WORDED DIFFERENTIAL EQUATIONS

(by separation of variables)

DIFFERENTIAL EQUATIONS

IN CONTEXT WITHOUT MODELLING

Question 1 (**)

The gradient of a curve satisfies

$$\frac{dy}{dx} = \frac{1}{3y^2(x-1)}, \quad x > 1.$$

Given the curve passes though the point $P(2,-1)$ and the point $Q(q,1)$, determine the exact value of q.

$$\boxed{q = 1 + e^2}$$

$$\frac{dy}{dx} = \frac{1}{3y^2(x-1)}$$

$$3y^2\,dy = \frac{1}{x-1}\,dx$$

$$\int 3y^2\,dy = \int \frac{1}{x-1}\,dx$$

$$y^3 = \ln(x-1) + C$$

when $x=2$ $y=-1$

$$(-1)^3 = \cancel{\ln 1} + C$$

$$\boxed{C=-1}$$

$$\therefore \boxed{y^3 = -1 + \ln(x-1)}$$

when $y=1$

$$1^3 = -1 + \ln(x-1)$$

$$2 = \ln(x-1)$$

$$e^2 = x-1$$

$$x = 1 + e^2$$

Question 2 (**+)

Water is draining out of a tank so that the height of the water, h m, in time t minutes, satisfies the differential equation

$$\frac{dh}{dt} = -k\sqrt{h},$$

where k is a positive constant.

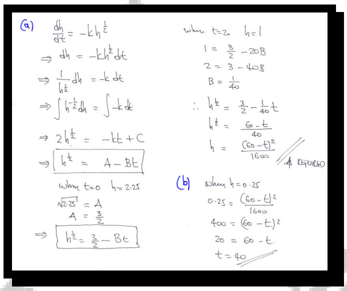
The initial height of the water is 2.25 m and 20 minutes later it drops to 1 m.

a) Show that the solution of the differential equation can be written as

$$h = \frac{(60-t)^2}{1600}.$$

b) Find after how long the height of the water drops to 0.25 m.

$\boxed{t = 40}$



(a) $\frac{dh}{dt} = -kh^{\frac{1}{2}}$

$\Rightarrow dh = -kh^{\frac{1}{2}}\,dt$

$\Rightarrow \frac{1}{h^{\frac{1}{2}}}\,dh = -k\,dt$

$\Rightarrow \int h^{-\frac{1}{2}}\,dh = \int -k\,dt$

$\Rightarrow 2h^{\frac{1}{2}} = -kt + C$

$\Rightarrow \boxed{h^{\frac{1}{2}} = A - Bt}$

when $t=0$ $h=2.25$

$\sqrt{2.25} = A$

$A = \frac{3}{2}$

$\Rightarrow \boxed{h^{\frac{1}{2}} = \frac{3}{2} - Bt}$

when $t=20$ $h=1$

$1 = \frac{3}{2} - 20B$

$2 = 3 - 40B$

$B = \frac{1}{40}$

$\therefore h^{\frac{1}{2}} = \frac{3}{2} - \frac{1}{40}t$

$h^{\frac{1}{2}} = \frac{60-t}{40}$

$h = \frac{(60-t)^2}{1600}$ // AS REQUIRED

(b) when $h=0.25$

$0.25 = \frac{(60-t)^2}{1600}$

$400 = (60-t)^2$

$20 = 60 - t$

$t = 40$

Question 3 (**+)

The radius, r mm, of a circular ink stain, t seconds after it was formed, satisfies the differential equation

$$\frac{dr}{dt} = \frac{k}{r}, \quad r \neq 0$$

where k is a positive constant.

The initial radius of the stain is 4 mm and 8 seconds later it has increased to 20 mm.

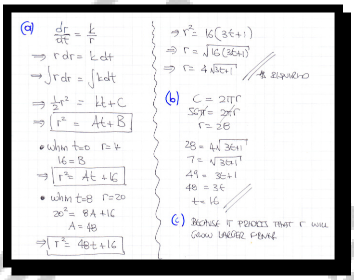
a) Solve the differential equation to show that

$$r = 4\sqrt{3t+1}.$$

b) Find the time when the stain will have a **circumference** of 56π.

c) Explain why this model is only likely to hold for small values of t.

$\boxed{t = 16}$



(a) $\frac{dr}{dt} = \frac{k}{r}$

$\Rightarrow r\,dr = k\,dt$

$\Rightarrow \int r\,dr = \int k\,dt$

$\Rightarrow \frac{1}{2}r^2 = kt + C$

$\Rightarrow \boxed{r^2 = At + B}$

• when $t=0$ $r=4$

$16 = B$

$\Rightarrow \boxed{r^2 = At + 16}$

• when $t=8$ $r=20$

$20^2 = 8A + 16$

$A = 48$

$\Rightarrow \boxed{r^2 = 48t + 16}$

$\Rightarrow r^2 = 16(3t+1)$

$\Rightarrow r = \sqrt{16(3t+1)}$

$\Rightarrow r = 4\sqrt{3t+1}$ // AS REQUIRED

(b) $C = 2\pi r$

$56\pi = 2\pi r$

$r = 28$

$28 = 4\sqrt{3t+1}$

$7 = \sqrt{3t+1}$

$49 = 3t+1$

$48 = 3t$

$t = 16$ //

(c) BECAUSE IT PREDICTS THAT r WILL GROW LARGER FOREVER

Question 4 **(**+)**

An entomologist believes that the population P insects in a colony, t weeks after it was first observed, obeys the differential equation

$$\frac{dP}{dt} = kP^2,$$

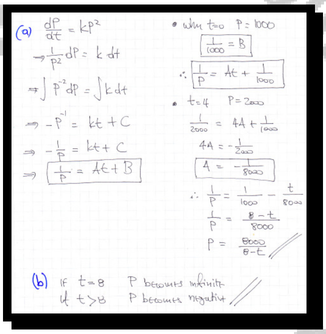
where k is a positive constant.

Initially 1000 insects were observed, and this population doubled after 4 weeks.

a) Find a solution of the differential equation, in the form $P = f(t)$.

b) Give <u>two different</u> reasons why the model can only work for small values of t.

$$\boxed{P = \frac{8000}{8-t}}$$



Question 5 (**+)

The area, A km^2, of an oil spillage on the surface of the sea, at time t hours after it was formed, satisfies the differential equation

$$\frac{dA}{dt} = \frac{A^{\frac{3}{2}}}{t^2}, \quad t > 0.$$

When $t = 1$, $A = 0.25$.

a) Find a solution of the differential equation, in the form $A = f(t)$.

b) Determine the largest area that the oil spillage will ever attain.

$$\boxed{A = \frac{4t^2}{(3t+1)^2}}, \quad \boxed{A_{\text{max}} \to \frac{4}{9}}$$

(a) $\frac{dA}{dt} = \frac{A^{\frac{3}{2}}}{t^2}$

$\Rightarrow \frac{1}{A^{\frac{3}{2}}}\,dA = \frac{1}{t^2}\,dt$

$\Rightarrow \int A^{-\frac{3}{2}}\,dA = \int t^{-2}\,dt$

$\Rightarrow -2A^{-\frac{1}{2}} = -t^{-1} + C$

$\Rightarrow A^{-\frac{1}{2}} = C + \frac{1}{2t}$

$\Rightarrow \boxed{\frac{1}{\sqrt{A}} = C + \frac{1}{2t}}$

when $t=1$ $A = 0.25$

$\frac{1}{\sqrt{0.25}} = C + \frac{1}{2}$

$\therefore C = \frac{3}{2}$

Hence

$\Rightarrow \frac{1}{\sqrt{A}} = \frac{3}{2} + \frac{1}{2t}$

$\Rightarrow \frac{1}{\sqrt{A}} = \frac{3t+1}{2t}$

$\Rightarrow \sqrt{A} = \frac{2t}{3t+1}$

$\Rightarrow A = \frac{4t^2}{(3t+1)^2}$

(b) $A = \frac{4t^2}{9t^2+6t+1}$

For very large t

$A \approx \frac{4t^2}{9t^2}$

$\therefore A \to \frac{4}{9}$

Question 6 (***)

The mass, m grams, of a burning candle, t hours after it was lit up, satisfies the differential equation

$$\frac{dm}{dt} = -k(m-10),$$

where k is a positive constant.

a) Solve the differential equation to show that

$$m = 10 + A\mathrm{e}^{-kt},$$

where A is a non-zero constant.

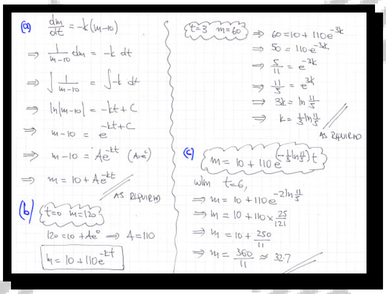
The initial mass of the candle was 120 grams, and 3 hours later its mass has halved.

b) Find the value of A and show further that

$$k = \frac{1}{3}\ln\left(\frac{11}{5}\right).$$

c) Calculate, correct to three significant figures, the mass of the candle after a further period of 3 hours has elapsed.

$\boxed{}$, $\boxed{A = 110}$, $\boxed{m \approx 32.7}$


(a) $\frac{dm}{dt} = -k(m-10)$

$\Rightarrow \frac{1}{m-10}\,dm = -k\,dt$

$\Rightarrow \int \frac{1}{m-10} = \int -k\,dt$

$\Rightarrow \ln|m-10| = -kt + C$

$\Rightarrow m - 10 = e^{-kt+C}$

$\Rightarrow m - 10 = Ae^{-kt}$ $(A = e^{C})$

$\Rightarrow m = 10 + Ae^{-kt}$ // AS REQUIRED

(b) $t=0$ $m=120$

$120 = 10 + Ae^{0} \Rightarrow A = 110$

$\boxed{m = 10 + 110e^{-kt}}$

$t=3$ $m=60$ $\Rightarrow 60 = 10 + 110e^{-3k}$

$\Rightarrow 50 = 110e^{-3k}$

$\Rightarrow \frac{5}{11} = e^{-3k}$

$\Rightarrow \frac{11}{5} = e^{3k}$

$\Rightarrow 3k = \ln\frac{11}{5}$

$\Rightarrow k = \frac{1}{3}\ln\frac{11}{5}$ // AS REQUIRED

(c) $m = 10 + 110e^{\left(-\frac{1}{3}\ln\frac{11}{5}\right)t}$

when $t=6$,

$\Rightarrow m = 10 + 110e^{-2\ln\frac{11}{5}}$

$\Rightarrow m = 10 + 110 \times \frac{25}{121}$

$\Rightarrow m = 10 + \frac{250}{11}$

$\Rightarrow m = \frac{360}{11} \approx 32.7$ //

Question 7 (***)

A radioactive isotope decays in such a way so that the number N of the radioactive nuclei present at time t days, satisfies the differential equation

$$\frac{dN}{dt} = -kN ,$$

where k is a positive constant.

a) Show clearly that

$$N = A\mathrm{e}^{-kt} ,$$

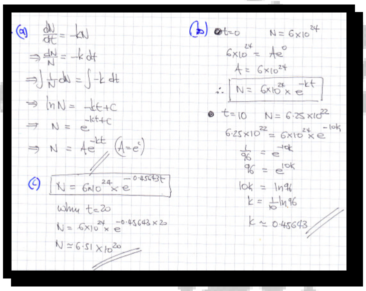
where A is a non zero constant.

Initially there were 6.00×10^{24} radioactive nuclei and 10 days later this number reduced to 6.25×10^{22}.

b) Show further that $k = 0.45643$, correct to five decimal places.

c) Calculate the number of the radioactive nuclei after a further period of 10 days has elapsed.

$\boxed{6.51\times10^{20}}$


(a) $\frac{dN}{dt} = -kN$

$\Rightarrow \frac{dN}{N} = -k\,dt$

$\Rightarrow \int \frac{1}{N}\,dN = \int -k\,dt$

$\Rightarrow \ln N = -kt + C$

$\Rightarrow N = e^{-kt+C}$

$\Rightarrow N = Ae^{-kt}$ $(A = e^{C})$

(b) • $t=0$ $N = 6\times10^{24}$

$6\times10^{24} = Ae^{0}$

$A = 6\times10^{24}$

$\therefore \boxed{N = 6\times10^{24}\times e^{-kt}}$

• $t=10$ $N = 6.25\times10^{22}$

$6.25\times10^{22} = 6\times10^{24}\times e^{-10k}$

$\frac{1}{96} = e^{-10k}$

$96 = e^{10k}$

$10k = \ln 96$

$k = \frac{1}{10}\ln 96$

$k \simeq 0.45643$

(c) $\boxed{N = 6\times10^{24}\times e^{-0.45643t}}$

when $t=20$

$N = 6\times10^{24}\times e^{-0.45643\times20}$

$N \simeq 6.51\times10^{20}$

Question 8 (***)

The number of fish x in a small lake at time t months after a certain instant, is modelled by the differential equation

$$\frac{dx}{dt} = x(1-kt),$$

where k is a positive constant.

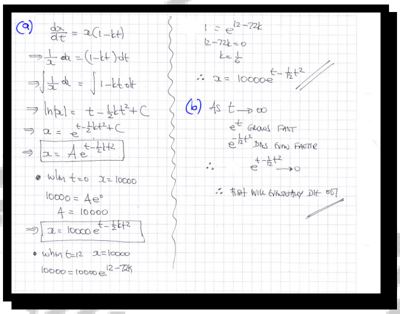
We may assume that x can be treated as a continuous variable.

It is estimated that there are 10000 fish in the lake when $t=0$ and 12 months later the number of fish returns back to 10000.

a) Find a solution of the differential equation, in the form $x = f(t)$.

b) Find the long term prospects for this population of fish.

$\boxed{x = 10000\mathrm{e}^{t-\frac{1}{12}t^2}}$, $\boxed{x \to 0}$



Question 9 (***)

The area, A km^2, of an oil spillage is growing in time t hours according the differential equation

$$\frac{dA}{dt} = \frac{4e^t}{\sqrt{A}}, \quad A > 0.$$

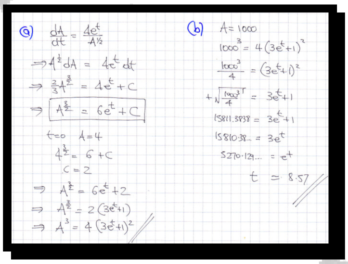
The initial area of the oil spillage was 4 km^2.

a) Solve the differential equation to show that

$$A^3 = 4\left(3e^t + 1\right)^2.$$

b) Find, to three significant figures, the value of t when the area of the spillage reaches 1000 km^2.

$\boxed{t \approx 8.57}$



Question 10 (***+)

A cylindrical tank of height 150 cm is full of oil which started leaking out from a small hole at the side of the tank.

Let h cm be the height of the oil still left in the tank, after leaking for t minutes, and assume the leaking can be modelled by the differential equation

$$\frac{dh}{dt} = -\frac{1}{4}(h-6)^{\frac{3}{2}}.$$

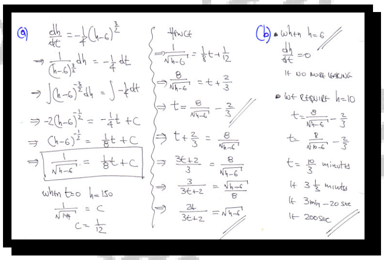
a) Solve the differential equation to show that …

i. … $t = \dfrac{8}{\sqrt{h-6}} - \dfrac{2}{3}$.

ii. … $\sqrt{h-6} = \dfrac{24}{3t+2}$.

b) State how high is the hole from the bottom of the tank and hence show further that it takes 200 seconds for the oil level to reach 4 cm above the level of the hole.

$\boxed{6\,\text{cm}}$



(a) $\frac{dh}{dt} = -\frac{1}{4}(h-6)^{\frac{3}{2}}$

$\Rightarrow \frac{1}{(h-6)^{\frac{3}{2}}}dh = -\frac{1}{4}dt$

$\Rightarrow \int (h-6)^{-\frac{3}{2}}dh = \int -\frac{1}{4}dt$

$\Rightarrow -2(h-6)^{-\frac{1}{2}} = -\frac{1}{4}t + C$

$\Rightarrow (h-6)^{-\frac{1}{2}} = \frac{1}{8}t + C$

$\Rightarrow \boxed{\frac{1}{\sqrt{h-6}} = \frac{1}{8}t + C}$

when $t=0$ $h=150$

$\frac{1}{\sqrt{144}} = C$

$C = \frac{1}{12}$

HENCE

$\Rightarrow \frac{1}{\sqrt{h-6}} = \frac{1}{8}t + \frac{1}{12}$

$\Rightarrow \frac{8}{\sqrt{h-6}} = t + \frac{2}{3}$

$\Rightarrow t = \frac{8}{\sqrt{h-6}} - \frac{2}{3}$

$\Rightarrow t + \frac{2}{3} = \frac{8}{\sqrt{h-6}}$

$\Rightarrow \frac{3t+2}{3} = \frac{8}{\sqrt{h-6}}$

$\Rightarrow \frac{3}{3t+2} = \frac{\sqrt{h-6}}{8}$

$\Rightarrow \frac{24}{3t+2} = \sqrt{h-6}$

(b) • when $h=6$

$\frac{dh}{dt} = 0$

ie NO MORE LEAKING

• WE REQUIRE $h=10$

$t = \frac{8}{\sqrt{h-6}} - \frac{2}{3}$

$t = \frac{8}{\sqrt{10-6}} - \frac{2}{3}$

$t = \frac{10}{3}$ minutes

ie $3\frac{1}{3}$ minutes

ie 3 min – 20 sec

ie 200 sec

Question 11 **(***+)**

Water is leaking out of a hole at the side of a tank.

Let the height of the water in the tank is y cm at time t minutes.

The rate at which the height of the water in the tank is decreasing is modelled by the differential equation

$$\frac{dy}{dt} = -6(y-7)^{\frac{2}{3}}.$$

When $t = 0$, $y = 132$.

a) Find how long it takes for the water level to drop from 132 cm to 34 cm.

The tank is filled up with water again to a height of 132 cm and allowed to leak out in exactly the same fashion as the one described in part **(a)**.

b) Determine how long it takes for the water to stop leaking.

$\boxed{}$, $\boxed{t = 1}$, $\boxed{t = 2.5}$

(a) $\frac{dy}{dt} = -6(y-7)^{\frac{2}{3}}$

$\Rightarrow \frac{1}{(y-7)^{\frac{2}{3}}}\,dy = -6\,dt$

$\Rightarrow \int (y-7)^{-\frac{2}{3}}\,dy = \int -6\,dt$

$\Rightarrow 3(y-7)^{\frac{1}{3}} = -6t + C$

$\boxed{(y-7)^{\frac{1}{3}} = A - 2t}$

when $t=0$ $y=132$

$(132-7)^{\frac{1}{3}} = A$

$\boxed{A = 5}$

$\boxed{(y-7)^{\frac{1}{3}} = 5 - 2t}$

when $y = 34$

$(34-7)^{\frac{1}{3}} = 5 - 2t$

$3 = 5 - 2t$

$2t = 2$

$t = 1$

(b) THIS OCCURS WHEN $\frac{dy}{dt} = 0$

(IE NO MORE CHANGE IN y)

HENCE $y = 7$

$0 = 5 - 2t$

$2t = 5$

$t = 2.5$

Question 12 (***+)

The speed, v ms^{-1}, of a skydiver falling through still air t seconds after jumping off a plane, can be modelled by the differential equation

$$8\frac{dv}{dt} = 80 - v.$$

The skydiver jumps off the plane with a downward speed of $5\ \text{ms}^{-1}$.

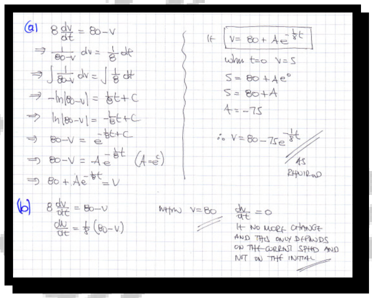
a) Solve the differential equation to show that

$$v = 80 - 75\text{e}^{-\frac{1}{8}t}.$$

b) Find the maximum possible speed that the skydiver can achieve and show that this speed is independent of the speed he jumps off the plane

You may assume that the skydiver cannot possible jump at a speed greater than his subsequent maximum speed.

$$\boxed{80\,\text{ms}^{-1}}$$


(a) $8\frac{dv}{dt} = 80 - v$

$\Rightarrow \frac{1}{80-v}\,dv = \frac{1}{8}\,dt$

$\Rightarrow \int \frac{1}{80-v}\,dv = \int \frac{1}{8}\,dt$

$\Rightarrow -\ln|80-v| = \frac{1}{8}t + C$

$\Rightarrow \ln|80-v| = -\frac{1}{8}t + C$

$\Rightarrow 80 - v = e^{-\frac{1}{8}t + C}$

$\Rightarrow 80 - v = Ae^{-\frac{1}{8}t}$ $(A = e^{C})$

$\Rightarrow 80 + Ae^{-\frac{1}{8}t} = v$

$\boxed{v = 80 + Ae^{-\frac{1}{8}t}}$

when $t=0$ $v=5$

$5 = 80 + Ae^{0}$

$5 = 80 + A$

$A = -75$

$\therefore v = 80 - 75e^{-\frac{1}{8}t}$ AS REQUIRED

(b) $8\frac{dv}{dt} = 80 - v$

$\frac{dv}{dt} = \frac{1}{8}(80 - v)$

when $v = 80$ $\frac{dv}{dt} = 0$

IT NO MORE CHANGE AND THIS ONLY DEPENDS ON THE CURRENT SPEED AND NOT ON THE INITIAL

Question 13 (***+)

A population P, in millions, at a given time t years, satisfies the differential equation

$$\frac{dP}{dt} = P(1-P).$$

Initially the population is one quarter of a million.

a) Solve the differential equation to show that

$$\frac{3P}{1-P} = e^{t}.$$

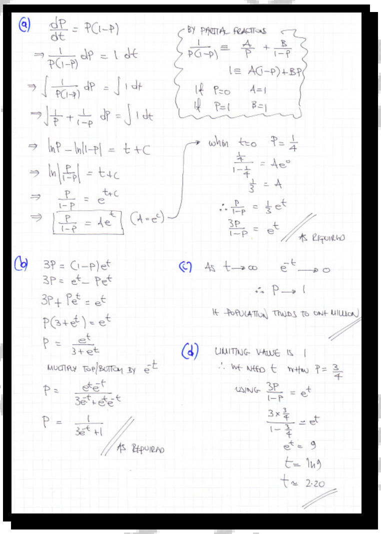
b) Show further that

$$P = \frac{1}{1+3e^{-t}}.$$

c) Show mathematically that the limiting value for this population is one million.

d) Find, to two decimal places, the time it takes for the population to reach three quarters of its limiting value.

$\boxed{}$, $\boxed{t \to \infty,\ \ P \to 1}$, $\boxed{t = \ln 9 \approx 2.20}$



Question 14 (***+)

The number of foxes N, in thousands, living within an urban area t years after a given instant, can be modelled by the differential equation

$$\frac{dN}{dt} = 2N - N^2, \quad t \geq 0.$$

Initially it is thought 1000 foxes lived within this urban area.

a) Find a solution of the differential equation, in the form $N = f(t)$.

b) Find the long term prospects of this population of foxes, as predicted by this model, clearly showing your reasoning.

$\boxed{}$, $\boxed{N = \dfrac{2}{1+e^{-2t}} \quad \text{or} \quad N = \dfrac{2e^{2t}}{e^{2t}+1}}$, $\boxed{\text{population} \to 2000}$

(a) $\frac{dN}{dt} = 2N - N^2$

$\Rightarrow dN = (2N - N^2)\,dt$

$\Rightarrow \frac{1}{2N-N^2}\,dN = 1\,dt$

$\Rightarrow \int \frac{1}{2N-N^2}\,dN = \int 1\,dt$

$\Rightarrow \int \frac{1}{N(2-N)}\,dN = \int 1\,dt$

• BY PARTIAL FRACTIONS

$\frac{1}{N(2-N)} \equiv \frac{A}{N} + \frac{B}{2-N}$

$\boxed{1 \equiv A(2-N) + BN}$

If $N=2 \Rightarrow 1 = 2B$, $B = \frac{1}{2}$

If $N=0 \Rightarrow 1 = 2A$, $A = \frac{1}{2}$

$\Rightarrow \int \frac{1/2}{N} + \frac{1/2}{2-N}\,dN = \int 1\,dt$

$\Rightarrow \int \frac{1}{N} + \frac{1}{2-N}\,dN = \int 2\,dt$

$\Rightarrow \ln N - \ln|2-N| = 2t + C$

$\Rightarrow \ln\left|\frac{N}{2-N}\right| = 2t + C$

$\Rightarrow \frac{N}{2-N} = e^{2t+C}$

$\Rightarrow \boxed{\frac{N}{2-N} = Ae^{2t}}$ $(A = e^C)$

$t=0 \quad N=1$ (POP IN THOUSANDS)

$\frac{1}{2-1} = Ae^0$

$A = 1$

$\therefore \frac{N}{2-N} = e^{2t}$

$\Rightarrow N = 2e^{2t} - Ne^{2t}$

$\Rightarrow N + Ne^{2t} = 2e^{2t}$

$\Rightarrow N(1+e^{2t}) = 2e^{2t}$

$\Rightarrow N = \frac{2e^{2t}}{1+e^{2t}}$

(b) As $t \to \infty$ $N \approx \frac{2e^{2t}}{e^{2t}} \approx 2$

ie POPULATION TENDS TO 2000

Question 15 (***+)

A machine is used to produce waves in the swimming pool of a water theme park.

Let x cm be the height of the wave produced above a certain level in the pool, and suppose it can be modelled by the differential equation

$$\frac{dx}{dt} = 2x\sin 2t \,, \quad t \geq 0 \,,$$

where t is the time in seconds.

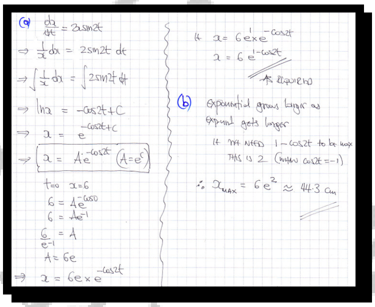
When $t = 0$, $x = 6$.

a) Solve the differential equation to show

$$x = 6\mathrm{e}^{1-\cos 2t} \,.$$

b) Find the maximum height of the wave.

$\boxed{}$, $\boxed{x_{\max} \approx 44.3 \text{ cm}}$



Question 16 (***+)

Food is placed in a preheated oven maintained at a constant temperature of $200\ ^\circ\text{C}$.

Let $\theta\ ^\circ\text{C}$ be temperature of the food t minutes after it was placed in the oven.

It is assumed that θ satisfies the differential equation

$$\frac{d\theta}{dt} = k\left(200-\theta\right),$$

where k is a positive constant.

a) Solve the differential equation to show that

$$\theta = 200 + A\,\mathrm{e}^{-kt},$$

where A is a non zero constant.

When a food item was placed in this oven it had a temperature of $20\ ^\circ\text{C}$ and 10 minutes later its temperature had risen to $120\ ^\circ\text{C}$.

b) Show further that $k \approx 0.0811$.

c) Find the value of t when the food item reaches a temperature of $160\ ^\circ\text{C}$.

$\boxed{}$, $\boxed{t \approx 18.55}$

(a) $\Rightarrow \frac{d\theta}{dt} = k(200-\theta)$

$\Rightarrow \frac{1}{200-\theta}\,d\theta = k\,dt$

$\Rightarrow \int \frac{1}{200-\theta}\,d\theta = \int k\,dt$

$\Rightarrow -\ln|200-\theta| = kt + C$

$\Rightarrow \ln|200-\theta| = -kt + C$

$\Rightarrow 200-\theta = e^{-kt+C}$

$\Rightarrow 200-\theta = e^{-kt} \times e^{C}$

$\Rightarrow 200-\theta = Be^{-kt}$ $(B=e^{C})$

$\Rightarrow 200 - Be^{-kt} = \theta$

$\Rightarrow \theta = 200 + Ae^{-kt}$ $(A=-B)$

(b) • $t=0$ $\theta=20$

$20 = 200 + Ae^{0}$

$20 = 200 + A$

$\boxed{A=-180}$ $\therefore \boxed{\theta = 200 - 180e^{-kt}}$

• $t=10$ $\theta=120$

$\Rightarrow 120 = 200 - 180e^{-10k}$

$\Rightarrow 180e^{-10k} = 80$

$\Rightarrow e^{-10k} = \frac{4}{9}$

$\Rightarrow e^{10k} = \frac{9}{4}$

$\Rightarrow 10k = \ln\frac{9}{4}$

$\Rightarrow k = \frac{1}{10}\ln\frac{9}{4} \approx 0.0811$

AS REQUIRED

(c) $\theta = 200 - 180e^{-0.0811t}$

$\Rightarrow 160 = 200 - 180e^{-0.0811t}$

$\Rightarrow 180e^{-0.0811t} = 40$

$\Rightarrow e^{-0.0811t} = \frac{2}{9}$

$\Rightarrow e^{0.0811t} = \frac{9}{2}$

$\Rightarrow 0.0811t = \ln\frac{9}{2}$

$\Rightarrow t = 18.55$

Question 17 (****)

Consider the following identity for t.

$$\frac{1}{t\left(t^2+1\right)} \equiv \frac{At+B}{t^2+1}+\frac{C}{t}.$$

a) Find the value of each of the constants A, B and C.

In a chemical reaction the mass, m grams, of the chemical produced at time t, in minutes, satisfies the differential equation

$$\frac{dm}{dt}=\frac{m}{t\left(t^2+1\right)}.$$

b) Find a general solution of the differential equation, in the form $m=f(t)$.

Two minutes after the reaction started the mass produced is 10 grams.

c) Calculate the mass which will be produced after a further period of 2 minutes.

d) Determine, in exact surd form, the maximum mass that will ever be produced by this chemical reaction.

$\boxed{}$, $\boxed{A=-1,\ B=0,\ C=1}$, $\boxed{m=\frac{kt}{\sqrt{t^2+1}}}$, $\boxed{\frac{20}{17}\sqrt{85}\approx 10.85}$, $\boxed{m_{\text{max}}=5\sqrt{5}}$

a) ADD THE R.H.S & COMPARE

$$\frac{1}{t(t^2+1)} \equiv \frac{At+B}{t^2+1}+\frac{C}{t}$$

$$\Rightarrow \frac{1}{t(t^2+1)} \equiv \frac{t(At+B)+C(t^2+1)}{(t^2+1)t}$$

$$\Rightarrow \boxed{1 \equiv At^2+Bt+Ct^2+C}$$

$C=1$ $B=0$ $A+C=0$, $A=-1$

b) SOLVING BY SEPARATION OF VARIABLES

$$\Rightarrow \frac{dm}{dt}=\frac{m}{t(t^2+1)}$$

$$\Rightarrow \frac{1}{m}\,dm = \frac{1}{t(t^2+1)}\,dt$$

$$\Rightarrow \int \frac{1}{m}\,dm = \int \frac{-t}{t^2+1}+\frac{1}{t}\,dt \qquad \times 2$$

$$\Rightarrow \int \frac{2}{m}\,dm = \int \frac{2}{t}-\frac{2t}{t^2+1}\,dt$$

$$\Rightarrow 2\ln m = 2\ln t - \ln(t^2+1) + \ln A$$

$$\frac{d}{dt}\left[\ln(t^2+1)\right]=\frac{1}{t^2+1}\times 2t$$

$$\Rightarrow \ln m^2 = \ln t^2 - \ln(t^2+1) + \ln A$$

$$\Rightarrow \ln m^2 = \ln\left(\frac{At^2}{t^2+1}\right)$$

$$\Rightarrow m^2 = \frac{At^2}{t^2+1}$$

$$\Rightarrow m = \frac{kt}{\sqrt{t^2+1}} \quad (m>0)$$

c) APPLY THE CONDITION $t=2$, $m=10$

$$\Rightarrow 10 = \frac{k\times 2}{\sqrt{5}}$$

$$\Rightarrow k = 5\sqrt{5}$$

$$\Rightarrow \boxed{m=\frac{5\sqrt{5}\,t}{\sqrt{t^2+1}}}$$

WHEN $t=4$

$$\Rightarrow m = \frac{5\sqrt{5}\times 4}{\sqrt{17}} = 20\sqrt{\frac{5}{17}} \approx 10.85$$

d) LOOKING AT THE SOLUTION

$$m=\frac{5\sqrt{5}\,t}{\sqrt{t^2+1}} \quad \text{AS } t\to\infty \quad \frac{t}{\sqrt{t^2+1}}\longrightarrow 1$$

$$m \longrightarrow 5\sqrt{5}$$

Question 18 (****)

The population of a herd of zebra, P thousands, in time t years is thought to be governed by the differential equation

$$\frac{dP}{dt} = \frac{1}{20}P(2P-1)\cos t \, .$$

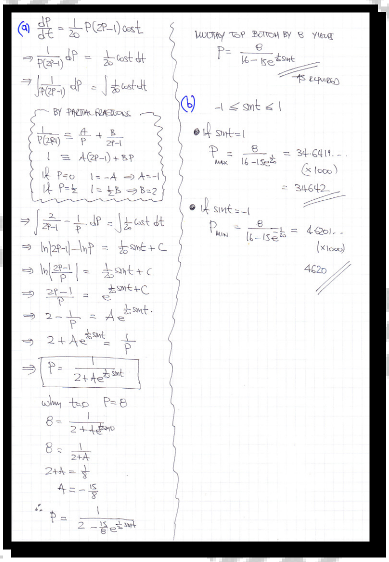
It is assumed that since P is large it can be modelled as a continuous variable, and its initial value is 8.

a) Solve the differential equation to show that

$$P = \frac{8}{16 - 15e^{\frac{1}{20}\sin t}} \, .$$

b) Find the maximum and minimum population of the herd.

$\boxed{}$, $\boxed{P_{max} = 34642}$, $\boxed{P_{min} = 4620}$



(a) $\frac{dP}{dt} = \frac{1}{20}P(2P-1)\cos t$

$\Rightarrow \frac{1}{P(2P-1)}\,dP = \frac{1}{20}\cos t\,dt$

$\Rightarrow \int \frac{1}{P(2P-1)}\,dP = \int \frac{1}{20}\cos t\,dt$

BY PARTIAL FRACTIONS

$\frac{1}{P(2P-1)} \equiv \frac{A}{P} + \frac{B}{2P-1}$

$1 \equiv A(2P-1) + BP$

If $P=0$ $\quad 1 = -A \Rightarrow A=-1$

If $P=\frac{1}{2}$ $\quad 1 = \frac{1}{2}B \Rightarrow B=2$

$\Rightarrow \int \frac{2}{2P-1} - \frac{1}{P}\,dP = \int \frac{1}{20}\cos t\,dt$

$\Rightarrow \ln|2P-1| - \ln P = \frac{1}{20}\sin t + C$

$\Rightarrow \ln\left|\frac{2P-1}{P}\right| = \frac{1}{20}\sin t + C$

$\Rightarrow \frac{2P-1}{P} = e^{\frac{1}{20}\sin t + C}$

$\Rightarrow 2 - \frac{1}{P} = Ae^{\frac{1}{20}\sin t}$

$\Rightarrow 2 + Ae^{\frac{1}{20}\sin t} = \frac{1}{P}$

$\Rightarrow \boxed{P = \frac{1}{2 + Ae^{\frac{1}{20}\sin t}}}$

when $t=0$ $\quad P=8$

$8 = \frac{1}{2 + Ae^{\frac{1}{20}\sin 0}}$

$8 = \frac{1}{2+A}$

$2 + A = \frac{1}{8}$

$A = -\frac{15}{8}$

$\therefore P = \frac{1}{2 - \frac{15}{8}e^{\frac{1}{20}\sin t}}$

MULTIPLY TOP BOTTOM BY 8 YIELDS

$P = \frac{8}{16 - 15e^{\frac{1}{20}\sin t}}$

AS REQUIRED

(b) $-1 \le \sin t \le 1$

- If $\sin t = 1$

$P_{MAX} = \frac{8}{16 - 15e^{\frac{1}{20}}} = 34.6419\ldots$ $(\times 1000)$

$= 34642$

- If $\sin t = -1$

$P_{MIN} = \frac{8}{16 - 15e^{-\frac{1}{20}}} = 4.6201\ldots$ $(\times 1000)$

4620

Question 19 **(****)**

A population p, in millions, is thought to obey the differential equation

$$\frac{dp}{dt} = kp\cos kt$$

where k is a positive constant, and t is measured in days from a certain instant.

