\,d\theta$

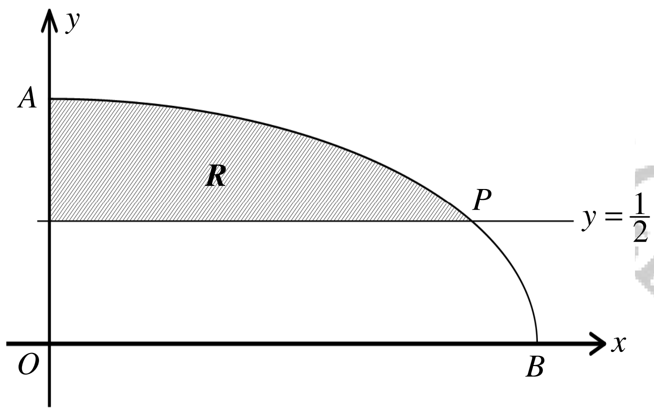
(b) $A = \int_0^{2\pi} (1-\cos\theta)^2\,d\theta = \int_0^{2\pi} 1 - 2\cos\theta + \cos^2\theta\,d\theta = \int_0^{2\pi} 1 - 2\cos\theta + \frac{1}{2} + \frac{1}{2}\cos2\theta\,d\theta$

$A = \int_0^{2\pi} \frac{3}{2} - 2\cos\theta + \frac{1}{2}\cos2\theta\,d\theta = \left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin2\theta\right]_0^{2\pi}$

$A = \left(\frac{3}{2}(2\pi) - \cancel{2\sin2\pi} + \cancel{\frac{1}{4}\sin4\pi}\right) - \left(0 - \cancel{2\sin0} + \cancel{\frac{1}{4}\sin0}\right)$

$\therefore A = 3\pi$

Question 9 (****)



The figure above shows the curve C, with parametric equations

$$x = 4\cos\theta, \quad y = \sin\theta \ , \ 0 \le \theta \le \frac{\pi}{2}.$$

The curve meets the coordinate axes at the points A and B. The straight line with equation $y = \frac{1}{2}$ meets C at the point P.

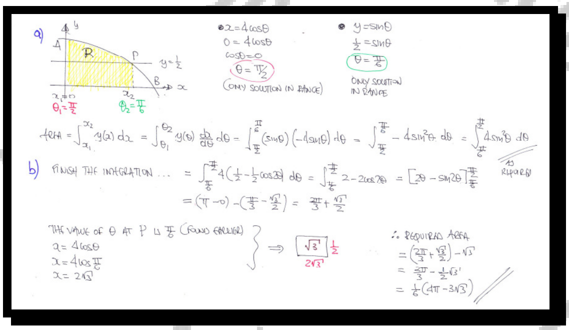
a) Show that the area under the arc of the curve between A and P, and the x axis, is given by the integral

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 4\sin^2\theta \ d\theta .$$

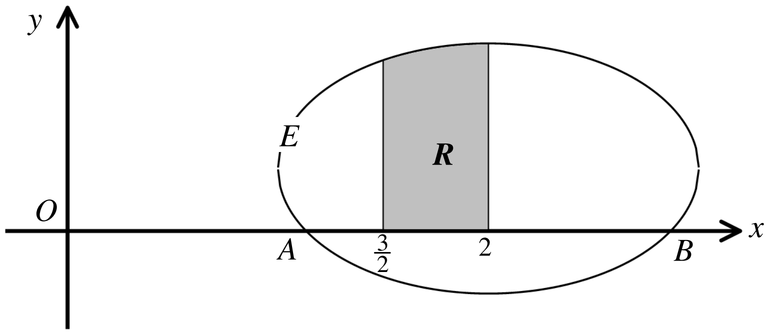
The shaded region R is bounded by C, the straight line with equation $y = \frac{1}{2}$ and the y axis.

b) Find an exact value for the area of R.

$$\boxed{\text{area} = \frac{1}{6}\left(4\pi - 3\sqrt{3}\right)}$$



Question 10 (****)



The figure above shows an ellipse E, given parametrically by

$$x = 2 - \cos\theta, \quad y = 1 + 2\sin\theta \ , \quad 0 \le \theta < 2\pi .$$

The ellipse crosses the x axis at the points A and B.

a) Find, as exact surds, the coordinates of A and the coordinates of B.

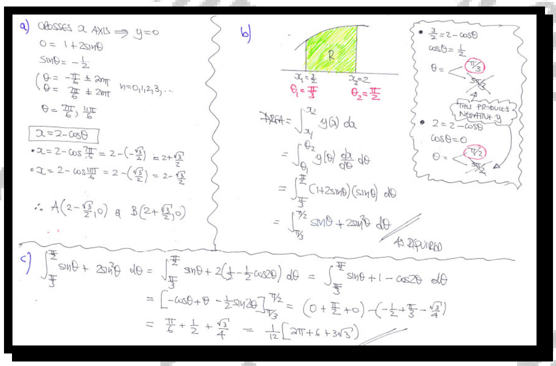
The finite region R is bounded by E, for which $y \ge 0$, the x axis and the straight lines with equations $x = \frac{3}{2}$ and $x = 2$.

b) Show that the area of R is given by

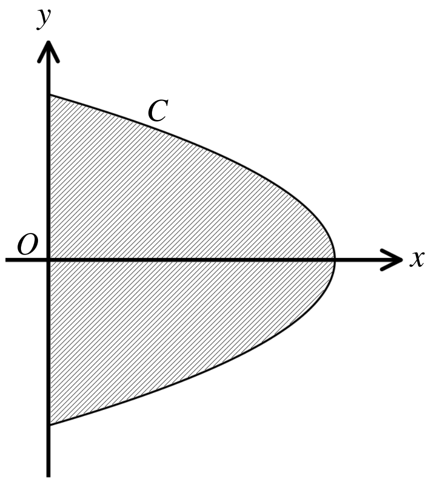
$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin\theta + 2\sin^2\theta \ d\theta .$$

c) Hence find the area of R.

$$\boxed{A\left(2-\frac{\sqrt{3}}{2}\right), B\left(2+\frac{\sqrt{3}}{2}\right)}, \quad \boxed{\text{area} = \frac{1}{12}\left(6+2\pi+3\sqrt{3}\right) \approx 1.46}$$



Question 11 (****)



The figure above shows the curve C, given parametrically by

$$x = 5\cos^2\theta, \quad y = 6\sin\theta \ , \ -\frac{\pi}{2} < \theta \le \frac{\pi}{2}.$$

The curve is symmetrical in the x axis.

The finite region bounded by C and the y axis is denoted by R.

a) Show that the area of R is given by

$$\int_0^{\frac{\pi}{2}} 120\sin^2\theta\cos\theta \ d\theta.$$

b) Hence find the area of R.

[continues overleaf]

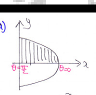
[continued from overleaf]

The region R is to be revolved by π radians in the x axis to form a solid of revolution S.

c) Show that the volume of S is 90π cubic units.

$\boxed{\text{area} = 40}$

(a)



$x = 0 \Rightarrow 5\cos^2\theta = 0$, $\cos\theta = 0$, $\theta = \frac{\pi}{2}$

$y = 0 \Rightarrow 6\sin\theta = 0$, $\sin\theta = 0$, $\theta = 0$

$$A = 2\int_{x_1}^{x_2} y(x)\,dx = 2\int_{\theta_1}^{\theta_2} y(\theta)\frac{dx}{d\theta}\,d\theta = 2\int_{\frac{\pi}{2}}^{0} 6\sin\theta(-10\cos\theta\sin\theta)\,d\theta$$

$$= \int_{0}^{\frac{\pi}{2}} 120\sin^2\theta\cos\theta\,d\theta$$

MINUS USED TO SWAP LIMITS

(b) $\text{INTEGRAL} = \left[40\sin^3\theta\right]_0^{\pi/2}$ ← BY REVERSE CHAIN RULE

$$= 40\sin^3\frac{\pi}{2} - 0$$

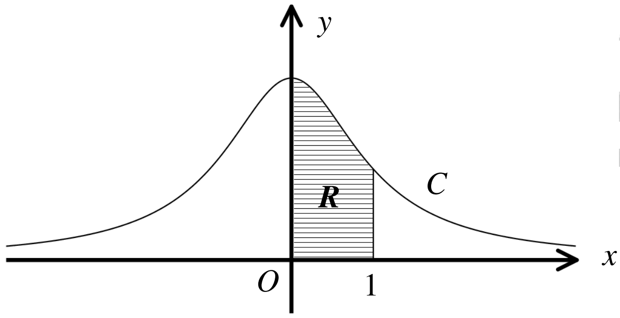
$$= 40$$

(c) $$V = \pi\int_{x_1}^{x_2}(y(x))^2\,dx = \pi\int_{\theta_1}^{\theta_2}(y(\theta))^2\frac{dx}{d\theta}\,d\theta = \pi\int_{\frac{\pi}{2}}^{0}(6\sin\theta)^2(-10\cos\theta\sin\theta)\,d\theta$$

$$= \pi\int_0^{\frac{\pi}{2}} 360\sin^3\theta\cos\theta\,d\theta = \pi\left[90\sin^4\theta\right]_0^{\pi/2}$$ ← BY REVERSE CHAIN RULE

$$= \pi\left[90\sin^4\frac{\pi}{2} - 0\right] = \pi \times 90 = 90\pi$$

Question 12 (****)



The figure above shows the curve C, defined by the parametric equations

$$x = \tan\theta, \quad y = \cos^2\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}.$$

The finite region R is bounded by C, the coordinate axes and the straight line with equation $x = 1$.

a) Determine the area of R.

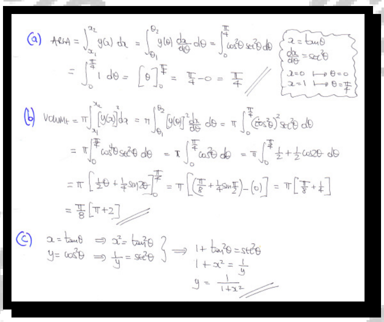
The region R is revolved by 2π radians in the x axis, forming a solid S.

b) Show that the volume of S is

$$\frac{\pi}{8}(\pi + 2).$$

c) Find a Cartesian equation of C, giving the answer in the form $y = f(x)$.

$\boxed{}$, $\boxed{\text{area} = \frac{\pi}{4}}$, $\boxed{y = \frac{1}{1+x^2}}$



(a) AREA $= \int_{x_1}^{x_2} y(x)\,dx = \int_{\theta_1}^{\theta_2} y(\theta)\frac{dx}{d\theta}\,d\theta = \int_0^{\frac{\pi}{4}} \cos^2\theta\sec^2\theta\,d\theta$

$= \int_0^{\frac{\pi}{4}} 1\,d\theta = [\theta]_0^{\frac{\pi}{4}} = \frac{\pi}{4} - 0 = \frac{\pi}{4}$

$x = \tan\theta$
$\frac{dx}{d\theta} = \sec^2\theta$
$x = 0 \mapsto \theta = 0$
$x = 1 \mapsto \theta = \frac{\pi}{4}$

(b) VOLUME $= \pi\int_{x_1}^{x_2}[y(x)]^2\,dx = \pi\int_{\theta_1}^{\theta_2}[y(\theta)]^2\frac{dx}{d\theta}\,d\theta = \pi\int_0^{\frac{\pi}{4}}(\cos^2\theta)^2\sec^2\theta\,d\theta$

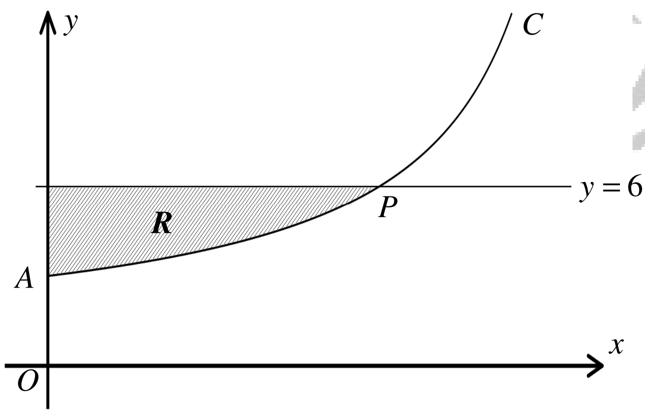
$= \pi\int_0^{\frac{\pi}{4}}\cos^4\theta\sec^2\theta\,d\theta = \pi\int_0^{\frac{\pi}{4}}\cos^2\theta\,d\theta = \pi\int_0^{\frac{\pi}{4}}\frac{1}{2}+\frac{1}{2}\cos 2\theta\,d\theta$

$= \pi\left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{4}} = \pi\left[\left(\frac{\pi}{8}+\frac{1}{4}\sin\frac{\pi}{2}\right)-(0)\right] = \pi\left[\frac{\pi}{8}+\frac{1}{4}\right]$

$= \frac{\pi}{8}[\pi+2]$

(c) $x = \tan\theta \Rightarrow x^2 = \tan^2\theta$
$y = \cos^2\theta \Rightarrow \frac{1}{y} = \sec^2\theta$
$\Rightarrow 1 + \tan^2\theta = \sec^2\theta$
$1 + x^2 = \frac{1}{y}$
$y = \frac{1}{1+x^2}$

Question 13 (****)



The figure above shows the curve C, with parametric equations

$$x = 6t\sin t, \quad y = 3\sec t \ , \ 0 \le t < \frac{\pi}{2}.$$

The curve meets the coordinate axes at the point A.

The line $y = 6$ meets C at the point P.

a) Show that the area **under** the arc of the curve between A and P, and the x axis is given by the integral

$$18\int_0^{\frac{\pi}{3}} t + \tan t \ \ dt .$$

The shaded region R is bounded by C, the line $y = 6$ and the y axis.

b) Show that the area of R is approximately 10.3 square units.

$\boxed{}$, $\boxed{\text{proof}}$

[solution overleaf]

a) FIND THE CARTESIAN CO-ORDINATES & THE VALUE OF t AT P & A

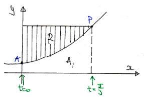
- $y = 3\sec t$
  $6 = 3\sec t$
  $2 = \sec t$
  $\cos t = \frac{1}{2}$
  $t = \frac{\pi}{3}$
- $x = 6t\sin t$
  $x = 6 \times \frac{\pi}{3} \times \sin\frac{\pi}{3}$
  $x = 2\pi \times \frac{\sqrt{3}}{2}$
  $x = \sqrt{3}\pi$

$\therefore P(\sqrt{3}\pi, 6)$ WITH $t = \frac{\pi}{3}$

$x = 0$
$t = 0$ (BY INSPECTION)

$\therefore A(0,3)$ WITH $t = 0$

LOOKING AT THE DIAGRAM



$$A_1 = \int_{x_1}^{x_2} y(x)\,dx$$

$$A_1 = \int_{t_1}^{t_2} y(t)\frac{dx}{dt}\,dt$$

$$A_1 = \int_0^{\frac{\pi}{3}} (3\sec t)\Big[\underbrace{6\sin t + 6t\cos t}_{\frac{dx}{dt}}\Big]\,dt$$

SIMPLIFYING THE INTEGRAND

$$A_1 = \int_0^{\frac{\pi}{3}} \frac{18}{\cos t}(\sin t + t\cos t)\,dt = 18\int_0^{\frac{\pi}{3}} \frac{\sin t}{\cos t} + \frac{t\cos t}{\cos t}\,dt$$

$$= 18\int_0^{\frac{\pi}{3}} \tan t + t\,dt$$

AS REQUIRED

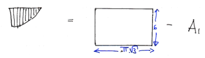
b) FINISHING THE INTEGRAL

$$A_1 = 18\int_0^{\pi/3} t + \tan t\,dt = 18\left[\frac{1}{2}t^2 + \ln|\sec t|\right]_0^{\frac{\pi}{3}}$$

$$= 18\left[\left[\frac{1}{2}\times\frac{\pi^2}{9} + \ln\left(\sec\frac{\pi}{3}\right)\right] - \left[0 + \ln(\cancel{\sec 0})\right]\right]$$

$$= 18\left[\frac{\pi^2}{18} + \ln 2\right] = \pi^2 + 18\ln 2$$

LOOKING AT THE PREVIOUS DIAGRAM



$= 6\pi\sqrt{3} - (18\ln 2 + \pi^2)$

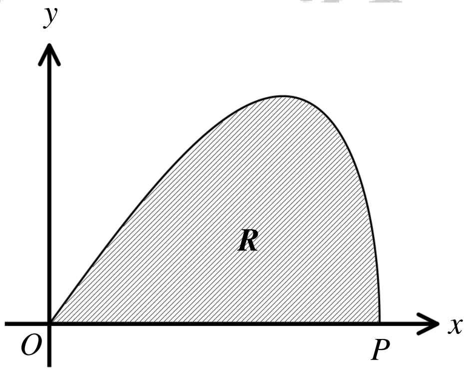
$= 6\pi\sqrt{3} - \pi^2 - 18\ln 2$

10.30213491

$\simeq 10.3$

AS REQUIRED

Question 14 (****)



The figure above shows the curve C with parametric equations

$$x = 5\cos t, \quad y = 3\sin 2t \ , \quad 0 \le t \le \frac{\pi}{2}.$$

The curve meets the x axis at the origin O and at the point P.

a) Find the value of t at O and at P.

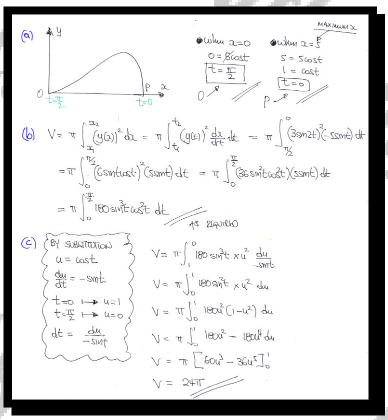
The finite region R bounded by C and the x axis is revolved by 2π radians in the x axis forming a solid of revolution S.

b) Show that the volume of S is given by the integral

$$\pi\int_0^{\frac{\pi}{2}} 180\sin^3 t\, \cos^2 t \ dt .$$

c) By using the substitution $u = \cos t$, or otherwise, find the volume of S.

$\boxed{\text{C4S}}$, $\boxed{t_O = \frac{\pi}{2},\ t_P = 0}$, $\boxed{\text{volume} = 24\pi}$


(a)

• when $x=0$: $0 = 5\cos t$, $\boxed{t = \frac{\pi}{2}}$ (O)

• when $x = 5$ (MAXIMUM x): $5 = 5\cos t$, $1 = \cos t$, $\boxed{t=0}$ (P)

(b) $V = \pi\int_{x_1}^{x_2} (y(x))^2\,dx = \pi\int_{t_1}^{t_2} (y(t))^2 \frac{dx}{dt}\,dt = \pi\int_{\pi/2}^{0} (3\sin 2t)^2(-5\sin t)\,dt$

$= \pi\int_0^{\pi/2} (6\sin t\cos t)^2(5\sin t)\,dt = \pi\int_0^{\frac{\pi}{2}} (36\sin^2 t\cos^2 t)(5\sin t)\,dt$

$= \pi\int_0^{\frac{\pi}{2}} 180\sin^3 t\cos^2 t\,dt$ AS REQUIRED

(c) BY SUBSTITUTION: $u = \cos t$, $\frac{du}{dt} = -\sin t$, $t=0 \mapsto u=1$, $t=\frac{\pi}{2} \mapsto u=0$, $dt = \frac{du}{-\sin t}$

$V = \pi\int_1^0 180\sin^3 t \times u^2 \frac{du}{-\sin t}$

$V = \pi\int_0^1 180\sin^2 t \times u^2\,du$

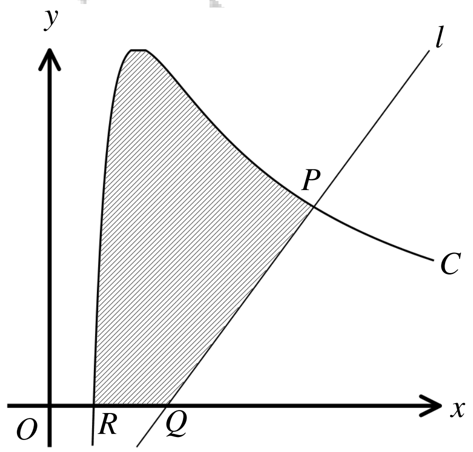
$V = \pi\int_0^1 180u^2(1-u^2)\,du$

$V = \pi\int_0^1 180u^2 - 180u^4\,du$

$V = \pi\left[60u^3 - 36u^5\right]_0^1$

$V = 24\pi$

Question 15 (****)



The figure above shows part of the curve C with parametric equations

$$x = \frac{6}{t}, \quad y = 6t - t^2 \ , \quad t \neq 0 .$$

The curve crosses the x axis at the point R.

The point $P(6,5)$ lies on C and the straight line l is the normal to C at P.

This normal crosses the x axis at the point Q.

a) Determine …

i. … the value of t at the points R and P.

ii. … an equation for l.

iii. … the coordinates of R and Q.

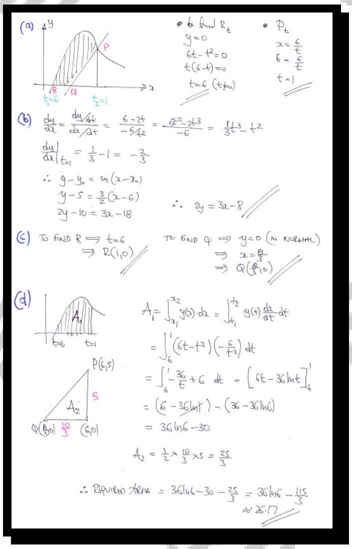
[continues overleaf]

[continued from overleaf]

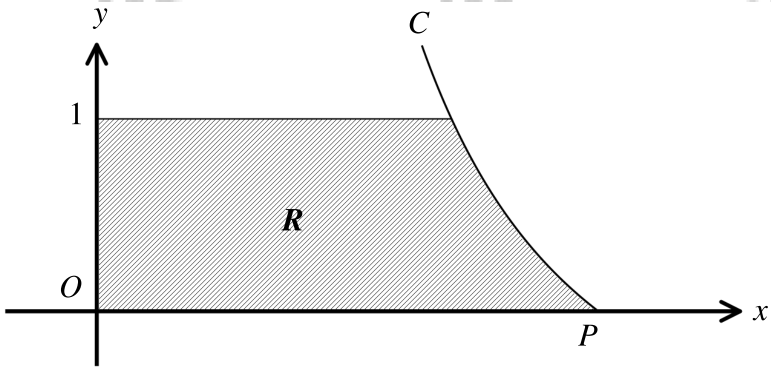
The finite region bounded by C, l and the x axis is shown shaded in the figure above.

b) Use parametric integration to find, correct to two decimal places, the area of this region.

$\boxed{}$, $\boxed{t_R = 6}$, $\boxed{t_P = 1}$, $\boxed{2y = 3x - 8}$, $\boxed{R(1,0)}$, $\boxed{Q\left(\frac{8}{3},0\right)}$, $\boxed{36\ln 6 - \frac{115}{3} \approx 26.17}$



Question 16 (****)



The figure above shows part of the curve C with parametric equations

$$x = \cos\theta, \quad y = \tan^2\theta \ , \quad 0 \le \theta < \frac{\pi}{2}.$$

The finite region R, shown shaded in the figure above, is bounded by C, the coordinate axes and the line $y = 1$.

a) Show that the area of R is given by the integral

$$\int_{\theta_1}^{\theta_2} 2\tan\theta\sec\theta \ d\theta,$$

for some appropriate limits θ_1 and θ_2.

b) Hence find an exact value for the area of R.

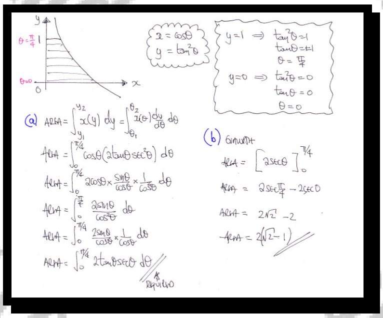
[continues overleaf]

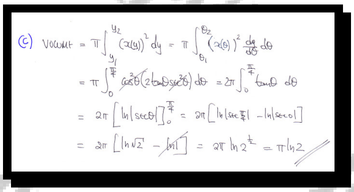
[continued from overleaf]

The finite region R is revolved by 2π radians in the y axis forming a solid of revolution S.

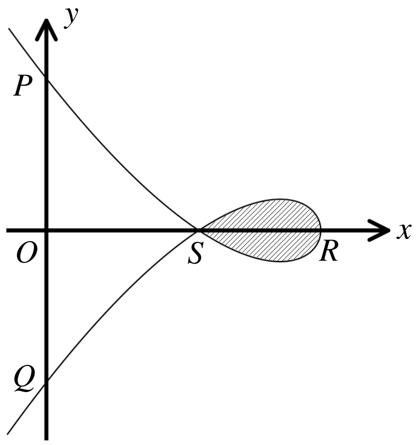
c) Show that the volume of S is exactly $\pi \ln 2$.

