INTEGRATION
VOLUME OF REVOLUTION
The figure above shows the graph of the curve with equation
\[ y = 4 - x^2. \]

The shaded region \( R \) is bounded by the curve and the \( x \) axis. The region \( R \) is rotated through \( 2\pi \) radians about the \( x \) axis to form a solid of revolution.

Show that the volume of the solid is \( \frac{512\pi}{15} \).

**proof**
Question 2 (**)

The figure above shows part of the graph of the curve with equation

\[ y = 1 + \frac{2}{x}, \quad x \neq 0. \]

The region \( R \), shown shaded in the figure above, is bounded by the curve, the straight lines with equations \( x = 1 \) and \( x = 2 \), and the \( x \) axis.

The region \( R \) is rotated through \( 360^\circ \) about the \( x \) axis to form a solid of revolution.

Show that the volume of the solid is

\[ \pi (3 + 4 \ln 2). \]
The figure above shows part of the graph of the curve with equation

\[ x = y\sqrt{1-y}, \quad y \leq 1. \]

The shaded region \( R \), bounded by the curve and the \( y \) axis is rotated through \( 2\pi \) radians about the \( y \) axis to form a solid of revolution.

Show that the volume of the solid is \( \frac{\pi}{12} \).

\[ \text{proof} \]
Question 4 (**+)**

The diagram above shows the graph of the curve with equation

$$y = \frac{6}{x+3}, \quad x \neq -3.$$ 

The region $R$, shown shaded in the figure above, is bounded by the curve, the coordinate axes and the straight lines with equations $x = -1$ and $x = 3$.

a) Show that the area of $R$ is exactly $6\ln 3$.

The region $R$ is rotated by $360^\circ$ about the $x$ axis to form a solid of revolution.

b) Show that the volume of the solid generated is $12\pi$. 

[Graph of the curve with shaded region $R$]
Question 5 (**+) 

The figure above shows the parabola with equation 

\[ y = x^2 - 11. \]

The shaded region \( R \), is bounded by the curve, the \( y \) axis and the horizontal lines with equations \( y = 5 \) and \( y = 14 \).

This region \( R \) is rotated through \( 360^\circ \) about the \( y \) axis to form a solid of revolution.

Show that the volume of the solid generated is \( \frac{369\pi}{2} \).

\textbf{proof}
Question 6 (**)

The figure above shows part of the curve with equation

\[ y = 2(x-1)^{\frac{3}{2}}. \]

The shaded region, labelled as \( R \), bounded by the curve, the \( x \) axis and the straight lines with equations \( x = 2 \) and \( x = 4 \).

This region is rotated by \( 2\pi \) radians in the \( x \) axis, to form a solid of revolution \( S \).

Show that the volume of \( S \) is \( 80\pi \).
Question 7 (**+)  

The figure above shows the graph of the curve $C$ with equation

$$y = x^2 + 4,$$

intercepted by the straight line $L$ with equation

$$y = 8.$$

The shaded region $R$, is bounded by $C$, the $y$-axis and $L$.

Show that when $R$ is rotated through $2\pi$ radians about the $y$ axis it will generate a volume of $8\pi$ cubic units.

---

proof

$$V = \pi \int_{x_1}^{x_2} (8 - (x^2 + 4))^2 \, dx$$

$$= \pi \int_{x_1}^{x_2} (64 - 16x^2 + x^4) \, dx$$

$$= \pi \left[ \frac{64x}{2} - \frac{16x^3}{3} + \frac{x^5}{5} \right]_{x_1}^{x_2}$$

$$= \pi \left[ 32x - \frac{8}{3}x^3 + \frac{1}{5}x^5 \right]_{x_1}^{x_2}$$

$$= \pi \left[ 32x - \frac{8}{3}x^3 + \frac{1}{5}x^5 \right]_{x_1}^{x_2}$$
Question 8 \((**+)**\)

The figure above shows the graph of the curve with equation

\[ y = \frac{2x+1}{\sqrt{x}}, \quad x > 0. \]

The shaded region \(R\) is bounded by the curve, the \(x\) axis and the straight lines with equations \(x = 1\) and \(x = 2\).

Find the volume that will be generated when \(R\) is rotated through \(360^\circ\) in the \(x\) axis.

Give the answer in the form \(\pi(a + b \ln 2)\), where \(a\) and \(b\) are integers.

\[ \pi(10 + \ln 2) \]
Question 9  (***)

The figure above shows part of the curve with equation

\[ y = x - \frac{1}{x}, \quad x \neq 0. \]

The shaded region bounded by the curve and the straight line with equation \( x = 2 \) is rotated by \( 360^\circ \) about the \( x \) axis to form a solid of revolution.