When $t = 0$, $p = p_0$.

a) Solve the differential equation to find p in terms of p_0, k and t.

The value of k is now assumed to be 3.

b) Calculate, correct to the nearest minute, the time for the population to reach p_0 again, for the first time since $t = 0$.

$\boxed{p = p_0 e^{\sin kt}}$, $\boxed{1508 \text{ min}}$

a) SOLVING BY SEPARATING VARIABLES

$\Rightarrow \frac{dp}{dt} = kp\cos kt$

$\Rightarrow dp = kp\cos kt \, dt$

$\Rightarrow \frac{1}{p} \, dp = k\cos kt \, dt$

$\Rightarrow \int \frac{1}{p} \, dp = \int k\cos kt \, dt$

$\Rightarrow \ln p = \sin kt + C$

p = population (millions)
t = time (days)
$t=0$, $p=p_0$

TIDY BEFORE APPLYING THE GIVEN CONDITION

$\Rightarrow p = e^{\sin kt + C}$

$\Rightarrow p = e^{\sin kt} \times e^{C}$

$\Rightarrow p = Ae^{\sin kt}$ $\quad (A = e^C)$

$t=0 \quad p=p_0 \Rightarrow p_0 = Ae^0$
$\Rightarrow A = p_0$

$\Rightarrow p = p_0 e^{\sin kt}$

b) TAKING $k=3$, THE SOLUTION BECOMES

$\Rightarrow p = p_0 e^{\sin 3t}$

$\Rightarrow p_0 = p_0 e^{\sin 3t}$

$\Rightarrow 1 = e^{\sin 3t}$

$\Rightarrow \ln 1 = \sin 3t$

$\Rightarrow \sin 3t = 0$

$\Rightarrow 3t = \dots -3\pi, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, \dots$

$\Rightarrow 3t = \pi$

$\Rightarrow t = \frac{\pi}{3}$ days

$\times 24$

$\Rightarrow t = 8\pi$ hours

$\times 60$

$\Rightarrow t = 480\pi$ minutes ≈ 1508 minutes

Question 20 (****)

Cars are attached to a giant wheel on a fairground ride, and they can be made to lower or rise in height as the wheel is turning around.

Let the height above ground of one such car be h metres, and let t be the time in seconds, since the ride starts.

It may be assumed that h satisfies the differential equation

$$\frac{dh}{dt} = \frac{3}{2}\sqrt{h}\sin\left(\frac{3t}{4}\right).$$

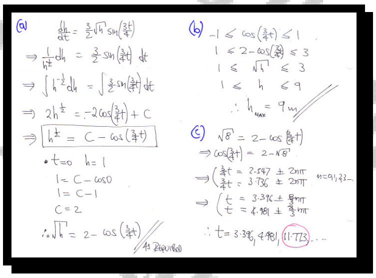
a) Solve the differential equation subject to the condition $t=0$, $h=1$, to show

$$\sqrt{h} = 2 - \cos\left(\frac{3t}{4}\right).$$

b) Find the greatest height of the car above ground.

c) Find the value of t when the car reaches a height of 8 metres above the ground <u>for the third time</u>, since the ride started.

$\boxed{h_{max} = 9}$, $\boxed{t \approx 11.77}$


(a)
$\frac{dh}{dt} = \frac{3}{2}\sqrt{h}\sin\left(\frac{3t}{4}\right)$
$\Rightarrow \frac{1}{h^{\frac{1}{2}}}\,dh = \frac{3}{2}\sin\left(\frac{3}{4}t\right)dt$
$\Rightarrow \int h^{-\frac{1}{2}}\,dh = \int \frac{3}{2}\sin\left(\frac{3}{4}t\right)dt$
$\Rightarrow 2h^{\frac{1}{2}} = -2\cos\left(\frac{3}{4}t\right) + C$
$\Rightarrow \boxed{h^{\frac{1}{2}} = C - \cos\left(\frac{3}{4}t\right)}$
• $t=0$ $h=1$
$1 = C - \cos 0$
$1 = C - 1$
$C = 2$
$\therefore \sqrt{h} = 2 - \cos\left(\frac{3}{4}t\right)$ // AS REQUIRED

(b)
$-1 \leq \cos\left(\frac{3}{4}t\right) \leq 1$
$1 \leq 2-\cos\left(\frac{3}{4}t\right) \leq 3$
$1 \leq \sqrt{h} \leq 3$
$1 \leq h \leq 9$
$\therefore h_{MAX} = 9\text{ m}$ //

(c)
$\sqrt{8} = 2 - \cos\left(\frac{3}{4}t\right)$
$\Rightarrow \cos\left(\frac{3}{4}t\right) = 2 - \sqrt{8}$
$\Rightarrow \begin{cases} \frac{3}{4}t = 2.547 \pm 2n\pi \\ \frac{3}{4}t = 3.736 \pm 2n\pi \end{cases}$ $n = 0,1,2,3\ldots$
$\Rightarrow \begin{cases} t = 3.396 \pm \frac{8}{3}n\pi \\ t = 4.981 \pm \frac{8}{3}n\pi \end{cases}$
$\therefore t = 3.396, 4.981, 11.773, \ldots$

Question 21 (****)

During a chemical reaction a compound is formed, whose mass m grams in time t minutes satisfies the differential equation

$$\frac{dm}{dt} = k(m-6)(m-3),$$

where k is a positive constant.

a) Solve the differential equation to show that

$$\frac{m-6}{m-3} = A\mathrm{e}^{3kt},$$

where A is a non zero constant.

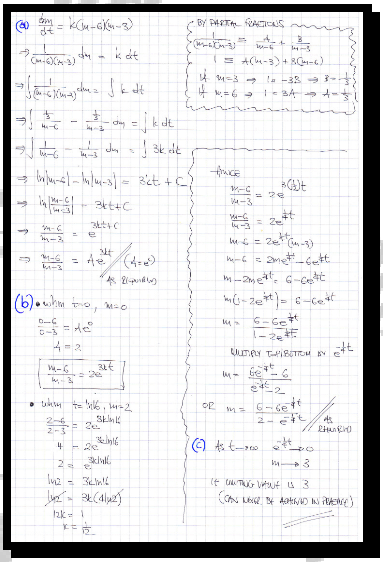
When the chemical reaction started there was no compound present, and when $t = \ln 16$ the mass of the compound has risen to 2 grams.

b) Show further that

$$m = \frac{6-6\mathrm{e}^{-\frac{1}{4}t}}{2-\mathrm{e}^{-\frac{1}{4}t}}.$$

c) Show that in practice, 3 grams of the compound can never be produced.

$\boxed{}$, $\boxed{\text{proof}}$


(a) $\frac{dm}{dt} = k(m-6)(m-3)$

$\Rightarrow \frac{1}{(m-6)(m-3)}\,dm = k\,dt$

$\Rightarrow \int \frac{1}{(m-6)(m-3)}\,dm = \int k\,dt$

$\Rightarrow \int \frac{\frac{1}{3}}{m-6} - \frac{\frac{1}{3}}{m-3}\,dm = \int k\,dt$

$\Rightarrow \int \frac{1}{m-6} - \frac{1}{m-3}\,dm = \int 3k\,dt$

$\Rightarrow \ln|m-6| - \ln|m-3| = 3kt + C$

$\Rightarrow \ln\left|\frac{m-6}{m-3}\right| = 3kt + C$

$\Rightarrow \frac{m-6}{m-3} = e^{3kt+C}$

$\Rightarrow \frac{m-6}{m-3} = Ae^{3kt}$ $(A = e^C)$ AS REQUIRED

BY PARTIAL FRACTIONS

$\frac{1}{(m-6)(m-3)} \equiv \frac{A}{m-6} + \frac{B}{m-3}$

$1 \equiv A(m-3) + B(m-6)$

If $m=3 \Rightarrow 1 = -3B \Rightarrow B = -\frac{1}{3}$

If $m=6 \Rightarrow 1 = 3A \Rightarrow A = \frac{1}{3}$

(b) • when $t=0$, $m=0$

$\frac{0-6}{0-3} = Ae^0$

$A = 2$

$\boxed{\frac{m-6}{m-3} = 2e^{3kt}}$

• when $t = \ln 16$, $m=2$

$\frac{2-6}{2-3} = 2e^{3k\ln 16}$

$4 = 2e^{3k\ln 16}$

$2 = e^{3k\ln 16}$

$\ln 2 = 3k\ln 16$

$\cancel{\ln 2} = 3k(4\cancel{\ln 2})$

$12k = 1$

$k = \frac{1}{12}$

HENCE

$\frac{m-6}{m-3} = 2e^{3(\frac{1}{12})t}$

$\frac{m-6}{m-3} = 2e^{\frac{1}{4}t}$

$m-6 = 2e^{\frac{1}{4}t}(m-3)$

$m-6 = 2me^{\frac{1}{4}t} - 6e^{\frac{1}{4}t}$

$m - 2me^{\frac{1}{4}t} = 6 - 6e^{\frac{1}{4}t}$

$m(1-2e^{\frac{1}{4}t}) = 6 - 6e^{\frac{1}{4}t}$

$m = \frac{6-6e^{\frac{1}{4}t}}{1-2e^{\frac{1}{4}t}}$

MULTIPLY TOP/BOTTOM BY $e^{-\frac{1}{4}t}$

$m = \frac{6e^{-\frac{1}{4}t}-6}{e^{-\frac{1}{4}t}-2}$

OR $m = \frac{6-6e^{-\frac{1}{4}t}}{2-e^{-\frac{1}{4}t}}$ AS REQUIRED

(c) AS $t \to \infty$ $e^{-\frac{1}{4}t} \to 0$

$m \to 3$

IE LIMITING VALUE IS 3

(CAN NEVER BE ACHIEVED IN PRACTICE)

Question 22 (****)

During a chemical reaction a compound is formed, whose mass y grams in time t minutes satisfies the differential equation

$$\frac{dy}{dt} = k(1-2y)(1-3y), \quad t \geq 0,$$

where k is a positive constant.

a) Solve the differential equation to show that

$$\ln\left|\frac{1-2y}{1-3y}\right| = kt + C,$$

where C is a constant.

When the chemical reaction started there was no compound present, and when $t = \ln 4$ the mass of the compound has risen to 0.25 grams.

b) Show further that

$$y = \frac{1-e^{-\frac{1}{2}t}}{3-2e^{-\frac{1}{2}t}}.$$

c) State, with justification, the limiting value of y as t gets large.

$\boxed{}$, $\boxed{t \to \infty,\ y \to \frac{1}{3}}$

a) SEPARATING THE VARIABLES

$$\Rightarrow \frac{dy}{dt} = k(1-2y)(1-3y)$$

$$\Rightarrow 1\,dy = k(1-2y)(1-3y)\,dt$$

$$\Rightarrow \frac{1}{(1-2y)(1-3y)}\,dy = k\,dt$$

BY PARTIAL FRACTIONS

$$\frac{1}{(1-2y)(1-3y)} \equiv \frac{A}{1-2y} + \frac{B}{1-3y}$$

$$\boxed{1 \equiv A(1-3y) + B(1-2y)}$$

- If $y = \frac{1}{2}$, $1 = -\frac{1}{2}A$, $\underline{A = -2}$
- If $y = \frac{1}{3}$, $1 = \frac{1}{3}B$, $\underline{B = 3}$

RETURNING TO THE O.D.E

$$\Rightarrow \int \frac{-2}{1-2y} + \frac{3}{1-3y}\,dy = \int k\,dt$$

$$\Rightarrow \ln|1-2y| - \ln|1-3y| = kt + C$$

$$\Rightarrow \ln\left|\frac{1-2y}{1-3y}\right| = kt + C$$

AS REQUIRED

b) WHEN $t=0$, $y=0$

$$\Rightarrow \cancel{\ln 1} = 0 + C$$

$$\Rightarrow \underline{C = 0}$$

$$\Rightarrow \ln\left|\frac{1-2y}{1-3y}\right| = kt \longrightarrow$$

WHEN $t = \ln 4$, $y = \frac{1}{4}$

$$\ln\left|\frac{1-\frac{1}{2}}{1-\frac{3}{4}}\right| = k\ln 4$$

$$\ln\left|\frac{1/2}{1/4}\right| = 2k\ln 2$$

$$\ln 2 = 2k\ln 2$$

$$1 = 2k$$

$$k = \frac{1}{2}$$

$$\Rightarrow \frac{1-2y}{1-3y} = e^{kt}$$

$$\Rightarrow 1-2y = (1-3y)e^{kt}$$

$$\Rightarrow 1-2y = e^{kt} - 3ye^{kt}$$

$$\Rightarrow 3ye^{kt} - 2y = e^{kt} - 1$$

$$\Rightarrow y(3e^{kt} - 2) = e^{kt} - 1$$

$$\Rightarrow \boxed{y = \frac{e^{kt}-1}{3e^{kt}-2}}$$

$$\Rightarrow y = \frac{e^{\frac{1}{2}t}-1}{3e^{\frac{1}{2}t}-2}$$

$$\Rightarrow y = \frac{e^{\frac{1}{2}t}e^{-\frac{1}{2}t} - 1 \times e^{-\frac{1}{2}t}}{3e^{\frac{1}{2}t}e^{-\frac{1}{2}t} - 2 \times e^{-\frac{1}{2}t}}$$

$$\Rightarrow y = \frac{1-e^{-\frac{1}{2}t}}{3-2e^{-\frac{1}{2}t}}$$

AS REQUIRED

c) AS $t \to +\infty$, $e^{-\frac{1}{2}t} \to 0$, SO $y \to \frac{1}{3}$

LIMITING VALUE OF $\frac{1}{3}$

Question 23 (****+)

During a chemical reaction a compound is formed, whose mass x grams, in time t minutes, satisfies the differential equation

$$\frac{dx}{dt} = k(4+x)(4-x)\mathrm{e}^{-t}, \quad t \geq 0,$$

where k is a positive constant.

When the chemical reaction started there was no compound present.

The limiting mass of the compound is 2 grams.

Find the value of t, when half the limiting mass of the compound has been produced.

$\boxed{\text{R}}$, $\boxed{t \approx 0.625}$

$\boxed{\frac{dx}{dt} = k(4+x)(4-x)e^{-t} \quad t=0 \quad x=0}$

SEPARATING VARIABLES

$$\Rightarrow \frac{1}{(4+x)(4-x)}\,dx = ke^{-t}\,dt$$

$$\Rightarrow \int_{x=0}^{x} \frac{1}{(4+x)(4-x)}\,dx = \int_{t=0}^{t} ke^{-t}\,dt$$

BY PARTIAL FRACTIONS IN THE LHS

$$\frac{1}{(4+x)(4-x)} \equiv \frac{A}{4+x} + \frac{B}{4-x}$$

$$1 \equiv A(4-x) + B(4+x)$$

- IF $x=4$: $1 = 8B$, $B = \frac{1}{8}$
- IF $x=-4$: $1 = 8A$, $A = \frac{1}{8}$

RETURNING TO THE O.D.E.

$$\Rightarrow \int_{x=0}^{x} \frac{\frac{1}{8}}{4+x} + \frac{\frac{1}{8}}{4-x}\,dx = \int_{t=0}^{t} ke^{-t}\,dt$$

$$\Rightarrow \int_{x=0}^{x} \frac{1}{4+x} + \frac{1}{4-x}\,dx = \int_{t=0}^{t} 8ke^{-t}$$

$$\Rightarrow \Big[\ln|4+x| - \ln|4-x|\Big]_0^x = \Big[-8ke^{-t}\Big]_0^t$$

$$\Rightarrow \left[\ln\left|\frac{4+x}{4-x}\right|\right]_0^x = \Big[8ke^{-t}\Big]_t^0$$

$$\Rightarrow \ln\left|\frac{4+x}{4-x}\right| - \cancel{\ln 1} = 8k - 8ke^{-t}$$

$$\Rightarrow \ln\left|\frac{4+x}{4-x}\right| = 8k(1-e^{-t})$$

NOW AS $t \to +\infty$ $\quad x \to 2$

$$\Rightarrow \ln\left(\frac{4+2}{4-2}\right) = 8k(1-0)$$

$$\Rightarrow \ln 3 = 8k$$

$$\Rightarrow \left(k = \tfrac{1}{8}\ln 3\right)$$

FINALLY WE HAVE

$$\Rightarrow \ln\left|\frac{4+x}{4-x}\right| = (\ln 3)(1-e^{-t})$$

$$\Rightarrow \ln\left|\frac{4+1}{4-1}\right| = (\ln 3)(1-e^{-t})$$

$$\Rightarrow \ln\frac{5}{3} = \ln 3\,(1-e^{-t})$$

$$\Rightarrow \frac{\ln 5/3}{\ln 3} = 1 - e^{-t}$$

$$\Rightarrow e^{-t} = 1 - \frac{\ln 5/3}{\ln 3}$$

$$\Rightarrow e^{-t} = 0.535026\ldots$$

$$\Rightarrow -t = \ln(0.535026\ldots)$$

$$t = 0.625$$

Question 24 (****+)

The equation of motion of a small raindrop falling freely in still air, released from rest, is given by

$$m\frac{dv}{dt} = mg - kv,$$

where m kg is the mass of the raindrop, v ms^{-1} is the speed of the raindrop t seconds after release, and g and k are positive constants.

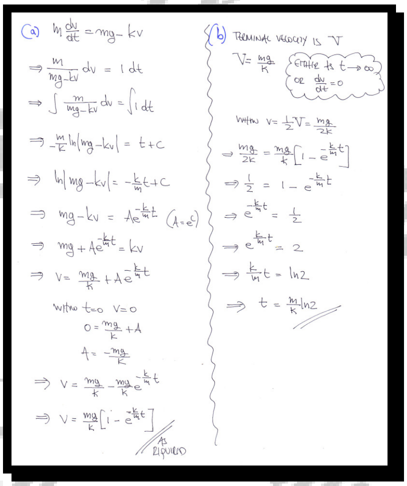
a) Solve the differential equation to show that

$$v = \frac{mg}{k}\left(1 - e^{-\frac{k}{m}t}\right).$$

The raindrop has a limiting speed V. (It is known as terminal velocity).

b) Show that the raindrop reaches a speed of $\frac{1}{2}V$ in time $\frac{m}{k}\ln 2$ seconds.

$\boxed{\text{proof}}$


(a) $m\frac{dv}{dt} = mg - kv$

$\Rightarrow \frac{m}{mg-kv}\,dv = 1\,dt$

$\Rightarrow \int \frac{m}{mg-kv}\,dv = \int 1\,dt$

$\Rightarrow -\frac{m}{k}\ln|mg-kv| = t + C$

$\Rightarrow \ln|mg-kv| = -\frac{k}{m}t + C$

$\Rightarrow mg - kv = Ae^{-\frac{k}{m}t}$ $(A = e^C)$

$\Rightarrow mg + Ae^{-\frac{k}{m}t} = kv$

$\Rightarrow v = \frac{mg}{k} + Ae^{-\frac{k}{m}t}$

when $t=0$ $v=0$

$0 = \frac{mg}{k} + A$

$A = -\frac{mg}{k}$

$\Rightarrow v = \frac{mg}{k} - \frac{mg}{k}e^{-\frac{k}{m}t}$

$\Rightarrow v = \frac{mg}{k}\left[1 - e^{-\frac{k}{m}t}\right]$

AS REQUIRED

(b) TERMINAL VELOCITY IS V

$V = \frac{mg}{k}$ EITHER AS $t \to \infty$ OR $\frac{dv}{dt} = 0$

when $v = \frac{1}{2}V = \frac{mg}{2k}$

$\Rightarrow \frac{mg}{2k} = \frac{mg}{k}\left[1 - e^{-\frac{k}{m}t}\right]$

$\Rightarrow \frac{1}{2} = 1 - e^{-\frac{k}{m}t}$

$\Rightarrow e^{-\frac{k}{m}t} = \frac{1}{2}$

$\Rightarrow e^{\frac{k}{m}t} = 2$

$\Rightarrow \frac{k}{m}t = \ln 2$

$\Rightarrow t = \frac{m}{k}\ln 2$

Question 25 (*****)

The population P of a colony of birds, in thousands, is assumed to vary according to the differential equation

$$\frac{dP}{dt} = P\mathrm{e}^{-0.5t}, \quad P > 0, \; t \geq 0,$$

where t is the time in years.

It is further assumed that P is large enough to be treated as a continuous variable.

Solve the differential equation to show that P will reach half its limiting value when

$$t = 2\ln\left(\frac{2}{\ln 2}\right).$$

$\boxed{}$, $\boxed{\text{proof}}$

$$\boxed{\frac{dP}{dt} = Pe^{-0.5t}, \quad t \geq 0, \quad P > 0}$$

SOLVING BY SEPARATING VARIABLES

$$\Rightarrow \frac{1}{P}\,dP = e^{-0.5t}\,dt$$

$$\Rightarrow \int \frac{1}{P}\,dP = \int e^{-0.5t}\,dt$$

$$\Rightarrow \ln P = -2e^{-0.5t} + C$$

$$\Rightarrow P = e^{-2e^{-0.5t} + C}$$

$$\Rightarrow P = e^{-2e^{-0.5t}} \times e^{C}$$

$$\Rightarrow P = Ae^{-2e^{-0.5t}} \qquad (A = e^{C})$$

NOW LET THE INITIAL POPULATION BE P_0, $P_0 > 0$

$$\Rightarrow P_0 = Ae^{-2}$$

$$\Rightarrow P_0e^{2}$$

HENCE WE HAVE

$$\Rightarrow P = P_0e^{2} \times e^{-2e^{-0.5t}} = P_0e^{2-2e^{-0.5t}}$$

THE LIMITING VALUE OF THIS POPULATION IS P_0e^2

SINCE AS $t \to \infty$ $e^{-0.5t} \to 0$

$$P \to P_0e^{2-2\times 0}$$

$$P \to P_0e^{2}$$

NOW WHEN $P = \frac{1}{2}P_0e^2$

$$\Rightarrow \frac{1}{2}P_0e^2 = P_0e^{2-2e^{-0.5t}}$$

$$\Rightarrow \frac{1}{2}\cancel{P_0e^2} = \cancel{P_0e^2} \times e^{-2e^{-0.5t}}$$

$$\Rightarrow 2 = e^{2e^{-0.5t}}$$

$$\Rightarrow \ln 2 = 2e^{-0.5t}$$

$$\Rightarrow \frac{\ln 2}{2} = e^{-0.5t}$$

$$\Rightarrow \frac{2}{\ln 2} = e^{0.5t}$$

$$\Rightarrow \ln\left(\frac{2}{\ln 2}\right) = 0.5t$$

$$\Rightarrow t = 2\ln\left(\frac{2}{\ln 2}\right)$$

AS REQUIRED

Question 26 (*****)

Water is leaking from a hole at the **side** of a water tank.

The tank has a height of 3 m and is initially full. It is thought that while the tank is leaking, the height, H m, of the water in the tank at time t hours, is governed by the differential equation

$$\frac{dH}{dt} = -k\,\mathrm{e}^{-0.1t},$$

where k is a positive constant.

The height of the water drops to 2 metres after 10 hours.

Find in **exact simplified** form ...

a) ... an expression for H in terms of t.

b) ... the height of the hole from the ground.

$\boxed{}$, $\boxed{H = \frac{2\mathrm{e}-3}{\mathrm{e}-1} + \frac{\mathrm{e}}{\mathrm{e}-1}\mathrm{e}^{-0.1t} \quad \text{or} \quad H = \frac{\mathrm{e}^{1-0.1t} + 2\mathrm{e} - 3}{\mathrm{e}-1}}$, $\boxed{\frac{2\mathrm{e}-3}{\mathrm{e}-1}}$

SOLVING THE O.D.E BY SEPARATION OF VARIABLES

$\Rightarrow \frac{dH}{dt} = -ke^{-0.1t}$

$\Rightarrow 1\,dH = -ke^{-0.1t}$

$\Rightarrow \int 1\,dH = \int -ke^{-0.1t}\,dt$

$\Rightarrow H = 10ke^{-0.1t} + C$

$\Rightarrow H = Ae^{-0.1t} + B$

WHEN $t=0$, $H=3$

$\Rightarrow 3 = A + B$

WHEN $t=10$, $H=2$

$\Rightarrow 2 = Ae^{-1} + B$

SOLVING TO FIND A & B

$\Rightarrow 1 = A - Ae^{-1}$ $\times e$

$\Rightarrow e = Ae - A$

$\Rightarrow e = A(e-1)$

$\Rightarrow A = \frac{e}{e-1}$

$\Rightarrow B = 3 - A = 3 - \frac{e}{e-1}$

$\Rightarrow B = \frac{3e-3-e}{e-1}$

$\Rightarrow B = \frac{2e-3}{e-1}$

HENCE THE SOLUTION OF THE O.D.E BECOMES

$H = \frac{e}{e-1}e^{-0.1t} + \frac{2e-3}{e-1}$

$H = \frac{e^{1-0.1t} + 2e - 3}{e-1}$ //

FINALLY WE HAVE

$\lim_{t \to \infty} H = \frac{2e-3}{e-1}$ // (AS $e^{-0.1t} \to 0$)

Question 27 (*****)

The equation of motion of a small raindrop falling freely in still air, released from rest, is given by

$$mv\frac{dv}{dx} = mg - kv,$$

where m kg is the mass of the raindrop, v ms^{-1} is the speed of the raindrop x metres below the point of release, and g and k are positive constants.

a) Solve the differential equation to show that

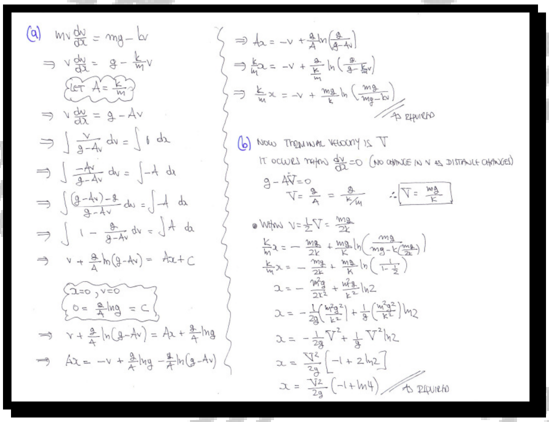
$$\frac{k}{m}x = \frac{mg}{k}\ln\left(\frac{mg}{mg-kv}\right) - v.$$

The raindrop has a limiting speed V.

(*It is known as terminal velocity*).

b) Show clearly that the raindrop reaches a speed of $\frac{1}{2}V$, after a falling through a distance of $\frac{V^2}{2g}(-1+\ln 4)$ metres.

proof



(a) $mv\frac{dv}{dx} = mg - kv$

$\Rightarrow v\frac{dv}{dx} = g - \frac{k}{m}v$

LET $A = \frac{k}{m}$

$\Rightarrow v\frac{dv}{dx} = g - Av$

$\Rightarrow \int \frac{v}{g-Av}\,dv = \int 1\,dx$

$\Rightarrow \int \frac{-Av}{g-Av}\,dv = \int -A\,dx$

$\Rightarrow \int \frac{(g-Av)-g}{g-Av}\,dv = \int -A\,dx$

$\Rightarrow \int 1 - \frac{g}{g-Av}\,dv = \int A\,dx$

$\Rightarrow v + \frac{g}{A}\ln(g-Av) = Ax + C$

$x=0,\ v=0$

$0 = \frac{g}{A}\ln g = C$

$\Rightarrow v + \frac{g}{A}\ln(g-Av) = Ax + \frac{g}{A}\ln g$

$\Rightarrow Ax = -v + \frac{g}{A}\ln g - \frac{g}{A}\ln(g-Av)$

$\Rightarrow Ax = -v + \frac{g}{A}\ln\left(\frac{g}{g-Av}\right)$

$\Rightarrow \frac{k}{m}x = -v + \frac{g}{\frac{k}{m}}\ln\left(\frac{g}{g-\frac{k}{m}v}\right)$

$\Rightarrow \frac{k}{m}x = -v + \frac{mg}{k}\ln\left(\frac{mg}{mg-kv}\right)$ // AS REQUIRED

(b) NOW TERMINAL VELOCITY IS V

IT OCCURS WHEN $\frac{dv}{dx}=0$ (NO CHANGE IN V AS DISTANCE CHANGES)

$g - AV = 0$

$V = \frac{g}{A} = \frac{g}{k/m}$ $\therefore \boxed{V = \frac{mg}{k}}$

• WHEN $v = \frac{1}{2}V = \frac{mg}{2k}$

$\frac{k}{m}x = -\frac{mg}{2k} + \frac{mg}{k}\ln\left(\frac{mg}{mg-k\left(\frac{mg}{2k}\right)}\right)$

$\frac{k}{m}x = -\frac{mg}{2k} + \frac{mg}{k}\ln\left(\frac{1}{1-\frac{1}{2}}\right)$

$x = -\frac{m^2g}{2k^2} + \frac{m^2g}{k^2}\ln 2$

$x = -\frac{1}{2g}\left(\frac{m^2g^2}{k^2}\right) + \frac{1}{g}\left(\frac{m^2g^2}{k^2}\right)\ln 2$

$x = -\frac{1}{2g}V^2 + \frac{1}{g}V^2\ln 2$

$x = \frac{V^2}{2g}\left[-1 + 2\ln 2\right]$

$x = \frac{V^2}{2g}(-1+\ln 4)$ // AS REQUIRED

Question 28 (*****)

The equation of motion of a small raindrop falling freely in still air, released from rest, is given by

$$m\frac{dv}{dt} = mg - kv^2,$$

where m kg is the mass of the raindrop, v ms^{-1} is the speed of the raindrop t seconds after release, and g and k are positive constants.

a) Solve the differential equation to show that

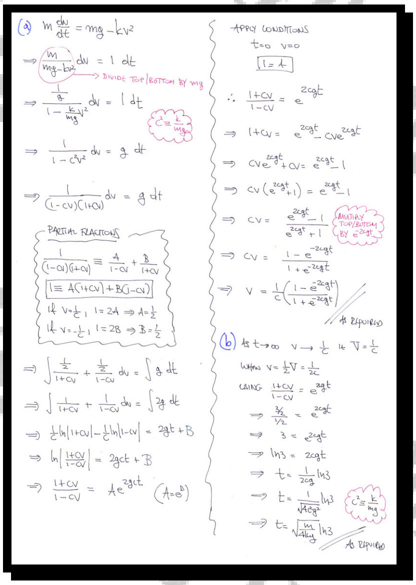
$$v = \frac{1}{c}\left(\frac{1-e^{-2cgt}}{1+e^{-2cgt}}\right), \text{ where } c^2 = \frac{k}{mg}.$$

The raindrop has a limiting speed V.

(*It is known as terminal velocity*).

b) Show that the raindrop reaches a speed of $\frac{1}{2}V$ in time $\sqrt{\frac{m}{4kg}}\ln 3$ seconds.

proof



(a) $m\frac{dv}{dt} = mg - kv^2$

$\Rightarrow \frac{m}{mg-kv^2}\,dv = 1\,dt$ → DIVIDE TOP/BOTTOM BY mg

$\Rightarrow \frac{\frac{1}{g}}{1-\frac{k}{mg}v^2}\,dv = 1\,dt$ $\quad c^2 \equiv \frac{k}{mg}$

$\Rightarrow \frac{1}{1-c^2v^2}\,dv = g\,dt$

$\Rightarrow \frac{1}{(1-cv)(1+cv)}\,dv = g\,dt$

PARTIAL FRACTIONS

$\frac{1}{(1-cv)(1+cv)} \equiv \frac{A}{1-cv} + \frac{B}{1+cv}$

$\boxed{1 \equiv A(1+cv) + B(1-cv)}$

If $v = \frac{1}{c}$, $1 = 2A \Rightarrow A = \frac{1}{2}$

If $v = -\frac{1}{c}$, $1 = 2B \Rightarrow B = \frac{1}{2}$

$\Rightarrow \int \frac{\frac{1}{2}}{1+cv} + \frac{\frac{1}{2}}{1-cv}\,dv = \int g\,dt$

$\Rightarrow \int \frac{1}{1+cv} + \frac{1}{1-cv}\,dv = \int 2g\,dt$

$\Rightarrow \frac{1}{c}\ln|1+cv| - \frac{1}{c}\ln|1-cv| = 2gt + B$

$\Rightarrow \ln\left|\frac{1+cv}{1-cv}\right| = 2gct + B$

$\Rightarrow \frac{1+cv}{1-cv} = Ae^{2gct} \quad (A = e^B)$

APPLY CONDITIONS

$t=0 \quad v=0$

$\boxed{1 = A}$

$\therefore \frac{1+cv}{1-cv} = e^{2cgt}$

$\Rightarrow 1 + cv = e^{2cgt} - cve^{2cgt}$

$\Rightarrow cve^{2cgt} + cv = e^{2cgt} - 1$

$\Rightarrow cv(e^{2cgt} + 1) = e^{2cgt} - 1$

$\Rightarrow cv = \frac{e^{2cgt} - 1}{e^{2cgt} + 1}$ MULTIPLY TOP/BOTTOM BY e^{-2cgt}

$\Rightarrow cv = \frac{1 - e^{-2cgt}}{1 + e^{-2cgt}}$

$\Rightarrow v = \frac{1}{c}\left(\frac{1 - e^{-2cgt}}{1 + e^{-2cgt}}\right)$

AS REQUIRED

(b) As $t \to \infty \quad v \to \frac{1}{c} \quad$ i.e. $V = \frac{1}{c}$

WHEN $v = \frac{1}{2}V = \frac{1}{2c}$

USING $\frac{1+cv}{1-cv} = e^{2cgt}$

$\Rightarrow \frac{3/2}{1/2} = e^{2cgt}$

$\Rightarrow 3 = e^{2cgt}$

$\Rightarrow \ln 3 = 2cgt$

$\Rightarrow t = \frac{1}{2cg}\ln 3$

$\Rightarrow t = \frac{1}{\sqrt{4c^2g^2}}\ln 3 \quad c^2 \equiv \frac{k}{mg}$

$\Rightarrow t = \sqrt{\frac{m}{4kg}}\ln 3$

AS REQUIRED

Question 29 (*****)

The equation of motion of a small raindrop falling freely in still air, released from rest, is given by

$$mv\frac{dv}{dx} = mg - kv^2,$$

where m kg is the mass of the raindrop, v ms^{-1} is the speed of the raindrop x metres below the point of release, and g and k are positive constants.

a) Solve the differential equation to show that

$$v^2 = c^2\left(1 - e^{-\frac{2k}{m}x}\right), \text{ where } c^2 = \frac{mg}{k}.$$

The raindrop has a limiting speed V.
(*It is known as terminal velocity*).

b) Show that the raindrop reaches a speed of $\frac{1}{2}V$, after covering a distance of $\frac{V^2}{2g}\ln\left(\frac{4}{3}\right)$ metres.

proof

(a) $mv\frac{dv}{dx} = mg - kv^2$

$\Rightarrow \frac{mv}{mg-kv^2}\,dv = 1\,dx$

$\Rightarrow \frac{v}{mg-kv^2}\,dv = \frac{1}{m}\,dx$

$\Rightarrow \frac{-2kv}{mg-kv^2}\,dv = \frac{-2k}{m}\,dx$

$\Rightarrow \int \frac{-2kv}{mg-kv^2}\,dv = \int -\frac{2k}{m}\,dx$

$\Rightarrow \ln\left|mg-kv^2\right| = -\frac{2k}{m}x + B$

$\Rightarrow mg - kv^2 = Ae^{-\frac{2k}{m}x} \quad (A = e^B)$

$\Rightarrow kv^2 = mg + Ae^{-\frac{2k}{m}x}$

$\Rightarrow v^2 = \frac{mg}{k} + Ae^{-\frac{2k}{m}x}$

$\Rightarrow \boxed{v^2 = c^2 + Ae^{-\frac{2k}{m}x}}$

APPLY CONDITION

$x=0 \quad v=0$

$0 = c^2 + A$

$A = -c^2$

$\Rightarrow v^2 = c^2 - c^2e^{-\frac{2k}{m}x}$

$\Rightarrow v^2 = c^2\left(1 - e^{-\frac{2k}{m}x}\right)$

AS REQUIRED

(b) As $x \to \infty$

$v \to c$

$\therefore V = c$

when $v = \frac{1}{2}V = \frac{1}{2}c$

$\Rightarrow \left(\frac{1}{2}c\right)^2 = c^2\left(1 - e^{-\frac{2k}{m}x}\right)$

$\Rightarrow \frac{1}{4} = 1 - e^{-\frac{2k}{m}x}$

$\Rightarrow e^{-\frac{2k}{m}x} = \frac{3}{4}$

$\Rightarrow e^{\frac{2k}{m}x} = \frac{4}{3}$

$\Rightarrow \frac{2k}{m}x = \ln\frac{4}{3}$

$\Rightarrow x = \frac{m}{2k}\ln\frac{4}{3}$

$\Rightarrow x = \frac{mg}{k} \times \frac{1}{2g}\ln\frac{4}{3}$

$\Rightarrow x = c^2 \times \frac{1}{2g}\ln\frac{4}{3}$

$\Rightarrow x = \frac{V^2}{2g}\ln\frac{4}{3}$

AS REQUIRED

DIFFERENTIAL EQUATIONS

IN CONTEXT WITH MODELLING

Question 1 (**)

The number of bacterial cells N on a laboratory dish is increasing, so that the <u>hourly</u> rate of increase is 5 times the number of the bacteria present at that time.

