DIFFERENTIAL EQUATIONS

(by separation of variables)

GENERAL SOLUTIONS

Question 1 (**)

Find a general solution of the differential equation

$$3y^2 \frac{dy}{dx} + 2x = 1$$

giving the answer in the form $y^3 = f(x)$.

$\boxed{}$, $\boxed{y^3 = A - x^2 + x}$

$$3y^2 \frac{dy}{dx} + 2x = 1$$
$$\Rightarrow 3y^2 \frac{dy}{dx} = 1 - 2x$$
$$\Rightarrow 3y^2\,dy = (1-2x)\,dx$$
$$\Rightarrow \int 3y^2\,dy = \int 1-2x\,dx$$
$$\Rightarrow y^3 = x - x^2 + C$$

Question 2 (**)

Find a general solution of the differential equation

$$\frac{dy}{dx} = xy, \quad x \neq 0, \quad y \neq 0,$$

giving the answer in the form $y = f(x)$.

$\boxed{y = A\mathrm{e}^{\frac{1}{2}x^2}}$

$$\frac{dy}{dx} = xy$$
$$\Rightarrow dy = xy\,dx$$
$$\Rightarrow \frac{1}{y}\,dy = x\,dx$$
$$\Rightarrow \int \frac{1}{y}\,dy = \int x\,dx$$
$$\Rightarrow \ln|y| = \frac{1}{2}x^2 + C$$
$$\Rightarrow y = e^{\frac{1}{2}x^2 + C}$$
$$\Rightarrow y = e^{\frac{1}{2}x^2} \times e^C$$
$$\Rightarrow y = Ae^{\frac{1}{2}x^2} \quad (A = e^C)$$

Question 3 (**+)

Find a general solution of the differential equation

$$\frac{dy}{dx} = (y+1)(1-2x), \quad y \neq -1.$$

giving the answer in the form $y = f(x)$.

$$\boxed{y = A\mathrm{e}^{x-x^2} - 1}$$

$$\frac{dy}{dx} = (y+1)(1-2x)$$
$$\Rightarrow dy = (y+1)(1-2x)\,dx$$
$$\Rightarrow \frac{1}{y+1}\,dy = (1-2x)\,dx$$
$$\Rightarrow \int \frac{1}{y+1}\,dy = \int 1-2x\,dx$$
$$\Rightarrow \ln|y+1| = x - x^2 + C$$
$$\Rightarrow y+1 = e^{x-x^2+C}$$
$$\Rightarrow y+1 = e^{x-x^2} \times e^{C}$$
$$\Rightarrow y+1 = Ae^{x-x^2} \quad (A = e^{C})$$
$$\Rightarrow y = -1 + Ae^{x-x^2}$$

Question 4 (**+)

Find a general solution of the differential equation

$$\frac{dy}{dx} = y\tan x, \; y > 0$$

giving the answer in the form $y = f(x)$.

$$\boxed{y = A\sec x}$$

$$\frac{dy}{dx} = y\tan x$$
$$\Rightarrow dy = y\tan x\,dx$$
$$\Rightarrow \frac{1}{y}\,dy = \tan x\,dx$$
$$\Rightarrow \int \frac{1}{y}\,dy = \int \tan x\,dx$$
$$\Rightarrow \ln|y| = \ln|\sec x| + C$$
$$\Rightarrow \ln|y| = \ln|\sec x| + \ln A$$
$$\Rightarrow \ln|y| = \ln|A\sec x|$$
$$\Rightarrow y = A\sec x$$

Question 5 (**+)

Find a general solution of the differential equation

$$\frac{dy}{dx} = 2\mathrm{e}^{x-y}$$

giving the answer in the form $y = f(x)$.

$$\boxed{y = \ln\left(2\mathrm{e}^{x} + C\right)}$$

$$\Rightarrow \frac{dy}{dx} = 2e^{x-y}$$
$$\Rightarrow dy = 2e^{x-y}dx$$
$$\Rightarrow dy = 2e^{x}\times e^{-y}\,dx$$
$$\Rightarrow \frac{1}{e^{-y}}\,dy = 2e^{x}\,dx$$
$$\Rightarrow \int \frac{1}{e^{-y}}\,dy = \int 2e^{x}\,dx$$
$$\Rightarrow \int e^{y}\,dy = \int 2e^{x}\,dx$$
$$\Rightarrow e^{y} = 2e^{x} + C$$
$$\Rightarrow y = \ln\left[2e^{x} + C\right]$$

Question 6 (***)

Find a general solution of the differential equation

$$x^2\frac{dy}{dx} = xy + y, \quad x \neq 0, \quad y \neq 0,$$

giving the answer in the form $y = f(x)$.

[*the final answer may not contain natural logarithms*]

$$\boxed{}\,,\ \boxed{y = Ax\mathrm{e}^{-\frac{1}{x}}}$$

$$x^2\frac{dy}{dx} = xy + y$$
$$\Rightarrow x^2\frac{dy}{dx} = y(x+1)$$
$$\Rightarrow x^2\,dy = y(x+1)\,dx$$
$$\Rightarrow \frac{1}{y}\,dy = \frac{x+1}{x^2}\,dx$$
$$\Rightarrow \int \frac{1}{y}\,dy = \int \frac{x+1}{x^2}$$
$$\Rightarrow \ln|y| = \int \frac{1}{x} + \frac{1}{x^2}\,dx$$
$$\Rightarrow \ln|y| = \int \frac{1}{x} + x^{-2}\,dx$$
$$\Rightarrow \ln|y| = \ln|x| - x^{-1} + C$$
$$\Rightarrow y = e^{\ln|x| - \frac{1}{x} + C}$$
$$\Rightarrow y = e^{\ln|x|}\times e^{-\frac{1}{x}}\times e^{C}$$
$$\Rightarrow y = Axe^{-\frac{1}{x}}$$

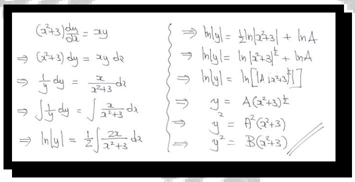
Question 7 (***+)

Find a general solution of the differential equation

$$\left(x^2+3\right)\frac{dy}{dx} = xy\,,\ \ y>0\,,$$

giving the answer in the form $y^2 = f(x)$.

$$\boxed{y^2 = A\left(x^2+3\right)}$$



$(x^2+3)\frac{dy}{dx} = xy$

$\Rightarrow (x^2+3)\,dy = xy\,dx$

$\Rightarrow \frac{1}{y}\,dy = \frac{x}{x^2+3}\,dx$

$\Rightarrow \int \frac{1}{y}\,dy = \int \frac{x}{x^2+3}\,dx$

$\Rightarrow \ln|y| = \frac{1}{2}\int \frac{2x}{x^2+3}\,dx$

$\Rightarrow \ln|y| = \frac{1}{2}\ln|x^2+3| + \ln A$

$\Rightarrow \ln|y| = \ln|x^2+3|^{\frac{1}{2}} + \ln A$

$\Rightarrow \ln|y| = \ln\left[\,|A\,(x^2+3)^{\frac{1}{2}}|\,\right]$

$\Rightarrow y = A(x^2+3)^{\frac{1}{2}}$

$\Rightarrow y^2 = A^2(x^2+3)$

$\Rightarrow y^2 = B(x^2+3)$

Question 8 (***+)

Show that a general solution of the differential equation

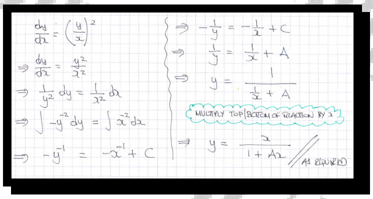
$$\frac{dy}{dx} = \left(\frac{y}{x}\right)^2,$$

is given by

$$y = \frac{x}{1+Ax}\,,$$

where A is an arbitrary constant.

$\boxed{\text{proof}}$



$\frac{dy}{dx} = \left(\frac{y}{x}\right)^2$

$\Rightarrow \frac{dy}{dx} = \frac{y^2}{x^2}$

$\Rightarrow \frac{1}{y^2}\,dy = \frac{1}{x^2}\,dx$

$\Rightarrow \int -y^{-2}\,dy = \int x^{-2}\,dx$

$\Rightarrow -y^{-1} = -x^{-1} + C$

$\Rightarrow -\frac{1}{y} = -\frac{1}{x} + C$

$\Rightarrow \frac{1}{y} = \frac{1}{x} + A$

$\Rightarrow y = \frac{1}{\frac{1}{x} + A}$

MULTIPLY TOP|BOTTOM OF FRACTION BY x

$\Rightarrow y = \frac{x}{1+Ax}$

AS REQUIRED

Question 9 (*+)**

Find a general solution of the differential equation

$$\frac{dy}{dx} = \frac{x e^{x}}{\sin y \cos y},$$

giving the answer in the form $f(x, y) = \text{constant}$.

$$\boxed{\cos 2y + 4e^{x}(x-1) = C \quad \text{or} \quad e^{x}(x-1) - \sin^{2} y = C \quad \text{or} \quad e^{x}(x-1) + \cos^{2} y = C}$$

$\frac{dy}{dx} = \frac{xe^{x}}{\sin y \cos y}$

$\Rightarrow \sin y \cos y \, dy = xe^{x} \, dx$

$\Rightarrow \int \sin y \cos y \, dy = \int xe^{x} \, dx$

$\Rightarrow \int \frac{1}{2}\sin 2y \, dy = xe^{x} - \int e^{x} \, dx$

$\Rightarrow -\frac{1}{4}\cos 2y = xe^{x} - e^{x} + C$

$\Rightarrow -\cos 2y = 4e^{x}(x-1) + C$

$\Rightarrow -\cos 2y - 4e^{x}(x-1) = C$

$\Rightarrow \cos 2y + e^{x}(x-1) = C$

BY PARTS

$x \to 1$

$e^{x} \to e^{x}$

Question 10 (*+)**

Find a general solution of the differential equation

$$\frac{dy}{dx}\cos^{2} x = y^{2} \sin^{2} x$$

giving the answer in the form $y = f(x)$.

$$\boxed{y = \frac{1}{C + x - \tan x}}$$

$\frac{dy}{dx}\cos^{2}x = y^{2}\sin^{2}x$

$\Rightarrow dy \cos^{2}x = y^{2}\sin^{2}x \, dx$

$\Rightarrow \frac{1}{y^{2}} \, dy = \frac{\sin^{2}x}{\cos^{2}x} \, dx$

$\Rightarrow \int y^{-2} \, dy = \int \tan^{2}x \, dx$

$\Rightarrow -y^{-1} = \int \sec^{2}x - 1 \, dx$

$\Rightarrow -\frac{1}{y} = \tan x - x + C$

$\Rightarrow \frac{1}{y} = x - \tan x + C$

$\Rightarrow y = \frac{1}{x - \tan x + C}$

Question 11 **(***+)**

Find a general solution of the differential equation

$$\sec 3x \frac{dy}{dx} = \cot^2 2y$$

giving the answer in the form $f(x, y) = c$.

$$\boxed{3\tan 2y - 6y - 2\sin 3x = C}$$

$$\sec 3x \frac{dy}{dx} = \cot^2 2y$$
$$\Rightarrow \frac{1}{\cot^2 2y}\,dy = \frac{1}{\sec 3x}\,dx$$
$$\Rightarrow \int \tan^2 2y \,dy = \int \cos 3x \,dx$$
$$\Rightarrow \int \sec^2 2y - 1 \,dy = \int \cos 3x \,dx$$
$$\Rightarrow \frac{1}{2}\tan 2y - y = \frac{1}{3}\sin 3x + C$$
$$\Rightarrow 3\tan 2y - 6y = 2\sin 3x + C$$
$$\Rightarrow 3\tan 2y - 6y - 2\sin 3x = C$$

Question 12 **(***+)**

Find a general solution of the differential equation

$$e^{2x} \frac{dy}{dx} = \operatorname{cosec}^2 y$$

giving the answer in the form $f(x, y) = c$.

$$\boxed{2x + 2e^{-2x} - \sin 2y = C}$$

$$e^{2x} \frac{dy}{dx} = \operatorname{cosec}^2 y$$
$$\Rightarrow e^{2x}\,dy = \operatorname{cosec}^2 y \,dx$$
$$\Rightarrow \frac{1}{\operatorname{cosec}^2 y}\,dy = \frac{1}{e^{2x}}\,dx$$
$$\Rightarrow \int \sin^2 y \,dy = \int e^{-2x}\,dx$$
$$\Rightarrow \int \frac{1}{2} - \frac{1}{2}\cos 2y \,dy = \int e^{-2x}\,dx$$
$$\Rightarrow \frac{1}{2}y - \frac{1}{4}\sin 2y = -\frac{1}{2}e^{-2x} + C$$
$$\Rightarrow 2y - \sin 2y = -2e^{-2x} + C$$
$$\Rightarrow 2y - \sin 2y + 2e^{-2x} = C$$

Question 13 (****)

Show that a general solution of the differential equation

$$5\frac{dy}{dx} = 2y^2 - 7y + 3$$

is given by

$$y = \frac{A e^x - 3}{2A e^x - 1},$$

where A is an arbitrary constant.

$\boxed{\text{proof}}$

$$5\frac{dy}{dx} = 2y^2 - 7y + 3$$

START BY SEPARATING VARIABLES

$$\Rightarrow 5\,dy = (2y^2 - 7y + 3)\,dx$$

$$\Rightarrow \frac{5}{2y^2 - 7y + 3}\,dy = 1\,dx$$

$$\Rightarrow \frac{5}{(2y-1)(y-3)}\,dy = 1\,dy$$

PARTIAL FRACTIONS ON THE L.H.S OF THE O.D.E

$$\Rightarrow \frac{5}{(2y-1)(y-3)} \equiv \frac{P}{2y-1} + \frac{Q}{y-3}$$

$$\Rightarrow \boxed{5 \equiv P(y-3) + Q(2y-1)}$$

- IF $y=3 \Rightarrow 5 = 5Q$
  $\Rightarrow Q = 1$
- IF $y=0 \Rightarrow 5 = -3P - Q$
  $\Rightarrow 5 = -3P - 1$
  $\Rightarrow 3P = -6$
  $\Rightarrow P = -2$

RETURNING TO THE O.D.E

$$\Rightarrow \int \frac{1}{y-3} - \frac{2}{2y-1}\,dy = \int 1\,dx$$

$$\Rightarrow \ln|y-3| - \ln|2y-1| = x + C$$

$$\Rightarrow \ln\left|\frac{y-3}{2y-1}\right| = x + C$$

$$\Rightarrow \frac{y-3}{2y-1} = e^{x+C}$$

$$\Rightarrow \frac{y-3}{2y-1} = Ae^x, \quad \text{WHERE } A = e^C$$

$$\Rightarrow y - 3 = 2Aye^x - Ae^x$$

$$\Rightarrow Ae^x - 3 = 2Aye^x - y$$

$$\Rightarrow Ae^x - 3 = y\left[2Ae^x - 1\right]$$

$$\Rightarrow y = \frac{Ae^x - 3}{2Ae^x - 1}$$

AS REQUIRED

Question 14 (**+)**

Show that a general solution of the differential equation

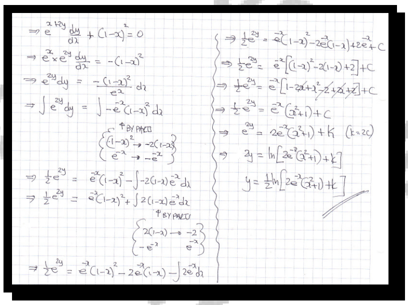
$$e^{x+2y}\frac{dy}{dx}+(1-x)^2=0$$

is given by

$$y=\frac{1}{2}\ln\left[2e^{-x}\left(x^2+1\right)+K\right],$$

where K is an arbitrary constant.

$\boxed{\text{proof}}$



$\Rightarrow e^{x+2y}\frac{dy}{dx}+(1-x)^2=0$

$\Rightarrow e^{x}\times e^{2y}\frac{dy}{dx}=-(1-x)^2$

$\Rightarrow e^{2y}\,dy=\frac{-(1-x)^2}{e^{x}}\,dx$

$\Rightarrow \int e^{2y}\,dy=\int -e^{-x}(1-x)^2\,dx$

BY PARTS

$\left\{\begin{array}{l}(1-x)^2 \to -2(1-x)\\ e^{-x} \to -e^{-x}\end{array}\right\}$

$\Rightarrow \frac{1}{2}e^{2y}=e^{-x}(1-x)^2-\int -2(1-x)e^{-x}\,dx$

$\Rightarrow \frac{1}{2}e^{2y}=e^{-x}(1-x)^2+\int 2(1-x)e^{-x}\,dx$

BY PARTS

$\left\{\begin{array}{ll}2(1-x) \to & -2\\ -e^{-x} & e^{-x}\end{array}\right\}$

$\Rightarrow \frac{1}{2}e^{2y}=e^{-x}(1-x)^2-2e^{-x}(1-x)-\int 2e^{-x}\,dx$

$\Rightarrow \frac{1}{2}e^{2y}=e^{-x}(1-x)^2-2e^{-x}(1-x)+2e^{-x}+C$

$\Rightarrow \frac{1}{2}e^{2y}=e^{-x}\left[(1-x)^2-2(1-x)+2\right]+C$

$\Rightarrow \frac{1}{2}e^{2y}=e^{-x}\left[1-\cancel{2x}+x^2\cancel{-2}+\cancel{2x}\cancel{+2}\right]+C$

$\Rightarrow \frac{1}{2}e^{2y}=e^{-x}(x^2+1)+C$

$\Rightarrow e^{2y}=2e^{-x}(x^2+1)+K \quad (K=2C)$

$\Rightarrow 2y=\ln\left[2e^{-x}(x^2+1)+K\right]$

$y=\frac{1}{2}\ln\left[2e^{-x}(x^2+1)+K\right]$

Question 15 (*****)

$$2x\frac{dy}{dx} = x - y + 3, \ x > 0.$$

Determine a general solution of the above differential equation, by using the substitution $u = y\sqrt{x}$.

$\boxed{}$, $\boxed{y = \frac{1}{3}x + 3 + Cx^{-\frac{1}{2}}}$

$$\boxed{2x\frac{dy}{dx} = x - y + 3, \quad x > 0}$$

- USING THE SUBSTITUTION GIVEN $u = yx^{\frac{1}{2}}$, IN REARRANGED FORM

$$y = x^{-\frac{1}{2}}u(x)$$

$$\frac{dy}{dx} = -\frac{1}{2}x^{-\frac{3}{2}}u + x^{-\frac{1}{2}}\frac{du}{dx}$$

- SUBSTITUTE INTO THE O.D.E

$$\Rightarrow 2x\left[-\frac{1}{2}x^{-\frac{3}{2}}u + x^{-\frac{1}{2}}\frac{du}{dx}\right] = x - x^{-\frac{1}{2}}u + 3$$

$$\Rightarrow \cancel{-x^{-\frac{1}{2}}u} + 2x^{\frac{1}{2}}\frac{du}{dx} = x \cancel{- x^{-\frac{1}{2}}u} + 3$$

$$\Rightarrow 2x^{\frac{1}{2}}\frac{du}{dx} = x + 3$$

$$\Rightarrow \frac{du}{dx} = \frac{1}{2}x^{\frac{1}{2}} + \frac{3}{2}x^{-\frac{1}{2}}$$

- DIRECT INTEGRATION

$$\Rightarrow u = \frac{1}{3}x^{\frac{3}{2}} + 3x^{\frac{1}{2}} + A$$

$$\Rightarrow yx^{\frac{1}{2}} = \frac{1}{3}x^{\frac{3}{2}} + 3x^{\frac{1}{2}} + A$$

$$\Rightarrow y = \frac{1}{3}x + 3 + \frac{A}{x^{\frac{1}{2}}}$$

Question 16 (*****)

By using the substitution $y = xu$, where $u = f(x)$, or otherwise, find a simplified general solution for the following differential equation.

$$x\frac{dy}{dx} = 2x^2 + 2xy + y.$$

$\boxed{}$, $\boxed{y = Axe^{-x} - x}$

$$\boxed{x\frac{dy}{dx} = 2x^2 + 2xy + y}$$

USING THE SUBSTITUTION GIVEN

$$\Rightarrow y = xu(x)$$
$$\Rightarrow \frac{dy}{dx} = 1 \times u(x) + x\frac{d}{dx}(u(x))$$
$$\Rightarrow \frac{dy}{dx} = u + x\frac{du}{dx}$$

SUBSTITUTE INTO THE O.D.E

$$\Rightarrow x\left[u + x\frac{du}{dx}\right] = 2x^2 + 2x(xu) + xu$$
$$\Rightarrow \cancel{xu} + x^2\frac{du}{dx} = 2x^2 + 2x^2u + \cancel{xu}$$
$$\Rightarrow x^2\frac{du}{dx} = 2x^2 + 2x^2u$$
$$\Rightarrow \frac{du}{dx} = 2 + 2u$$
$$\Rightarrow \int \frac{1}{u+1}\,du = \int 2\,dx$$
$$\Rightarrow \ln|u+1| = 2x + C$$
$$\Rightarrow u + 1 = e^{2x+C}$$
$$\Rightarrow u = -1 + e^{2x} \times e^{C}$$
$$\Rightarrow u = -1 + Ae^{2x} \quad (A = e^{C})$$
$$\Rightarrow \frac{y}{x} = Ae^{2x} - 1$$
$$\Rightarrow y = Axe^{2x} - x$$

ALTERNATIVE WITHOUT THE SUBSTITUTION

$$\Rightarrow x\frac{dy}{dx} = 2x^2 + 2xy + y$$
$$\Rightarrow x\frac{dy}{dx} - 2xy - y = 2x^2$$
$$\Rightarrow \frac{dy}{dx} - 2y - \frac{y}{x} = 2x$$
$$\Rightarrow \frac{dy}{dx} + y\left(-2 - \frac{1}{x}\right) = 2x$$

INTEGRATING FACTOR

$$e^{\int -2 - \frac{1}{x}\,dx} = e^{-2x - \ln x} = e^{-2x} \times e^{-\ln x} = e^{-2x} \times e^{\ln\frac{1}{x}} = \frac{1}{x}e^{-2x}$$

HENCE WE OBTAIN

$$\Rightarrow \frac{d}{dx}\left[y\frac{1}{x}e^{-2x}\right] = 2x\left(\frac{1}{x}e^{-2x}\right)$$
$$\Rightarrow \frac{d}{dx}\left[\frac{y}{x}e^{-2x}\right] = 2e^{-2x}$$
$$\Rightarrow \frac{y}{x}e^{-2x} = \int 2e^{-2x}\,dx$$
$$\Rightarrow \frac{y}{x}e^{-2x} = -e^{-2x} + A$$
$$\Rightarrow ye^{-2x} = -xe^{-2x} + Ax$$
$$\Rightarrow y = -x + Axe^{2x}$$

AS BEFORE.

Question 17 (*****)

Use differentiation to find a simplified general solution for the following differential equation.

$$\left(x^2-1\right)\left(\frac{dy}{dx}\right)^2-2xy\left(\frac{dy}{dx}\right)+y^2=1.$$

$$\boxed{\left(A+y\sqrt{x^2-1}\right)\left(y+Bx+C\right)=0}$$

$$\boxed{(x^2-1)\left(\frac{dy}{dx}\right)^2-2xy\left(\frac{dy}{dx}\right)+y^2=1}$$

- DIFFERENTIATE THE DIFFERENTIAL EQUATION W.R.T x

$$\Rightarrow 2x\left(\frac{dy}{dx}\right)^2-(x^2-1)\times 2\left(\frac{dy}{dx}\right)^1\frac{d^2y}{dx^2}-2y\frac{dy}{dx}-2x\left(\frac{dy}{dx}\right)\left(\frac{dy}{dx}\right)-2xy\frac{d^2y}{dx^2}+2y\left(\frac{dy}{dx}\right)=0$$

$$\Rightarrow \cancel{2x\left(\frac{dy}{dx}\right)^2}-2(x^2-1)\frac{dy}{dx}\frac{d^2y}{dx^2}-\cancel{2y\frac{dy}{dx}}-\cancel{2x\left(\frac{dy}{dx}\right)^2}-2xy\frac{d^2y}{dx^2}+\cancel{2y\frac{dy}{dx}}=0$$

$$\Rightarrow -2(x^2-1)\frac{d^2y}{dx^2}\frac{dy}{dx}-2xy\frac{d^2y}{dx^2}=0$$

$$\Rightarrow -\frac{d^2y}{dx^2}\left[2(x^2-1)\frac{dy}{dx}+2xy\right]=0$$

- EITHER $\frac{d^2y}{dx^2}=0$
- OR $\cancel{2}(x^2-1)\frac{dy}{dx}+\cancel{2}xy=0$

$$\Rightarrow \frac{1}{y}\,dy=-\frac{x}{x^2-1}\,dx$$

$$\Rightarrow \ln y=-\tfrac{1}{2}\ln|x^2-1|+\ln A$$

$$\Rightarrow \ln y=\ln A-\ln(x^2-1)^{\frac{1}{2}}$$

$$\Rightarrow \ln y=\ln\frac{A}{\sqrt{x^2-1}}$$

$$\Rightarrow y=\frac{A}{\sqrt{x^2-1}}$$

$$\Rightarrow y\sqrt{x^2-1}-A=0 \quad \text{OR} \quad \underline{\underline{y\sqrt{x^2-1}+A=0}}$$

- SIMILARLY

$$\frac{d^2y}{dx^2}=0$$

$$\frac{dy}{dx}=B$$

$$y=Bx+C$$

$$y-Bx-C=0 \quad \text{OR} \quad \underline{\underline{y+Bx+C=0}}$$

- HENCE

$$\left(y+Bx+C\right)\left(y\sqrt{x^2-1}+A\right)=0$$

Question 18 (*****)

A circle touches the x axis at the origin O.