$\boxed{\theta_1 = 0}$, $\boxed{\theta_2 = \frac{\pi}{4}}$, $\boxed{\text{area} = 2\left(\sqrt{2}-1\right)}$





Question 17 (****)



The figure above shows the **re-entrant** curve C with parametric equations

$$x = 27 - 3t^2, \quad y = 5t\left(4 - t^2\right), \quad t \in \mathbb{R}.$$

The curve meets the y axis at P and Q, and the x axis at R and S.

a) Determine …

i. … the value of t at the points P, Q, R and S.

ii. … the Cartesian coordinates of the points P, Q, R and S.

b) Given that C is symmetrical about the x axis, show that the area enclosed by the loop of C, shown shaded in the figure above, is 256 square units.

c) Find a Cartesian equation of C, in the form $y^2 = f(x)$.

$\boxed{\text{N-C}}$, $\boxed{P(0,75), t=-3}$, $\boxed{Q(0,-75), t=3}$, $\boxed{R(27,0), t=0}$, $\boxed{S(15,0), t=\pm 2}$,

$\boxed{y^2 = \frac{25}{27}(27-x)(x-15)^2}$

[solution overleaf]

a) AT P & Q $x=0$

$27-3t^2=0$

$27=3t^2$

$t^2=9$

$t = 3, -3$

$t=3 \quad x=0 \quad y=-75$

$t=-3 \quad x=0 \quad y=75$

AT S & R, $y=0$

$0 = 5t(4-t^2)$

$0 = 5t(2-t)(2+t)$

$t = 0, 2, -2$

$t=0 \quad x=27 \quad y=0$

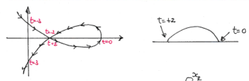
$t=2 \quad x=15 \quad y=0$

$t=-2 \quad x=15 \quad y=0$

$\therefore \quad t_P=-3, \ t_S=\mp 2, \ t_R=0, \ t_Q=3$

$\therefore \quad P(0,75) \quad Q(0,-75) \quad R(27,0) \quad S(15,0)$

b) LOOKING AT THE DIAGRAM & CONSIDERING THE TOP HALF OF THE LOOP



TOTAL LOOP AREA = 2 × TOP HALF $= 2\int_{x_S}^{x_R} y(x)\,dx$

$= 2\int_{t_S}^{t_R} y(t)\frac{dx}{dt}\,dt$

$= 2\int_{2}^{0} 5t(4-t^2)(-6t)\,dt$

$= 60\int_{2}^{0} -t^2(4-t^2)\,dt = 60\int_{0}^{2} t^2(4-t^2)\,dt$

$= 60\int_{0}^{2} 4t^2-t^4\,dt = 60\left[\frac{4}{3}t^3-\frac{1}{5}t^5\right]_0^2$

$= \left[80t^3-12t^5\right]_0^2 = (640-384)-(0)$

$= 256$ AS REQUIRED

c) WORK AS FOLLOWS

• $y = 5t(4-t^2)$

$y^2 = 25t^2(4-t^2)^2$

×27

$27y^2 = 27\left[25t^2(4-t^2)^2\right]$

$27y^2 = 75t^2(12-3t^2)^2$

$27y^2 = 25(27-x)(12-(27-x))^2$

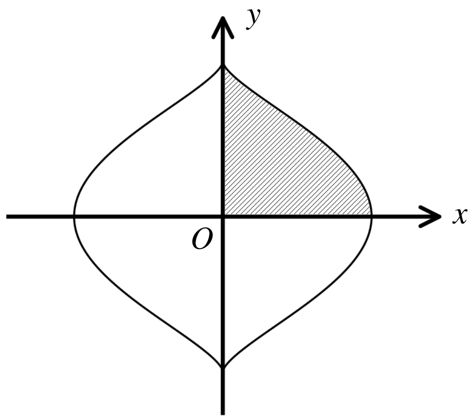
$27y^2 = 25(27-x)(x-15)^2$

$y^2 = \frac{25}{27}(27-x)(x-15)^2$

• $x = 27-3t^2$

$3t^2 = 27-x$

Question 18 **(****)**



The figure above shows the curve C with parametric equations

$$x = \cos^3\theta, \quad y = 12\sin\theta \ , \quad 0 \le \theta < 2\pi \ .$$

The finite region in the first quadrant, bounded by C and the coordinate axes is shown shaded in the figure above. The curve is symmetrical in both the x and in the y axis.

a) Show that the area of the shaded region is given by the integral

$$36\int_0^{\frac{\pi}{2}} \sin^2\theta\cos^2\theta \ d\theta \ .$$

b) Use trigonometric identities to show that

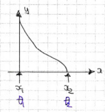
$$\cos^2\theta\sin^2\theta \equiv \frac{1}{8}\left(1-\cos 4\theta\right).$$

c) Hence find, in terms of π, the **total** area enclosed by C.

$\boxed{}$, $\boxed{9\pi}$

[solution overleaf]

a) LOOKING AT THE DIAGRAM



TO FIND θ_1 SET $x=0$

$0 = \cos^3\theta$

$\cos\theta = 0$

$\theta = \frac{\pi}{2} \leftarrow \theta_1$, $\cancel{\frac{3\pi}{2}}$ THIS PRODUCES $y=-12$

TO FIND θ_2, SET $y=0$

$12\sin\theta = 0$

$\sin\theta = 0$

$\theta = 0, \cancel{\pi}$ THIS PRODUCES $x=-1$; $0 \leftarrow \theta_2$

SETTING UP AN INTEGRAL

$$\text{AREA} = \int_{x_1}^{x_2} y(x)\,dx = \int_{\theta_1}^{\theta_2} y(\theta)\frac{dx}{d\theta}\,d\theta = \int_{\frac{\pi}{2}}^{0} 12\sin\theta\underbrace{\left(-3\cos^2\theta\sin\theta\right)}_{\frac{dx}{d\theta}}d\theta$$

$$= \int_{\frac{\pi}{2}}^{0} -36\cos^2\theta\sin^2\theta\,d\theta = +36\int_{0}^{\frac{\pi}{2}}\cos^2\theta\sin^2\theta\,d\theta = 36\int_{0}^{\frac{\pi}{2}}\sin^2\theta\cos^2\theta\,d\theta$$

AS REQUIRED

b) USING $\cos2\theta \equiv 2\cos^2\theta - 1$ & $\cos2\theta \equiv 1 - 2\sin^2\theta$

$$\cos^2\theta\sin^2\theta = \left(\frac{1}{2}+\frac{1}{2}\cos2\theta\right)\left(\frac{1}{2}-\frac{1}{2}\cos2\theta\right)$$

DIFFERENCE OF SQUARES

$$= \frac{1}{4} - \frac{1}{4}\cos^2 2\theta$$

NOW REAPPLY THE IDENTITY $\cos2\theta = 2\cos^2\theta - 1$ AS $\cos4\theta = 2\cos^2 2\theta - 1$

$$= \frac{1}{4} - \frac{1}{4}\cos^2 2\theta = \frac{1}{4} - \frac{1}{4}\left(\frac{1}{2}+\frac{1}{2}\cos4\theta\right) = \frac{1}{4} - \frac{1}{8} - \frac{1}{8}\cos4\theta$$

$$= \frac{1}{8} - \frac{1}{8}\cos4\theta = \frac{1}{8}(1-\cos4\theta)$$

AS REQUIRED

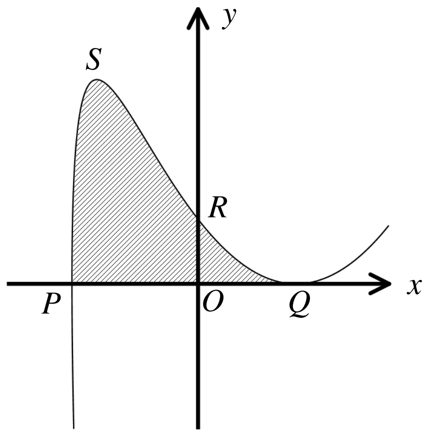
c) INTEGRATE, EVALUATE AND MULTIPLY BY 4 TO FIND THE TOTAL AREA

$$\text{AREA} = 4\int_{0}^{\frac{\pi}{2}} 36\sin^2\theta\cos^2\theta\,d\theta = 4\int_{0}^{\frac{\pi}{2}} 36\times\frac{1}{8}(1-\cos4\theta)\,d\theta$$

$$= \int_{0}^{\frac{\pi}{2}} 18 - 18\cos4\theta\,d\theta = \left[18\theta - \frac{9}{2}\cos4\theta\right]_{0}^{\pi/2}$$

$$= \left(9\pi - \cancel{\tfrac{9}{2}}\right) - \left(0 - \cancel{\tfrac{9}{2}}\right) = 9\pi$$

Question 19 (****)



The figure above shows part of the curve with parametric equations

$$x = t^2 - 9, \ \ y = t(4-t)^2, \ t \in \mathbb{R}.$$

The curve meets the x axis at the points P and Q, and the y axis at the points R and T. The point T is not shown in the figure.

a) Find the coordinates of each of the points P, Q, R and T.

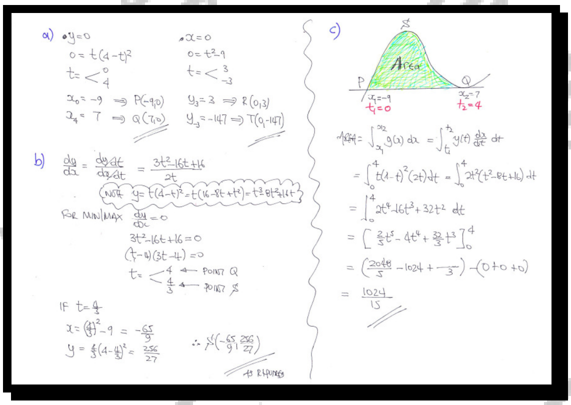
The point S is a stationary point of the curve.

b) Show that the coordinates of S are $\left(-\frac{65}{9}, \frac{256}{27}\right)$.

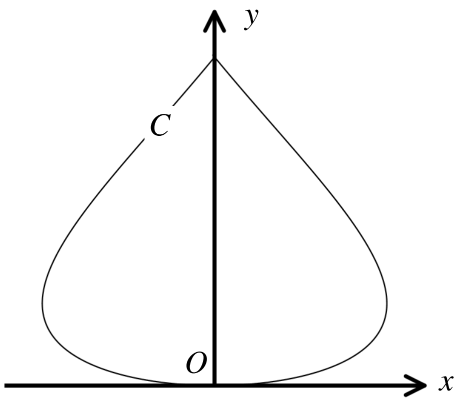
The region bounded by the curve and the x axis is shown shaded in the figure above.

c) Determine an exact value for the area of the shaded region.

$\boxed{}$, $\boxed{P(-9,0), Q(7,0), R(0,3), T(0,-147)}$, $\boxed{\text{area} = \frac{1024}{15}}$



Question 20 (****)



The figure above shows the curve C with parametric equations

$$x = \sin t \, , \;\; y = t^2 \, , \;\; 0 \le t \le 2\pi \, .$$

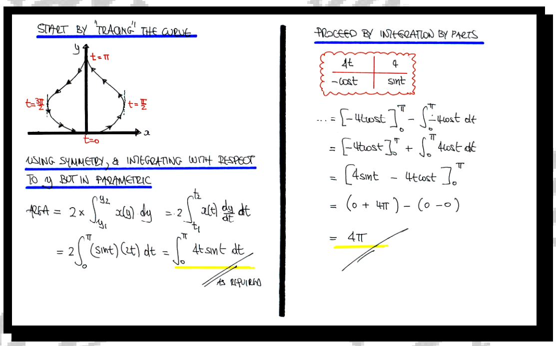
It is given that C is symmetrical about the y axis.

Show that the area enclosed by C can be found by the integral

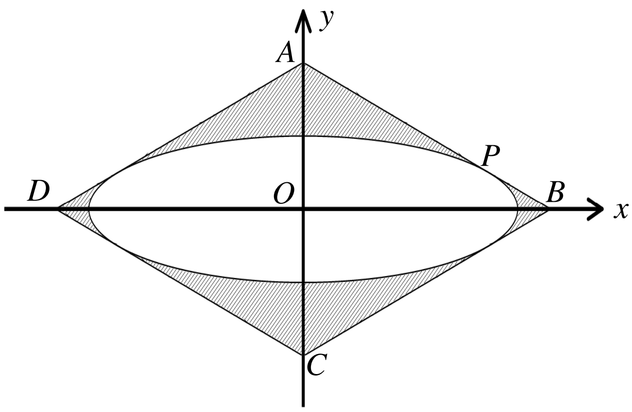
$$\int_0^{\pi} 4t \sin t \;\, dt \, ,$$

and hence find an exact value for this area.

$\boxed{}$, $\boxed{4\pi}$



Question 21 **(****)**



The figure above shows the design of a pendant $ABCD$ in the shape of a rhombus, made up of two different types of metal.

The innermost part of the design is enclosed by a curve and is made of silver. The rest of the design is made of gold.

The design is symmetrical about both the x and the y axis.

The innermost part of the design is modelled by an ellipse, given parametrically by

$$x = 12\cos\theta, \ y = 6\sin\theta, \ 0 \le \theta < 2\pi .$$

a) Use integration to show that the area enclosed by the ellipse is exactly 72π.

[continues overleaf]

[continued from overleaf]

The point P lies on the ellipse where $\theta = \dfrac{\pi}{6}$.

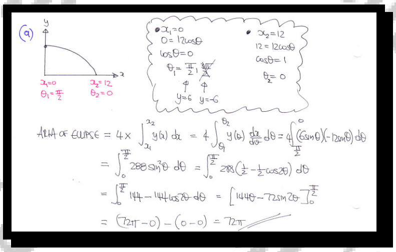
The straight line AB is the tangent to the ellipse at P.

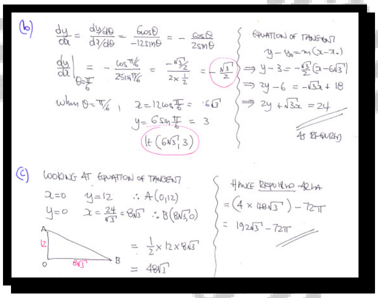
b) Show that the equation of the tangent AB can be written as

$$2y + \sqrt{3}x = 24.$$

c) Hence find an exact value for the area of the pendant that is made up of gold.

$\boxed{}$, $\boxed{\text{area} = 192\sqrt{3} - 72\pi}$





Question 22 (****)

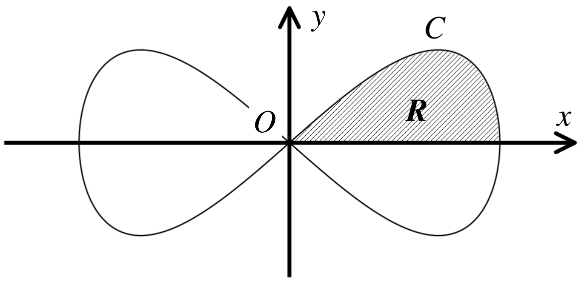


figure 1

The figure 1 above, shows the curve C with parametric equations

$$x = 6\cos t, \quad y = 12\sin 2t, \quad 0 \le t \le 2\pi .$$

The curve is symmetrical in the x axis and in the y axis.

The region R, shown shaded on the figure 1, is bounded by the part of C in the first quadrant and the coordinate axes.

a) Show that the area of R is given by

$$\int_0^{\frac{\pi}{2}} 144\cos t\sin^2 t \ \ dt .$$

b) Hence find the area enclosed by C in all four quadrants.

[continues overleaf]

[continued from overleaf]

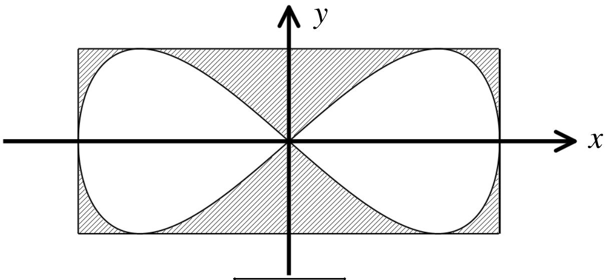


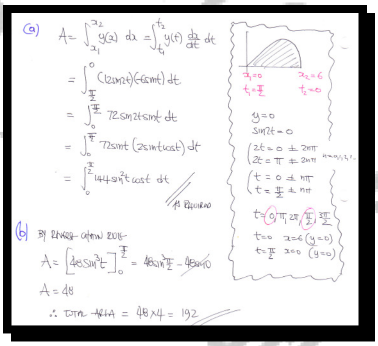
figure 2

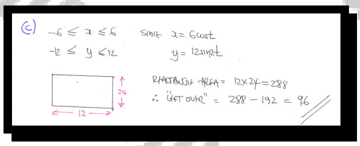
The area enclosed by the entire curve is to be cut out of a piece of rectangular card, as shown in the figure 2. This is modelled by a rectangle whose sides are tangents to the curve, parallel to the coordinate axes.

The area of the card left over after the curve was cut out is shown shaded in figure 2.

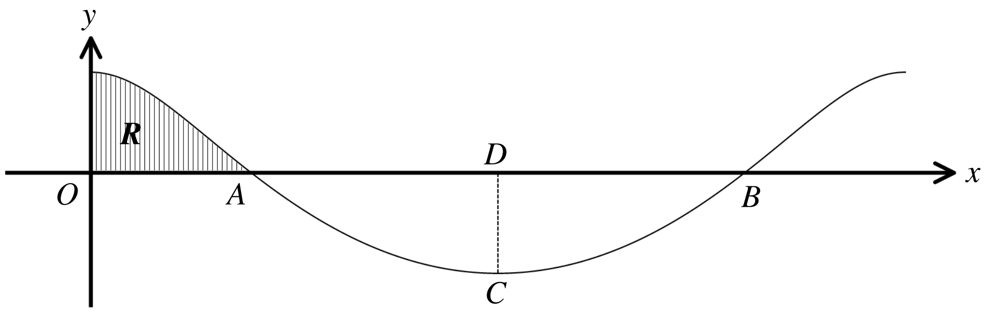
c) Show that the area of the card left over is exactly 96 square units.

$\boxed{}$, $\boxed{\text{area} = 192}$





Question 23 (****)



The diagram above shows the curve defined by the parametric equations

$$x = 4\theta - \sin\theta, \quad y = 2\cos\theta, \text{ for } 0 \le \theta < 2\pi.$$

The curve crosses the x axis at points A and B. The point C is the minimum point on the curve and CD is perpendicular to the x axis and a line of symmetry for the curve.

a) Find the exact coordinates of A, B and C.

b) Show that an equation of the tangent to the curve at the point A is given by

$$x + 2y = 2\pi - 1.$$

[continues overleaf]

[continued from overleaf]

c) Show further that the area of the region R bounded by the curve and the coordinate axes is given by

$$\int_0^{\frac{\pi}{2}} 8\cos\theta - 2\cos^2\theta \ d\theta .$$

d) Determine an exact value for this integral.

$\boxed{A(2\pi-1,0)}$, $\boxed{B(6\pi+1,0)}$, $\boxed{C(4\pi,-2)}$, $\boxed{8-\frac{\pi}{2}}$

(a) $x = 4\theta - \sin\theta$
$y = 2\cos\theta$

$y=0$
$2\cos\theta = 0$
$\cos\theta = 0$
$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$

$\therefore\ x_1 = 4\left(\frac{\pi}{2}\right) - \sin\frac{\pi}{2} = 2\pi - 1$
$x_2 = 4\left(\frac{3\pi}{2}\right) - \sin\frac{3\pi}{2} = 6\pi + 1$

$\therefore\ A(2\pi-1,0)$ & $B(6\pi+1)$

BY SYMMETRY

$C\left(\frac{(2\pi-1)+(6\pi+1)}{2}, -2\right)$ ← $y = 2\cos\theta$ MIN OF -2

$C(4\pi,-2)$

(b) $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-2\sin\theta}{4-\cos\theta}$

$\left.\frac{dy}{dx}\right|_{A\ (\theta=\frac{\pi}{2})} = \frac{-2\sin\frac{\pi}{2}}{4-\cancel{\cos\frac{\pi}{2}}} = \frac{-2}{4} = -\frac{1}{2}$

• HENCE $m=-\frac{1}{2}$ $A(2\pi-1,0)$
TANGENT
$\Rightarrow y - y_0 = m(x-x_0)$
$\Rightarrow y - 0 = -\frac{1}{2}(x-(2\pi-1))$
$\Rightarrow 2y = -x + (2\pi - 1)$
$\Rightarrow 2y + x = 2\pi - 1$

(c) AREA $= \int_{x_1}^{x_2} y\ dx = \int_{\theta_1}^{\theta_2} y(\theta)\frac{dx}{d\theta} = \int_0^{\frac{\pi}{2}} (2\cos\theta)(4-\cos\theta)\ d\theta$ ← FOUND EARLIER ← by inspection

$= \int_0^{\frac{\pi}{2}} 8\cos\theta - 2\cos^2\theta\ d\theta$

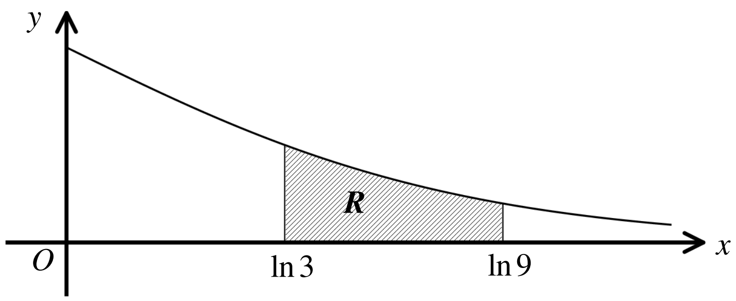
AS REQUIRED

(d) $\int_0^{\pi/2} 8\cos\theta - 2\cos^2\theta\ d\theta = \int_0^{\pi/2} 8\cos\theta - 2\left(\frac{1}{2}+\frac{1}{2}\cos 2\theta\right) d\theta$

$= \int_0^{\pi/2} 8\cos\theta - 1 - \cos 2\theta\ d\theta = \left[8\sin\theta - \frac{1}{2}\sin 2\theta - \theta\right]_0^{\pi/2}$

$= \left(8 - 0 - \frac{\pi}{2}\right) - 0 = 8 - \frac{\pi}{2}$

Question 24 (****)



The figure above shows the curve C with parametric equations

$$x = \ln(t+1), \quad y = \frac{2}{t+2}, \quad t \in \mathbb{R},\ t \geq 0 .$$

The finite region R, shown shaded in the figure above, is bounded by C, the straight lines with equations $x = \ln 3$ and $x = \ln 9$ and the x axis.

a) Show that the area R is given by the integral

$$I = \int_2^8 \frac{2}{(t+1)(t+2)}\, dt .$$

b) Find an exact value for the above integral.

c) Show that a Cartesian equation of C is

$$y = \frac{2}{e^x + 1} .$$

[continues overleaf]

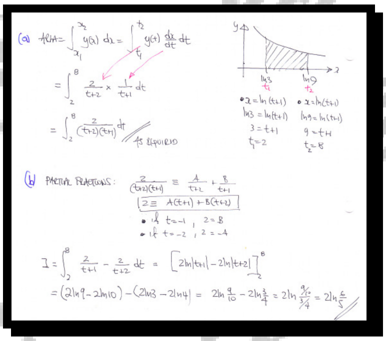
[continued from overleaf]

d) Use the Cartesian equation of C and the substitution $u = \mathrm{e}^x + 1$ to show that the area of R can also be found by the integral

$$J = \int_4^{10} \frac{2}{u(u-1)}\, du\,.$$

e) Without evaluating J, show that $J = I$.