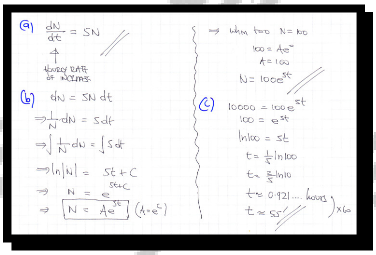
Initially 100 bacteria were placed on the dish.

a) Form a suitable differential equation to model this problem.

b) Find the solution of this differential equation.

c) Find to the nearest minute, the time taken for the bacteria to reach 10000.

$$\boxed{\frac{dN}{dt} = 5N}, \quad \boxed{N = 100e^{5t}}, \quad \boxed{55 \text{ minutes}}$$



(a) $\frac{dN}{dt} = 5N$

HOURLY RATE OF INCREASE

(b) $dN = 5N\,dt$

$\Rightarrow \frac{1}{N}\,dN = 5\,dt$

$\Rightarrow \int \frac{1}{N}\,dN = \int 5\,dt$

$\Rightarrow \ln|N| = 5t + C$

$\Rightarrow N = e^{5t+C}$

$\Rightarrow \boxed{N = Ae^{5t}}$ $(A = e^{C})$

$\Rightarrow$ when $t=0$ $N=100$

$100 = Ae^{0}$

$A = 100$

$N = 100e^{5t}$

(c) $10000 = 100e^{5t}$

$100 = e^{5t}$

$\ln 100 = 5t$

$t = \frac{1}{5}\ln 100$

$t = \frac{2}{5}\ln 10$

$t \approx 0.921\ldots$ hours

$t \approx 55'$ ×60

Question 2 **(**)**

The gradient at any point (x, y) on a curve $y = f(x)$ is proportional to the square root of the y coordinate of that point.

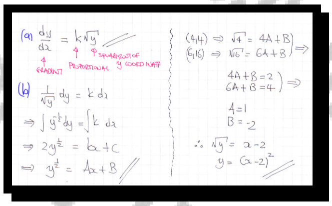
a) Form a suitable differential equation to model this problem.

b) Find a general solution of this differential equation, in terms of suitable constants.

The curve passes though the points $P(4,4)$ and $Q(6,16)$.

c) Find a solution to the differential equation in the form $y = f(x)$.

$\boxed{\frac{dy}{dx} = k\sqrt{y}}$, $\boxed{\sqrt{y} = Ax + B}$, $\boxed{y = (x-2)^2}$



(a) $\frac{dy}{dx} = k\sqrt{y}$

GRADIENT PROPORTIONAL SQUAREROOT OF y COORDINATE

(b) $\frac{1}{\sqrt{y}}\,dy = k\,dx$

$\Rightarrow \int y^{-\frac{1}{2}}\,dy = \int k\,dx$

$\Rightarrow 2y^{\frac{1}{2}} = kx + C$

$\Rightarrow y^{\frac{1}{2}} = Ax + B$

$(4,4) \Rightarrow \sqrt{4} = 4A + B$

$(6,16) \Rightarrow \sqrt{16} = 6A + B$

$4A + B = 2$

$6A + B = 4$

$A = 1$

$B = -2$

$\therefore \sqrt{y} = x - 2$

$y = (x-2)^2$

Question 3 (**)

A certain brand of car is valued at £V at time t years from new.

A model for the value of the car assumes that the rate of decrease of its value is proportional to its value at that time.

a) By forming and solving a suitable differential equation, show that

$$V = A\mathrm{e}^{-kt},$$

where A and k are positive constants.

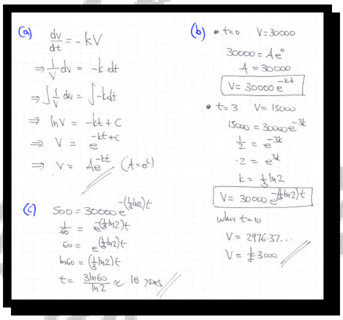
The value of one such car when new is £30000 and this value halves after 3 years.

b) Find, to the nearest £100, the value of one such car after 10 years.

One such car is to be scrapped when its value drops below £500.

c) Find after how many years this car is to be scrapped.

$\boxed{£3000}$, $\boxed{t \approx 17.7 \approx 18}$



(a) $\frac{dV}{dt} = -kV$

$\Rightarrow \frac{1}{V}\,dV = -k\,dt$

$\Rightarrow \int \frac{1}{V}\,dV = \int -k\,dt$

$\Rightarrow \ln V = -kt + C$

$\Rightarrow V = e^{-kt+C}$

$\Rightarrow V = Ae^{-kt}$ $(A = e^C)$

(b) • $t=0$ $V=30000$

$30000 = Ae^0$

$A = 30000$

$\boxed{V = 30000e^{-kt}}$

• $t=3$ $V=15000$

$15000 = 30000e^{-3k}$

$\frac{1}{2} = e^{-3k}$

$2 = e^{3k}$

$k = \frac{1}{3}\ln 2$

$\boxed{V = 30000e^{-(\frac{1}{3}\ln 2)t}}$

when $t=10$

$V = 2976.37\ldots$

$V = £3000$

(c) $500 = 30000e^{-(\frac{1}{3}\ln 2)t}$

$\frac{1}{60} = e^{-(\frac{1}{3}\ln 2)t}$

$60 = e^{(\frac{1}{3}\ln 2)t}$

$\ln 60 = (\frac{1}{3}\ln 2)t$

$t = \frac{3\ln 60}{\ln 2} \approx 18$ years

Question 4 (**+)

Water is leaking out of a tank from a tap which is located 5 cm from the bottom of the tank.

The height of the water, h cm, is decreasing at a rate proportional to square root of the difference of the height of the water and the height of the tap.

a) Model this problem with a differential equation involving h, the time t in minutes and a suitable proportionality constant.

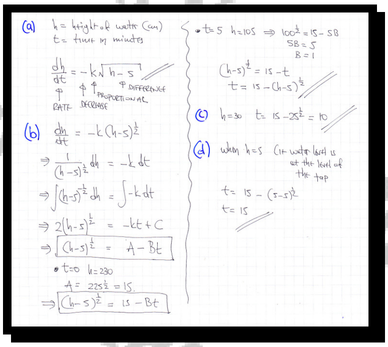
The initial height of the water in the tank is 230 cm and 5 minutes later it has dropped to 105 cm.

b) Find a solution of the differential equation of part **(a)**, in the form $t = f(h)$.

c) Calculate the time taken for the height of the water to fall to 30 cm.

d) State how many minutes it takes for the tank to stop leaking.

$\boxed{\frac{dh}{dt} = -k\sqrt{h-5}}$, $\boxed{t = 15 - \sqrt{h-5}}$, $\boxed{10 \text{ minutes}}$, $\boxed{15 \text{ minutes}}$



(a) h = height of water (cm)
t = time in minutes

$\frac{dh}{dt} = -k\sqrt{h-5}$

RATE DECREASE PROPORTIONAL DIFFERENCE

(b) $\frac{dh}{dt} = -k(h-5)^{\frac{1}{2}}$

$\Rightarrow \frac{1}{(h-5)^{\frac{1}{2}}} dh = -k \, dt$

$\Rightarrow \int (h-5)^{-\frac{1}{2}} dh = \int -k \, dt$

$\Rightarrow 2(h-5)^{\frac{1}{2}} = -kt + C$

$\Rightarrow \boxed{(h-5)^{\frac{1}{2}} = A - Bt}$

• $t=0 \quad h=230$

$A = 225^{\frac{1}{2}} = 15$

$\Rightarrow \boxed{(h-5)^{\frac{1}{2}} = 15 - Bt}$

• $t=5 \quad h=105 \Rightarrow 100^{\frac{1}{2}} = 15 - 5B$
$5B = 5$
$B = 1$

$(h-5)^{\frac{1}{2}} = 15 - t$

$t = 15 - (h-5)^{\frac{1}{2}}$

(c) $h=30 \quad t = 15 - 25^{\frac{1}{2}} = 10$

(d) when $h=5$ (the water level is at the level of the tap)

$t = 15 - (5-5)^{\frac{1}{2}}$

$t = 15$

Question 5 (***)

A laboratory dish with 100 bacterial cells is placed under observation and 65 minutes later this number has increased to 900 cells.

Let y be the number of bacterial cells present in the dish after t minutes, and assume that y can be treated as a continuous variable.

The rate at which the bacterial cells reproduce is inversely proportional to the square root of the number of the bacterial cells present.

a) Form a differential equation in terms of y, t and a proportionality constant k.

b) Solve the differential equation to show

$$y^{\frac{3}{2}} = At + B,$$

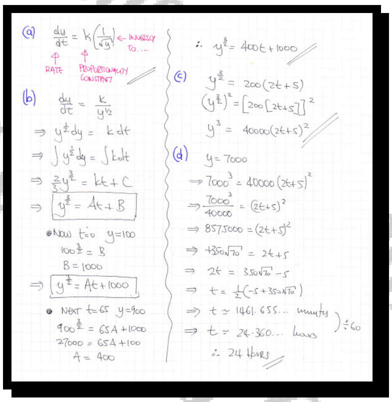
where A and B are constants to be found.

c) Show that the solution to this problem can be written as

$$y^3 = 40000(2t+5)^2.$$

d) Calculate, to the nearest **hour**, the time when the number of bacterial cells reaches 7000.

$$\boxed{\frac{dy}{dt} = \frac{k}{\sqrt{y}}}, \boxed{A = 400}, \boxed{B = 1000}, \boxed{t \approx 24}$$


(a) $\frac{dy}{dt} = k\left(\frac{1}{\sqrt{y}}\right)$ ← INVERSELY TO...
RATE PROPORTIONALITY CONSTANT

(b) $\frac{dy}{dt} = \frac{k}{y^{1/2}}$
$\Rightarrow y^{\frac{1}{2}}\,dy = k\,dt$
$\Rightarrow \int y^{\frac{1}{2}}\,dy = \int k\,dt$
$\Rightarrow \frac{2}{3}y^{\frac{3}{2}} = kt + C$
$\Rightarrow \boxed{y^{\frac{3}{2}} = At + B}$
• Now $t=0$ $y=100$
$100^{\frac{3}{2}} = B$
$B = 1000$
$\Rightarrow \boxed{y^{\frac{3}{2}} = At + 1000}$
• Next $t=65$ $y=900$
$900^{\frac{3}{2}} = 65A + 1000$
$27000 = 65A + 100$
$A = 400$
$\therefore y^{\frac{3}{2}} = 400t + 1000$

(c) $y^{\frac{3}{2}} = 200(2t+5)$
$\left(y^{\frac{3}{2}}\right)^2 = \left[200[2t+5]\right]^2$
$y^3 = 40000(2t+5)^2$

(d) $y = 7000$
$\Rightarrow 7000^3 = 40000(2t+5)^2$
$\Rightarrow \frac{7000^3}{40000} = (2t+5)^2$
$\Rightarrow 8575000 = (2t+5)^2$
$\Rightarrow +350\sqrt{70} = 2t+5$
$\Rightarrow 2t = 350\sqrt{70} - 5$
$\Rightarrow t = \frac{1}{2}(-5 + 350\sqrt{70})$
$\Rightarrow t \approx 1461.655\ldots$ minutes $\div 60$
$\Rightarrow t \approx 24.360\ldots$ hours
$\therefore$ 24 HOURS

Question 6 (***)

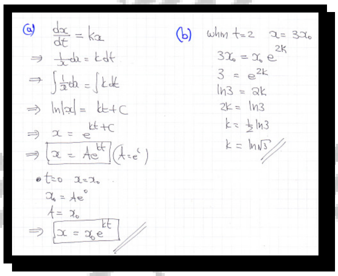
The number x of bacterial cells in time t hours, after they were placed on a laboratory dish, is increasing at the rate proportional to the number of the bacterial cells present at that time.

a) If x_0 is the initial number of the bacterial cells and k is a positive constant, show that

$$x = x_0 \mathrm{e}^{kt}.$$

b) If the number of bacteria triples in 2 hours, show that $k = \ln\sqrt{3}$.

$\boxed{\text{proof}}$



Question 7 (***)

During a car service, the motor oil is drained out of the engine.

The rate, in $\text{cm}^3\,\text{s}^{-1}$, at which the oil is drained out, is proportional to the volume, V cm^3, of the oil still left inside the engine.

a) Form a differential equation involving V, the time t in seconds and a proportionality constant k.

Initially there were $4000\ \text{cm}^3$ of oil in the engine.

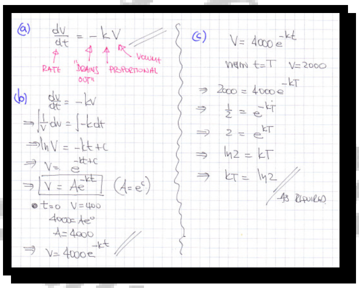
b) Find a solution of the differential equation, giving the answer in terms of k.

It takes T seconds to drain half the oil out of the engine.

c) Show clearly that

$$kT = \ln 2 .$$

$\boxed{\frac{dV}{dt} = -kV}$, $\boxed{V = 4000\,\mathrm{e}^{-kt}}$



Question 8 (***)

A species of tree is growing in height and the typical maximum height it can reach in its lifetime is 12 m.

The rate of growth of its height, $H \text{ m}$, is proportional to the difference between its height and the maximum height it can reach.

When a tree of this species was planted, it was 1 m in height and at that instant the tree was growing at the rate of 0.1 m per month.

a) Show clearly that

$$110\frac{dH}{dt} = 12 - H,$$

where t is the time, measured in months, since the tree was planted.

b) Determine a simplified solution for the above differential equation, giving the answer in the form $H = f(t)$.

c) Find, correct to 2 decimal places, the height of the tree after 5 **years**.

d) Calculate, correct to the nearest year, the number of years it will take for the tree to reach a height of 11 m.

$\boxed{}$, $\boxed{H = 12 - 11e^{-\frac{1}{110}t}}$, $\boxed{5.62 \text{ m}}$, $\boxed{t = 110\ln 11 \text{ months} \approx 22 \text{ years}}$

a) FORMING THE DIFFERENTIAL EQUATION FROM THE INFORMATION GIVEN

$$\underbrace{\frac{dH}{dt}}_{\text{RATE}} = \underbrace{+}_{\text{INCREASE}}\underbrace{k}_{\text{PROPORTIONAL}}(\overbrace{12}^{\text{MAX HEIGHT}} - \underbrace{H}_{\text{HEIGHT}})$$

H = height of tree (m)
t = time (months)
$H_{MAX} = 12 \text{ m}$
$t=0$
$H=1$
$\left.\frac{dH}{dt}\right|_{\substack{t=0\\H=1}} = 0.1$

- APPLY CONDITION $\left.\frac{dH}{dt}\right|_{H=1} = 0.1$

$0.1 = k(12-1)$
$k = \frac{1}{110}$

- $\frac{dH}{dt} = \frac{1}{110}(12-H)$

$110\frac{dH}{dt} = 12 - H$ // AS REQUIRED

b) SOLVING THE O.D.E. BY SEPARATING VARIABLES

$\Rightarrow 110\,dH = (12-H)\,dt$
$\Rightarrow \frac{110}{12-H}\,dH = 1\,dt$
$\Rightarrow \int \frac{110}{12-H}\,dH = \int 1\,dt$
$\Rightarrow -110\ln|12-H| = t + C$
$\Rightarrow \ln|12-H| = -\frac{1}{110}t + C$
$\Rightarrow 12 - H = e^{-\frac{1}{110}t + C}$
$\Rightarrow 12 - H = e^{-\frac{1}{110}t} \times e^{C}$
$\Rightarrow 12 - H = Ae^{-\frac{1}{110}t}$ $(A = e^{C})$
$\Rightarrow H = 12 + Ae^{-\frac{1}{110}t}$

APPLY THE CONDITION $t=0$ $H=1$

$\Rightarrow 1 = 12 + A$
$\Rightarrow A = -11$

$\therefore H = 12 - 11e^{-\frac{1}{110}t}$ //

c) WHEN $t = 60$ (5 YEARS = 60 MONTHS)

$\Rightarrow H = 12 - 11e^{-\frac{1}{110}\times 60}$
$\Rightarrow H = 12 - 11e^{-\frac{6}{11}}$
$\Rightarrow H \simeq 5.62 \text{ m}$ //

d) WHEN $H = 11$

$\Rightarrow 11 = 12 - 11e^{-\frac{1}{110}t}$
$\Rightarrow 11e^{-\frac{1}{110}t} = 1$
$\Rightarrow e^{-\frac{1}{110}t} = \frac{1}{11}$
$\Rightarrow e^{\frac{1}{110}t} = 11$
$\Rightarrow \frac{1}{110}t = \ln 11$
$\Rightarrow t = 110\ln 11 \approx 263.76\ldots \text{ months} \xrightarrow{\div 12} \approx 22 \text{ YEARS}$

Question 9 (***)

An area of neglected lawn is treated with weed killer. Before the treatment started, the area covered by the weed was 75 m^2 and two days later it has reduced to 33.7 m^2.

Let the area of the lawn covered with weed be $A \text{ m}^2$, t days after it was treated.

The rate at which the area covered by the weed is decreasing, is proportional to the area still covered by the weed.

By forming and solving a suitable differential equation, express A in terms of t.

$\boxed{}$, $\boxed{A \approx 75e^{-0.4t}}$

COLLECTING ALL THE RELEVANT INFORMATION

$$\frac{dA}{dt} = -kA$$

RATE DECREASE PROPORTIONAL AREA STILL COVERED BY WEED

$t=0 \quad A=75$
$t=2 \quad A=33.7$

SOLVE BY SEPARATION OF VARIABLES

$\Rightarrow dA = -kA\,dt$

$\Rightarrow \frac{1}{A}\,dA = -k\,dt$

$\Rightarrow \int \frac{1}{A}\,dA = \int -k\,dt$

$\Rightarrow \ln A = -kt + C \qquad (A>0)$

$\Rightarrow A = e^{-kt+C}$

$\Rightarrow A = e^{-kt} \times e^{C}$

$\Rightarrow A = Ce^{-kt} \qquad (e^{C} = C)$

APPLY CONDITIONS

$t=0,\ A=75 \Rightarrow 75 = Ce^{0}$

$\Rightarrow 75 = C$

$\Rightarrow A = 75e^{-kt}$

WHEN $t=2 \quad A=33.7 \Rightarrow 33.7 = 75e^{-k \times 2}$

$\Rightarrow e^{-2k} = \frac{337}{750}$

$\Rightarrow e^{2k} = \frac{750}{337}$

$\Rightarrow 2k = \ln\left(\frac{750}{337}\right)$

$\Rightarrow k = \frac{1}{2}\ln\left(\frac{750}{337}\right)$

$\Rightarrow k = 0.3999995\ldots$

$\therefore A = 75e^{-0.4t}$

Question 10 (***+)

The number, x thousands, of reported cases of an infectious disease, t months after it was first reported, is now dropping. The rate at which it is dropping is proportional to the square of the number of the reported cases.

It is assumed that x can be treated as a continuous variable.

a) Form a differential equation in terms of x, t and a proportionality constant k.

Initially there were 2500 reported cases and one month later they had dropped to 1600 cases.

b) Solve the differential equation to show that

$$x = \frac{40}{9t+16}.$$

c) Find after how many months there will be 250 reported cases.

$\boxed{\dfrac{dx}{dt} = -kx^2}$, $\boxed{t = 16}$

a)

$$\boxed{\frac{dx}{dt} = -k\,x^2}$$

SQUARE

RATE DROP PROPORTIONAL

b) SOLVING THE DIFFERENTIAL EQUATION BY SEPARATING VARIABLES

$\Rightarrow dx = -kx^2\,dt$

$\Rightarrow -\frac{1}{x^2}\,dx = k\,dt$

$\Rightarrow \int -\frac{1}{x^2}\,dx = \int k\,dt$

$\Rightarrow \boxed{\frac{1}{x} = kt + C}$

APPLY THE CONDITION $t=0$, $x = 2.5$ (2500 CASES)

$\Rightarrow \frac{1}{2.5} = C$

$\Rightarrow C = \frac{2}{5}$

$\Rightarrow \boxed{\frac{1}{x} = kt + \frac{2}{5}}$

APPLY THE CONDITION $t=1$, $x=1.6$ (1600 CASES)

$\Rightarrow \frac{1}{1.6} = k + \frac{2}{5}$

$\Rightarrow \frac{5}{8} = k + \frac{2}{5}$

$\Rightarrow k = \frac{9}{40}$

$\Rightarrow \boxed{\frac{1}{x} = \frac{9}{40}t + \frac{2}{5}}$

TIDY UP FURTHER

$\frac{1}{x} = \frac{9t}{40} + \frac{2}{5}$

$\frac{1}{x} = \frac{9t}{40} + \frac{16}{40}$

$\frac{1}{x} = \frac{9t+16}{40}$

$x = \frac{40}{9t+16}$ // AS REQUIRED

c) FINALLY WHEN $x = 0.25$ (250 CASES)

$\Rightarrow \frac{1}{4} = \frac{40}{9t+16}$

$\Rightarrow 9t+16 = 160$

$\Rightarrow 9t = 144$

$\Rightarrow t = 16$ //

Question 11 (***+)

The value of a machine, in thousands of pounds, t years after it was purchased is denoted by $\pounds V$.

The value of this machine at any given time is depreciating at rate proportional to its value **squared**, at that time.

a) Given that the initial value of the machine was £12000, show that

$$V = \frac{12}{at+1},$$

where a is a positive constant.

b) Given further that the machine depreciated by £4000 two years after it was bought, find its value after a **further** period of ten years has elapsed.

$\boxed{}$, $\boxed{\pounds 3000}$

a) COLLECTING ALL THE INFORMATION

$$\frac{dV}{dt} = -kV^2$$

RATE DEPRECIATING PROPORTIONAL VALUE SQUARED

V = value, in thousands
t = time, in years
t = 0, V = 12

SOLVING BY SEPARATING VARIABLES

$\Rightarrow dV = -kV^2\,dt$

$\Rightarrow -\frac{1}{V^2}\,dV = k\,dt$

$\Rightarrow \int -\frac{1}{V^2}\,dV = \int k\,dt$

$\Rightarrow \boxed{\frac{1}{V} = kt + C}$

APPLY CONDITION $t=0$, $V=12$

$\Rightarrow \frac{1}{12} = C$

$\Rightarrow \frac{1}{V} = kt + \frac{1}{12}$

$\Rightarrow V = \frac{1}{kt + \frac{1}{12}}$ MULTIPLY TOP & BOTTOM OF THE FRACTION IN THE R.H.S BY 12

$\Rightarrow V = \frac{12}{12kt+1}$

$\Rightarrow \boxed{V = \frac{12}{at+1}}$ // AS REQUIRED

b) USING THE FINAL CONDITION

$t=0 \quad V=8 \leftarrow \pounds 12000 - \pounds 4000$

$\Rightarrow 8 = \frac{12}{2a+1}$

$\Rightarrow 16a + 8 = 12$

$\Rightarrow 16a = 4$

$\Rightarrow a = \frac{1}{4}$

REWRITING THE FORMULA

$\Rightarrow V = \frac{12}{\frac{1}{4}t+1}$

$\Rightarrow V = \frac{12}{\frac{1}{4}\times 12+1}$ (A FURTHER PERIOD ....)

$\Rightarrow V = 3$

$\therefore \pounds 3000$

Question 12 (***+)

Water is leaking out of a hole at the bottom of a tank.

Let the height of the water in the tank be y cm at time t minutes.

At any given time after the leaking started, the height of the water in the tank is decreasing at a rate proportional to the cube root of the height of the water in the tank.

When $t = 0$, $y = 125$ and when $t = 3$, $y = 64$.

By forming and solving a differential equation, find the value of y when $t = 7\frac{7}{12}$.

$\boxed{\text{MP2}}$, $\boxed{y = 3.375}$

FORMING A DIFFERENTIAL EQUATION

$$\frac{dy}{dt} = -k\,y^{\frac{1}{3}}$$

RATE

LEAKING

PROPORTIONAL

CUBE ROOT OF THE HEIGHT

y = HEIGHT (cm)
t = TIME (min)

SOLVING BY SEPARATING VARIABLES

$$\Rightarrow dy = -k y^{\frac{1}{3}}\,dt$$

$$\Rightarrow \frac{1}{y^{\frac{1}{3}}}\,dy = -k\,dt$$

$$\Rightarrow \int y^{-\frac{1}{3}}\,dy = \int -k\,dt$$

$$\Rightarrow \frac{3}{2}y^{\frac{2}{3}} = -kt + C$$

$$\Rightarrow \boxed{y^{\frac{2}{3}} = At + B} \qquad \left(A = -\frac{2k}{3},\ B = \frac{2C}{3}\right)$$

APPLY THE CONDITIONS GIVEN

$$t=0,\ y=125 \Rightarrow 125^{\frac{2}{3}} = B$$

$$\Rightarrow B = 25$$

$$\Rightarrow y^{\frac{2}{3}} = At + 25$$

$$t=3,\ y=64 \Rightarrow 64^{\frac{2}{3}} = A\times 3 + 25$$

$$\Rightarrow 16 = 3A + 25$$

$$\Rightarrow -9 = 3A$$

$$\Rightarrow A = -3$$

FINALLY USING THE FORMULA OBTAINED

$$\Rightarrow y^{\frac{2}{3}} = 25 - 3t$$

$$\Rightarrow y^{\frac{2}{3}} = 25 - 3\left(7 + \frac{7}{12}\right)$$

$$\Rightarrow y^{\frac{2}{3}} = \frac{9}{4}$$

$$\Rightarrow \left(y^{\frac{2}{3}}\right)^{\frac{3}{2}} = \left(\frac{9}{4}\right)^{\frac{3}{2}}$$

$$\Rightarrow y = \frac{27}{8} = 3.375$$

Question 13 (***+)

Water is pouring into a container at a constant rate of $600 \text{ cm}^3\text{s}^{-1}$ and is leaking from a hole at the base of the container at the rate of $\frac{3V}{4} \text{ cm}^3\text{s}^{-1}$, where $V \text{ cm}^3$ is the volume of the water in the container.

a) Show clearly that

$$-4\frac{dV}{dt} = 3V - 2400,$$

where t is the time measured in seconds.

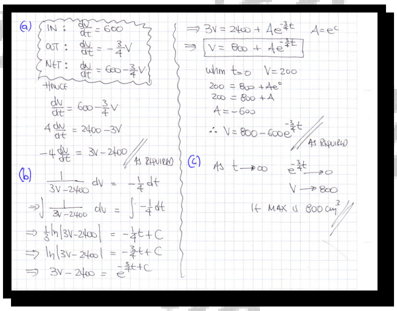
Initially there were 200 cm^3 of water in the container.

b) Show further that

$$V = 800 - 600\,\mathrm{e}^{-\frac{3}{4}t}.$$

c) State the maximum volume that the water in the container will ever attain.

$\boxed{V_{max} = 800}$


(a) IN : $\frac{dV}{dt} = 600$
OUT : $\frac{dV}{dt} = -\frac{3}{4}V$
NET : $\frac{dV}{dt} = 600 - \frac{3}{4}V$
HENCE
$\frac{dV}{dt} = 600 - \frac{3}{4}V$
$4\frac{dV}{dt} = 2400 - 3V$
$-4\frac{dV}{dt} = 3V - 2400$ // AS REQUIRED

(b) $\frac{1}{3V-2400}\,dV = -\frac{1}{4}\,dt$
$\Rightarrow \int \frac{1}{3V-2400}\,dV = \int -\frac{1}{4}\,dt$
$\Rightarrow \frac{1}{3}\ln|3V-2400| = -\frac{1}{4}t + C$
$\Rightarrow \ln|3V-2400| = -\frac{3}{4}t + C$
$\Rightarrow 3V - 2400 = e^{-\frac{3}{4}t + C}$
$\Rightarrow 3V = 2400 + Ae^{-\frac{3}{4}t}$ $A = e^C$
$\Rightarrow \boxed{V = 800 + Ae^{-\frac{3}{4}t}}$
when $t=0$ $V=200$
$200 = 800 + Ae^0$
$200 = 800 + A$
$A = -600$
$\therefore V = 800 - 600e^{-\frac{3}{4}t}$ // AS REQUIRED

(c) As $t \to \infty$ $e^{-\frac{3}{4}t} \to 0$
$V \to 800$
$\therefore$ MAX IS 800 cm^3 //

Question 14 (***+)

A body is moving and its distance, x metres, is measured from a fixed point O at different times, t seconds.

The body is moving in such a way, so that the rate of change of its distance x is inversely proportional to its distance x at that time.

When $t = 0$, $x = 50$ and when $t = 4$, $x = 30$.

Determine the time it takes for the body to reach O.

$\boxed{}$, $\boxed{t = 6.25}$

FORM A DIFFERENTIAL EQUATION

$$\frac{dx}{dt} = k\frac{1}{x}$$

RATE

PROPORTIONALITY CONSTANT

INVERSELY PROPORTIONAL TO x

x = DISTANCE FROM O
t = TIME

$t=0$, $x=50$
$t=4$, $x=30$

SOLVING BY SEPARATING VARIABLES

$\Rightarrow dx = \frac{k}{x}\,dt$

$\Rightarrow x\,dx = k\,dt$

$\Rightarrow \int x\,dx = \int k\,dt$

$\Rightarrow \frac{1}{2}x^2 = kt + C$

$\Rightarrow x^2 = At + B$

APPLY CONDITIONS TO FIND THE CONSTANTS

- $t=0$, $x=50$ $\Rightarrow 50^2 = B$
  $\Rightarrow B = 2500$
  $\Rightarrow x^2 = At + 2500$

- $t=4$, $x=30$ $\Rightarrow 30^2 = A\times 4 + 2500$
  $\Rightarrow 900 = 4A + 2500$
  $\Rightarrow 4A = -1600$
  $\Rightarrow A = -400$
  $\Rightarrow x^2 = -400t + 2500$

- WHEN $x=0$ $\Rightarrow 0^2 = -400t + 2500$
  $\Rightarrow 400t = 2500$
  $\Rightarrow 4t = 25$
  $\Rightarrow t = 6.25$

Question 15 (***+)

The value of a computer system V, in hundred of £, t years from when it was new, is depreciating at a rate proportional to its value <u>cubed</u>, at that time t.

The value of the computer system when it was new, was £1000.

a) By forming and solving a differential equation, show that

$$\frac{1}{V^2} = At + \frac{1}{100},$$

where A is a positive constant.

b) Given that the value of the computer system halves after one year, find the value of t when the system is worth £250.

$\boxed{}$, $\boxed{t = 5}$

a) FORM A DIFFERENTIAL EQUATION FROM THE GIVEN INFORMATION

$$\frac{dV}{dt} = -kV^3$$

RATE DEPRECIATES PROPOR VALUE CUBED

V = VALUE, IN £100
t = TIME, IN YEARS
t=0, V=10

SOLVING BY SEPARATIONS OF VARIABLES

$\Rightarrow dV = -kV^3\,dt$

$\Rightarrow -\frac{1}{V^3}\,dV = k\,dt$

$\Rightarrow \int -V^{-3}\,dV = \int k\,dt$

$\Rightarrow \frac{1}{2}V^{-2} = kt + c$

$\Rightarrow \frac{1}{2V^2} = kt + C$

APPLY THE GIVEN CONDITION t=0 V=10 (£1000)

$\Rightarrow \frac{1}{2\times 10^2} = C$

$\Rightarrow C = \frac{1}{200}$

$\Rightarrow \frac{1}{2V^2} = kt + \frac{1}{200}$

$\Rightarrow \frac{1}{V^2} = 2kt + \frac{1}{100}$ (LET $A = 2k$)

$\Rightarrow \frac{1}{V^2} = At + \frac{1}{100}$ AS REQUIRED

b) USING THE FACT THAT WHEN t=1 V=5 (£500/HALF)

$\Rightarrow \frac{1}{5^2} = A\times 1 + \frac{1}{100}$

$\Rightarrow \frac{1}{25} = A + \frac{1}{100}$

$\Rightarrow A = \frac{3}{100}$

$\Rightarrow \frac{1}{V^2} = \frac{3}{100}t + \frac{1}{100}$

FINALLY WHEN V=2.5 (VAL=£250)

$\Rightarrow \frac{1}{2.5^2} = \frac{3}{100}t + \frac{1}{100}$

$\Rightarrow \frac{1}{6.25} = \frac{3}{100}t + \frac{1}{100}$ $\times 100$

$\Rightarrow 16 = 3t + 1$

$\Rightarrow 3t = 15$

$\Rightarrow t = 5$

Question 16 (***+)

Water is pouring into a container at a constant rate of 0.05 m^3 per hour and is leaking from a hole at the base of the container at the rate of $\frac{4V}{5}$ m^3 per hour, where V m^3 is the volume of the water in the container.

a) Show clearly that

$$-20\frac{dV}{dt} = 16V - 1,$$

where t is the time measured in hours.

Initially there were 4 m^3 of water in the container.

b) Show further that

$$V = \frac{1}{16}\left(1 + 63e^{-\frac{4}{5}t}\right).$$

c) State, with justification, the minimum volume that the water in the container will ever attain.

$\boxed{}$, $\boxed{V_{\text{min}} = \frac{1}{16}}$

a) SETTING A DIFFERENTIAL EQUATION

IN FLOW : $\frac{dV}{dt} = 0.05$

OUT FLOW : $\frac{dV}{dt} = -\frac{4V}{5}$

NET FLOW : $\frac{dV}{dt} = 0.05 - \frac{4V}{5}$

$\Rightarrow \frac{dV}{dt} = \frac{1}{20} - \frac{4V}{5}$ $\times(-20)$

$\Rightarrow -20\frac{dV}{dt} = -1 + 16V$

$\Rightarrow -20\frac{dV}{dt} = 16V - 1$ // AS REQUIRED

b) SOLVING BY SEPARATING VARIABLES

$\Rightarrow -20\,dV = (16V-1)\,dt$

$\Rightarrow -\frac{20}{16V-1}\,dV = 1\,dt$

$\Rightarrow \int \frac{-20}{16V-1}\,dV = \int 1\,dt$

$\Rightarrow -\frac{5}{4}\ln|16V-1| = t + C$

$\Rightarrow \ln|16V-1| = -\frac{4}{5}t + C$

$\Rightarrow 16V - 1 = e^{-\frac{4}{5}t + C}$

$\Rightarrow 16V - 1 = e^{-\frac{4}{5}t} \times e^{C}$

$\Rightarrow 16V = 1 + Ae^{-\frac{4}{5}t}$ $(A = e^{C})$

$\Rightarrow V = \frac{1}{16} + Ae^{-\frac{4}{5}t}$

APPLY THE INITIAL CONDITION $t=0$ $V=4$

$\Rightarrow 4 = \frac{1}{16} + A$

$\Rightarrow A = \frac{63}{16}$

$\Rightarrow V = \frac{1}{16} + \frac{63}{16}e^{-\frac{4}{5}t}$

$\Rightarrow V = \frac{1}{16}\left[1 + 63e^{-\frac{4}{5}t}\right]$ // AS REQUIRED

c) AS $t \to \infty$, $e^{-\frac{4}{5}t} \to 0$

$\therefore V \to \frac{1}{16} = 0.0625$

$\therefore$ THE VOLUME WILL TEND TO $0.0625\ \text{m}^3$ (62.5 litres)

Question 17 (****)

A population P, in millions, at a given time t years, is growing at a rate equal to the product of the population squared and the difference of the population from one million.