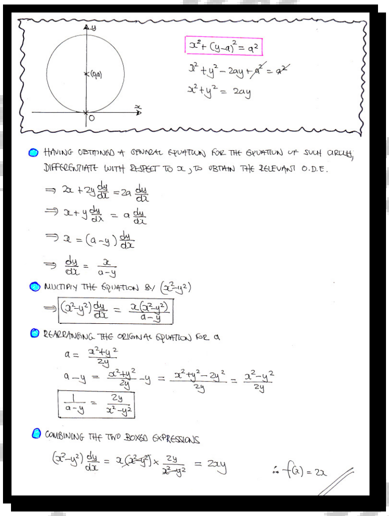
It is further given that the equation of such a circle satisfies the differential equation

$$\left(x^2 - y^2\right)\frac{dy}{dx} = y\,f(x),$$

for some function f.

Use an algebraic method to find an expression for $f(x)$.

$\boxed{f(x) = 2x}$



$\boxed{x^2 + (y-a)^2 = a^2}$

$x^2 + y^2 - 2ay + \cancel{a^2} = \cancel{a^2}$

$x^2 + y^2 = 2ay$

HAVING OBTAINED A GENERAL EQUATION FOR THE EQUATION OF SUCH CIRCLES, DIFFERENTIATE WITH RESPECT TO x, TO OBTAIN THE RELEVANT O.D.E.

$\Rightarrow 2x + 2y\frac{dy}{dx} = 2a\frac{dy}{dx}$

$\Rightarrow x + y\frac{dy}{dx} = a\frac{dy}{dx}$

$\Rightarrow x = (a-y)\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{x}{a-y}$

MULTIPLY THE EQUATION BY (x^2-y^2)

$\Rightarrow \boxed{(x^2-y^2)\frac{dy}{dx} = \frac{x(x^2-y^2)}{a-y}}$

REARRANGING THE ORIGINAL EQUATION FOR a

$a = \frac{x^2+y^2}{2y}$

$a - y = \frac{x^2+y^2}{2y} - y = \frac{x^2+y^2-2y^2}{2y} = \frac{x^2-y^2}{2y}$

$\boxed{\frac{1}{a-y} = \frac{2y}{x^2-y^2}}$

COMBINING THE TWO BOXED EXPRESSIONS

$(x^2-y^2)\frac{dy}{dx} = x\cancel{(x^2-y^2)} \times \frac{2y}{\cancel{x^2-y^2}} = 2xy$

$\therefore f(x) = 2x$

Question 19 (*****)

The non zero functions $u(x)$ and $v(x)$ satisfy the integral equations

$$\int u(x)\,dx = ux^2 \qquad \text{and} \qquad \int u(x)v(x)\,dx = \left[\int u(x)\,dx\right]\left[\int v(x)\,dx\right].$$

Determine, in terms of an arbitrary constant, a simplified expression for $u(x)$ and a similar expression for $[v(x)]^2$.

$\boxed{\text{SP}}$, $\boxed{u(x) = \dfrac{A\mathrm{e}^{-\frac{1}{x}}}{x^2}}$, $\boxed{[v(x)]^2 = \dfrac{B}{(1-x)^2(1-x^2)}}$

- STARTING WITH

$$\int u\,dx = ux^2$$

- DIFFERENTIATE W.R.T x

$$\Rightarrow \frac{d}{dx}\int u\,dx = \frac{d}{dx}(ux^2)$$
$$\Rightarrow u = \frac{du}{dx}x^2 + u(2x)$$
$$\Rightarrow x^2\frac{du}{dx} = u - 2ux$$
$$\Rightarrow x^2\frac{du}{dx} = u(1-2x)$$
$$\Rightarrow \frac{1}{u}\,du = \frac{1-2x}{x^2}\,dx$$
$$\Rightarrow \int \frac{1}{u}\,du = \int \frac{1}{x^2} - \frac{2}{x}\,dx$$
$$\Rightarrow \ln|u| = -\frac{1}{x} - 2\ln|x| + C$$
$$\Rightarrow u = e^{C - \frac{1}{x} + \ln x^{-2}}$$
$$\Rightarrow u = e^{C} \times e^{-\frac{1}{x}} \times e^{\ln\frac{1}{x^2}}$$
$$\Rightarrow u = \frac{A}{x^2}e^{-\frac{1}{x}}$$

- NEXT WE PROCEED WITH

$$\int uv\,dx = \left[\int u\,dx\right]\left[\int v\,dx\right]$$

- DIFFERENTIATING W.R.T x

$$\Rightarrow \frac{d}{dx}\int uv\,dx = \frac{d}{dx}\left[\left[\int u\,dx\right]\left[\int v\,dx\right]\right]$$
$$\Rightarrow uv = \frac{d}{dx}\left[\int u\,dx\right] \times \int v\,dx + \int u\,dx \times \frac{d}{dx}\left[\int v\,dx\right]$$
$$\Rightarrow uv = u\int v\,dx + v\int u\,dx$$
$$\Rightarrow uv = u\int v\,dx + v(ux^2)$$
$$\Rightarrow v = \int v\,dx + vx^2$$
$$\Rightarrow v - vx^2 = \int v\,dx$$
$$\Rightarrow v(1-x^2) = \int v\,dx$$

DIFFERENTIATE W.R.T x AGAIN

$$\Rightarrow \frac{dv}{dx}(1-x^2) + v(-2x) = \frac{d}{dx}\int v\,dx$$
$$\Rightarrow \frac{dv}{dx}(1-x^2) - 2vx = v$$
$$\Rightarrow \frac{dv}{dx}(1-x^2) = v + 2vx$$
$$\Rightarrow \frac{dv}{dx}(1-x^2) = v(1+2x)$$

- SEPARATING VARIABLES AND INTEGRATING

$$\Rightarrow \int \frac{1}{v}\,dv = \int \frac{2x+1}{1-x^2}\,dx$$
$$\Rightarrow \ln|v| = \int \frac{2x+1}{(1-x)(1+x)}\,dx$$

- PARTIAL FRACTIONS BY INSPECTION (COVER UP)

$$\Rightarrow \ln|v| = \int \frac{\frac{3}{2}}{1-x} - \frac{\frac{1}{2}}{1+x}\,dx$$
$$\Rightarrow 2\ln|v| = \int \frac{3}{1-x} - \frac{1}{1+x}\,dx$$
$$\Rightarrow \ln v^2 = -3\ln|1-x| - \ln|1+x| + \ln A$$
$$\Rightarrow \ln v^2 = \ln\left|\frac{B}{(1-x)^3(1+x)}\right|$$
$$\Rightarrow v^2 = \frac{B}{(1-x)^3(1+x)}$$

OR

$$v^2 = \frac{B}{(1-x)^2(1-x^2)}$$

Question 20 (*****)

The positive solution of the quadratic equation $x^2 - x - 1 = 0$ is denoted by ϕ, and is commonly known as the golden section or golden number.

a) Show, with a detailed method, that $F(x) = f(\phi)x^{g(\phi)}$ is a solution of the differential equation,

$$F'(x) = F^{-1}(x),$$

where f and g are constant expressions of ϕ, to be found in simplified form.

b) Verify the answer obtained in part **(a)** satisfies the differential equation, by differentiation and function inversion.

[You may assume that $F(x)$ is differentiable and invertible]

$\boxed{}$, $\boxed{F(x) = \left(\dfrac{1}{\phi}\right)^{\frac{1}{\phi}} x^{\phi} = \phi^{1-\phi}x^{\phi}}$

ASSUME A SOLUTION OF THE FORM $y = Ax^r$, WHERE A IS A CONSTANT & r IS ALSO A CONSTANT — FURTHER ASSUME THAT "FUNCTION" IS SMOOTH AND INVERTIBLE

- $y = Ax^r$

$\Rightarrow \underbrace{\dfrac{dy}{dx}}_{F'(x)} = rAx^{r-1}$

- $y = Ax^r$

$\Rightarrow \dfrac{y}{A} = x^r$

$\Rightarrow \left(\dfrac{y}{A}\right)^{\frac{1}{r}} = (x^r)^{\frac{1}{r}}$

$\Rightarrow x = \left(\dfrac{1}{A}\right)^{\frac{1}{r}} y^{\frac{1}{r}}$ SWAP x & y

$\Rightarrow y = \left(\dfrac{1}{A}\right)^{\frac{1}{r}} x^{\frac{1}{r}}$

$\Rightarrow y = (A^{-1})^{\frac{1}{r}} x^{\frac{1}{r}}$

$\Rightarrow \underbrace{y}_{F^{-1}(x)} = A^{-\frac{1}{r}} x^{\frac{1}{r}}$

SETTING EQUAL TO ONE ANOTHER, AS IN THE O.D.E

$\Rightarrow rAx^{r-1} = A^{-\frac{1}{r}}x^{\frac{1}{r}}$

$\Rightarrow \dfrac{x^{r-1}}{x^{\frac{1}{r}}} = \dfrac{A^{-\frac{1}{r}}}{rA}$

$\Rightarrow x^{r-1-\frac{1}{r}} = \dfrac{1}{r}A^{-1-\frac{1}{r}}$

NOW R.H.S IS A CONSTANT $\Rightarrow$ L.H.S MUST ALSO BE A CONSTANT

$\Rightarrow$ EXPONENT OF x MUST BE ZERO

$\Rightarrow r - 1 - \frac{1}{r} = 0$

BUT $r - 1 - \frac{1}{r} = 0$ CAN BE REARRANGED TO SEVERAL FORMS

LET $r = \phi$

$\phi = 1 + \frac{1}{\phi}$ $\quad \phi^2 - \phi - 1 = 0$, $\quad \phi^2 = \phi + 1$, ETC

BUT AS THE LHS IS A CONSTANT, THE ONLY CONSTANT IT CAN BE IS "ONE" AND THUS THE R.H.S MUST ALSO BE "ONE"

$\Rightarrow \frac{1}{r}A^{-1-\frac{1}{r}} = 1$

$\Rightarrow \frac{1}{\phi}A^{-1-\frac{1}{\phi}} = 1$

$\Rightarrow A^{-(1+\frac{1}{\phi})} = \phi$

$\Rightarrow A^{-\phi} = \phi$

$\Rightarrow A^{\phi} = \frac{1}{\phi}$

$\Rightarrow A = \left(\frac{1}{\phi}\right)^{\frac{1}{\phi}}$ OR EQUIVALENT $(\phi^{-1})^{\frac{1}{\phi}} = \phi^{-\frac{1}{\phi}} = \phi^{1-\phi} = \phi^{1-(\phi^2-1)} = \phi^{2-\phi^2}$ ETC

$\therefore F(x) = \phi^{1-\phi}x^{\phi}$

b) DIFFERENTIATING $F(x) = \phi^{1-\phi}x^{\phi}$

$F'(x) = \phi\,\phi^{1-\phi}x^{\phi-1} = \phi^{2-\phi}x^{\phi-1}$

WE MAY HAVE TO MANIPULATE THIS FURTHER AT A LATER STAGE

INVERTING $F(x)$

$\Rightarrow y = \phi^{1-\phi}x^{\phi}$

$\Rightarrow \dfrac{y}{\phi^{1-\phi}} = x^{\phi}$

$\Rightarrow \dfrac{(y)^{\frac{1}{\phi}}}{(\phi)^{\frac{1-\phi}{\phi}}} = (x^{\phi})^{\frac{1}{\phi}}$

$\Rightarrow x = \phi^{-\frac{1-\phi}{\phi}}y^{\frac{1}{\phi}}$

$\Rightarrow F^{-1}(x) = \phi^{\frac{\phi-1}{\phi}}x^{\frac{1}{\phi}}$

LOOKING AT THE POWERS OF x, STARTING WITH $F^{-1}(x)$

$\frac{1}{\phi} = \phi - 1$ (SINCE $\phi = 1 + \frac{1}{\phi}$)

LOOKING AT THE CONSTANT, STARTING WITH THE EXPONENT AT $F^{-1}(x)$

$\frac{\phi-1}{\phi} = 1 - \frac{1}{\phi} = 1 - (\phi - 1) = 2 - \phi$

$\therefore \phi^{2-\phi}x^{\phi-1} = \phi^{\frac{\phi-1}{\phi}}x^{\frac{1}{\phi}}$

$\therefore F'(x) = F^{-1}(x)$

SPECIFIC SOLUTIONS

Question 1 (**)

Solve the differential equation

$$\frac{dy}{dx} + \frac{4x}{y} = 0, \quad y \neq 0,$$

subject to the condition $y = 2$ at $x = 0$.

Give the answer in the form $f(x, y) = \text{constant}$.

$$\boxed{4x^2 + y^2 = 4}$$

$$\frac{dy}{dx} + \frac{4x}{y} = 0$$
$$\Rightarrow \frac{dy}{dx} = -\frac{4x}{y}$$
$$\Rightarrow y\,dy = -4x\,dx$$
$$\Rightarrow \int y\,dy = \int -4x\,dx$$
$$\Rightarrow \frac{1}{2}y^2 = -2x^2 + C$$
$$\Rightarrow y^2 = -4x^2 + C \leftarrow \text{gen sol}$$

{when $x=0$ $y=2$}

$4 = 0 + C \quad \therefore C = 4$

Hence

$$y^2 = -4x^2 + 4$$
$$y^2 + 4x^2 = 4$$

Question 2 (**)

Solve the differential equation

$$\frac{dy}{dx} = \frac{\cos 2x}{y}, \quad x > 0, \quad y > 0,$$

subject to the condition $y = 6$ at $x = \frac{\pi}{4}$, giving the answer in the form $y^2 = f(x)$.

$$\boxed{y^2 = 35 + \sin 2x}$$

$$\frac{dy}{dx} = \frac{\cos 2x}{y}$$
$$\Rightarrow y\,dy = \cos 2x\,dx$$
$$\Rightarrow \int y\,dy = \int \cos 2x\,dx$$
$$\Rightarrow \frac{1}{2}y^2 = \frac{1}{2}\sin 2x + C$$
$$\Rightarrow \boxed{y^2 = \sin 2x + C}$$

when $x = \frac{\pi}{4}$ $y = 6$

$36 = \sin\frac{\pi}{2} + C$

$36 = 1 + C$

$C = 35$

$\therefore y^2 = \sin 2x + 35$

Question 3 (**)

Solve the differential equation

$$\frac{dy}{dx} = 6xy^2,$$

with $y = 1$ at $x = 2$, giving the answer in the form $y = f(x)$.

$$\boxed{y = \frac{1}{13 - 3x^2}}$$

$$\frac{dy}{dx} = 6xy^2$$
$$\Rightarrow dy = 6xy^2\,dx$$
$$\Rightarrow \frac{1}{y^2}\,dy = 6x\,dx$$
$$\Rightarrow \int y^{-2}\,dy = \int 6x\,dx$$
$$\Rightarrow -y^{-1} = 3x^2 + C$$
$$\Rightarrow -\frac{1}{y} = 3x^2 + C$$
$$\frac{1}{y} = C - 3x^2$$
$$y = \frac{1}{C - 3x^2} \leftarrow \text{gen sol}$$
when $x = 2$ $y = 1$
$$1 = \frac{1}{C - 12} \Rightarrow C - 12 = 1 \Rightarrow \boxed{C = 13}$$
$$\therefore y = \frac{1}{13 - 3x^2}$$

Question 4 (**)

Solve the differential equation

$$\frac{dy}{dx} = \frac{3\sin 3x}{y},$$

subject to the condition $y = 3$ at $x = \frac{\pi}{3}$, giving the answer in the form $y^2 = f(x)$.

$$\boxed{y^2 = 7 - 2\cos 3x}$$

$$\frac{dy}{dx} = \frac{3\sin 3x}{y}$$
$$\Rightarrow y\,dy = 3\sin 3x\,dx$$
$$\Rightarrow \int y\,dy = \int 3\sin 3x\,dx$$
$$\Rightarrow \frac{1}{2}y^2 = -\cos 3x + C$$
$$\Rightarrow \boxed{y^2 = -2\cos 3x + C}$$
APPLY CONDITION $x = \frac{\pi}{3}$, $y = 3$
$$9 = -2\cos\pi + C$$
$$9 = 2 + C$$
$$\boxed{C = 7}$$
$$\therefore y^2 = 7 - 2\cos 3x$$

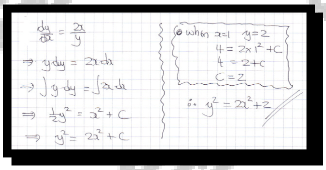
Question 5 (**)

Solve the differential equation

$$\frac{dy}{dx} = \frac{2x}{y},$$

with $y = 2$ at $x = 1$, giving the answer in the form $y^2 = f(x)$.

$$\boxed{y^2 = 2x^2 + 2}$$



$\frac{dy}{dx} = \frac{2x}{y}$

$\Rightarrow y\,dy = 2x\,dx$

$\Rightarrow \int y\,dy = \int 2x\,dx$

$\Rightarrow \frac{1}{2}y^2 = x^2 + C$

$\Rightarrow y^2 = 2x^2 + C$

when $x=1$ $y=2$

$4 = 2\times 1^2 + C$

$4 = 2 + C$

$C = 2$

$\therefore y^2 = 2x^2 + 2$

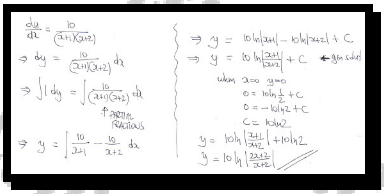
Question 6 (**)

Solve the differential equation

$$\frac{dy}{dx} = \frac{10}{(x+1)(x+2)},$$

subject to the condition $y = 0$ at $x = 0$, giving the answer in the form $y = f(x)$.

$$\boxed{y = 10\ln\left|\frac{2x+2}{x+2}\right|}$$



$\frac{dy}{dx} = \frac{10}{(x+1)(x+2)}$

$\Rightarrow dy = \frac{10}{(x+1)(x+2)}\,dx$

$\Rightarrow \int 1\,dy = \int \frac{10}{(x+1)(x+2)}\,dx$

PARTIAL FRACTIONS

$\Rightarrow y = \int \frac{10}{x+1} - \frac{10}{x+2}\,dx$

$\Rightarrow y = 10\ln|x+1| - 10\ln|x+2| + C$

$\Rightarrow y = 10\ln\left|\frac{x+1}{x+2}\right| + C$ ← gen. solut.

when $x=0$ $y=0$

$0 = 10\ln\frac{1}{2} + C$

$0 = -10\ln 2 + C$

$C = 10\ln 2$

$y = 10\ln\left|\frac{x+1}{x+2}\right| + 10\ln 2$

$y = 10\ln\left|\frac{2x+2}{x+2}\right|$

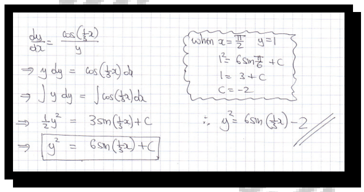
Question 7 (**)

Solve the differential equation

$$\frac{dy}{dx} = \frac{\cos\left(\frac{1}{3}x\right)}{y},$$

subject to the condition $y = 1$ at $x = \frac{\pi}{2}$, giving the answer in the form $y^2 = f(x)$.

$$\boxed{y^2 = 6\sin\left(\tfrac{1}{3}x\right) - 2}$$



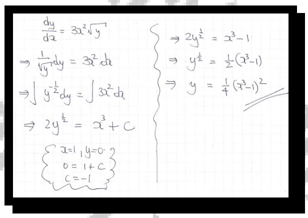
Question 8 (**)

Solve the differential equation

$$\frac{dy}{dx} = 3x^2\sqrt{y}$$

subject to the condition $y = 0$ at $x = 1$, giving the answer in the form $y = f(x)$.

$$\boxed{y = \tfrac{1}{4}\left(x^3 - 1\right)^2}$$



Question 9 **(**)**

Solve the differential equation

$$\frac{dy}{dx} = \sqrt{\frac{y}{x+1}}, \qquad y \neq 0,\ x \neq -1,$$

subject to the condition $y = 9$ at $x = 8$, giving the answer in the form $y = f(x)$.

$$\boxed{y = x + 1}$$

$\frac{dy}{dx} = \sqrt{\frac{y}{x+1}}$

$\Rightarrow \frac{dy}{dx} = \frac{y^{1/2}}{(x+1)^{1/2}}$

$\Rightarrow \frac{1}{y^{1/2}}\,dy = \frac{1}{(x+1)^{1/2}}\,dx$

$\Rightarrow \int y^{-\frac{1}{2}}\,dy = \int (x+1)^{-\frac{1}{2}}\,dx$

$\Rightarrow 2y^{\frac{1}{2}} = 2(x+1)^{\frac{1}{2}} + C$

$\Rightarrow \boxed{y^{\frac{1}{2}} = (x+1)^{\frac{1}{2}} + C}$

• when $x = 8$, $y = 9$

$3 = 3 + C$

$C = 0$.

$y^{\frac{1}{2}} = (x+1)^{\frac{1}{2}}$

$y = x + 1$

Question 10 **(**)**

Solve the differential equation

$$\frac{dy}{dx} = 2x\sqrt{2y-1}, \qquad y > \frac{1}{2}$$

subject to the condition $y = \frac{1}{2}$ at $x = 0$, giving the answer in the form $y = f(x)$.

$$\boxed{y = \frac{1}{2}\left(x^4 + 1\right)}$$

$\frac{dy}{dx} = 2x\sqrt{2y-1}$

$\Rightarrow \frac{1}{(2y-1)^{\frac{1}{2}}}\,dy = 2x\,dx$

$\Rightarrow \int (2y-1)^{-\frac{1}{2}}\,dy = \int 2x\,dx$

$\Rightarrow \boxed{(2y-1)^{\frac{1}{2}} = x^2 + C}$

$x = 0$ $y = \frac{1}{2}$

$0 = 0 + C$

$\Leftarrow C = 0$

$\therefore (2y-1)^{\frac{1}{2}} = x^2$

$2y - 1 = x^4$

$2y = x^4 + 1$

$y = \frac{1}{2}(x^4 + 1)$

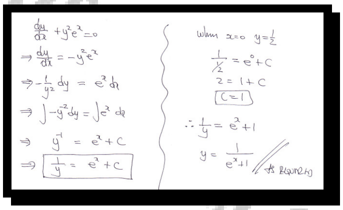
Question 11 (**)

Solve the differential equation

$$\frac{dy}{dx} + y^2 \mathrm{e}^x = 0,$$

subject to the condition $y = \frac{1}{2}$ at $x = 0$, giving the answer in the form $y = f(x)$.

$$\boxed{y = \frac{1}{\mathrm{e}^x + 1}}$$



$\frac{dy}{dx} + y^2 e^x = 0$
$\Rightarrow \frac{dy}{dx} = -y^2 e^x$
$\Rightarrow -\frac{1}{y^2}\,dy = e^x\,dx$
$\Rightarrow \int -y^{-2}\,dy = \int e^x\,dx$
$\Rightarrow y^{-1} = e^x + C$
$\Rightarrow \boxed{\frac{1}{y} = e^x + C}$
when $x=0$ $y=\frac{1}{2}$
$\frac{1}{\frac{1}{2}} = e^0 + C$
$2 = 1 + C$
$\boxed{C=1}$
$\therefore \frac{1}{y} = e^x + 1$
$y = \frac{1}{e^x+1}$

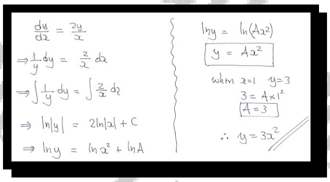
Question 12 (**)

$$\frac{dy}{dx} = \frac{2y}{x}, \quad x > 0, \ y > 0.$$

Show that the solution of the above differential equation subject to the boundary condition $y = 3$ at $x = 1$, is given by

$$y = 3x^2.$$

$$\boxed{\text{proof}}$$



$\frac{dy}{dx} = \frac{2y}{x}$
$\Rightarrow \frac{1}{y}\,dy = \frac{2}{x}\,dx$
$\Rightarrow \int \frac{1}{y}\,dy = \int \frac{2}{x}\,dx$
$\Rightarrow \ln|y| = 2\ln|x| + C$
$\Rightarrow \ln y = \ln x^2 + \ln A$
$\ln y = \ln(Ax^2)$
$\boxed{y = Ax^2}$
when $x=1$ $y=3$
$3 = A \times 1^2$
$\boxed{A=3}$
$\therefore y = 3x^2$

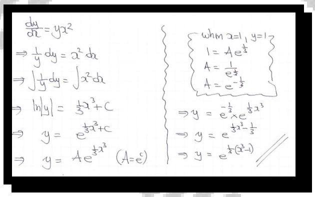
Question 13 (+)**

Solve the differential equation

$$\frac{dy}{dx} = yx^2, \; x \neq 0, \; y \neq 0,$$

subject to the condition $y = 1$ at $x = 1$, giving the answer in the form $y = f(x)$.

$$\boxed{y = e^{\frac{1}{3}\left(x^3 - 1\right)}}$$



$\frac{dy}{dx} = yx^2$

$\Rightarrow \frac{1}{y}\,dy = x^2\,dx$

$\Rightarrow \int \frac{1}{y}\,dy = \int x^2\,dx$

$\Rightarrow \ln|y| = \frac{1}{3}x^3 + C$

$\Rightarrow y = e^{\frac{1}{3}x^3 + C}$

$\Rightarrow y = Ae^{\frac{1}{3}x^3} \quad (A = e^C)$

when $x=1$, $y=1$

$1 = Ae^{\frac{1}{3}}$

$A = \frac{1}{e^{\frac{1}{3}}}$

$A = e^{-\frac{1}{3}}$

$\Rightarrow y = e^{-\frac{1}{3}} \times e^{\frac{1}{3}x^3}$

$\Rightarrow y = e^{\frac{1}{3}x^3 - \frac{1}{3}}$

$\Rightarrow y = e^{\frac{1}{3}(x^3-1)}$

Question 14 (**+)

Solve the differential equation

$$\frac{dy}{dx} = y^2\sqrt{x}, \quad x \neq 0, \quad y \neq 0,$$

with $y = -2$ at $x = 1$.