Show that this volume is \( \frac{5\pi}{6} \).

\[ \text{proof} \]

\[ \text{\hspace{1cm}} \]
Question 10  (***)

The curve $C$ has equation

$$y = 2 + \frac{1}{x}, \quad x > 0.$$  

The region bounded by $C$, the $x$ axis and the lines $x = \frac{1}{2}, \quad x = 2$ is rotated through $360^\circ$ about the $x$ axis.

Show that the volume of the solid formed is

$$\pi \left( \frac{15}{2} + 8 \ln 2 \right).$$  

[proof]
Question 11  (***)
The curve $C$ has equation

$$y = \sqrt{x + \frac{4}{\sqrt{x}}}, \quad x > 0.$$ 

The region bounded by $C$, the $x$ axis and the lines $x = 1$, $x = 4$ is rotated through $360^\circ$ about the $x$ axis.

Show that the volume of the solid formed is

$$\frac{\pi}{2} (63 + 64\ln 2).$$

proof

Question 12  (***)
The curve $C$ has equation

$$y = x^2 - 3x.$$ 

The region bounded by $C$ and the $x$ axis is rotated through $2\pi$ radians in the $x$ axis.

Find the exact volume of the solid formed.

$$\frac{81\pi}{10}$$
Question 13 (***)

The curve $C$ has equation

$$y = x^{ \frac{3}{2} } \sqrt{\ln x}, \quad x > 0.$$ 

The region bounded by $C$, the $x$ axis and the straight lines with equations $x = 1$ and $x = e$ is rotated through $360^\circ$ about the $x$ axis.

Use integration by parts to show that the volume of the solid formed is

$$\frac{1}{16} \pi \left(3e^4 + 1\right).$$
The curve $C$ has equation

$$y = \sqrt{x + 1}, \quad x > -1.$$  

The region $R$ is bounded by $C$, the $y$ axis and the straight line with equation $y = 4$ is rotated through $360^\circ$ about the $y$ axis to form a solid of revolution.

Show that the volume of the solid is $\frac{828\pi}{5}$. 

**proof**
The graph below shows the curve with equation
\[ y = \frac{3}{\sqrt{6x + 1}}, \quad x \neq -\frac{1}{6}. \]

The region \( R \), shown in the figure shaded, is bounded by the curve, the coordinate axes and the straight line with equation \( x = 4 \).

a) Show that the area of \( R \) is 4 square units.

The shaded region \( R \) is rotated by \( 2\pi \) radians about the \( x \) axis to form a solid of revolution.

b) Show that the volume of the solid generated is \( 3\pi \ln 5 \).
Question 16 (***)

The figure above shows part of the graph of the curve with equation

\[ y = \frac{6}{\sqrt{x}} - 3, \quad x > 0. \]

The point \( B \) lies on the curve where \( x = 1 \).

The shaded region \( R \) is bounded by the curve, the coordinate axes and a straight line segment \( AB \), where \( AB \) is parallel to the \( x \) axis. The region \( R \) is rotated through \( 2\pi \) radians in the \( y \) axis to form a solid of revolution.

Show that the volume of this solid is \( 14\pi \).

\[
\text{[proof]}
\]

Created by T. Madas
Question 17  (***)

The curve $C$ has equation

$$y = x + \frac{1}{x^2}, \quad x > 0.$$  

The region bounded by $C$, the $x$ axis and the lines $x = 1$, $x = 2$ is rotated through $360^\circ$ about the $x$ axis.

Show that the volume of the solid formed is

$$\pi \left( \frac{21}{8} + 2 \ln 2 \right).$$

\[\text{proof}\]
Question 18  (***)

The figure above shows the graph of the curve with equation

\[ y = 1 + \cos 2x, \quad 0 \leq x \leq \frac{\pi}{2}. \]

a) Show clearly that

\[ (1 + \cos 2x)^2 = \frac{3}{2} + 2\cos 2x + \frac{1}{2}\cos 4x. \]

The shaded region bounded by the curve and the coordinate axes is rotated by 2π radians about the x-axis to form a solid of revolution.

b) Show that the volume of the solid is

\[ \frac{3}{4} \pi^2. \]
The figure above shows the graph of the curve with equation

\[ y = 1 + \frac{6}{2x+1}, \quad x \neq -\frac{1}{2}. \]

**a)** Show that

\[
\left(1 + \frac{6}{2x+1}\right)^2 \equiv 1 + \frac{A}{2x+1} + \frac{B}{(2x+1)^2},
\]

where \(A\) and \(B\) are constants to be found.