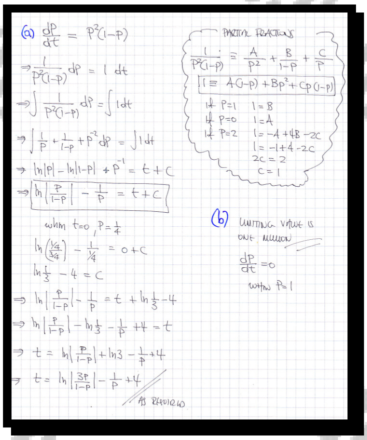
Initially the population is one quarter of a million.

a) Form and solve a differential equation to show that

$$t = \ln\left|\frac{3P}{1-P}\right| - \frac{1}{P} + 4 .$$

b) State the limiting value for this population.

$\boxed{}$, $\boxed{\frac{dP}{dt} = P^2(1-P)}$, $\boxed{t \to \infty,\ P \to 1}$


(a) $\frac{dP}{dt} = P^2(1-P)$

$\Rightarrow \frac{1}{P^2(1-P)}\,dP = 1\,dt$

$\Rightarrow \int \frac{1}{P^2(1-P)}\,dP = \int 1\,dt$

$\Rightarrow \int \frac{1}{P} + \frac{1}{1-P} + P^{-2}\,dP = \int 1\,dt$

$\Rightarrow \ln|P| - \ln|1-P| + P^{-1} = t + C$

$\Rightarrow \boxed{\ln\left|\frac{P}{1-P}\right| - \frac{1}{P} = t + C}$

PARTIAL FRACTIONS

$\frac{1}{P^2(1-P)} \equiv \frac{A}{P^2} + \frac{B}{1-P} + \frac{C}{P}$

$\boxed{1 \equiv A(1-P) + BP^2 + CP(1-P)}$

If $P=1$: $1 = B$

If $P=0$: $1 = A$

If $P=2$: $1 = -A + 4B - 2C$

$1 = -1 + 4 - 2C$

$2C = 2$

$C = 1$

when $t=0$, $P=\frac{1}{4}$

$\ln\left(\frac{1/4}{3/4}\right) - \frac{1}{1/4} = 0 + C$

$\ln\frac{1}{3} - 4 = C$

$\Rightarrow \ln\left|\frac{P}{1-P}\right| - \frac{1}{P} = t + \ln\frac{1}{3} - 4$

$\Rightarrow \ln\left|\frac{P}{1-P}\right| - \ln\frac{1}{3} - \frac{1}{P} + 4 = t$

$\Rightarrow t = \ln\left|\frac{P}{1-P}\right| + \ln 3 - \frac{1}{P} + 4$

$\Rightarrow t = \ln\left|\frac{3P}{1-P}\right| - \frac{1}{P} + 4$

AS REQUIRED

(b) LIMITING VALUE IS ONE MILLION

$\frac{dP}{dt} = 0$ when $P = 1$

Question 18 (****)

In a laboratory a dangerous chemical is stored in a cylindrical drum of height 160 cm which is initially full.

One day the drum was found leaking and when this was first discovered, the level of the chemical had dropped to 100 cm, and at that instant the level of the chemical was found to be dropping at the rate of 0.25 cm per minute.

In order to assess the contamination level in the laboratory, it is required to find the length of time that the leaking has been taking place.

It is assumed that the rate at which the height of the chemical was dropping is proportional to the square root of its height.

a) Form a suitable differential equation to model the above problem, where the time, in minutes, is measured from the instant that the leaking was discovered.

b) Find a solution of the differential equation and use it to calculate, in hours and minutes, for how long the leaking has been taking place.

$\boxed{}$, $\boxed{\dfrac{dh}{dt} = -\dfrac{1}{40}\sqrt{h}}$, $\boxed{3 \text{ hours } 32 \text{ minutes}}$

a) FORMING A DIFFERENTIAL EQUATION

$$\frac{dh}{dt} = -k h^{\frac{1}{2}}$$

RATE LEAK PROPORTIONAL SQUARE ROOT OF HEIGHT

h = HEIGHT OF CHEMICAL (cm)
t = TIME IN MINUTES

$t = 0$, $h = 100$

$$\left.\frac{dh}{dt}\right|_{\substack{t=0\\h=100}} = -0.25$$

APPLY THE CONDITION $\left.\frac{dh}{dt}\right|_{h=100} = -0.25$

$$-0.25 = -k \times 100^{\frac{1}{2}}$$
$$10k = 0.25$$
$$k = \frac{1}{40}$$

$$\therefore \frac{dh}{dt} = -\frac{1}{40}h^{\frac{1}{2}}$$

b) SEPARATING VARIABLES TO OBTAIN

$$\Rightarrow \frac{dh}{dt} = -\frac{1}{40}h^{\frac{1}{2}}$$
$$\Rightarrow dh = -\frac{1}{40}h^{\frac{1}{2}}\,dt$$
$$\Rightarrow \frac{1}{h^{\frac{1}{2}}}\,dh = -\frac{1}{40}\,dt$$
$$\Rightarrow \int h^{-\frac{1}{2}}\,dh = \int -\frac{1}{40}\,dt$$
$$\Rightarrow 2h^{\frac{1}{2}} = -\frac{1}{40}t + C$$

APPLY THE CONDITION $t=0$ $h = 100$ cm

$$\Rightarrow 2 \times 100^{\frac{1}{2}} = -\frac{1}{40} \times 0 + C$$
$$\Rightarrow C = 20$$
$$\Rightarrow 2h^{\frac{1}{2}} = 20 - \frac{1}{40}t$$

THE LEAKING STARTED WHEN $h = 160$ cm

$$\Rightarrow 2 \times \sqrt{160} = 20 - \frac{1}{40}t$$
$$\Rightarrow 80\sqrt{160} = 800 - t$$
$$\Rightarrow t = 800 - 80\sqrt{160}$$
$$\Rightarrow t = -211.92\ldots \text{ minutes}$$
$$\Rightarrow t = -3 \text{ HOURS } \& \text{ 32 MINUTES, BEFORE } t = 0$$

$\therefore$ DRUM HAS BEEN LEAKING FOR 3 HOURS & 32 MINUTES

Question 19 (****)

A grass lawn has an area of 225 m^2 and has become host to a parasitic weed.

Let $A \text{ m}^2$ be the area covered by the parasitic weed, t days after it was first noticed.

The rate at which A is growing is proportional to the square root of the area of the lawn already covered by the weed.

Initially the parasitic weed has spread to an area of 1 m^2, and **at that instant** the parasitic weed is growing at the rate of 0.25 m^2 per day.

By forming and solving a suitable differential equation, calculate after how many days, the weed will have spread to the entire lawn.

$\boxed{\quad}$, $\boxed{\dfrac{dA}{dt} = \dfrac{1}{4}\sqrt{A}}$, $\boxed{t = 8\sqrt{A} - 8}$, $\boxed{112 \text{ days}}$

FORMING A DIFFERENTIAL EQUATION

$$\frac{dA}{dt} = +k\sqrt{A}$$

RATE GROWING PROPORTIONAL SQUARE ROOT OF THE AREA ALREADY COVERED

A = AREA COVERED (m^2)
t = TIME (days)

APPLY STRAIGHT AWAY THE CONDITION $\left.\frac{dA}{dt}\right|_{\substack{t=0\\A=1}} = 0.25$

$\Rightarrow 0.25 = k \times \sqrt{1}$

$\Rightarrow k = \frac{1}{4}$

$\Rightarrow \frac{dA}{dt} = \frac{1}{4}A^{\frac{1}{2}}$

SOLVE BY SEPARATION OF VARIABLES

$\Rightarrow dA = \frac{1}{4}A^{\frac{1}{2}}\,dt$

$\Rightarrow \frac{1}{A^{\frac{1}{2}}}\,dA = \frac{1}{4}\,dt$

$\Rightarrow \int A^{-\frac{1}{2}}\,dA = \int \frac{1}{4}\,dt$

$\Rightarrow 2A^{\frac{1}{2}} = \frac{1}{4}t + C$

APPLY THE CONDITION $t=0$, $A=1$

$\Rightarrow 2 \times 1^{\frac{1}{2}} = 0 + C$

$\Rightarrow C = 2$

$\Rightarrow 2A^{\frac{1}{2}} = \frac{1}{4}t + 2$

WHEN THE WHOLE LAWN IS INFECTED WHEN $A = 225$

$\Rightarrow 2 \times 225^{\frac{1}{2}} = \frac{1}{4}t + 2$

$\Rightarrow 2 \times 15 = \frac{1}{4}t + 2$

$\Rightarrow 28 = \frac{1}{4}t$

$\Rightarrow t = 112 \text{ days}$

Question 20 (****)

A snowball is melting and its shape remains spherical at all times.

The volume of the snowball, V cm^3, is decreasing at constant rate.

Let t be the time in hours since the snowball's radius was 18 cm.

Ten hours later its radius has reduced to 9 cm.

Show that the volume V of the melting snowball satisfies

$$V = 97.2\pi(80 - 7t),$$

and hence find the value of t when the radius of the snowball has reduced to 4.5 cm.

$\left[\text{volume of a sphere of radius } r \text{ is given by } \frac{4}{3}\pi r^3\right]$

$\boxed{}$, $\boxed{t = 11.25}$

START BY FORMING A DIFFERENTIAL EQUATION

$$\frac{dV}{dt} = -k$$

RATE

MELTING/DECREASING

CONSTANT RATE

V = VOLUME OF SNOWBALL (cm^3)
t = TIME (HOURS)

• $t=0$, $r=18$
$V = \frac{4}{3} \times \pi \times 18^3$
$V = 7776\pi$

• $t=10$, $r=9$
$V = \frac{4}{3} \times \pi \times 9^3$
$V = 972\pi$

SEPARATING VARIABLES

$\Rightarrow dV = -k\,dt$

$\Rightarrow \int 1\,dV = \int -k\,dt$

$\Rightarrow V = -kt + C$

APPLY $t=0$, $V = 7776\pi$

$\Rightarrow 7776\pi = C$

$\Rightarrow V = 7776\pi - kt$

APPLY $t=10$, $V = 972\pi$

$\Rightarrow 972\pi = -10k + 7776\pi$

$\Rightarrow 10k = 6804\pi$

$\Rightarrow k = 680.4\pi$

$\Rightarrow V = 7776\pi - 680.4\pi t$

$\Rightarrow V = 97.2\pi[80 - 7t]$ // AS REQUIRED

FINALLY WHEN $r = 4.5$ cm

• $V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(4.5)^3 = 121.5\pi$ cm^3

$\Rightarrow 121.5\pi = 97.2\pi(80 - 7t)$

$\Rightarrow 1.25 = 80 - 7t$

$\Rightarrow 7t = 78.75$

$\Rightarrow t = 11.25$ HOURS

Question 21 (****)

The temperature in a bathroom is maintained at the constant value of 20°C and the water in a hot bath is left to cool down.

The rate, in °C per second, at which the temperature of the water in the bath, T °C, is cooling down, is proportional to the difference in the temperature between the bathwater and the room.

Initially the bathwater had a temperature of 40°C, and at that instant was cooling down at the rate of 0.005°C per second.

Let t be the time in seconds, since the bathwater was left to cool down.

a) Show that

$$\frac{dT}{dt} = -\frac{1}{4000}(T-20).$$

b) Solve the differential equation of part **(a)**, to find, correct to the nearest minute, after how long the temperature of the bathwater will drop to 36°C.

$\boxed{}$, $\boxed{15 \text{ minutes}}$

a) FORMING A DIFFERENTIAL EQUATION

$$\frac{dT}{dt} = -k(T-20)$$

RATE — COOLING — PROPORTIONAL — DIFFERENCE BETWEEN...

T = TEMPERATURE OF WATER (°C)
t = TIME (sec)

$t=0$, $T=40$

$\left.\frac{dT}{dt}\right|_{\substack{t=0\\T=40}} = -0.005$

APPLY THE CONDITION $\left.\frac{dT}{dt}\right|_{T=40} = -0.005$

$$\Rightarrow -0.005 = -k(40-20)$$
$$\Rightarrow -0.005 = -20k$$
$$\Rightarrow k = -\frac{1}{4000}$$
$$\therefore \frac{dT}{dt} = -\frac{1}{4000}(T-20)$$

AS REQUIRED

b) SOLVE BY SEPARATION OF VARIABLES

$$\Rightarrow \frac{dT}{dt} = -\frac{1}{4000}(T-20)$$
$$\Rightarrow \frac{1}{T-20}\,dT = -\frac{1}{4000}\,dt$$
$$\Rightarrow \int \frac{1}{T-20}\,dT = \int -\frac{1}{4000}\,dt$$
$$\Rightarrow \ln|T-20| = -\frac{1}{4000}t + C$$
$$\Rightarrow T-20 = e^{-\frac{1}{4000}t + C}$$
$$\Rightarrow T-20 = e^{-\frac{1}{4000}t} \times e^{C}$$
$$\Rightarrow T = 20 + Ae^{-\frac{1}{4000}t} \qquad (A = e^{C})$$

APPLY THE CONDITION $t=0$, $T=40$

$$\Rightarrow 40 = 20 + Ae^{0}$$
$$\Rightarrow A = 20$$
$$\Rightarrow T = 20 + 20e^{-\frac{1}{4000}t}$$

FINALLY WHEN $T=36$

$$\Rightarrow 36 = 20 + 20e^{-\frac{1}{4000}t}$$
$$\Rightarrow 16 = 20e^{-\frac{1}{4000}t}$$
$$\Rightarrow \frac{4}{5} = e^{-\frac{1}{4000}t}$$
$$\Rightarrow e^{\frac{1}{4000}t} = \frac{5}{4}$$
$$\Rightarrow \frac{1}{4000}t = \ln\left(\frac{5}{4}\right)$$
$$\Rightarrow t = 4000\ln(1.25)$$
$$\Rightarrow t \simeq 892.57\ldots \text{ (sec)}$$
$$\Rightarrow t \simeq 14.876\ldots \text{ (min)}$$

$\therefore$ APPROX 15 MIN

Question 22 (****)

The gradient at a point on the curve C with equation $y = f(x)$ is proportional to the product of its x and y coordinates. The gradient at the point $(2,6)$ is $\frac{3}{2}$.

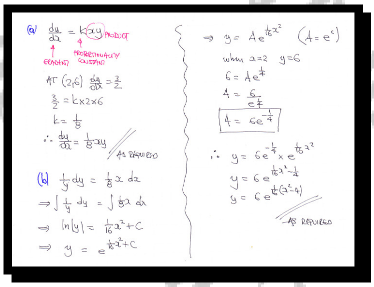
a) Show that

$$\frac{dy}{dx} = \frac{xy}{8}.$$

b) Solve the differential equation to show that

$$y = 6e^{\frac{1}{16}\left(x^2-4\right)}.$$

proof



(a) $\frac{dy}{dx} = k\,xy$ PRODUCT

GRADIENT PROPORTIONALITY CONSTANT

AT $(2,6)$ $\frac{dy}{dx} = \frac{3}{2}$

$\frac{3}{2} = k \times 2 \times 6$

$k = \frac{1}{8}$

$\therefore \frac{dy}{dx} = \frac{1}{8}xy$ AS REQUIRED

(b) $\frac{1}{y}\,dy = \frac{1}{8}x\,dx$

$\Rightarrow \int \frac{1}{y}\,dy = \int \frac{1}{8}x\,dx$

$\Rightarrow \ln|y| = \frac{1}{16}x^2 + C$

$\Rightarrow y = e^{\frac{1}{16}x^2 + C}$

$\Rightarrow y = Ae^{\frac{1}{16}x^2}$ $(A = e^C)$

when $x=2$ $y=6$

$6 = Ae^{\frac{1}{4}}$

$A = \frac{6}{e^{\frac{1}{4}}}$

$\boxed{A = 6e^{-\frac{1}{4}}}$

$\therefore y = 6e^{-\frac{1}{4}} \times e^{\frac{1}{16}x^2}$

$y = 6e^{\frac{1}{16}x^2 - \frac{1}{4}}$

$y = 6e^{\frac{1}{16}(x^2-4)}$ AS REQUIRED

Question 23 (****)

Hot tea in a cup has a temperature T °C at time t minutes and it is left to cool in a room of constant temperature T_0.

Newton's Law of cooling asserts that the rate at which a body cools is directly proportional to the excess temperature of the body and the temperature of its immediate surroundings.

a) Assuming the tea cooling in the cup obeys this law, form a differential equation in terms of T, T_0, t and a proportionality constant k.

b) Show clearly that

$$T = T_0 + A\mathrm{e}^{-kt},$$

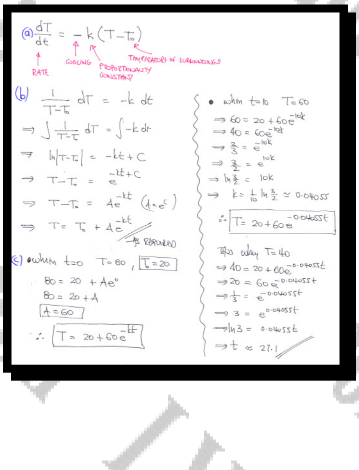
where A is a constant.

Initially the temperature of the tea is 80 °C and 10 minutes later is 60 °C.

The room temperature remains constant at 20 °C.

c) Find the value of t when the tea reaches a temperature of 40 °C.

$\boxed{\frac{dT}{dt} = -k\left(T - T_0\right)}$, $\boxed{t \approx 27.1}$



(a) $\frac{dT}{dt} = -k(T - T_0)$

RATE

COOLING

PROPORTIONALITY CONSTANT

TEMPERATURE OF SURROUNDINGS

(b) $\frac{1}{T-T_0}\,dT = -k\,dt$

$\Rightarrow \int \frac{1}{T-T_0}\,dT = \int -k\,dt$

$\Rightarrow \ln|T-T_0| = -kt + C$

$\Rightarrow T - T_0 = e^{-kt+C}$

$\Rightarrow T - T_0 = Ae^{-kt} \quad (A = e^C)$

$\Rightarrow T = T_0 + Ae^{-kt}$

AS REQUIRED

(c) • when $t=0$ $T=80$, $\boxed{T_0 = 20}$

$80 = 20 + Ae^0$

$80 = 20 + A$

$\boxed{A = 60}$

$\therefore \boxed{T = 20 + 60e^{-kt}}$

• when $t=10$ $T=60$

$\Rightarrow 60 = 20 + 60e^{-10k}$

$\Rightarrow 40 = 60e^{-10k}$

$\Rightarrow \frac{2}{3} = e^{-10k}$

$\Rightarrow \frac{3}{2} = e^{10k}$

$\Rightarrow \ln\frac{3}{2} = 10k$

$\Rightarrow k = \frac{1}{10}\ln\frac{3}{2} \simeq 0.04055$

$\therefore \boxed{T = 20 + 60e^{-0.04055t}}$

Thus when $T=40$

$\Rightarrow 40 = 20 + 60e^{-0.04055t}$

$\Rightarrow 20 = 60e^{-0.04055t}$

$\Rightarrow \frac{1}{3} = e^{-0.04055t}$

$\Rightarrow 3 = e^{0.04055t}$

$\Rightarrow \ln 3 = 0.04055t$

$\Rightarrow t \approx 27.1$

Question 24 (****)

At time t hours, the rate of decay of the mass, x kg, of a radioactive substance is directly proportional to the mass present at that time. Initially the mass is x_0.

a) By forming and solving a suitable differential equation, show that

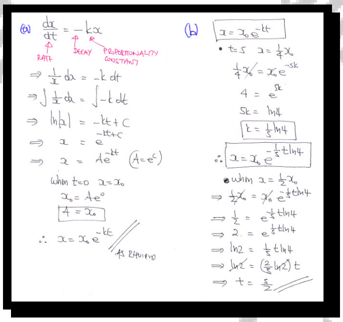
$$x = x_0 \mathrm{e}^{-kt},$$

where k is a positive constant.

When $t = 5$, $x = \frac{1}{4}x_0$.

b) Find the value of t when $x = \frac{1}{2}x_0$.

$$\boxed{t = \frac{5}{2}}$$



Question 25 (****)

At a given instant a lake is thought to contain 20000 fish and the following model is assumed for times t weeks after that instant.

The number of fish N, in tens of thousands is increasing at a rate $0.2N$, fish are dying at a rate $0.1N^2$ and fish are harvested at the constant rate of 1000 per week.

a) Show clearly that

$$\frac{dN}{dt} = -\frac{1}{10}(N-1)^2 .$$

b) Solve the above differential equation giving the answer in the form $N = f(t)$.

c) Find after how many weeks the number of fish will drop to 16250.

d) State the long term prospects for the fish population.

$\boxed{}$, $\boxed{N = \frac{t+20}{t+10}}$, $\boxed{6}$, $\boxed{\text{population} \to 10000}$

a) FORMING THE O.D.E

N = NUMBER OF FISH (10,000) t=0 N=2 (20000 FISH)
t = TIME IN WEEKS

$$\Rightarrow \frac{dN}{dt} = +0.2N - 0.1N^2 - 0.1$$

RATE OF CHANGE OF FISH (10000 PER WEEK) "BIRTHS" "DEATHS" "FISHED OUT"

$$\Rightarrow \frac{dN}{dt} = -\frac{1}{10}\left[N^2 - 2N + 1\right]$$

$$\Rightarrow \frac{dN}{dt} = -\frac{1}{10}(N-1)^2$$ AS REQUIRED

SOLVING BY SEPARATING VARIABLES

$$\Rightarrow dN = -\frac{1}{10}(N-1)^2\,dt$$

$$\Rightarrow \frac{1}{(N-1)^2}\,dN = -\frac{1}{10}\,dt$$

$$\Rightarrow \int (N-1)^{-2}\,dN = \int -\frac{1}{10}\,dt$$

$$\Rightarrow -(N-1)^{-1} = -\frac{1}{10}t + C$$

$$\Rightarrow \frac{1}{N-1} = \frac{1}{10}t + C$$

APPLY CONDITION t=0 N=2

$$\Rightarrow \frac{1}{2-1} = C$$

$$\Rightarrow C = 1$$

$$\Rightarrow \frac{1}{N-1} = \frac{1}{10}t + 1$$

$$\Rightarrow \frac{10}{N-1} = t + 10$$

$$\Rightarrow \frac{10}{t+10} = N - 1$$

$$\Rightarrow N = \frac{10}{t+10} + 1 = \frac{10 + (t+10)}{t+10}$$

$$\Rightarrow N = \frac{t+20}{t+10}$$

c) N = 1.625 (16250 FISH)

$$\Rightarrow 1.625 = \frac{t+20}{t+10}$$

$$\Rightarrow 1.625t + 16.25 = t + 10$$

$$\Rightarrow 0.625t = 3.75$$

$$\Rightarrow t = 6$$

d) AS $t \to \infty$ $\quad \frac{t+20}{t+10} \to 1$

$\therefore$ POPULATION WILL SETTLE TO 10000

Question 26 (****)

An object is moving in such a way so that its coordinates relative to a fixed origin O are given by

$$x = 4\cos t - 3\sin t + 1, \qquad y = 3\cos t + 4\sin t - 1,$$

where t is the time in seconds.

Initially the object was at the point with coordinates $(5, 2)$.

a) Show that the motion of the particle is governed by the differential equation

$$\frac{dy}{dx} = \frac{1-x}{1+y}.$$

b) Find, in exact form, the possible values of the y coordinate of the object when its x coordinate is 2.

$\boxed{}$, $\boxed{y = -1 \pm 2\sqrt{6}}$

a) DIFFERENTIATING THE GIVEN PARAMETRIC EQUATIONS

- $x = 4\cos t - 3\sin t + 1$ — $\frac{dx}{dt} = -4\sin t - 3\cos t$
- $y = 3\cos t + 4\sin t - 1$ — $\frac{dy}{dt} = -3\sin t + 4\cos t$

$$\Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-3\sin t + 4\cos t}{-4\sin t - 3\cos t}$$

$$\Rightarrow \frac{dy}{dx} = \frac{4\cos t - 3\sin t}{-(3\cos t + 4\sin t)}$$

BUT $x - 1 = 4\cos t - 3\sin t$
$y + 1 = 3\cos t + 4\sin t$

$$\Rightarrow \frac{dy}{dx} = \frac{x-1}{-(y+1)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1-x}{1+y} \quad \text{// AS REQUIRED}$$

b) SOLVING BY SEPARATION OF VARIABLES

$$\Rightarrow dy = \frac{1-x}{1+y}\,dx$$

$$\Rightarrow (1+y)\,dy = (1-x)\,dx$$

$$\Rightarrow \int 1+y \, dy = \int 1-x \, dx$$

$$\Rightarrow y + \tfrac{1}{2}y^2 = x - \tfrac{1}{2}x^2 + C$$

$$\Rightarrow 2y + y^2 = 2x - x^2 + C$$

APPLY THE CONDITION $(5,2)$

$$\Rightarrow (2\times 2) + 2^2 = 2\times 5 - 5^2 + C$$

$$\Rightarrow 8 = -15 + C$$

$$\Rightarrow C = 23$$

$$\Rightarrow y^2 + 2y + x^2 - 2x = 23 \quad \text{(I.E. A CIRCLE)}$$

FINALLY $x = 2$

$$\Rightarrow y^2 + 2y + 4 - 4 = 23$$

$$\Rightarrow y^2 + 2y = 23$$

$$\Rightarrow y^2 + 2y + 1 = 24$$

$$\Rightarrow (y+1)^2 = 24$$

$$\Rightarrow y + 1 = \pm\sqrt{24}$$

$$\Rightarrow y = \begin{cases} -1 + 2\sqrt{6} \\ -1 - 2\sqrt{6} \end{cases} \quad //$$

Question 27 (****+)

In a cold winter morning when the temperature of the air is $10°\text{C}$, Ben the builder pours a cup of coffee out of his flask.

Let x be the temperature of the coffee, in $°\text{C}$, t minutes after it was poured.

The rate at which the temperature of the coffee is decreasing is proportional to the square of the difference between the temperature of the coffee and the air temperature.

The initial temperature of the coffee is $80°\text{C}$ and ten minutes later the temperature of the coffee has dropped to $40°\text{C}$.

By forming and solving a suitable differential equation show that

$$x = \frac{20t + 1200}{2t + 15},$$

and hence find after how many minutes the coffee will have a temperature of $20°\text{C}$.

$\boxed{}$, $\boxed{t = 45}$

FORMING A DIFFERENTIAL EQUATION

x = COFFEE TEMPERATURE (°C)
t = TIME (MINUTES)
$t = 0$, $x = 80$
$t = 10$, $x = 40$

$$\frac{dx}{dt} = -k(x-10)^2$$

RATE
DECREASING
PROPORTIONAL
"DIFFERENCE ... SQUARED"

SOLVING BY SEPARATION OF VARIABLES

$$\Rightarrow \frac{dx}{dt} = -k(x-10)^2$$
$$\Rightarrow dx = -k(x-10)^2\,dt$$
$$\Rightarrow \frac{1}{(x-10)^2}\,dx = -k\,dt$$
$$\Rightarrow \int (x-10)^{-2}\,dx = \int -k\,dt$$
$$\Rightarrow -(x-10)^{-1} = -kt + C$$
$$\Rightarrow \frac{1}{x-10} = At + B$$

APPLY THE CONDITIONS GIVEN

$$t=0,\ x=80 \Rightarrow \frac{1}{70} = B$$
$$\Rightarrow \frac{1}{x-10} = At + \frac{1}{70}$$
$$t=10,\ x=40 \Rightarrow \frac{1}{30} = 10A + \frac{1}{70}$$

$$\Rightarrow A = \frac{1}{525}$$
$$\Rightarrow \frac{1}{x-10} = \frac{1}{525}t + \frac{1}{70}$$
$\times 1050$
$$\Rightarrow \frac{1050}{x-10} = 2t + 15$$
$$\Rightarrow \frac{1050}{2t+15} = x - 10$$
$$\Rightarrow x = \frac{1050}{2t+15} + 10$$
$$\Rightarrow x = \frac{1050 + 10(2t+15)}{2t+15}$$
$$\Rightarrow x = \frac{20t + 1200}{2t+15}$$
AS REQUIRED

FINALLY WHEN $x = 20$

$$\frac{1050}{x-10} = 2t + 15$$ (FROM EARLIER)
$$\frac{1050}{20-10} = 2t + 15$$
$$105 = 2t + 15$$
$$90 = 2t$$
$$t = 45$$

Question 28 (****+)

Mould is spreading on a wall of area 20 m^2 and when it was first noticed 2 m^2 of the wall was already covered by this mould.

Let A, in m^2, represent the area of the wall covered by the mould, after time t weeks.

The rate at which A is changing is proportional to the product of the area covered by the mould and the area of the wall not yet covered by the mould.

After a further period of 2 weeks the area of the wall covered by the mould is 4 m^2.

By forming and solving a suitable differential equation, show that

$$A = \frac{20}{1+9\left(\frac{2}{3}\right)^t}.$$

[ ], [proof]

FORMING A DIFFERENTIAL EQUATION

- A = AREA COVERED BY MOULD (m^2)
- t = TIME (WEEKS)

$$\frac{dA}{dt} = +kA(20-A)$$

RATE

INCREASE/SPREAD

PROPORTIONAL

AREA COVERED BY MOULD

AREA NOT YET COVERED

SOLVING BY SEPARATING VARIABLES

$$\Rightarrow dA = kA(20-A)\,dt$$
$$\Rightarrow \frac{1}{A(20-A)}\,dA = k\,dt$$
$$\Rightarrow \int \frac{1}{A(20-A)}\,dA = \int k\,dt$$

PROCEED BY PARTIAL FRACTIONS

$$\frac{1}{A(20-A)} \equiv \frac{P}{A} + \frac{Q}{20-A}$$
$$\boxed{1 \equiv P(20-A) + QA}$$

- IF $A=0$: $20P = 1$, $P = \frac{1}{20}$
- IF $A=20$: $20Q = 1$, $Q = \frac{1}{20}$

RETURNING TO THE INTEGRAL

$$\Rightarrow \int \frac{\frac{1}{20}}{A} + \frac{\frac{1}{20}}{20-A}\,dA = \int k\,dt$$
$$\Rightarrow \int \frac{1}{A} + \frac{1}{20-A}\,dA = \int 20k\,dt$$
$$\Rightarrow \ln A - \ln|20-A| = 20kt + C$$
$$\Rightarrow \ln\left|\frac{A}{20-A}\right| = 20kt + C$$
$$\Rightarrow \frac{A}{20-A} = e^{20kt+C}$$
$$\Rightarrow \frac{A}{20-A} = e^{20kt} \times e^{C}$$
$$\Rightarrow \frac{A}{20-A} = Be^{\alpha t} \quad (\alpha = 20k,\ B = e^C)$$

APPLY CONDITION $t=0$, $A=2$

$$\Rightarrow \frac{2}{20-2} = Be^0$$
$$\Rightarrow B = \frac{1}{9}$$
$$\Rightarrow \frac{A}{20-A} = \frac{1}{9}e^{\alpha t}$$
$$\Rightarrow \frac{9A}{20-A} = e^{\alpha t}$$

APPLY CONDITION $t=2$, $A=4$

$$\Rightarrow \frac{36}{20-4} = e^{2\alpha}$$
$$\Rightarrow \frac{9}{4} = e^{2\alpha}$$
$$\Rightarrow 2\alpha = \ln\frac{9}{4}$$
$$\Rightarrow \alpha = \frac{1}{2}\ln\frac{9}{4}$$
$$\Rightarrow \alpha = \ln\left(\frac{9}{4}\right)^{\frac{1}{2}}$$
$$\Rightarrow \alpha = \ln\frac{3}{2}$$

REWRITE & TIDY THE SOLUTION

$$\Rightarrow \frac{9A}{20-A} = e^{\alpha t}$$
$$\Rightarrow \frac{9A}{20-A} = e^{(\ln\frac{3}{2})t}$$
$$\Rightarrow \frac{9A}{20-A} = \left(\frac{3}{2}\right)^t$$
$$\Rightarrow (20-A)\left(\frac{3}{2}\right)^t = 9A$$
$$\Rightarrow 20\left(\frac{3}{2}\right)^t - A\left(\frac{3}{2}\right)^t = 9A$$
$$\Rightarrow 20\left(\frac{3}{2}\right)^t = 9A + A\left(\frac{3}{2}\right)^t$$
$$\Rightarrow A\left[9 + \left(\frac{3}{2}\right)^t\right] = 20\left(\frac{3}{2}\right)^t$$
$$\Rightarrow A = \frac{20\left(\frac{3}{2}\right)^t}{\left(\frac{3}{2}\right)^t + 9}$$
$$\Rightarrow A = \frac{20\left(\frac{3}{2}\right)^t\left(\frac{2}{3}\right)^t}{\left(\frac{3}{2}\right)^t\left(\frac{2}{3}\right)^t + 9\left(\frac{2}{3}\right)^t}$$
$$\Rightarrow A = \frac{20}{1+9\left(\frac{2}{3}\right)^t}$$

// AS REQUIRED

Question 29 (****+)

The mass of a radioactive isotope decays at a rate proportional to the mass of the isotope present.

The half life of the isotope is 80 years.

Determine the percentage of the original amount which remains after 50 years.

$\boxed{}$, $\boxed{64.8\%}$

FORMING A DIFFERENTIAL EQUATION

$$\frac{dm}{dt} = -km$$

RATE DECAYING PROPORTIONAL MASS OF ISOTOPE PRESENT

m = MASS (SAY IN kg)
t = TIME (YEARS)

$t=0 \quad m=M$
$t=80 \quad m=\frac{1}{2}M$

SEPARATING VARIABLES

$\Rightarrow dm = -km\,dt$

$\Rightarrow \frac{1}{m}\,dm = -k\,dt$

$\Rightarrow \int \frac{1}{m}\,dm = \int -k\,dt$

$\Rightarrow \ln|m| = -kt + C$

$\Rightarrow m = e^{-kt+C}$

$\Rightarrow m = e^{-kt} \times e^{C}$

$\Rightarrow m = Ae^{-kt} \qquad (A = e^{C})$

APPLY CONDITION $m=M$, AT $t=0$

$\Rightarrow M = Ae^{0}$

$\Rightarrow A = M$

$\Rightarrow m = Me^{-kt}$

"HALF LIFE OF 80" $\Rightarrow t=80 \quad m=\frac{1}{2}M$

$\Rightarrow \frac{1}{2}M = Me^{-80k}$

$\Rightarrow \frac{1}{2} = e^{-80k}$

$\Rightarrow 2 = e^{80k}$

$\Rightarrow \left(e^{10k}\right)^{8} = 2$

$\Rightarrow e^{10k} = 2^{\frac{1}{8}} \approx 1.0905\ldots \qquad (\text{OR } k \simeq 0.0086643\ldots)$

FINALLY WITH $t=50$

$\Rightarrow m = Me^{-kt}$

$\Rightarrow m = Me^{-50k}$

$\Rightarrow m = M\left(e^{10k}\right)^{-5}$

$\Rightarrow m = M \times 2^{-\frac{5}{8}}$

$\therefore$ PROPORTION WHICH REMAINS $= \frac{2^{-\frac{5}{8}}\cancel{M}}{\cancel{M}} = \frac{1}{2^{\frac{5}{8}}} = 0.648$

$\simeq 64.8\%$

Question 30 (****+)

A small forest with an area of 25 km^2 has caught fire.

Let A, in km^2, be the area of the forest destroyed by the fire, t hours after the fire was first noticed.

The rate at which the forest is destroyed is proportional to the difference between the total area of the forest squared, and the area of the forest destroyed squared.

When the fire was first noticed 7 km^2 of the forest had been destroyed and **at that instant** the rate at which the area of the forest was destroyed was 7.2 km^2 per hour.

a) Show clearly that

$$50\frac{dA}{dt} = \frac{5}{8}\left(625 - A^2\right).$$

b) Solve the differential equation to obtain

$$\frac{25+A}{25-A} = \frac{16}{9}e^{\frac{5}{8}t}.$$

c) Show further that 14 km^2 of the forest will be destroyed, approximately 66 minutes after the fire was first noticed.

$\boxed{}$, $\boxed{\text{proof}}$

a) FORMING THE DIFFERENTIAL EQUATION

A = AREA OF FOREST DESTROYED (km^2)
t = TIME (IN HOURS)
$t=0$, $A=7$, $\left.\frac{dA}{dt}\right|_{\substack{t=0\\A=7}} = 7.2$

$\frac{dA}{dt} = +k\left(25^2 - A^2\right)$ ← DIFFERENCE BETWEEN...