Give the answer in the form $y = \dfrac{A}{1 + Bx^{\frac{3}{2}}}$, where A and B are integers.

$$\boxed{y = \frac{6}{1 - 4x^{\frac{3}{2}}}}$$

SOLVE BY SEPARATION OF VARIABLES

$\Rightarrow \frac{dy}{dx} = y^2 x^{\frac{1}{2}}$

$\Rightarrow \frac{1}{y^2}\,dy = x^{\frac{1}{2}}\,dx$

$\Rightarrow \int y^{-2}\,dy = \int x^{\frac{1}{2}}\,dx$

$\Rightarrow -y^{-1} = \frac{2}{3}x^{\frac{3}{2}} + C$

$\Rightarrow -\frac{1}{y} = \frac{2}{3}x^{\frac{3}{2}} + C$

$\Rightarrow \boxed{\frac{1}{y} = -\frac{2}{3}x^{\frac{3}{2}} + C}$

APPLY CONDITION $(1,-2)$

$\Rightarrow -\frac{1}{2} = -\frac{2}{3} + C$

$\Rightarrow C = \frac{1}{6}$

REARRANGE TO THE REQUIRED ANSWER

$\Rightarrow \frac{1}{y} = -\frac{2}{3}x^{\frac{3}{2}} + \frac{1}{6}$

$\Rightarrow \frac{1}{y} = \frac{1}{6} - \frac{4}{6}x^{\frac{3}{2}}$

$\Rightarrow \frac{1}{y} = \frac{1 - 4x^{\frac{3}{2}}}{6}$

$\Rightarrow y = \frac{6}{1 - 4x^{3/2}}$

// ie $A = 6$, $B = -4$

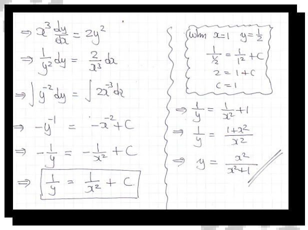
Question 15 **(**+)**

Solve the differential equation

$$x^3 \frac{dy}{dx} = 2y^2$$

subject to the condition $y = \frac{1}{2}$ at $x = 1$, giving the answer in the form $y = f(x)$.

$$\boxed{y = \frac{x^2}{x^2 + 1}}$$



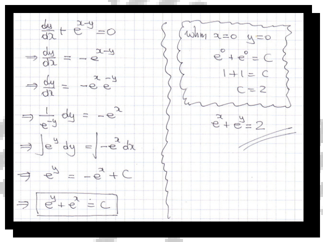
Question 16 **(**+)**

Solve the differential equation

$$\frac{dy}{dx} + e^{x-y} = 0,$$

subject to $y = 0$ at $x = 0$, giving the answer in the form $f(x, y) = \text{constant}$.

$$\boxed{e^x + e^y = 2}$$



Question 17 **(**+)**

Solve the differential equation

$$\frac{dy}{dx} = xy\,\mathrm{e}^{x}, \quad x > 0, \quad y > 0,$$

subject to boundary condition $y = \mathrm{e}$ at $x = 1$.

Give the answer in the form $\ln y = f(x)$.

$$\boxed{\ln y = x\,\mathrm{e}^{x} - \mathrm{e}^{x} + 1}$$

$\frac{dy}{dx} = xye^{x}$

$\Rightarrow dy = xye^{x}\,dx$

$\Rightarrow \frac{1}{y}\,dy = xe^{x}\,dx$

$\Rightarrow \int \frac{1}{y}\,dy = \int xe^{x}\,dx$

BY PARTS

$\begin{Bmatrix} x & 1 \\ e^{x} & e^{x} \end{Bmatrix}$

$\Rightarrow \ln|y| = xe^{x} - \int e^{x}\,dx + C$

$\Rightarrow \ln y = xe^{x} - e^{x} + C$

$y>0$

gm solution

$x=1 \quad y=e$

$\ln e = \cancel{e} - \cancel{e} + C$

$1 = C$

$\ln y = xe^{x} - e^{x} + 1$

Question 18 **(**+)**

Solve the differential equation

$$\frac{dy}{dx} = -\frac{\sqrt{4y+1}}{x^{2}}$$

subject to the condition, $y = 2$ at $x = \frac{2}{3}$, giving the answer in the form $y = f(x)$.

$$\boxed{y = \frac{1}{x^{2}} - \frac{1}{4}}$$

$\frac{dy}{dx} = -\frac{\sqrt{4y+1}}{x^{2}}$

$\Rightarrow \frac{1}{\sqrt{4y+1}}\,dy = -\frac{1}{x^{2}}\,dx$

$\Rightarrow \int (4y+1)^{-\frac{1}{2}}\,dy = \int -\frac{1}{x^{2}}\,dx$

$\Rightarrow \frac{1}{2}(4y+1)^{\frac{1}{2}} = \frac{1}{x} + C$

$\Rightarrow \boxed{(4y+1)^{\frac{1}{2}} = \frac{2}{x} + C}$

Apply $x = \frac{2}{3} \quad y = 2$

$9^{\frac{1}{2}} = \frac{2}{2/3} + C$

$3 = 3 + C$

$C = 0$

Thus

$(4y+1)^{\frac{1}{2}} = \frac{2}{x}$

$4y+1 = \frac{4}{x^{2}}$

$4y = \frac{4}{x^{2}} - 1$

$y = \frac{1}{x^{2}} - \frac{1}{4}$

Question 19 (+)**

Solve the differential equation

$$\frac{dy}{dx} = \frac{2y^2}{x^3}$$

subject to the condition $y = -1$ at $x = 1$, giving the answer in the form $y = f(x)$.

$$\boxed{y = \frac{x^2}{1-2x^2}}$$

$$\frac{dy}{dx} = \frac{2y^2}{x^3}$$

$$\Rightarrow \frac{1}{y^2}\,dy = \frac{2}{x^3}\,dx$$

$$\Rightarrow \int \frac{1}{y^2}\,dy = \int \frac{2}{x^3}\,dx$$

$$\Rightarrow \int y^{-2}\,dy = \int 2x^{-3}\,dx$$

$$\Rightarrow -y^{-1} = -x^{-2} + C$$

$$\Rightarrow -\frac{1}{y} = -\frac{1}{x^2} + C$$

$$\Rightarrow \boxed{\frac{1}{y} = \frac{1}{x^2} + C}$$

APPLY CONDITION $x=1, y=-1$

$$-1 = 1 + C$$

$$C = -2$$

$$\therefore \frac{1}{y} = \frac{1}{x^2} - 2$$

$$\frac{1}{y} = \frac{1-2x^2}{x^2}$$

$$y = \frac{x^2}{1-2x^2}$$

Question 20 (**+)

Given that $y = 2$ at $x = 0$, solve the differential equation

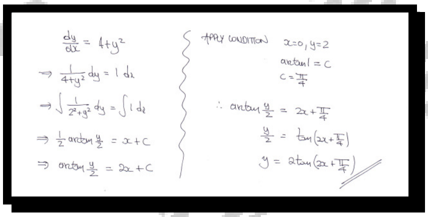
$$\frac{dy}{dx} = 4 + y^2,$$

giving the answer in the form $y = f(x)$.

You may assume that

$$\int \frac{1}{a^2 + x^2}\, dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C.$$

$$\boxed{y = 2\tan\left(2x + \frac{\pi}{4}\right)}$$



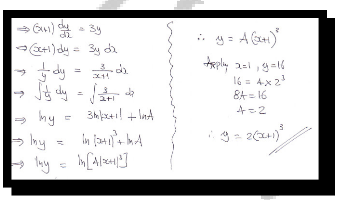
Question 21 (***)

$$(x+1)\frac{dy}{dx} = 3y,\quad y > 0.$$

Solve the differential equation subject to the condition $y = 16$ at $x = 1$, to show that

$$y = 2(x+1)^3.$$

$\boxed{\text{proof}}$



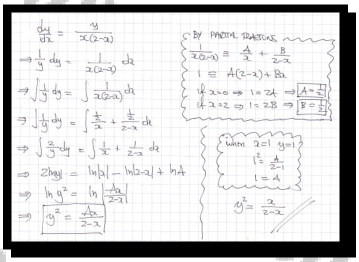
Question 22 (***)

Solve the differential equation

$$\frac{dy}{dx} = \frac{y}{x(2-x)}, \quad y > 0$$

subject to the condition $y = 1$ at $x = 1$, giving the answer in the form $y^2 = f(x)$.

$$\boxed{}, \quad \boxed{y^2 = \frac{x}{2-x}}$$



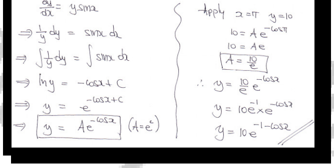
Question 23 (***)

Find the solution of the differential equation

$$\frac{dy}{dx} = y \sin x, \quad y > 0$$

subject to the condition $y = 10$ at $x = \pi$, giving the answer in the form $y = f(x)$.

$$\boxed{y = 10\mathrm{e}^{-1-\cos x}}$$



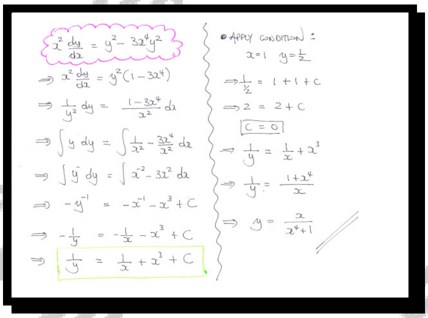
Question 24 (***)

Solve the differential equation

$$x^2 \frac{dy}{dx} = y^2 - 3x^4 y^2$$

subject to the condition $y = \frac{1}{2}$ at $x = 1$, giving the answer in the form $y = f(x)$.

$\boxed{}$, $\boxed{y = \dfrac{x}{x^4+1}}$



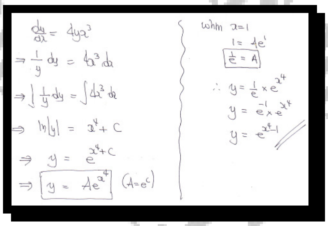
Question 25 (***)

Solve the differential equation

$$\frac{dy}{dx} = 4yx^3, \quad y \neq 0$$

subject to the condition $y = 1$ at $x = 1$, giving the answer in the form $y = f(x)$.

$\boxed{y = \mathrm{e}^{x^4 - 1}}$



Question 26 (***)

Solve the differential equation

$$\left(1-x^2\right)\frac{dy}{dx} = y(x+1), \quad y \neq 0,\ x \neq \pm 1,$$

subject to the condition $y=2$ at $x=\frac{1}{2}$, giving the answer in the form $y=f(x)$.

$$\boxed{y=\frac{1}{1-x}}$$

$(1-x^2)\frac{dy}{dx} = y(x+1)$

$\Rightarrow \frac{1}{y}\,dy = \frac{x+1}{1-x^2}\,dx$

$\Rightarrow \int \frac{1}{y}\,dy = \int \frac{\cancel{x+1}}{(1-x)\cancel{(1+x)}}\,dx$

$\Rightarrow \ln y = -\ln|1-x| + \ln A$

$\Rightarrow \ln y = \ln\left|\frac{A}{1-x}\right|$

$\Rightarrow \boxed{y = \frac{A}{1-x}}$

When $x=\frac{1}{2}$, $y=2$

$\Rightarrow 2 = \frac{A}{1-\frac{1}{2}}$

$\Rightarrow 2 = \frac{A}{\frac{1}{2}}$

$\Rightarrow \boxed{A=1}$

$\therefore\ y = \frac{1}{1-x}$

Question 27 (***)

Solve the differential equation

$$\frac{dy}{dx} = \frac{2x\ln x}{y}, \quad x>0,\ y>0$$

subject to the condition $y=2\mathrm{e}$ at $x=\mathrm{e}$, giving the answer in the form $y^2=f(x)$.

$$\boxed{y^2 = x^2(2\ln x - 1) + 3\mathrm{e}^2}$$

$\frac{dy}{dx} = \frac{2x\ln x}{y}$

$\Rightarrow y\,dy = 2x\ln x\,dx$

$\Rightarrow \int y\,dy = \int 2x\ln x\,dx$

PARTS

$\ln x \to \frac{1}{x}$

$x^2 \to 2x$

$\Rightarrow \frac{1}{2}y^2 = x^2\ln|x| - \int x\,dx$

$\Rightarrow \frac{1}{2}y^2 = x^2\ln|x| - \frac{1}{2}x^2 + C$

$\Rightarrow \boxed{y^2 = 2x^2\ln|x| - x^2 + C}$

APPLY $x=\mathrm{e}$ $y=2\mathrm{e}$

$4\mathrm{e}^2 = 2\mathrm{e}^2 - \mathrm{e}^2 + C$

$3\mathrm{e}^2 = C$

$\therefore\ y^2 = 2x^2\ln|x| - x^2 + 3\mathrm{e}^2$

Question 28 (***)

Solve the differential equation

$$\frac{dy}{dx} = 4xy - 3yx^2$$

subject to the condition $y = 1$ at $x = 2$, giving the answer in the form $y = f(x)$.

$\boxed{}$, $\boxed{y = \mathrm{e}^{2x^2 - x^3}}$

FACTORIZE THE R.H.S & SEPARATE VARIABLES

$\Rightarrow \frac{dy}{dx} = 4xy - 3yx^2$

$\Rightarrow \frac{dy}{dx} = xy(4-3x)$

$\Rightarrow \frac{1}{y}\,dy = x(4-3x)\,dx$

$\Rightarrow \int \frac{1}{y}\,dy = \int 4x - 3x^2\,dx$

$\Rightarrow \ln|y| = 2x^2 - x^3 + C$

$\Rightarrow y = e^{2x^2 - x^3 + C}$

$\Rightarrow y = e^{2x^2 - x^3} \times e^{C}$

$\Rightarrow \boxed{y = Ae^{2x^2 - x^3}}$

APPLY THE BOUNDARY CONDITION (1,2)

$\Rightarrow 1 = Ae^{8-8}$

$\Rightarrow 1 = Ae^{0}$

$\Rightarrow A = 1$

$\therefore y = e^{2x^2 - x^3}$

Question 29 (***)

Solve the differential equation

$$\frac{dy}{dx} = \frac{10 - y}{5}$$

subject to the condition $y = 1$ at $x = 0$, giving the answer in the form $y = f(x)$.

$\boxed{y = 10 - 9\mathrm{e}^{-\frac{1}{5}x}}$

$\frac{dy}{dx} = \frac{10-y}{5}$

$\Rightarrow \frac{1}{10-y}\,dy = \frac{1}{5}\,dx$

$\Rightarrow \int \frac{1}{10-y}\,dy = \int \frac{1}{5}\,dx$

$\Rightarrow -\ln|10-y| = \frac{1}{5}x + C$

$\Rightarrow \ln|10-y| = -\frac{1}{5}x + C$

$\Rightarrow 10 - y = e^{-\frac{1}{5}x + C}$

$\Rightarrow 10 - y = Ae^{-\frac{1}{5}x}$ $(A = e^{C})$

$\Rightarrow 10 - Ae^{-\frac{1}{5}x} = y$

when $x=0$ $y=1$

$10 - Ae^{0} = 1$

$10 - A = 1$

$A = 9$

$\therefore y = 10 - 9e^{-\frac{1}{5}x}$

Question 30 (***)

Show that the solution of the differential equation

$$\frac{dy}{dx} = \frac{\sqrt{2y-1}}{x^2}, \quad x \neq 0, y > \frac{1}{2}$$

subject to the condition $y = 1$ at $x = 1$, is given by

$$y = \frac{5x^2 - 4x + 1}{2x^2}.$$

proof

$$\frac{dy}{dx} = \frac{\sqrt{2y-1}}{x^2}$$

$$\Rightarrow \frac{1}{\sqrt{2y-1}}\,dy = \frac{1}{x^2}\,dx$$

$$\Rightarrow \int (2y-1)^{-\frac{1}{2}}\,dy = \int x^{-2}\,dx$$

$$\Rightarrow (2y-1)^{\frac{1}{2}} = -x^{-1} + C$$

$$\Rightarrow \boxed{\sqrt{2y-1} = C - \frac{1}{x}}$$

Apply $x=1, y=1$

$1 = C - 1$

$\boxed{C = 2}$

$$\Rightarrow \sqrt{2y-1} = 2 - \frac{1}{x}$$

$$\Rightarrow 2y - 1 = 4 - \frac{4}{x} + \frac{1}{x^2}$$

$$\Rightarrow 2y = 5 - \frac{4}{x} + \frac{1}{x^2}$$

$$\Rightarrow 2y = \frac{5x^2 - 4x + 1}{x^2}$$

$$\Rightarrow y = \frac{5x^2 - 4x + 1}{2x^2}$$

AS REQUIRED

Question 31 (***+)

Solve the differential equation

$$x(x+2)\frac{dy}{dx} = y, \quad x > 0, \quad y > 0$$

subject to the condition $y = 2$ at $x = 2$, giving the answer in the form $y^2 = f(x)$.

$\boxed{}$, $\boxed{y^2 = \dfrac{8x}{x+2}}$

SEPARATING VARIABLES

$$\Rightarrow x(x+2)\frac{dy}{dx} = y$$

$$\Rightarrow x(x+2)\,dy = y\,dx$$

$$\Rightarrow \frac{1}{y}\,dy = \frac{1}{x(x+2)}\,dx$$

BY PARTIAL FRACTIONS, FOR THE R.H.S

$$\frac{1}{x(x+2)} \equiv \frac{A}{x} + \frac{B}{x+2}$$

$$\boxed{1 \equiv A(x+2) + Bx}$$

- $x=0 \Rightarrow 1 = 2A$
  $\Rightarrow A = \frac{1}{2}$
- $x=-2 \Rightarrow 1 = -2B$
  $\Rightarrow B = -\frac{1}{2}$

RETURNING TO THE O.D.E, & INTEGRATING SUBJECT TO (2,2)

$$\Rightarrow \int_{y=2}^{y} \frac{1}{y}\,dy = \int_{x=2}^{x} \frac{\frac{1}{2}}{x} - \frac{\frac{1}{2}}{x+2}\,dx$$

$$\Rightarrow \int_{y=2}^{y} \frac{2}{y}\,dy = \int_{x=2}^{x} \frac{1}{x} - \frac{1}{x+2}\,dx$$

$$\Rightarrow \Big[2\ln|y|\Big]_{y=2}^{y} = \Big[\ln|x| - \ln|x+2|\Big]_{x=2}^{x}$$

$$\Rightarrow 2\ln|y| - 2\ln 2 = \left(\ln|x| - \ln|x+2|\right) - \left(\ln 2 - \ln 4\right)$$

$$\Rightarrow \ln y^2 - \ln 4 = \ln\left|\frac{x}{x+2}\right| - \ln\frac{1}{2}$$

$$\Rightarrow \ln\left(\frac{y^2}{4}\right) = \ln\left|\frac{x}{x+2}\right| + \ln 2$$

$$\Rightarrow \ln\left(\frac{y^2}{4}\right) = \ln\left|\frac{2x}{x+2}\right|$$

$$\Rightarrow \frac{y^2}{4} = \frac{2x}{x+2}$$

$$\Rightarrow y^2 = \frac{8x}{x+2}$$

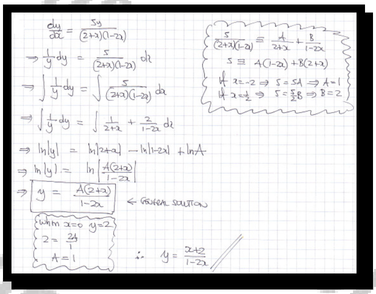
Question 32 (***+)

Solve the differential equation

$$\frac{dy}{dx} = \frac{5y}{(2+x)(1-2x)}$$

subject to the condition $y = 2$ at $x = 0$, giving the answer in the form $y = f(x)$.

$\boxed{}$, $\boxed{y = \dfrac{2+x}{1-2x}}$



$$\frac{dy}{dx} = \frac{5y}{(2+x)(1-2x)}$$
$$\Rightarrow \frac{1}{y}\,dy = \frac{5}{(2+x)(1-2x)}\,dx$$
$$\Rightarrow \int \frac{1}{y}\,dy = \int \frac{5}{(2+x)(1-2x)}\,dx$$
$$\Rightarrow \int \frac{1}{y}\,dy = \int \frac{1}{2+x} + \frac{2}{1-2x}\,dx$$
$$\Rightarrow \ln|y| = \ln|2+x| - \ln|1-2x| + \ln A$$
$$\Rightarrow \ln|y| = \ln\left|\frac{A(2+x)}{1-2x}\right|$$
$$\Rightarrow \boxed{y = \frac{A(2+x)}{1-2x}} \leftarrow \text{GENERAL SOLUTION}$$

$$\frac{5}{(2+x)(1-2x)} \equiv \frac{A}{2+x} + \frac{B}{1-2x}$$
$$5 \equiv A(1-2x) + B(2+x)$$
If $x=-2 \Rightarrow 5 = 5A \Rightarrow A = 1$
If $x=\frac{1}{2} \Rightarrow 5 = \frac{5}{2}B \Rightarrow B = 2$

when $x=0$ $y=2$
$$2 = \frac{2A}{1}$$
$$A = 1$$
$$\therefore\ y = \frac{x+2}{1-2x}$$

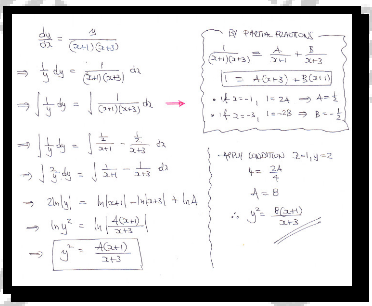
Question 33 (***+)

Solve the differential equation

$$\frac{dy}{dx} = \frac{y}{(x+1)(x+3)}, \quad y > 0, \ x > -1$$

subject to the condition $y = 2$ at $x = 1$, giving the answer in the form $y^2 = f(x)$.

$$\boxed{y^2 = \frac{8(x+1)}{x+3}}$$



$$\frac{dy}{dx} = \frac{y}{(x+1)(x+3)}$$
$$\Rightarrow \frac{1}{y}\,dy = \frac{1}{(x+1)(x+3)}\,dx$$
$$\Rightarrow \int \frac{1}{y}\,dy = \int \frac{1}{(x+1)(x+3)}\,dx \Rightarrow$$
$$\Rightarrow \int \frac{1}{y}\,dy = \int \frac{\frac{1}{2}}{x+1} - \frac{\frac{1}{2}}{x+3}\,dx$$
$$\Rightarrow \int \frac{2}{y}\,dy = \int \frac{1}{x+1} - \frac{1}{x+3}\,dx$$
$$\Rightarrow 2\ln|y| = \ln|x+1| - \ln|x+3| + \ln A$$
$$\Rightarrow \ln y^2 = \ln\left|\frac{A(x+1)}{x+3}\right|$$
$$\Rightarrow \boxed{y^2 = \frac{A(x+1)}{x+3}}$$

BY PARTIAL FRACTIONS
$$\frac{1}{(x+1)(x+3)} \equiv \frac{A}{x+1} + \frac{B}{x+3}$$
$$\boxed{1 \equiv A(x+3) + B(x+1)}$$
- If $x=-1$, $1 = 2A \Rightarrow A = \frac{1}{2}$
- If $x=-3$, $1 = -2B \Rightarrow B = -\frac{1}{2}$

APPLY CONDITION $x=1, y=2$
$$4 = \frac{2A}{4}$$
$$A = 8$$
$$\therefore\ y^2 = \frac{8(x+1)}{x+3}$$

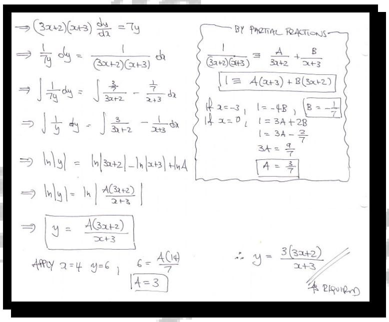
Question 34 (*+)**

$$\left(3x+2\right)\left(x+3\right)\frac{dy}{dx}=7y\,,\;\; y>0\,,\; x>-3\,.$$

Show that the solution of the above differential equation subject to the boundary condition, $y=6$ at $x=4$, is given by

$$y=\frac{3\left(3x+2\right)}{x+3}\,.$$

$\boxed{\text{proof}}$



$\Rightarrow (3x+2)(x+3)\frac{dy}{dx}=7y$

$\Rightarrow \frac{1}{7y}\,dy=\frac{1}{(3x+2)(x+3)}\,dx$

$\Rightarrow \int \frac{1}{7y}\,dy=\int \frac{\frac{3}{7}}{3x+2}-\frac{\frac{1}{7}}{x+3}\,dx$

$\Rightarrow \int \frac{1}{y}\,dy=\int \frac{3}{3x+2}-\frac{1}{x+3}\,dx$

$\Rightarrow \ln|y|=\ln|3x+2|-\ln|x+3|+\ln A$

$\Rightarrow \ln|y|=\ln\left|\frac{A(3x+2)}{x+3}\right|$

$\Rightarrow \boxed{y=\frac{A(3x+2)}{x+3}}$

APPLY $x=4$ $y=6$, $6=\frac{A(14)}{7}$ $\boxed{A=3}$

BY PARTIAL FRACTIONS

$\frac{1}{(3x+2)(x+3)}\equiv\frac{A}{3x+2}+\frac{B}{x+3}$

$\boxed{1\equiv A(x+3)+B(3x+2)}$

If $x=-3$, $1=-4B$, $\boxed{B=-\frac{1}{7}}$

If $x=0$, $1=3A+2B$

$1=3A-\frac{2}{7}$

$3A=\frac{9}{7}$

$\boxed{A=\frac{3}{7}}$

$\therefore\; y=\frac{3(3x+2)}{x+3}$

AS REQUIRED

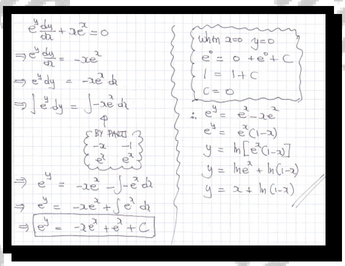
Question 35 (*+)**

Solve the differential equation

$$e^{y}\frac{dy}{dx}+xe^{x}=0\,,\;\; x<1$$

subject to the condition $y=0$ at $x=0$, giving the answer in the form $y=f\left(x\right)$.