$\boxed{\quad}$, $\boxed{2\ln\left(\frac{6}{5}\right)}$



(a) AREA $= \int_{x_1}^{x_2} y(x)\,dx = \int_{t_1}^{t_2} y(t)\frac{dx}{dt}\,dt$

$= \int_2^8 \frac{2}{t+2} \times \frac{1}{t+1}\,dt$

$= \int_2^8 \frac{2}{(t+2)(t+1)}\,dt$ AS REQUIRED

• $x = \ln(t+1)$, $\ln 3 = \ln(t+1)$, $3 = t+1$, $t_1 = 2$

• $x = \ln(t+1)$, $\ln 9 = \ln(t+1)$, $9 = t+1$, $t_2 = 8$

(b) PARTIAL FRACTIONS: $\frac{2}{(t+2)(t+1)} \equiv \frac{A}{t+2} + \frac{B}{t+1}$

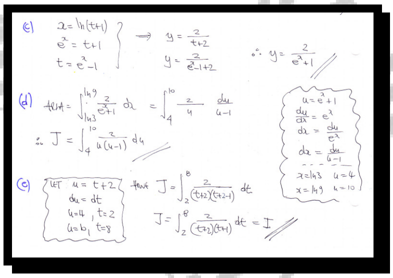
$\boxed{2 \equiv A(t+1) + B(t+2)}$

• if $t = -1$, $2 = B$

• if $t = -2$, $2 = -A$

$I = \int_2^8 \frac{2}{t+1} - \frac{2}{t+2}\,dt = \Big[2\ln|t+1| - 2\ln|t+2|\Big]_2^8$

$= (2\ln 9 - 2\ln 10) - (2\ln 3 - 2\ln 4) = 2\ln\frac{9}{10} - 2\ln\frac{3}{4} = 2\ln\frac{9/10}{3/4} = 2\ln\frac{6}{5}$



(c) $x = \ln(t+1)$, $e^x = t+1$, $t = e^x - 1$ $\Rightarrow$ $y = \frac{2}{t+2}$, $y = \frac{2}{e^x - 1 + 2}$ $\therefore y = \frac{2}{e^x+1}$

(d) AREA $= \int_{\ln 3}^{\ln 9} \frac{2}{e^x+1}\,dx = \int_4^{10} \frac{2}{u}\,\frac{du}{u-1}$

$\therefore J = \int_4^{10} \frac{2}{u(u-1)}\,du$

$u = e^x + 1$, $\frac{du}{dx} = e^x$, $dx = \frac{du}{e^x}$, $dx = \frac{du}{u-1}$

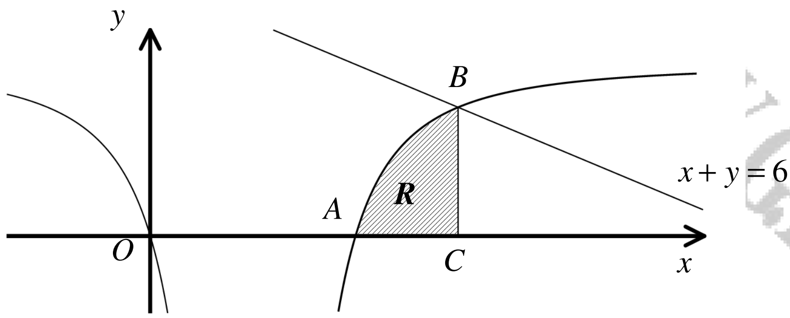
$x = \ln 3$, $u = 4$; $x = \ln 9$, $u = 10$

(e) LET $u = t+2$, $du = dt$, $u = 4$, $t = 2$; $u = 10$, $t = 8$

then $J = \int_2^8 \frac{2}{(t+2)(t+2-1)}\,dt$

$J = \int_2^8 \frac{2}{(t+2)(t+1)}\,dt = I$

Question 25 (****)



The figure above shows two sections of a curve with parametric equations

$$x = \frac{2}{t} + 1, \quad y = 4 - t^2, \text{ for } t \in \mathbb{R},\ t \neq 0.$$

The curve crosses the x axis at the origin O and at the point A.

The straight line with equation $x + y = 6$ intersects the curve at the point B.

a) Find the value of t at the point A.

b) Determine the coordinates of B.

The line BC is parallel to the y axis.

The finite region R, bounded by the curve, the y axis and the line BC is revolved by 2π radians to form a solid of revolution S.

c) Use integration in parametric form to find an exact value for the volume of S.

$\boxed{t = 2}$, $\boxed{B(3,3)}$, $\boxed{\text{volume} = \frac{14\pi}{3}}$

(a) $x = \frac{2}{t} + 1$, $y = 4 - t^2$

when $y = 0$: $0 = 4 - t^2$, $0 = (2-t)(2+t)$, $t = -2$ or 2, $x = 0$ ← ORIGIN, $x = 2$ ← POINT A

$\therefore t = 2$

(b) $x + y = 6$, $x = \frac{2}{t} + 1$, $y = 4 - t^2$ SOLVING SIMULTANEOUSLY

$\left(\frac{2}{t} + 1\right) + (4 - t^2) = 6$

$-t^2 + \frac{2}{t} - 1 = 0$

$-t^3 + 2 - t = 0$

$t^3 + t - 2 = 0$

BY INSPECTION $t = 1$ IS A SOLUTION

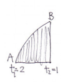
$(t-1)(t^2 + t + 2) = 0$

IRREDUCIBLE $b^2 - 4ac = -7$

$t = 1$ IS THE ONLY SOLUTION

$\therefore B\left(\frac{2}{1} + 1, 4 - 1^2\right)$ IE $B(3,3)$

(c) $\text{VOLUME} = \pi \int_{x_1}^{x_2} (y(x))^2 \, dx = \pi \int_{t_1}^{t_2} (y(t))^2 \frac{dx}{dt} \, dt = \pi \int_2^1 (4 - t^2)^2 \left(-\frac{2}{t^2}\right) dt$



$= \pi \int_1^2 \frac{2}{t^2}(16 - 8t^2 + t^4) \, dt$

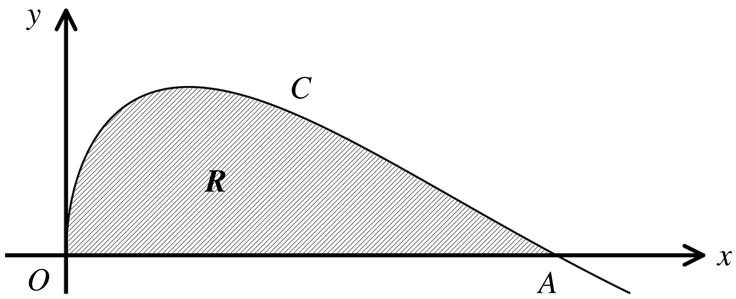
$= 2\pi \int_1^2 \frac{16}{t^2} - 8 + t^2 \, dt$

$= 2\pi \left[-\frac{16}{t} - 8t + \frac{1}{3}t^3\right]_1^2$

$= 2\pi \left[\left(-8 - 16 + \frac{8}{3}\right) - \left(-16 - 8 + \frac{1}{3}\right)\right]$

$= \frac{14\pi}{3}$

Question 26 (****)



The figure above shows the curve C with parametric equations

$$x = t^2, \quad y = 4\sin 2t \;, \quad t \in \mathbb{R},\ t \geq 0 .$$

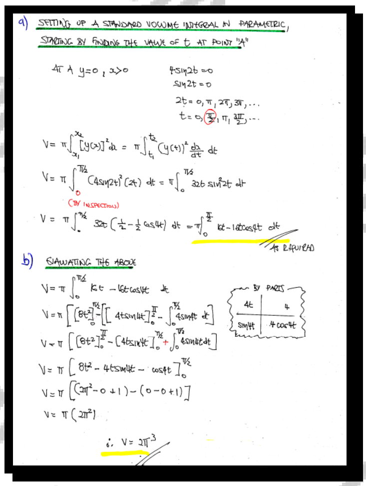
The curve crosses the x axis for the first time at the point A. The finite region R, shown shaded in the figure above, is bounded by C and the part of the x axis from the origin O to the point A. This region is revolved about the x axis to form a solid of revolution S.

a) Show that the volume of S is given by the integral

$$I = \pi \int_0^{\frac{\pi}{2}} 16t - 16\cos 4t \;\; dt .$$

b) Hence find an exact value for the volume of S.

$\boxed{}$, $\boxed{2\pi^3}$



a) SETTING UP A STANDARD VOLUME INTEGRAL IN PARAMETRIC, STARTING BY FINDING THE VALUE OF t AT POINT "A"

AT A $y=0,\ x>0$

$4\sin 2t = 0$

$\sin 2t = 0$

$2t = 0, \pi, 2\pi, 3\pi, \dots$

$t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \dots$

$V = \pi \int_{x_1}^{x_2} [y(x)]^2 dx = \pi \int_{t_1}^{t_2} (y(t))^2 \frac{dx}{dt}\, dt$

$V = \pi \int_0^{\pi/2} (4\sin 2t)^2 (2t)\, dt = \pi \int_0^{\pi/2} 32t \sin^2 2t\, dt$

(BY INSPECTION)

$V = \pi \int_0^{\pi/2} 32t\left(\frac{1}{2} - \frac{1}{2}\cos 4t\right) dt = \pi \int_0^{\frac{\pi}{2}} 16t - 16t\cos 4t\, dt$

AS REQUIRED

b) EVALUATING THE ABOVE

$V = \pi \int_0^{\pi/2} 16t - 16t\cos 4t\, dt$

BY PARTS

$4t$	4
$\sin 4t$	$4\cos 4t$

$V = \pi\left[\left[8t^2\right]_0^{\pi/2} - \left[\left[4t\sin 4t\right]_0^{\frac{\pi}{2}} - \int_0^{\pi/2} 4\sin 4t\, dt\right]\right]$

$V = \pi\left[\left[8t^2\right]_0^{\frac{\pi}{2}} - \left[4t\sin 4t\right]_0^{\pi/2} + \int_0^{\pi/2} 4\sin 4t\, dt\right]$

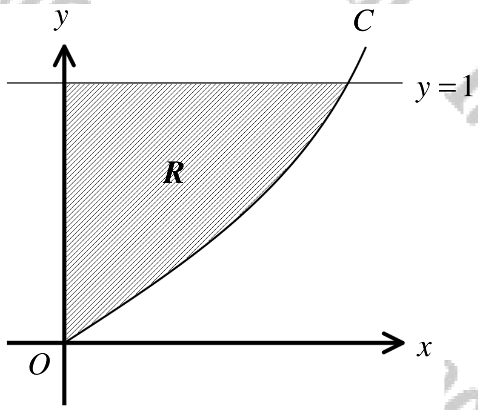
$V = \pi\left[8t^2 - 4t\sin 4t - \cos 4t\right]_0^{\pi/2}$

$V = \pi\left[(2\pi^2 - 0 + 1) - (0 - 0 + 1)\right]$

$V = \pi(2\pi^2)$

$\therefore V = 2\pi^3$

Question 27 (****)



The figure above shows the curve C with parametric equations

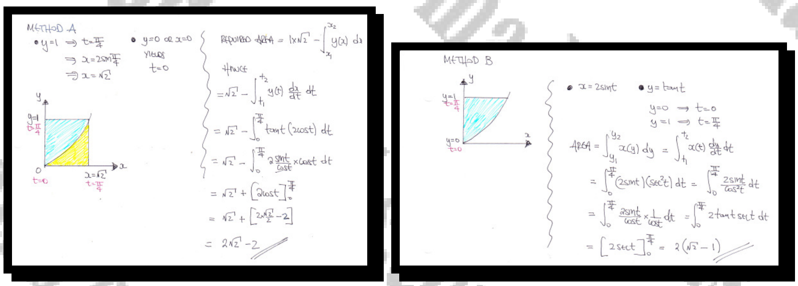
$$x = 2\sin t, \quad y = \tan t \ , \ \text{for} \ \ 0 \le t < \frac{\pi}{2}.$$

The curve passes through the origin O.

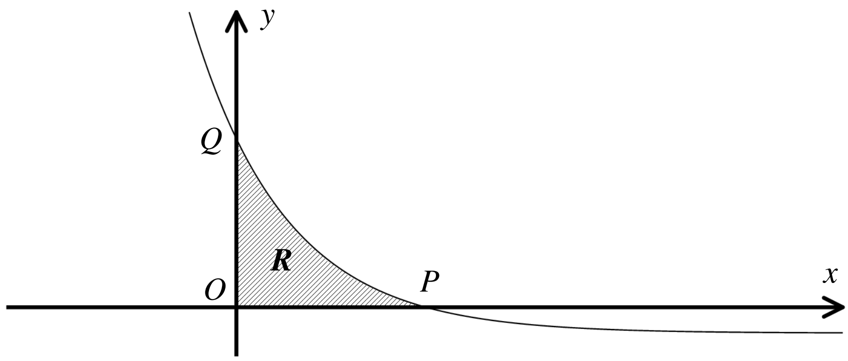
The region R is bounded by C, the y axis and the line $y = 1$.

Use integration in parametric form to find an exact value for the area of R.

$\boxed{\text{area} = 2\sqrt{2} - 2}$



Question 28 (****)



The figure above shows the graph of the curve with parametric equations

$$x = 2 - \tfrac{1}{4}t, \quad y = 2^{t} - 2, \quad t \in \mathbb{R}.$$

The curve meets the x axis at the point P and the y axis at the point Q.

a) Find the coordinates of P and Q.

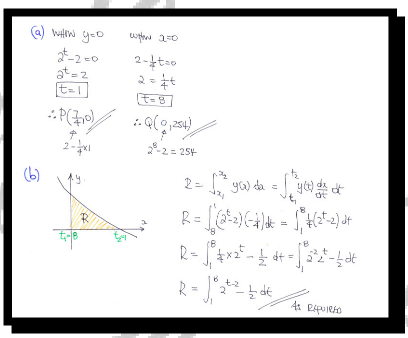
The finite region R is bounded by the curve and the coordinate axes, and is shown shaded in the figure above.

b) Show that the area R is given by the integral

$$\int_{1}^{8} 2^{t-2} - \tfrac{1}{2} \; dt.$$

c) Hence find an exact value for R.

$\boxed{}$, $\boxed{P\left(\tfrac{7}{4},0\right)}$, $\boxed{Q(0,254)}$, $\boxed{\dfrac{127}{2\ln 2} - \dfrac{7}{2}}$


(a) when $y=0$ when $x=0$

$2^t - 2 = 0$ $2 - \frac{1}{4}t = 0$

$2^t = 2$ $2 = \frac{1}{4}t$

$\boxed{t=1}$ $\boxed{t=8}$

$\therefore P\left(\frac{7}{4},0\right)$ $\therefore Q(0,254)$

$2-\frac{1}{4}\times 1$ $2^8 - 2 = 254$

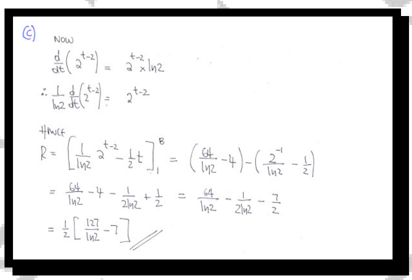
(b)

$R = \int_{x_1}^{x_2} y(x)\,dx = \int_{t_1}^{t_2} y(t)\frac{dx}{dt}\,dt$

$R = \int_{8}^{1} (2^t - 2)\left(-\frac{1}{4}\right)dt = \int_{1}^{8} \frac{1}{4}(2^t - 2)\,dt$

$R = \int_{1}^{8} \frac{1}{4}\times 2^t - \frac{1}{2}\,dt = \int_{1}^{8} 2^{-2}2^t - \frac{1}{2}\,dt$

$R = \int_{1}^{8} 2^{t-2} - \frac{1}{2}\,dt$ as required


(c) Now

$\frac{d}{dt}\left(2^{t-2}\right) = 2^{t-2}\times\ln 2$

$\therefore \frac{1}{\ln 2}\frac{d}{dt}\left(2^{t-2}\right) = 2^{t-2}$

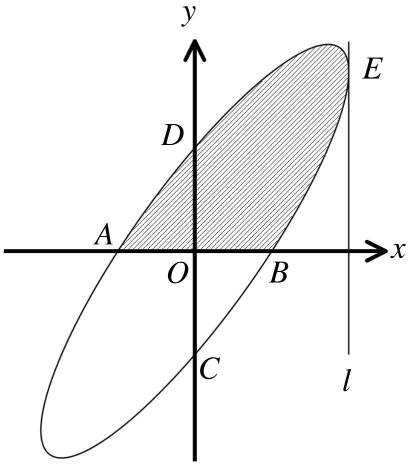
Hence

$R = \left[\frac{1}{\ln 2}2^{t-2} - \frac{1}{2}t\right]_1^8 = \left(\frac{64}{\ln 2} - 4\right) - \left(\frac{2^{-1}}{\ln 2} - \frac{1}{2}\right)$

$= \frac{64}{\ln 2} - 4 - \frac{1}{2\ln 2} + \frac{1}{2} = \frac{64}{\ln 2} - \frac{1}{2\ln 2} - \frac{7}{2}$

$= \frac{1}{2}\left[\frac{127}{\ln 2} - 7\right]$

Question 29 (****)



The figure above shows an ellipse with parametric equations

$$x = 2\cos\theta, \;\; y = 6\sin\left(\theta + \frac{\pi}{3}\right), \;\; -\pi \le \theta < \pi .$$

The curve meets the coordinate axes at the points A, B, C and D.

a) Find the coordinates of the points A, B, C and D.

The straight line l is the tangent to the ellipse at the point E.

b) State the equation of l, given that l is parallel to the y axis.

c) Find the value of θ at the point E.

[continues overleaf]

[continued from overleaf]

The finite region bounded by the ellipse and the x axis for which $y \geq 0$ is shown shaded in the figure above.

d) Show that the area of this region is given by the integral

$$\int_{-\frac{\pi}{3}}^{\frac{2\pi}{3}} 3 - 3\cos 2\theta + 3\sqrt{3}\sin 2\theta \ d\theta .$$

e) Hence find the area of the shaded region.

$\boxed{}$, $\boxed{A(-1,0),\ B(1,0),\ C(-3,0),\ D(-3,0)}$, $\boxed{x=2}$, $\boxed{\theta_A = 0}$, $\boxed{\text{area} = 3\pi}$

(a) when $x=0$: $0 = 2\cos\theta$, $\cos\theta = 0$, $\theta = \frac{\pi}{2}, -\frac{\pi}{2}$

$\therefore y = 6\sin\left(\frac{\pi}{2}+\frac{\pi}{3}\right) = 3$

$y = 6\sin\left(-\frac{\pi}{2}+\frac{\pi}{3}\right) = -3$

$\therefore D(0,3)$ & $C(0,-3)$

when $y=0$: $0 = 6\sin\left(\theta+\frac{\pi}{3}\right)$, $\sin\left(\theta+\frac{\pi}{3}\right) = 0$

$\theta + \frac{\pi}{3} = 0, \pi$

$\theta = -\frac{\pi}{3}, \frac{2\pi}{3}$

$x = 2\cos\left(-\frac{\pi}{3}\right) = 1$

$x = 2\cos\left(\frac{2\pi}{3}\right) = -1$

$\therefore A(-1,0)$ & $B(1,0)$

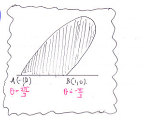
(b) $x = 2\cos\theta$, $-\pi \leq \theta < \pi \implies x_{MAX} = 2$

$\therefore$ TANGENT: $x = 2$

(c) $2 = 2\cos\theta$

$\cos\theta = 1$ $\quad \therefore \theta_A = 0$ (ONLY VALUE IN THIS RANGE)

(d)



AS CURVE IS ENTIRELY ABOVE x AXIS INTEGRATE CONTINUOUSLY FROM $\frac{2\pi}{3}$ TO $-\frac{\pi}{3}$

HENCE $A = \int_{x_1}^{x_2} y(x)\, dx$

$A = \int_{\theta_1}^{\theta_2} y(\theta)\frac{dx}{d\theta}\, d\theta$

IN THIS CASE

$A = \int_{\frac{2\pi}{3}}^{-\frac{\pi}{3}} 6\sin\left(\theta+\frac{\pi}{3}\right)(-2\sin\theta)\, d\theta = \int_{\frac{2\pi}{3}}^{-\frac{\pi}{3}} -12\sin\theta\sin\left(\theta+\frac{\pi}{3}\right) d\theta$

$= \int_{-\frac{\pi}{3}}^{\frac{2\pi}{3}} 12\sin\theta\left[\sin\theta\cos\frac{\pi}{3} + \cos\theta\sin\frac{\pi}{3}\right] d\theta$

$= \int_{-\frac{\pi}{3}}^{\frac{2\pi}{3}} 12\sin\theta\left(\frac{1}{2}\sin\theta + \frac{\sqrt{3}}{2}\cos\theta\right) d\theta = \int_{-\frac{\pi}{3}}^{\frac{2\pi}{3}} 6\sin^2\theta + 6\sqrt{3}\sin\theta\cos\theta\, d\theta$

$= \int_{-\frac{\pi}{3}}^{\frac{2\pi}{3}} 6\left(\frac{1}{2}-\frac{1}{2}\cos 2\theta\right) + 3\sqrt{3}\sin 2\theta\, d\theta = \int_{-\frac{\pi}{3}}^{\frac{2\pi}{3}} 3 - 3\cos 2\theta + 3\sqrt{3}\sin 2\theta\, d\theta$

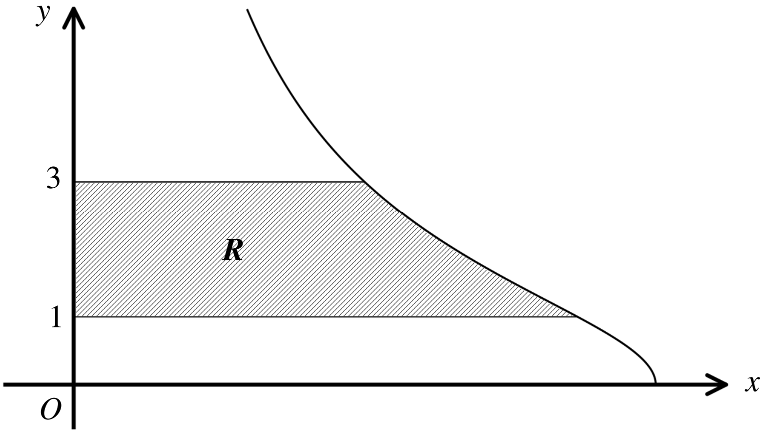
AS REQUIRED

(e) $A = \left[3\theta - \frac{3}{2}\sin 2\theta - \frac{3\sqrt{3}}{2}\cos 2\theta\right]_{-\frac{\pi}{3}}^{\frac{2\pi}{3}}$

$A = \left[2\pi - \frac{3}{2}\times\left(-\frac{\sqrt{3}}{2}\right) - \frac{3}{2}\sqrt{3}\left(-\frac{1}{2}\right)\right] - \left[-\pi - \frac{3}{2}\left(-\frac{\sqrt{3}}{2}\right) - \frac{3\sqrt{3}}{2}\left(-\frac{1}{2}\right)\right]$

$A = \left[2\pi + \cancel{\frac{3}{4}\sqrt{3}} + \cancel{\frac{3}{4}\sqrt{3}}\right] - \left[-\pi + \cancel{\frac{3}{4}\sqrt{3}} + \cancel{\frac{3}{4}\sqrt{3}}\right] = 3\pi$

Question 30 (****)



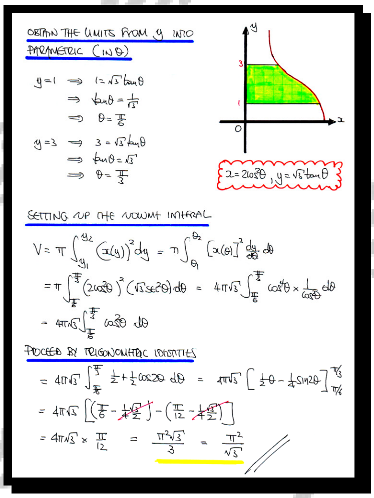
The figure above shows the curve with parametric equations

$$x = 2\cos^2\theta, \quad y = \sqrt{3}\tan\theta, \quad 0 \le \theta < \frac{\pi}{2}.$$

The finite region R shown shaded in the figure, bounded by the curve, the y axis, and the straight lines with equations $y = 1$ and $y = 3$.

Use integration in parametric to show that the volume of the solid formed when R is fully revolved about the y axis is $\dfrac{\pi^2}{\sqrt{3}}$.

$\boxed{}$, $\boxed{\text{proof}}$


OBTAIN THE LIMITS FROM y INTO PARAMETRIC (IN θ)

$y=1 \Rightarrow 1 = \sqrt{3}\tan\theta \Rightarrow \tan\theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6}$

$y=3 \Rightarrow 3 = \sqrt{3}\tan\theta \Rightarrow \tan\theta = \sqrt{3} \Rightarrow \theta = \frac{\pi}{3}$

$x = 2\cos^2\theta, \ y = \sqrt{3}\tan\theta$

SETTING UP THE VOLUME INTEGRAL

$$V = \pi\int_{y_1}^{y_2} (x(y))^2\,dy = \pi\int_{\theta_1}^{\theta_2} [x(\theta)]^2 \frac{dy}{d\theta}\,d\theta$$

$$= \pi\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (2\cos^2\theta)^2(\sqrt{3}\sec^2\theta)\,d\theta = 4\pi\sqrt{3}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos^4\theta \times \frac{1}{\cos^2\theta}\,d\theta$$

$$= 4\pi\sqrt{3}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos^2\theta\,d\theta$$

PROCEED BY TRIGONOMETRIC IDENTITIES

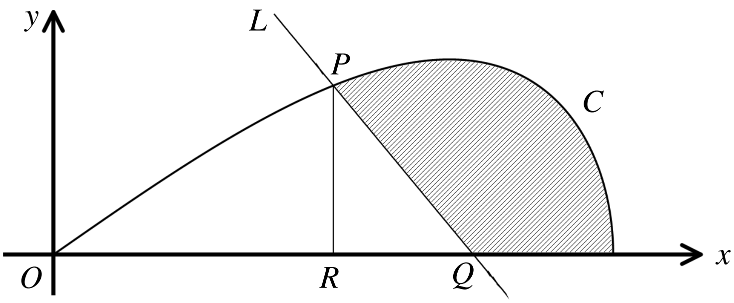
$$= 4\pi\sqrt{3}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{2} + \frac{1}{2}\cos 2\theta\,d\theta = 4\pi\sqrt{3}\left[\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_{\pi/6}^{\pi/3}$$

$$= 4\pi\sqrt{3}\left[\left(\frac{\pi}{6} - \cancel{\frac{1}{4}\frac{\sqrt{3}}{2}}\right) - \left(\frac{\pi}{12} - \cancel{\frac{1}{4}\frac{\sqrt{3}}{2}}\right)\right]$$

$$= 4\pi\sqrt{3} \times \frac{\pi}{12} = \frac{\pi^2\sqrt{3}}{3} = \frac{\pi^2}{\sqrt{3}}$$

23 HARD QUESTIONS

Question 1 (****+)



The figure above shows the curve C with parametric equations

$$x = 6\cos t, \quad y = 3\sin 2t \ , \quad 0 \le t \le \frac{\pi}{2} .$$

The point P lies on C where $x = 3$.

a) Find the y coordinate of P.