RATE; AREA OF THE FOREST BEING DESTROYED IS INCREASING; PROPORTIONAL; AREA OF FOREST SQUARED; AREA OF THE FOREST DESTROYED, SQUARED

APPLY THE CONDITION $\left.\frac{dA}{dt}\right|_{A=7} = 7.2$

$\Rightarrow 7.2 = k\left(25^2 - 7^2\right)$
$\Rightarrow 7.2 = 576k$
$\Rightarrow k = \frac{1}{80}$
$\Rightarrow \frac{dA}{dt} = \frac{1}{80}\left(625 - A^2\right)$ $\times 50$
$\Rightarrow 50\frac{dA}{dt} = \frac{5}{8}\left(625 - A^2\right)$ AS REQUIRED

b) SEPARATING VARIABLES

$\Rightarrow 50\,dA = \frac{5}{8}\left(625 - A^2\right)dt$
$\Rightarrow \frac{50}{625 - A^2}\,dA = \frac{5}{8}\,dt$
$\Rightarrow \int \frac{50}{(25+A)(25-A)}\,dA = \int \frac{5}{8}\,dt$

OBTAIN THE PARTIAL FRACTIONS

$\frac{50}{(25+A)(25-A)} \equiv \frac{P}{25+A} + \frac{Q}{25-A}$

$50 \equiv P(25 - A) + Q(25+A)$

• IF $A = 25$: $50 = 50Q$, $Q = 1$
• IF $A = -25$: $50 = 50P$, $P = 1$

RETURNING TO THE INTEGRAL

$\Rightarrow \int \frac{1}{25+A} + \frac{1}{25-A}\,dA = \int \frac{5}{8}\,dt$
$\Rightarrow \ln|25+A| - \ln|25-A| = \frac{5}{8}t + C$
$\Rightarrow \ln\left|\frac{25+A}{25-A}\right| = \frac{5}{8}t + C$
$\Rightarrow \frac{25+A}{25-A} = e^{\frac{5}{8}t + C}$
$\Rightarrow \frac{25+A}{25-A} = e^{\frac{5}{8}t} \times e^{C}$
$\Rightarrow \frac{25+A}{25-A} = Be^{\frac{5}{8}t}$ $(B = e^C)$

APPLY THE CONDITION $t=0$ $A=7$

$\Rightarrow \frac{25+7}{25-7} = B$
$\Rightarrow B = \frac{32}{18}$
$\Rightarrow B = \frac{16}{9}$
$\Rightarrow \frac{25+A}{25-A} = \frac{16}{9}e^{\frac{5}{8}t}$ AS REQUIRED

c) FINALLY WHEN $A = 14$

$\Rightarrow \frac{25+14}{25-14} = \frac{16}{9}e^{\frac{5}{8}t}$
$\Rightarrow \frac{39}{11} = \frac{16}{9}e^{\frac{5}{8}t}$
$\Rightarrow \frac{351}{176} = e^{\frac{5}{8}t}$
$\Rightarrow \frac{5}{8}t = 0.6903022...$
$\Rightarrow t \approx 1.1044...$ HOURS
$\Rightarrow t \approx 66.269...$ MINUTES

IT APPROX 66 MINUTES

Question 31 (****+)

The initial population of a city is 1 million.

Let P be the number of inhabitants in millions, t be the time in years, and treat P as a continuous variable.

The rate at which the population of this city is growing per year, is proportional to the product of its population and the difference of its population from 3 million.

a) By forming and solving a differential equation, show that

$$\frac{2P}{3-P} = e^{at},$$

where a is a positive constant.

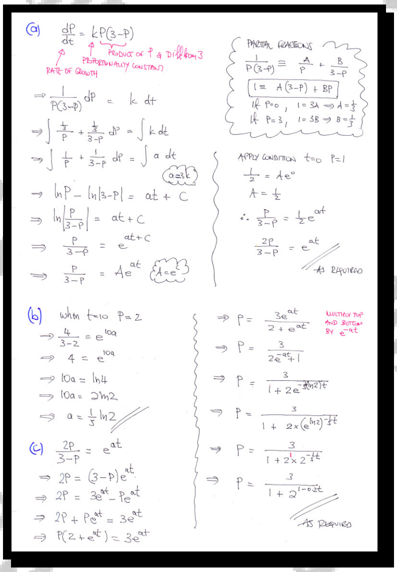
The city doubles its population to 2 million, after ten years.

b) Find the value of a in terms of $\ln 2$.

c) Rearrange the answer in part **(a)** to show that

$$P = \frac{3}{1+2^{1-0.2t}}.$$

$\boxed{}$, $\boxed{a = \frac{1}{5}\ln 2}$



(a) $\frac{dP}{dt} = kP(3-P)$

RATE OF GROWTH

PROPORTIONALITY CONSTANT

PRODUCT OF P & DIFF from 3

$\Rightarrow \frac{1}{P(3-P)}\,dP = k\,dt$

$\Rightarrow \int \frac{\frac{1}{3}}{P} + \frac{\frac{1}{3}}{3-P}\,dP = \int k\,dt$

$\Rightarrow \int \frac{1}{P} + \frac{1}{3-P}\,dP = \int a\,dt$ $\quad a=3k$

$\Rightarrow \ln P - \ln|3-P| = at + C$

$\Rightarrow \ln\left|\frac{P}{3-P}\right| = at + C$

$\Rightarrow \frac{P}{3-P} = e^{at+C}$

$\Rightarrow \frac{P}{3-P} = Ae^{at}$ $\quad A=e^C$

PARTIAL FRACTIONS

$\frac{1}{P(3-P)} \equiv \frac{A}{P} + \frac{B}{3-P}$

$\boxed{1 \equiv A(3-P) + BP}$

If $P=0$, $1=3A \Rightarrow A=\frac{1}{3}$

If $P=3$, $1=3B \Rightarrow B=\frac{1}{3}$

APPLY CONDITION $t=0$ $P=1$

$\frac{1}{2} = Ae^0$

$A = \frac{1}{2}$

$\therefore \frac{P}{3-P} = \frac{1}{2}e^{at}$

$\frac{2P}{3-P} = e^{at}$

AS REQUIRED

(b) when $t=10$ $P=2$

$\Rightarrow \frac{4}{3-2} = e^{10a}$

$\Rightarrow 4 = e^{10a}$

$\Rightarrow 10a = \ln 4$

$\Rightarrow 10a = 2\ln 2$

$\Rightarrow a = \frac{1}{5}\ln 2$

(c) $\frac{2P}{3-P} = e^{at}$

$\Rightarrow 2P = (3-P)e^{at}$

$\Rightarrow 2P = 3e^{at} - Pe^{at}$

$\Rightarrow 2P + Pe^{at} = 3e^{at}$

$\Rightarrow P(2+e^{at}) = 3e^{at}$

$\Rightarrow P = \frac{3e^{at}}{2+e^{at}}$ MULTIPLY TOP AND BOTTOM BY e^{-at}

$\Rightarrow P = \frac{3}{2e^{-at}+1}$

$\Rightarrow P = \frac{3}{1+2e^{-\frac{1}{5}(\ln 2)t}}$

$\Rightarrow P = \frac{3}{1+2\times(e^{\ln 2})^{-\frac{1}{5}t}}$

$\Rightarrow P = \frac{3}{1+2^1\times 2^{-\frac{1}{5}t}}$

$\Rightarrow P = \frac{3}{1+2^{1-0.2t}}$

AS REQUIRED

Question 32 (****+)

A variable x decreases with time t, both in suitable units, at a rate directly proportional to the value of x^3 at that time.

If the value of x is half of its initial value when $t = 3$, determine the value of t when x has reduced to 20% of its initial value.

$\boxed{}$, $\boxed{t = 24}$

START WITH THE OBVIOUS O.D.E

$$\frac{dx}{dt} = -kx^3$$

SOLVE BY SEPARATING VARIABLES

$$-\frac{1}{x^3}\,dx = k\,dt$$

$$\int -x^{-3}\,dx = \int k\,dt$$

$$\frac{1}{2}x^{-2} = kt + C$$

$$\frac{1}{2x^2} = kt + C$$

MULTIPLY BY 2, AND RELABEL CONSTANTS

$$\frac{1}{x^2} = At + B$$

$$\boxed{x^2 = \frac{1}{At + B}}$$

NOW WE NEED AN ARBITRARY STARTING VALUE, SAY $t=0$, $x = x_0$

$$x_0^2 = \frac{1}{0 + B}$$

$$B = \frac{1}{x_0^2}$$

$$x^2 = \frac{1}{At + \frac{1}{x_0^2}}$$

MULTIPLY TOP & BOTTOM BY x_0^2

$$x^2 = \frac{x_0^2}{Ax_0^2 t + 1}$$

RELABEL Ax_0^2 AS D

$$\boxed{x^2 = \frac{x_0^2}{Dt + 1}}$$

NEXT APPLY CONDITION WHEN $t=3$ $x = \frac{1}{2}x_0$ (HALF)

$$\Rightarrow \left(\frac{1}{2}x_0\right)^2 = \frac{x_0^2}{D \times 3 + 1}$$

$$\Rightarrow \frac{1}{4}x_0^2 = \frac{x_0^2}{3D + 1}$$

$$\Rightarrow \frac{1}{4} = \frac{1}{3D + 1}$$

$$\Rightarrow 3D + 1 = 4$$

$$\Rightarrow D = 1$$

$$\therefore \boxed{x^2 = \frac{x_0^2}{t + 1}}$$

THUS WHEN $x = \frac{1}{5}x_0$ (20%)

$$\left(\frac{1}{5}x_0\right)^2 = \frac{x_0^2}{t + 1}$$

$$\frac{1}{25}x_0^2 = \frac{x_0^2}{t + 1}$$

$$\frac{1}{25} = \frac{1}{t + 1}$$

$$t + 1 = 25$$

$$t = 24$$

Question 33 (****+)

An object is released from rest from a great height and allowed to fall down through still air, all the way to the ground.

Let v ms^{-1} be the velocity of the object t seconds after it was released.

The velocity of the object is increasing at the constant rate of 10 ms^{-1} every second.

At the same time due to the air resistance its velocity is decreasing at a rate proportional its velocity at that time.

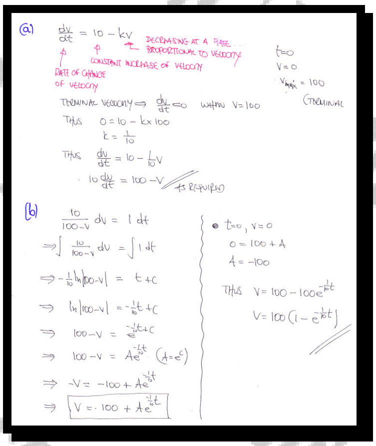
The maximum velocity that the particle can achieve is 100 ms^{-1}.

Show clearly that …

a) … $10\dfrac{dv}{dt} = 100 - v$.

b) … $v = 100\left(1 - \mathrm{e}^{-0.1t}\right)$

$\boxed{\text{proof}}$



(a) $\frac{dv}{dt} = 10 - kv$

DECREASING AT A RATE PROPORTIONAL TO VELOCITY

CONSTANT INCREASE OF VELOCITY

RATE OF CHANGE OF VELOCITY

$t=0$

$V=0$

$V_{max} = 100$

(TERMINAL

TERMINAL VELOCITY $\Rightarrow \frac{dv}{dt} = 0$ WHEN $V = 100$

THUS $0 = 10 - k \times 100$

$k = \frac{1}{10}$

THUS $\frac{dv}{dt} = 10 - \frac{1}{10}V$

$10\frac{dv}{dt} = 100 - V$ AS REQUIRED

(b) $\frac{10}{100-V}\,dV = 1\,dt$

$\Rightarrow \int \frac{10}{100-V}\,dV = \int 1\,dt$

$\Rightarrow -\frac{1}{10}\ln|100-V| = t + C$

$\Rightarrow \ln|100-V| = -\frac{1}{10}t + C$

$\Rightarrow 100 - V = e^{-\frac{1}{10}t + C}$

$\Rightarrow 100 - V = Ae^{-\frac{1}{10}t}$ $(A = e^C)$

$\Rightarrow -V = -100 + Ae^{-\frac{1}{10}t}$

$\Rightarrow \boxed{V = 100 + Ae^{-\frac{1}{10}t}}$

• $t=0$, $V=0$

$0 = 100 + A$

$A = -100$

THUS $V = 100 - 100e^{-\frac{1}{10}t}$

$V = 100\left(1 - e^{-\frac{1}{10}t}\right)$

Question 34 (****+)

An object is placed on the still water of a lake and allowed to fall down through the water to the bottom of the lake.

Let v ms^{-1} be the velocity of the object t seconds after it was released.

The velocity of the object is increasing at the constant rate of 9.8 ms^{-1} every second.

At the same time due to the resistance of the water its velocity is decreasing at a rate proportional to the square of its velocity at that time.

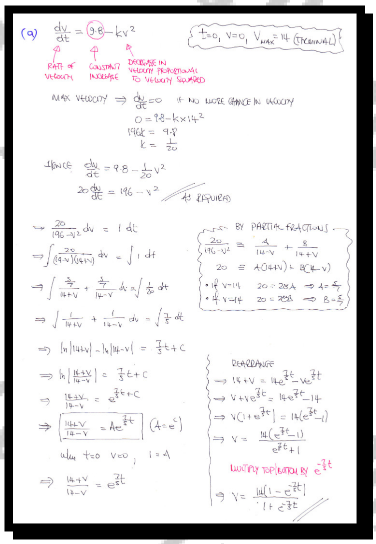
The maximum velocity that the particle can achieve is 14 ms^{-1}.

Show clearly that …

a) … $20\dfrac{dv}{dt} = 196 - v^2$.

b) … $v = 14\left(\dfrac{1-e^{-1.4t}}{1+e^{-1.4t}}\right)$

$\boxed{\text{proof}}$



(a) $\dfrac{dv}{dt} = 9.8 - kv^2$

$\{t=0,\ v=0,\ V_{MAX}=14 \text{ (TERMINAL)}\}$

RATE OF VELOCITY

CONSTANT INCREASE

DECREASE IN VELOCITY PROPORTIONAL TO VELOCITY SQUARED

MAX VELOCITY $\Rightarrow \dfrac{dv}{dt}=0$ IF NO MORE CHANGE IN VELOCITY

$0 = 9.8 - k\times 14^2$

$196k = 9.8$

$k = \frac{1}{20}$

HENCE $\dfrac{dv}{dt} = 9.8 - \frac{1}{20}v^2$

$20\dfrac{dv}{dt} = 196 - v^2$ // AS REQUIRED

$\Rightarrow \dfrac{20}{196-v^2}\,dv = 1\,dt$

$\Rightarrow \displaystyle\int \frac{20}{(14-v)(14+v)}\,dv = \int 1\,dt$

$\Rightarrow \displaystyle\int \frac{\frac{5}{7}}{14+v} + \frac{\frac{5}{7}}{14-v}\,dv = \int \frac{1}{20}\,dt$

$\Rightarrow \displaystyle\int \frac{1}{14+v} + \frac{1}{14-v}\,dv = \int \frac{7}{5}\,dt$

$\Rightarrow \ln|14+v| - \ln|14-v| = \frac{7}{5}t + C$

$\Rightarrow \ln\left|\dfrac{14+v}{14-v}\right| = \frac{7}{5}t + C$

$\Rightarrow \dfrac{14+v}{14-v} = e^{\frac{7}{5}t+C}$

$\Rightarrow \boxed{\dfrac{14+v}{14-v} = Ae^{\frac{7}{5}t}}\ (A=e^C)$

when $t=0$ $v=0$, $1=A$

$\Rightarrow \dfrac{14+v}{14-v} = e^{\frac{7}{5}t}$

BY PARTIAL FRACTIONS

$\dfrac{20}{196-v^2} \equiv \dfrac{A}{14-v} + \dfrac{B}{14+v}$

$20 \equiv A(14+v) + B(14-v)$

- if $v=14$ $20=28A \Rightarrow A=\frac{5}{7}$
- if $v=-14$ $20=28B \Rightarrow B=\frac{5}{7}$

REARRANGE

$\Rightarrow 14+v = 14e^{\frac{7}{5}t} - ve^{\frac{7}{5}t}$

$\Rightarrow v + ve^{\frac{7}{5}t} = 14e^{\frac{7}{5}t} - 14$

$\Rightarrow v(1+e^{\frac{7}{5}t}) = 14(e^{\frac{7}{5}t}-1)$

$\Rightarrow v = \dfrac{14(e^{\frac{7}{5}t}-1)}{e^{\frac{7}{5}t}+1}$

MULTIPLY TOP/BOTTOM BY $e^{-\frac{7}{5}t}$

$\Rightarrow v = \dfrac{14(1-e^{-\frac{7}{5}t})}{1+e^{-\frac{7}{5}t}}$ //

Question 35 (****+)

Water is pouring into a container at a constant rate of $200 \text{ cm}^3\text{s}^{-1}$ and is leaking from a hole at the base of the container at a rate proportional the volume V of the water already in the container.

a) Form a differential equation connecting the volume $V \text{ cm}^3$, the time t in seconds and a proportionality constant k.

b) Show that a general solution of the differential equation is given by

$$V = \frac{200}{k} + A\mathrm{e}^{-kt},$$

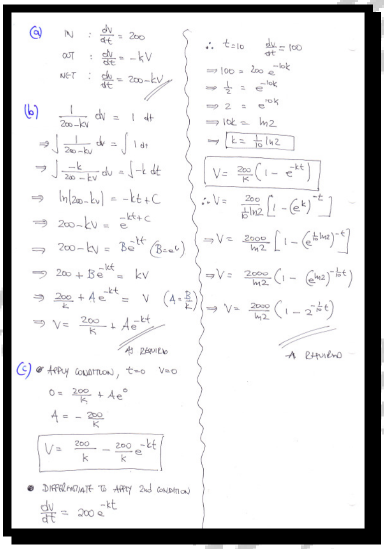
where A is a constant.

The container was initially empty and after 10 seconds the volume of the water V is increasing at the rate of $100 \text{ cm}^3\text{s}^{-1}$.

c) Show further that

$$V = \frac{2000}{\ln 2}\left(1 - 2^{-\frac{1}{10}t}\right).$$

$\boxed{}$, $\boxed{\frac{dV}{dt} = 200 - kV}$


(a) IN : $\frac{dV}{dt} = 200$
OUT : $\frac{dV}{dt} = -kV$
NET : $\frac{dV}{dt} = 200 - kV$
(b) $\frac{1}{200-kV}\,dV = 1\,dt$
$\Rightarrow \int \frac{1}{200-kV}\,dV = \int 1\,dt$
$\Rightarrow \int \frac{-k}{200-kV}\,dV = \int -k\,dt$
$\Rightarrow \ln|200-kV| = -kt + C$
$\Rightarrow 200 - kV = e^{-kt+C}$
$\Rightarrow 200 - kV = Be^{-kt} \quad (B = e^C)$
$\Rightarrow 200 + Be^{-kt} = kV$
$\Rightarrow \frac{200}{k} + Ae^{-kt} = V \quad \left(A = \frac{B}{k}\right)$
$\Rightarrow V = \frac{200}{k} + Ae^{-kt}$
AS REQUIRED
(c) • APPLY CONDITION, $t=0 \quad V=0$
$0 = \frac{200}{k} + Ae^0$
$A = -\frac{200}{k}$
$\boxed{V = \frac{200}{k} - \frac{200}{k}e^{-kt}}$
• DIFFERENTIATE TO APPLY 2nd CONDITION
$\frac{dV}{dt} = 200e^{-kt}$
$\therefore \; t = 10 \quad \frac{dV}{dt} = 100$
$\Rightarrow 100 = 200e^{-10k}$
$\Rightarrow \frac{1}{2} = e^{-10k}$
$\Rightarrow 2 = e^{10k}$
$\Rightarrow 10k = \ln 2$
$\Rightarrow \boxed{k = \frac{1}{10}\ln 2}$
$\boxed{V = \frac{200}{k}\left(1 - e^{-kt}\right)}$
$\therefore V = \frac{200}{\frac{1}{10}\ln 2}\left[1 - \left(e^{k}\right)^{-t}\right]$
$\Rightarrow V = \frac{2000}{\ln 2}\left[1 - \left(e^{\frac{1}{10}\ln 2}\right)^{-t}\right]$
$\Rightarrow V = \frac{2000}{\ln 2}\left(1 - \left(e^{\ln 2}\right)^{-\frac{1}{10}t}\right)$
$\Rightarrow V = \frac{2000}{\ln 2}\left(1 - 2^{-\frac{1}{10}t}\right)$
AS REQUIRED

Question 36 (****+)

There are $20{,}000$ chickens in a farm and some of them have been infected by a virus. Let x be the number of infected chickens **in thousands**, and t the time in hours since the infection was first discovered.

The rate at which chickens are infected is proportional to the product of the number of chickens infected and the number of chickens not yet infected.

a) Form a differential equation in terms of x, t and a proportionality constant k.

When the disease was first discovered 4000 chickens were infected, and chickens were infected at the rate of 32 chickens per hour.

b) Solve the differential equation to show that

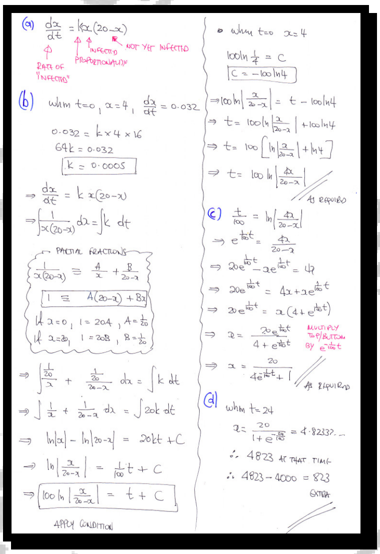
$$t = 100\ln\left[\frac{4x}{20-x}\right].$$

c) Rearrange the answer in part **(b)** to show further that

$$x = \frac{20}{1+4e^{-0.01t}}.$$

d) If a vet cannot attend the farm for 24 hours, since the infection was first discovered, find how many extra chickens will be infected by the time the vet arrives.

$\boxed{}$, $\boxed{\frac{dx}{dt} = kx(20-x)}$, $\boxed{823}$



(a) $\frac{dx}{dt} = kx(20-x)$

RATE OF "INFECTED"

PROPORTIONALITY

INFECTED

NOT YET INFECTED

(b) when $t=0$, $x=4$, $\frac{dx}{dt} = 0.032$

$0.032 = k \times 4 \times 16$

$64k = 0.032$

$\boxed{k = 0.0005}$

$\Rightarrow \frac{dx}{dt} = kx(20-x)$

$\Rightarrow \int \frac{1}{x(20-x)}\,dx = \int k\,dt$

PARTIAL FRACTIONS

$\frac{1}{x(20-x)} \equiv \frac{A}{x} + \frac{B}{20-x}$

$\boxed{1 \equiv A(20-x) + Bx}$

If $x=0$, $1 = 20A$, $A = \frac{1}{20}$

If $x=20$, $1 = 20B$, $B = \frac{1}{20}$

$\Rightarrow \int \frac{\frac{1}{20}}{x} + \frac{\frac{1}{20}}{20-x}\,dx = \int k\,dt$

$\Rightarrow \int \frac{1}{x} + \frac{1}{20-x}\,dx = \int 20k\,dt$

$\Rightarrow \ln|x| - \ln|20-x| = 20kt + C$

$\Rightarrow \ln\left|\frac{x}{20-x}\right| = \frac{1}{100}t + C$

$\Rightarrow \boxed{100\ln\left|\frac{x}{20-x}\right| = t + C}$

APPLY CONDITION

• when $t=0$ $x=4$

$100\ln\frac{1}{4} = C$

$\boxed{C = -100\ln 4}$

$\Rightarrow 100\ln\left|\frac{x}{20-x}\right| = t - 100\ln 4$

$\Rightarrow t = 100\ln\left|\frac{x}{20-x}\right| + 100\ln 4$

$\Rightarrow t = 100\left[\ln\left|\frac{x}{20-x}\right| + \ln 4\right]$

$\Rightarrow t = 100\ln\left|\frac{4x}{20-x}\right|$

AS REQUIRED

(c) $\frac{t}{100} = \ln\left|\frac{4x}{20-x}\right|$

$\Rightarrow e^{\frac{1}{100}t} = \frac{4x}{20-x}$

$\Rightarrow 20e^{\frac{1}{100}t} - xe^{\frac{1}{100}t} = 4x$

$\Rightarrow 20e^{\frac{1}{100}t} = 4x + xe^{\frac{1}{100}t}$

$\Rightarrow 20e^{\frac{1}{100}t} = x(4 + e^{\frac{1}{100}t})$

$\Rightarrow x = \frac{20e^{\frac{1}{100}t}}{4 + e^{\frac{1}{100}t}}$

MULTIPLY TOP/BOTTOM BY $e^{-\frac{1}{100}t}$

$\Rightarrow x = \frac{20}{4e^{-\frac{1}{100}t} + 1}$

AS REQUIRED

(d) when $t = 24$

$x = \frac{20}{1 + e^{-\frac{24}{100}}} = 4.82337\ldots$

$\therefore$ 4823 AT THAT TIME

$\therefore 4823 - 4000 = 823$

EXTRA

Question 37 (****+)

An unstable substance Z decomposes into two different substances X and Y, and at the same time X and Y recombine to reform substance Z. Two parts of Z decompose to one part of X and one part of Y, and at the same time one part of X and one part of Y recombine to reform two parts of Z. As a result at any given time the mass of X and Y are equal.

The rate at which the mass of Z reduces, due to decomposition, is k times the mass of Z present. The rate at which the mass of Z increases, due to reforming, is $4k$ times the product of the masses of X and Y.

Initially there are 6 grams of Z only.

a) Show that if x grams is the mass of X present, t seconds after the reaction started, then

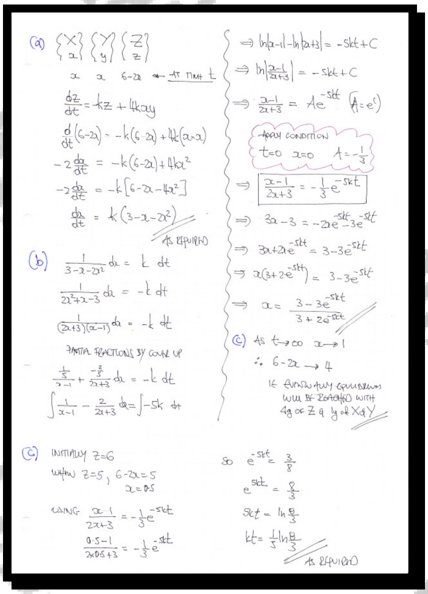
$$\frac{dx}{dt} = k\left(3 - x - 2x^2\right).$$

b) Find a solution of the above differential equation, in the form $x = f(t)$.

c) Find the limiting values of the three substances.

d) Show that when the mass of Z is 5 grams, $kt = \frac{1}{5}\ln\left(\frac{8}{3}\right)$.

$$\boxed{x = \frac{3-3e^{-5kt}}{3+2e^{-5kt}}}, \quad \boxed{x \mapsto 1, \;\; y \mapsto 1, \;\; z \mapsto 4}$$



Question 38 (*****)

Fungus is spreading on a wall and when it was first noticed $\frac{1}{3}$ of the wall was already covered by this fungus.

Let x represent the proportion of the wall not yet covered by the fungus, t weeks after the fungus was first required. The rate at which x is changing is proportional to the square root of the proportion of the wall not yet covered by the fungus.

When the fungus was first noticed it was spreading at rate that if it this rate was to remain constant from that instant onwards the fungus would have covered the entire wall in 4 weeks.

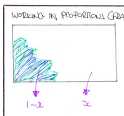
Determine the proportion of the wall covered by the fungus, 4 weeks after it was first noticed.

$\boxed{}$, $\boxed{\frac{5}{6}}$

- $\frac{dx}{dt} = -k\sqrt{x}$

AREA OF THE CLEAR PART OF THE WALL IS DECREASING

WORKING IN PROPORTIONS (FRACTIONS)



$t=0,\ x=\frac{2}{3},\ \frac{dx}{dt} = -\frac{1}{6}$

Week 1 $\frac{1}{6}$	Week 2 $\frac{1}{6}$	ALREADY COVERED $\frac{1}{6}$
Week 3 $\frac{1}{6}$	Week 4 $\frac{1}{6}$	$\frac{1}{6}$

$1 - \frac{1}{3} = \frac{2}{3}$

$\frac{2}{3} \div 4 = \frac{1}{6}$

IF IT WAS TO REMAIN CONSTANT

- APPLY THE CONDITION $x = \frac{2}{3},\ \frac{dx}{dt} = -\frac{1}{6}$ BEFORE SOLVING THE O.D.E

$$-\frac{1}{6} = -k\sqrt{\frac{2}{3}}$$
$$\frac{1}{6} = k\sqrt{\frac{2}{3}}$$
$$k = \frac{1}{6}\sqrt{\frac{3}{2}}$$
$$k = \frac{1}{6}\frac{\sqrt{3}}{\sqrt{2}}$$
$$k = \frac{1}{6}\frac{\sqrt{6}}{2}$$
$$\boxed{k = \frac{1}{12}\sqrt{6}}$$

- SOLVING THE ODE BY SEPARATING VARIABLES; WE REQUIRE x WHEN $t=4$

$$\Rightarrow \frac{dx}{dt} = -\frac{1}{12}\sqrt{6}\sqrt{x}$$
$$\Rightarrow \frac{1}{\sqrt{x}}\,dx = -\frac{\sqrt{6}}{12}\,dt$$
$$\Rightarrow \int_{x=\frac{2}{3}}^{x} x^{-\frac{1}{2}}\,dx = \int_{t=0}^{t=4} -\frac{\sqrt{6}}{12}\,dt$$
$$\Rightarrow \left[2x^{\frac{1}{2}}\right]_{\frac{2}{3}}^{x} = \left[-\frac{\sqrt{6}}{12}t\right]_{0}^{4}$$
$$\Rightarrow 2x^{\frac{1}{2}} - 2\left(\frac{2}{3}\right)^{\frac{1}{2}} = -\frac{\sqrt{6}}{12}(4)$$
$$\Rightarrow 2x^{\frac{1}{2}} - 2\sqrt{\frac{2}{3}} = -\frac{\sqrt{6}}{3}$$
$$\Rightarrow 2x^{\frac{1}{2}} - 2\frac{\sqrt{2}}{\sqrt{3}} = -\frac{\sqrt{6}}{3}$$
$$\Rightarrow 2x^{\frac{1}{2}} - \frac{2}{3}\sqrt{6} = -\frac{1}{3}\sqrt{6}$$
$$\Rightarrow 2\sqrt{x} = \frac{1}{3}\sqrt{6}$$
$$\Rightarrow 4x = \frac{1}{9}\times 6$$
$$\Rightarrow 4x = \frac{2}{3}$$
$$\Rightarrow x = \frac{1}{6}$$ ← PROPORTION WITHOUT FUNGUS

$\therefore \frac{5}{6}$ OF THE WALL IS COVERED BY THE FUNGUS

Question 39 (*****)

The mass of a radioactive isotope decays at a rate proportional to the mass of the isotope present at that instant.

The half life of the isotope is 12 days.

Show that the proportion of the original amount of the isotope left after a period of 30 days is $\frac{1}{8}\sqrt{2}$.

[ ] , [proof]

$$\frac{dm}{dt} = -km$$

m = MASS PRESENT
t = TIME IN DAYS

IMPLIED CONDITIONS
$t=0 \quad m=m_0$ (ARBITRARY)
$t=12 \quad m=\frac{1}{2}m_0$

- SEPERATING VARIABLES & SOLVING

$\Rightarrow \frac{1}{m}\,dm = -k\,dt$

$\Rightarrow \int \frac{1}{m}\,dm = \int -k\,dt$

$\Rightarrow \ln m = -kt + C$

$\Rightarrow m = e^{-kt+C}$

$\Rightarrow \boxed{m = Ae^{-kt}} \quad (A=e^C)$

- APPLY THE FIRST CONDITION, $t=0 \quad m=m_0$

$m_0 = Ae^0$

$\boxed{A = m_0}$

$\Rightarrow \boxed{m = m_0e^{-kt}}$

- APPLY THE SECOND CONDITION, $t=12 \quad m=\frac{1}{2}m_0$

$\frac{1}{2}m_0 = m_0e^{-12k}$

$\frac{1}{2} = e^{-12k}$

$2 = e^{12k}$

$12k = \ln 2$

$\boxed{k = \frac{1}{12}\ln 2}$

$\Rightarrow m = m_0e^{-kt}$

$\Rightarrow m = m_0e^{-\left(\frac{1}{12}\ln 2\right)t}$

$\Rightarrow m = m_0\left(e^{\ln 2}\right)^{-\frac{1}{12}t}$

$\Rightarrow \boxed{m = m_0 \times 2^{-\frac{t}{12}}}$

- WHEN $t=30$

$m = m_0 \times 2^{-\frac{30}{12}}$

$m = m_0 \times 2^{-\frac{5}{2}}$

$m = m_0 \times \frac{1}{2^{5/2}}$

$m = m_0 \times \frac{1}{(\sqrt{2})^5}$

$m = m_0 \times \frac{1}{4\sqrt{2}}$

$m = m_0 \times \frac{\sqrt{2}}{4\sqrt{2}\sqrt{2}}$

$m = m_0 \times \frac{\sqrt{2}}{8}$

I.E. THE PROPORTION LEFT OF THE ORIGINAL m_0 IS $\frac{1}{8}\sqrt{2}$

Question 40 (*****)

A shop stays open for 8 hours every Sunday and its sales, £ x, t hours after the shop opens are modelled as follows.

The rate at which the sales are made, is **directly proportional** to the time left until the shop closes and **inversely proportional** to the sales already made until that time.

Two hours after the shop opens it has made sales worth £336 and sales are made at the rate of £72 per hour.

a) Show clearly that

$$x\frac{dx}{dt} = 4032(8-t).$$

b) Solve the differential equation to show

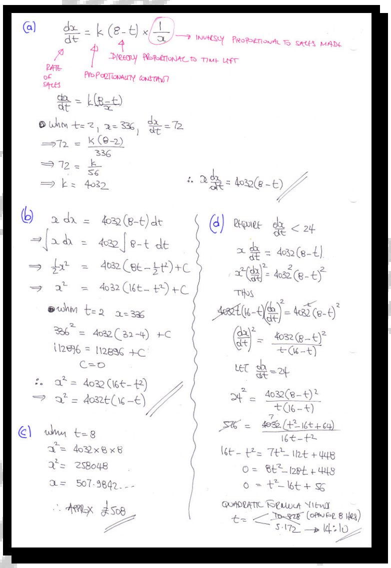
$$x^2 = 4032t(16-t).$$

c) Find, to the nearest £, the Sunday sales of the shop according to this model.

The shop opens on Sundays at 09.00. The owner knows that the shop is not profitable once the rate at which it makes sales drops under £24 per hour.

d) By squaring the differential equation of part **(a)**, find to the nearest minute, the time the shop should close on Sundays.

$\boxed{}$, $\boxed{£508}$, $\boxed{14.10}$



(a) $\frac{dx}{dt} = k(8-t) \times \frac{1}{x}$ → INVERSELY PROPORTIONAL TO SALES MADE

RATE OF SALES

PROPORTIONALITY CONSTANT

DIRECTLY PROPORTIONAL TO TIME LEFT

$\frac{dx}{dt} = k\frac{(8-t)}{x}$

• when $t=2$, $x=336$, $\frac{dx}{dt}=72$

$\Rightarrow 72 = \frac{k(8-2)}{336}$

$\Rightarrow 72 = \frac{k}{56}$

$\Rightarrow k = 4032$

$\therefore x\frac{dx}{dt} = 4032(8-t)$

(b) $x\,dx = 4032(8-t)\,dt$

$\Rightarrow \int x\,dx = 4032\int 8-t\,dt$

$\Rightarrow \frac{1}{2}x^2 = 4032(8t-\frac{1}{2}t^2)+C$

$\Rightarrow x^2 = 4032(16t-t^2)+C$

• when $t=2$ $x=336$

$336^2 = 4032(32-4)+C$

$112896 = 112896 + C$

$C=0$

$\therefore x^2 = 4032(16t-t^2)$

$\Rightarrow x^2 = 4032t(16-t)$

(c) when $t=8$

$x^2 = 4032\times 8\times 8$

$x^2 = 258048$

$x = 507.9842...$

$\therefore$ APPROX £508

(d) REQUIRE $\frac{dx}{dt} < 24$

$x\frac{dx}{dt} = 4032(8-t)$

$x^2\left(\frac{dx}{dt}\right)^2 = 4032^2(8-t)^2$

THUS

$\cancel{4032}t(16-t)\left(\frac{dx}{dt}\right)^2 = 4032^{\cancel{2}}(8-t)^2$

$\left(\frac{dx}{dt}\right)^2 = \frac{4032(8-t)^2}{t(16-t)}$

LET $\frac{dx}{dt}=24$

$24^2 = \frac{4032(8-t)^2}{t(16-t)}$

$\cancel{576} = \frac{\cancel{4032}^{7}(t^2-16t+64)}{16t-t^2}$

$16t-t^2 = 7t^2-112t+448$

$0 = 8t^2-128t+448$

$0 = t^2-16t+56$

QUADRATIC FORMULA YIELDS

$t = $ ~~10.828~~ (OPEN FOR 8 HRS)

$5.172 \rightarrow 14{:}10$

Question 41 (*****)

At time $t=0$, one litre of a certain liquid chemical is added to a tank containing 20 litres of water. The chemical reacts with the water forming a gas, and as a result of this reaction both the volumes of the water and the chemical are reduced.