$\boxed{y=x+\ln\left(1-x\right)}$



$e^{y}\frac{dy}{dx}+xe^{x}=0$

$\Rightarrow e^{y}\frac{dy}{dx}=-xe^{x}$

$\Rightarrow e^{y}\,dy=-xe^{x}\,dx$

$\Rightarrow \int e^{y}\,dy=\int -xe^{x}\,dx$

BY PARTS

$-x \quad -1$

$e^{x} \quad e^{x}$

$\Rightarrow e^{y}=-xe^{x}-\int -e^{x}\,dx$

$\Rightarrow e^{y}=-xe^{x}+\int e^{x}\,dx$

$\Rightarrow \boxed{e^{y}=-xe^{x}+e^{x}+C}$

when $x=0$ $y=0$

$e^{0}=0+e^{0}+C$

$1=1+C$

$C=0$

$\therefore\; e^{y}=e^{x}-xe^{x}$

$e^{y}=e^{x}(1-x)$

$y=\ln\left[e^{x}(1-x)\right]$

$y=\ln e^{x}+\ln(1-x)$

$y=x+\ln(1-x)$

Question 36 (*+)**

Solve the differential equation

$$(2x-3)(x-1)\frac{dy}{dx} = y(2x-1), \quad y \neq 0,$$

subject to the condition $y=10$ at $x=2$, giving the answer in the form $y = f(x)$.

$\boxed{}$, $\boxed{y = \dfrac{10(2x-3)^2}{x-1}}$

$$(2x-3)(x-1)\frac{dy}{dx} = y(2x-1)$$
$$\Rightarrow \frac{1}{y}\,dy = \frac{2x-1}{(2x-3)(x-1)}\,dx$$
$$\Rightarrow \int \frac{1}{y}\,dy = \int \frac{4}{2x-3} - \frac{1}{x-1}\,dx$$
$$\Rightarrow \ln|y| = 2\ln|2x-3| - \ln|x-1| + \ln A$$
$$\Rightarrow \ln|y| = \ln|2x-3|^2 - \ln|x-1| + \ln A$$
$$\Rightarrow \ln|y| = \ln\left|\frac{A(2x-3)^2}{x-1}\right|$$
$$\Rightarrow \boxed{y = \frac{A(2x-3)^2}{x-1}}$$

BY PARTIAL FRACTIONS

$$\frac{2x-1}{(2x-3)(x-1)} \equiv \frac{A}{2x-3} + \frac{B}{x-1}$$
$$2x-1 \equiv A(x-1) + B(2x-3)$$
If $x=1$, $1=-B \Rightarrow B=-1$

If $x=\frac{3}{2}$, $2=\frac{1}{2}A \Rightarrow A=4$

when $x=2$ $y=10$

$10 = \frac{A}{1}$

$A=10$

$$\therefore y = \frac{10(2x-3)^2}{x-1}$$

Question 37 (*+)**

Solve the differential equation

$$e^x \frac{dy}{dx} = \frac{x}{\sin 2y}, \quad 0 < y < \pi,$$

subject to $y = \frac{\pi}{4}$ at $x=0$, giving the answer in the form $\cos f(y) = g(x)$.

$\boxed{\cos 2y = 2xe^{-x} + 2e^{-x} - 2}$

$$e^x \frac{dy}{dx} = \frac{x}{\sin 2y}$$
$$\Rightarrow \sin 2y\,dy = \frac{x}{e^x}\,dx$$
$$\Rightarrow \int \sin 2y\,dy = \int xe^{-x}\,dx$$

BY PARTS: $x \to 1$, $-e^{-x} \to e^{-x}$

$$\Rightarrow -\tfrac{1}{2}\cos 2y = -xe^{-x} - \int -e^{-x}\,dx$$
$$\Rightarrow -\tfrac{1}{2}\cos 2y = -xe^{-x} + \int e^{-x}\,dx$$
$$\Rightarrow -\tfrac{1}{2}\cos 2y = -xe^{-x} - e^{-x} + C$$
$$\Rightarrow \cos 2y = 2xe^{-x} + 2e^{-x} + C$$
$$\Rightarrow \boxed{\cos 2y = 2e^{-x}(x+1) + C}$$

when $x=0$ $y=\frac{\pi}{4}$

$\cos\frac{\pi}{2} = 2e^{0}(1) + C$

$0 = 2 + C$

$C = -2$

$$\therefore \cos 2y = 2e^{-x}(x+1) - 2$$

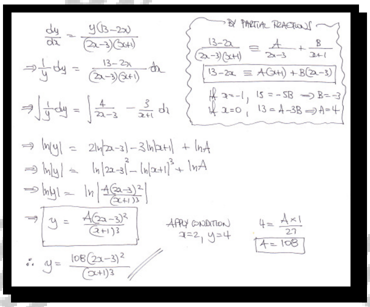
Question 38 (***+)

Solve the differential equation

$$\frac{dy}{dx}=\frac{y(13-2x)}{(2x-3)(x+1)},\quad y\neq 0,$$

subject to the condition $y=4$ at $x=2$, giving the answer in the form $y=f(x)$.

$$\boxed{y=\frac{108(2x-3)^2}{(x+1)^3}}$$



$$\frac{dy}{dx}=\frac{y(13-2x)}{(2x-3)(x+1)}$$
$$\Rightarrow \frac{1}{y}\,dy=\frac{13-2x}{(2x-3)(x+1)}\,dx$$
$$\Rightarrow \int\frac{1}{y}\,dy=\int\frac{4}{2x-3}-\frac{3}{x+1}\,dx$$

BY PARTIAL FRACTIONS

$$\frac{13-2x}{(2x-3)(x+1)}\equiv\frac{A}{2x-3}+\frac{B}{x+1}$$
$$\boxed{13-2x\equiv A(x+1)+B(2x-3)}$$
If $x=-1$, $15=-5B\Rightarrow B=-3$
If $x=0$, $13=A-3B\Rightarrow A=4$

$$\Rightarrow \ln|y|=2\ln|2x-3|-3\ln|x+1|+\ln A$$
$$\Rightarrow \ln|y|=\ln|2x-3|^2-\ln|x+1|^3+\ln A$$
$$\Rightarrow \ln|y|=\ln\left|\frac{A(2x-3)^2}{(x+1)^3}\right|$$
$$\Rightarrow \boxed{y=\frac{A(2x-3)^2}{(x+1)^3}}$$

APPLY CONDITION $x=2,\ y=4$: $4=\frac{A\times 1}{27}$, $\boxed{A=108}$

$$\therefore\ y=\frac{108(2x-3)^2}{(x+1)^3}$$

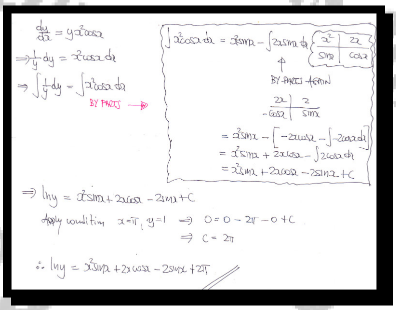
Question 39 (***+)

Solve the differential equation

$$\frac{dy}{dx}=yx^2\cos x,\quad x>0,\ y>0$$

subject to the condition $y=1$ at $x=\pi$, giving the answer in the form $\ln y=f(x)$.

$$\boxed{\ln y=x^2\sin x+2x\cos x-2\sin x+2\pi}$$



$$\frac{dy}{dx}=yx^2\cos x$$
$$\Rightarrow \frac{1}{y}\,dy=x^2\cos x\,dx$$
$$\Rightarrow \int\frac{1}{y}\,dy=\int x^2\cos x\,dx$$

BY PARTS

$$\int x^2\cos x\,dx=x^2\sin x-\int 2x\sin x\,dx$$

x^2	$2x$
$\sin x$	$\cos x$

BY PARTS AGAIN

$2x$	2
$-\cos x$	$\sin x$

$$=x^2\sin x-\left[-2x\cos x-\int -2\cos x\,dx\right]$$
$$=x^2\sin x+2x\cos x-\int 2\cos x\,dx$$
$$=x^2\sin x+2x\cos x-2\sin x+C$$

$$\Rightarrow \ln y=x^2\sin x+2x\cos x-2\sin x+C$$

Apply condition $x=\pi,\ y=1$ $\Rightarrow 0=0-2\pi-0+C$ $\Rightarrow C=2\pi$

$$\therefore\ \ln y=x^2\sin x+2x\cos x-2\sin x+2\pi$$

Question 40 (***+)

$$2y\frac{dy}{dx}=\frac{1}{x+3},\ \ y\neq 0,$$

Show that the solution of the above differential equation, subject to the boundary condition $y=1$ at $x=1$, can be written as

$$y^2=\ln\left|\frac{e(x+3)}{4}\right|.$$

proof

$$2y\frac{dy}{dx}=\frac{1}{x+3}$$
$$\Rightarrow 2y\,dy=\frac{1}{x+3}\,dx$$
$$\Rightarrow \int 2y\,dy=\int\frac{1}{x+3}\,dx$$
$$\Rightarrow y^2=\ln|x+3|+C$$

when $x=1,\ y=1$
$1^2=\ln 4+C$
$C=1-\ln 4$

$$\Rightarrow y^2=\ln|x+3|+1-\ln 4$$
$$\Rightarrow y^2=\ln|x+3|+\ln e-\ln 4$$
$$\Rightarrow y^2=\ln\left|\frac{e(x+3)}{4}\right|$$

Question 41 (***+)

$$\frac{dy}{dx}\cos^2 4x=y,\ \ y>0.$$

Show that the solution of the above differential equation subject to the boundary condition $y=e^3$ at $x=\frac{\pi}{16}$ is given by

$$y=e^{\frac{1}{4}(11+\tan 4x)}.$$

, proof

$$\cos^2 4x\frac{dy}{dx}=y$$
$$\Rightarrow \frac{1}{y}\,dy=\frac{1}{\cos^2 4x}\,dx$$
$$\Rightarrow \int\frac{1}{y}\,dy=\int\sec^2 4x\,dx$$
$$\Rightarrow \ln|y|=\frac{1}{4}\tan 4x+C$$
$$\Rightarrow y=e^{\frac{1}{4}\tan 4x+C}$$
$$\Rightarrow \boxed{y=Ae^{\frac{1}{4}\tan 4x}}\ (A=e^C)$$

APPLY CONDITION $x=\frac{\pi}{16},\ y=e^3$
$e^3=Ae^{\frac{1}{4}}$
$A=e^{\frac{11}{4}}$

$$\therefore y=e^{\frac{11}{4}}\times e^{\frac{1}{4}\tan 4x}$$
$$y=e^{\frac{11}{4}+\frac{1}{4}\tan 4x}$$
$$y=e^{\frac{1}{4}(11+\tan 4x)}$$

Question 42 (*+)**

Show that if $y = a$ at $t = 0$, the solution of the differential equation

$$\frac{dy}{dt} = \omega\left(a^2 - y^2\right)^{\frac{1}{2}},$$

where a and ω are positive constants, can be written as

$$y = a\cos\omega t.$$

You may assume that

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \arcsin\left(\frac{x}{a}\right) + C.$$

$\boxed{\text{proof}}$

$$\frac{dx}{dt} = \omega\left(a^2 - x^2\right)^{\frac{1}{2}}$$
$$\Rightarrow \frac{1}{\left(a^2 - x^2\right)^{\frac{1}{2}}}\,dx = \omega\,dt$$
$$\Rightarrow \int \frac{1}{\left(a^2 - x^2\right)^{\frac{1}{2}}}\,dx = \int \omega\,dt$$
$$\Rightarrow \arcsin\frac{x}{a} = \omega t + c$$
$$\Rightarrow \frac{x}{a} = \sin(\omega t + c)$$
$$\Rightarrow x = a\sin(\omega t + c)$$

when $t = 0$ $x = a$

$$a = a\sin c$$
$$1 = \sin c$$
$$c = \frac{\pi}{2}$$

So $x = a\sin\left(\omega t + \frac{\pi}{2}\right)$

$$x = a\left[\cancel{\sin\omega t\cos\frac{\pi}{2}} + \cos\omega t\sin\frac{\pi}{2}\right]$$

$$x = a\cos\omega t$$

As Required

$\cos\frac{\pi}{2} = 0$

$\sin\frac{\pi}{2} = 1$

Question 43 (*+)**

Solve the differential equation

$$y e^{y^2}\frac{dy}{dx} = e^{2x}, \quad x \neq 0, \; y \neq 0,$$

subject to the condition $y = 2$ at $x = 2$, giving the answer in the form $y^2 = f(x)$.

$\boxed{y^2 = 2x}$

$$y e^{y^2}\frac{dy}{dx} = e^{2x}$$
$$\Rightarrow y e^{y^2}\,dy = e^{2x}\,dx$$
$$\Rightarrow \int y e^{y^2}\,dy = \int e^{2x}\,dx$$

BY REVERSE CHAIN RULE

$$\Rightarrow \frac{1}{2}e^{y^2} = \frac{1}{2}e^{2x} + C$$
$$\Rightarrow \boxed{e^{y^2} = e^{2x} + C}$$

when $x = 2$ $y = 2$

$$e^4 = e^4 + C$$
$$\boxed{C = 0}$$
$$\therefore e^{y^2} = e^{2x}$$
$$y^2 = 2x$$

Question 44 (***+)

$$\frac{1}{(y-2)(y+1)} \equiv \frac{P}{(y-2)} + \frac{Q}{(y+1)}, \quad y \neq -1,\ 2$$

a) Find the value of each of the constants P and Q.

b) Hence, show that the solution of the differential equation

$$\frac{dy}{dx} = x^2 (y-2)(y+1)$$

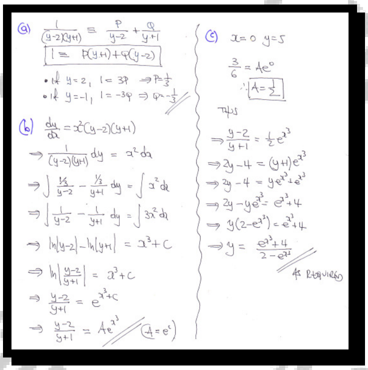
can be written as

$$\frac{y-2}{y+1} = A\mathrm{e}^{x^3}, \text{ where } A \text{ is a constant.}$$

c) Given further that $y = 5$ when $x = 0$, show clearly that

$$y = \frac{4 + \mathrm{e}^{x^3}}{2 - \mathrm{e}^{x^3}}.$$

$$\boxed{P = \frac{1}{3},\ Q = -\frac{1}{3}}$$



(a) $\frac{1}{(y-2)(y+1)} \equiv \frac{P}{y-2} + \frac{Q}{y+1}$

$\boxed{1 \equiv P(y+1) + Q(y-2)}$

• if $y=2$, $1 = 3P \Rightarrow P = \frac{1}{3}$

• if $y=-1$, $1 = -3Q \Rightarrow Q = -\frac{1}{3}$

(b) $\frac{dy}{dx} = x^2(y-2)(y+1)$

$\Rightarrow \frac{1}{(y-2)(y+1)}\,dy = x^2\,dx$

$\Rightarrow \int \frac{1/3}{y-2} - \frac{1/3}{y+1}\,dy = \int x^2\,dx$

$\Rightarrow \int \frac{1}{y-2} - \frac{1}{y+1}\,dy = \int 3x^2\,dx$

$\Rightarrow \ln|y-2| - \ln|y+1| = x^3 + C$

$\Rightarrow \ln\left|\frac{y-2}{y+1}\right| = x^3 + C$

$\Rightarrow \frac{y-2}{y+1} = e^{x^3+C}$

$\Rightarrow \frac{y-2}{y+1} = Ae^{x^3}$ $(A = e^C)$

(c) $x=0$ $y=5$

$\frac{3}{6} = Ae^0$

$\therefore \boxed{A = \frac{1}{2}}$

Thus

$\Rightarrow \frac{y-2}{y+1} = \frac{1}{2}e^{x^3}$

$\Rightarrow 2y - 4 = (y+1)e^{x^3}$

$\Rightarrow 2y - 4 = ye^{x^3} + e^{x^3}$

$\Rightarrow 2y - ye^{x^3} = e^{x^3} + 4$

$\Rightarrow y(2 - e^{x^3}) = e^{x^3} + 4$

$\Rightarrow y = \frac{e^{x^3}+4}{2-e^{x^3}}$

AS REQUIRED

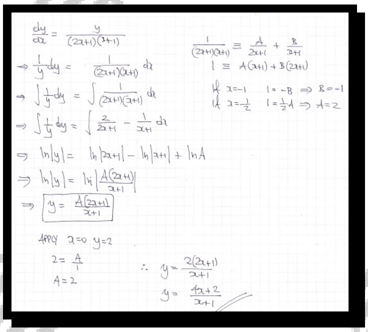
Question 45 (*+)**

Solve the differential equation

$$\frac{dy}{dx} = \frac{y}{(2x+1)(x+1)}, \quad x > -\frac{1}{2}, \quad y > 0$$

subject to the condition $y = 2$ at $x = 0$, giving the answer in the form $y = f(x)$.

$$\boxed{y = \frac{4x+2}{x+1}}$$



$\frac{dy}{dx} = \frac{y}{(2x+1)(x+1)}$

$\frac{1}{(2x+1)(x+1)} \equiv \frac{A}{2x+1} + \frac{B}{x+1}$

$1 \equiv A(x+1) + B(2x+1)$

If $x=-1$ $\quad 1 = -B \Rightarrow B = -1$

If $x = -\frac{1}{2}$ $\quad 1 = \frac{1}{2}A \Rightarrow A = 2$

$\Rightarrow \frac{1}{y}\,dy = \frac{1}{(2x+1)(x+1)}\,dx$

$\Rightarrow \int \frac{1}{y}\,dy = \int \frac{1}{(2x+1)(x+1)}\,dx$

$\Rightarrow \int \frac{1}{y}\,dy = \int \frac{2}{2x+1} - \frac{1}{x+1}\,dx$

$\Rightarrow \ln|y| = \ln|2x+1| - \ln|x+1| + \ln A$

$\Rightarrow \ln|y| = \ln\left|\frac{A(2x+1)}{x+1}\right|$

$\Rightarrow \boxed{y = \frac{A(2x+1)}{x+1}}$

APPLY $x=0$ $y=2$

$2 = \frac{A}{1}$

$A = 2$

$\therefore y = \frac{2(2x+1)}{x+1}$

$y = \frac{4x+2}{x+1}$

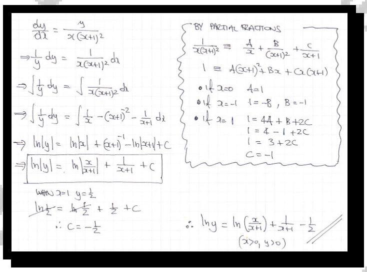
Question 46 (*+)**

Solve the differential equation

$$\frac{dy}{dx} = \frac{y}{x(x+1)^2}, \quad x > 0, \; y > 0$$

subject to the condition $y = \frac{1}{2}$ at $x = 1$, giving the answer in the form $\ln y = f(x)$.

$$\boxed{\ln y = \ln\left(\frac{x}{x+1}\right) + \frac{1}{x+1} - \frac{1}{2}}$$



$\frac{dy}{dx} = \frac{y}{x(x+1)^2}$

BY PARTIAL FRACTIONS

$\frac{1}{x(x+1)^2} \equiv \frac{A}{x} + \frac{B}{(x+1)^2} + \frac{C}{x+1}$

$1 \equiv A(x+1)^2 + Bx + Cx(x+1)$

• if $x=0$ $\quad A=1$

• if $x=-1$ $\quad 1=-B$, $B=-1$

• if $x=1$ $\quad 1 = 4A + B + 2C$

$1 = 4 - 1 + 2C$

$1 = 3 + 2C$

$C = -1$

$\Rightarrow \frac{1}{y}\,dy = \frac{1}{x(x+1)^2}\,dx$

$\Rightarrow \int \frac{1}{y}\,dy = \int \frac{1}{x(x+1)^2}\,dx$

$\Rightarrow \int \frac{1}{y}\,dy = \int \frac{1}{x} - (x+1)^{-2} - \frac{1}{x+1}\,dx$

$\Rightarrow \ln|y| = \ln|x| + (x+1)^{-1} - \ln|x+1| + C$

$\Rightarrow \boxed{\ln|y| = \ln\left|\frac{x}{x+1}\right| + \frac{1}{x+1} + C}$

WHEN $x=1$ $y=\frac{1}{2}$

$\cancel{\ln\frac{1}{2}} = \cancel{\ln\frac{1}{2}} + \frac{1}{2} + C$

$\therefore C = -\frac{1}{2}$

$\therefore \ln y = \ln\left(\frac{x}{x+1}\right) + \frac{1}{x+1} - \frac{1}{2}$

$(x>0, y>0)$

Question 47 **(***+)**

Solve the differential equation

$$\frac{dy}{dx} = 3x\mathrm{e}^{3x+y}$$

subject to the condition $y = 0$ at $x = 0$, giving the answer in the form $\mathrm{e}^{y} = f(x)$.

$$\boxed{\mathrm{e}^{y} = \frac{3}{\mathrm{e}^{3x} - 3x\mathrm{e}^{3x} + 2}}$$

$\frac{dy}{dx} = 3x e^{3x+y}$

$\Rightarrow \frac{dy}{dx} = 3x e^{3x} e^{y}$

$\Rightarrow \frac{1}{e^{y}} dy = 3x e^{3x} dx$

$\Rightarrow \int e^{-y} dy = \int 3x e^{3x} dx$

PARTS
$x \rightarrow 1$
$e^{3x} \rightarrow 3e^{3x}$

$\Rightarrow -e^{-y} = xe^{3x} - \int e^{3x} dx$

$\Rightarrow -e^{-y} = xe^{3x} - \frac{1}{3}e^{3x} + C$

$\Rightarrow \boxed{e^{-y} = \frac{1}{3}e^{3x} - xe^{3x} + C}$

$x=0 \quad y=0$

$1 = \frac{1}{3} + C$

$C = \frac{2}{3}$

$\therefore e^{-y} = \frac{1}{3}e^{3x} - xe^{3x} + \frac{2}{3}$

$e^{y} = \frac{1}{\frac{1}{3}e^{3x} - xe^{3x} + \frac{2}{3}}$

$e^{y} = \frac{3}{e^{3x} - 3xe^{3x} + 2}$

Question 48 **(***+)**

$$-5\frac{dy}{dx} = 2y - 150, \quad y < 75$$

Solve the above differential equation, given that when $x = 0$, $y = 275$.

Give the answer in the form $y = f(x)$.