The shaded region, labelled as \(R\), bounded by the curve, the coordinate axes and the line \(x = 1\) is rotated by \(2\pi\) radians in the \(x\) axis to form a solid of revolution.

**b)** Show further that the volume generated is

\[ \pi(13 + 6 \ln 3). \]

\[ A = 12, \quad B = 36 \]
The figure above shows the graph of the curve \( C \) with equation
\[
y = x^2 + 2,
\]
interacted by the straight line \( L \) with equation
\[
x + y = 4.
\]
The point \( A \) is the intersection of \( C \) and \( L \). The point \( B \) is the point where \( L \) meets the \( x \) axis.

The region \( R \), shown shaded in the figure above, is bounded by \( C \), \( L \) and the coordinate axes. This region is rotated by 360° in the \( x \) axis, forming a solid of revolution \( S \).

Find an exact value for the volume of \( S \).

\[
\int_0^2 \pi (x^2 + 2)^2 - \pi (4-x)^2 \, dx = \frac{218}{15} \pi
\]
The figure above shows part of the graph of the curve \( C \) with equation

\[
y = 2 - \frac{1}{2x-1}, \quad x \neq \frac{1}{2}.
\]

The shaded region bounded by \( C \) and the straight lines with equations \( x = 1 \) and \( x = 2 \), is rotated by 360° about the \( x \) axis, forming a solid of revolution.

Show that the volume of the solid is

\[
\pi \left( \frac{13}{3} - 2\ln 3 \right).
\]

**proof**

\[
\pi \left( \frac{13}{3} - 2\ln 3 \right) = \pi \left( \frac{13}{3} - 2 \cdot 1.0986 \right) = \pi \left( \frac{13}{3} - 2.1972 \right) = \pi \left( \frac{13}{3} - 2.1972 \right).
\]
The figure above shows the graph of the equation

$$y = 4 - 2\sqrt{x}, \quad x \geq 0.$$ 

The shaded region $R$, bounded by the curve and the coordinate axes, is rotated through 4 right angles about the $y$ axis to form a solid of revolution.

Show that the volume generated is $\frac{64\pi}{5}$.

**proof**
Question 23 (***+)

The figure below shows the graph of the curve $C$ with equation

$$y = \ln x, \ x > 0,$$

intersected by the horizontal straight line $L$ with equation

$$y = 2.$$

The shaded region $R$, bounded by $C$, $L$ and the coordinate axes, is rotated through $2\pi$ radians in the $y$ axis to form a solid of revolution.

Show that the volume of the solid is

$$\frac{1}{2}\pi(e^4 - 1).$$

\[ \text{proof} \]
Question 24  (***)

The figure above shows part of the curve \( C \) with equation

\[ y = 2 - \frac{x^2}{4}, \quad x > 0. \]

The curve crosses the \( x \)-axis at the point \( P \).

The shaded region bounded by the curve, the straight line with equation \( x = 1 \) and the \( x \)-axis is rotated by 360° about the \( x \)-axis to form a solid of revolution.

Show that the volume of the solid is \( \frac{71\pi}{80} \).

\[\text{proof}\]

\[\text{Created by T. Madas}\]
Question 25 (***++)

The curve $C$ lies entirely above the $x$ axis and has equation

$$y = 1 + \frac{1}{2\sqrt{x}}, \ x \geq 0.$$

a) Show that

$$y^2 = 1 + \frac{1}{\sqrt{x}} + \frac{1}{4x}.$$

The region $R$ is bounded by the curve, the $x$ axis and the straight lines with equations $x=1$ and $x=4$.

b) Show that when $R$ is rotated by $360^\circ$ about the $x$ axis, the solid generated has a volume

$$\pi \left(5 + \ln \sqrt{2}\right).$$

proof
Question 26 (***)

The figure above shows part of the curve with equation

\[ y = \frac{x}{\sqrt{x^3 + 2}}, \quad x^3 > -2. \]

The shaded region \( R \), bounded by the curve, the \( x \) axis and the straight line with equation \( x = 1 \), is rotated by \( 360^\circ \) about the \( x \) axis to form a solid of revolution.

Show that the solid has a volume of

\[ \frac{\pi}{3} \ln \left( \frac{3}{2} \right). \]
Question 27 (***+)

The figure above shows the graph of the curve $C$ with equation

$$y = 4\sqrt{x}e^x, \quad x \geq 0.$$  

The shaded region $R$ bounded by the curve, the $x$ axis and the vertical straight line with equation $x = \ln 2$, is rotated by $2\pi$ radians in the $x$ axis, forming a solid of revolution $S$.