At time t minutes since the chemical reaction started, the respective volumes of the chemical and the water used in the reaction, are $(1-v)$ litres and $4(1-v)$ litres.

The rate at which the volume of the chemical in the tank reduces, is proportional to the product of the volume of the chemical and the volume of the water, still left in the tank.

Given that 2 minutes after the reaction started the volume of the chemical remaining is $\frac{4}{19}$ of a litre, show that

$$2^t = \frac{v+4}{5v}.$$

$\boxed{}$, $\boxed{\text{proof}}$

IF $1-V$ IS THE VOLUME OF THE CHEMICAL AT TIME t, USED IN THE REACTION THEN V IS THE AMOUNT OF CHEMICAL LEFT $[1-(1-V)]$

IF $4(1-V)$ IS THE VOLUME OF THE WATER, AT TIME t, USED IN THE REACTION THE AMOUNT OF WATER LEFT IS $20-4(1-V)$

$$\frac{dv}{dt} = -kv\left[20-4(1-v)\right]$$

REDUCES

VOLUME OF WATER LEFT

VOLUME OF CHEMICAL LEFT

RATE AT WHICH THE VOLUME OF THE CHEMICAL

$$\Rightarrow \frac{dv}{dt} = -kv\left[20-4+4v\right]$$

$$\Rightarrow \frac{dv}{dt} = -kv\left[4v+16\right]$$

$$\Rightarrow \frac{dv}{dt} = -kv(v+4)$$

ABSORB 4 INTO k

$$\Rightarrow \frac{1}{v(v+4)}\,dv = -k\,dt$$

PARTIAL FRACTIONS BY INSPECTION OR COVER UP

$$\Rightarrow \frac{\frac{1}{4}}{v} + \frac{-\frac{1}{4}}{v+4}\,dv = -k\,dt$$

$$\Rightarrow \frac{1}{v} - \frac{1}{v+4}\,dv = -k\,dt$$

MULTIPLIED BY 4 & ABSORBED INTO k AGAIN

- NOW WHEN $t=0$ $\quad v=1$
  $t=2$ $\quad v=\frac{4}{19}$

- INTEGRATE USING THE INITIAL CONDITION

$$\Rightarrow \int_{v=1}^{v} \frac{1}{v} - \frac{1}{v+4}\,dv = \int_{t=0}^{t} -k\,dt$$

$$\Rightarrow \Big[\ln v - \ln|v+4|\Big]_1^v = \Big[-kt\Big]_0^t$$

$$\Rightarrow \ln v - \ln|v+4| + \ln 5 = -kt.$$

$$\Rightarrow \ln\frac{5v}{v+4} = -kt.$$

$$\Rightarrow \frac{5v}{v+4} = e^{-kt}$$

- APPLY $t=2$ $\quad v=\frac{4}{19}$

$$\Rightarrow \frac{5\left(\frac{4}{19}\right)}{\frac{4}{19}+4} = e^{-2k}$$

$$\Rightarrow \frac{20}{4+76} = e^{-2k}$$

$$\Rightarrow \frac{20}{80} = e^{-2k}$$

$$\Rightarrow \frac{1}{4} = e^{-2k}$$

$$\Rightarrow 4 = e^{2k}$$

$$\Rightarrow \ln 4 = 2k$$

$$\Rightarrow k = \ln 2$$

- FINALLY WE MAY TIDY

$$\Rightarrow \frac{5v}{v+4} = e^{-kt}$$

$$\Rightarrow \frac{v+4}{5v} = e^{kt}$$

$$\Rightarrow \frac{v+4}{5v} = \left(e^{k}\right)^t$$

$$\Rightarrow \frac{v+4}{5v} = \left(e^{\ln 2}\right)^t$$

$$\Rightarrow \frac{v+4}{5v} = 2^t$$

Question 42 (*****)

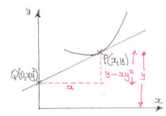
At every point $P(x,y)$ which lie on the curve C, with equation $y = f(x)$, the y intercept of the tangent to C at P has coordinates $(0, xy^2)$.

Given further that the point $Q(1,1)$ also lies on C determine an equation for C, giving the answer in the form $y = f(x)$.

You might find the expression for $\frac{d}{dx}\left(\frac{x}{y}\right)$ useful in this question.

$$\boxed{y = \frac{2x}{1+x^2}}$$

START BY DRAWING A DIAGRAM



THE DIFFERENTIAL EQUATION IS

$$\frac{dy}{dx} = \frac{y - xy^2}{x}$$

LOOKING AT THE HINT GIVEN

$$\frac{d}{dx}\left[\frac{x}{y}\right] = \frac{y \times 1 - x\frac{dy}{dx}}{y^2}$$

REARRANGING THE EQUATION

$$\Rightarrow x\frac{dy}{dx} = y - xy^2$$

$$\Rightarrow xy^2 = y - x\frac{dy}{dx}$$

$$\Rightarrow x = \frac{y - x\frac{dy}{dx}}{y^2}$$

$$\Rightarrow x = \frac{d}{dx}\left(\frac{x}{y}\right)$$

$$\Rightarrow \frac{x}{y} = \int x \, dx$$

$$\Rightarrow \frac{x}{y} = \frac{1}{2}x^2 + C$$

$$\Rightarrow x = \frac{1}{2}x^2y + Cy$$

APPLY CONDITION $(1,1)$

$$\Rightarrow 1 = \frac{1}{2} + C$$

$$\Rightarrow C = \frac{1}{2}$$

HENCE WE OBTAIN

$$\Rightarrow x = \frac{1}{2}x^2y + \frac{1}{2}y$$

$$\Rightarrow 2x = x^2y + y$$

$$\Rightarrow 2x = y(x^2+1)$$

$$\Rightarrow y = \frac{2x}{x^2+1}$$

Question 43 (*****)

In a chemical reaction two substances X and Y bind together to form a third substance Z. In terms of their masses in grams, 1 part of substance X binds with 3 parts of substance Y to form 4 parts of substance Z.

Let z grams be the mass of substance Z formed, t minutes after the reaction started.

The rate at which Z forms is directly proportional to the product of the masses of X and Y, present at that instant.

Initially there were 10 grams of substance X, 10 grams of substance Y and none of substance Z, and the initial rate of formation of Z was 1.6 grams per minute.

a) Show clearly that

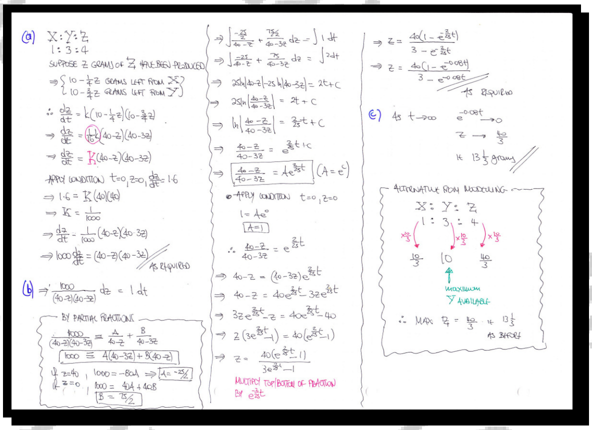
$$1000\frac{dz}{dt} = (40-z)(40-3z).$$

b) Solve the differential equation to show that

$$z = \frac{40\left(1-e^{-0.08t}\right)}{3-e^{-0.08t}}.$$

c) State, with justification, the maximum mass of the substance Z that can ever be produced.

$\boxed{}$, $\boxed{t\to\infty,\ z\to\frac{40}{3}}$



Question 44 (*****)

A shop opens on Saturdays at 09.00 and stays open for 9 hours has its sales modelled as follows.

The rate at which the sales are made, is **directly proportional** to the time left until the shop closes and **inversely proportional** to the sales already made until that time.

One hour after the shop opens it has made sales worth £500 and at that instant sales are made at the rate of £2000 per hour.

The owner knows that the shop is not profitable once the rate at which it makes sales drops under £200 per hour.

Use a detailed method to find the time the shop should close on Saturdays.

$\boxed{\text{SPX-I}}$, $\boxed{14.00}$

FORMING A DIFFERENTIAL EQUATION

x = SALES (IN £1000)
t = TIME (HOURS SINCE 09:00)
$t=0$ AT 09:00 (OPENING)
$t=9$ AT 18:00 (CLOSES)

$t=1$
$x=0.5$ (£500)
$\left.\frac{dx}{dt}\right|_{\substack{t=1\\x=0.5}} = 2$

$$\underbrace{\frac{dx}{dt}}_{\text{RATE OF SALES}} = \underbrace{k}_{\text{PROPORTIONALITY CONSTANT}}\underbrace{(9-t)}_{\text{DIRECTLY PROPORTIONAL TO THE TIME LEFT UNTIL IT CLOSES}} \times \underbrace{\frac{1}{x}}_{\text{INVERSELY PROPORTIONAL}}$$

OPENS FOR 9 HOURS

APPLY THE CONDITION $\left.\frac{dx}{dt}\right|_{\substack{t=1\\x=0.5}} = 2$

$\Rightarrow 2 = k(9-1) \times \frac{1}{0.5}$

$\Rightarrow 2 = 8k \times 2$

$\Rightarrow 8k = 1$

$\Rightarrow k = \frac{1}{8}$

$\Rightarrow \frac{dx}{dt} = \frac{1}{8}\,\frac{9-t}{x}$

SOLVE BY SEPARATING VARIABLES

$\Rightarrow 8x\,dx = (9-t)\,dt$

$\Rightarrow \int_{x=\frac{1}{2}}^{x} 8x\,dx = \int_{t=1}^{t} 9-t\,dt$

$\Rightarrow \left[4x^2\right]_{\frac{1}{2}}^{x} = \left[-\frac{1}{2}(9-t)^2\right]_{1}^{t}$

$\Rightarrow 4x^2 - 1 = -\frac{1}{2}(9-t)^2 + \frac{1}{2}\times 64$

$\Rightarrow 4x^2 = 33 - \frac{1}{2}(9-t)^2$

$\Rightarrow 8x^2 = 66 - (9-t)^2$

COMBINING EQUATIONS TO ELIMINATE x

• O.D.E: $8x\frac{dx}{dt} = 9-t$

$64x^2\left(\frac{dx}{dt}\right)^2 = (9-t)^2$

• SOLUTION: $8x^2 = 66-(9-t)^2$

$64x^2 = 8\left[66-(9-t)^2\right]$

DIVIDING

$\Rightarrow \left(\frac{dx}{dt}\right)^2 = \frac{(9-t)^2}{8(66-(9-t)^2)}$

$\frac{dx}{dt} = 0.2 = \frac{1}{5}$ (£200 PER HOUR)

$\Rightarrow \frac{1}{25} = \frac{(9-t)^2}{8(66-(9-t)^2)}$

$\Rightarrow \frac{8}{25} = \frac{T^2}{66-T^2}$ $(T = 9-t)$

$\Rightarrow 25T^2 = 8(66-T^2)$

$\Rightarrow 25T^2 = 480 + 48 - 8T^2$

$\Rightarrow 33T^2 = 528$

$\Rightarrow T^2 = \frac{528}{33}$

$\Rightarrow T^2 = \frac{330+165+33}{33}$

$\Rightarrow T^2 = 16$

$\Rightarrow T = \pm 4$

$\Rightarrow 9-t = 4$ or -4

$\Rightarrow t = 5$ or 13

$\Rightarrow t = 14{:}00$ or $\cancel{22{:}00}$ SHOP CLOSED

Question 45 (*****)

Initially a tank contains 25 litres of fresh water.

At time $t=0$ salt water of concentration 0.2 kg of salt per litre begins to pour into the tank, at the rate of 1 litre per minute, and at the same time the salt water mix begins to leave the tank at the rate of 1.5 litre per minute.

The concentration of the salt water mix in the tank is thereafter maintained uniform, by constant stirring.

Let x kg be the mass of salt dissolved in the water in the tank, at time t minutes.

a) Show by detailed workings that

$$\frac{dx}{dt}=\frac{1}{5}-\frac{3x}{50-t}.$$

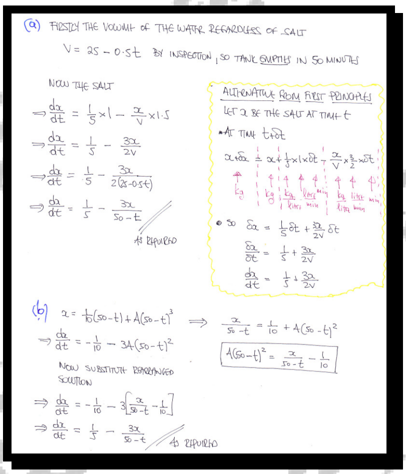
b) Verify by differentiation that the general solution of the differential equation of part **(a)**, is

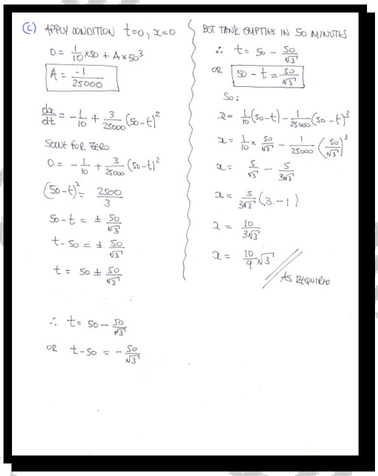
$$x=\frac{1}{10}(50-t)+A(50-t)^3,$$

where A is an arbitrary constant.

c) Determine as an exact simplified surd the maximum value of x.

$\boxed{}$, $\boxed{x_{max}=\frac{10}{9}\sqrt{3}}$





Question 46 (*****)

The point P lies on the curve C with equation $y = f(x)$.

It is further given that C passes through the origin O and lies in the first quadrant.

The normal to C at P meets the x axis at the point A.

The point B is the foot of the perpendicular of P onto the x axis.

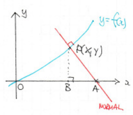
Given that for all positions of P,

$$|OA|^2 = 9|OB|,$$

determine in simplified form an equation of C.

$\boxed{}$, $\boxed{y = \sqrt{4x^{\frac{3}{2}} - x^2}}$

- LET THE POINT P HAVE VARIABLE CO-ORDINATES (X, Y)



- THEN GRADIENT AT P IS GIVEN BY

$$\left.\frac{dy}{dx}\right|_P = \frac{dY}{dX}$$

- EQUATION OF THE NORMAL AT $P(X, Y)$ IS GIVEN BY

$$y - Y = -\frac{1}{\frac{dY}{dX}}(x - X)$$

$$y - Y = -\frac{dX}{dY}(x - X)$$

WHEN $y = 0$ (AT POINT A)

$$-Y = -\frac{dX}{dY}(x - X)$$

$$Y\frac{dY}{dX} = x - X$$

$$x = X + Y\frac{dY}{dX} \quad \leftarrow x \text{ CO-ORD OF A}$$

- NOW $|OA|^2 = 9|OB|$

$$\left(X + Y\frac{dY}{dX}\right)^2 = 9X$$

$$X + Y\frac{dY}{dX} = +3X^{\frac{1}{2}}$$

$$Y\frac{dY}{dX} = 3X^{\frac{1}{2}} - X$$

$$\Rightarrow \int Y\,dY = \int 3X^{\frac{1}{2}} - X\,dX$$

$$\Rightarrow \frac{1}{2}Y^2 = 2X^{\frac{3}{2}} - \frac{1}{2}X^2 + C$$

$$\Rightarrow Y^2 = 4X^{\frac{3}{2}} - X^2 + C$$

AS CURVE PASSES THROUGH THE ORIGIN $C = 0$

$$\therefore Y^2 = 4X^{\frac{3}{2}} - X^2$$

BUT CURVE LIES IN THE FIRST QUADRANT

$$Y = \sqrt{4X^{\frac{3}{2}} - X^2}$$

Question 47 (*****)

An unstable substance Z decomposes into two different substances X and Y, and at the same time X and Y recombine to reform substance Z. Two parts of Z decompose to one part of X and one part of Y, and at the same time one part of X and one part of Y recombine to reform two parts of Z. As a result at any given time the mass of X and Y are equal.

The rate at which the mass of Z reduces, due to decomposition, is k times the mass of Z present. The rate at which the mass of Z increases, due to reforming, is $4k$ times the product of the masses of X and Y.

Initially there are 6 grams of Z only.

Show that when the mass of Z is 5 grams, $kt = \frac{1}{5}\ln\left(\frac{8}{3}\right)$.

proof

- LET THE CORRESPONDING MASSES OF X, Y, Z BE x, y, z.
- AT TIME t THE MASSES OF THE THREE SUBSTANCES PRESENT ARE

Y	Y	Z	
x	y	z	← AT TIME t
x	x	$6-2x$	← A TIME t, IN TERM OF x

- FORMING A DIFFERENTIAL EQUATION

$$\Rightarrow \frac{dz}{dt} = -kz + 4kxy$$
$$\Rightarrow \frac{d}{dt}(6-2x) = -k(6-2x) + 4kx \times x$$
$$\Rightarrow -2\frac{dx}{dt} = -k\left[6-2x-4x^2\right]$$
$$\Rightarrow -\frac{dx}{dt} = k(2x^2+x-3)$$

- SEPARATE VARIABLES

$$\Rightarrow \frac{1}{2x^2+x-3}\,dx = -k\,dt$$
$$\Rightarrow \frac{1}{(2x+3)(x-1)}\,dx = -k\,dt$$

- PARTIAL FRACTIONS BY COVER UP (INSPECTION)

$$\Rightarrow \frac{\frac{1}{5}}{x-1} + \frac{-\frac{2}{5}}{2x+3}\,dx = -k\,dt$$
$$\Rightarrow \int \frac{1}{x-1} - \frac{2}{2x+3}\,dx = \int -5k\,dt$$

- INTEGRATING SUBJECT TO CONDITIONS $t=0$ $x=0$ (ONLY z) AND WE REQUIRE THE TIME WHEN $z=5$, IF $x=\frac{1}{2}$ (USING $z=6-2x$)

$$\Rightarrow \int_{x=0}^{x=\frac{1}{2}} \frac{1}{x-1} - \frac{2}{2x+3}\,dx = \int_{t=0}^{t} -5k\,dt$$
$$\Rightarrow \Big[\ln|x-1| - \ln|2x+3|\Big]_0^{\frac{1}{2}} = \Big[-5kt\Big]_0^t$$
$$\Rightarrow \left(\ln\tfrac{1}{2} - \ln 4\right) - \left(\cancel{\ln 1} - \ln 3\right) = -5kt.$$
$$\Rightarrow \ln\tfrac{1}{2} - \ln 4 + \ln 3 = -5kt$$
$$\Rightarrow 5kt = \ln 4 - \ln 3 - \ln\tfrac{1}{2}$$
$$\Rightarrow 5kt = \ln 4 - \ln 3 + \ln 2$$
$$\Rightarrow 5kt = \ln\frac{8}{3}$$
$$\Rightarrow kt = \frac{1}{5}\ln\frac{8}{3}$$

Question 48 (*****)

The point P and the point $R(0,1)$ lie on the curve with equation

$$f(y) = g(x), \ |y| \le 1.$$

The tangent to the curve at P meets the x axis at the point Q.

Given that $|PQ| = 1$ for all possible positions of P on this curve, determine the equation of this curve, in the form $f(y) = g(x)$.

The final answer may not contain natural logarithms.

$\boxed{}$, $\boxed{\dfrac{y\,e^{\sqrt{1-y^2}}}{1+\sqrt{1-y^2}} = e^{\pm x}}$

START BY MODELLING THE PROBLEM

- LET $P(X,Y)$ LIE ON THE CURVE
- THE EQUATION OF THE TANGENT AT P IS GIVEN BY

$$y - Y = \left.\frac{dy}{dx}\right|_P (x - X)$$

- x INTERCEPT $\Rightarrow y = 0$

$$\Rightarrow -Y = \left.\frac{dy}{dx}\right|_P (x - X)$$

$$\Rightarrow \left(\left.\frac{dx}{dy}\right|_P\right)(-Y) = x - X$$

$$\Rightarrow x = X - Y\left.\frac{dx}{dy}\right|_P$$

$$\therefore \boxed{Q\left[X - Y\left.\frac{dx}{dy}\right|_P, 0\right]}$$

NEXT WE REQUIRE AN EXPRESSION FOR $|PQ|$

$$\Rightarrow |PQ| = \sqrt{\left[X - \left(X - Y\left.\frac{dx}{dy}\right|_P\right)\right]^2 + [Y-0]^2}$$

$$\Rightarrow |PQ| = \sqrt{Y^2\left(\left.\frac{dx}{dy}\right|_P\right)^2 + Y^2}$$

$$\Rightarrow |PQ| = |Y|\sqrt{\left(\left.\frac{dx}{dy}\right|_P\right)^2 + 1}$$

$$\Rightarrow |PQ|^2 = Y^2\left[\left(\left.\frac{dx}{dy}\right|_P\right)^2 + 1\right]$$

$$\Rightarrow 1 = Y^2\left[\left(\left.\frac{dx}{dy}\right|_P\right)^2 + 1\right]$$

$$\Rightarrow \frac{1}{Y^2} = \left(\left.\frac{dx}{dy}\right|_P\right)^2 + 1$$

$$\Rightarrow \frac{1}{Y^2} - 1 = \left(\left.\frac{dx}{dy}\right|_P\right)^2$$

$$\Rightarrow \frac{1-Y^2}{Y^2} = \left(\left.\frac{dx}{dy}\right|_P\right)^2$$

$$\Rightarrow \left(\left.\frac{dy}{dx}\right|_P\right)^2 = \frac{Y^2}{1-Y^2}$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_P = \pm\frac{Y}{\sqrt{1-Y^2}}$$

REWRITE THE O.D.E ALL IN "CAPITALS"

$$\Rightarrow \frac{dY}{dX} = \pm\frac{Y}{\sqrt{1-Y^2}}$$

$$\Rightarrow \frac{\sqrt{1-Y^2}}{Y}\,dY = \pm 1\,dX$$

$$\Rightarrow \int \frac{\sqrt{1-Y^2}}{Y}\,dY = \int \pm 1\,dX$$

USING A TRIGONOMETRIC SUBSTITUTION ON THE LHS

$$\boxed{\begin{array}{l} Y = \sin\theta \quad (\theta = \arcsin Y) \\ dY = \cos\theta\,d\theta \\ \cos\theta = +\sqrt{1-Y^2} \end{array}}$$

$$\Rightarrow \int \frac{\cos\theta}{\sin\theta}(\cos\theta\,d\theta) = \pm X + C$$

$$\Rightarrow \int \frac{\cos^2\theta}{\sin\theta}\,d\theta = \pm X + C$$

$$\Rightarrow \int \frac{1-\sin^2\theta}{\sin\theta}\,d\theta = \pm X + C$$

$$\Rightarrow \int \operatorname{cosec}\theta - \sin\theta\,d\theta = \pm X + C$$

$$\Rightarrow \cos\theta - \ln|\operatorname{cosec}\theta + \cot\theta| = \pm X + C$$

$$\Rightarrow \cos\theta - \ln\left|\frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta}\right| = \pm X + C$$

$$\Rightarrow \cos\theta - \ln\left|\frac{1+\cos\theta}{\sin\theta}\right| = \pm X + C$$

$$\Rightarrow \sqrt{1-Y^2} - \ln\left|\frac{1+\sqrt{1-Y^2}}{Y}\right| = \pm X + C$$

APPLY CONDITION $(0,1)$

$$0 - \ln 1 = C$$

$$\boxed{C = 0}$$

"FINAL TIDY" TO ELIMINATE LOGARITHMS

$$\Rightarrow \sqrt{1-Y^2} - \ln\left|\frac{1+\sqrt{1-Y^2}}{Y}\right| = \pm X$$

$$\Rightarrow \sqrt{1-Y^2} \pm X = \ln\left|\frac{1+\sqrt{1-Y^2}}{Y}\right|$$

$$\Rightarrow e^{\sqrt{1-Y^2} \pm X} = \frac{1+\sqrt{1-Y^2}}{Y}$$

$$\Rightarrow e^{\sqrt{1-Y^2}} \times e^{\pm X} = \frac{1+\sqrt{1-Y^2}}{Y}$$

$$\Rightarrow e^{\pm X} = \frac{1+\sqrt{1-Y^2}}{Y e^{\sqrt{1-Y^2}}}$$

OR $\boxed{\dfrac{1+\sqrt{1-y^2}}{y e^{\sqrt{1-y^2}}} = e^{\pm x}}$ OR $\boxed{\dfrac{y e^{\sqrt{1-y^2}}}{1+\sqrt{1-y^2}} = e^{\pm x}}$

DIFFERENTIAL EQUATIONS

WITH RELATED VARIABLES

Question 1 (***)

A container is in the shape of a hollow right circular cylinder of base radius 50 cm and height 100 cm.

The container is filled with water and is standing upright on horizontal ground. Water is leaking out of a hole on the side of the container which is 1 cm above the ground.

Let h cm be the height of the water in the container, where h is measured from the ground, and t minutes be the time from the instant since $h = 100$.

The rate at which the volume of the water is decreasing is directly proportional to the square root of the height of the water in the container.

a) By relating the volume and the height of the water in the container, show that

$$\frac{dh}{dt} = -Ah^{\frac{1}{2}},$$

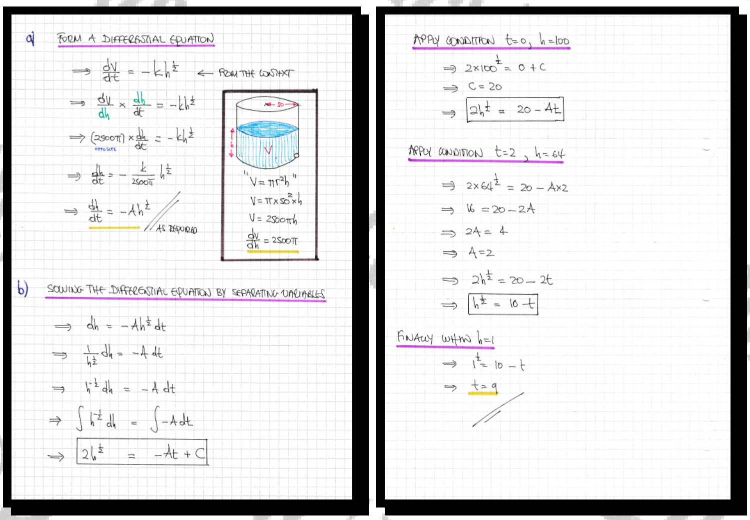
where A is a positive constant.

$$\left[\text{volume of a cylinder of radius } r \text{ and height } h \text{ is given by } \pi r^2 h\right]$$

When $t = 2$, $h = 64$.

b) Determine the value of t, by which no more water leaks out of the container.

$\boxed{}$, $\boxed{t = 9}$



a) FORM A DIFFERENTIAL EQUATION

$\Rightarrow \frac{dV}{dt} = -kh^{\frac{1}{2}}$ ← FROM THE CONTEXT

$\Rightarrow \frac{dV}{dh} \times \frac{dh}{dt} = -kh^{\frac{1}{2}}$

$\Rightarrow (2500\pi) \times \frac{dh}{dt} = -kh^{\frac{1}{2}}$

OPPOSITE

$\Rightarrow \frac{dh}{dt} = -\frac{k}{2500\pi} h^{\frac{1}{2}}$

$\Rightarrow \frac{dh}{dt} = -Ah^{\frac{1}{2}}$ // AS REQUIRED

"$V = \pi r^2 h$"

$V = \pi \times 50^2 \times h$

$V = 2500\pi h$

$\frac{dV}{dh} = 2500\pi$

b) SOLVING THE DIFFERENTIAL EQUATION BY SEPARATING VARIABLES

$\Rightarrow dh = -Ah^{\frac{1}{2}}\, dt$

$\Rightarrow \frac{1}{h^{\frac{1}{2}}}\, dh = -A\, dt$

$\Rightarrow h^{-\frac{1}{2}}\, dh = -A\, dt$

$\Rightarrow \int h^{-\frac{1}{2}}\, dh = \int -A\, dt$

$\Rightarrow \boxed{2h^{\frac{1}{2}} = -At + C}$

APPLY CONDITION $t = 0$, $h = 100$

$\Rightarrow 2 \times 100^{\frac{1}{2}} = 0 + C$

$\Rightarrow C = 20$

$\Rightarrow \boxed{2h^{\frac{1}{2}} = 20 - At}$

APPLY CONDITION $t = 2$, $h = 64$

$\Rightarrow 2 \times 64^{\frac{1}{2}} = 20 - A \times 2$

$\Rightarrow 16 = 20 - 2A$

$\Rightarrow 2A = 4$

$\Rightarrow A = 2$

$\Rightarrow 2h^{\frac{1}{2}} = 20 - 2t$

$\Rightarrow \boxed{h^{\frac{1}{2}} = 10 - t}$

FINALLY WHEN $h = 1$

$\Rightarrow 1^{\frac{1}{2}} = 10 - t$

$\Rightarrow t = 9$ //

Question 2 (***)

The shape of a weather balloon remains spherical at all times. It is filled with a special type of gas and is floating at very high altitude.

The rate at which the volume of the balloon is decreasing is directly proportional to the square of the surface area of the balloon at that instant.

Let r m be the radius of the balloon, t hours since $r = 5$.

a) By relating the volume, the surface area and the radius of the weather balloon show that

$$\frac{dr}{dt} = -kr^2,$$

where k is a positive constant.

[volume of a sphere of radius r is given by $\frac{4}{3}\pi r^3$]

[surface area of a sphere of radius r is given by $4\pi r^2$]

When $t = 10$, $r = 4.8$.

b) Determine the value of t when $r = 4$.

$\boxed{}$, $\boxed{t = 60}$

a) FORMING A DIFFERENTIAL EQUATION

$\Rightarrow \frac{dV}{dt} = -cA^2$ (C = PROPORTIONALITY CONSTANT)

$\Rightarrow \frac{dV}{dr} \times \frac{dr}{dt} = -c\left(4\pi r^2\right)^2$

$\Rightarrow \cancel{4\pi r^2} \times \frac{dr}{dt} = -c\left(4\pi r^2\right)^{\cancel{2}}$

$\Rightarrow \frac{dr}{dt} = -c\left(4\pi r^2\right)$

$\Rightarrow \frac{dr}{dt} = -(4\pi c)r^2$

$\Rightarrow \frac{dr}{dt} = -kr^2$ // AS REQUIRED

t = TIME (hours)
A = SURFACE AREA (m^2)
r = RADIUS (m)
V = VOLUME (m^3)

$t=0$, $r=5$

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 4\pi r^2$

b) SOLVING THE O.D.E BY SEPARATING VARIABLES

$\Rightarrow dr = -kr^2\,dt$

$\Rightarrow -\frac{1}{r^2}\,dr = k\,dt$

$\Rightarrow \int -\frac{1}{r^2}\,dr = \int k\,dt$

$\Rightarrow \boxed{\frac{1}{r} = kt + B}$

APPLY THE INITIAL CONDITION, $t=0$ $r=5$

$\Rightarrow \frac{1}{5} = 0 + B$

$\Rightarrow B = \frac{1}{5}$

$\Rightarrow \boxed{\frac{1}{r} = kt + \frac{1}{5}}$

APPLY BOUNDARY CONDITION $t=10$, $r=4.8$

$\Rightarrow \frac{1}{4.8} = k \times 10 + \frac{1}{5}$

$\Rightarrow \frac{5}{24} = 10k + \frac{1}{5}$

$\Rightarrow 10k = \frac{1}{120}$

$\Rightarrow k = \frac{1}{1200}$

$\therefore \boxed{\frac{1}{r} = \frac{t}{1200} + \frac{1}{5}}$

FINALLY WE OBTAIN

when $r=4$

$\frac{1}{4} = \frac{t}{1200} + \frac{1}{5}$ $\times 1200$

$300 = t + 240$

$t = 60$ //

Question 3 (***+)

At time t seconds, a spherical balloon has radius r cm and volume V cm^3.

Air is pumped into the balloon so that its volume is increasing at a rate inversely proportional to its volume at that time.

a) Show clearly that

$$\frac{dr}{dt} = \frac{A}{r^5},$$

where A is a positive constant.

The initial radius of the balloon is 2 cm and when $t = 1$ it has increased to 3 cm.

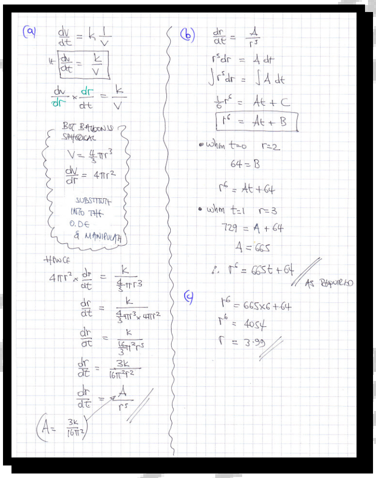
b) Show further that

$$r^6 = 665t + 64.$$

c) Find the value of r when $t = 6$.

$\left[\text{volume of a sphere of radius } r \text{ is given by } \frac{4}{3}\pi r^3\right]$

$\boxed{}$, $\boxed{r \approx 3.99}$



Question 4 (***+)

A container is the shape of a hollow **inverted** right circular cone has base radius 20 cm and height 80 cm. The container is filled with water and is supported in an upright position.

Water is leaking out of a hole at the vertex of the cone.

Let h cm be the height of the water in the container, where h is measured from the vertex of the cone, and t minutes be the time from the instant since $h = 80$.

The rate at which the volume of the water is decreasing is directly proportional to the height of the water in the container.

a) By relating the measurements of the container to that of the volume of the water in the container, show that

$$\frac{dh}{dt} = -\frac{A}{h},$$

where A is a positive constant.

$\left[\text{volume of a cone of radius } r \text{ and height } h \text{ is given by } \tfrac{1}{3}\pi r^2 h\right]$

When $t = 1$, $h = 78$.

b) Determine the value of t by which all the water would have leaked out of the container.