$\boxed{}$, $\boxed{y = 75 + 200\mathrm{e}^{-\frac{2}{5}x}}$

$-5\frac{dy}{dx} = 2y - 150$

$\Rightarrow \frac{-5}{2y-150} dy = 1\, dx$

$\Rightarrow \int \frac{-5}{2y-150} dy = \int 1\, dx$

$\Rightarrow -\frac{5}{2}\ln|2y-150| = x + C$

$\Rightarrow \ln|2y-150| = -\frac{2}{5}x + C$

$\Rightarrow 2y - 150 = e^{-\frac{2}{5}x + C}$

$\Rightarrow 2y - 150 = Ae^{-\frac{2}{5}x} \quad (A = e^{C})$

$\Rightarrow 2y = 150 + Ae^{-\frac{2}{5}x}$

$\Rightarrow \boxed{y = 75 + Ae^{-\frac{2}{5}x}}$

when $x=0 \quad y=275$

$275 = 75 + Ae^{0}$

$A = 200$

$\therefore y = 75 + 200e^{-\frac{2}{5}x}$

Question 49 (***+)

$$\left(1+x^2\right)\frac{dy}{dx} = x(1+y), \text{ with } y=0 \text{ at } x=0.$$

Show that the solution of the above differential equation is

$$y = \left(1+x^2\right)^{\frac{1}{2}} - 1.$$

$\boxed{\text{proof}}$

$$\left(1+x^2\right)\frac{dy}{dx} = x(1+y)$$
$$\Rightarrow \frac{1}{1+y}\,dy = \frac{x}{1+x^2}\,dx$$
$$\Rightarrow \int \frac{1}{1+y}\,dy = \frac{1}{2}\int \frac{2x}{1+x^2}\,dx$$
$$\Rightarrow \ln|1+y| = \frac{1}{2}\ln|1+x^2| + \ln A$$
$$\Rightarrow \ln|1+y| = \ln\left(1+x^2\right)^{\frac{1}{2}} + \ln A$$
$$\Rightarrow \ln|1+y| = \ln\left[A\left(1+x^2\right)^{\frac{1}{2}}\right]$$
$$\Rightarrow 1+y = A\left(1+x^2\right)^{\frac{1}{2}}$$
$$\Rightarrow y = A\left(1+x^2\right)^{\frac{1}{2}} - 1$$

• when $x=0$ $y=0$

$0 = A - 1$

$A = 1$

$$\therefore y = \left(1+x^2\right)^{\frac{1}{2}} - 1$$

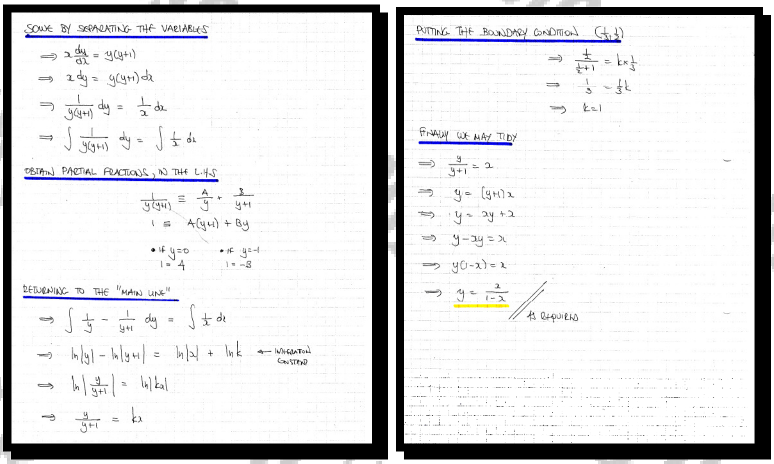
Question 50 (***+)

$$x\frac{dy}{dx}=y(y+1),\ x>0,\ y>0$$

Show that the solution of the above differential equation subject to the boundary condition $y=\frac{1}{2}$ at $x=\frac{1}{3}$, is given by

$$y=\frac{x}{1-x}.$$

$\boxed{}$, $\boxed{\text{proof}}$



SOLVE BY SEPARATING THE VARIABLES

$\Rightarrow x\frac{dy}{dx}=y(y+1)$

$\Rightarrow x\,dy=y(y+1)\,dx$

$\Rightarrow \frac{1}{y(y+1)}\,dy=\frac{1}{x}\,dx$

$\Rightarrow \int\frac{1}{y(y+1)}\,dy=\int\frac{1}{x}\,dx$

OBTAIN PARTIAL FRACTIONS, IN THE L.H.S

$\frac{1}{y(y+1)}\equiv\frac{A}{y}+\frac{B}{y+1}$

$1\equiv A(y+1)+By$

• If $y=0$: $1=A$ • If $y=-1$: $1=-B$

RETURNING TO THE "MAIN LINE"

$\Rightarrow \int\frac{1}{y}-\frac{1}{y+1}\,dy=\int\frac{1}{x}\,dx$

$\Rightarrow \ln|y|-\ln|y+1|=\ln|x|+\ln k$ ← INTEGRATION CONSTANT

$\Rightarrow \ln\left|\frac{y}{y+1}\right|=\ln|kx|$

$\Rightarrow \frac{y}{y+1}=kx$

PUTTING THE BOUNDARY CONDITION $\left(\frac{1}{3},\frac{1}{2}\right)$

$\Rightarrow \frac{\frac{1}{2}}{\frac{1}{2}+1}=k\times\frac{1}{3}$

$\Rightarrow \frac{1}{3}=\frac{1}{3}k$

$\Rightarrow k=1$

FINALLY WE MAY TIDY

$\Rightarrow \frac{y}{y+1}=x$

$\Rightarrow y=(y+1)x$

$\Rightarrow y=xy+x$

$\Rightarrow y-xy=x$

$\Rightarrow y(1-x)=x$

$\Rightarrow y=\frac{x}{1-x}$ // AS REQUIRED

Question 51 (***+)

$$x\frac{dy}{dx} = y\left(1+2x^2\right),\ x>0,\ y>0$$

Show that the solution of the above differential equation subject to $y = \mathrm{e}$ at $x = 1$, is

$$y = x\mathrm{e}^{x^2}.$$

$\boxed{\text{proof}}$

$x\frac{dy}{dx} = y(1+2x^2)$

$\Rightarrow \frac{1}{y}\,dy = \frac{1+2x^2}{x}\,dx$

$\Rightarrow \int \frac{1}{y}\,dy = \int \frac{1}{x} + 2x\,dx$

$\Rightarrow \ln y = \ln x + x^2 + C$

$\Rightarrow y = e^{\ln x + x^2 + C}$

$\Rightarrow y = e^{\ln x} \times e^{x^2} \times e^{C}$

$\Rightarrow \boxed{y = Axe^{x^2}}$

when $x=1$ $y=e$

$e = A \times 1 \times e^1$

$e = Ae$

$\boxed{1 = A}$

$\therefore y = xe^{x^2}$

Question 52 (****)

$$\frac{dy}{dx} = \frac{y^2 - 1}{x}, \quad x > 0, \quad y > 0$$

Show that the solution of the above differential equation subject to $y = 2$ at $x = 1$, is

$$y = \frac{3 + x^2}{3 - x^2}.$$

$\boxed{}$, $\boxed{\text{proof}}$

SOLVE BY SEPARATING VARIABLES

$$\Rightarrow \frac{dy}{dx} = \frac{y^2-1}{x}$$

$$\Rightarrow \frac{1}{y^2-1}\,dy = \frac{1}{x}\,dx$$

$$\Rightarrow \int \frac{1}{(y-1)(y+1)}\,dy = \int \frac{1}{x}\,dx$$

PROCEED BY PARTIAL FRACTIONS

$$\frac{1}{(y-1)(y+1)} \equiv \frac{A}{y-1} + \frac{B}{y+1}$$

$$1 \equiv A(y+1) + B(y-1)$$

- IF $y=1$: $1 = 2A$, $A = \frac{1}{2}$
- IF $y=-1$: $1 = -2B$, $B = -\frac{1}{2}$

RETURNING TO THE O.D.E.

$$\Rightarrow \int \frac{\frac{1}{2}}{y-1} - \frac{\frac{1}{2}}{y+1}\,dy = \int \frac{1}{x}$$

$$\Rightarrow \int \frac{1}{y-1} - \frac{1}{y+1}\,dy = \int \frac{2}{x}$$

$$\Rightarrow \ln|y-1| - \ln|y+1| = 2\ln|x| + \ln A$$

$$\Rightarrow \ln\left|\frac{y-1}{y+1}\right| = \ln|Ax^2|$$

$$\Rightarrow \frac{y-1}{y+1} = Ax^2$$

APPLY CONDITION (1,2)

$$\frac{2-1}{2+1} = A \times 1^2$$

$$\frac{1}{3} = A$$

$$\therefore \frac{y-1}{y+1} = \frac{1}{3}x^2$$

REARRANGING

$$\Rightarrow \frac{y-1}{y+1} = \frac{x^2}{3}$$

$$\Rightarrow 3y - 3 = x^2 y + x^2$$

$$\Rightarrow 3y - yx^2 = 3 + x^2$$

$$\Rightarrow y(3 - x^2) = 3 + x^2$$

$$\Rightarrow y = \frac{3+x^2}{3-x^2}$$

AS REQUIRED

Question 53 (****)

$$e^x \frac{dy}{dx} + y^2 = xy^2, \; x > 0, \; y > 0$$

Show that the solution of the above differential equation subject to $y = e$ at $x = 1$, is

$$y = \frac{1}{x}e^x.$$

$\boxed{}$, $\boxed{\text{proof}}$

START BY REGROUPING TERMS

$$\Rightarrow e^x \frac{dy}{dx} + y^2 = xy^2$$
$$\Rightarrow e^x \frac{dy}{dx} = xy^2 - y^2$$
$$\Rightarrow e^x \frac{dy}{dx} = y^2(x-1)$$

SEPARATE THE VARIABLES

$$\Rightarrow e^x \, dy = y^2(x-1)\, dx$$
$$\Rightarrow \frac{1}{y^2}\, dy = \frac{x-1}{e^x}\, dx$$
$$\Rightarrow \int y^{-2}\, dy = \int (x-1)e^{-x}\, dx$$

INTEGRATION BY PARTS

$x-1$	1
$-e^{-x}$	e^{-x}

$$\Rightarrow -y^{-1} = -e^{-x}(x-1) - \int -e^{-x}\, dx$$
$$\Rightarrow -\frac{1}{y} = -e^{-x}(x-1) + \int e^{-x}\, dx$$
$$\Rightarrow -\frac{1}{y} = -e^{-x}(x-1) - e^{-x} + C$$
$$\Rightarrow \frac{1}{y} = e^{-x}(x-1) + e^{-x} + C$$
$$\Rightarrow \frac{1}{y} = xe^{-x} - \cancel{e^{-x}} + \cancel{e^{-x}} + C$$
$$\Rightarrow \frac{1}{y} = xe^{-x} + C$$

APPLY THE BOUNDARY CONDITION $x=1$ $y=e$

$$\frac{1}{e} = 1 \times e^{-1} + C$$
$$\frac{1}{e} = \frac{1}{e} + C$$
$$C = 0$$

$$\therefore \frac{1}{y} = xe^{-x}$$
$$y = \frac{1}{xe^{-x}}$$
$$y = \frac{1}{x}e^x$$

AS REQUIRED

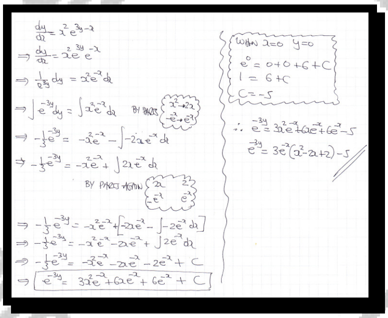
Question 54 (****)

Solve the differential equation

$$\frac{dy}{dx} = x^2 e^{3y-x}$$

subject to the condition $y = 0$ at $x = 0$, giving the answer in the form $e^{f(y)} = g(x)$.

$$\boxed{e^{-3y} = 3e^{-x}\left(x^2 + 2x + 2\right) - 5}$$



$\frac{dy}{dx} = x^2 e^{3y-x}$

$\Rightarrow \frac{dy}{dx} = x^2 e^{3y} e^{-x}$

$\Rightarrow \frac{1}{e^{3y}} dy = x^2 e^{-x} dx$

$\Rightarrow \int e^{-3y} dy = \int x^2 e^{-x} dx$ BY PARTS $x^2 \to 2x$, $-e^{-x} \to e^{-x}$

$\Rightarrow -\frac{1}{3}e^{-3y} = -x^2 e^{-x} - \int -2x e^{-x} dx$

$\Rightarrow -\frac{1}{3}e^{-3y} = -x^2 e^{-x} + \int 2x e^{-x} dx$

BY PARTS AGAIN $2x$, 2; $-e^{-x}$, e^{-x}

$\Rightarrow -\frac{1}{3}e^{-3y} = -x^2 e^{-x} + \left[-2x e^{-x} - \int -2e^{-x} dx\right]$

$\Rightarrow -\frac{1}{3}e^{-3y} = -x^2 e^{-x} - 2x e^{-x} + \int 2e^{-x} dx$

$\Rightarrow -\frac{1}{3}e^{-3y} = -x^2 e^{-x} - 2x e^{-x} - 2e^{-x} + C$

$\Rightarrow \boxed{e^{-3y} = 3x^2 e^{-x} + 6x e^{-x} + 6e^{-x} + C}$

WHEN $x = 0$ $y = 0$

$e^0 = 0 + 0 + 6 + C$

$1 = 6 + C$

$C = -5$

$\therefore e^{-3y} = 3x^2 e^{-x} + 6x e^{-x} + 6e^{-x} - 5$

$e^{-3y} = 3e^{-x}(x^2 + 2x + 2) - 5$

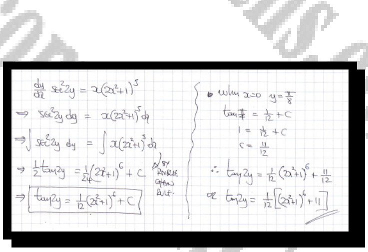
Question 55 (****)

Solve the differential equation

$$\frac{1}{x}\frac{dy}{dx} = \left(2x^2 + 1\right)^5 \cos^2 2y$$

subject to $x = 0$, $y = \frac{\pi}{8}$, giving the answer in the form $\tan f(y) = g(x)$

$$\boxed{\tan 2y = \frac{1}{12}\left[\left(2x^2 + 1\right)^6 + 11\right]}$$



$\frac{dy}{dx} \sec^2 2y = x(2x^2+1)^5$

$\Rightarrow \sec^2 2y \, dy = x(2x^2+1)^5 dx$

$\Rightarrow \int \sec^2 2y \, dy = \int x(2x^2+1)^5 dx$

$\Rightarrow \frac{1}{2}\tan 2y = \frac{1}{24}(2x^2+1)^6 + C$ BY REVERSE CHAIN RULE.

$\Rightarrow \boxed{\tan 2y = \frac{1}{12}(2x^2+1)^6 + C}$

when $x = 0$ $y = \frac{\pi}{8}$

$\tan\frac{\pi}{4} = \frac{1}{12} + C$

$1 = \frac{1}{12} + C$

$C = \frac{11}{12}$

$\therefore \tan 2y = \frac{1}{12}(2x^2+1)^6 + \frac{11}{12}$

or $\tan 2y = \frac{1}{12}\left[(2x^2+1)^6 + 11\right]$

Question 56 **(****)**

Solve the differential equation

$$(2x+1)-\frac{dy}{dx}(2x-1)^3=0$$

subject to the condition $y=0$ at $x=0$, giving the answer as $y=f(x)$.

$$\boxed{y=-\frac{x}{(2x-1)^2}}$$

30 $(2x+1)-\frac{dy}{dx}(2x-1)^3=0$

$\Rightarrow (2x+1)=\frac{dy}{dx}(2x-1)^3$

$\Rightarrow \frac{2x+1}{(2x-1)^3}\,dx = 1\,dy$

$\Rightarrow \int 1\,dy = \int \frac{2x+1}{(2x-1)^3}\,dx$

$\Rightarrow y = \int \frac{2x+1}{(2x-1)^3}\,dx$

BY SUBSTITUTION OR PARTIAL FRACTIONS

$\frac{2x+1}{(2x-1)^3} \equiv \frac{A}{(2x-1)^3}+\frac{B}{(2x-1)^2}+\frac{C}{2x-1}$

$\boxed{2x+1 \equiv A + B(2x-1) + C(2x-1)^2}$

If $x=\frac{1}{2} \Rightarrow \boxed{2=A}$

If $x=0 \Rightarrow 1 = A-B+C$

$\boxed{B-C=1}$

If $x=1 \Rightarrow 3 = A+B+C$

$\boxed{B+C=1}$

$\therefore \boxed{C=0}, \boxed{B=1}$

$\Rightarrow y = \int 2(2x-1)^{-3}+(2x-1)^{-2}\,dx$

$\Rightarrow y = -\frac{1}{2}(2x-1)^{-2}-\frac{1}{2}(2x-1)^{-1}+C$

$\Rightarrow \boxed{y = C-\frac{1}{2(2x-1)^2}-\frac{1}{2(2x-1)}}$

$\Rightarrow$ when $x=0$ $y=0$

$0 = C-\frac{1}{2}+\frac{1}{2} \quad \therefore C=0$

HENCE:

$y=-\frac{1}{2(2x-1)^2}-\frac{1}{2(2x-1)}$

$y=\frac{-1}{2(2x-1)^2}-\frac{2x-1}{2(2x-1)^2}$

$y=\frac{-2x}{2(2x-1)^2}$

$y=-\frac{x}{(2x-1)^2}$

[SUBSTITUTION PRODUCES THE FINAL ANSWER WITH FAR LESS ALGEBRA]

Question 57 (****)

$$\frac{dy}{dx} = \frac{xy}{x^2 - 3x + 2}, \quad x, y > 2$$

Solve the differential equation above, subject to the boundary condition $y = \frac{1}{3}$ at $x = 3$, to show that

$$y = \frac{2(x-2)^2}{3(x-1)}.$$

$\boxed{\text{proof}}$

$\frac{dy}{dx} = \frac{xy}{x^2-3x+2}$

$\frac{1}{y}\,dy = \frac{x}{(x-1)(x-2)}\,dx$

BY PARTIAL FRACTIONS

$\frac{x}{(x-1)(x-2)} \equiv \frac{A}{x-1} + \frac{B}{x-2}$

$\boxed{x \equiv A(x-2) + B(x-1)}$

- If $x=1$ $\quad 1=-A$
- If $x=2$ $\quad B=2$

$\Rightarrow \int \frac{1}{y}\,dy = \int \frac{2}{x-2} - \frac{1}{x-1}\,dx$

$\Rightarrow \ln y = 2\ln|x-2| - \ln|x-1| + \ln A$

$\Rightarrow \ln y = \ln(x-2)^2 - \ln(x-1) + \ln A$ $\quad (x>2)$

$\Rightarrow \ln y = \ln\left[\frac{A(x-2)^2}{x-1}\right]$

Hence

$\boxed{y = \frac{A(x-2)^2}{x-1}}$

when $x=3$ $\quad y=\frac{1}{3}$

$\frac{1}{3} = \frac{A}{2}$

$A = \frac{2}{3}$

$\therefore y = \frac{2(x-2)^2}{x-1}$

AS REQUIRED

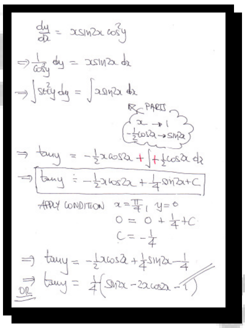
Question 58 **(****)**

Solve the differential equation

$$\frac{dy}{dx} = x\sin 2x\cos^2 y$$

subject to the condition $x = \frac{\pi}{4}$, $y = 0$, giving the answer in the form $\tan y = f(x)$.

$$\boxed{\tan y = \frac{1}{4}\left(\sin 2x - 2x\cos 2x - 1\right)}$$



$$\frac{dy}{dx} = x\sin 2x\cos^2 y$$
$$\Rightarrow \frac{1}{\cos^2 y}\,dy = x\sin 2x\,dx$$
$$\Rightarrow \int \sec^2 y\,dy = \int x\sin 2x\,dx$$

BY PARTS

$$x \to 1$$
$$-\tfrac{1}{2}\cos 2x \to \sin 2x$$

$$\Rightarrow \tan y = -\tfrac{1}{2}x\cos 2x + \int +\tfrac{1}{2}\cos 2x\,dx$$
$$\Rightarrow \boxed{\tan y = -\tfrac{1}{2}x\cos 2x + \tfrac{1}{4}\sin 2x + C}$$

APPLY CONDITION $x = \frac{\pi}{4}$, $y = 0$

$$0 = 0 + \tfrac{1}{4} + C$$
$$C = -\tfrac{1}{4}$$

$$\Rightarrow \tan y = -\tfrac{1}{2}x\cos 2x + \tfrac{1}{4}\sin 2x - \tfrac{1}{4}$$

OR

$$\Rightarrow \tan y = \tfrac{1}{4}\left(\sin 2x - 2x\cos 2x - 1\right)$$

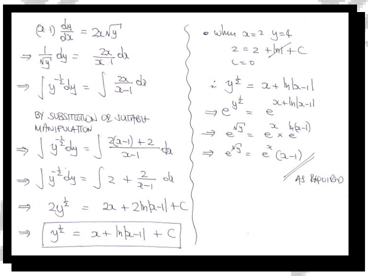
Question 59 **(****)**

$$(x-1)\frac{dy}{dx} = 2x\sqrt{y}$$

Solve the differential equation above, subject to the boundary condition $y = 4$ at $x = 2$, to show that

$$e^{\sqrt{y}} = e^{x}(x-1).$$

$\boxed{\text{proof}}$



$$(x-1)\frac{dy}{dx} = 2x\sqrt{y}$$
$$\Rightarrow \frac{1}{\sqrt{y}}\,dy = \frac{2x}{x-1}\,dx$$
$$\Rightarrow \int y^{-\frac{1}{2}}\,dy = \int \frac{2x}{x-1}\,dx$$

BY SUBSTITUTION OR SUITABLE MANIPULATION

$$\Rightarrow \int y^{-\frac{1}{2}}\,dy = \int \frac{2(x-1)+2}{x-1}\,dx$$
$$\Rightarrow \int y^{-\frac{1}{2}}\,dy = \int 2 + \frac{2}{x-1}\,dx$$
$$\Rightarrow 2y^{\frac{1}{2}} = 2x + 2\ln|x-1| + C$$
$$\Rightarrow \boxed{y^{\frac{1}{2}} = x + \ln|x-1| + C}$$

• when $x=2$ $y=4$

$$2 = 2 + \cancel{\ln 1} + C$$
$$C = 0$$

$$\therefore y^{\frac{1}{2}} = x + \ln|x-1|$$
$$\Rightarrow e^{y^{\frac{1}{2}}} = e^{x+\ln|x-1|}$$
$$\Rightarrow e^{\sqrt{y}} = e^{x} \times e^{\ln(x-1)}$$
$$\Rightarrow e^{\sqrt{y}} = e^{x}(x-1)$$

AS REQUIRED

Question 60 (****)

$$\frac{dy}{dx}\sec x = y^2 - y$$

Solve the differential equation above, subject to the boundary condition, $y = \frac{1}{2}$ at $x = 0$, to show that

$$y = \frac{1}{1+e^{\sin x}}.$$

$\boxed{}$, $\boxed{\text{proof}}$

SEPARATE VARIABLES AND INTEGRATE

$$\Rightarrow \frac{dy}{dx}\sec x = y^2 - y$$

$$\Rightarrow dy \sec x = (y^2 - y)\,dx$$

$$\Rightarrow \frac{1}{y^2-y}\,dy = \frac{1}{\sec x}\,dx$$

$$\Rightarrow \int \frac{1}{y^2-y}\,dy = \int \cos x \,dx$$

THE L.H.S REQUIRES PARTIAL FRACTIONS

$$\frac{1}{y^2-y} = \frac{1}{y(y-1)} = \text{BY INSPECTION (OR FULL METHOD)} \ldots = \frac{1}{y-1} - \frac{1}{y}$$

RETURNING TO THE DIFFERENTIAL EQUATION

$$\Rightarrow \int \frac{1}{y-1} - \frac{1}{y}\,dy = \int \cos x\,dx$$

$$\Rightarrow \ln|y-1| - \ln|y| = \sin x + C$$

$$\Rightarrow \ln\left|\frac{y-1}{y}\right| = \sin x + C$$

$$\Rightarrow \frac{y-1}{y} = e^{\sin x + C}$$

$$\Rightarrow \frac{y-1}{y} = e^{\sin x} \times e^{C}$$

$$\Rightarrow \frac{y-1}{y} = Ae^{\sin x} \qquad (A = e^{C})$$

APPLY CONDITION $x=0$, $y=\frac{1}{2}$

$$\frac{\frac{1}{2}-1}{\frac{1}{2}} = Ae^{\sin 0}$$

$$-1 = A$$

THUS WE FINALLY HAVE

$$\frac{y-1}{y} = -e^{\sin x}$$

$$1 - \frac{1}{y} = -e^{\sin x}$$

$$1 + e^{\sin x} = \frac{1}{y}$$

$$y = \frac{1}{1+e^{\sin x}} \quad // \text{ AS REQUIRED}$$

Question 61 **(****)**

Solve the differential equation

$$\frac{dy}{dx} = \frac{y(1-x)}{(1+x)(1+x^2)},$$

subject to the condition $x=0$, $y=1$, giving the answer in the form $y^2 = f(x)$.

$$\boxed{y^2 = \frac{(1+x)^2}{1+x^2}}$$

$$\frac{dy}{dx} = \frac{y(1-x)}{(1+x)(1+x^2)}$$
$$\Rightarrow \frac{1}{y}\,dy = \frac{1-x}{(1+x)(1+x^2)}$$
$$\Rightarrow \int \frac{1}{y}\,dy = \int \frac{1}{1+x} - \frac{x}{x^2+1}\,dx$$
$$\Rightarrow \int \frac{1}{y}\,dy = \int \frac{1}{1+x} - \frac{1}{2}\times\frac{2x}{x^2+1}\,dx$$
$$\Rightarrow \ln|y| = \ln|1+x| - \frac{1}{2}\ln|x^2+1| + \ln A$$
$$\Rightarrow 2\ln|y| = 2\ln|1+x| - \ln|x^2+1| + \ln B$$
$$\Rightarrow \ln y^2 = \ln(1+x)^2 - \ln(x^2+1) + \ln B$$
$$\Rightarrow \ln y^2 = \ln\left[\frac{B(1+x)^2}{x^2+1}\right]$$
$$\Rightarrow \boxed{y^2 = \frac{B(1+x)^2}{1+x^2}}$$

Apply $\left.\begin{matrix} x=0 \\ y=1 \end{matrix}\right\} \Rightarrow 1 = B \quad \therefore y^2 = \frac{(1+x)^2}{1+x^2}$

PARTIAL FRACTIONS

$$\frac{1-x}{(1+x)(1+x^2)} \equiv \frac{A}{1+x} + \frac{Bx+C}{x^2+1}$$
$$\boxed{1-x \equiv A(x^2+1) + (1+x)(Bx+C)}$$

If $x=-1$, $2=2A \Rightarrow A=1$

If $x=0$, $1=A+C \Rightarrow C=0$

If $x=1$, $0=2A+2B \Rightarrow B=-1$

Question 62 **(****)**

Solve the differential equation

$$\left(1+x^2\right)\frac{dy}{dx} = x\left(1-y^2\right), \quad y \neq \pm 1,$$

subject to the condition $y=0$ at $x=0$, giving the answer in the form $y = f(x)$.

$$\boxed{y = \frac{x^2}{x^2+2}}$$

$$(1+x^2)\frac{dy}{dx} = x(1-y^2)$$
$$\Rightarrow \frac{1}{1-y^2}\,dy = \frac{x}{1+x^2}\,dx$$
$$\Rightarrow \int \frac{1}{(1-y)(1+y)}\,dy = \int \frac{x}{1+x^2}\,dx$$
$$\Rightarrow \int \frac{\frac{1}{2}}{1-y} + \frac{\frac{1}{2}}{1+y}\,dy = \int \frac{x}{1+x^2}\,dx$$
$$\Rightarrow \int \frac{1}{1-y} + \frac{1}{1+y}\,dy = \int \frac{2x}{1+x^2}\,dx$$
$$\Rightarrow -\ln|1-y| + \ln|1+y| = \ln|1+x^2| + \ln A$$
$$\Rightarrow \ln\left|\frac{1+y}{1-y}\right| = \ln|A(1+x^2)|$$
$$\Rightarrow \boxed{\frac{1+y}{1-y} = A(1+x^2)}$$

$x=0$ $y=0$

$1 = A$

BY PARTIAL FRACTIONS

$$\frac{1}{(1-y)(1+y)} \equiv \frac{A}{1-y} + \frac{B}{1+y}$$
$$1 \equiv A(1+y) + B(1-y)$$

If $y=1$ $1=2A \Rightarrow A=\frac{1}{2}$

If $y=-1$ $1=2B \Rightarrow B=\frac{1}{2}$

HENCE

$$\frac{1+y}{1-y} = 1+x^2$$
$$1+y = (1-y)(1+x^2)$$
$$1+y = (1+x^2) - y(1+x^2)$$
$$y + y(1+x^2) = (1+x^2) - 1$$
$$y\left(1+(1+x^2)\right) = x^2$$
$$y = \frac{x^2}{2+x^2}$$