The line L is the normal to the curve at P. This normal meets the x axis at Q.

b) Show that an equation of L is

$$2y + 2x\sqrt{3} = 9\sqrt{3} .$$

[continues overleaf]

[continued from overleaf]

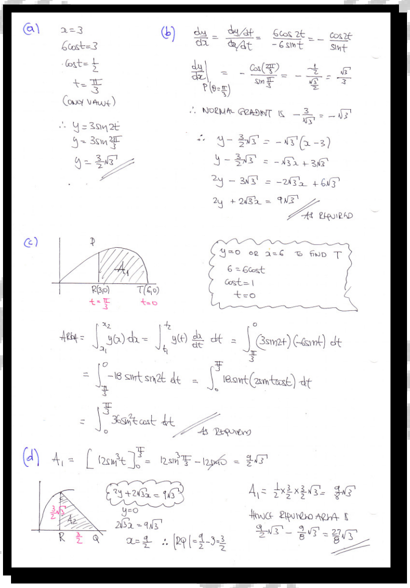
The line PR is parallel to the y axis.

c) Show that the area of the finite region bounded by C, the line PR and the x axis is given by the integral

$$\int_0^{\frac{\pi}{3}} 36\sin^2 t\cos t \ dt .$$

d) Hence find an exact value for the area of the **shaded region**, bounded by C, the normal L and the x axis

$$\boxed{\frac{3\sqrt{3}}{2}}, \ \boxed{\text{area} = \frac{27\sqrt{3}}{8}}$$



Question 2 (****+)

A curve has parametric equations

$$x = \frac{1-e^{4-4t}}{4}, \qquad y = \frac{2t}{e^{2t}}, \quad t \in \mathbb{R}.$$

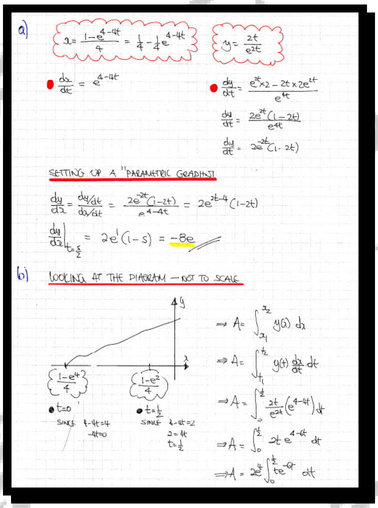
a) Find the gradient at the point on the curve where $t = \frac{5}{2}$.

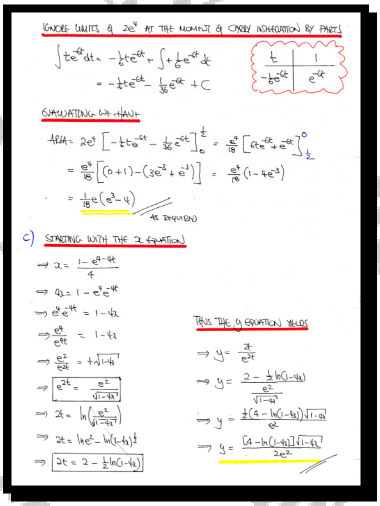
b) Show that the finite area bounded by the curve, the x axis, and the straight lines with equations $x = \frac{1-e^4}{4}$ and $x = \frac{1-e^2}{4}$, is exactly

$$\frac{1}{18}e\left(e^3 - 4\right).$$

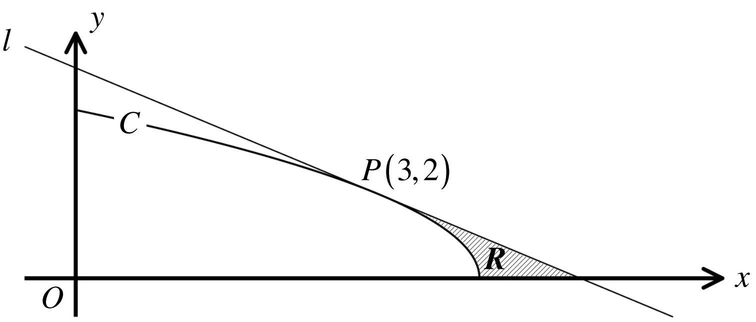
c) Determine a Cartesian equation of the curve in the form $y = f(x)$.

$\boxed{\text{SYN}}$, $\boxed{\left.\frac{dy}{dx}\right|_{t=\frac{5}{2}} = -8e}$, $\boxed{y = \frac{\left[4 - \ln(1-4x)\right]\sqrt{1-4x}}{2e^2}}$





Question 3 (****+)



The figure above shows the curve C with parametric equations

$$x = 4\cos^2\theta \ , \ y = 4\sin\theta , \quad 0 \le \theta \le \frac{\pi}{2} .$$

The point $P(3,2)$ lies on C. The straight line l is the tangent to C at P.

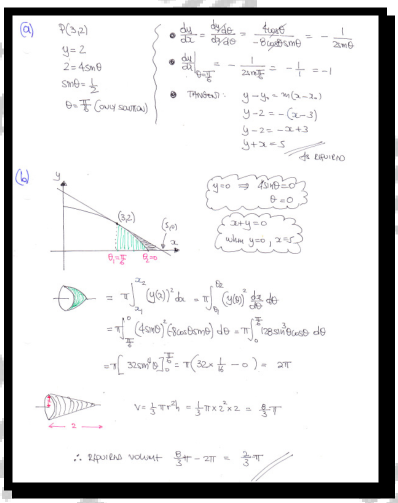
a) Show that an equation of l is

$$x + y = 5 .$$

The finite region R is bounded by C, l and the x axis. This region is to be revolved by 2π radians about the x axis to form a solid S.

b) Find an exact value for the volume of S.

$$\boxed{V = \frac{2\pi}{3}}$$


(a) $P(3,2)$

$y = 2$

$2 = 4\sin\theta$

$\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}$ (ONLY SOLUTION)

• $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{4\cos\theta}{-8\cos\theta\sin\theta} = -\frac{1}{2\sin\theta}$

• $\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{6}} = -\frac{1}{2\sin\frac{\pi}{6}} = -\frac{1}{1} = -1$

• TANGENT: $y - y_0 = m(x - x_0)$

$y - 2 = -(x-3)$

$y - 2 = -x + 3$

$y + x = 5$ AS REQUIRED

(b) $(3,2)$ $(5,0)$ $\theta_1 = \frac{\pi}{6}$ $\theta_2 = 0$

$y = 0 \Rightarrow 4\sin\theta = 0$, $\theta = 0$

$x + y = 0$ when $y = 0$, $x = 5$

$= \pi\int_{x_1}^{x_2}(y(x))^2\,dx = \pi\int_{\theta_1}^{\theta_2}(y(\theta))^2\frac{dx}{d\theta}\,d\theta$

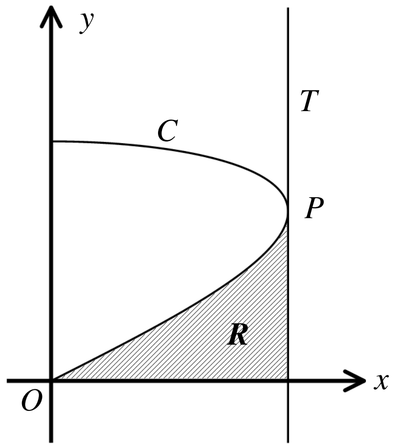
$= \pi\int_{\frac{\pi}{6}}^{0}(4\sin\theta)^2(-8\cos\theta\sin\theta)\,d\theta = \pi\int_0^{\frac{\pi}{6}}128\sin^3\theta\cos\theta\,d\theta$

$= \pi\left[32\sin^4\theta\right]_0^{\frac{\pi}{6}} = \pi\left(32\times\frac{1}{16} - 0\right) = 2\pi$

$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\times2^2\times2 = \frac{8}{3}\pi$

$\therefore$ REQUIRED VOLUME $\frac{8}{3}\pi - 2\pi = \frac{2}{3}\pi$

Question 4 (****+)



The figure above shows the curve C, given parametrically by

$$x = 3\sin 2\theta, \quad y = \cos\theta \ , \text{ for } \ 0 \le \theta \le \frac{\pi}{2}.$$

a) Find an expression for $\frac{dy}{dx}$ in terms of θ.

The line T is parallel to the y axis and is a tangent to C at the point P.

b) Show that $\theta = \frac{\pi}{4}$ at P.

[continues overleaf]

[continued from overleaf]

The finite region R bounded by C , T and the x axis.

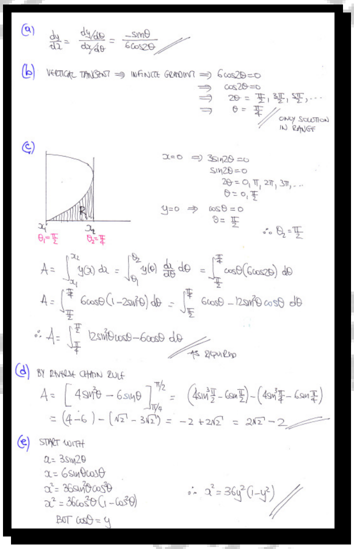
c) Show that the area of R is given by

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 12\sin^2\theta\cos\theta - 6\cos\theta \ d\theta .$$

d) Hence find the area of R .

e) Find a Cartesian equation for C in the form $x^2 = f(y)$.

$\boxed{\frac{dy}{dx} = -\frac{\sin\theta}{6\cos 2\theta}}$, $\boxed{\text{area} = 2\sqrt{2} - 2}$, $\boxed{x^2 = 36y^2\left(1 - y^2\right)}$



(a) $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-\sin\theta}{6\cos 2\theta}$

(b) VERTICAL TANGENT $\Rightarrow$ INFINITE GRADIENT $\Rightarrow 6\cos 2\theta = 0$
$\Rightarrow \cos 2\theta = 0$
$\Rightarrow 2\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots$
$\Rightarrow \theta = \frac{\pi}{4}$ ONLY SOLUTION IN RANGE

(c)

$x = 0 \Rightarrow 3\sin 2\theta = 0$
$\sin 2\theta = 0$
$2\theta = 0, \pi, 2\pi, 3\pi, \dots$
$\theta = 0, \frac{\pi}{2}$
$y = 0 \Rightarrow \cos\theta = 0$
$\theta = \frac{\pi}{2}$
$\therefore \theta_2 = \frac{\pi}{2}$

$A = \int_{x_1}^{x_2} y(x)\,dx = \int_{\theta_1}^{\theta_2} y(\theta)\frac{dx}{d\theta}\,d\theta = \int_{\frac{\pi}{2}}^{\frac{\pi}{4}} \cos\theta(6\cos 2\theta)\,d\theta$

$A = \int_{\frac{\pi}{2}}^{\frac{\pi}{4}} 6\cos\theta(1 - 2\sin^2\theta)\,d\theta = \int_{\frac{\pi}{2}}^{\frac{\pi}{4}} 6\cos\theta - 12\sin^2\theta\cos\theta\,d\theta$

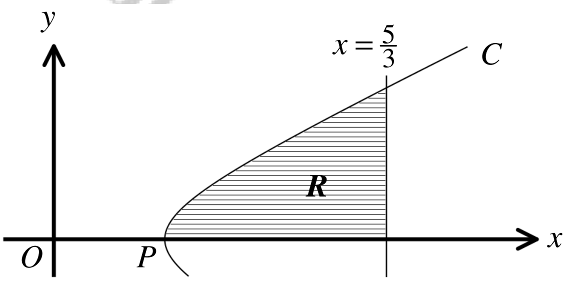
$\therefore A = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 12\sin^2\theta\cos\theta - 6\cos\theta\,d\theta$ AS REQUIRED

(d) BY REVERSE CHAIN RULE

$A = \left[4\sin^3\theta - 6\sin\theta\right]_{\pi/4}^{\pi/2} = \left(4\sin^3\frac{\pi}{2} - 6\sin\frac{\pi}{2}\right) - \left(4\sin^3\frac{\pi}{4} - 6\sin\frac{\pi}{4}\right)$
$= (4 - 6) - (\sqrt{2} - 3\sqrt{2}) = -2 + 2\sqrt{2} = 2\sqrt{2} - 2$

(e) START WITH
$x = 3\sin 2\theta$
$x = 6\sin\theta\cos\theta$
$x^2 = 36\sin^2\theta\cos^2\theta$
$x^2 = 36\cos^2\theta(1 - \cos^2\theta)$
BUT $\cos\theta = y$
$\therefore x^2 = 36y^2(1 - y^2)$

Question 5 (****+)



The figure above shows part of the curve C with parametric equations

$$x = t + \frac{1}{4t}, \quad y = t - \frac{1}{4t}, \quad t > 0.$$

The curve crosses the x axis at P.

a) Determine the coordinates of P.

b) Show that the gradient at any point on C is given by

$$\frac{dy}{dx} = \frac{4t^2+1}{4t^2-1}.$$

c) By considering $x+y$ and $x-y$ find a Cartesian equation for C.

[continues overleaf]

[continued from overleaf]

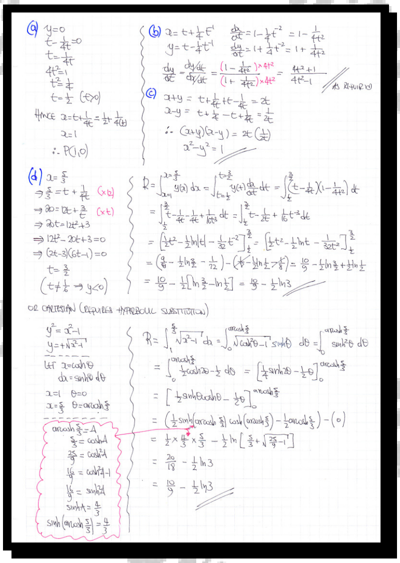
The region R bounded by C, the line $x = \frac{5}{3}$ and the x axis is shown shaded in the above figure.

d) Show that the area of R is given by

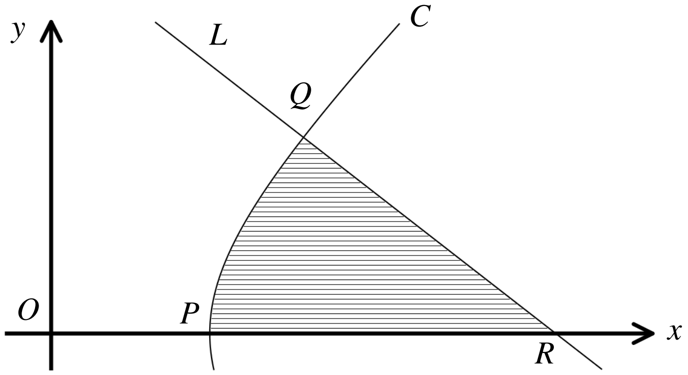
$$\int_{\frac{1}{2}}^{\frac{3}{2}} \left(t - \frac{1}{4t}\right)\left(1 - \frac{1}{4t^2}\right)\, dt .$$

e) Hence calculate an exact value for the area of R.

$\boxed{P(1,0)}$, $\boxed{x^2 - y^2 = 1}$, $\boxed{\text{Area} = \frac{10}{9} - \frac{1}{2}\ln 3}$



Question 6 (****+)



The figure above shows part of the curve C with parametric equations

$$x = 2t + \frac{1}{t}, \quad y = 2t - \frac{1}{t}, \quad t > 0.$$

The curve crosses the x axis at P and the point Q is such so that $t = 2$.

The straight line L is a normal to C at Q.

a) Determine the exact coordinates of P.

b) Show that the gradient at any point on C is given by

$$\frac{dy}{dx} = \frac{2t^2 + 1}{2t^2 - 1}.$$

[continues overleaf]

[continued from overleaf]

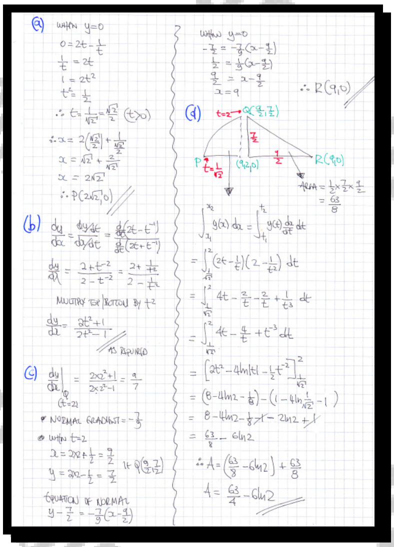
The normal L crosses the x axis at R.

The finite region bounded by C, L and the x axis, shown shaded in the figure, has area A.

c) Find the coordinates of R.

d) Calculate an exact value for A.

$\boxed{P\left(2\sqrt{2},0\right)}$, $\boxed{R(9,0)}$, $\boxed{A=\frac{63}{4}-6\ln 2}$



(a) WHEN $y=0$

$0=2t-\frac{1}{t}$

$\frac{1}{t}=2t$

$1=2t^2$

$t^2=\frac{1}{2}$

$\therefore t=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}\ (t>0)$

$\therefore x=2\left(\frac{\sqrt{2}}{2}\right)+\frac{1}{\frac{\sqrt{2}}{2}}$

$x=\sqrt{2}+\frac{2}{\sqrt{2}}$

$x=2\sqrt{2}$

$\therefore P(2\sqrt{2},0)$

(b) $\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{\frac{d}{dt}(2t-t^{-1})}{\frac{d}{dt}(2t+t^{-1})}$

$\frac{dy}{dx}=\frac{2+t^{-2}}{2-t^{-2}}=\frac{2+\frac{1}{t^2}}{2-\frac{1}{t^2}}$

MULTIPLY TOP/BOTTOM BY t^2

$\frac{dy}{dx}=\frac{2t^2+1}{2t^2-1}$

AS REQUIRED

(c) $\left.\frac{dy}{dx}\right|_{Q\ (t=2)}=\frac{2\times 2^2+1}{2\times 2^2-1}=\frac{9}{7}$

• NORMAL GRADIENT $=-\frac{7}{9}$

• WHEN $t=2$

$x=2\times 2+\frac{1}{2}=\frac{9}{2}$

$y=2\times 2-\frac{1}{2}=\frac{7}{2}$ i.e. $Q\left(\frac{9}{2},\frac{7}{2}\right)$

EQUATION OF NORMAL

$y-\frac{7}{2}=-\frac{7}{9}\left(x-\frac{9}{2}\right)$

WHEN $y=0$

$-\frac{7}{2}=-\frac{7}{9}\left(x-\frac{9}{2}\right)$

$\frac{1}{2}=\frac{1}{9}\left(x-\frac{9}{2}\right)$

$\frac{9}{2}=x-\frac{9}{2}$

$x=9$

$\therefore R(9,0)$

(d)

AREA $=\frac{1}{2}\times\frac{7}{2}\times\frac{9}{2}=\frac{63}{8}$

$\int_{x_1}^{x_2} y(x)\,dx=\int_{t_1}^{t_2} y(t)\frac{dx}{dt}\,dt$

$=\int_{\frac{1}{\sqrt{2}}}^{2}\left(2t-\frac{1}{t}\right)\left(2-\frac{1}{t^2}\right)dt$

$=\int_{\frac{1}{\sqrt{2}}}^{2} 4t-\frac{2}{t}-\frac{2}{t}+\frac{1}{t^3}\,dt$

$=\int_{\frac{1}{\sqrt{2}}}^{2} 4t-\frac{4}{t}+t^{-3}\,dt$

$=\left[2t^2-4\ln|t|-\frac{1}{2}t^{-2}\right]_{\frac{1}{\sqrt{2}}}^{2}$

$=\left(8-4\ln 2-\frac{1}{8}\right)-\left(1-4\ln\frac{1}{\sqrt{2}}-1\right)$

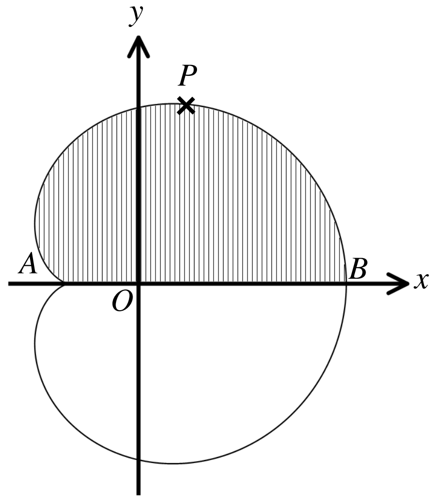
$=8-4\ln 2-\frac{1}{8}-\cancel{1}-2\ln 2+\cancel{1}$

$=\frac{63}{8}-6\ln 2$

$\therefore A=\left(\frac{63}{8}-6\ln 2\right)+\frac{63}{8}$

$A=\frac{63}{4}-6\ln 2$

Question 7 (****+)



The figure above shows a curve known as a Cardioid which is symmetrical about the x axis.

The curve crosses the x axis at the points $A(-2,0)$ and $B(6,0)$.

The point P is the maximum point of the curve.

The parametric equations of this Cardioid are

$$x = 4\cos\theta + 2\cos 2\theta, \quad y = 4\sin\theta + 2\sin 2\theta, \quad 0 \le \theta < 2\pi .$$

a) Find a simplified expression for $\frac{dy}{dx}$, in terms of θ, and hence find the exact coordinates of P.

b) Show that the area of the top half of this Cardioid, shown shaded in the figure, is given by the integral

$$\int_0^{\pi} 16\sin^2\theta + 24\sin\theta\sin 2\theta + 8\sin^2 2\theta \; d\theta ,$$

and hence find the exact value of the area enclosed by the Cardioid.

$\boxed{}$, $\boxed{\frac{dy}{dx} = -\frac{\cos\theta + \cos 2\theta}{\sin\theta + \sin 2\theta}}$, $\boxed{P\left(1, 3\sqrt{3}\right)}$, $\boxed{\text{area} = 24\pi}$

[solution overleaf]

a) STANDARD DIFFERENTIATION

$$\frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{4\cos\theta+4\cos2\theta}{-4\sin\theta-4\sin2\theta}=-\frac{\cos\theta+\cos2\theta}{\sin\theta+\sin2\theta}$$

SOLVING FOR ZERO (NUMERATOR MUST EQUAL ZERO)

$\Rightarrow \cos\theta+\cos2\theta=0$

$\Rightarrow \cos\theta+2\cos^2\theta-1$

$\Rightarrow 2\cos^2\theta+\cos\theta-1=0$

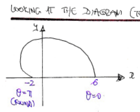
$\Rightarrow (2\cos\theta-1)(\cos\theta+1)=0$

$\Rightarrow \cos\theta = \begin{cases} -1 \\ \frac{1}{2} \end{cases}$ ← CUSP AT A YIELDS $A(-2,0)$

YIELDS $\begin{cases} \pi/3 & \text{if } P(1, 3\sqrt{3}) \\ 5\pi/3 & \text{if } \cancel{(1, -3\sqrt{3})} \end{cases}$

↑ SYMMETRICALLY AT THE BOTTOM

b) LOOKING AT THE DIAGRAM (TOP HALF)



BY INSPECTION LOOKING AT $0 \leqslant \theta < \pi$

$x_0 = 4\times1+2\times1=6$

$y_0 = 4\times0+2\times0=0$

$K(6,0)$

$$\text{AREA}=\int_{x_1}^{x_2} y(x)\,dx$$

$$=\int_{\theta_1}^{\theta_2} y(\theta)\frac{dx}{d\theta}\,d\theta$$

$$=\int_{\pi}^{0}(4\sin\theta+2\sin2\theta)(-4\sin\theta-4\sin2\theta)\,d\theta$$

$$=\int_{\pi}^{0}-8(2\sin\theta+\sin2\theta)(\sin\theta+\sin2\theta)\,d\theta$$

$$=\int_{0}^{\pi}+8(2\sin^2\theta+3\sin\theta\sin2\theta+\sin^2 2\theta\,d\theta$$

FINALLY WE HAVE THE REQUIRED ANSWER

$$\text{AREA OF "TOP HALF"}=\int_0^{\pi}16\sin^2\theta+24\sin\theta\sin2\theta+8\sin^2 2\theta\,d\theta$$

$$=\int_0^{\pi}16\sin^2\theta+24\sin\theta(2\sin\theta\cos\theta)+8\sin^2 2\theta\,d\theta$$

$$=\int_0^{\pi}16\sin^2\theta+8\sin^2 2\theta+48\sin^2\theta\cos\theta\,d\theta$$

$$=\int_0^{\pi}16\left[\tfrac{1}{2}-\tfrac{1}{2}\cos2\theta\right]+8\left(\tfrac{1}{2}-\tfrac{1}{2}\cos4\theta\right)+48\sin^2\theta\cos\theta\,d\theta$$

$$=\int_0^{\pi}12+8\cos2\theta+4\cos4\theta+48\sin^2\theta\cos\theta\,d\theta$$

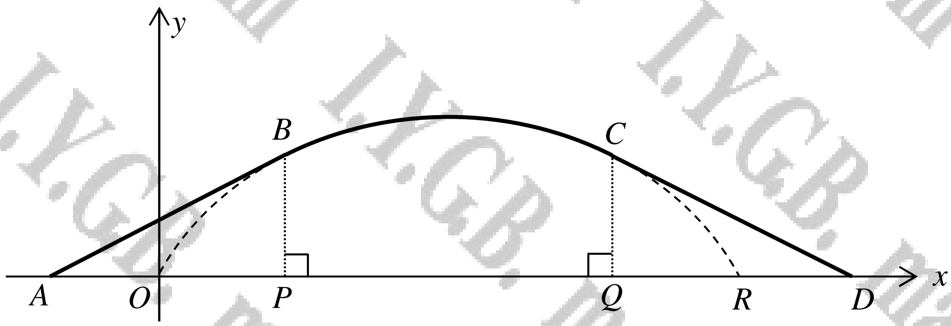
$$=\left[12\theta+4\sin2\theta+\sin4\theta+16\sin^3\theta\right]_0^{\pi}$$

$$=(12\pi+0+0+0)-(0+0+0+0)$$

$$=12\pi$$

$\therefore$ TOTAL AREA $=24\pi$

Question 8 (****+)



The figure above shows a **symmetrical** design for a suspension bridge arch $ABCD$.