$\boxed{}$, $\boxed{t \approx 20.25...}$

a) FORMING A DIFFERENTIAL EQUATION

$\Rightarrow \frac{dV}{dt} = -kh$

$\Rightarrow \frac{dV}{dh} \times \frac{dh}{dt} = -kh$

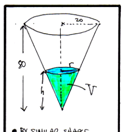
$\Rightarrow \left(\frac{1}{16}\pi h^2\right)\frac{dh}{dt} = -kh$

$\Rightarrow \pi h^2 \frac{dh}{dt} = -16kh$

$\Rightarrow \frac{dh}{dt} = -\frac{16kh}{\pi h^2}$

$\Rightarrow \frac{dh}{dt} = -\left(\frac{16k}{\pi}\right)\frac{1}{h}$

$\Rightarrow \frac{dh}{dt} = -\frac{A}{h}$ // AS REQUIRED



• BY SIMILAR SHAPES

$\frac{r}{h} = \frac{20}{80}$

$r = \frac{1}{4}h$

• VOLUME OF WATER, V

$V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3}\pi\left(\frac{1}{4}h\right)^2 h$

$V = \frac{1}{48}\pi h^3$

$\frac{dV}{dh} = \frac{1}{16}\pi h^2$

b) SOLVE BY SEPARATION OF VARIABLES

$\Rightarrow h\,dh = -A\,dt$

$\Rightarrow \int h\,dh = \int -A\,dt$

$\Rightarrow \frac{1}{2}h^2 = -At + B$

APPLY THE INITIAL CONDITION, IMPLICITLY GIVEN AS $t=0$ $h=80$

$\rightarrow \frac{1}{2}\times 80^2 = -A\times 0 + B$

$\Rightarrow B = 3200$

$\Rightarrow \boxed{\frac{1}{2}h^2 = 3200 - At}$

APPLY THE BOUNDARY CONDITION, $t=1$ $h=78$

$\Rightarrow \frac{1}{2}\times 78^2 = -A\times 1 + 3200$

$\Rightarrow 3042 = -A + 3200$

$\Rightarrow A = 158$

$\therefore \boxed{\frac{1}{2}h^2 = 3200 - 158t}$

FINALLY FOR "TOTAL LEAKAGE", $h=0$

$\Rightarrow 0 = 3200 - 158t$

$\Rightarrow 158t = 3200$

$\Rightarrow t = \frac{1600}{79} \approx 20.25\ldots$

$\simeq 20\frac{1}{4}$ MINUTES //

Question 5 **(****)**

Water is drained from a large hole at the bottom of a tank of height 4 m.

Let $V \text{ m}^3$ and $x \text{ m}$ be the volume and the height of the water in the tank, respectively, at time t minutes since the water started draining out.

Suppose further that the shape of the tank is such so that V and x are related by

$$V = \frac{5}{3}x^3.$$

The rate at which the volume of the water is drained is proportional to the square root of its height, so that it can be modelled by the differential equation

$$\frac{dV}{dt} = -kx^{\frac{1}{2}},$$

where k is a positive constant.

a) Given that it takes 32 minutes to empty the full tank, show that …

i. … $5x^{\frac{3}{2}}\frac{dx}{dt} = -k$.

ii. … $t = 32 - x^{\frac{5}{2}}$.

[continues overleaf]

[continued from overleaf]

When the tank is completely empty, water begins to pour in at the constant rate of 0.5 m^3 per minute and continues to drain out at the same rate as before.

b) Show further that …

i. … $\dfrac{dx}{dt} = \dfrac{1-4\sqrt{x}}{10x^2}$.

ii. … the height of the water cannot exceed $\dfrac{1}{16}$ of a metre.

proof

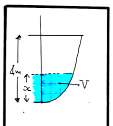
a) USING THE O.D.E GIVEN

(I) $\dfrac{dV}{dt} = -kx^{\frac{1}{2}}$

$\dfrac{dV}{dx} \times \dfrac{dx}{dt} = -kx^{\frac{1}{2}}$

$5x^2 \dfrac{dx}{dt} = -kx^{\frac{1}{2}}$

$5x^{\frac{3}{2}} \dfrac{dx}{dt} = -k$ // AS REQUIRED



$V = \frac{5}{3}x^3$

$\dfrac{dV}{dx} = 5x^2$

(II) SOLVING BY SEPARATION OF VARIABLES

$\Rightarrow 5x^{\frac{3}{2}}\,dx = -k\,dt$

$\Rightarrow \int 5x^{\frac{3}{2}}\,dx = \int -k\,dt$

$\Rightarrow \boxed{2x^{\frac{5}{2}} = -kt + C}$

WHEN $t=0$, $x=4$ (GIVEN)

$\Rightarrow 2 \times 4^{\frac{5}{2}} = -k \times 0 + C$

$\Rightarrow C = 64$

$\Rightarrow \boxed{2x^{\frac{5}{2}} = 64 - kt}$

WHEN $t=32$, $x=0$ (TANK IS EMPTY IN 32 MINUTES)

$\Rightarrow 2 \times 0^{\frac{5}{2}} = 64 - k \times 32$

$\Rightarrow 0 = 64 - 32k$

$\Rightarrow k = 2$

$\Rightarrow 2x^{\frac{5}{2}} = 64 - 2t$

$\Rightarrow 2t = 64 - 2x^{\frac{5}{2}}$

$\Rightarrow t = 32 - x^{\frac{5}{2}}$ // AS REQUIRED

b) REMODELLING THE O.D.E BY LOOKING AT THE ORIGINAL O.D.E

$\dfrac{dV}{dt} = \underbrace{-kx^{\frac{1}{2}}}_{\text{LEAKAGE ONLY}}$ NOW $\dfrac{dV}{dt} = \underset{\text{LEAKAGE AS BEFORE}}{-kx^{\frac{1}{2}}} + \underset{\text{WATER IN AT CONSTANT RATE}}{\frac{1}{2}}$

(I) RELATING THE RATES AS BEFORE

$\Rightarrow \dfrac{dV}{dt} = \dfrac{1}{2} - kx^{\frac{1}{2}}$

$\Rightarrow \dfrac{dV}{dx} \times \dfrac{dx}{dt} = \dfrac{1}{2} - 2x^{\frac{1}{2}}$

$\Rightarrow 5x^2 \dfrac{dx}{dt} = \dfrac{1}{2} - 2x^{\frac{1}{2}}$

$\Rightarrow 10x^2 \dfrac{dx}{dt} = 1 - 4x^{\frac{1}{2}}$

$\Rightarrow \dfrac{dx}{dt} = \dfrac{1 - 4x^{\frac{1}{2}}}{10x^2}$ // AS REQUIRED

(II) FINALLY THE HEIGHT WILL NO LONGER CHANGE WHEN $\dfrac{dx}{dt} = 0$

$\dfrac{1 - 4x^{\frac{1}{2}}}{10x^2} = 0 \Rightarrow 1 - 4x^{\frac{1}{2}} = 0$

$\Rightarrow 4x^{\frac{1}{2}} = 1$

$\Rightarrow x^{\frac{1}{2}} = \dfrac{1}{4}$

$\Rightarrow x = \dfrac{1}{16}$ ← LIMITING VALUE FOR x

AS THE HEIGHT STARTS FROM $x=0$ & ITS LIMITING VALUE IS $\frac{1}{16}$, IT CANNOT EXCEED $\frac{1}{16}$ //

Question 6 (****)

Gas is kept in a sealed container whose volume, V cm^3, can be varied as needed.

The pressure of the gas P, in suitable units, is such so that at any given time the product of P and V remains constant.

The container is heated up so that the volume of the gas begins to expand at a rate inversely proportional to the volume of the gas at that instant.

Let t, in seconds, be the time since the volume began to expand.

a) Show clearly that

$$\frac{dP}{dt} = -AP^3 ,$$

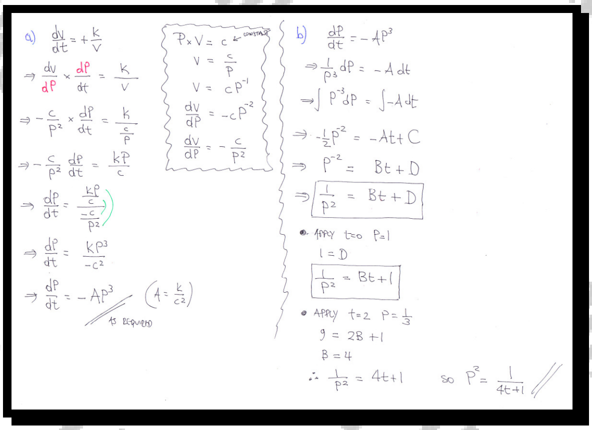
where A is a positive constant.

When $t=0$, $P=1$ and when $t=2$, $P=\frac{1}{3}$.

b) Solve the differential equation to show that

$$P^2 = \frac{1}{4t+1} .$$

$\boxed{}$, $\boxed{\text{proof}}$



a) $\frac{dV}{dt} = +\frac{k}{V}$

$\Rightarrow \frac{dV}{dP} \times \frac{dP}{dt} = \frac{k}{V}$

$\Rightarrow -\frac{c}{P^2} \times \frac{dP}{dt} = \frac{k}{\frac{c}{P}}$

$\Rightarrow -\frac{c}{P^2}\frac{dP}{dt} = \frac{kP}{c}$

$\Rightarrow \frac{dP}{dt} = \frac{\frac{kP}{c}}{\frac{-c}{P^2}}$

$\Rightarrow \frac{dP}{dt} = \frac{kP^3}{-c^2}$

$\Rightarrow \frac{dP}{dt} = -AP^3$ AS REQUIRED $\left(A = \frac{k}{c^2}\right)$

$P \times V = c$ ← CONSTANT

$V = \frac{c}{P}$

$V = cP^{-1}$

$\frac{dV}{dP} = -cP^{-2}$

$\frac{dV}{dP} = -\frac{c}{P^2}$

b) $\frac{dP}{dt} = -AP^3$

$\Rightarrow \frac{1}{P^3}\,dP = -A\,dt$

$\Rightarrow \int P^{-3}\,dP = \int -A\,dt$

$\Rightarrow -\frac{1}{2}P^{-2} = -At + C$

$\Rightarrow P^{-2} = Bt + D$

$\Rightarrow \boxed{\frac{1}{P^2} = Bt + D}$

• APPLY $t=0$ $P=1$

$1 = D$

$\boxed{\frac{1}{P^2} = Bt + 1}$

• APPLY $t=2$ $P=\frac{1}{3}$

$9 = 2B + 1$

$B = 4$

$\therefore \frac{1}{P^2} = 4t+1$ so $P^2 = \frac{1}{4t+1}$ //

Question 7 (****)

A cylindrical tank has constant radius of 0.9 m.

The volume, V m^3, of the water in the tank has height h m.

Water can be poured into the tank from a tap at the top of the tank and can be drained out of a tap at the base of the tank, which are initially both turned off.

Water then starts pouring in at the constant rate of 0.36π m^3 per minute and at the same time water begins to drain out at the rate of $0.45\pi h$ m^3 per minute.

a) Given further that t is measured from the instant when both taps were turned on, show that

$$9\frac{dh}{dt} = 4 - 5h.$$

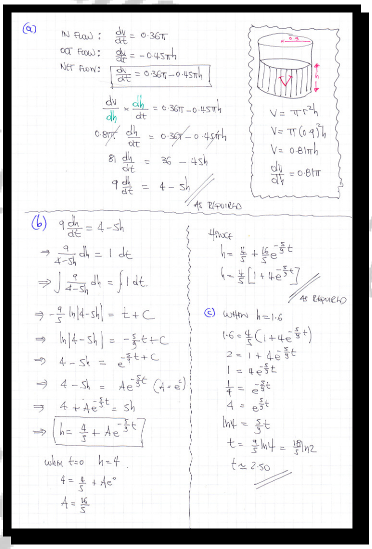
Initially the water in the tank has a height of 4 m.

b) Solve the above differential equation to show that

$$h = \frac{4}{5}\left(1 + 4e^{-\frac{5}{9}t}\right).$$

c) Find the value of t when $h = 1.6$.

$\boxed{}$, $\boxed{t = \frac{18}{5}\ln 2 \approx 2.50}$


(a)
IN FLOW: $\frac{dV}{dt} = 0.36\pi$
OUT FLOW: $\frac{dV}{dt} = -0.45\pi h$
NET FLOW: $\boxed{\frac{dV}{dt} = 0.36\pi - 0.45\pi h}$

$\frac{dV}{dh} \times \frac{dh}{dt} = 0.36\pi - 0.45\pi h$

$0.81\cancel{\pi}\frac{dh}{dt} = 0.36\cancel{\pi} - 0.45\cancel{\pi}h$

$81\frac{dh}{dt} = 36 - 45h$

$9\frac{dh}{dt} = 4 - 5h$ // AS REQUIRED

0.9, h, V

$V = \pi r^2 h$
$V = \pi(0.9)^2 h$
$V = 0.81\pi h$
$\frac{dV}{dh} = 0.81\pi$

(b) $9\frac{dh}{dt} = 4 - 5h$

$\Rightarrow \frac{9}{4-5h}\,dh = 1\,dt$

$\Rightarrow \int \frac{9}{4-5h}\,dh = \int 1\,dt$

$\Rightarrow -\frac{9}{5}\ln|4-5h| = t + C$

$\Rightarrow \ln|4-5h| = -\frac{5}{9}t + C$

$\Rightarrow 4 - 5h = e^{-\frac{5}{9}t + C}$

$\Rightarrow 4 - 5h = Ae^{-\frac{5}{9}t} \quad (A = e^C)$

$\Rightarrow 4 + Ae^{-\frac{5}{9}t} = 5h$

$\Rightarrow \boxed{h = \frac{4}{5} + Ae^{-\frac{5}{9}t}}$

when $t=0$ $h=4$

$4 = \frac{4}{5} + Ae^0$

$A = \frac{16}{5}$

HENCE

$h = \frac{4}{5} + \frac{16}{5}e^{-\frac{5}{9}t}$

$h = \frac{4}{5}\left[1 + 4e^{-\frac{5}{9}t}\right]$ // AS REQUIRED

(c) WHEN $h = 1.6$

$1.6 = \frac{4}{5}\left(1 + 4e^{-\frac{5}{9}t}\right)$

$2 = 1 + 4e^{-\frac{5}{9}t}$

$1 = 4e^{-\frac{5}{9}t}$

$\frac{1}{4} = e^{-\frac{5}{9}t}$

$4 = e^{\frac{5}{9}t}$

$\ln 4 = \frac{5}{9}t$

$t = \frac{9}{5}\ln 4 = \frac{18}{5}\ln 2$

$t \simeq 2.50$ //

Question 8 (****+)

The shape of a weather balloon remains spherical at all times. It is filled with a special type of gas and is floating at very high altitude. Gas started escaping from its valve so that the rate at which its surface area is decreasing is directly proportional to the square of its surface area at that time.

Let V m^3 be the volume of the balloon, t hours since $V = 1000$.

a) By relating the volume, the surface area and the radius of the weather balloon show that

$$\frac{dV}{dt} = -kV^{\frac{5}{3}},$$

where k is a positive constant.

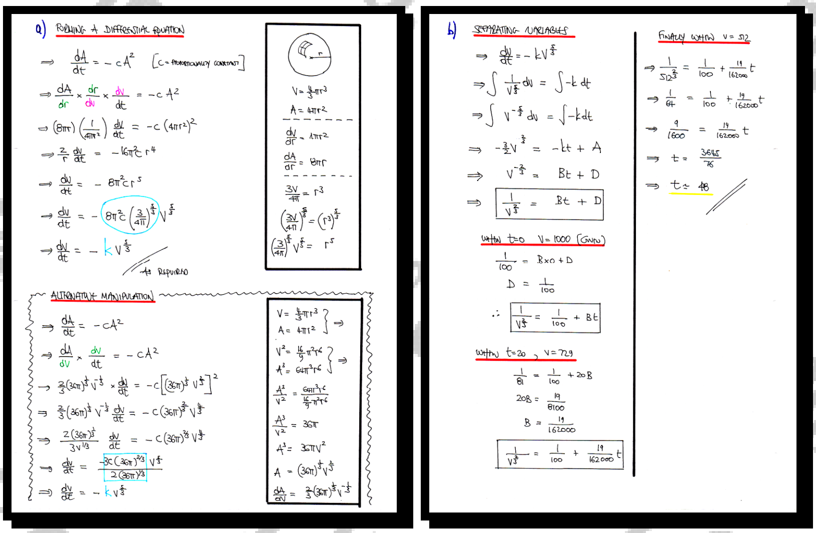
[volume of a sphere of radius r is given by $\frac{4}{3}\pi r^3$]

[surface area of a sphere of radius r is given by $4\pi r^2$]

When $t = 20$, $V = 729$.

b) Determine the value of t when $V = 512$.

$\boxed{}$, $\boxed{t \approx 48}$



Question 9 (****+)

A water tank has the shape of a hollow inverted hemisphere with a radius of 1 m.

It can be shown by calculus that when the depth of the water in the tank is h m, its volume, V m^3, is given by the formula

$$V = \frac{1}{3}\pi h^2 (3-h).$$

Water is leaking from a hole at the bottom of the tank, in m^3 per hour, at a rate proportional to the volume of the water left in the tank at that time.

a) Show clearly that

$$\frac{dh}{dt} = -\frac{kh(3-h)}{3(2-h)},$$

where k is a positive constant.

The water tank is initially full.

b) Solve the differential equation to show further that

$$3h^2 - h^3 = 2e^{-kt}.$$

$\boxed{}$, $\boxed{\text{proof}}$

a) FORMING AN O.D.E.

$\Rightarrow \frac{dV}{dt} = -kV$

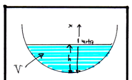
$\Rightarrow \frac{dV}{dh} \times \frac{dh}{dt} = -kV$

$\Rightarrow \cancel{\pi}(2h-h^2)\frac{dh}{dt} = -k \times \frac{1}{3}\cancel{\pi}h^2(3-h)$

$\Rightarrow h(2-h)\frac{dh}{dt} = -\frac{1}{3}kh^2(3-h)$

$\Rightarrow (2-h)\frac{dh}{dt} = -\frac{1}{3}kh(3-h)$

$\Rightarrow \frac{dh}{dt} = -\frac{kh(3-h)}{3(2-h)}$ // AS REQUIRED



$V = \frac{1}{3}\pi h^2(3-h)$

$V = \pi(h^2 - \frac{1}{3}h^3)$

$\frac{dV}{dh} = \pi(2h-h^2)$

b) SEPARATING VARIABLES

$\Rightarrow \frac{3(2-h)}{h(3-h)}\,dh = -k\,dt$

$\Rightarrow \int \frac{3(2-h)}{h(3-h)}\,dh = \int -k\,dt.$

OBTAIN THE PARTIAL FRACTIONS

$\frac{3(2-h)}{h(3-h)} \equiv \frac{A}{h} + \frac{B}{3-h}$

$3(2-h) \equiv A(3-h) + Bh$

• IF $h=3$: $-3 = 3B$, $B=-1$

• IF $h=0$: $6 = 3A$, $A=2$

$\frac{3(2-h)}{h(3-h)} = \frac{2}{h} - \frac{1}{3-h}$

CARRYING OUT THE INTEGRATIONS

$\Rightarrow \int \frac{2}{h} - \frac{1}{3-h}\,dh = -kt + C$

$\Rightarrow 2\ln|h| + \ln|3-h| = -kt + C$

$\Rightarrow \ln h^2 + \ln|3-h| = -kt + C$

$\Rightarrow \ln|h^2(3-h)| = -kt + C$

$\Rightarrow h^2(3-h) = e^{-kt+C}$

$\Rightarrow 3h^2 - h^3 = e^{-kt} \times e^{C}$

$\Rightarrow 3h^2 - h^3 = Ae^{-kt}$

APPLY THE INITIAL CONDITION $t=0$, $h=1$ (INITIALLY FULL)

$\Rightarrow 3 - 1 = Ae^0$

$\Rightarrow A = 2$

$\therefore 3h^2 - h^3 = 2e^{-kt}$ // AS REQUIRED

Question 10 (****+)

A large water tank is in the shape of a cuboid with a rectangular base measuring 10 m by 5 m, and a height of 5 m.

Let h m be the height of the water in the tank and t the time in hours.

At a certain instant, water begins to pour into the tank at the constant rate of 50 m^3 per hour and at the same time water begins to drain from a tap at the bottom of the tank at the rate of $10h \text{ m}^3$ per hour.

a) Show clearly that

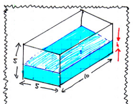
$$5\frac{dh}{dt} = 5 - h.$$

b) Show further that it takes $5\ln 3$ hours for the height of the water to rise from 2 m to 4 m.

$\boxed{}$, $\boxed{\text{proof}}$

a) FORMING AN O.D.E

- IN: $\frac{dV}{dt} = 50$
- OUT $\frac{dV}{dt} = -10h$
- NET $\frac{dV}{dt} = 50 - 10h$



V = VOLUME OF WATER

$V = 5 \times 10 \times h$

$V = 50h$

$\frac{dV}{dh} = 50$

RELATING V & h

$$\frac{dV}{dh} \times \frac{dh}{dt} = 50 - 10h$$

$$50 \times \frac{dh}{dt} = 50 - 10h$$

$$5\frac{dh}{dt} = 5 - h$$

AS REQUIRED

b) SOLVING BY SEPARATION OF VARIABLES

$\Rightarrow 5\,dh = (5-h)\,dt$

$\Rightarrow \int \frac{1}{5-h}\,dh = \int \frac{1}{5}\,dt$

$\Rightarrow -\ln|5-h| = \frac{1}{5}t + C$

$\Rightarrow \ln|5-h| = -\frac{1}{5}t + C$

$\Rightarrow 5-h = e^{-\frac{1}{5}t+C}$

$\Rightarrow 5-h = e^{-\frac{1}{5}t} \times e^{C}$

$\Rightarrow 5-h = Ae^{-\frac{1}{5}t}$ $(A = e^{C})$

$\Rightarrow 5 + Ae^{-\frac{1}{5}t} = h$

$\rightarrow \boxed{h = 5 + Ae^{-\frac{1}{5}t}}$

WHEN $t=0$ $h=2$ (ARBITRARY START OF CLOCK)

$2 = 5 + Ae^{0}$

$2 = 5 + A$

$A = -3$

$\therefore \boxed{h = 5 - 3e^{-\frac{1}{5}t}}$

WHEN $h=4$

$4 = 5 - 3e^{-\frac{1}{5}t}$

$3e^{-\frac{1}{5}t} = 1$

$e^{-\frac{1}{5}t} = \frac{1}{3}$

$e^{\frac{1}{5}t} = 3$

$\frac{1}{5}t = \ln 3$

$t = 5\ln 3$

Question 11 (****+)

Water is leaking out of a hole at the base of a cylindrical barrel with constant cross sectional area and a height of 1 m.

It is given that t minutes after the leaking started, the volume of the water left in the barrel is $V \text{ m}^3$, and its height is $h \text{ m}$.

It is assumed that the water is leaking out, in m^3 per minute, at a rate proportional to the square root of the volume of the water left in the barrel.

a) Show clearly that

$$\frac{dh}{dt} = -B\sqrt{h},$$

where B is a positive constant.

The barrel was initially full and 5 minutes later half its contents have leaked out.

b) Solve the differential equation to show that

$$\sqrt{h} = 1 - \frac{1}{10}\left(2-\sqrt{2}\right)t.$$

c) Show further that

$$t = 5\left(2+\sqrt{2}\right)\left(1-\sqrt{h}\right).$$

d) If T is the time taken for the barrel to empty, find h when $t = \frac{1}{2}T$.

$\boxed{h = \frac{1}{4}}$

a) WE ARE GIVEN THAT

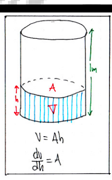
$\Rightarrow \frac{dV}{dt} = -kV^{\frac{1}{2}}$

$\Rightarrow \frac{dV}{dh} \times \frac{dh}{dt} = -kV^{\frac{1}{2}}$

$\Rightarrow A\frac{dh}{dt} = -k(Ah)^{\frac{1}{2}}$

$\Rightarrow \frac{dh}{dt} = -k\frac{A^{\frac{1}{2}}h^{\frac{1}{2}}}{A}$

$\Rightarrow \frac{dh}{dt} = -Bh^{\frac{1}{2}} \quad \left(B = \frac{kA^{\frac{1}{2}}}{A}\right)$



$V = Ah$

$\frac{dV}{dh} = A$

b) SOLVING THE O.D.E BY SEPARATING VARIABLES

$\Rightarrow \frac{1}{h^{\frac{1}{2}}}\,dh = -B\,dt$

$\Rightarrow \int h^{-\frac{1}{2}}\,dh = \int -B\,dt$

$\Rightarrow 2h^{\frac{1}{2}} = -Bt + C$

$\Rightarrow \boxed{h^{\frac{1}{2}} = Pt + Q}$

• APPLYING CONDITION $t=0,\ h=1 \Rightarrow Q=1$

$\boxed{h^{\frac{1}{2}} = 1 + Pt}$

• APPLY CONDITION $t=5\ h=\frac{1}{2}$

$\sqrt{\frac{1}{2}} = 1 + 5P$

$\frac{\sqrt{2}}{2} = 1 + 5P$

$\sqrt{2} = 2 + 10P$

$10P = \sqrt{2} - 2$

$P = \frac{1}{10}(\sqrt{2}-2)$

$\therefore h^{\frac{1}{2}} = 1 - \frac{1}{10}(2-\sqrt{2})t$

REARRANGING & TIDYING UP

$\sqrt{h} = 1 - \frac{1}{10}(2-\sqrt{2})t$

$\frac{1}{10}(2-\sqrt{2})t = 1 - \sqrt{h}$

$(2-\sqrt{2})t = 10(1-\sqrt{h})$

$t = \frac{10(1-\sqrt{h})}{2-\sqrt{2}}$

$t = \frac{10(2+\sqrt{2})(1-\sqrt{h})}{(2-\sqrt{2})(2+\sqrt{2})}$

$t = \frac{10(2+\sqrt{2})(1-\sqrt{h})}{4-2}$

$t = \frac{10(2+\sqrt{2})(1-\sqrt{h})}{2}$

$t = 5(2+\sqrt{2})(1-\sqrt{h})$ AS REQUIRED

d) FIND T, I.E THE VALUE OF t WHEN $h=0$

$T = 5(2+\sqrt{2})(1-\sqrt{0})$

$T = 5(2+\sqrt{2})$

$\therefore \frac{1}{2}T = \frac{5}{2}(2+\sqrt{2})$

FINALLY USING

$\Rightarrow \sqrt{h} = 1 - \frac{1}{10}(2-\sqrt{2})t$

$\Rightarrow \sqrt{h} = 1 - \frac{1}{10}(2-\sqrt{2}) \times \frac{5}{2}(2+\sqrt{2})$

$\Rightarrow \sqrt{h} = 1 - \frac{1}{4}(2-\sqrt{2})(2+\sqrt{2})$

$\Rightarrow \sqrt{h} = 1 - \frac{1}{4}(4-2)$

$\Rightarrow \sqrt{h} = 1 - \frac{1}{4} \times 2$

$\Rightarrow \sqrt{h} = 1 - \frac{1}{2}$

$\Rightarrow \sqrt{h} = \frac{1}{2}$

$\Rightarrow h = \frac{1}{4}$

Question 12 (****+)

A snowball is melting and its shape remains spherical at all times. The volume of the snowball, V cm^3, is decreasing at a rate proportional to its surface area.

Let t be the time in hours since the snowball's surface area was 4 m^2.

Sixteen hours later its surface area has reduced to 2.25 m^2.

By forming and solving a suitable differential equation, determine the value of t by which the snowball would have completely melted.

$\left[\text{volume of a sphere of radius } r \text{ is given by } \frac{4}{3}\pi r^3\right]$

$\left[\text{surface area of a sphere of radius } r \text{ is given by } 4\pi r^2\right]$

$\boxed{t = 64}$

FORMING A DIFFERENTIAL EQUATION

$$\frac{dV}{dt} = -kA$$

RATE OF VOLUME, DECREASING, PROPORTIONALITY CONSTANT, SURFACE AREA

V = VOLUME
A = SURFACE AREA
r = RADIUS
t = TIME (IN HOURS)

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 4\pi r^2$

$A = 4\pi r^2$

$\frac{dA}{dr} = 8\pi r$

$\frac{dr}{dA} = \frac{1}{8\pi r}$

$A = 4\pi r^2$

$r^2 = \frac{A}{4\pi}$

$r = \sqrt{\frac{A}{4\pi}}$

$\frac{1}{r} = \frac{\sqrt{4\pi}}{A^{\frac{1}{2}}}$

RELATING THE VARIABLES IN AN "OPTIMUM" WAY

$\Rightarrow \frac{dV}{dr} \times \frac{dr}{dA} \times \frac{dA}{dt} = -kA$

$\Rightarrow 4\pi r^2 \times \frac{1}{8\pi r^2} \times \frac{dA}{dt} = -kA$

$\Rightarrow \frac{1}{2} r \frac{dA}{dt} = -kA$

$\Rightarrow \frac{dA}{dt} = -\frac{2kA}{r}$

$\Rightarrow \frac{dA}{dt} = -2kA \times \frac{\sqrt{4\pi}}{A^{\frac{1}{2}}}$

$\Rightarrow \boxed{\frac{dA}{dt} = -BA^{\frac{1}{2}}}$

SOLVING THE O.D.E BY SEPARATION OF VARIABLES

$\Rightarrow \frac{1}{A^{\frac{1}{2}}} dA = -B \, dt$

$\Rightarrow \int A^{-\frac{1}{2}} dA = \int -B \, dt$

$\Rightarrow 2A^{\frac{1}{2}} = -Bt + C$

$\Rightarrow \boxed{\sqrt{A} = Dt + E}$

- APPLY CONDITION $t=0, A=4 \Rightarrow \boxed{E=2}$

$\Rightarrow \boxed{\sqrt{A} = Dt + 2}$

- APPLY THE NEXT CONDITION $t=16, A=2.25$

$\sqrt{2.25} = D \times 16 + 2$

$1.5 = 16D + 2$

$16D = -0.5$

$\boxed{D = -\frac{1}{32}}$

$\Rightarrow \boxed{\sqrt{A} = 2 - \frac{1}{32}t}$

FINALLY WHEN THE SNOWBALL MELTS $A=0$

$\Rightarrow 0 = 2 - \frac{1}{32}t$

$\frac{1}{32}t = 2$

$t = 64$ // AS REQUIRED

ALTERNATIVE APPROACH

STARTING FROM THE DIFFERENTIAL EQUATION

$$\frac{dV}{dt} = -kA$$

OBTAIN FIRST A RELATIONSHIP BETWEEN THE VOLUME AND THE SURFACE AREA OF A SPHERE

$\left.\begin{aligned} V &= \tfrac{4}{3}\pi r^3 \\ A &= 4\pi r^2 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} 3V &= 4\pi r^3 \\ A &= 4\pi r^2 \end{aligned}\right\} \Rightarrow \text{DIVIDE} \quad \boxed{\frac{3V}{A} = r}$

THUS

$\Rightarrow A = 4\pi r^2 = 4\pi\left(\frac{3V}{A}\right)^2 = 4\pi\left(\frac{9V^2}{A^2}\right) = \frac{36\pi V^2}{A^2}$

$\Rightarrow A = \frac{36\pi V^2}{A^2}$

$\Rightarrow A^3 = 36\pi V^2$

$\Rightarrow \boxed{A = \sqrt[3]{36\pi}\, V^{\frac{2}{3}}}$

CONVERT THE BOUNDARY CONDITIONS

$t=0, A=4$

$4 = (36\pi)^{\frac{1}{3}} V^{\frac{2}{3}}$

$V^{\frac{2}{3}} = \frac{4}{(36\pi)^{\frac{1}{3}}}$

$V^{\frac{1}{3}} = \frac{2}{(36\pi)^{\frac{1}{6}}}$

$t=16, A=2.25$

$2.25 = (36\pi)^{\frac{1}{3}} V^{\frac{2}{3}}$

$V^{\frac{2}{3}} = \frac{2.25}{(36\pi)^{\frac{1}{3}}}$

$V^{\frac{1}{3}} = \frac{1.5}{(36\pi)^{\frac{1}{6}}}$

HENCE WE SOLVE THE RESULTING O.D.E

$\Rightarrow \frac{dV}{dt} = -kA$

$\Rightarrow \frac{dV}{dt} = -k\left[\sqrt[3]{36\pi}\, V^{\frac{2}{3}}\right]$

$\Rightarrow \frac{dV}{dt} = -kV^{\frac{2}{3}}$

$\Rightarrow \frac{1}{V^{\frac{2}{3}}} dV = -k \, dt$

$\Rightarrow \int V^{-\frac{2}{3}} dV = \int -k \, dt$

$\Rightarrow 3V^{\frac{1}{3}} = -kt + F$

$\Rightarrow \boxed{V^{\frac{1}{3}} = Qt + P}$

APPLYING THE CONVERTED BOUNDARY CONDITIONS

- $t=0, V^{\frac{1}{3}} = \frac{2}{(36\pi)^{\frac{1}{3}}}$

$\therefore \boxed{P = \frac{2}{(36\pi)^{\frac{1}{6}}}}$

$\boxed{V^{\frac{1}{3}} = Qt + \frac{2}{(36\pi)^{\frac{1}{6}}}}$

- $t=16, V^{\frac{1}{3}} = \frac{1.5}{(36\pi)^{\frac{1}{6}}}$

$\frac{1.5}{(36\pi)^{\frac{1}{6}}} = 16Q + \frac{2}{(36\pi)^{\frac{1}{6}}}$

$16Q = -\frac{0.5}{(36\pi)^{\frac{1}{6}}}$

$\therefore \boxed{Q = -\frac{1}{32(36\pi)^{\frac{1}{6}}}}$

$\boxed{V^{\frac{1}{3}} = -\frac{1}{32(36\pi)^{\frac{1}{6}}}t + \frac{2}{(36\pi)^{\frac{1}{6}}}}$

WHEN THE SNOWBALL MELTS, $V=0$

$\Rightarrow 0 = -\frac{1}{32(36\pi)^{\frac{1}{6}}}t + \frac{2}{(36\pi)^{\frac{1}{6}}}$

$\Rightarrow 0 = -\frac{1}{32}t + 2$

$\Rightarrow \frac{1}{32}t = 2$

$\Rightarrow t = 64$ // AS REQUIRED

Question 13 (****+)

Water is pouring into a long vertical cylinder at a constant rate of $2400 \text{ cm}^3\text{s}^{-1}$ and leaking out of a hole at the base of the cylinder at a rate proportional to the square root of the height of the water already in the cylinder.

The cylinder has constant cross sectional area of 4800 cm^2.

a) Show that, if H is the height of the water in the cylinder, in cm, at time t seconds, then

$$\frac{dH}{dt} = \frac{1}{2} - B\sqrt{H},$$

where B is positive constant.

The cylinder was initially empty and when the height of the water in the cylinder reached 16 cm water was **leaking out of the hole**, at the rate of $120 \text{ cm}^3\text{s}^{-1}$.

b) Show clearly that

$$\frac{dH}{dt} = \frac{80 - \sqrt{H}}{160}.$$

c) Use the substitution $u = 80 - \sqrt{H}$, to find

$$\int \frac{1}{80 - \sqrt{H}} \, dH.$$

[continues overleaf]

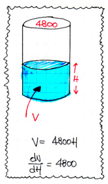
[continued from overleaf]

d) Solve the differential equation in part **(b)** to find, to the nearest minute, the time it takes to fill the cylinder from empty to a height of 4 **metres.**

$\boxed{}$, $\boxed{-2\sqrt{H}-160\ln\left|80-\sqrt{H}\right|+C}$, $\boxed{t\approx 16}$

a) SETTING UP A MODEL

- IN FLOW $\frac{dV}{dt}=2400$
- OUT FLOW $\frac{dV}{dt}=-kH^{\frac{1}{2}}$
- NET FLOW $\frac{dV}{dt}=2400-kH^{\frac{1}{2}}$



$V=4800H$

$\frac{dV}{dH}=4800$

RELATING VARIABLES H & V

$\Rightarrow \frac{dV}{dH}\times\frac{dH}{dt}=2400-kH^{\frac{1}{2}}$

$\Rightarrow 4800\frac{dH}{dt}=2400-kH^{\frac{1}{2}}$

$\Rightarrow \frac{dH}{dt}=\frac{1}{2}-\frac{k}{4800}H^{\frac{1}{2}}$

$\Rightarrow \frac{dH}{dt}=\frac{1}{2}-BH^{\frac{1}{2}}$ // $\left(B=\frac{k}{4800}=\text{CONSTANT}\right)$

AS REQUIRED

b) USING THE CONDITION GIVEN: $H=16$, $\frac{dH}{dt}=-120$

$\Rightarrow -120=-k\times16^{\frac{1}{2}}$ (OUTFLOW ONLY)

$\Rightarrow -120=-4k$

$\Rightarrow k=30$

$\Rightarrow B=\frac{k}{4800}=\frac{30}{4800}=\frac{1}{160}$

$\therefore \frac{dH}{dt}=\frac{1}{2}-\frac{1}{160}H^{\frac{1}{2}}$

$\frac{dH}{dt}=\frac{80-H^{\frac{1}{2}}}{160}$ // AS REQUIRED

c) USING THE SUBSTITUTION GIVEN

$u=80-\sqrt{H}$

$\sqrt{H}=80-u$

$H=(80-u)^2$

$\frac{dH}{du}=2(80-u)^1(-1)$

$\frac{dH}{du}=-2(80-u)$

$\frac{dH}{du}=2(u-80)$

$dH=2(u-80)\,du$

$\int\frac{1}{80-\sqrt{H}}\,dH=\int\frac{1}{u}\times2(u-80)\,du$

$=\int\frac{2u-160}{u}\,du=\int 2-\frac{160}{u}\,du$

$=2u-160\ln|u|+C$

$=2(80-\sqrt{H})-160\ln|80-\sqrt{H}|+C$

$=160-2\sqrt{H}-160\ln|80-\sqrt{H}|+C$

$=-2\sqrt{H}-160\ln|80-\sqrt{H}|+C$ //

d) SEPARATING VARIABLES

$\Rightarrow \frac{dH}{dt}=\frac{80-\sqrt{H}}{160}$

$\Rightarrow \int\frac{1}{80-\sqrt{H}}\,dH=\int\frac{1}{160}\,dt$) PART (c)

$\Rightarrow -2\sqrt{H}-160\ln|80-\sqrt{H}|=\frac{1}{160}t+C$

APPLY CONDITION $t=0$ $H=0$ $\Rightarrow C=-160\ln80$

$\Rightarrow -2\sqrt{H}-160\ln|80-\sqrt{H}|=\frac{1}{160}t-160\ln80$

FINALLY WHEN $H=4\text{m}=400\text{cm}$

$\Rightarrow -2\times20-160\ln(80-20)=\frac{1}{160}t-160\ln80$

$\Rightarrow \frac{1}{160}t=160\ln80-160\ln60-40$

$\Rightarrow t\approx 964.66105\ldots$ seconds $\approx$ 16 minutes //

Question 14 (****+)

At time t seconds, a spherical balloon has radius r cm and surface area S cm^2.