Question 63 **(****)**

$$(1+x)\frac{dy}{dx} = y(1-x), \ y>0, \ x>-1.$$

Solve the above given differential equation, subject to the boundary condition $y=1$ at $x=0$, to show that

$$y=(x+1)^2 e^{-x}.$$

$\boxed{}$, $\boxed{\text{proof}}$

SEPARATE VARIABLES & INTEGRATE

$\Rightarrow (1+x)\frac{dy}{dx} = y(1-x)$

$\Rightarrow (1+x)dy = y(1-x)\,dx$

$\Rightarrow \frac{1}{y}\,dy = \frac{1-x}{1+x}\,dx$

$\Rightarrow \int \frac{1}{y}\,dy = \int \frac{1-x}{1+x}\,dx$

INTEGRATE THE R.H.S BY THE SUBSTITUTION $u=1+x$, OR MANIPULATION

$\Rightarrow \int \frac{1}{y}\,dy = \int \frac{2-(1+x)}{1+x}\,dx$

$\Rightarrow \int \frac{1}{y}\,dy = \int \frac{2}{1+x} - \frac{1+x}{1+x}\,dx$

$\Rightarrow \int \frac{1}{y}\,dy = \int \frac{2}{1+x} - 1\,dx$

$\Rightarrow \ln|y| = 2\ln|1+x| - x + C$

$\Rightarrow y = e^{2\ln|1+x| - x + C}$

$\Rightarrow y = e^{2\ln|1+x|} \times e^{-x} \times e^{C}$ A

$\Rightarrow y = e^{\ln(1+x)^2} \times e^{-x} \times A$

$\Rightarrow \boxed{y = Ae^{-x}(1+x)^2}$

APPLY CONDITION $(0,1)$

$1 = Ae^{0} \times 1^2$

$A=1$

$\therefore \ y=(x+1)^2 e^{-x}$ // AS REQUIRED

Question 64 (****+)

Solve the differential equation

$$\frac{dy}{dx} = 24\cos^2 y \cos^3 x$$

subject to the condition $y = \frac{\pi}{4}$ at $x = \frac{\pi}{6}$, giving the answer in the form $\tan y = f(x)$.

$$\boxed{\tan y = 24\sin x - 8\sin^3 x - 10}$$

$\frac{dy}{dx} = 24\cos^2 y \cos^3 x$

$\Rightarrow \frac{1}{\cos^2 y}\,dy = 24\cos^3 x\,dx$

$\Rightarrow \int \sec^2 y\,dy = \int 24\cos^3 x\,dx$

$\Rightarrow \tan y = \int 24\cos^2 x\cos x\,dx$

$\Rightarrow \tan y = \int 24(1-\sin^2 x)\cos x\,dx$

$\Rightarrow \tan y = \int 24\cos x - 24\sin^2 x\cos x\,dx$

$\Rightarrow \boxed{\tan y = 24\sin x - 8\sin^3 x + C}$

APPLY CONDITION

$x = \frac{\pi}{6} \quad y = \frac{\pi}{4}$

$\tan\frac{\pi}{4} = 24\sin\frac{\pi}{6} - 8\sin^3\frac{\pi}{6} + C$

$1 = 12 - 1 + C$

$C = -10$

$\therefore$

$\tan y = 24\sin x - 8\sin^3 x - 10$

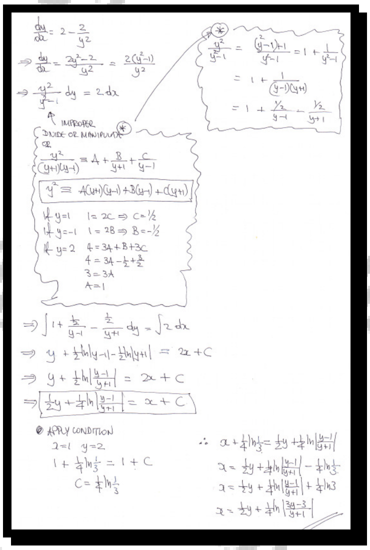
Question 65 (****+)

Solve the differential equation

$$\frac{dy}{dx} = 2 - \frac{2}{y^2},$$

subject to the condition $y = 2$ at $x = 1$, giving the answer in the form $x = f(y)$.

$$\boxed{x = \frac{1}{2}y + \frac{1}{4}\ln\left|\frac{3y-3}{y+1}\right|}$$



$\frac{dy}{dx} = 2 - \frac{2}{y^2}$

$\Rightarrow \frac{dy}{dx} = \frac{2y^2-2}{y^2} = \frac{2(y^2-1)}{y^2}$

$\Rightarrow \frac{y^2}{y^2-1}\,dy = 2\,dx$

IMPROPER

DIVIDE OR MANIPULATE (*)

OR

$\frac{y^2}{(y+1)(y-1)} \equiv A + \frac{B}{y+1} + \frac{C}{y-1}$

$\boxed{y^2 \equiv A(y+1)(y-1) + B(y-1) + C(y+1)}$

If $y=1$ $\quad 1 = 2C \Rightarrow C = \frac{1}{2}$

If $y=-1$ $\quad 1 = 2B \Rightarrow B = -\frac{1}{2}$

If $y=2$ $\quad 4 = 3A + B + 3C$

$4 = 3A - \frac{1}{2} + \frac{3}{2}$

$3 = 3A$

$A = 1$

(*)

$\frac{y^2}{y^2-1} = \frac{(y^2-1)+1}{y^2-1} = 1 + \frac{1}{y^2-1}$

$= 1 + \frac{1}{(y-1)(y+1)}$

$= 1 + \frac{\frac{1}{2}}{y-1} - \frac{\frac{1}{2}}{y+1}$

$\Rightarrow \int 1 + \frac{\frac{1}{2}}{y-1} - \frac{\frac{1}{2}}{y+1}\,dy = \int 2\,dx$

$\Rightarrow y + \frac{1}{2}\ln|y-1| - \frac{1}{2}\ln|y+1| = 2x + C$

$\Rightarrow y + \frac{1}{2}\ln\left|\frac{y-1}{y+1}\right| = 2x + C$

$\Rightarrow \boxed{\frac{1}{2}y + \frac{1}{4}\ln\left|\frac{y-1}{y+1}\right| = x + C}$

• APPLY CONDITION

$x=1 \quad y=2$

$1 + \frac{1}{4}\ln\frac{1}{3} = 1 + C$

$C = \frac{1}{4}\ln\frac{1}{3}$

$\therefore \; x + \frac{1}{4}\ln\frac{1}{3} = \frac{1}{2}y + \frac{1}{4}\ln\left|\frac{y-1}{y+1}\right|$

$x = \frac{1}{2}y + \frac{1}{4}\ln\left|\frac{y-1}{y+1}\right| - \frac{1}{4}\ln\frac{1}{3}$

$x = \frac{1}{2}y + \frac{1}{4}\ln\left|\frac{y-1}{y+1}\right| + \frac{1}{4}\ln 3$

$x = \frac{1}{2}y + \frac{1}{4}\ln\left|\frac{3y-3}{y+1}\right|$

Question 66 (****+)

$$xy + (1+x)\frac{dy}{dx} = y \, .$$

Solve the differential equation subject to the condition $y = 3$ at $x = 0$, to show that

$$y = 3(1+x)^2 \mathrm{e}^{-x} \, .$$

proof

$$xy + (1+x)\frac{dy}{dx} = y$$
$$\Rightarrow (1+x)\frac{dy}{dx} = y - xy$$
$$\Rightarrow (1+x)\frac{dy}{dx} = y(1-x)$$
$$\Rightarrow \frac{1}{y}\,dy = \frac{1-x}{1+x}\,dx$$
$$\Rightarrow \int \frac{1}{y}\,dy = \int \frac{2-(1+x)}{1+x}\,dx$$
$$\Rightarrow \int \frac{1}{y}\,dy = \int \frac{2}{1+x} - 1\,dx$$
$$\Rightarrow \ln|y| = 2\ln|x+1| - x + C$$
$$\Rightarrow \ln|y| = \ln|x+1|^2 - x + C$$
$$\Rightarrow y = e^{\ln|x+1|^2 - x + C}$$
$$\Rightarrow y = e^{\ln|x+1|^2} \times e^{-x} \times e^{C}$$
$$\Rightarrow \boxed{y = A e^{-x} (x+1)^2}$$

APPLY CONDITION

$x = 0 \quad y = 3$

$3 = A e^{0}$

$A = 3$

$\therefore y = 3(x+1)^2 e^{-x}$

Question 67 (****+)

$$\frac{dy}{dx}\cot x = 1 - y^2 .$$

Solve the differential equation above, subject to the boundary condition $y = 0$ at $x = \frac{\pi}{4}$, to show that

$$y = \frac{1 - 2\cos^2 x}{1 + 2\cos^2 x} .$$

$\boxed{}$, $\boxed{\text{proof}}$

SEPARATING VARIABLES

$\Rightarrow \frac{dy}{dx}\cot x = 1 - y^2$

$\Rightarrow dy \cot x = (1-y^2)\,dx$

$\Rightarrow \frac{1}{1-y^2}\,dy = \frac{1}{\cot x}\,dx$

$\Rightarrow \int \frac{1}{(1-y)(1+y)}\,dy = \int \tan x\,dx$

PROCEED WITH PARTIAL FRACTIONS

$\frac{1}{(1-y)(1+y)} \equiv \frac{A}{1-y} + \frac{B}{1+y}$

$1 \equiv A(1+y) + B(1-y)$

- IF $y=1$: $1 = 2A$, $A = \frac{1}{2}$
- IF $y=-1$: $1 = 2B$, $B = \frac{1}{2}$

RETURNING TO THE O.D.E.

$\Rightarrow \int \frac{\frac{1}{2}}{1+y} + \frac{\frac{1}{2}}{1-y}\,dy = \int \tan x\,dx$

$\Rightarrow \int \frac{1}{1+y} + \frac{1}{1-y}\,dy = \int 2\tan x\,dx$

$\int \tan x\,dx = \ln|\sec x| + C$

INTEGRATING BOTH SIDES SUBJECT TO THE BOUNDARY CONDITION $(\frac{\pi}{4}, 0)$

$\Rightarrow \Big[\ln|1+y| - \ln|1-y|\Big]_{y=0}^{y=y} = \Big[2\ln|\sec x|\Big]_{x=\frac{\pi}{4}}^{x=x}$

$\Rightarrow \Big[\ln|1+y| - \ln|1-y|\Big] - \cancel{\Big[\ln 1 - \ln 1\Big]} = 2\ln|\sec x| - 2\ln\left(\sec\frac{\pi}{4}\right)$

$\Rightarrow \ln\left|\frac{1+y}{1-y}\right| = 2\ln|\sec x| - 2\ln(\sqrt{2})$

$\Rightarrow \ln\left|\frac{1+y}{1-y}\right| = \ln(\sec^2 x) - \ln 2$

$\Rightarrow \ln\left|\frac{1+y}{1-y}\right| = \ln\left(\frac{\sec^2 x}{2}\right)$

$\Rightarrow \frac{1+y}{1-y} = \frac{\sec^2 x}{2}$

$\Rightarrow 2 + 2y = \sec^2 x - y\sec^2 x$

$\Rightarrow 2y + y\sec^2 x = \sec^2 x - 2$

$\Rightarrow y(2 + \sec^2 x) = \sec^2 x - 2$

$\Rightarrow y = \frac{\sec^2 x - 2}{2 + \sec^2 x}$

$\Rightarrow y = \frac{\sec^2 x\cos^2 x - 2\cos^2 x}{2\cos^2 x + \sec^2 x\cos^2 x}$

$\Rightarrow y = \frac{1 - 2\cos^2 x}{2\cos^2 x + 1}$

$\Rightarrow y = \frac{1 - 2\cos^2 x}{1 + 2\cos^2 x}$ // AS REQUIRED

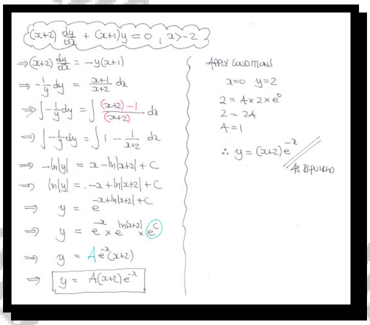
Question 68 (****+)

$$(x+2)\frac{dy}{dx} + y(x+1) = 0, \ x > -2.$$

Solve the differential equation above, subject to the boundary condition $y = 2$ at $x = 0$, to show that

$$y = (x+2)e^{-x}.$$

proof



$(x+2)\frac{dy}{dx} + (x+1)y = 0, \ x > -2$

$\Rightarrow (x+2)\frac{dy}{dx} = -y(x+1)$

$\Rightarrow -\frac{1}{y}\,dy = \frac{x+1}{x+2}\,dx$

$\Rightarrow \int -\frac{1}{y}\,dy = \int \frac{(x+2)-1}{(x+2)}\,dx$

$\Rightarrow \int -\frac{1}{y}\,dy = \int 1 - \frac{1}{x+2}\,dx$

$\Rightarrow -\ln|y| = x - \ln|x+2| + C$

$\Rightarrow \ln|y| = -x + \ln|x+2| + C$

$\Rightarrow y = e^{-x+\ln|x+2|+C}$

$\Rightarrow y = e^{-x} \times e^{\ln|x+2|} \times e^{C}$

$\Rightarrow y = Ae^{-x}(x+2)$

$\Rightarrow \boxed{y = A(x+2)e^{-x}}$

APPLY CONDITIONS

$x=0 \quad y=2$

$2 = A \times 2 \times e^{0}$

$2 = 2A$

$A = 1$

$\therefore y = (x+2)e^{-x}$

AS REQUIRED

Question 69 (**+)**

A curve $y = f(x)$ satisfies the differential equation

$$y = 1 - \frac{dy}{dx}\frac{x+1}{(x-1)(x+2)}, \quad y > 1,\, x > -1$$

a) Solve the differential equation to show that

$$\ln(y-5) + \frac{1}{2}x^2 + 4x - 2\ln(x+1) = C .$$

When $x = 0$, $y = 2$.

b) Show further that

$$y = 1 + (x+1)^2 e^{-\frac{1}{2}x^2} .$$

$\boxed{\text{proof}}$

(a) $y = 1 - \frac{dy}{dx}\frac{x+1}{(x-1)(x+2)}$

$\Rightarrow \frac{dy}{dx}\frac{x+1}{(x-1)(x+2)} = 1 - y$

$\Rightarrow \frac{1}{1-y}\,dy = \frac{(x-1)(x+2)}{x+1}\,dx$

$\Rightarrow \int \frac{1}{1-y}\,dy = \int \frac{x^2+x-2}{x+1}\,dx$

BY SUBSTITUTION $u = x+1$ OR BY ALGEBRAIC MANIPULATION

$\Rightarrow \int \frac{1}{1-y}\,dy = \int \frac{x(x+1)-2}{x+1}\,dx$

$\Rightarrow \int \frac{1}{1-y}\,dy = \int x - \frac{2}{x+1}\,dx$

$\Rightarrow -\ln|1-y| = \frac{1}{2}x^2 - 2\ln|x+1| + C$

BUT $y>1$ & $x>-1$

$\Rightarrow -\ln(y-1) = \frac{1}{2}x^2 - 2\ln(x+1) + C$

$\Rightarrow \ln(y-1) = -\frac{1}{2}x^2 + 2\ln(x+1) + C$

$\Rightarrow \ln(y-1) + \frac{1}{2}x^2 - 2\ln(x+1) = C$

AS REQUIRED

(b) when $x=0$ $y=2$

$\ln 1 + 0 - 2\ln 1 = C$

$C = 0$

$\therefore \ln(y-1) + \frac{1}{2}x^2 - 2\ln(x+1) = 0$

$\Rightarrow \ln(y-1) = 2\ln(x+1) - \frac{1}{2}x^2$

$\Rightarrow \ln(y-1) = \ln(x+1)^2 - \frac{1}{2}x^2$

$\Rightarrow y-1 = e^{\ln(x+1)^2 - \frac{1}{2}x^2}$

$\Rightarrow y-1 = e^{\ln(x+1)^2} \times e^{-\frac{1}{2}x^2}$

$\Rightarrow y-1 = (x+1)^2 \times e^{-\frac{1}{2}x^2}$

$\Rightarrow y = 1 + (x+1)^2 e^{-\frac{1}{2}x^2}$

AS REQUIRED

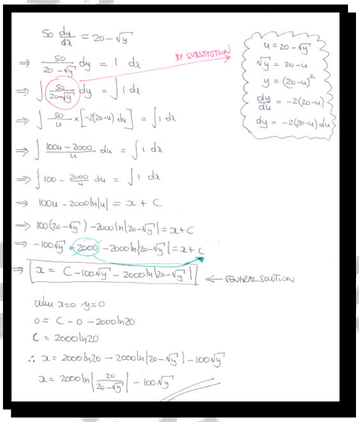
Question 70 (****+)

Solve the differential equation

$$50\frac{dy}{dx} = 20 - \sqrt{y},$$

given that when $x = 0$, $y = 0$, giving the answer in the form $x = f(y)$.

$$\boxed{x = 2000\ln\left|\frac{20}{20-\sqrt{y}}\right| - 100\sqrt{y}}$$



$50\frac{dy}{dx} = 20 - \sqrt{y}$

$\Rightarrow \frac{50}{20-\sqrt{y}}\,dy = 1\,dx$

BY SUBSTITUTION

$u = 20 - \sqrt{y}$

$\sqrt{y} = 20 - u$

$y = (20-u)^2$

$\frac{dy}{du} = -2(20-u)$

$dy = -2(20-u)\,du$

$\Rightarrow \int \frac{50}{20-\sqrt{y}}\,dy = \int 1\,dx$

$\Rightarrow \int \frac{50}{u} \times \left[-2(20-u)\,du\right] = \int 1\,dx$

$\Rightarrow \int \frac{100u - 2000}{u}\,du = \int 1\,dx$

$\Rightarrow \int 100 - \frac{2000}{u}\,du = \int 1\,dx$

$\Rightarrow 100u - 2000\ln|u| = x + C$

$\Rightarrow 100(20-\sqrt{y}) - 2000\ln|20-\sqrt{y}| = x + C$

$\Rightarrow -100\sqrt{y} + \cancel{2000} - 2000\ln|20-\sqrt{y}| = x + C$

$\Rightarrow \boxed{x = C - 100\sqrt{y} - 2000\ln|20-\sqrt{y}|}$ ← GENERAL SOLUTION

when $x=0$ $y=0$

$0 = C - 0 - 2000\ln 20$

$C = 2000\ln 20$

$\therefore x = 2000\ln 20 - 2000\ln|20-\sqrt{y}| - 100\sqrt{y}$

$x = 2000\ln\left|\frac{20}{20-\sqrt{y}}\right| - 100\sqrt{y}$

Question 71 (****+)

Solve the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 1,$$

given that $y = -\frac{1}{4}$ and $\frac{dy}{dx} = 1$ at $x = 0$, giving the answer in the form $y = f(x)$.

MP , $\boxed{y = \frac{1}{2}\left[2x - e^{-2x}\right]}$

USING A SUBSTITUTION

$$u = \frac{dy}{dx} \qquad \frac{du}{dx} = \frac{d^2y}{dx^2}$$

HENCE THE O.D.E. TRANSFORMS TO

$$\Rightarrow \frac{d^2y}{dx^2} + \frac{2dy}{dx} = 1$$
$$\Rightarrow \frac{du}{dx} + 2u = 1$$
$$\Rightarrow \frac{du}{dx} = 1 - 2u$$
$$\Rightarrow \frac{1}{1-2u}\,du = 1\,dx$$
$$\Rightarrow \int \frac{1}{1-2u}\,du = \int 1\,dx$$
$$\Rightarrow -\frac{1}{2}\ln|1-2u| = x + C$$
$$\Rightarrow \ln|1-2u| = -2x + A$$
$$\Rightarrow 1 - 2u = e^{A-2x}$$
$$\Rightarrow 1 - 2u = Ce^{-2x}$$
$$\Rightarrow 1 + Ce^{-2x} = 2u$$
$$\Rightarrow u = \frac{1}{2} + Ce^{-2x}$$

REVERSING THE TRANSFORMATION

$$\frac{dy}{dx} = \frac{1}{2} + Ce^{-2x}$$

$x = 0$ $\frac{dy}{dx} = 1$

$1 = \frac{1}{2} + C$

$C = \frac{1}{2}$

$$\frac{dy}{dx} = \frac{1}{2} + \frac{1}{2}e^{-2x}$$
$$y = \frac{1}{2}x - \frac{1}{4}e^{-2x} + k$$

APPLY CONDITION $x = 0$, $y = -\frac{1}{4}$

$-\frac{1}{4} = 0 - \frac{1}{4} + k$

$k = 0$

$$\therefore y = \frac{1}{2}x - \frac{1}{4}e^{-2x}$$

Question 72 (****+)

A curve $y = f(x)$ satisfies the differential equation

$$\frac{dy}{dx} = \frac{k(9-x)}{y}, \quad y > 0, \quad 0 \le x \le 9,$$

where k is a positive constant.

It is further given that $y = \frac{1}{2}$, $\frac{dy}{dx} = 2$ at $x = 1$.

Find the possible values of x when $\frac{dy}{dx} = \frac{1}{5}$.

$\boxed{}$, $\boxed{x = 5 \cup x = 13}$

$\boxed{\frac{dy}{dx} = \frac{k(9-x)}{y} \text{ SUBJECT TO } y = \frac{1}{2}, \frac{dy}{dx} = 2 \text{ AT } x = 1}$

SUBSTITUTE THE GIVEN CONDITIONS INTO THE O.D.E TO OBTAIN k

$$\Rightarrow \frac{1}{2} = \frac{k(9-1)}{2}$$
$$\Rightarrow 1 = 8k$$
$$\Rightarrow k = \frac{1}{8}$$

SOLVE THE O.D.E BY SEPARATION OF VARIABLES

$$\Rightarrow y\,dy = k(9-x)\,dx$$
$$\Rightarrow \int y\,dy = \int k(9-x)\,dx$$
$$\Rightarrow \frac{1}{2}y^2 = -k(9-x)^2 \times \frac{1}{2} + C$$
$$\Rightarrow y^2 = C - k(9-x)^2$$

APPLY $x = 1$, $y = \frac{1}{2}$

$$\Rightarrow \frac{1}{4} = C - \frac{1}{8}(9-1)^2$$
$$\Rightarrow \frac{1}{4} = C - 8$$
$$\Rightarrow C = 8 + \frac{1}{4} = \frac{33}{4}$$

$$\therefore \quad y^2 = \frac{33}{4} - \frac{1}{8}(9-x)^2$$

NOW SETTING $\frac{dy}{dx} = \frac{1}{5}$ INTO THE O.D.E

$$\Rightarrow \frac{1}{5} = \frac{1}{8}\frac{(9-x)}{y}$$
$$\Rightarrow 8y = 5(9-x)$$
$$\Rightarrow y = \frac{5}{8}(9-x)$$

NOW WITH $\frac{dy}{dx} = \frac{1}{5}$ $\quad y = \frac{5}{8}(9-x)$

$$y^2 = \frac{25}{64}(9-x)^2$$

THUS WE NOW HAVE

$$\left.\begin{array}{l} y^2 = \frac{33}{4} - \frac{1}{8}(9-x)^2 \\ y^2 = \frac{25}{64}(9-x)^2 \end{array}\right\} \Rightarrow \text{COMBINING}$$

$$\Rightarrow \frac{33}{4} - \frac{1}{8}(9-x)^2 = \frac{25}{64}(9-x)^2 \quad \times 64$$
$$\Rightarrow 33 \times 16 - 8(9-x)^2 = 25(9-x)^2$$
$$\Rightarrow 33 \times 16 = 33(9-x)^2$$
$$\Rightarrow 16 = (9-x)^2$$
$$\Rightarrow 9 - x = \begin{cases} 4 \\ -4 \end{cases}$$
$$\Rightarrow x - 9 = \begin{cases} -4 \\ 4 \end{cases}$$
$$\Rightarrow x = \begin{cases} 5 \\ 13 \end{cases}$$

Question 73 (****+)

A curve passes through the point with coordinates $\left[1\,,\ \log_2\left(\log_2 \mathrm{e}\right)\right]$ and its gradient function satisfies

$$\frac{dy}{dx} = 2^y\,, \quad x \in \mathbb{R}\,, \quad x < 2\,.$$

Find the equation of the curve in the form $y = f(x)$

$\square$, $\boxed{y = -\log_2\left[(2-x)\ln 2\right]}$

REWRITE IN TERMS OF THE EXPONENTIAL FUNCTION & SEPARATE VARIABLES

$\Rightarrow \frac{dy}{dx} = 2^y$

$\Rightarrow \frac{dy}{dx} = e^{\ln(2^y)}$

$\Rightarrow \frac{dy}{dx} = e^{y\ln 2}$

$\Rightarrow dy = e^{y\ln 2}\,dx$

$\Rightarrow \frac{1}{e^{y\ln 2}}\,dy = 1\,dx$

$\Rightarrow \int e^{-y\ln 2}\,dy = \int 1\,dx$

$\Rightarrow \frac{1}{-\ln 2}e^{-y\ln 2} = x + C$

$\Rightarrow e^{-y\ln 2} = (-\ln 2)(x+C)$

$\Rightarrow \frac{1}{e^{y\ln 2}} = (A-x)\ln 2$

$\Rightarrow e^{y\ln 2} = \frac{1}{(A-x)\ln 2}$

NOW IT IS BETTER TO MANIPULATE FURTHER BEFORE APPLYING THE BOUNDARY CONDITION $\left[1, \log_2(\log_2 e)\right]$

$\Rightarrow 2^y = \frac{1}{(A-x)\ln 2}$

$\Rightarrow 2^{\log_2(\log_2 e)} = \frac{1}{(A-1)\ln 2}$

$\Rightarrow \log_2 e = \frac{1}{(A-1)\ln 2}$

$\Rightarrow \frac{\log_e e}{\log_e 2} = \frac{1}{(A-1)\ln 2}$

$\Rightarrow \frac{1}{\cancel{\ln 2}} = \frac{1}{(A-1)\cancel{\ln 2}}$

$\Rightarrow A - 1 = 1$

$\Rightarrow A = 2$

$\Rightarrow 2^y = \frac{1}{(2-x)\ln 2}$

$\Rightarrow \log_2 2^y = \log_2\left[\frac{1}{(2-x)\ln 2}\right]$

$\Rightarrow y = -\log_2\left[(2-x)\ln 2\right]$

Question 74 (*****)

The function $y = f(x)$ satisfies the differential equation

$$\frac{dy}{dx} = \frac{2xy(y+1)}{\sin^2\left(x + \frac{1}{6}\pi\right)},$$

subject to the condition $y = 1$ at $x = 0$.