The curve $OBCR$ is a cycloid with parametric equations

$$x = 6(2\theta - \sin 2\theta), \quad y = 6(1 - \cos 2\theta), \quad 0 \le \theta \le \pi .$$

a) Show clearly that

$$\frac{dy}{dx} = \cot\theta .$$

[continues overleaf]

[continued from overleaf]

The arch design consists of the curved part BC and the straight lines AB and CD.

The straight line AB is a tangent to the cycloid at the point B where $\theta = \frac{\pi}{3}$, and similarly the straight line CD is a tangent to the cycloid at the point C where $\theta = \frac{2\pi}{3}$.

b) Show further that …

i. … the tangent to the cycloid at B meets the x axis at $x = 4\pi - 12\sqrt{3}$.

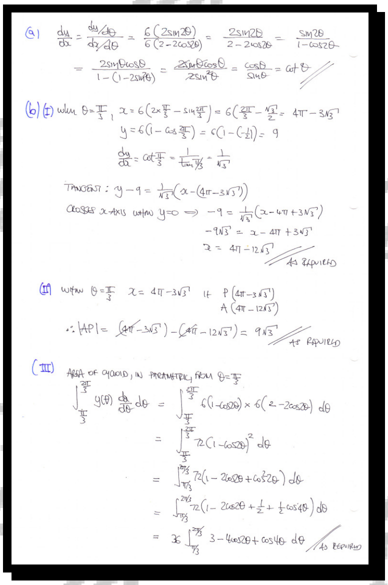
ii. … the length of AP is $9\sqrt{3}$.

iii. ... the area between the x axis and the part of the cycloid between B and C is given by

$$36\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} 3 - 4\cos 2\theta + \cos 4\theta \; d\theta .$$

c) Hence find an exact value for the area enclosed by $ABCD$ and the x axis.

$\boxed{A2}$, $\boxed{\text{area} = 36\pi + 162\sqrt{3}}$



(a) $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{6(2\sin 2\theta)}{6(2-2\cos 2\theta)} = \frac{2\sin 2\theta}{2-2\cos 2\theta} = \frac{\sin 2\theta}{1-\cos 2\theta}$

$= \frac{2\sin\theta\cos\theta}{1-(1-2\sin^2\theta)} = \frac{2\sin\theta\cos\theta}{2\sin^2\theta} = \frac{\cos\theta}{\sin\theta} = \cot\theta$

(b) (I) when $\theta = \frac{\pi}{3}$, $x = 6\left(2\times\frac{\pi}{3} - \sin\frac{2\pi}{3}\right) = 6\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right) = 4\pi - 3\sqrt{3}$

$y = 6\left(1-\cos\frac{2\pi}{3}\right) = 6\left(1-\left(-\frac{1}{2}\right)\right) = 9$

$\frac{dy}{dx} = \cot\frac{\pi}{3} = \frac{1}{\tan\pi/3} = \frac{1}{\sqrt{3}}$

TANGENT: $y - 9 = \frac{1}{\sqrt{3}}\left(x - (4\pi - 3\sqrt{3})\right)$

CROSSES x-AXIS WHEN $y=0 \Rightarrow -9 = \frac{1}{\sqrt{3}}(x - 4\pi + 3\sqrt{3})$

$-9\sqrt{3} = x - 4\pi + 3\sqrt{3}$

$x = 4\pi - 12\sqrt{3}$ AS REQUIRED

(II) WHEN $\theta = \frac{\pi}{3}$ $x = 4\pi - 3\sqrt{3}$ i.e. $P(4\pi - 3\sqrt{3})$, $A(4\pi - 12\sqrt{3})$

$\therefore |AP| = (\cancel{4\pi} - 3\sqrt{3}) - (\cancel{4\pi} - 12\sqrt{3}) = 9\sqrt{3}$ AS REQUIRED

(III) AREA OF CYCLOID, IN PARAMETRIC, FROM $\theta = \frac{\pi}{3}$

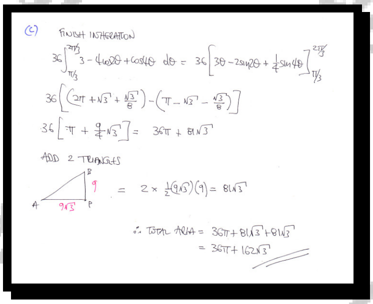
$\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} y(\theta)\frac{dx}{d\theta}\,d\theta = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} 6(1-\cos 2\theta)\times 6(2-2\cos 2\theta)\,d\theta$

$= \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} 72(1-\cos 2\theta)^2\,d\theta$

$= \int_{\pi/3}^{2\pi/3} 72(1 - 2\cos 2\theta + \cos^2 2\theta)\,d\theta$

$= \int_{\pi/3}^{2\pi/3} 72\left(1 - 2\cos 2\theta + \frac{1}{2} + \frac{1}{2}\cos 4\theta\right)d\theta$

$= 36\int_{\pi/3}^{2\pi/3} 3 - 4\cos 2\theta + \cos 4\theta\,d\theta$ AS REQUIRED



(c) FINISH INTEGRATION

$36\int_{\pi/3}^{2\pi/3} 3 - 4\cos 2\theta + \cos 4\theta\,d\theta = 36\left[3\theta - 2\sin 2\theta + \frac{1}{4}\sin 4\theta\right]_{\pi/3}^{2\pi/3}$

$36\left[\left(2\pi + \sqrt{3} + \frac{\sqrt{3}}{8}\right) - \left(\pi - \sqrt{3} - \frac{\sqrt{3}}{8}\right)\right]$

$36\left[\pi + \frac{9}{4}\sqrt{3}\right] = 36\pi + 81\sqrt{3}$

ADD 2 TRIANGLES

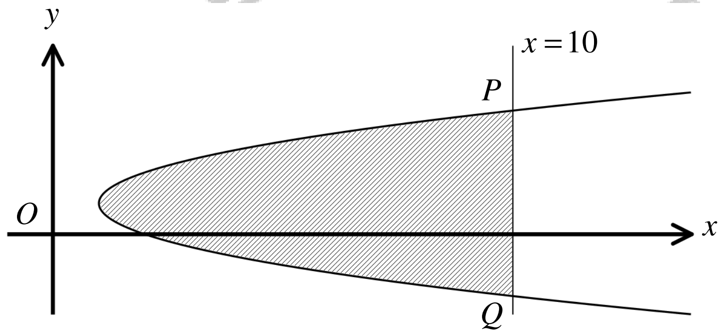
B, 9, A, $9\sqrt{3}$, P

$= 2\times\frac{1}{2}(9\sqrt{3})(9) = 81\sqrt{3}$

$\therefore$ TOTAL AREA $= 36\pi + 81\sqrt{3} + 81\sqrt{3}$

$= 36\pi + 162\sqrt{3}$

Question 9 (****+)



The figure above shows the curve with parametric equations

$$x = t^2 + 1\ , \quad y = 2t + 2\ , \quad t \in \mathbb{R}\ .$$

The straight line with equation $x = 10$ meets the curve at the points P and Q.

The area of the finite region bounded by the curve and the straight line with equation $x = 10$ is shown shaded in the figure above.

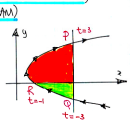
Show that this area is given by

$$8\int_0^3 t^2\ dt\ ,$$

and hence find its value.

$\boxed{\text{area} = 72}$

START BY DETERMINING THE VALUE OF t, AT P, Q & R (DIAGRAM)



- $y = 0$
  $2t + 2 = 0$
  $t = -1 \leftarrow R(2,0)$
- $x = 10$
  $t^2 + 1 = 10$
  $t^2 = 9$
  $t = -3 \leftarrow Q(10,-4)$, $\quad t = 3 \leftarrow P(10,8)$

INTEGRATING IN PARAMETRIC IN "ONE GO"

$$\text{AREA} = \int_{x_1}^{x_2} y(x)\,dx = \int_{t_1}^{t_2} y(t)\frac{dx}{dt}\,dt$$

$$= \int_{-3}^{3} (2t+2)(2t)\,dt = \int_{-3}^{3} 4t^2 + \cancel{4t}\,dt$$

EVEN (×2) ODD

$$= 2\int_0^3 4t^2\,dt = 8\int_0^3 t^2\,dt$$ AS REQUIRED

$$= \left[\tfrac{8}{3}t^3\right]_0^3 = \left(\tfrac{8}{3}\times 27\right) - 0$$

$$= 72$$

ALTERNATIVE BY SPLITTING THE AREA IN 2

$$\text{"ORANGE AREA"} = \int_{-1}^{3} 4t^2 + 4t\,dt$$

$$\text{"GREEN AREA"} = -\int_{-1}^{-3} 4t^2 + 4t\,dt$$

AS THE AREA IS BELOW THE x AXIS THE MINUS WILL MAKE IT POSITIVE

HENCE THE TOTAL AREA CAN BE FOUND

$$\text{TOTAL AREA} = \int_{-1}^{3} 4t^2 + 4t\,dt - \int_{-1}^{-3} 4t^2 + 4t\,dt$$

$$= \int_{-1}^{3} 4t^2 + 4t\,dt + \int_{-3}^{-1} 4t^2 + 4t\,dt$$

$$= \int_{-3}^{3} 4t^2 + \cancel{4t}\,dt$$

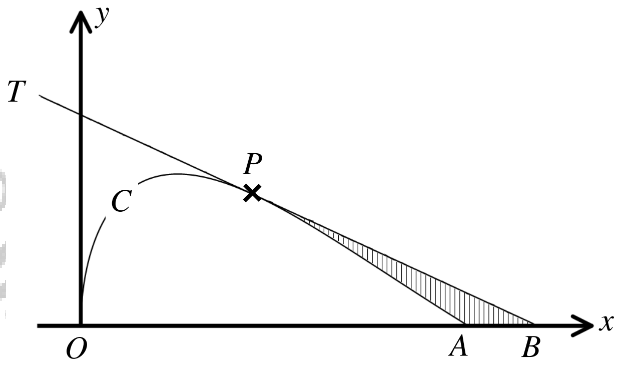
ODD IN A SYMMETRICAL DOMAIN

EVEN IN A SYMMETRICAL DOMAIN $\Rightarrow \times 2$

$$= 2\int_{-3}^{3} 4t^2\,dt$$

$= \ldots$ AS BEFORE.

Question 10 (****+)



The figure above shows the curve with parametric equations

$$x = t^2, \quad y = \sin t, \quad 0 \le t \le \pi .$$

The curve crosses the x axis at the origin O and at the point A.

The point P lies on the curve where $t = \frac{2}{3}\pi$.

The straight line T is a tangent to the curve at P.

Show that the area of the finite region bounded by the curve, the tangent T and the x axis, shown shaded in the figure above, is

$$\tfrac{1}{3}\left(3\sqrt{3} - \pi\right).$$

proof

START COLLECTING AUXILIARY INFORMATION

- WHEN $y=0$
  $\sin t = 0$
  $t = 0 \leftarrow$ ORIGIN, $t = \pi \leftarrow A(\pi^2, 0)$
- WHEN $t = \frac{2\pi}{3}$
  $x = \left(\frac{2\pi}{3}\right)^2 = \frac{4}{9}\pi^2$
  $y = \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$
  $P\left(\frac{4}{9}\pi^2, \frac{\sqrt{3}}{2}\right)$

FIND THE EQUATION OF THE TANGENT AT P

- $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos t}{2t}$
- $\left.\frac{dy}{dx}\right|_{t=\frac{2\pi}{3}} = \frac{\cos\frac{2\pi}{3}}{2 \times \frac{2\pi}{3}} = \frac{-\frac{1}{2}}{\frac{4\pi}{3}} = -\frac{3}{8\pi}$
- $y - y_0 = m(x - x_0)$
  $y - \frac{\sqrt{3}}{2} = -\frac{3}{8\pi}\left(x - \frac{4}{9}\pi^2\right)$

WHEN $y=0$ (TO FIND B)

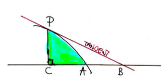
$\Rightarrow -\frac{\sqrt{3}}{2} = -\frac{3}{8\pi}\left(x - \frac{4}{9}\pi^2\right)$

$\Rightarrow \frac{8\pi\sqrt{3}}{6} = x - \frac{4}{9}\pi^2$

$\Rightarrow x = \frac{4}{9}\pi^2 + \frac{4}{3}\pi\sqrt{3}$

$\therefore B\left(\frac{4}{9}\pi^2 + \frac{4}{3}\pi\sqrt{3}, 0\right)$

LOOKING AT THE DIAGRAM BELOW



- $|PC| = \frac{\sqrt{3}}{2}$
- $|BC| = \left(\cancel{\frac{4}{9}\pi^2} + \frac{4}{3}\pi\sqrt{3}\right) - \cancel{\frac{4}{9}\pi^2} = \frac{4}{3}\sqrt{3}\pi$

- AREA OF $\overset{\triangle}{BCP} = \frac{1}{2}|PC||BC| = \frac{1}{2} \times \frac{\sqrt{3}}{2} \times \frac{4}{3}\sqrt{3}\pi = \pi$
- AREA UNDER THE PARAMETRIC CURVE (SHOWN IN GREEN)

$\text{AREA} = \int_{x_C}^{x_A} y(x)\,dx = \int_{t_P}^{t_A} y(t)\frac{dx}{dt}\,dt$

$= \int_{\frac{2}{3}\pi}^{\pi} (\sin t)(2t)\,dt = \int_{\frac{2}{3}\pi}^{\pi} 2t\sin t\,dt$

INTEGRATION BY PARTS

$2t$	2
$-\cos t$	$\sin t$

$\ldots = \left[-2t\cos t\right]_{\frac{2\pi}{3}}^{\pi} - \int_{\frac{2\pi}{3}}^{\pi} -2\cos t\,dt$

$= \left[-2t\cos t\right]_{\frac{2\pi}{3}}^{\pi} + \int_{\frac{2\pi}{3}}^{\pi} 2\cos t\,dt$

$= \left[2\sin t - 2t\cos t\right]_{\frac{2}{3}\pi}^{\pi}$

$= (0 + 2\pi) - \left(\sqrt{3} + \frac{2\pi}{3}\right) = \frac{4}{3}\pi - \sqrt{3}$

FINALLY THE REQUIRED AREA

$\pi - \left(\frac{4}{3}\pi - \sqrt{3}\right) = -\frac{\pi}{3} + \sqrt{3} = \frac{1}{3}\left(3\sqrt{3} - \pi\right)$

Question 11 (****+)

A curve lies entirely above the x axis and has parametric equations

$$x = \sin^2 t\,, \qquad y = 4\tan^3 t\,, \qquad 0 \le t < \tfrac{1}{2}\pi\,.$$

The finite region R is bounded by the curve, the x axis and the straight line with equation $x = \frac{1}{2}$.

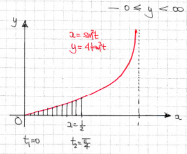
Use integration in parametric to find the exact area of R.

$\boxed{}$, $\boxed{10 - 3\pi}$

START WITH A DIAGRAM — BY INSPECTION $t=0 \Rightarrow (0,0)$

— $0 \le x < 1$

— $0 \le y < \infty$



- $x=0 \longmapsto t=0$
- $x=\frac{1}{2}$ $\quad \frac{1}{2} = \sin^2 t$, $\quad \sin t = \pm\frac{1}{\sqrt{2}}$, $\quad t = +\frac{\pi}{4}$

SETTING UP A PARAMETRIC INTEGRAL

$$\text{AREA} = \int_{x_1}^{x_2} y(x)\,dx = \int_{t_1}^{t_2} y(t)\frac{dx}{dt}\,dt = \int_0^{\frac{\pi}{4}} 4\tan^3 t\,(2\sin t\cos t\,dt)$$

$$= \int_0^{\frac{\pi}{4}} 8\tan^3 t\sin t\cos t\,dt = 8\int_0^{\frac{\pi}{4}} \frac{\sin^3 t}{\cos^3 t}\times \sin t\cos t\,dt$$

$$= 8\int_0^{\frac{\pi}{4}} \frac{\sin^4 t}{\cos^2 t}\,dt = 8\int_0^{\frac{\pi}{4}} \frac{(1-\cos^2 t)^2}{\cos^2 t}\,dt$$

EXPAND AND SPLIT THE FRACTION

$$\text{AREA} = 8\int_0^{\frac{\pi}{4}} \frac{1 - 2\cos^2 t + \cos^4 t}{\cos^2 t}\,dt = 8\int_0^{\frac{\pi}{4}} \sec^2 t - 2 + \cos^2 t\,dt$$

$$= 8\int_0^{\frac{\pi}{4}} \sec^2 t - 2 + \left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2t\right)dt$$

FINALLY TIDY & EVALUATE

$$\text{AREA} = 8\int_0^{\frac{\pi}{4}} \sec^2 t - \tfrac{3}{2} + \tfrac{1}{2}\cos 2t\,dt$$

$$= 8\left[\tan t - \tfrac{3}{2}t + \tfrac{1}{4}\sin 2t\right]_0^{\frac{\pi}{4}}$$

$$= 8\left[\left(1 - \tfrac{3}{2}\times\tfrac{\pi}{4} + \tfrac{1}{4}\right) - 0\right]$$

$$= 8\left[\tfrac{5}{4} - \tfrac{3\pi}{8}\right]$$

$$= 10 - 3\pi$$

Question 12 (****+)

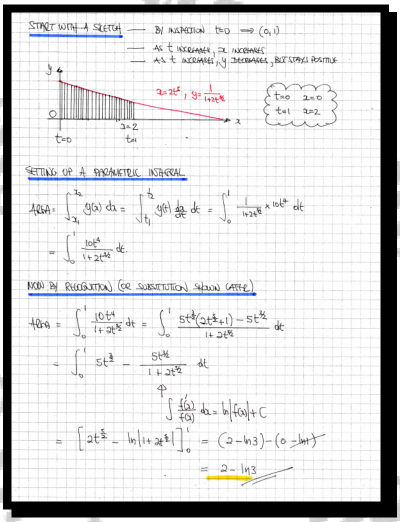
A curve lies entirely above the x axis and has parametric equations

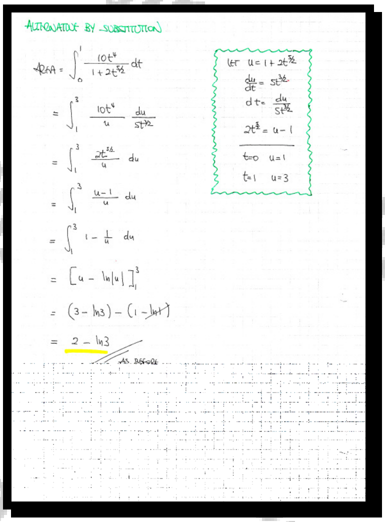
$$x = 2t^5, \qquad y = \frac{1}{1+2t^{\frac{5}{2}}}, \qquad t \geq 0.$$

The finite region R is bounded by the curve, the x axis, the y axis and the straight line with equation $x = 2$.

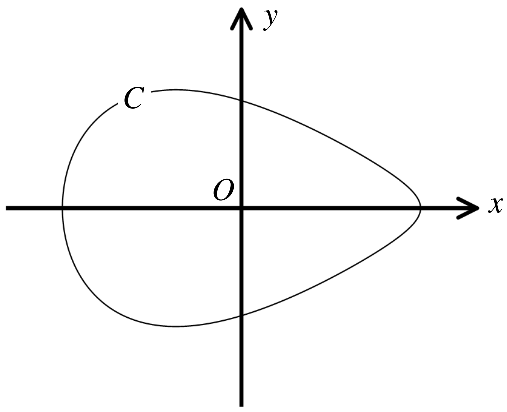
Use integration in parametric to find the exact area of R.

$\boxed{}$, $\boxed{2-\ln 3}$





Question 13 **(****+)**



The figure above shows the closed curve C with parametric equations

$$x = \cos\theta\,,\quad y = \sin\theta - \tfrac{1}{4}\sin 2\theta\,,\quad 0 \le \theta < 2\pi\,.$$

The curve is symmetrical about the x axis.

The finite region enclosed by C is revolved by π radians about the x axis, forming a solid of revolution S.

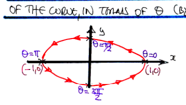
Show that the volume of S is given by

$$\pi\int_0^{\pi} \sin^3\theta\left(1-\tfrac{1}{2}\cos\theta\right)^2\,d\theta\,,$$

and by using the substitution $u = \cos\theta$, or otherwise, determine an exact value for the volume of S.

$\boxed{\frac{7}{5}\pi}$

FIRST DETERMINE THE "ORIENTATION / TRACING" OF THE CURVE, IN TERMS OF θ (BY INSPECTION)



USING THE SYMMETRY OF THE CURVE AND REVOLVING THE TOP HALF BY 2π

$$V = \pi\int_{x_1}^{x_2}[y(x)]^2\,dx = \pi\int_{\theta_1}^{\theta_2}[y(\theta)]^2\frac{dx}{d\theta}\,d\theta$$

$$= \pi\int_{\pi}^{0}\left(\sin\theta - \tfrac{1}{4}\sin 2\theta\right)^2(-\sin\theta)\,d\theta$$

$$= \pi\int_0^{\pi}\left(\sin\theta - \tfrac{1}{4}\times 2\sin\theta\cos\theta\right)^2\sin\theta\,d\theta$$

$$= \pi\int_0^{\pi}\sin^2\theta\left(1-\tfrac{1}{2}\cos\theta\right)^2\sin\theta\,d\theta$$

$$\therefore V = \pi\int_0^{\pi}\sin^3\theta\left(1-\tfrac{1}{2}\cos\theta\right)^2\,d\theta$$

NOW LET $u = \cos\theta \implies \frac{du}{d\theta} = -\sin\theta$

$$\implies d\theta = -\frac{du}{\sin\theta}$$

θ	0	π
u	1	-1

$$\implies V = \pi\int_1^{-1}\sin^{\cancel{3}\,2}\theta\left(1-\tfrac{1}{2}u\right)^2\left(-\frac{du}{\cancel{\sin\theta}}\right)$$

$$\implies V = \pi\int_{-1}^{1}\sin^2\theta\left(1-\tfrac{1}{2}u\right)^2\,du$$

$$\implies V = \pi\int_{-1}^{1}(1-\cos^2\theta)\left(1-\tfrac{1}{2}u\right)^2\,du$$

$$\implies V = \pi\int_{-1}^{1}(1-u^2)\left(1-\tfrac{1}{2}u\right)^2\,du$$

$$\implies V = \pi\int_{-1}^{1}(1-u^2)\left(1-u+\tfrac{1}{4}u^2\right)\,du$$

MULTIPLY OUT & THROW AWAY ODD PARTS AS THE DOMAIN IS SYMMETRICAL

$$\implies V = \pi\int_{-1}^{1}1\cancel{-u}+\tfrac{1}{4}u^2-u^2\cancel{+u^3}-\tfrac{1}{4}u^4\,du$$

$$\implies V = \pi\int_{-1}^{1}1-\tfrac{3}{4}u^2-\tfrac{1}{4}u^4\,du$$

EVEN PARTS $\times 2$

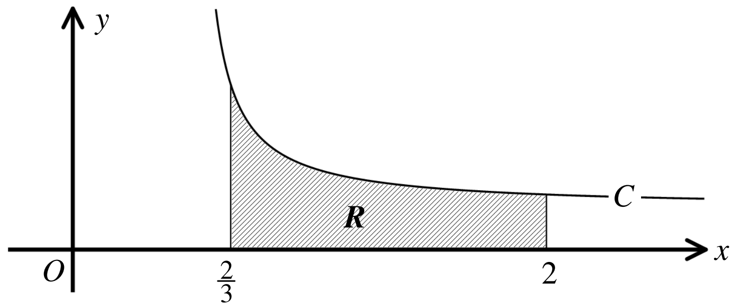
$$\implies V = \pi\int_0^{1}2-\tfrac{3}{2}u^2-\tfrac{1}{2}u^4\,du$$

$$\implies V = \pi\left[2u-\tfrac{1}{2}u^3-\tfrac{1}{10}u^5\right]_0^1$$

$$\implies V = \pi\left[2-\tfrac{1}{2}-\tfrac{1}{10}\right]$$

$$\implies V = \tfrac{7}{5}\pi$$

Question 14 **(****+)**



The figure above shows the curve with parametric equations

$$x = \frac{1}{1+t}, \quad y = \frac{1}{1-t}, \quad -1 < t < 1.$$

The region R, shown shaded in the figure, is bounded by the curve, the x axis and the straight lines with equations $x = \frac{2}{3}$ and $x = 2$.

a) Show that the area of R can be found by the parametric integral

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{(1-t)(1+t)^2}\, dt,$$

and hence find the exact area of R.

b) Determine a Cartesian equation of the curve, in the form $y = f(x)$, and by evaluating a suitable integral in Cartesian verify the answer given to part (**a**).