The surface area of the balloon is increasing at a constant rate of 24π cm^{2}s^{-1}.

a) Show that

$$\frac{dr}{dt} = \frac{3}{r}.$$

At time t seconds the balloon has volume V cm^3.

b) By considering $\frac{dV}{dr} \times \frac{dr}{dt}$, show further that

$$\frac{dV}{dt} = \sqrt[3]{1296\pi^2 V}.$$

c) Solve the differential equation of part **(b)** to show

$$V^{\frac{2}{3}} = \frac{2}{3}\left(1296\pi^2\right)^{\frac{1}{3}} t + \text{constant}.$$

d) Given that the initial volume of the balloon was 64π cm^3, find an exact simplified value of V when $t = \sqrt[3]{36}$.

$\left[\text{volume of a sphere of radius } r \text{ is given by } \frac{4}{3}\pi r^3\right]$

$\left[\text{surface area of a sphere of radius } r \text{ is given by } 4\pi r^2\right]$

$\boxed{}$, $\boxed{V = 80\pi\sqrt{10} \approx 795}$

[solution overleaf]

a) PROCEED AS FOLLOWS

$$\Rightarrow \frac{dS}{dt} = 24\pi$$

$$\Rightarrow \frac{dS}{dr} \times \frac{dr}{dt} = 24\pi$$

$$\Rightarrow \cancel{8}\pi r \times \frac{dr}{dt} = \cancel{24}\pi$$

$$\Rightarrow \frac{dr}{dt} = \frac{24}{8r}$$

$$\Rightarrow \frac{dr}{dt} = \frac{3}{r}$$

SURFACE AREA OF SPHERE

$$S = 4\pi r^2$$

$$\frac{dS}{dr} = 8\pi r$$

b) RELATING THE VOLUME NEXT

$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$$

$$\frac{dV}{dt} = 4\pi r^2 \times \frac{3}{r}$$

$$\frac{dV}{dt} = 12\pi r$$

$$\frac{dV}{dt} = \sqrt[3]{(12\pi r)^3}$$

$$\frac{dV}{dt} = \sqrt[3]{1728\pi^3 r^3}$$

$$\frac{dV}{dt} = \sqrt[3]{1728\pi^3 \times \frac{3V}{4\pi}}$$

$$\frac{dV}{dt} = \sqrt[3]{1296\pi^2 V}$$

AS REQUIRED

VOLUME OF A SPHERE

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dr} = 4\pi r^2$$

$$r^3 = \frac{3V}{4\pi}$$

c) SEPARATING VARIABLES & SOLVING THE O.D.E

$$\Rightarrow \frac{dV}{dt} = (1296\pi^2)^{\frac{1}{3}} V^{\frac{1}{3}}$$

$$\Rightarrow \frac{1}{V^{\frac{1}{3}}}\, dV = (1296\pi^2)^{\frac{1}{3}}\, dt$$

$$\Rightarrow \int V^{-\frac{1}{3}}\, dV = \int (1296\pi^2)^{\frac{1}{3}}\, dt$$

$$\Rightarrow \frac{3}{2}V^{\frac{2}{3}} = (1296\pi^2)^{\frac{1}{3}} t + C$$

$$\Rightarrow V^{\frac{2}{3}} = \frac{2}{3}(1296\pi^2)^{\frac{1}{3}} t + C$$

AS REQUIRED

d) FINALLY WHEN $t = \sqrt[3]{36} = 36^{\frac{1}{3}}$, GIVEN THAT $t=0$ $V = 64\pi$

$$(64\pi)^{\frac{2}{3}} = C$$

$$C = 16\pi^{\frac{2}{3}}$$

$$\therefore V^{\frac{2}{3}} = \frac{2}{3}(1296\pi^2)^{\frac{1}{3}} t + 16\pi^{\frac{2}{3}}$$

$$\Rightarrow V^{\frac{2}{3}} = \frac{2}{3}(1296\pi^2)^{\frac{1}{3}} \times 36^{\frac{1}{3}} + 16\pi^{\frac{2}{3}}$$

$$\Rightarrow V^{\frac{2}{3}} = \frac{2}{3}(46656\pi^2)^{\frac{1}{3}} + 16\pi^{\frac{2}{3}}$$

$$\Rightarrow V^{\frac{2}{3}} = \frac{2}{3} \times 36\pi^{\frac{2}{3}} + 16\pi^{\frac{2}{3}}$$

$$\Rightarrow V^{\frac{2}{3}} = 24\pi^{\frac{2}{3}} + 16\pi^{\frac{2}{3}}$$

$$\Rightarrow V^{\frac{2}{3}} = 40\pi^{\frac{2}{3}}$$

$$\Rightarrow V = \left(40\pi^{\frac{2}{3}}\right)^{\frac{3}{2}}$$

$$\Rightarrow V = 40^{\frac{3}{2}}\pi$$

$$\Rightarrow V = 40\sqrt{40}\,\pi$$

Question 15 (****+)

A large cylindrical water tank has a height of 16 m and a horizontal cross section of constant area 20 m^2.

Water is pouring into the tank at a constant rate of 10 m^3 per hour and leaking out of a tap at the base of the tank at a rate $\sqrt{x} \text{ m}^3$ per hour, where x is the height of the water in the tank, in m, at time t hours.

a) Show that

$$20\frac{dx}{dt} = 10 - \sqrt{x}.$$

The water in the cylinder had an initial height of 9 m.

b) Solve the differential equation of part **(a)** to find, correct to the nearest hour, the time it takes to fill up the tank.

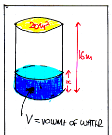
$\boxed{}$, $\boxed{t \approx 22}$

a) START MODELLING AS FOLLOWS

IN FLOW: $\frac{dV}{dt} = 10$

OUT FLOW: $\frac{dV}{dt} = -\sqrt{x}$

NET FLOW: $\frac{dV}{dt} = 10 - \sqrt{x}$



$V = 20x$

$\frac{dV}{dx} = 20$

$t = 0, \ x = 9$

RELATING VARIABLES, x & V

$$\frac{dV}{dx} \times \frac{dx}{dt} = 10 - \sqrt{x}$$

$$20\frac{dx}{dt} = 10 - \sqrt{x}$$ AS REQUIRED

b) SOLVING THE O.D.E. BY SEPARATING VARIABLES, & INPUTTING THE INITIAL CONDITION & THE REQUIRED ANSWER AS LIMITS

$$\Rightarrow 20\,dx = (10 - \sqrt{x})\,dt$$

$$\Rightarrow \frac{20}{10 - \sqrt{x}}\,dx = 1\,dt$$

$$\Rightarrow \int_{x=9}^{x=16} \frac{20}{10-\sqrt{x}}\,dx = \int_{t=0}^{t} 1\,dt$$

BY SUBSTITUTION ON THE INTEGRAL ON THE L.H.S

- $u = 10 - \sqrt{x}$
  $\sqrt{x} = 10 - u$
  $x = (10-u)^2$
  $\frac{dx}{du} = -2(10-u)$
  $dx = -2(10-u)$
- LIMITS $x = 9 \mapsto u = 7$
  $x = 16 \mapsto u = 6$

$$\Rightarrow \int_{u=7}^{u=6} \frac{20}{u}\left[-2(10-u)\,du\right] = \left[t\right]_0^t$$

$$\Rightarrow \int_7^6 \frac{-40(10-u)}{u}\,du = t - 0$$

$$\Rightarrow \int_7^6 \frac{40(u-10)}{u}\,du = t$$

$$\Rightarrow t = 40\int_7^6 \frac{u-10}{u}\,du$$

$$\Rightarrow t = 40\int_7^6 1 - \frac{10}{u}\,du$$

$$\Rightarrow t = 40\left[u - 10\ln|u|\right]_7^6$$

$$\Rightarrow t = 40\left[(6 - 10\ln 6) - (7 - 10\ln 7)\right]$$

$$\Rightarrow t = 40\left[10\ln 7 - 10\ln 6 - 1\right]$$

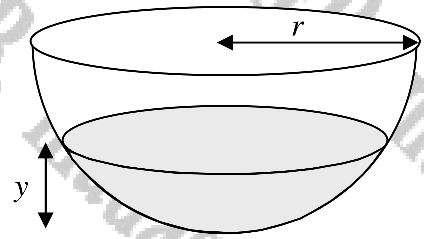
$$\Rightarrow t \approx 21.66027193$$

$$\Rightarrow t \simeq 22 \text{ HOURS}$$

Question 16 (*****)

A water tank has the shape of a hollow inverted hemisphere of radius r cm.

The tank has a hole at the bottom which allows the water to drain out.



Let V, in cm^3, and y, in cm, be the volume and the height of the water in the tank, respectively, at time t seconds.

At time $t = 0$ the empty tank is placed under a running water tap. The rate at which the volume of the water in the tank is changing is proportional to the difference between the tank's constant diameter and the height of the water at that instant.

It can be shown by calculus that V and y are related by

$$V = \frac{1}{3}\pi\left(3ry^2 - y^3\right).$$

a) Show clearly that …

i. … $\dfrac{dy}{dt} = \dfrac{k}{\pi y}$,

where k is a positive constant.

ii. … the time it takes to fill the tank is $\dfrac{\pi r^2}{2k}$ seconds.

[continues overleaf]

[continued from overleaf]

When the tank is full the running tap is instantly turned off but the water in the tank continues to leak out from the hole at the bottom.

b) Show it takes three times as long to empty the tank than it took to fill it up.

[ ] , [proof]

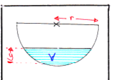
a) I) $\boxed{\frac{dV}{dt} = +k(2r-y) \quad k>0}$

$\Rightarrow \frac{dV}{dy} \times \frac{dy}{dt} = k(2r-y)$

$\Rightarrow \pi(2ry-y^2)\frac{dy}{dt} = k(2r-y)$

$\Rightarrow \pi y\cancel{(2r-y)}\frac{dy}{dt} = k\cancel{(2r-y)}$

$\Rightarrow \boxed{\frac{dy}{dt} = \frac{k}{\pi y}}$



$V = \frac{1}{3}\pi(3ry^2 - y^3)$

$\frac{dV}{dy} = \frac{1}{3}\pi(6ry - 3y^2)$

$\frac{dV}{dy} = \pi(2ry - y^2)$

II) SOLVING THE O.D.E BY SEPARATING VARIABLES

$\int_{y=0}^{y=r} y \, dy = \int_{t=0}^{t} \frac{k}{\pi} \, dt$

$\left[\frac{1}{2}y^2\right]_0^r = \left[\frac{k}{\pi}t\right]_0^t$

$\frac{1}{2}r^2 = \frac{k}{\pi}t$

$\boxed{t = \frac{\pi r^2}{2k}}$

b) NOW REMODEL THE O.D.E

$\frac{dV}{dt} = k(2r-y) = 2kr - ky$

WATER GOING IN (CONSTANT)

WATER LEAKING (PROPORTIONAL TO y)

THUS THE O.D.E WHICH NOW MODELS THE PROBLEM IS

$\boxed{\frac{dV}{dt} = -ky \;, \quad k>0 \quad \text{SUBJECT TO } t=0 \quad y=r}$

SWITCHING INTO y & t BY RELATING V & y AS BEFORE

$\Rightarrow \frac{dV}{dt} = -ky$

$\Rightarrow \frac{dV}{dy} \times \frac{dy}{dt} = -ky$

$\Rightarrow \pi\cancel{y}(2r-y)\frac{dy}{dt} = -k\cancel{y}$

$\Rightarrow (2r-y)\,dy = -\frac{k}{\pi}\,dt$

INTEGRATING SUBJECT TO $t=0, y=r$ & REQUIRING t WHEN $y=0$

$\Rightarrow \int_{y=r}^{y=0} (2r-y)\,dy = \int_{t=0}^{t} -\frac{k}{\pi}\,dt$

$\Rightarrow \left[2ry - \frac{1}{2}y^2\right]_{y=r}^{y=0} = \left[-\frac{k}{\pi}t\right]_0^t$

$\Rightarrow 0 - \left(2r^2 - \frac{1}{2}r^2\right) = -\frac{kt}{\pi}$

$\Rightarrow \frac{kt}{\pi} = \frac{3}{2}r^2$

$\Rightarrow t = \frac{3r^2}{2} \times \frac{\pi}{k}$

$\Rightarrow t = 3\left(\frac{\pi r^2}{2k}\right)$

$\therefore$ IT TAKES $\frac{\pi r^2}{2k}$ TO FILL UP

IT TAKES $3\left(\frac{\pi r^2}{2k}\right)$ TO EMPTY

i.e. 3 TIMES AS LONG

Question 17 (*****)

A large water tank is in the shape of a cuboid with a rectangular base measuring 10 m by 5 m, and a height of 5 m.

Let h m be the height of the water in the tank and t the time in hours.

At a certain instant, water begins to pour into the tank at the constant rate of 50 m^3 per hour and at the same time water begins to drain from a tap at the bottom of the tank at the rate of $10h$ m^3 per hour.

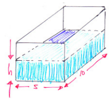
Show that it takes $5\ln 3$ hours for the height of the water to rise from 2 m to 4 m.

$\boxed{}$, $\boxed{\text{proof}}$

IN: $\frac{dV}{dt} = 50$

OUT: $\frac{dV}{dt} = -10h$

NET: $\frac{dV}{dt} = 50 - 10h$



RELATE THE VOLUME AND THE HEIGHT

$V = 50h$

$\frac{dV}{dh} = 50$

$\Rightarrow \frac{dV}{dh} \times \frac{dh}{dt} = 50 - 10h$

$\Rightarrow 50\frac{dh}{dt} = 50 - 10h$

$\Rightarrow \boxed{5\frac{dh}{dt} = 5 - h}$

SEPARATE VARIABLES & SOLVE O.D.E

$\Rightarrow \frac{5}{5-h}\,dh = 1\,dt$

• USE INITIAL CONDITION OF $t=0$ $h=2$, AFTER INTEGRATION

$\Rightarrow \int \frac{5}{5-h}\,dh = \int 1\,dt$

$\Rightarrow -5\ln|5-h| = t + C$

$\Rightarrow \ln|5-h| = C - \frac{1}{5}t$

$\Rightarrow 5-h = e^{C-\frac{1}{5}t}$

$\Rightarrow 5-h = Ae^{-\frac{1}{5}t}$

$\Rightarrow 5 + Ae^{-\frac{1}{5}t} = h$

$\Rightarrow \boxed{h = 5 + Ae^{-\frac{1}{5}t}}$

when $t=0$ $h=2$: $\Rightarrow 2 = 5 + Ae^0$

$\Rightarrow 2 = 5 + A$

$\Rightarrow \boxed{A = -3}$

$\Rightarrow \boxed{h = 5 - 3e^{-\frac{1}{5}t}}$

when $h=4$

$\Rightarrow 4 = 5 - 3e^{-\frac{1}{5}t}$

$\Rightarrow 3e^{-\frac{1}{5}t} = 1$

$\Rightarrow e^{-\frac{1}{5}t} = \frac{1}{3}$

$\Rightarrow e^{\frac{1}{5}t} = 3$

$\Rightarrow \frac{1}{5}t = \ln 3$

$\Rightarrow t = 5\ln 3$ //

OR USE "FULL LIMITS" IN THE INTEGRAL

$\Rightarrow \int_{h=2}^{h=4} \frac{5}{5-h}\,dh = \int_{t=t_1}^{t=t_2} 1\,dt$

$\Rightarrow \left[-5\ln|5-h|\right]_2^4 = \left[t\right]_{t_1}^{t_2}$

$\Rightarrow \cancel{-5\ln 1} + 5\ln 3 = t_2 - t_1$

ie $t_2 - t_1 = 5\ln 3$ //

Question 18 (*****)

Water is leaking out of a hole at the base of a cylindrical barrel with constant cross sectional area and a height of 1 m.

It is given that t minutes after the leaking started, the volume of the water left in the barrel is $V\text{ m}^3$, and its height is $h\text{ m}$.

It is assumed that the water is leaking out, in m^3 per minute, at a rate proportional to the square root of the volume of the water left in the barrel. The barrel was initially full and 5 minutes later half its contents have leaked out.

If T is the time taken for the barrel to empty, find h when $t=\frac{1}{2}T$.

$\boxed{h=\frac{1}{4}}$

WE ARE GIVEN THAT

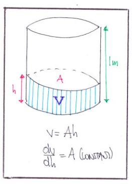
$$\frac{dV}{dt}=-kV^{\frac{1}{2}}$$

$$\Rightarrow \frac{dV}{dh}\times\frac{dh}{dt}=-kV^{\frac{1}{2}}$$

$$\Rightarrow A\frac{dh}{dt}=-k(Ah)^{\frac{1}{2}}$$

$$\Rightarrow \frac{dh}{dt}=-\frac{k}{A^{\frac{1}{2}}}h^{\frac{1}{2}}$$

$$\Rightarrow \boxed{\frac{dh}{dt}=-Bh^{\frac{1}{2}}}\ \left(\text{WHERE } B=\frac{k}{A^{\frac{1}{2}}}\right)$$



$V=Ah$

$\frac{dV}{dh}=A$ (CONSTANT)

SOLVING THE O.D.E BY SEPARATING VARIABLES

$$\Rightarrow \frac{1}{h^{\frac{1}{2}}}dh=-B\,dt$$

$$\Rightarrow \int h^{-\frac{1}{2}}dh=\int -B\,dt$$

$$\Rightarrow 2h^{\frac{1}{2}}=-Bt+C$$

$$\Rightarrow \boxed{h^{\frac{1}{2}}=Pt+Q}$$

- APPLY CONDITION $t=0$ $h=1 \Rightarrow Q=1$

$$\Rightarrow \boxed{h^{\frac{1}{2}}=1+Pt}$$

- APPLY CONDITION $t=5$, $h=\frac{1}{2}$

$$\sqrt{\tfrac{1}{2}}=1+5P$$

$$\frac{\sqrt{2}}{2}=1+5P$$

$$\sqrt{2}=2+10P$$

$$10P=2-\sqrt{2}$$

$$P=\frac{1}{10}(\sqrt{2}-2)$$

$$\therefore \boxed{h^{\frac{1}{2}}=1-\frac{1}{10}(2-\sqrt{2})t}$$

TIDY THE SOLUTION FURTHER BY SOLVING FOR t

$$\Rightarrow \sqrt{h}=1-\frac{1}{10}(2-\sqrt{2})t$$

$$\Rightarrow \frac{1}{10}(2-\sqrt{2})t=1-\sqrt{h}$$

$$\Rightarrow (2-\sqrt{2})t=10(1-\sqrt{h})$$

$$\Rightarrow t=\frac{10(1-\sqrt{h})}{2-\sqrt{2}}$$

$$\Rightarrow t=\frac{10(1-\sqrt{h})(2+\sqrt{2})}{(2-\sqrt{2})(2+\sqrt{2})}=\frac{10(2+\sqrt{2})(1-\sqrt{h})}{2}$$

$$\Rightarrow \boxed{t=5(2+\sqrt{2})(1-\sqrt{h})}$$

FINALLY FIND THE TIME IT TAKES TO EMPTY THE BARREL

WHEN $h=0$ $T=5(2+\sqrt{2})$

USING $\boxed{\sqrt{h}=1-\frac{1}{10}(2-\sqrt{2})t}$ WITH $t=\frac{1}{2}T=\frac{5}{2}(2+\sqrt{2})$

$$\Rightarrow \sqrt{h}=1-\frac{1}{10}(2-\sqrt{2})\times\frac{5}{2}(2+\sqrt{2})$$

$$\Rightarrow \sqrt{h}=1-\frac{1}{4}(2-\sqrt{2})(2+\sqrt{2})$$

$$\Rightarrow \sqrt{h}=1-\frac{1}{4}(4-2)$$

$$\Rightarrow \sqrt{h}=\frac{1}{2}$$

$$\Rightarrow h=\frac{1}{4}$$

Question 19 (*****)

Water is leaking out of a hole at the base of a cylindrical barrel with constant cross sectional area and a height of H m.

It is given that t minutes after the leaking started, the volume of the water left in the barrel is V m^3, and the height of the water is h m.

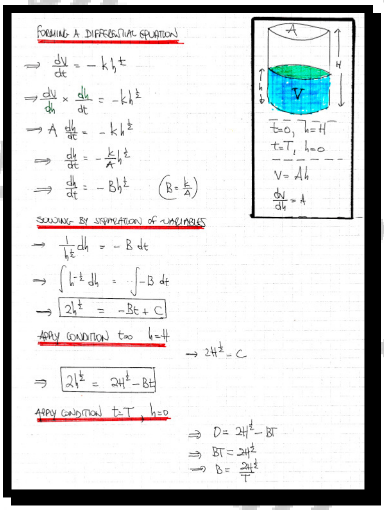
It is assumed that the water is leaking out, in m^3 per minute, at a rate proportional to the square root of the height of the water still left in the barrel.

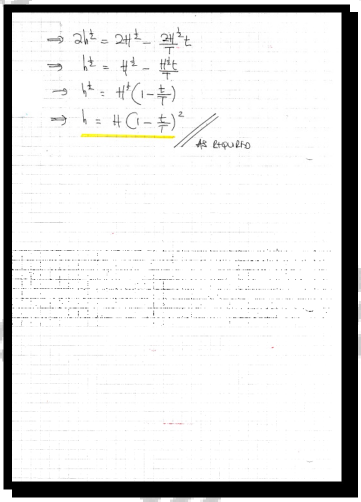
The barrel was initially full and T minutes later all the water has leaked out.

Show by a complete calculus method that

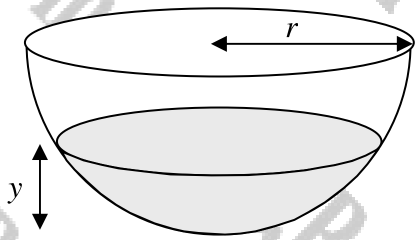
$$h = H\left(1-\frac{t}{T}\right)^2, \quad 0 \le t \le T.$$

$\boxed{}$, $\boxed{\text{proof}}$





Question 20 (*****)



A water tank has the shape of a hollow inverted hemisphere of radius r cm.

The tank has a hole at the bottom which allows the water to drain out.

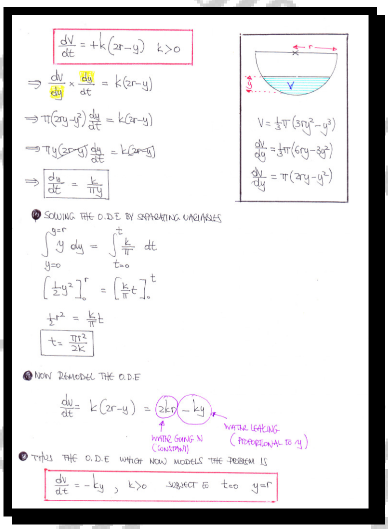
Let V, in cm^3, and y, in cm, be the volume and the height of the water in the tank, respectively, at time t seconds.

At time $t = 0$ the empty tank is placed under a running water tap. The rate at which the volume of the water in the tank is changing is proportional to the difference between the tank's constant diameter and the height of the water at that instant. When the tank is full the running tap is instantly turned off but the water in the tank continues to leak out from the hole at the bottom.

Show it takes three times as long to empty the tank than it took to fill it up.

You may use without proof that V *and* y *are related by* $V = \frac{1}{3}\pi\left(3ry^2 - y^3\right)$.

$\boxed{}$, $\boxed{\text{proof}}$



$\boxed{\frac{dV}{dt} = +k(2r-y) \quad k>0}$

$V = \frac{1}{3}\pi(3ry^2 - y^3)$

$\frac{dV}{dy} = \frac{1}{3}\pi(6ry - 3y^2)$

$\frac{dV}{dy} = \pi(2ry - y^2)$

$\Rightarrow \frac{dV}{dy} \times \frac{dy}{dt} = k(2r-y)$

$\Rightarrow \pi(2ry - y^2)\frac{dy}{dt} = k(2r-y)$

$\Rightarrow \pi y\cancel{(2r-y)}\frac{dy}{dt} = k\cancel{(2r-y)}$

$\Rightarrow \boxed{\frac{dy}{dt} = \frac{k}{\pi y}}$

SOLVING THE O.D.E BY SEPARATING VARIABLES

$\int_{y=0}^{y=r} y \, dy = \int_{t=0}^{t} \frac{k}{\pi} \, dt$

$\left[\frac{1}{2}y^2\right]_0^r = \left[\frac{k}{\pi}t\right]_0^t$

$\frac{1}{2}r^2 = \frac{k}{\pi}t$

$\boxed{t = \frac{\pi r^2}{2k}}$

NOW REMODEL THE O.D.E

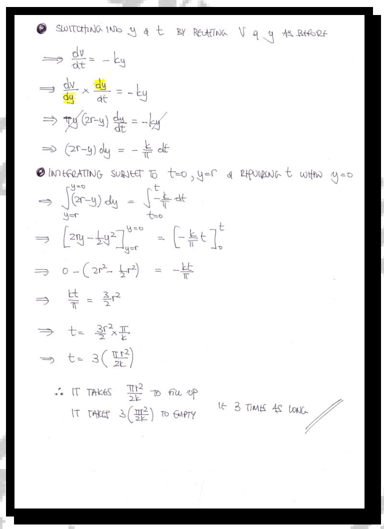
$\frac{dV}{dt} = k(2r-y) = 2kr - ky$

WATER GOING IN (CONSTANT)

WATER LEAKING (PROPORTIONAL TO y)

THUS THE O.D.E WHICH NOW MODELS THE PROBLEM IS

$\boxed{\frac{dV}{dt} = -ky, \quad k>0 \quad \text{SUBJECT TO } t=0 \quad y=r}$



SWITCHING INTO y & t BY RELATING V & y AS BEFORE

$\Rightarrow \frac{dV}{dt} = -ky$

$\Rightarrow \frac{dV}{dy} \times \frac{dy}{dt} = -ky$

$\Rightarrow \cancel{\pi y}(2r-y)\frac{dy}{dt} = -k\cancel{y}$

$\Rightarrow (2r-y)\,dy = -\frac{k}{\pi}\,dt$

INTEGRATING SUBJECT TO $t=0, y=r$ & REQUIRING t WHEN $y=0$

$\Rightarrow \int_{y=r}^{y=0} (2r-y)\,dy = \int_{t=0}^{t} -\frac{k}{\pi}\,dt$

$\Rightarrow \left[2ry - \frac{1}{2}y^2\right]_{y=r}^{y=0} = \left[-\frac{k}{\pi}t\right]_0^t$

$\Rightarrow 0 - \left(2r^2 - \frac{1}{2}r^2\right) = -\frac{kt}{\pi}$

$\Rightarrow \frac{kt}{\pi} = \frac{3}{2}r^2$

$\Rightarrow t = \frac{3r^2}{2} \times \frac{\pi}{k}$

$\Rightarrow t = 3\left(\frac{\pi r^2}{2k}\right)$

$\therefore$ IT TAKES $\frac{\pi r^2}{2k}$ TO FILL UP

IT TAKES $3\left(\frac{\pi r^2}{2k}\right)$ TO EMPTY

IE 3 TIMES AS LONG

Question 21 (*****)

A water tank has the shape of a hollow inverted hemisphere with a radius of 1 m, which is initially full.

Water is leaking from a hole at the bottom of the tank, in m^3 per hour, at a rate proportional to the volume of the water left in the tank at that time.

Show that if the height of the water in the tank is h m, then

$$3h^2 - h^3 = 2\mathrm{e}^{-kt},$$

where k is a positive constant.

$$\left[\text{volume of a sphere of radius } r \text{ is given by } \tfrac{4}{3}\pi r^3\right]$$

$\boxed{}$, $\boxed{\text{proof}}$

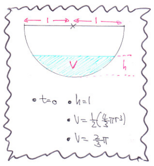
- $\boxed{\frac{dV}{dt} = -kV}$

$\Rightarrow \frac{1}{V}\,dV = -k\,dt$

$\Rightarrow \int \frac{1}{V}\,dV = \int -k\,dt$

$\Rightarrow \ln|V| = -kt + C$

$\Rightarrow V = Ae^{-kt} \quad (A = e^C)$



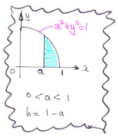
- APPLY CONDITION $t=0,\ V = \frac{2\pi}{3}$

$\frac{2\pi}{3} = A$

$\Rightarrow \boxed{V = \frac{2\pi}{3}e^{-kt}}$

- WE NEED TO EXPRESS V IN TERMS OF h

VOLUME OF REVOLUTION OF THE BLUE SECTION IS GIVEN BY



$V = \pi \int_{x=a}^{x=1} y^2\,dx = \pi \int_a^1 1 - x^2\,dx$

$= \pi\left[x - \frac{1}{3}x^3\right]_a^1 = \pi\left[\left(1-\frac{1}{3}\right) - \left(a - \frac{1}{3}a^3\right)\right]$

$= \pi\left[\frac{2}{3} - a + \frac{1}{3}a^3\right]$

$\Rightarrow V = \pi\left[\frac{2}{3} - (1-h) + \frac{1}{3}(1-h)^3\right]$

$\Rightarrow V = \pi\left[\frac{2}{3} - 1 + h + \frac{1}{3}\left[1 - 3h + 3h^2 - h^3\right]\right]$

$\Rightarrow V = \pi\left[\cancel{-\frac{1}{3}} + \cancel{h} + \cancel{\frac{1}{3}} \cancel{-h} + h^2 - \frac{1}{3}h^3\right]$

$\Rightarrow V = \pi\left[h^2 - \frac{1}{3}h^3\right]$

- THUS RETURNING TO THE MAIN PROBLEM

$\Rightarrow V = \frac{2\pi}{3}e^{-kt}$

$\Rightarrow \pi\left(h^2 - \frac{1}{3}h^3\right) = \frac{2\pi}{3}e^{-kt}$

$\Rightarrow h^2 - \frac{1}{3}h^3 = \frac{2}{3}e^{-kt}$

$\Rightarrow 3h^2 - h^3 = 2e^{-kt}$

AS REQUIRED

Question 22 (*****)

At time t seconds, a spherical balloon has radius r cm, surface area S cm^2 and volume V cm^3. The surface area of the balloon is increasing at a constant rate of $24\pi \ \text{cm}^2\text{s}^{-1}$.

Show that

$$\frac{dV}{dt} = \sqrt[3]{1296\pi^2 V},$$

and given further that the initial volume of the balloon was $64\pi \ \text{cm}^3$, find an exact simplified value for V when $t = \sqrt[3]{36}$.

[volume of a sphere of radius r is given by $\frac{4}{3}\pi r^3$]

[surface area of a sphere of radius r is given by $4\pi r^2$]

SP-Q, $\boxed{V = 80\pi\sqrt{10} \approx 795}$

STARTING WITH RELATED RATES OF CHANGE

$\Rightarrow \frac{dr}{dt} = \frac{dr}{dS} \times \frac{dS}{dt}$

$\Rightarrow \frac{dr}{dt} = \frac{1}{8\pi r} \times 24\pi$

$\Rightarrow \boxed{\frac{dr}{dt} = \frac{3}{r}}$

SURFACE AREA OF A SPHERE

$S = 4\pi r^2$

$\frac{dS}{dr} = 8\pi r$

$\frac{dr}{dS} = \frac{1}{8\pi r}$

NEXT RELATING THE VOLUME RATE OF CHANGE

$\Rightarrow \frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$

$\Rightarrow \frac{dV}{dt} = 4\pi r^2 \times \frac{3}{r}$

$\Rightarrow \frac{dV}{dt} = 12\pi r$

$\Rightarrow \frac{dV}{dt} = 12\pi \left(\frac{3V}{4\pi}\right)^{\frac{1}{3}}$

$\Rightarrow \frac{dV}{dt} = \left[(12\pi)^3\right]^{\frac{1}{3}} \left[\frac{3V}{4\pi}\right]^{\frac{1}{3}}$

$\Rightarrow \frac{dV}{dt} = (1728\pi^3)^{\frac{1}{3}} \left(\frac{3V}{4\pi}\right)^{\frac{1}{3}}$

$\Rightarrow \boxed{\frac{dV}{dt} = (1296\pi^2 V)^{\frac{1}{3}}}$

VOLUME OF A SPHERE

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 4\pi r^2$

ALSO REARRANGING

$3V = 4\pi r^3$

$\frac{3V}{4\pi} = r^3$

$r = \left(\frac{3V}{4\pi}\right)^{\frac{1}{3}}$

SOLVING BY SEPARATION OF VARIABLES

$\Rightarrow \frac{dV}{dt} = (1296\pi^2)^{\frac{1}{3}} V^{\frac{1}{3}}$

$\Rightarrow \frac{1}{V^{\frac{1}{3}}} dV = (1296\pi^2)^{\frac{1}{3}} dt$

$\Rightarrow \int V^{-\frac{1}{3}} dV = \int (1296\pi^2)^{\frac{1}{3}} dt$

$\Rightarrow \frac{3}{2} V^{\frac{2}{3}} = (1296\pi^2)^{\frac{1}{3}} t + C$

$\Rightarrow \boxed{V^{\frac{2}{3}} = \frac{2}{3}(1296\pi^2)^{\frac{1}{3}} t + B}$ ← CONSTANT

APPLYING CONDITIONS

when $t=0$, $V = 64\pi$

$(64\pi)^{\frac{2}{3}} = B$

$B = 16\pi^{2/3}$

HENCE WE OBTAIN

$\Rightarrow V^{\frac{2}{3}} = \frac{2}{3}(1296\pi^2)^{\frac{1}{3}} t + 16\pi^{\frac{2}{3}}$

$\Rightarrow V^{\frac{2}{3}} = \frac{2}{3}(1296)^{\frac{1}{3}} \times \pi^{\frac{2}{3}} \times 36^{\frac{1}{3}} + 16\pi^{\frac{2}{3}}$

$\Rightarrow V^{\frac{2}{3}} = \frac{2}{3} \times (6^4)^{\frac{1}{3}} \times (6^2)^{\frac{1}{3}} \times \pi^{\frac{2}{3}} + 16\pi^{\frac{2}{3}}$

$\Rightarrow V^{\frac{2}{3}} = \frac{2}{3} \times 6^{\frac{4}{3}} \times 6^{\frac{2}{3}} \times \pi^{\frac{2}{3}} + 16\pi^{\frac{2}{3}}$

$\Rightarrow V^{\frac{2}{3}} = \frac{2}{3} \times 6^2 \times \pi^{\frac{2}{3}} + 16\pi^{\frac{2}{3}}$

$\Rightarrow V^{\frac{2}{3}} = 24\pi^{\frac{2}{3}} + 16\pi^{\frac{2}{3}}$

$\Rightarrow V^{\frac{2}{3}} = 40\pi^{\frac{2}{3}}$

$\Rightarrow V = 40^{\frac{3}{2}}\pi$

$\Rightarrow V = 40\sqrt{40}\,\pi$

$\Rightarrow V = 80\sqrt{10}\,\pi$

$\simeq 795$