Find the exact value of y when $x = \frac{\pi}{12}$.

$\boxed{\text{MP2-S}}$, $\boxed{y = \frac{1}{e^{\frac{1}{6}\pi} - 1}}$

SOLVE THE O.D.E. BY SEPARATING VARIABLES

$$\Rightarrow \frac{dy}{dx} = \frac{2xy(y+1)}{\sin^2(x+\frac{\pi}{6})}$$

$$\Rightarrow \frac{1}{y(y+1)}\,dy = \frac{2x}{\sin^2(x+\frac{\pi}{6})}\,dx$$

$$\Rightarrow \int \frac{1}{y(y+1)}\,dy = \int 2x\,\text{cosec}^2(x+\tfrac{\pi}{6})\,dx$$

THE L.H.S REQUIRES PARTIAL FRACTIONS (BY INSPECTION) AND THE R.H.S INTEGRATION BY PARTS

$2x$	2
$-\cot(x+\frac{\pi}{6})$	$\text{cosec}^2(x+\frac{\pi}{6})$

$$\Rightarrow \int \frac{1}{y} - \frac{1}{y+1}\,dy = -2x\cot(x+\tfrac{\pi}{6}) - \int -2\cot(x+\tfrac{\pi}{6})\,dx$$

$$\Rightarrow \ln|y| - \ln|y+1| = -2x\cot(x+\tfrac{\pi}{6}) + \int 2\cot(x+\tfrac{\pi}{6})\,dx$$

$$\Rightarrow \ln|y| - \ln|y+1| = -2x\cot(x+\tfrac{\pi}{6}) + 2\ln|\sin(x+\tfrac{\pi}{6})| + C$$

$$\int \cot x\,dx = \ln|\sin x| + C$$

APPLY CONDITION $x=0,\ y=1$

$$\cancel{\ln 1} - \ln 2 = 0 + 2\ln(\sin\tfrac{\pi}{6}) + C$$

$$-\ln 2 = 2\ln\tfrac{1}{2} + C$$

$$-\ln 2 = -2\ln 2 + C$$

$$C = \ln 2$$

THUS WE NOW HAVE

$$\ln|y| - \ln|y+1| = \ln 2 - 2x\cot(x+\tfrac{\pi}{6}) + 2\ln|\sin(x+\tfrac{\pi}{6})|$$

WHEN $x = \frac{\pi}{12}$

$$\Rightarrow \ln\left|\frac{y}{y+1}\right| = \ln 2 - 2\times\left(\tfrac{\pi}{12}\right)\cot\tfrac{\pi}{4} + 2\ln\left(\sin\tfrac{\pi}{4}\right)$$

$$\Rightarrow \ln\left|\frac{y}{y+1}\right| = \ln 2 - \tfrac{\pi}{6} + 2\ln\left(\tfrac{1}{\sqrt{2}}\right)$$

$$\Rightarrow \ln\left|\frac{y}{y+1}\right| = \ln 2 - \tfrac{\pi}{6} + 2\ln 2^{-\frac{1}{2}}$$

$$\Rightarrow \ln\left|\frac{y}{y+1}\right| = \cancel{\ln 2} - \tfrac{\pi}{6} - \cancel{\ln 2}$$

$$\Rightarrow \frac{y}{y+1} = e^{-\frac{\pi}{6}}$$

$$\Rightarrow \frac{y+1}{y} = e^{\frac{\pi}{6}}$$

$$\Rightarrow 1 + \frac{1}{y} = e^{\frac{\pi}{6}}$$

$$\Rightarrow \frac{1}{y} = e^{\frac{\pi}{6}} - 1$$

$$\Rightarrow y = \frac{1}{e^{\frac{\pi}{6}} - 1}$$

Question 75 (*****)

Determine, in the form $y = f(x)$, a simplified solution for the following differential equation.

$$\frac{dy}{dx}\cos x + 4y^2 \sin x = \sin x \,, \quad y = \frac{15}{34} \text{ at } x = \frac{1}{3}\pi \,.$$

$\square$, $\boxed{y = \dfrac{\sec^4 x - 1}{2\left(\sec^4 x + 1\right)}}$

$\boxed{\frac{dy}{dx}\cos x + 4y^2\sin x = \sin x \text{ , SUBJECT TO } y = \frac{15}{34} \text{ at } x = \frac{\pi}{3}}$

- ATTEMPTING TO SEPARATE VARIABLES BY FACTORIZATION

$\Rightarrow \frac{dy}{dx}\cos x = \sin x - 4y^2\sin x$

$\Rightarrow \frac{dy}{dx}\cos x = \sin x(1-4y^2)$

$\Rightarrow \frac{dy}{dx} = (1-4y^2)\tan x$

$\Rightarrow \frac{1}{1-4y^2}\,dy = \tan x\,dx$

$\Rightarrow \int_{y=\frac{15}{34}}^{y} \underbrace{\frac{1}{(1+2y)(1-2y)}}_{\text{PARTIAL FRACTIONS BY INSPECTION (COVER-UP)}}\,dy = \int_{x=\frac{\pi}{3}}^{x} \tan x$

$\Rightarrow \int_{y=\frac{15}{34}}^{y} \frac{\frac{1}{2}}{1+2y} + \frac{\frac{1}{2}}{1-2y}\,dy = \int_{x=\frac{\pi}{3}}^{x} \tan x\,dx$

$\Rightarrow \left[\frac{1}{4}\ln|1+2y| - \frac{1}{4}\ln|1-2y|\right]_{y=\frac{15}{34}}^{y} = \left[\ln|\sec x|\right]_{\pi/3}^{x}$

$\Rightarrow \left[\ln|1+2y| - \ln|1-2y|\right]_{y=\frac{15}{34}}^{y} = \left[4\ln|\sec x|\right]_{\pi/3}^{x}$

$\Rightarrow \left[\ln\left|\frac{1+2y}{1-2y}\right|\right]_{y=\frac{15}{34}}^{y} = \left[\ln(\sec^4 x)\right]_{x=\pi/3}^{x}$

$\Rightarrow \left[\frac{1+2y}{1-2y}\right]_{y=\frac{15}{34}}^{y} = \left[\sec^4 x\right]_{x=}^{x}$

$\Rightarrow \frac{1+2y}{1-2y} - \frac{1+\frac{15}{34}\times 2}{1-2\times\frac{15}{34}} = \sec^4 x - \sec^4\frac{\pi}{3}$

$\Rightarrow \frac{1+2y}{1-2y} - \frac{17+15}{17-15} = \sec^4 x - 2^4$

$\Rightarrow \frac{1+2y}{1-2y} - \cancel{\frac{32}{2}} = \sec^4 x \cancel{-16}$

- REARRANGING TO A FINAL ANSWER

$\Rightarrow 1+2y = \sec^4 x - 2y\sec^4 x$

$\Rightarrow 2y + 2y\sec^4 x = \sec^4 x - 1$

$\Rightarrow 2y(1+\sec^4 x) = \sec^4 x - 1$

$\Rightarrow y = \frac{\sec^4 x - 1}{2(\sec^4 x + 1)}$

OR

$y = \frac{1-\cos^4 x}{2(1+\cos^4 x)}$

Question 76 (*****)

The function $v = f(t)$ satisfies the differential equation

$$\frac{dv}{dt} = k\left[\frac{1}{v} - \frac{1}{h}\right],$$

where k and h are non zero constants.

Given that $v = h-1$ at $t = 0$, solve the differential equation to show that

$$(h-v)\mathrm{e}^{\frac{k}{h}t+v} = A$$

where A is a non zero constant.

proof

$$\frac{dv}{dt} = k\left(\frac{1}{v} - \frac{1}{h}\right), \quad v = h-1 \text{ AT } t=0$$

SEPARATE THE VARIABLES AS FOLLOWS

$$\Rightarrow \frac{dv}{dt} = k\left(\frac{h-v}{vh}\right)$$

$$\Rightarrow \frac{dv}{dt} = \frac{k}{h}\left(\frac{h-v}{v}\right)$$

$$\Rightarrow \frac{v}{h-v}\,dv = \frac{k}{h}\,dt$$

$$\Rightarrow \frac{-v}{h-v}\,dv = -\frac{k}{h}\,dt$$

$$\Rightarrow \frac{(h-v)-h}{h-v}\,dv = -\frac{k}{h}\,dt$$

$$\Rightarrow \left[1 - \frac{h}{h-v}\right]dv = -\frac{k}{h}\,dt$$

INTEGRATING

$$\int 1 - \frac{h}{h-v}\,dv = \int -\frac{k}{h}\,dt$$

$$v + \ln(h-v) = -\frac{k}{h}t + C$$

APPLY CONDITIONS, $t=0 \quad v=h-1$

$$\Rightarrow h-1+\cancel{\ln[h-(h-1)]} = C$$

$$\Rightarrow C = h-1$$

$$\Rightarrow v + \ln(h-v) = h-1-\frac{k}{h}t$$

$$\Rightarrow e^{v+\ln(h-v)} = e^{h-1-\frac{k}{h}t}$$

$$\Rightarrow e^{v}\cdot e^{\ln(h-v)} = e^{h-1}\cdot e^{-\frac{k}{h}t}$$

$$\Rightarrow e^{v}\cdot(h-v) = e^{h-1}\,e^{-\frac{k}{h}t}$$

$$\Rightarrow e^{v}\cdot e^{\frac{k}{h}t}\cdot(h-v) = e^{h-1}$$

$$\Rightarrow e^{v+\frac{k}{h}t}(h-v) = A$$

Question 77 (*****)

The function $y = f(x)$ satisfies the differential equation

$$\frac{dy}{dx} = k\left[\frac{1}{h} - \frac{1}{x}\right],$$

where k and h are non zero constants.

It is further given that $y = -1$, $\frac{dy}{dx} = -1$, $\frac{d^2y}{dx^2} = 2$ at $x = -1$.

Solve the differential equation to show that

$$e^{y-x} = (y+2)^2.$$

$\boxed{}$, $\boxed{\text{proof}}$

$$\frac{dy}{dx} = k\left(\frac{1}{h} + \frac{1}{y}\right),$$

SUBJECT TO $y=-1$, $\frac{dy}{dx}=-1$, $\frac{d^2y}{dx^2}=2$ AT $x=-1$

- START BY OBTAINING THE 2ND DERIVATIVE FIRST

$$\frac{dy}{dx} = \frac{k}{h} + \frac{k}{y}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{k}{h} + \frac{k}{y}\right) = 0 + k\frac{d}{dx}\left(\frac{1}{y}\right)$$

$$\frac{d^2y}{dx^2} = k\left(-\frac{1}{y^2}\right)\frac{dy}{dx}$$

- APPLY CONDITION $y=-1$, $\frac{dy}{dx}=-1$, $\frac{d^2y}{dx^2}=2$

$$\Rightarrow 2 = k\left(-\frac{1}{1}\right)(-1)$$

$$\Rightarrow k=2$$

- APPLY CONDITION $y=-1$ $\frac{dy}{dx}=-1$ INTO THE O.D.E

$$\Rightarrow -1 = \frac{2}{h} + \frac{2}{-1}$$

$$\Rightarrow -1 = \frac{2}{h} - 2$$

$$\Rightarrow 1 = \frac{2}{h}$$

$$\Rightarrow h = 2$$

- THE O.D.E CAN NOW BE REWRITTEN AS

$$\Rightarrow \frac{dy}{dx} = 2\left(\frac{1}{2} + \frac{1}{y}\right)$$

$$\Rightarrow \frac{dy}{dx} = 1 + \frac{2}{y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y+2}{y}$$

$$\Rightarrow \frac{y}{y+2}\,dy = 1\,dx$$

$$\Rightarrow \frac{(y+2)-2}{y+2}\,dy = 1\,dx$$

$$\Rightarrow \left(1 - \frac{2}{y+2}\right)dy = 1\,dx$$

- INTEGRATE SUBJECT TO THE CONDITION $x=-1$ $y=-1$

$$\Rightarrow \int_{-1}^{y} 1 - \frac{2}{y+2}\,dy = \int_{-1}^{x} 1\,dx$$

$$\Rightarrow \Big[y - 2\ln|y+2|\Big]_{-1}^{y} = \Big[x\Big]_{-1}^{x}$$

$$\Rightarrow \Big[y - 2\ln(y+2)\Big] - \cancel{\Big[-1\Big]} = x - \cancel{(-1)}$$

$$\Rightarrow \ln e^{y} - \ln(y+2)^2 = x$$

$$\Rightarrow \ln\left[\frac{e^{y}}{(y+2)^2}\right] = x$$

$$\Rightarrow \frac{e^{y}}{(y+2)^2} = e^{x}$$

$$\Rightarrow e^{y-x} = (y+2)^2$$

Question 78 (*****)

The function $y = f(x)$ satisfies the differential equation

$$\frac{d}{dx}\left(yx^2\right) = \frac{dy}{dx}\frac{d}{dx}\left(x^2\right), \quad x > 0$$

subject to the condition $y = 4$ at $x = 3$.

Find a simplified expression for $y = f(x)$.

$\boxed{}$, $\boxed{y = \frac{4}{(x-2)^2}}$

$$\frac{d}{dx}\left(yx^2\right) = \frac{dy}{dx}\frac{dx^2}{dx}, \quad x>0$$

- DIFFERENTIATE THE LHS (PRODUCT RULE)

$$x^2\frac{dy}{dx} + y(2x) = \frac{dy}{dx} \times 2x$$

- DIVIDE BY x, AS $x>0$

$$x\frac{dy}{dx} + 2y = 2\frac{dy}{dx}$$

$$(x-2)\frac{dy}{dx} = -2y$$

$$\frac{1}{y}\,dy = \frac{-2}{x-2}\,dx$$

$$\int \frac{1}{y}\,dy = \int \frac{-2}{x-2}\,dx$$

$$\ln y = \ln A - 2\ln(x-2)$$

$$\ln y = \ln \frac{A}{(x-2)^2}$$

$$y = \frac{A}{(x-2)^2}$$

- APPLY CONDITION $(3,4)$

$$4 = \frac{A}{(3-2)^2}$$

$$A = 4$$

- HENCE

$$y = \frac{4}{(x-2)^2}$$

Question 79 (*****)

Use appropriate techniques to solve the following differential equation.

$$\frac{d^2y}{dx^2} = -\frac{144}{y^3}, \qquad y(0) = 6, \qquad \left.\frac{dy}{dx}\right|_{x=0} = 0.$$

$\boxed{}$, $\boxed{\frac{x^2}{9} + \frac{y^2}{36} = 1}$

$$\boxed{\frac{d^2y}{dx^2} = -\frac{144}{y^3} \quad \text{WITH } x=0,\ y=6,\ \frac{dy}{dx}=0}$$

LET $p = \frac{dy}{dx}$ SO WE HAVE

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{144}{y^3}$$
$$\Rightarrow \frac{d}{dx}\left(\frac{dy}{dx}\right) = -\frac{144}{y^3}$$
$$\Rightarrow \frac{dp}{dx} = -\frac{144}{y^3}$$
$$\Rightarrow \frac{dp}{dy} \times \frac{dy}{dx} = -\frac{144}{y^3}$$
$$\Rightarrow \frac{dp}{dy} \times p = -\frac{144}{y^3}$$
$$\Rightarrow p\,dp = -\frac{144}{y^3}$$
$$\Rightarrow \int p\,dp = \int -\frac{144}{y^3}\,dy$$

INTEGRATING BOTH SIDES WE GET

$$\Rightarrow \frac{1}{2}p^2 = \frac{72}{y^2} + A$$
$$\Rightarrow p^2 = \frac{144}{y^2} + B$$

$x=6,\ p=0 \Rightarrow 0 = 4 + B$
$\Rightarrow B = -4$

$$\therefore\ p^2 = \frac{144}{y^2} - 4$$

$$\Rightarrow p^2 = \frac{144 - 4y^2}{y^2}$$
$$\Rightarrow p = \pm\frac{\sqrt{144-4y^2}}{y}$$
$$\Rightarrow \frac{dy}{dx} = \pm\frac{\sqrt{144-4y^2}}{y}$$

MANIPULATE AS FOLLOWS

$$\Rightarrow \frac{dx}{dy} = \pm\frac{y}{(144-4y^2)^{\frac{1}{2}}} = \pm y(144-4y^2)^{-\frac{1}{2}}$$

SEPARATE VARIABLES AND INTEGRATE BY INSPECTION

$$\Rightarrow \int 1\,dx = \int \pm y(144-4y^2)^{-\frac{1}{2}}$$
$$\Rightarrow x = \pm\frac{1}{4}(144-4y^2)^{\frac{1}{2}} + C$$

APPLY $x=0,\ y=6$
$0 = \pm 0 + C$
$C = 0$

FINALLY WE HAVE

$$\Rightarrow x^2 = \left[\pm\frac{1}{4}(144-4y^2)^{\frac{1}{2}}\right]^2$$
$$\Rightarrow x^2 = \frac{1}{16}(144-4y^2)$$
$$\Rightarrow 16x^2 = 144 - 4y^2$$
$$\Rightarrow 16x^2 + 4y^2 = 144 \quad \text{OR} \quad \frac{x^2}{9} + \frac{y^2}{36} = 1$$

Question 80 (*****)

Solve the differential equation

$$\frac{d^2y}{dx^2}+4\left(\frac{dy}{dx}\right)^2=1,$$

given that $y=0$ and $\frac{dy}{dx}=\frac{1}{6}$ at $x=0$, giving the answer in the form $y=f(x)$.

$\boxed{}$, $\boxed{y=\frac{1}{4}\ln\left[\frac{1+2e^{4x}}{3}\right]-\frac{1}{2}x}$

$$\boxed{\frac{d^2y}{dx^2}+4\left(\frac{dy}{dx}\right)^2=1}$$

USE A SUBSTITUTION

LET $\boxed{p=\frac{dy}{dx} \quad \frac{dp}{dx}=\frac{d^2y}{dx^2}}$

$\Rightarrow \frac{dp}{dx}+4p^2=1$

$\Rightarrow \frac{dp}{dx}=1-4p^2$

$\Rightarrow \frac{1}{1-4p^2}\,dp=1\,dx$

$\Rightarrow \frac{1}{(1-2p)(1+2p)}\,dp=1\,dx$

PARTIAL FRACTIONS BY INSPECTION (COVER UP)

$\Rightarrow \left[\frac{\frac{1}{2}}{1-2p}+\frac{\frac{1}{2}}{1+2p}\right]dp=1\,dx$

$\Rightarrow \left[\frac{1}{1+2p}-\frac{1}{1-2p}\right]dp=2\,dx$

$\Rightarrow \int \frac{1}{1+2p}-\frac{1}{1-2p}\,dp=\int 2\,dx$

$\Rightarrow \frac{1}{2}\ln|1+2p|-\frac{1}{2}\ln|1-2p|=2x+C$

$\Rightarrow \ln\left|\frac{1+2p}{1-2p}\right|=4x+C$

$\Rightarrow \frac{1+2p}{1-2p}=e^{4x+C}$

$\Rightarrow \frac{1+2p}{1-2p}=Ae^{4x} \quad (A=e^c)$

REARRANGE FOR $p=\frac{dy}{dx}$

$\Rightarrow 1+2p=(1-2p)Ae^{4x}$

$\Rightarrow 1+2p=Ae^{4x}-2Ape^{4x}$

$\Rightarrow 2p+2Ape^{4x}=Ae^{4x}-1$

$\Rightarrow 2p(1+Ae^{4x})=Ae^{4x}-1$

$\Rightarrow p=\frac{1}{2}\left(\frac{Ae^{4x}-1}{Ae^{4x}+1}\right)$

$\Rightarrow \boxed{\frac{dy}{dx}=\frac{1}{2}\left(\frac{Ae^{4x}-1}{Ae^{4x}+1}\right)}$

APPLY CONDITION $x=0$, $\frac{dy}{dx}=\frac{1}{6}$

$\frac{1}{6}=\frac{1}{2}\left(\frac{A-1}{A+1}\right)$

$\frac{1}{3}=\frac{A-1}{A+1}$

$A+1=3A-3$

$4=2A$

$\boxed{A=2}$

$\Rightarrow \frac{dy}{dx}=\frac{1}{2}\left(\frac{2e^{4x}-1}{2e^{4x}+1}\right)$

$\Rightarrow y=\frac{1}{2}\int \frac{2e^{4x}-1}{2e^{4x}+1}\,dx$

BY SUBSTITUTION

$u=2e^{4x}+1$

$2e^{4x}=u-1$

$\frac{du}{dx}=8e^{4x}$

$dx=\frac{du}{8e^{4x}}$

$dx=\frac{du}{4(u-1)}$

$\Rightarrow y=\frac{1}{2}\int \frac{(u-1)-1}{(u-1)+1}\times\frac{du}{4(u-1)}$

$\Rightarrow y=\frac{1}{8}\int \frac{u-2}{u(u-1)}\,du$

PARTIAL FRACTIONS AGAIN

$\Rightarrow y=\frac{1}{8}\int \frac{2}{u}-\frac{1}{u-1}\,du$

$\Rightarrow y=\frac{1}{8}\left[2\ln|u|-\ln|u-1|\right]+D$

$\Rightarrow y=\frac{1}{8}\ln\left|\frac{u^2}{u-1}\right|+D$

$\Rightarrow y=\frac{1}{8}\ln\left[\frac{(2e^{4x}+1)^2}{2e^{4x}}\right]+D$

$\Rightarrow y=\frac{1}{8}\ln\left[\frac{2e^{4x}+1}{e^{2x}}\right]^2+E$

$\Rightarrow y=\frac{1}{4}\ln\left[\frac{2e^{4x}+1}{e^{2x}}\right]+E$

$\Rightarrow y=\frac{1}{4}\ln(2e^{4x}+1)-\frac{1}{4}\ln e^{2x}+E$

$\Rightarrow y=\frac{1}{4}\ln(2e^{4x}+1)-\frac{1}{2}x+E$

APPLY THE LAST CONDITION

$x=0, y=0$

$0=\frac{1}{4}\ln 3-0+E$

$E=-\frac{1}{4}\ln 3$

$\Rightarrow y=\frac{1}{4}\ln(2e^{4x}+1)-\frac{1}{4}\ln 3-\frac{1}{2}x$

$\Rightarrow y=\frac{1}{4}\ln\left(\frac{2e^{4x}+1}{3}\right)-\frac{1}{2}x$

Question 81 (*****)

Solve the following differential equation

$$\frac{d^2y}{dx^2} = x\left(\frac{dy}{dx}\right)^2,$$

given further $y = 0$, $\dfrac{dy}{dx} = 2$ at $x = 0$.

Give the answer in the form $x = f(y)$.

$$\boxed{}\ , \quad \boxed{x = \frac{e^y - 1}{e^y + 1} = \tanh\left(\tfrac{1}{2}x\right)}$$

$$\boxed{\frac{d^2y}{dx^2} = x\left(\frac{dy}{dx}\right)^2 \quad \text{GIVEN THAT } y=0,\ \frac{dy}{dx}=2 \text{ AT } x=0}$$

- LET $v = \frac{dy}{dx} \Rightarrow \frac{dv}{dx} = \frac{d^2y}{dx^2}$
- HENCE THE O.D.E CAN BE WRITTEN AS

$\Rightarrow \frac{dv}{dx} = xv^2$

$\Rightarrow \frac{1}{v^2}\,dv = x\,dx$

$\Rightarrow -\frac{1}{v} = \frac{1}{2}x^2 + C$

$\Rightarrow \frac{1}{v} = C - \frac{1}{2}x^2$

$\Rightarrow v = \frac{1}{C - \frac{1}{2}x^2}$

$\Rightarrow v = \frac{2}{A - x^2}$

$\Rightarrow \frac{dy}{dx} = \frac{2}{A - x^2}$

- APPLY CONDITION $x=0$ $\frac{dy}{dx} = 2$
  $2 = \frac{2}{A-0}$
  $A = 1$

$\Rightarrow \frac{dy}{dx} = \frac{2}{1-x^2}$

$\Rightarrow \frac{dy}{dx} = \frac{2}{(1-x)(1+x)}$

- PARTIAL FRACTIONS BY INSPECTION (COVER UP)

$\Rightarrow \frac{dy}{dx} = \frac{1}{1+x} + \frac{1}{1-x}$

$\Rightarrow \int 1\,dy = \int \left(\frac{1}{1+x} + \frac{1}{1-x}\right) dx$

$\Rightarrow y = \ln|1+x| - \ln|1-x| + \ln B$

$\Rightarrow y = \ln\left|\frac{B(1+x)}{1-x}\right|$

$\Rightarrow e^y = \frac{B(1+x)}{1-x}$

- APPLY CONDITION $x=0$ $y=0$ $\Rightarrow \boxed{B=0}$

$\Rightarrow e^y = \frac{1+x}{1-x}$

$\Rightarrow e^y - xe^y = 1 + x$

$\Rightarrow e^y - 1 = xe^y + x$

$\Rightarrow e^y - 1 = x(e^y + 1)$

$\Rightarrow x = \frac{e^y - 1}{e^y + 1}$ //

[OR $x = \tanh\frac{y}{2}$]

Question 82 (*****)

$$\frac{dy}{dx} = \sqrt{\frac{y^4 - y^2}{x^4 - x^2}}, \quad x > 0, \ y > 0.$$

Find the solution of the above differential equation subject to the boundary condition $y = \frac{2}{\sqrt{3}}$ at $x = 2$.