$\boxed{\text{area} = \frac{2}{3} + \frac{1}{2}\ln 3}$

a) START BY CONVERTING THE LIMITS

- $x = \frac{2}{3} \Rightarrow \frac{1}{1+t} = \frac{2}{3} \Rightarrow t+1 = \frac{3}{2} \Rightarrow t = \frac{1}{2}$
- $x = 2 \Rightarrow \frac{1}{1+t} = 2 \Rightarrow t+1 = \frac{1}{2} \Rightarrow t = -\frac{1}{2}$

SET UP THE INTEGRAL FOR THE AREA FROM CARTESIAN INTO PARAMETRIC

$$\text{AREA} = \int_{x_1}^{x_2} y(x)\, dx = \int_{t_1}^{t_2} y(t)\frac{dx}{dt}\, dt$$

$$= \int_{\frac{1}{2}}^{-\frac{1}{2}} \frac{1}{1-t}\left(-\frac{1}{(1+t)^2}\right) dt = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{1-t}\left(+\frac{1}{(1+t)^2}\right) dt$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{(1-t)(1+t)^2}\, dt$$

AS REQUIRED

BY PARTIAL FRACTIONS

$$\frac{1}{(1-t)(1+t)^2} \equiv \frac{A}{1-t} + \frac{B}{(1+t)^2} + \frac{C}{1+t}$$

$$\boxed{1 \equiv A(1+t)^2 + B(1-t) + C(1+t)(1-t)}$$

- IF $t = 1$: $1 = 4A$, $A = \frac{1}{4}$
- IF $t = -1$: $1 = 2B$, $B = \frac{1}{2}$
- IF $t = 0$: $1 = A + B + C$, $1 = \frac{1}{4} + \frac{1}{2} + C$, $C = \frac{1}{4}$

FINALLY THE AREA CAN BE FOUND

$$\Rightarrow \text{AREA} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\frac{1}{4}}{1-t} + \frac{\frac{1}{4}}{1+t} + \frac{\frac{1}{2}}{(1+t)^2}\, dt$$

$$\Rightarrow \text{AREA} = \left[\tfrac{1}{4}\ln|1+t| - \tfrac{1}{4}\ln|1-t| - \frac{\frac{1}{2}}{1+t}\right]_{-\frac{1}{2}}^{\frac{1}{2}}$$

$$\Rightarrow \text{AREA} = \frac{1}{4}\left[\ln(1+t) - \ln(1-t) - \frac{2}{t+1}\right]_{-\frac{1}{2}}^{\frac{1}{2}}$$

$$\Rightarrow \text{AREA} = \frac{1}{4}\left[\left(\ln\tfrac{3}{2} - \ln\tfrac{1}{2} - \tfrac{4}{3}\right) - \left(\ln\tfrac{1}{2} - \ln\tfrac{3}{2} - 4\right)\right]$$

$$\Rightarrow \text{AREA} = \frac{1}{4}\left[2\ln\tfrac{3}{2} - 2\ln\tfrac{1}{2} + \tfrac{8}{3}\right]$$

$$\Rightarrow \text{AREA} = \frac{1}{2}\left[\ln\tfrac{3}{2} - \ln\tfrac{1}{2} + \tfrac{4}{3}\right]$$

$$\Rightarrow \text{AREA} = \frac{1}{2}\left[\ln 3 + \tfrac{4}{3}\right]$$

$$\Rightarrow \text{AREA} = \frac{2}{3} + \frac{1}{2}\ln 3$$

b) ELIMINATE THE PARAMETER

$$\Rightarrow x = \frac{1}{t+1}$$

$$\Rightarrow t+1 = \frac{1}{x}$$

$$\Rightarrow t = \frac{1}{x} - 1 \quad \Rightarrow y = \frac{1}{1 - \left(\frac{1}{x} - 1\right)}$$

$$\Rightarrow y = \frac{1}{2 - \frac{1}{x}}$$

$$\Rightarrow y = \frac{x}{2x-1}$$

RE-ATTEMPTING THE AREA BY CARTESIAN INTEGRATION

$$\Rightarrow \text{AREA} = \int_{x_1}^{x_2} y(x)\, dx = \int_{\frac{2}{3}}^{2} \frac{x}{2x-1}\, dx$$

BY SUBSTITUTION OR MANIPULATIONS

$$\Rightarrow \text{AREA} = \frac{1}{2}\int_{\frac{2}{3}}^{2} \frac{2x}{2x-1}\, dx = \frac{1}{2}\int \frac{(2x-1)+1}{(2x-1)}\, dx$$

$$\Rightarrow \text{AREA} = \frac{1}{2}\int_{\frac{2}{3}}^{2} 1 + \frac{1}{2x-1}\, dx$$

$$\Rightarrow \text{AREA} = \frac{1}{2}\left[x + \tfrac{1}{2}\ln|2x-1|\right]_{\frac{2}{3}}^{2}$$

$$\Rightarrow \text{AREA} = \frac{1}{2}\left[\left(2 + \tfrac{1}{2}\ln 3\right) - \left(\tfrac{2}{3} + \tfrac{1}{2}\ln\tfrac{1}{3}\right)\right]$$

$$\Rightarrow \text{AREA} = \frac{1}{2}\left[\tfrac{4}{3} + \tfrac{1}{2}\ln 3 - \tfrac{1}{2}\ln\tfrac{1}{3}\right]$$

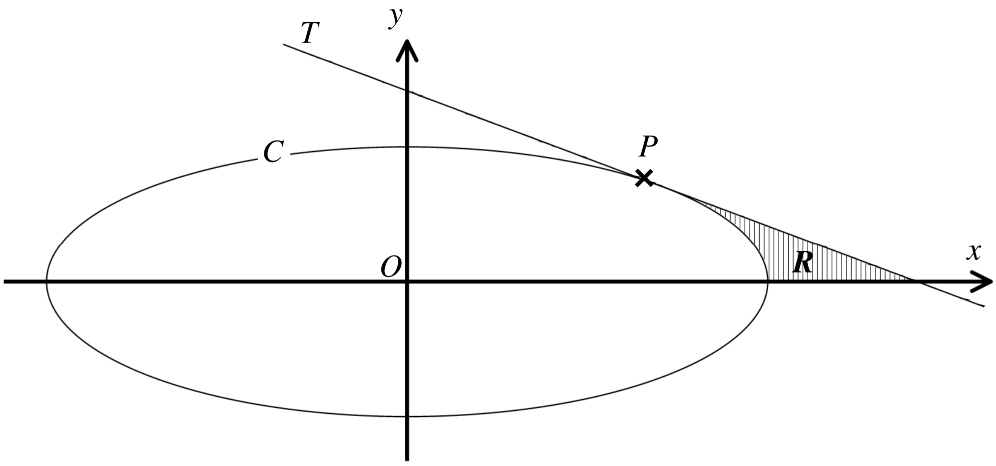
$$\Rightarrow \text{AREA} = \frac{1}{2}\left[\tfrac{4}{3} + \tfrac{1}{2}\ln 3 + \tfrac{1}{2}\ln 3\right]$$

$$\Rightarrow \text{AREA} = \frac{1}{2}\left[\tfrac{4}{3} + \ln 3\right]$$

$$\Rightarrow \text{AREA} = \frac{2}{3} + \frac{1}{2}\ln 3$$

AS BEFORE

Question 15 (****+)



The figure above shows the ellipse with parametric equations

$$x = 8\cos\theta, \; y = 4\sin\theta, \; 0 \le \theta < 2\pi.$$

The point P lies on the ellipse, where $\theta = \frac{1}{4}\pi$.

The straight line T is a tangent to the ellipse at P.

The finite region R, shown shaded in the figure, is bounded by the ellipse, the tangent T and the x axis.

Find an exact value for the area of R.

$\boxed{\text{area} = 16 - 4\pi}$

START BY FINDING THE GRADIENT AT P

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{4\cos\theta}{-8\sin\theta} = -\frac{1}{2}\cot\theta$$

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{4}} = -\frac{1}{2}\cot\frac{\pi}{4} = -\frac{1}{2}$$

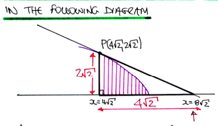
OBTAIN THE EQUATION OF THE TANGENT

- $x\big|_{\frac{\pi}{4}} = 8\cos\frac{\pi}{4} = 8 \times \frac{\sqrt{2}}{2} = 4\sqrt{2}$
- $y\big|_{\frac{\pi}{4}} = 4\sin\frac{\pi}{4} = 4 \times \frac{\sqrt{2}}{2} = 2\sqrt{2}$

I.E $P[4\sqrt{2}, 2\sqrt{2}]$

$$y - y_0 = m(x - x_0)$$
$$y - 2\sqrt{2} = -\frac{1}{2}(x - 4\sqrt{2})$$
$$2y - 4\sqrt{2} = -x + 4\sqrt{2}$$
$$2y + x = 8\sqrt{2}$$

NEXT WE FIND THE AREA OF THE TRIANGLE IN THE FOLLOWING DIAGRAM

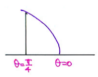


BY SETTING y=0 IN THE EQUATION OF THE TANGENT

$$\overset{\Delta}{\text{AREA}} = \frac{1}{2} \times 2\sqrt{2} \times 4\sqrt{2} = 8$$

FINALLY THE AREA SHOWN SHADED IN THE ABOVE DIAGRAM (AREA BETWEEN ELLIPSE & x AXIS)

FIRSTLY BY INSPECTION, THE ELLIPSE MEETS THE x AXIS AT $x = 8$, IE $\theta = 0$



$$\text{AREA} = \int_{x_1}^{x_2} y(x)\,dx = \int_{\theta_1}^{\theta_2} y(\theta)\frac{dx}{d\theta}\,d\theta$$

$$\text{AREA} = \int_{\frac{\pi}{4}}^{0} 4\sin\theta(-8\sin\theta)\,d\theta$$

$$\text{AREA} = \int_{0}^{\frac{\pi}{4}} +32\sin^2\theta\,d\theta = \int_0^{\frac{\pi}{4}} 32\left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right)d\theta$$

$$\text{AREA} = \int_0^{\frac{\pi}{4}} 16 - 16\cos 2\theta\,d\theta = \left[16\theta - 8\sin 2\theta\right]_0^{\frac{\pi}{4}}$$

$$\text{AREA} = (4\pi - 8) - (0)$$

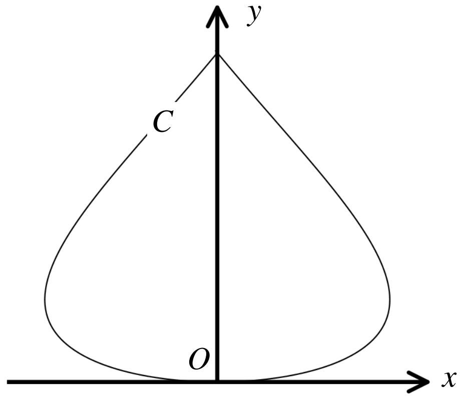
$$\text{AREA} = 4\pi - 8$$

FINALLY THE REQUIRED AREA CAN BE FOUND

$\Rightarrow$ AREA OF TRIANGLE − AREA UNDER ELLIPSE $= 8 - (4\pi - 8)$

$= 16 - 4\pi$

Question 16 (****+)



The figure above shows the curve C with parametric equations

$$x = \sin t, \quad y = t^2, \quad 0 \le t \le 2\pi.$$

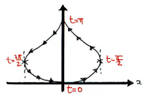
It is given that C is symmetrical about the y axis.

The region bounded by C is to be revolved about the y axis by π radians to form a solid of revolution with volume V.

By considering a suitable integral in parametric, or otherwise, find an exact value for this volume.

$\boxed{}$, $\boxed{\frac{1}{2}\pi^3}$

START BY TRACING THE CURVE, TO OBTAIN THE VALUES OF t AT DIFFERENT POINTS



SET A VOLUME INTEGRAL IN y (PARAMETRIC) BY REVOLVING THE "R.H.S" OF THE CURVE.

$$V = \pi\int_{y_1}^{y_2}[x(y)]^2\,dy = \pi\int_{t_1}^{t_2}[x(t)]^2\frac{dy}{dt}\,dt$$

$$= \pi\int_0^{\pi}(\sin t)^2(2t)\,dt = \pi\int_0^{\pi}2t\sin^2 t\,dt$$

PROCEED BY TRIGONOMETRIC IDENTITIES, FOLLOWED BY INTEGRATION BY PARTS

$$\dots = \pi\int_0^{\pi}2t\left(\tfrac{1}{2}-\tfrac{1}{2}\cos 2t\right)dt$$

$$= \pi\int_0^{\pi} t - t\cos 2t\ dt$$

$$= \pi\int_0^{\pi} t\ dt + \pi\int_0^{\pi} -t\cos 2t\ dt$$

$-t$	-1
$\frac{1}{2}\sin 2t$	$\cos 2t$

$$= \pi\left[\left[\tfrac{1}{2}t^2\right]_0^{\pi} + \left[-\tfrac{1}{2}t\sin 2t\right]_0^{\pi} - \int_0^{\pi}-\tfrac{1}{2}\sin 2t\,dt\right]$$

$$= \pi\left[\left[\tfrac{1}{2}t^2 - \tfrac{1}{2}t\sin 2t\right]_0^{\pi} + \int_0^{\pi}\tfrac{1}{2}\sin 2t\,dt\right]$$

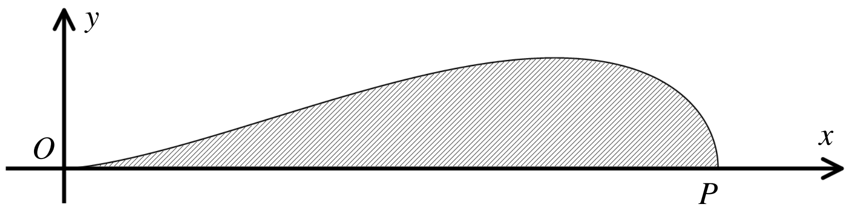
FINISHING OFF THE LAST INTEGRATION & EVALUATING

$$= \pi\left[\tfrac{1}{2}t^2 - \tfrac{1}{2}t\sin 2t - \tfrac{1}{4}\cos 2t\right]_0^{\pi}$$

$$= \pi\left[\left(\tfrac{1}{2}\pi^2 - 0 \cancel{-\tfrac{1}{4}}\right) - \left(0 - 0 \cancel{-\tfrac{1}{4}}\right)\right]$$

$$= \tfrac{1}{2}\pi^3$$

Question 17 (****+)



The figure above shows the curve with parametric equations

$$x = 2 + 2\sin\theta\,,\ \ y = 2\cos\theta + \sin 2\theta\,,\ -\tfrac{1}{2}\pi \le \theta \le \tfrac{1}{2}\pi\,.$$

The curve meets the x axis at the origin O and at the point P.

The finite region bounded by the curve and the x axis is rotated by 2π radians in the x axis, forming a solid of revolution S.

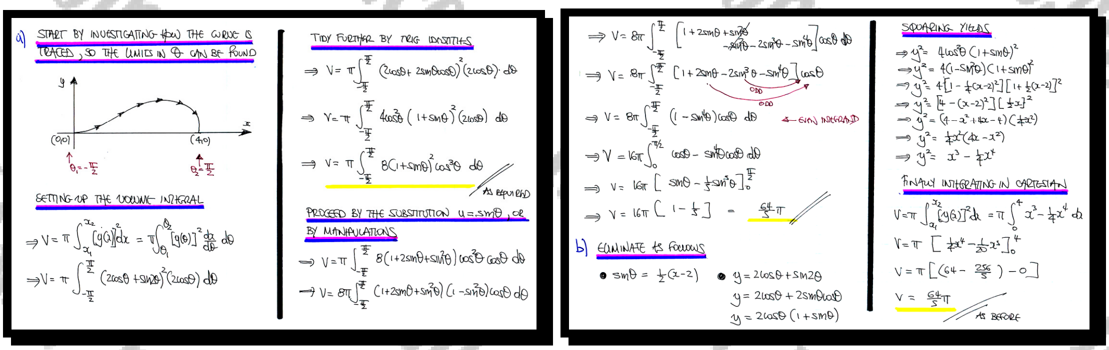
a) Show that the volume of S is given by

$$\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 8(1+\sin\theta)^2\cos^3\theta\ d\theta\,,$$

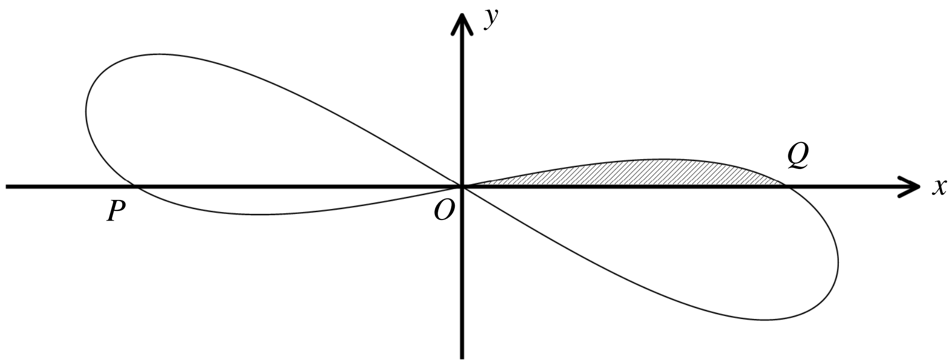
and hence find its exact value.

b) Determine a Cartesian equation of the curve and by forming and evaluating an appropriate integral in Cartesian verify the answer for the volume of S, found in part **(a)**.

$\boxed{}$, $\boxed{V = \frac{64}{5}\pi}$



Question 18 (****+)



The figure above shows a curve with parametric equations

$$x = \cos\theta\,,\;\; y = \sin 2\theta - \cos\theta\,,\;\; 0 \le \theta < 2\pi\,.$$

The curve, which has rotational symmetry about the origin O, crosses the x axis at the points P, Q and O.

The finite region bounded by the curve, for which $x \ge 0$, $y \ge 0$, and the x axis is shown shaded in the figure.

Show, with detailed workings, that …

i. … the area of shaded region is $\frac{5}{24}$.

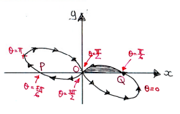
ii. … the area enclosed by the two loops of the curve is $\frac{8}{3}$.

iii. … a Cartesian equation of the curve is

$$4x^2\left(1-x^2\right) = (x+y)^2\,.$$

$\boxed{}$, $\boxed{\text{proof}}$

DETERMINE, BY INSPECTION, THE DIRECTION IN WHICH THE CURVE IS TRACED



$y = 0$, $\quad \sin 2\theta - \cos\theta = 0$

$2\sin\theta\cos\theta - \cos\theta = 0$

$\cos\theta(2\sin\theta - 1)$

• $\cos\theta = 0$ & • $\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{2}$ $\quad \theta = \frac{\pi}{6}$

$\theta = \frac{3\pi}{2}$ $\quad \theta = \frac{5\pi}{6}$

a) SETTING UP AN INTEGRAL TO FIND THE AREA OF THE SHADED REGION

$\Rightarrow \text{AREA} = \int_{x_1}^{x_2} y(x)\,dx = \int_{\theta_1}^{\theta_2} y(\theta)\frac{dx}{d\theta}\,d\theta$

$\Rightarrow \text{AREA} = \int_{\theta=\frac{\pi}{2}}^{\theta=\frac{\pi}{6}} (\sin 2\theta - \cos\theta)(-\sin\theta)\,d\theta$

$\Rightarrow \text{AREA} = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (2\sin\theta\cos\theta - \cos\theta)(+\sin\theta)\,d\theta$

$\Rightarrow \text{AREA} = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 2\sin^2\theta\cos\theta - \cos\theta\sin\theta\,d\theta$

BY REVERSE CHAIN RULE (INSPECTION)

$\Rightarrow \text{AREA} = \left[\frac{2}{3}\sin^3\theta - \frac{1}{2}\sin^2\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$

$\Rightarrow \text{AREA} = \left[\frac{2}{3} - \frac{1}{2}\right] - \left[\frac{1}{12} - \frac{1}{8}\right]$

$\Rightarrow \text{AREA} = \frac{5}{24}$ AS REQUIRED

b) NEXT FIND THE AREA FOR WHICH $x \ge 0$, $y \le 0$

$\text{AREA} = \int_{\theta=\frac{3\pi}{2}}^{\theta=\frac{\pi}{6}} \underbrace{2\sin^2\theta\cos\theta - \sin\theta\cos\theta}_{\text{MANIPULATED EARLIER}}\,d\theta$

THE AREA IS BELOW THE x AXIS, THIS HOWEVER WILL TURN OUT POSITIVE IN PARAMETRIC

$\Rightarrow$ "AREA UNDER" $= \int_{\frac{3\pi}{2}}^{\frac{5\pi}{6}} 2\sin^2\theta\cos\theta - \sin\theta\cos\theta\,d\theta$

$\Rightarrow$ "AREA UNDER" $= \left[\frac{2}{3}\sin^3\theta - \frac{1}{2}\sin^2\theta\right]_{\frac{3\pi}{2}}^{\frac{5\pi}{6}}$

$\Rightarrow$ "AREA UNDER" $= \left(\frac{1}{12} - \frac{1}{8}\right) - \left(-\frac{2}{3} - \frac{1}{2}\right)$

$\Rightarrow$ "AREA UNDER" $= \frac{9}{8}$

[WE WOULD HAVE GOT THE SAME IF WE INTEGRATED $\frac{5\pi}{6}$ TO $\frac{3\pi}{2}$]

$\therefore$ AREA OF ONE LOOP IS $\frac{9}{8} + \frac{5}{24} = \frac{4}{3}$

TOTAL AREA IS $2 \times \frac{4}{3} = \frac{8}{3}$ AS REQUIRED

c) PROCEED AS FOLLOWS

$y = \sin 2\theta - \cos\theta$

$y = 2\sin\theta\cos\theta - \cos\theta$

$y = 2\sin\theta\cos\theta - x$

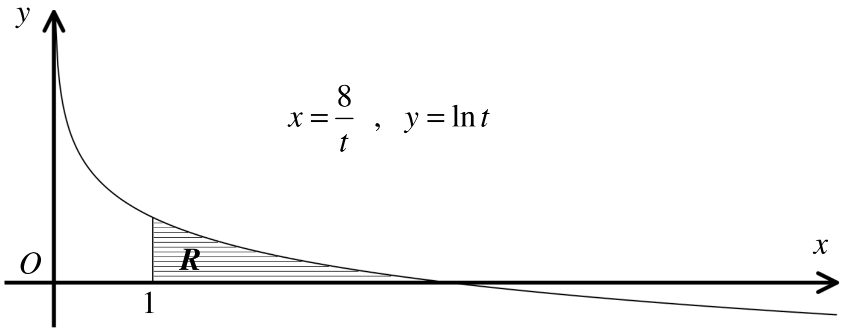
$y + x = 2\sin\theta\cos\theta$

$(y+x)^2 = 4\sin^2\theta\cos^2\theta$

$(y+x)^2 = 4\cos^2\theta(1-\cos^2\theta)$

$(y+x)^2 = 4x^2(1-x^2)$

Question 19 (****+)



The figure above shows part of the curve with parametric equations

$$x = \frac{8}{t}, \ y = \ln t, \ t > 0$$

The finite region R is bounded by the curve, the x axis and the straight line with equation $x = 1$.

(1)

a) Show that the area of R is given by

$$\int_{t_1}^{t_2} \frac{8 \ln t}{t^2} \, dt$$ **(5)**

where the t_1 and t_2 are constants to be found.

b) Evaluate the above parametric integral to determine, in exact simplified form, the area of R.

c) Find a Cartesian equation of the curve and hence verify the answer of part **(b)**.

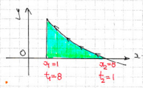
$\boxed{t_1 = 1}$, $\boxed{t_2 = 8}$, $\boxed{7 - 3\ln 2}$

[solution overleaf]

a) START BY FINDING THE x INTERCEPT OF THE CURVE AND THE VALUE OF t AT 1

- $y=0$
- $\ln t = 0$, $t = e^0$, $t = 1$
- $x = \frac{8}{t}$, $x = 8$

- $x = 1$, $1 = \frac{8}{t}$, $t = 8$ I.E THE CURVE IS TRACED "BACKWARDS"



$$\text{AREA} = \int_{x_1}^{x_2} y(x)\,dx = \int_{t_1}^{t_2} y(t)\frac{dx}{dt}\,dt = \int_8^1 (\ln t)\left(-\frac{8}{t^2}\right)dt$$

$$= \int_8^1 -\frac{8\ln t}{t^2}\,dt = \int_1^8 \frac{8\ln t}{t^2}\,dt$$ AS REQUIRED

b) FOLLOW BY INTEGRATION BY PARTS

$$\int_1^8 \frac{8\ln t}{t^2}\,dt = \left[-\frac{8\ln t}{t}\right]_1^8 - \int_1^8 -\frac{8}{t^2}\,dt$$

$8\ln t$	$\frac{8}{t}$
$-\frac{1}{t}$	$\frac{1}{t^2}$

$$= \left[-\frac{8\ln t}{t}\right]_1^8 + \int_1^8 \frac{8}{t^2}\,dt$$

$$= \left[-\frac{8\ln t}{t} - \frac{8}{t}\right]_1^8$$

$$= \left[\frac{8}{t} + \frac{8}{t}\ln t\right]_8^1$$

$$= (8 + \cancel{\ln 1}) - (1 + \ln 8)$$

$$= 7 - \ln 8$$

$$= 7 - 3\ln 2$$

c) SWITCHING INTO CARTESIAN

$$\left.\begin{array}{l} x = \frac{8}{t} \\ y = \ln t \end{array}\right\} \Rightarrow t = \frac{8}{x} \Rightarrow y = \ln\frac{8}{x}$$

$$\text{AREA} = \int_{x_1}^{x_2} y(x)\,dx = \int_1^8 \ln\left(\frac{8}{x}\right)dx = \int_1^8 -\ln\left(\frac{x}{8}\right)dx$$

$$= \int_8^1 \ln\left(\tfrac{1}{8}x\right)dx = \int_8^1 1\times\ln\left(\tfrac{1}{8}x\right)dx$$

INTEGRATION BY PARTS AGAIN

$\ln\left(\frac{1}{8}x\right)$	$\frac{1}{x}$
x	1

$$\ldots = \left[x\ln\left(\tfrac{1}{8}x\right)\right]_8^1 - \int_8^1 1\,dx$$

$$= \left[x\ln\left(\tfrac{1}{8}x\right) - x\right]_8^1$$

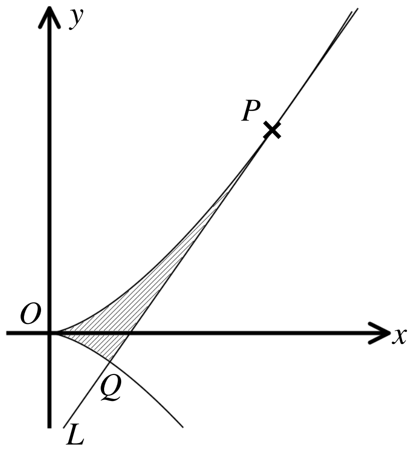
$$= \left(1\times\ln\left(\tfrac{1}{8}\right) - 1\right) - \left(8\cancel{\ln 1} - 8\right)$$

$$= \ln\left(\tfrac{1}{8}\right) - 1 + 8$$

$$= 7 + \ln\left(\tfrac{1}{8}\right)$$

$$= 7 - \ln 8$$ AS BEFORE

Question 20 (****+)



A semi-cubical parabola C, which consists of two sections meeting at the origin O, has parametric equations

$$x = t^2, \quad y = t^3, \quad t \in \mathbb{R}.$$

The point P lies on C where $t = 2$.

The straight line L is the tangent to C at P and the point Q is where L re-intersects C, as shown in the figure.

a) Find the coordinates of Q.

b) Determine the area of the finite region bounded by C and L, shown shaded in the figure above.

$\boxed{}$, $\boxed{Q(1,-1)}$, $\boxed{\text{area} = 2.7}$

a) START BY FINDING THE EQUATION OF L

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2}{2t} = \frac{3}{2}t$

$\left.\frac{dy}{dx}\right|_{t=2} = 3$

$x|_{t=2} = 4$ & $y|_{t=2} = 8$ ∴ $P(4,8)$

$\Rightarrow y - y_0 = m(x - x_0)$

$\Rightarrow y - 8 = 3(x - 4)$

$\Rightarrow y - 8 = 3x - 12$

$\Rightarrow y = 3x - 4$

SOLVING SIMULTANEOUSLY WITH CURVE

$\Rightarrow y = 3x - 4$

$\Rightarrow t^3 = 3t^2 - 4$

$\Rightarrow t^3 - 3t^2 + 4 = 0$

FACTORIZE BY INSPECTION

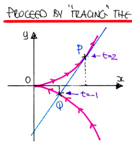
$\Rightarrow (t-2)^2(t+1) = 0$

$t=2$, POINT OF TANGENCY $P(4,8)$; Q

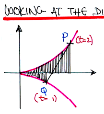
$\Rightarrow t = -1$

$\Rightarrow Q(1,-1)$

b) PROCEED BY "TRACING" THE CURVE

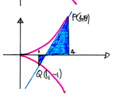


LOOKING AT THE DIAGRAM BELOW



● SHADED AREA $= \int_{t_1}^{t_2} y(t)\frac{dx}{dt}\,dt = \int_{-1}^{2} t^3(2t)\,dt = \int_{-1}^{2} 2t^4\,dt$

$= \left[\frac{2}{5}t^5\right]_{-1}^{2} = \frac{64}{5} - \left(-\frac{2}{5}\right) = \frac{66}{5}$



● EQUATION OF TANGENT CROSSES THE x AXIS AT ... $y = 3x - 4$, $0 = 3x - 4$, $x = \frac{4}{3}$

● "TOP" TRIANGLE $= \frac{1}{2} \times 8 \times \left(4 - \frac{4}{3}\right) = \frac{32}{3}$

● "BOTTOM" TRIANGLE $= \frac{1}{2} \times 1 \times \left(\frac{4}{3} - 1\right) = \frac{1}{6}$

THE REQUIRED AREA CAN NOW BE FOUND

"AREA UNDER CURVE" − "TOP TRIANGLE" + "BOTTOM TRIANGLE" $= \frac{66}{5} - \frac{32}{3} + \frac{1}{6} = \frac{27}{10} = 2.7$

Question 21 (*****)

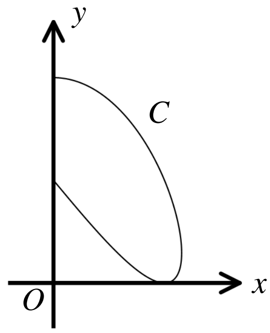


figure 1

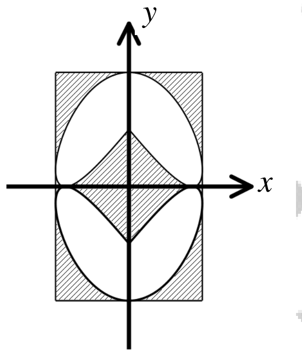


figure 2

Figure 1 above, shows the curve with parametric equations

$$x = \sin 2\theta,\ \ y = 1 - \sin 3\theta,\ \ 0 \le \theta \le \tfrac{1}{2}\pi .$$

Figure 2 above shows a glass design. It consists of the curve of figure 1, reflected successively in the x and y axis.

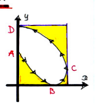
The resulting design fits snugly inside a rectangle, whose sides are tangents to the curve and its reflections, parallel to the coordinate axes. The region inside the 4 loops of the curve is made of clear glass while the region inside the rectangle but outside the 4 loops of C is made of yellow glass.

Determine the area of the yellow glass.

$\boxed{}$, $\boxed{\text{yellow area} = \frac{16}{5}}$

START BY "TRACING" THE CURVE

- "MAX x" $= 1 \Rightarrow \theta = \frac{\pi}{4}$
- "MAX y" $= 2 \Rightarrow \sin 3\theta = -1$
  $3\theta = \frac{3\pi}{2}$
  $\theta = \frac{\pi}{2}$



- $\theta = 0$ YIELDS $(0,1)$
- $x = 0$ YIELDS $\theta = 0, \frac{\pi}{2}$
- $y = 0$ YIELDS $\theta = \frac{\pi}{6}$

$x = \sin 2\theta$
$y = 1 - \sin 3\theta$
$0 \le \theta \le \frac{\pi}{2}$

HENCE BY INSPECTION

	A	B	C	D
θ	0	$\pi/6$	$\pi/4$	$\pi/2$
x	0	$\sqrt{3}/2$	1	0
y	1	0	$1-\frac{\sqrt{2}}{2}$	2

NEXT WE FIND THE AREA ENCLOSED BY THE LOOP IN THE FIRST QUADRANT

$$\text{AREA} = \int_{\theta_1}^{\theta_2} y(\theta)\frac{dx}{d\theta}\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} (1 - \sin 3\theta)(2\cos 2\theta)\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} 2\cos 2\theta - 2\sin 3\theta\cos 2\theta\,d\theta$$

USING TRIGONOMETRIC IDENTITIES

$\sin(3\theta + 2\theta) = \sin 5\theta = \sin 3\theta\cos 2\theta + \cos 3\theta \sin 2\theta$
$\sin(3\theta - 2\theta) = \sin\theta = \sin 3\theta\cos 2\theta - \cos 3\theta\cos 2\theta$

ADDING YIELDS

$$\boxed{\sin 5\theta + \sin\theta = 2\sin 3\theta\cos 2\theta}$$

RESUMING THE INTEGRATION

$$\ldots = \int_0^{\frac{\pi}{2}} 2\cos 2\theta - (\sin\theta + \sin 5\theta)\,d\theta$$

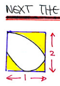
$$= \int_0^{\frac{\pi}{2}} 2\cos 2\theta - \sin\theta - \sin 5\theta\,d\theta$$

$$= \Big[\sin 2\theta + \cos\theta + \tfrac{1}{5}\cos 5\theta\Big]_0^{\frac{\pi}{2}}$$

$$= (0 + 0 + 0) - (0 + 1 + \tfrac{1}{5})$$

$$= \frac{6}{5}$$ ← AREA OF THE "LOOP"

NEXT THE RECTANGLE

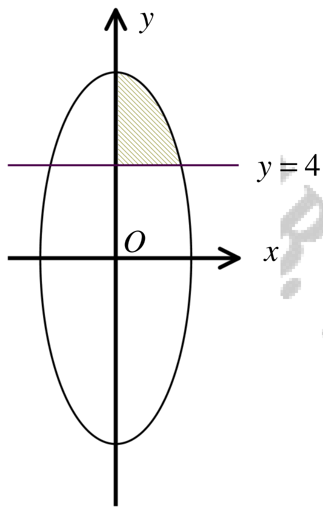


HENCE THE REQUIRED AREA OF THE YELLOW GLASS

$$\text{YELLOW GLASS} = 4 \times \left(2 - \frac{6}{5}\right)$$

$$= \frac{16}{5}$$

Question 22 (****+)

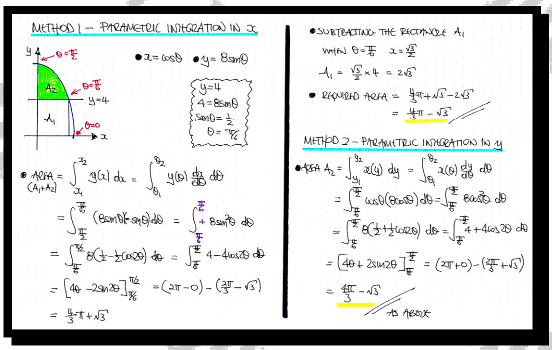


A curve is defined in terms of a parameter θ, by the following equations.

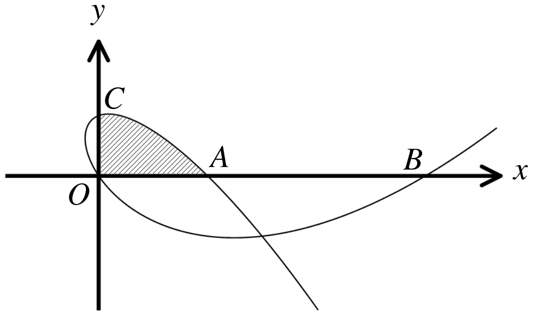
$$x = \cos\theta,\ \ y = 8\sin\theta,\ \ 0 \le \theta < 2\pi .$$

Determine an exact value for the area of the finite region bounded by the curve, the y axis for which $y \ge 0$, and the straight line with equation $y = 4$ for which $y \ge 4$.

$\boxed{}$, $\boxed{\text{area} = \frac{4}{3}\pi - \sqrt{3}}$



Question 23 (****+)



The figure above shows a curve with parametric equations

$$x = t^2 + 2t, \quad y = t^3 - 9t, \quad t \in \mathbb{R}.$$

The curve meets the coordinate axes at the origin O and at the points A, B and C.

a) Determine the coordinates of A, B and C.

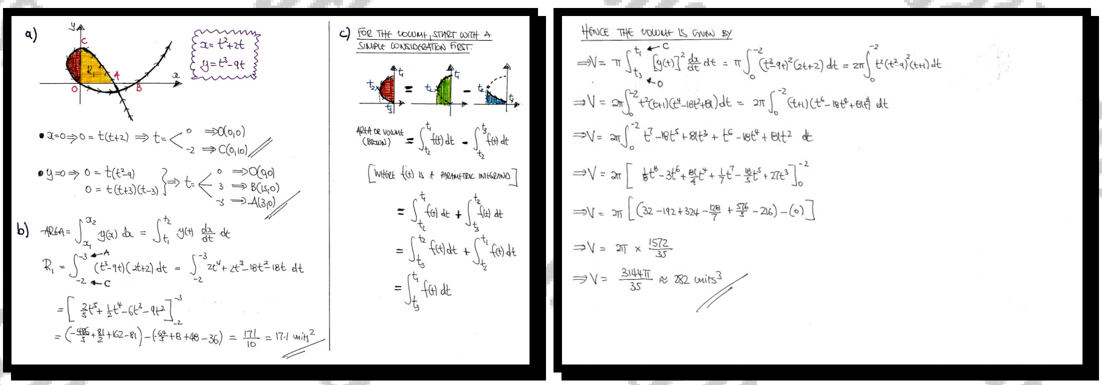
The finite region R, shown shaded in the figure, is bounded by the curve and the coordinate axes.

b) Find the area of R.

The finite region bounded by the curve and the y axis, for which $x < 0$, is revolved by 2π radians about the x axis, forming a solid S.

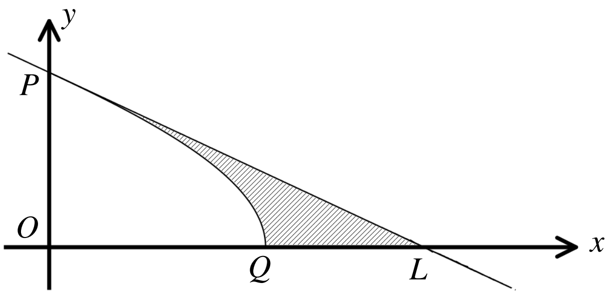
c) Calculate the volume of S.

$\boxed{A(3,0),\ B(15,0),\ C(0,10)}$, $\boxed{\text{area} = 17.1}$, $\boxed{\text{volume} \approx 282}$



8 ENRICHMENT QUESTIONS

Question 1 (*****)



The figure above shows the curve C with parametric equations

$$x = 4\cos 3\theta, \quad y = 4\sin\theta, \quad 0 \le \theta \le \frac{1}{6}\pi .$$

The curve meets the coordinate axes at $P(0,2)$ and at $Q(4,0)$.

The straight line L is the tangent to C at the point P.

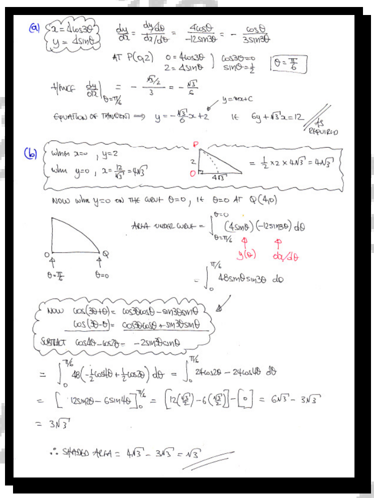
a) Show that an equation of L is

$$6y + x\sqrt{3} = 12 .$$

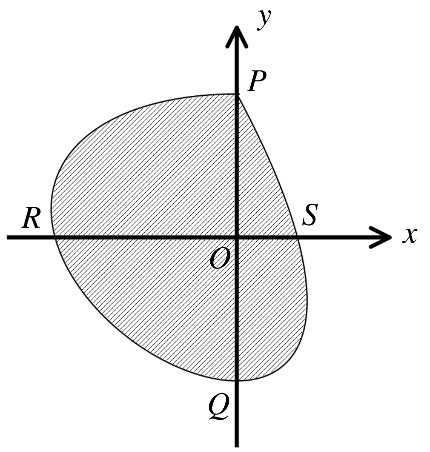
The finite region bounded by the curve C the tangent L and the x axis is shown shaded in the above figure.

b) Show further that the area of this region is exactly $\sqrt{3}$ square units.

$\boxed{}$, $\boxed{\text{proof}}$



Question 2 **(*****)**



The figure above shows the curve C with parametric equations

$$x = t\sin t, \quad y = \cos t, \quad 0 \le t < 2\pi .$$

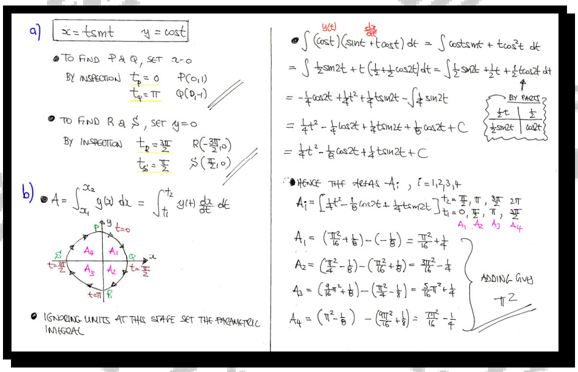
The curve meets the coordinate axes at the points P, Q, R and S.

a) Find the value of the parameter t at each the points P, Q, R and S.

The finite region bounded by the curve C is shown shaded in the above figure.

b) Show that the area of this region is exactly π^2 square units.

$\boxed{}$, $\boxed{t_P = 0,\ t_S = \dfrac{\pi}{2},\ t_Q = \pi,\ t_R = \dfrac{3\pi}{2}}$



a) $\boxed{x = t\sin t \qquad y = \cos t}$

- TO FIND P & Q, SET $x=0$

BY INSPECTION $t_P = 0$ $P(0,1)$

$t_Q = \pi$ $Q(0,-1)$

- TO FIND R & S, SET $y=0$

BY INSPECTION $t_R = \frac{3\pi}{2}$ $R\left(-\frac{3\pi}{2},0\right)$

$t_S = \frac{\pi}{2}$ $S\left(\frac{\pi}{2},0\right)$

b) • $A = \int_{x_1}^{x_2} y(x)\,dx = \int_{t_1}^{t_2} y(t)\frac{dx}{dt}\,dt$

P y t=0 A4 A1 S Q x $t=\frac{3\pi}{2}$ A3 A2 $t=\frac{\pi}{2}$ $t=\pi$ R

- IGNORING UNITS AT THIS STAGE SET THE PARAMETRIC INTEGRAL

• $\int \overset{y(t)}{(\cos t)}\overset{\frac{dx}{dt}}{(\sin t + t\cos t)}\,dt = \int \cos t\sin t + t\cos^2 t\,dt$

$= \int \frac{1}{2}\sin 2t + t\left(\frac{1}{2}+\frac{1}{2}\cos 2t\right)dt = \int \frac{1}{2}\sin 2t + \frac{1}{2}t + \frac{1}{2}t\cos 2t\,dt$

$= -\frac{1}{4}\cos 2t + \frac{1}{4}t^2 + \frac{1}{4}t\sin 2t - \int \frac{1}{4}\sin 2t$

$= \frac{1}{4}t^2 - \frac{1}{4}\cos 2t + \frac{1}{4}t\sin 2t + \frac{1}{8}\cos 2t + C$

$= \frac{1}{4}t^2 - \frac{1}{8}\cos 2t + \frac{1}{4}t\sin 2t + C$

BY PARTS

$\frac{1}{2}t$	$\frac{1}{2}$
$\frac{1}{2}\sin 2t$	$\cos 2t$

• HENCE THE AREAS A_i, $i = 1,2,3,4$

$A_i = \left[\frac{1}{4}t^2 - \frac{1}{8}\cos 2t + \frac{1}{4}t\sin 2t\right]_{t_1 = 0,\ \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2}}^{t_2 = \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2},\ 2\pi}$ A_1 A_2 A_3 A_4

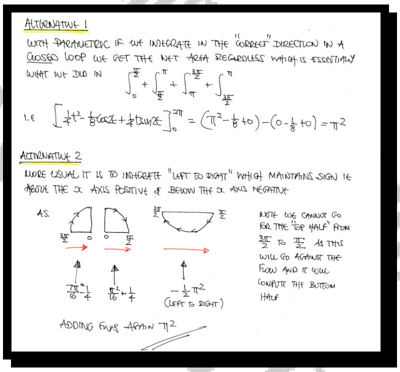
$A_1 = \left(\frac{\pi^2}{16}+\frac{1}{8}\right) - \left(-\frac{1}{8}\right) = \frac{\pi^2}{16}+\frac{1}{4}$

$A_2 = \left(\frac{\pi^2}{4}-\frac{1}{8}\right) - \left(\frac{\pi^2}{16}+\frac{1}{8}\right) = \frac{3\pi^2}{16}-\frac{1}{4}$

$A_3 = \left(\frac{9}{16}\pi^2+\frac{1}{8}\right) - \left(\frac{\pi^2}{4}-\frac{1}{8}\right) = \frac{5}{16}\pi^2+\frac{1}{4}$

$A_4 = \left(\pi^2-\frac{1}{8}\right) - \left(\frac{9\pi^2}{16}+\frac{1}{8}\right) = \frac{7\pi^2}{16}-\frac{1}{4}$

ADDING GIVES π^2



ALTERNATIVE 1

WITH PARAMETRIC IF WE INTEGRATE IN THE "CORRECT" DIRECTION IN A CLOSED LOOP WE GET THE NET AREA REGARDLESS WHICH IS ESSENTIALLY WHAT WE DID IN

$$\int_0^{\frac{\pi}{2}} + \int_{\frac{\pi}{2}}^{\pi} + \int_{\pi}^{\frac{3\pi}{2}} + \int_{\frac{3\pi}{2}}^{\pi}$$

I.E $\left[\frac{1}{4}t^2 - \frac{1}{8}\cos 2t + \frac{1}{4}t\sin 2t\right]_0^{2\pi} = \left(\pi^2 - \frac{1}{8} + 0\right) - \left(0 - \frac{1}{8} + 0\right) = \pi^2$

ALTERNATIVE 2

MORE USUAL IT IS TO INTEGRATE "LEFT TO RIGHT" WHICH MAINTAINS SIGN IE ABOVE THE x AXIS POSITIVE & BELOW THE x AXIS NEGATIVE

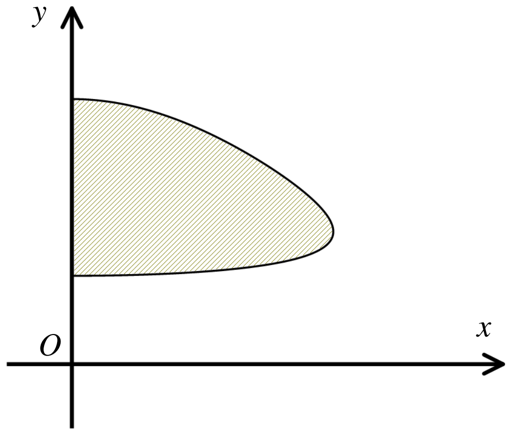
AS $\frac{3\pi}{2}$ 0 0 $\frac{\pi}{2}$ $\frac{3\pi}{2}$ $\frac{\pi}{2}$

$\frac{7\pi^2}{16}-\frac{1}{4}$ $\quad \frac{\pi^2}{16}+\frac{1}{4}$ $\quad -\frac{1}{2}\pi^2$ (LEFT TO RIGHT)

NOTE WE CANNOT GO FOR THE "TOP HALF" FROM $\frac{3\pi}{2}$ TO $\frac{\pi}{2}$ AS THIS WILL GO AGAINST THE FLOW AND IT WILL COMPUTE THE BOTTOM HALF

ADDING GIVES AGAIN π^2

Question 3 (*****)



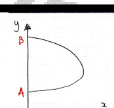
The figure above shows a curve given parametrically by

$$x = \sin 3t, \quad 1 + y\cos 3t = 2y, \quad t \in \mathbb{R}, \quad 0 \le t \le \tfrac{1}{3}\pi .$$

The finite region bounded by the curve and the y axis, shown shaded in the figure is revolved by 2π radians about the y axis, forming a solid of revolution.