Give the answer in the form $y = \frac{2x}{f(x)}$, where $f(x)$ is a function to be found.

$\boxed{}$, $\boxed{f(x) = \sqrt{3} + \sqrt{x^2 - 1}}$

SOLVE THE O.D.E BY SEPARATION OF VARIABLES

$$\frac{dy}{dx} = \sqrt{\frac{y^4-y^2}{x^4-x^2}} = \frac{|y|\sqrt{y^2-1}}{|x|\sqrt{x^2-1}} = \frac{y\sqrt{y^2-1}}{x\sqrt{x^2-1}} \quad \text{AS } x, y > 0$$

$$\Rightarrow \int \frac{1}{y\sqrt{y^2-1}}\,dy = \int \frac{1}{x\sqrt{x^2-1}}\,dx$$

IDENTICAL SUBSTITUTION OR DIRECTLY RECOGNISING THE DERIVATIVE OF "arcsec"

LET $x = \sec\theta$

$dx = \sec\theta\tan\theta\,d\theta$

$$\int \frac{1}{x\sqrt{x^2-1}}\,dx = \int \frac{1}{\sec\theta\sqrt{\sec^2\theta-1}}(\sec\theta\tan\theta)\,d\theta = \int \frac{\cancel{\sec\theta\tan\theta}}{\cancel{\sec\theta\tan\theta}}\,d\theta$$

$$= \int 1\,d\theta = \theta + C = \operatorname{arcsec}x + C$$

RETURNING TO THE O.D.E.

$$\Rightarrow \operatorname{arcsec}y = \operatorname{arcsec}x + C$$

APPLY CONDITION $\left(2, \frac{2}{\sqrt{3}}\right)$

$$\operatorname{arcsec}\frac{2}{\sqrt{3}} = \operatorname{arcsec}2 + C$$

$$\frac{\pi}{6} = \frac{\pi}{3} + C$$

$$C = -\frac{\pi}{6}$$

$$\Rightarrow \operatorname{arcsec}y = \operatorname{arcsec}x - \frac{\pi}{6}$$

$$\Rightarrow \cos(\operatorname{arcsec}y) = \cos\left(\operatorname{arcsec}x - \frac{\pi}{6}\right)$$

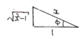
$$\Rightarrow \cos(\operatorname{arcsec}y) = \cos(\operatorname{arcsec}x)\cos\frac{\pi}{6} + \sin(\operatorname{arcsec}x)\sin\frac{\pi}{6}$$

NEXT REMOVE THE "arcsec"

$\operatorname{arcsec}x = \phi$

$\sec\phi = x$

$\cos\phi = \frac{1}{x}$ & $\sin\phi = \frac{\sqrt{x^2-1}}{x}$



$\therefore \cos(\operatorname{arcsec}x) = \frac{1}{x}$ $\qquad \sin(\operatorname{arcsec}x) = \frac{\sqrt{x^2-1}}{x}$

$\cos(\operatorname{arcsec}y) = \frac{1}{y}$

RETURNING TO THE O.D.E

$$\frac{1}{y} = \frac{1}{x} \times \frac{\sqrt{3}}{2} + \frac{\sqrt{x^2-1}}{x} \times \frac{1}{2}$$

$$\frac{1}{y} = \frac{\sqrt{3}}{2x} + \frac{\sqrt{x^2-1}}{2x}$$

$$\frac{1}{y} = \frac{\sqrt{3} + \sqrt{x^2-1}}{2x}$$

$$y = \frac{2x}{\sqrt{3} + \sqrt{x^2-1}} \quad //$$

Question 83 (*****)

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 1 .$$

Given that $y = \frac{dy}{dx} = 0$ at $x = 0$, show that

$$y = -x + \ln\left[\tfrac{1}{2}\left(1 + e^{2x}\right)\right].$$

$\boxed{}$, $\boxed{\text{proof}}$

$\boxed{\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 1 \quad \text{SUBJECT TO} \quad y = \frac{dy}{dx} = 0 \ \text{AT } x = 0}$

- REDUCE THE O.D.E INTO A FIRST ORDER AS FOLLOWS

$\Rightarrow \frac{d}{dx}\left[\frac{dy}{dx}\right] + \left[\frac{dy}{dx}\right]^2 = 1$ LET $p = \frac{dy}{dx}$

$\Rightarrow \frac{dp}{dx} + p^2 = 1$

$\Rightarrow \frac{dp}{dx} = 1 - p^2$

- SEPARATING VARIABLES & INTEGRATE SUBJECT TO $x=0$, $\frac{dy}{dx} = p = 0$

$\Rightarrow \frac{1}{1-p^2}\,dp = 1\,dx$

$\Rightarrow \int_0^p \frac{1}{1-p^2}\,dp = \int_0^x 1\,dx$

$\Rightarrow \int_0^p \frac{1}{(1+p)(1-p)}\,dp = \int_0^x 1\,dx$

- PARTIAL FRACTIONS BY INSPECTION (COVER UP) ON THE LHS

$\Rightarrow \int_0^p \frac{\frac{1}{2}}{1+p} + \frac{\frac{1}{2}}{1-p}\,dp = \int_0^x 1\,dx.$

$\Rightarrow \int_0^p \frac{1}{1+p} + \frac{1}{1-p}\,dp = \int_0^x 2\,dx$

$\Rightarrow \Big[\ln|1+p| + \ln|1-p|\Big]_0^p = \Big[2x\Big]_0^x$

$\Rightarrow \Big[\ln|1+p| + \ln|1-p|\Big] - \Big[\cancel{\ln 1} + \cancel{\ln 1}\Big] = 2x$

$\Rightarrow \ln\left|\frac{1+p}{1-p}\right| = 2x$

$\Rightarrow \frac{1+p}{1-p} = e^{2x}$

$\Rightarrow 1 + p = e^{2x} - pe^{2x}$

$\Rightarrow pe^{2x} + p = e^{2x} - 1$

$\Rightarrow p(e^{2x}+1) = e^{2x} - 1$

$\Rightarrow p = \frac{e^{2x}-1}{e^{2x}+1}$

$\Rightarrow \boxed{\frac{dy}{dx} = \frac{e^{2x}-1}{e^{2x}+1}}$

$\Rightarrow \int_{y=0}^{y} 1\,dy = \int_{x=0}^{x} \frac{e^{2x}-1}{e^{2x}+1}\,dx$

$\Rightarrow \Big[y\Big]_0^y = \int_2^{e^{2x}+1} \frac{u-2}{u}\,\frac{du}{2(u-1)}$

$\Rightarrow 2y = \int_2^{e^{2x}+1} \frac{u-2}{u(u-1)}\,du$

SUBSTITUTION
- $u = e^{2x}+1$
- $e^{2x} = u - 1$

$\frac{du}{dx} = 2e^{2x}$

$dx = \frac{du}{2e^{2x}}$

- $dx = \frac{du}{2(u-1)}$

& $x=0$, $u=2$

$x=x$, $u = e^{2x}+1$

PARTIAL FRACTIONS

$\Rightarrow 2y = \int_2^{e^{2x}+1} \frac{2}{u} - \frac{1}{u-1}\,du$

$\Rightarrow 2y = \Big[2\ln|u| - \ln|u-1|\Big]_2^{e^{2x}+1}$

$\Rightarrow 2y = \Big[2\ln(e^{2x}+1) - \ln(e^{2x})\Big] - \Big[2\ln 2 - \cancel{\ln 1}\Big]$

$\Rightarrow 2y = 2\ln(e^{2x}+1) - 2x - 2\ln 2$

$\Rightarrow y = \ln(e^{2x}+1) - \ln 2 - x$

$\Rightarrow y = \ln\left[\tfrac{1}{2}(e^{2x}+1)\right] - x$

- VARIATION USING HYPERBOLIC FUNCTIONS FROM THIS EXPRESSION ONWARDS

$\Rightarrow \frac{dy}{dx} = \frac{e^{2x}-1}{e^{2x}+1}$

$\Rightarrow \frac{dy}{dx} = \tanh x$

$\Rightarrow \int_{y=0}^{y} 1\,dy = \int_{x=0}^{x} \frac{\sinh x}{\cosh x}\,dx$

$\Rightarrow \Big[y\Big]_0^y = \Big[\ln(\cosh x)\Big]_0^x$

$\Rightarrow y = \ln(\cosh x) - \cancel{\ln 1}$

- WHICH MERGES SINCE

$\Rightarrow y = \ln\left[\tfrac{1}{2}e^{x} + \tfrac{1}{2}e^{-x}\right] = \ln\left[e^{-x}\left[\tfrac{1}{2}e^{2x} + \tfrac{1}{2}\right]\right]$

$= \ln e^{-x} + \ln\left[\tfrac{1}{2}e^{2x} + \tfrac{1}{2}\right] = -x + \ln\left[\tfrac{1}{2}(e^{2x}+1)\right]$

Question 84 (*****)

The non zero function $f(x)$ satisfies the integral equation

$$\sqrt{\int f(x)\,dx} = \int \sqrt{f(x)}\,dx\,, \qquad f(0) = \frac{1}{4}.$$

Use the substitution $f(x) = \left(\frac{dy}{dx}\right)^2$, to find a simplified expression for $f(x)$.

$\boxed{}$, $\boxed{f(x) = \frac{1}{4}e^{4x}}$

$$\sqrt{\int f(x)\,dx} = \int \sqrt{f(x)}\,dx \qquad x=0 \quad f(x)=\frac{1}{4}$$

- START WITH THE SUBSTITUTION GIVEN

$$\Rightarrow \sqrt{\int \left(\frac{dy}{dx}\right)^2 dx} = \int \sqrt{\left(\frac{dy}{dx}\right)^2}\,dx$$

$$\Rightarrow \sqrt{\int \left(\frac{dy}{dx}\right)^2 dx} = \int \frac{dy}{dx}\,dx$$

- INTEGRATE THE R.H.S

$$\Rightarrow \sqrt{\int \left(\frac{dy}{dx}\right)^2 dx} = y + k$$

- SQUARING BOTH SIDES

$$\Rightarrow \int \left(\frac{dy}{dx}\right)^2 dx = (y+k)^2$$

- DIFFERENTIATE BOTH SIDES WITH RESPECT TO x

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = 2(y+k)\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = 2(y+k)$$

$$\Rightarrow \frac{1}{y+k}\,dy = 2\,dx$$

$$\Rightarrow \int \frac{1}{y+k}\,dy = \int 2\,dx$$

$$\Rightarrow \ln|y+k| = 2x + C$$

$$\Rightarrow y + k = e^{2x+C}$$

$$\Rightarrow y + k = Ae^{2x} \quad (A = e^C)$$

$$\Rightarrow y = Ae^{2x} + k$$

- APPLY CONDITION $\quad x=0,\ \left(\frac{dy}{dx}\right)^2 = f(x) = \frac{1}{4}$

$\qquad x = 0 \quad \frac{dy}{dx} = \frac{1}{2}$

$$\Rightarrow \frac{dy}{dx} = 2Ae^{2x}$$

$$\Rightarrow \frac{1}{2} = 2Ae^0$$

$$\Rightarrow A = \frac{1}{4}$$

- FINALLY WE HAVE

$$f(x) = \left(\frac{dy}{dx}\right)^2$$

$$f(x) = \left(2Ae^{2x}\right)^2$$

$$f(x) = \left(2 \times \frac{1}{4}e^{2x}\right)^2$$

$$f(x) = \left(\frac{1}{2}e^{2x}\right)^2$$

$$f(x) = \frac{1}{4}e^{4x}$$

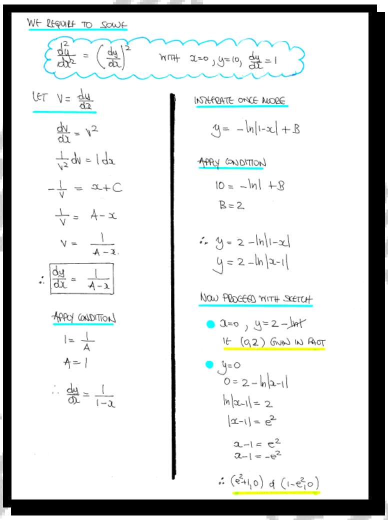
Question 85 (*****)

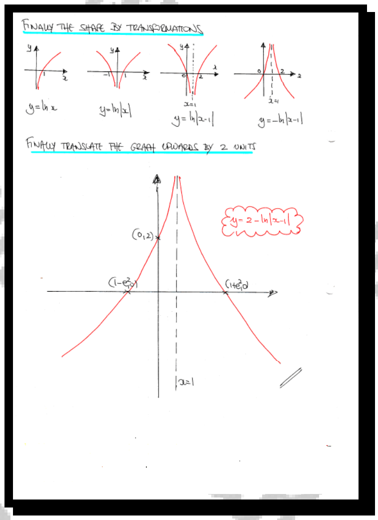
It is required to sketch the curve with equation $y = f(x)$, defined over the set of real numbers, in the greatest domain.

The curve has the property that at every point on the curve, the second derivative equals to the first derivative **squared**.

Showing all the relevant details, sketch the graph of $y = f(x)$, given further that the curve passes through the point $(0,2)$ and the gradient at that point is 1.

$\boxed{}$, $\boxed{\text{graph}}$





Question 86 (*****)

It is given that a function with equation $y = f(x)$ is a solution of the following differential equation.

$$\left(1-x^2\right)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + y = 0 .$$

Show with a clear method that

$$\int_1^x f(x)\ dx = \left(x^2-1\right)f'(x) .$$

proof

$$\boxed{(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + y = 0}$$

- NOTICE THAT THERE IS AN EXACT DIFFERENTIAL IN THE L.H.S

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] = -2x\frac{dy}{dx} + (1-x^2)\frac{d^2y}{dx^2}$$

- HENCE, WE REWRITE THE O.D.E AS

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + y = 0$$

$$-\frac{d}{dx}\left[(x^2-1)\frac{dy}{dx}\right] + y = 0$$

$$y = \frac{d}{dx}\left[(x^2-1)\frac{dy}{dx}\right]$$

- INTEGRATE BOTH SIDES W.R.T x, FROM 1 TO x

$$\int_0^1 y\ dx = \int_1^x \frac{d}{dx}\left[(x^2-1)\frac{dy}{dx}\right]\ dx$$

$$\int_0^1 y\ dx = \left[(x^2-1)\frac{dy}{dx}\right]_1^x$$

$$\int_0^1 y\ dx = (x^2-1)\frac{dy}{dx} - 0$$

- OR WRITTEN IN f NOTATION

$$\int_0^1 f(x)\ dx = (x^2-1)f'(x)$$

AS REQUIRED

ALTERNATIVE

$$\boxed{(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + y = 0}$$

- $y = f(x)$ IS A SOLUTION OF THE O.D.E

$$\Rightarrow (1-x^2)f''(x) - 2xf'(x) + f(x) = 0$$

$$\Rightarrow f(x) = 2xf'(x) + (x^2-1)f''(x)$$

- INTEGRATE THE O.D.E. W.R.T x, FROM 1 TO x

$$\Rightarrow \int_1^x f(x)\ dx = 2\int_1^x xf'(x)\ dx + \int_1^x (x^2-1)f''(x)\ dx$$

BY PARTS

x^2-1	$2x$
$f'(x)$	$f''(x)$

$$\Rightarrow \int_1^x f(x)\ dx = \cancel{2\int_1^x xf'(x)\ dx} + \left[(x^2-1)f'(x)\right]_1^x - \cancel{2\int_1^x xf'(x)\ dx}$$

$$\Rightarrow \int_1^x f(x)\ dx = (x^2-1)f'(x) - 0$$

$$\Rightarrow \int_1^x f(x)\ dx = (x^2-1)f'(x)$$

AS REQUIRED

Question 87 (*****)

Solve the following differential equation

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + 2y\frac{dy}{dx} = 0, \quad y(0) = 2, \quad \frac{dy}{dx}(0) = -\frac{1}{2}.$$

Give the answer in the form $y^2 = f(x)$.

$\boxed{}$, $\boxed{y^2 = 3 + e^{-2x}}$

$$\boxed{y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + 2y\frac{dy}{dx} = 0 \qquad x=0,\ y=2,\ \frac{dy}{dx} = -\frac{1}{2}}$$

BEFORE WE ATTEMPT A SUBSTITUTION, WE OBSERVE THAT THE TERMS OF THIS O.D.E. RESEMBLE DIFFERENTIALS

$$\frac{d}{dx}\left(y\frac{dy}{dx}\right) = \frac{dy}{dx}\times\frac{dy}{dx} + y\times\frac{d^2y}{dx^2} = y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2$$

$$\frac{d}{dx}\left(y^2\right) = 2y\frac{dy}{dx}$$

HENCE WE MAY REWRITE AS

$$\Rightarrow \frac{d}{dx}\left[y\frac{dy}{dx} + y^2\right] = 0$$

$$\Rightarrow y\frac{dy}{dx} + y^2 = C$$

APPLY CONDITION $y=2$, $\frac{dy}{dx} = -\frac{1}{2}$

$$\Rightarrow 2\left(-\tfrac{1}{2}\right) + 2^2 = C$$

$$\Rightarrow C = 3$$

$$\Rightarrow y\frac{dy}{dx} + y^2 = 3$$

PROCEED BY SEPARATION OF VARIABLES

$$\Rightarrow y\frac{dy}{dx} = 3 - y^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{3-y^2}{y}$$

$$\Rightarrow \frac{y}{3-y^2}\,dy = 1\,dx$$

$$\Rightarrow \int \frac{-2y}{3-y^2}\,dy = \int -2\,dx$$

$$\Rightarrow \ln|3-y^2| = -2x + A$$

$$\Rightarrow 3 - y^2 = e^{-2x+A}$$

$$\Rightarrow 3 - y^2 = e^{-2x}\times e^{A}$$

$$\Rightarrow 3 - y^2 = Be^{-2x}$$

$$\Rightarrow y^2 = 3 + Be^{-2x}$$

APPLY FINAL CONDITION, $x=0$ $y=2$

$$\Rightarrow 4 = 3 + Be^{0}$$

$$\Rightarrow B = 1$$

$$\therefore\ y^2 = 3 + e^{-2x}$$

NOTE THAT THE SUBSTITUTION $y = \frac{dp}{dx} \Rightarrow \frac{d^2y}{dx^2} = y\frac{dp}{dy}$ YIELDS

$$yp\frac{dp}{dy} + p^2 + 2yp = 0$$

$$\frac{dp}{dy} + \frac{p}{y} = -2$$

BY INTEGRATING FACTOR & CONDITION $y=2$ $p = \frac{dy}{dx} = -\frac{1}{2}$ WE OBTAIN

$$\frac{dy}{dx} = -y + \frac{3}{y}$$

$$\frac{dy}{dx} = \frac{3-y^2}{y}$$

WHICH MERGES WITH THIS SOLUTION

Question 88 (*****)

The function with equation $y = f(x)$ satisfies the differential equation

$$\frac{d^2y}{dx^2} = \frac{2}{2x-1}\left(1 - \frac{dy}{dx}\right), \quad y(0) = 1, \quad \frac{dy}{dx}(0) = -1.$$

Solve the above differential equation giving the answer in the form $y = f(x)$.

$\boxed{}$, $\boxed{y = x + 1 + \ln|2x-1|}$

- START BY REARRANGING THE O.D.E AS FOLLOWS

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{2}{2x-1}\left(1 - \frac{dy}{dx}\right)$$

$$\Rightarrow (2x-1)\frac{d^2y}{dx^2} = 2\left(1 - \frac{dy}{dx}\right)$$

$$\Rightarrow (2x-1)\frac{d^2y}{dx^2} = 2 - 2\frac{dy}{dx}$$

$$\Rightarrow (2x-1)\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 2$$

- BY INSPECTION THE L.H.S IS A PERFECT DIFFERENTIAL

$$\Rightarrow \frac{d}{dx}\left[(2x-1)\frac{dy}{dx}\right] = 2$$

$$\Rightarrow (2x-1)\frac{dy}{dx} = 2x + A$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x+A}{2x-1}$$

- NOW $\frac{dy}{dx} = -1$ AT $x=0$, GIVES $A=1$

$$\Rightarrow \frac{dy}{dx} = \frac{2x+1}{2x-1}$$

$$\Rightarrow y = \int \frac{2x+1}{2x-1}\,dx$$

$$\Rightarrow y = \int \frac{(2x-1)+2}{2x-1}\,dx$$

$$\Rightarrow y = \int 1 + \frac{2}{2x-1}\,dx$$

$$\Rightarrow y = x + \ln|2x-1| + B$$

NOW IF $x=0, y=1 \Rightarrow B=1$

$$\therefore y = x + \ln|2x-1| + 1$$

Question 89 (*****)

The positive solution of the quadratic equation $x^2 - x - 1 = 0$ is denoted by ϕ, and is commonly known as the golden section or golden number.

a) Show, with a detailed method, that $F(x) = f(\phi)\, x^{g(\phi)}$ is a solution of the differential equation,

$$F'(x) = F^{-1}(x),$$

where f and g are constant expressions of ϕ, to be found in simplified form.

b) Verify the answer obtained in part **(a)** satisfies the differential equation, by differentiation and function inversion.

[You may assume that $F(x)$ is differentiable and invertible]

V, , $\boxed{F(x) = \left(\frac{1}{\phi}\right)^{\frac{1}{\phi}} x^{\phi} = \phi^{1-\phi} x^{\phi}}$

ASSUME A SOLUTION OF THE FORM $y = Ax^r$, WHERE A IS A CONSTANT & r IS ALSO A CONSTANT — FURTHER ASSUME THAT "FUNCTION" IS SMOOTH AND INVERTIBLE

- $y = Ax^r$

$\Rightarrow \frac{dy}{dx} = rAx^{r-1}$

$F'(x)$

- $y = Ax^r$

$\Rightarrow \frac{y}{A} = x^r$

$\Rightarrow \left(\frac{y}{A}\right)^{\frac{1}{r}} = (x^r)^{\frac{1}{r}}$

$\Rightarrow x = \left(\frac{1}{A}\right)^{\frac{1}{r}} y^{\frac{1}{r}}$

SWAP x & y

$\Rightarrow y = \left(\frac{1}{A}\right)^{\frac{1}{r}} x^{\frac{1}{r}}$

$\Rightarrow y = (A^{-1})^{\frac{1}{r}} x^{\frac{1}{r}}$

$\Rightarrow y = A^{-\frac{1}{r}} x^{\frac{1}{r}}$

$F^{-1}(x)$

SETTING EQUAL TO ONE ANOTHER, AS IN THE O.D.E

$\Rightarrow rAx^{r-1} = A^{-\frac{1}{r}} x^{\frac{1}{r}}$

$\Rightarrow \frac{x^{r-1}}{x^{\frac{1}{r}}} = \frac{A^{-\frac{1}{r}}}{rA}$

$\Rightarrow x^{r-1-\frac{1}{r}} = \frac{1}{r} A^{-1-\frac{1}{r}}$

NOW R.H.S IS A CONSTANT $\Rightarrow$ L.H.S MUST ALSO BE A CONSTANT

$\Rightarrow$ EXPONENT OF x MUST BE ZERO

$\Rightarrow r - 1 - \frac{1}{r} = 0$

BUT $r - 1 - \frac{1}{r} = 0$ CAN BE REARRANGED TO SEVERAL FORMS

LET $r = \phi$

$\phi = 1 + \frac{1}{\phi}$ $\quad \phi^2 - \phi - 1 = 0$

$\phi^2 = \phi + 1$

ETC

BUT AS THE LHS IS A CONSTANT, THE ONLY CONSTANT IT CAN BE IS "ONE" AND THUS THE R.H.S MUST ALSO BE "ONE"

$\Rightarrow \frac{1}{r} A^{-1-\frac{1}{r}} = 1$

$\Rightarrow \frac{1}{\phi} A^{-1-\frac{1}{\phi}} = 1$

$\Rightarrow A^{-(1+\frac{1}{\phi})} = \phi$

$\Rightarrow A^{-\phi} = \phi$

$\Rightarrow A^{\phi} = \frac{1}{\phi}$

$\Rightarrow A = \left(\frac{1}{\phi}\right)^{\frac{1}{\phi}}$ OR EQUIVALENT $(\phi^{-1})^{\frac{1}{\phi}} = \phi^{-\frac{1}{\phi}}$

$= \phi^{1-\phi}$

$= \phi^{1-(\phi^2-1)}$

$= \phi^{2-\phi^2}$

ETC

$\therefore F(x) = \phi^{1-\phi} x^{\phi}$

b) DIFFERENTIATING $F(x) = \phi^{1-\phi} x^{\phi}$

$F'(x) = \phi\, \phi^{1-\phi} x^{\phi-1} = \phi^{2-\phi} x^{\phi-1}$

WE MAY HAVE TO MANIPULATE THIS FURTHER AT A LATER STAGE

INVERTING $F(x)$

$\Rightarrow y = \phi^{1-\phi} x^{\phi}$

$\Rightarrow \frac{y}{\phi^{1-\phi}} = x^{\phi}$

$\Rightarrow \frac{(y)^{\frac{1}{\phi}}}{(\phi)^{\frac{1-\phi}{\phi}}} = (x^{\phi})^{\frac{1}{\phi}}$

$\Rightarrow x = \phi^{-\frac{1-\phi}{\phi}} y^{\frac{1}{\phi}}$

$\Rightarrow F^{-1}(x) = \phi^{\frac{\phi-1}{\phi}} x^{\frac{1}{\phi}}$

LOOKING AT THE POWERS OF x, STARTING WITH $F^{-1}(x)$

$\frac{1}{\phi} = \phi - 1 \quad \left(\text{SINCE } \phi = 1 + \frac{1}{\phi}\right)$

LOOKING AT THE CONSTANT, STARTING WITH THE EXPONENT AT $F^{-1}(x)$

$\frac{\phi-1}{\phi} = 1 - \frac{1}{\phi} = 1 - (\phi - 1) = 2 - \phi$

$\therefore \phi^{2-\phi} x^{\phi-1} = \phi^{\frac{\phi-1}{\phi}} x^{\frac{1}{\phi}}$

$\therefore F'(x) = F^{-1}(x)$