Determine an exact simplified value for the volume of this solid.

$\boxed{}$, $\boxed{4\pi[\ln 3 - 1]}$



$$\boxed{x = \sin 3t, \quad 1 + y\cos 3t = 2y}$$

WHEN $t=0$ $\quad 1 + y = 2y$, $\quad y = 1 \leftarrow B$

WHEN $t = \frac{\pi}{3}$ $\quad 1 - y = 2y$, $\quad y = \frac{1}{3} \leftarrow A$

- REARRANGE THE y EQUATION & DIFFERENTIATE

$\Rightarrow 1 = 2y - y\cos 3t$

$\Rightarrow 1 = y(2 - \cos 3t)$

$\Rightarrow y = \frac{1}{2-\cos 3t} = (2-\cos 3t)^{-1}$

$\Rightarrow \frac{dy}{dt} = -(2-\cos 3t)^{-2} \times 3\sin 3t$

$\Rightarrow \frac{dy}{dt} = \frac{-3\sin 3t}{(2-\cos 3t)^2}$

- VOLUME OF REVOLUTION NEXT ABOUT y IS GIVEN BY

$\Rightarrow V = \pi \int_{y_1}^{y_2} [x(y)]^2 \, dy$

$\Rightarrow V = \pi \int_{t_1}^{t_2} [x(t)]^2 \frac{dy}{dt} \, dt$

$\Rightarrow V = \pi \int_{\frac{\pi}{3}}^{0} (\sin 3t)^2 \left[\frac{-3\sin 3t}{(2-\cos 3t)^2}\right] dt$

$\Rightarrow V = \pi \int_{0}^{\frac{\pi}{3}} \frac{3\sin^3 3t}{(2-\cos 3t)^2} \, dt$

- BY SUBSTITUTION

$u = 2 - \cos 3t$

$\frac{du}{dt} = 3\sin 3t$

$dt = \frac{du}{3\sin 3t}$

$t = 0 \mapsto u = 1$

$t = \frac{\pi}{3} \mapsto u = 3$

- TIDYING UP WE OBTAIN

$\Rightarrow V = \pi \int_{1}^{3} \frac{\cancel{3}\sin^{\cancel{3}} 3t}{u^2} \frac{du}{\cancel{3\sin 3t}}$

$\Rightarrow V = \pi \int_{1}^{3} \frac{\sin^2 3t}{u^2} \, du$

$\Rightarrow V = \pi \int_{1}^{3} \frac{1 - \cos^2 3t}{u^2} \, du$

$\Rightarrow V = \pi \int_{1}^{3} \frac{1 - (2-u)^2}{u^2} \, du$

$\Rightarrow V = \pi \int_{1}^{3} \frac{1 - (u^2 - 4u + 4)}{u^2} \, du$

$\Rightarrow V = \pi \int_{1}^{3} \frac{-u^2 + 4u - 3}{u^2} \, du$

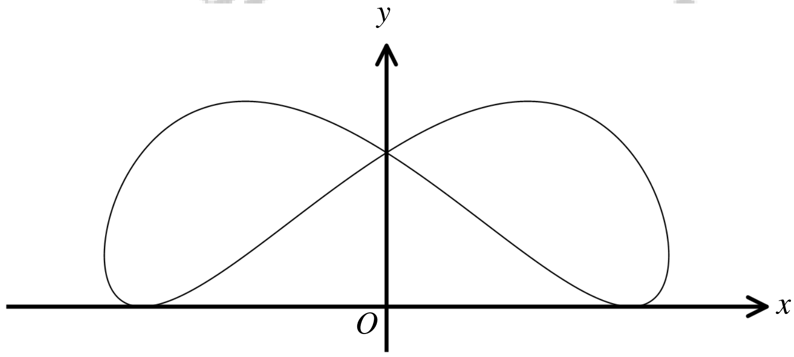
$\Rightarrow V = \pi \int_{1}^{3} -1 + \frac{4}{u} - \frac{3}{u^2} \, du$

$\Rightarrow V = \pi \left[-u + 4\ln u + \frac{3}{u} \right]_1^3$

$\Rightarrow V = \pi \left[(-3 + 4\ln 3 + 1) - (-1 + \cancel{4\ln 1} + 3) \right]$

$\Rightarrow V = \pi [4\ln 3 - 4] = 4\pi(\ln 3 - 1)$

Question 4 (*****)

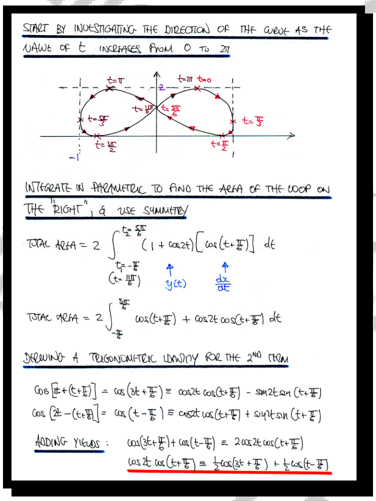


The figure above shows the curve with parametric equations

$$x = \sin\left(t + \frac{\pi}{6}\right), \quad y = 1 + \cos 2t, \quad 0 \le t < 2\pi .$$

Given that the curve is symmetrical about the y axis, show that the area enclosed by the two loops of the curve is $\dfrac{4\sqrt{3}}{3}$.

[ ], [proof]



START BY INVESTIGATING THE DIRECTION OF THE CURVE AS THE VALUE OF t INCREASES FROM 0 TO 2π

INTEGRATE IN PARAMETRIC TO FIND THE AREA OF THE LOOP ON THE "RIGHT", & USE SYMMETRY

$$\text{TOTAL AREA} = 2\int_{t_1=-\frac{\pi}{6}\ \left(t=\frac{11\pi}{6}\right)}^{t_2=\frac{5\pi}{6}} \underbrace{(1+\cos 2t)}_{y(t)}\underbrace{\left[\cos\left(t+\frac{\pi}{6}\right)\right]}_{\frac{dx}{dt}}\,dt$$

$$\text{TOTAL AREA} = 2\int_{-\frac{\pi}{6}}^{\frac{5\pi}{6}} \cos\left(t+\frac{\pi}{6}\right) + \cos 2t\cos\left(t+\frac{\pi}{6}\right)\,dt$$

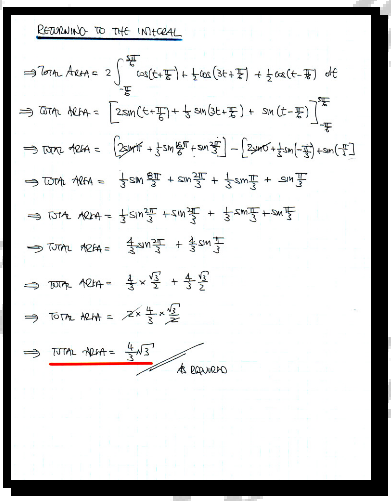
DERIVING A TRIGONOMETRIC IDENTITY FOR THE 2ND TERM

$$\cos\left[2t+\left(t+\frac{\pi}{6}\right)\right] = \cos\left(3t+\frac{\pi}{6}\right) \equiv \cos 2t\cos\left(t+\frac{\pi}{6}\right) - \sin 2t\sin\left(t+\frac{\pi}{6}\right)$$

$$\cos\left[2t-\left(t+\frac{\pi}{6}\right)\right] = \cos\left(t-\frac{\pi}{6}\right) \equiv \cos 2t\cos\left(t+\frac{\pi}{6}\right) + \sin 2t\sin\left(t+\frac{\pi}{6}\right)$$

ADDING YIELDS: $\cos\left(3t+\frac{\pi}{6}\right) + \cos\left(t-\frac{\pi}{6}\right) \equiv 2\cos 2t\cos\left(t+\frac{\pi}{6}\right)$

$$\cos 2t\cos\left(t+\frac{\pi}{6}\right) \equiv \frac{1}{2}\cos\left(3t+\frac{\pi}{6}\right) + \frac{1}{2}\cos\left(t-\frac{\pi}{6}\right)$$



RETURNING TO THE INTEGRAL

$$\Rightarrow \text{TOTAL AREA} = 2\int_{-\frac{\pi}{6}}^{\frac{5\pi}{6}} \cos\left(t+\frac{\pi}{6}\right) + \frac{1}{2}\cos\left(3t+\frac{\pi}{6}\right) + \frac{1}{2}\cos\left(t-\frac{\pi}{6}\right)\,dt$$

$$\Rightarrow \text{TOTAL AREA} = \left[2\sin\left(t+\frac{\pi}{6}\right) + \frac{1}{3}\sin\left(3t+\frac{\pi}{6}\right) + \sin\left(t-\frac{\pi}{6}\right)\right]_{-\frac{\pi}{6}}^{\frac{5\pi}{6}}$$

$$\Rightarrow \text{TOTAL AREA} = \left[\cancel{2\sin\pi} + \frac{1}{3}\sin\frac{16\pi}{6} + \sin\frac{2\pi}{3}\right] - \left[\cancel{2\sin 0} + \frac{1}{3}\sin\left(-\frac{\pi}{3}\right) + \sin\left(-\frac{\pi}{3}\right)\right]$$

$$\Rightarrow \text{TOTAL AREA} = \frac{1}{3}\sin\frac{8\pi}{3} + \sin\frac{2\pi}{3} + \frac{1}{3}\sin\frac{\pi}{3} + \sin\frac{\pi}{3}$$

$$\Rightarrow \text{TOTAL AREA} = \frac{1}{3}\sin\frac{2\pi}{3} + \sin\frac{2\pi}{3} + \frac{1}{3}\sin\frac{\pi}{3} + \sin\frac{\pi}{3}$$

$$\Rightarrow \text{TOTAL AREA} = \frac{4}{3}\sin\frac{2\pi}{3} + \frac{4}{3}\sin\frac{\pi}{3}$$

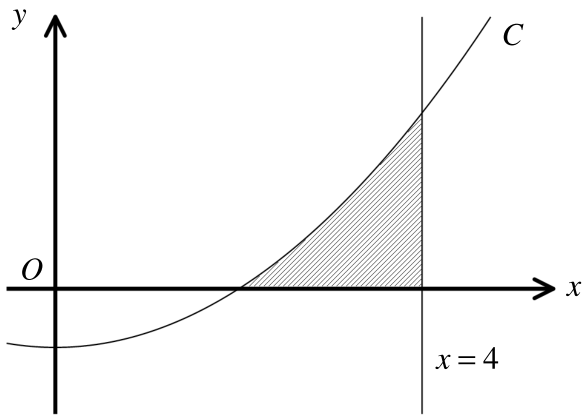
$$\Rightarrow \text{TOTAL AREA} = \frac{4}{3}\times\frac{\sqrt{3}}{2} + \frac{4}{3}\frac{\sqrt{3}}{2}$$

$$\Rightarrow \text{TOTAL AREA} = \cancel{2}\times\frac{4}{3}\times\frac{\sqrt{3}}{\cancel{2}}$$

$$\Rightarrow \text{TOTAL AREA} = \frac{4}{3}\sqrt{3}$$

AS REQUIRED

Question 5 (*****)



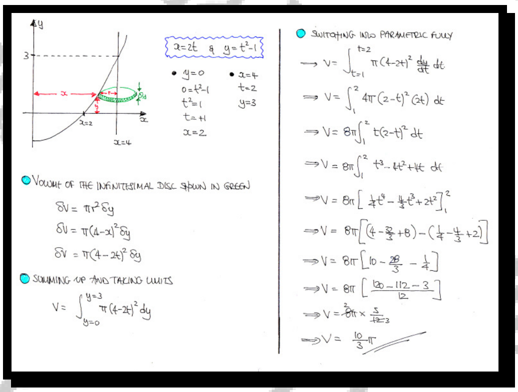
The figure above shows the curve C with parametric equations

$$x = 2t, \quad y = t^2 - 1, \quad t \in \mathbb{R}.$$

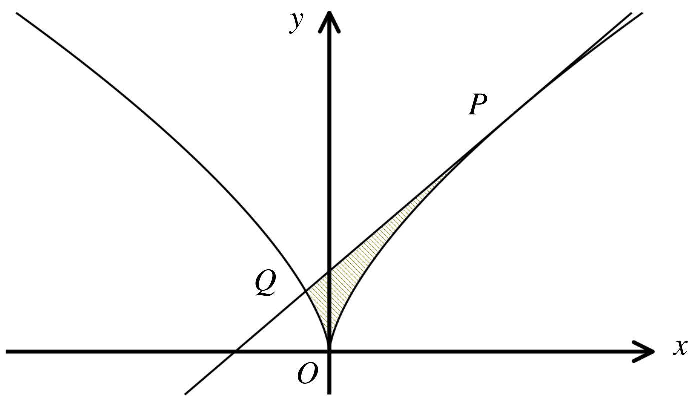
The finite region, bounded by C, the x axis and the line $x = 4$ is revolved by 2π radians about the line $x = 4$, to form a solid of revolution S.

Find an exact value for the volume of S.

$\boxed{}$, $\boxed{\text{volume} = \dfrac{10\pi}{3}}$



Question 6 (*****)



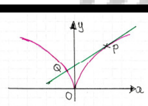
The figure above shows the curve with parametric equations

$$x = t^3, \quad y = t^2, \quad t \in \mathbb{R}.$$

The tangent to the curve at the point P meets the curve again at the point Q.

Given that the area of the finite region bounded by the curve and the tangent, shown shaded in the above figure, is $2\frac{7}{10}$ square units, determine the coordinates of P.

$\boxed{}$, $\boxed{P(8,4)}$



$x = t^3 \quad y = t^2$

- LET THE CO-ORDINATES OF $P(p^3, p^2)$ IE $t = p$ AT P

$\Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{3t^2} = \frac{2}{3t}$

$\Rightarrow \left.\frac{dy}{dx}\right|_{t=p} = \frac{2}{3p}$

- EQUATION OF THE TANGENT IS GIVEN BY

$\Rightarrow y - p^2 = \frac{2}{3p}(x - p^3)$

$\Rightarrow 3yp - 3p^3 = 2x - 2p^3$

- SOLVING SIMULTANEOUSLY WITH THE CURVE TO FIND Q

$\Rightarrow 3(t^2)p - 3p^3 = 2(t^3) - 2p^3$

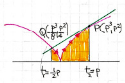
$\Rightarrow 3pt^2 = 2t^3 + p^3$

$\Rightarrow 2t^3 - 3pt^2 + p^3 = 0$

- AS $t = p$ IS A REPEATED SOLUTION AT P

$\Rightarrow (t-p)^2(2t+p) = 0$

$\Rightarrow t = \begin{cases} p & \leftarrow \text{POINT } P(p^3, p^2) \\ -\frac{1}{2}p & \leftarrow \text{POINT } Q(-\frac{1}{8}p^3, \frac{1}{4}p^2) \end{cases}$



- THE "YELLOW AREA" CAN BE FOUND AS

$\int_{x_1}^{x_2} y(x)\,dx = \int_{t_1}^{t_2} y(t)\frac{dx}{dt}\,dt$

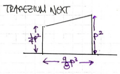
$= \int_{-\frac{1}{2}p}^{p} t^2(3t^2)\,dt$

$= \int_{-\frac{1}{2}p}^{p} 3t^4\,dt$

$= \frac{3}{5}\left[t^5\right]_{-\frac{1}{2}p}^{p} = \frac{3}{5}\left[p^5 - \left(-\frac{1}{2}p\right)^5\right] = \frac{3}{5}\left[p^5 + \frac{1}{32}p^5\right]$

$= \frac{3}{5} \times \frac{33}{32}p^5$

- TRAPEZIUM NEXT



$= \frac{\frac{1}{4}p^2 + p^2}{2} \times \frac{9}{8}p^3 = \frac{5}{8}p^2 \times \frac{9}{8}p^3 = \frac{45}{64}p^5$

- FINALLY WE HAVE

$\Rightarrow \frac{45}{64}p^5 - \frac{3 \times 33}{5 \times 32}p^5 = 2\frac{7}{10}$

$\Rightarrow \frac{45}{64}p^5 - \frac{99}{5 \times 32}p^5 = \frac{27}{10}$ $\quad) \div 9$

$\Rightarrow \frac{5}{64}p^5 - \frac{11}{5 \times 32}p^5 = \frac{3}{10}$ $\quad) \times 64$

$\Rightarrow 5p^5 - \frac{22}{5}p^5 = \frac{3 \times 64}{10}$ $\quad) \times 10$

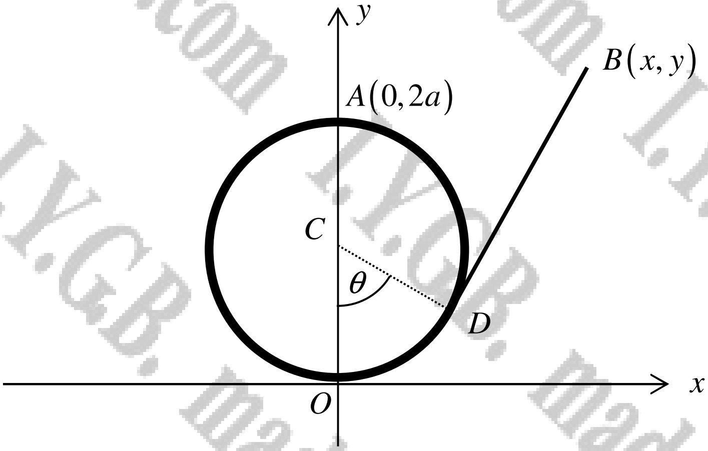
$\Rightarrow 50p^5 - 44p^5 = 192$

$p^5 = 32$

$p = 2$

$\therefore P(8,4)$

Question 7 (*****)



The figure above shows a set of coordinate axes superimposed with a circular cotton reel of radius a and centre at $C(0,3a)$.

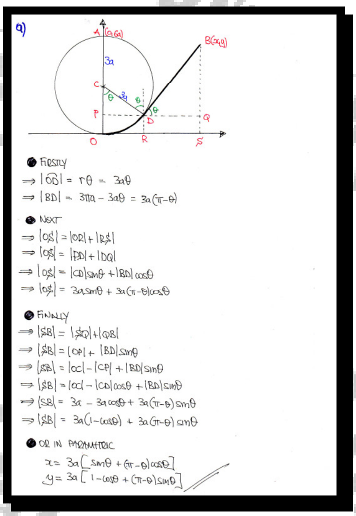
A piece of cotton thread, of length $3\pi a$, is fixed at one end at O and is being unwound from around the circumference of the fixed circular reel. The free end of the cotton thread is marked as the point $B(x,y)$ which was originally at $A(0,6a)$.

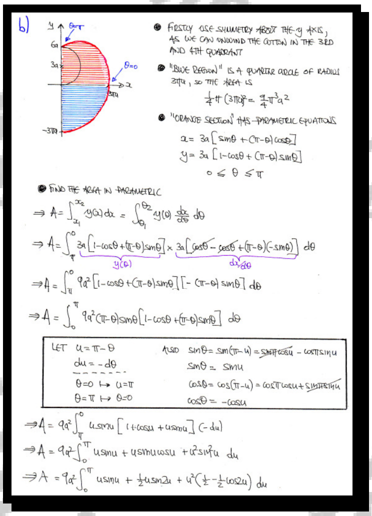
The unwound part of the cotton thread BD is kept straight and θ is the angle OCD as shown in the figure above.

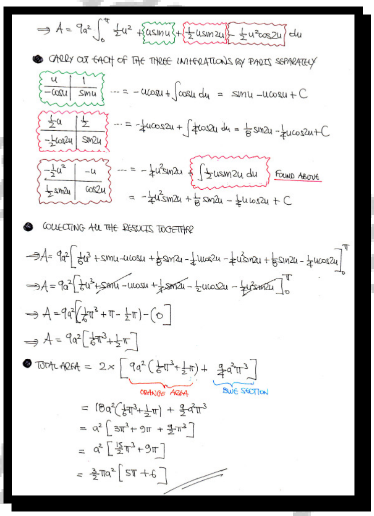
a) Determine the parametric equations that satisfy the locus of $B(x,y)$, as the cotton thread is unwound in the fashion described, for which $x>0$, $y>0$.

b) Find the total area enclosed by the curve traced by B, in the entire x-y plane.

$\boxed{x=3a\left[\sin\theta+(\pi-\theta)\cos\theta\right],\ y=3a\left[1-\cos\theta+(\pi-\theta)\sin\theta\right]}$, $\boxed{\text{area}=\frac{3}{2}\pi a^2\left(5\pi^2+6\right)}$







Question 8 (*****)

A curve C has equation

$$x^2 + xy + y^2 = 1. \quad , \quad 0 \le x \le 3.$$

By seeking a suitable parameterization of C in the form

$$x = A\cos\theta + B\sin\theta \quad \text{and} \quad y = A\cos\theta - B\sin\theta,$$

where A and B are suitable constants,

determine the area of the finite region in the first quadrant, bounded by the curve and the coordinate axes.

You may assume that the curve does not intersect itself.

$\boxed{}$, $\boxed{\text{area} = \frac{1}{9}\pi\sqrt{3}}$

$x^2 + xy + x^2 = 1$

$\Rightarrow x^2 + 2xy + y^2 - xy = 1$

$\Rightarrow (x+y)^2 - xy = 1$

TRY $x = \cos\theta + \sin\theta$

$y = \cos\theta - \sin\theta$

THEN WE HAVE

$(\cos\theta + \cancel{\sin\theta} + \cos\theta - \cancel{\sin\theta})^2 - (\cos\theta + \sin\theta)(\cos\theta - \sin\theta)$

$= (2\cos\theta)^2 - (\cos^2\theta - \sin^2\theta)$

$= 4\cos^2\theta - \cos^2\theta + \sin^2\theta$

$= 3\cos^2\theta + \sin^2\theta$

THUS ADJUST TO $\frac{1}{\sqrt{3}}$ IE $x = \frac{1}{\sqrt{3}}\cos\theta + \sin\theta$

$y = \frac{1}{\sqrt{3}}\cos\theta - \sin\theta$

CHECK

$\left(\frac{1}{\sqrt{3}}\cos\theta + \cancel{\sin\theta} + \frac{1}{\sqrt{3}}\cos\theta - \cancel{\sin\theta}\right)^2 - \left(\frac{1}{\sqrt{3}}\cos\theta + \sin\theta\right)\left(\frac{1}{\sqrt{3}}\cos\theta - \sin\theta\right)$

$= \left(\frac{2}{\sqrt{3}}\cos\theta\right)^2 - \left(\frac{1}{3}\cos^2\theta - \sin^2\theta\right)$

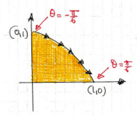
$= \frac{4}{3}\cos^2\theta - \frac{1}{3}\cos^2\theta + \sin^2\theta$

$= \cos^2\theta + \sin^2\theta$

$= 1$

NEXT START WITH A SKETCH

$x^2 + xy + y^2 = 1$



BY INSPECTION

- IF $\theta = \frac{\pi}{6}$, $x = \frac{1}{\sqrt{3}}\cos\frac{\pi}{6} + \sin\frac{\pi}{6}$

$x = \frac{1}{\sqrt{3}}\frac{\sqrt{3}}{2} + \frac{1}{2}$

$x = 1$

& $y = \frac{1}{\sqrt{3}}\cos\frac{\pi}{6} - \sin\frac{\pi}{6} = 0$

- SIMILARLY IF $\theta = -\frac{\pi}{6}$ $\quad x = \frac{1}{\sqrt{3}}\cos\left(-\frac{\pi}{6}\right) + \sin\left(-\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}\frac{\sqrt{3}}{2} - \frac{1}{2} = 0$

$y = \frac{1}{\sqrt{3}}\cos\left(-\frac{\pi}{6}\right) - \sin\left(-\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}\frac{\sqrt{3}}{2} - \left(-\frac{1}{2}\right) = 1$

FINALLY THE REQUIRED AREA IS

$\int_{x_1}^{x_2} y(x)\,dx = \int_{\theta_1}^{\theta_2} y(\theta)\frac{dx}{d\theta}\,d\theta = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \underbrace{\left(\frac{1}{\sqrt{3}}\cos\theta - \sin\theta\right)}_{y(\theta)}\underbrace{\left(-\frac{1}{\sqrt{3}}\sin\theta + \cos\theta\right)}_{dx/d\theta}\,d\theta$

$= \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \underbrace{\cancel{-\frac{1}{3}\cos\theta\sin\theta}}_{\text{ODD}} + \frac{1}{\sqrt{3}}\cos^2\theta + \frac{1}{\sqrt{3}}\sin^2\theta - \underbrace{\cancel{\cos\theta\sin\theta}}_{\text{ODD}}\,d\theta$

$= \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{\sqrt{3}}\left[\cos^2\theta + \sin^2\theta\right]d\theta = \frac{1}{\sqrt{3}}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} 1\,d\theta$

$= \frac{\sqrt{3}}{3} \times \left[\theta\right]_{-\pi/6}^{\pi/6} = \frac{\sqrt{3}}{3}\left[\frac{\pi}{6} - \left(-\frac{\pi}{6}\right)\right]$

$= \frac{\sqrt{3}}{3} \times \frac{\pi}{3} = \frac{\pi\sqrt{3}}{9}$