Find an exact value for the volume of $S$, giving the answer in the form $\pi(a + b\ln 2)$ where $a$ and $b$ are integers.

$$\pi(-12 + 32\ln 2)$$
Question 28  (***)

The figure above shows part of the graph of the curve with equation

\[ y = \frac{12}{x}, \ x \neq 0. \]

The points A and C lie on the curve where \( x = 1 \) and \( x = 4 \), respectively. The point B is such so that AB is parallel to the x axis and BC is parallel to the y axis.

The region R, shown shaded in the figure above, is bounded by the curve and the straight line segments AB and BC. This region is rotated by 2\( \pi \) radians in the x axis, forming a solid of revolution S.

Find the exact value for the volume of S.

\[ 324\pi \]
Question 29 (***)

The figure above shows part of the graph of the hyperbola $C$ with equation

$$x^2 - y^2 = 16$$

The hyperbola crosses the $x$ axis at $P(4,0)$, the point $R(5,3)$ lies on $C$ and the point $Q(11,0)$ lies on the $x$ axis.

The shaded region bounded by the curve, the $x$ axis and the straight line segment $RQ$ is rotated by $2\pi$ radians in the $x$ axis, forming a solid of revolution $S$.

Find an exact value for the volume of $S$.

$$\frac{67\pi}{3}$$
Question 30  (***)

The figure above shows part of the curve $C$, with equation $y = 2\sin 2x + 3\cos 2x$.

a) Show that

$$y^2 = A + B\cos 4x + C\sin 4x,$$

where $A$, $B$ and $C$ are constants.

The shaded region $R$ is bounded by the curve, the line $x = \frac{\pi}{4}$ and the coordinate axes.

b) Find the area of $R$.

The region $R$ is rotated by $2\pi$ radians in the $x$ axis forming a solid of revolution $S$.

c) Show that the volume of $S$ is

$$\frac{\pi}{8}(13\pi + 24).$$

\[
\boxed{A = \frac{13}{2}, B = \frac{5}{2}, C = 6, \text{ area } = 2.5}
\]
The point \( P \) lies on the curve with equation \( y = x^2, \quad x \geq 0. \)

The straight line \( L_1 \) is parallel to the \( x \) axis and passes through \( P \). The finite region \( R_1 \) is bounded by the curve, \( L_1 \) and the \( y \) axis.

The straight line \( L_2 \) is parallel to the \( y \) axis and passes through \( P \). The finite region \( R_2 \) is bounded by the curve, \( L_2 \) and the \( x \) axis.

When \( R_1 \) is fully revolved about the \( y \) axis the volume of the solid formed is equal to the volume of the solid formed when \( R_2 \) is fully revolved about the \( x \) axis.

Determine the \( x \) coordinate of \( P \).

\[
\square, \quad x = \frac{5}{2}
\]
The figure above shows part of the curve with equation

\[ y = \sec x + 4 \cos x. \]

The shaded region, labelled \( R \), bounded by the curve, the coordinate axes and the straight line with equation \( x = \frac{\pi}{6} \) is rotated by \( 2\pi \) radians in the \( x \) axis to form a solid of revolution.

Show that the solid has a volume of

\[ \frac{\pi}{3} \left( 8\pi + 7\sqrt{3} \right). \]
Question 33  (***)

The figures above show part of the parabola with equation

\[ y = x^2. \]

The shaded region, shown in Figure 1, is bounded by the curve, the \( x \) axis and the line \( x = 4 \). This region is revolved by \( 2\pi \) radians about the \( x \) axis, to form a solid of revolution.

\[ \text{a)} \quad \text{Show that the solid has a volume of } \frac{1024\pi}{5}. \]

The shaded region, shown in Figure 2, is bounded by the curve, the \( y \) axis and a horizontal line originating from a point on the parabola where \( x = 4 \). This region is revolved by \( 2\pi \) radians about the \( y \) axis, to form a solid of revolution.

\[ \text{b)} \quad \text{Show that the solid has a volume of } 128\pi. \]

\[ \text{c)} \quad \text{Hence find the value of the volume generated when the region shown in figure 1 is revolved by } 2\pi \text{ radians about the } y \text{ axis.} \]

\[ 128\pi \]
The figure above shows part of the curve $C$ with equation

$$y = \frac{x + 1}{\sqrt{x - 1}}, \quad x \geq 1.$$  

The shaded region $R$ is bounded by the curve, the $x$ axis and the straight lines with equations $x = 2$ and $x = 6$. The region $R$ is rotated by $360^\circ$ about the $x$ axis to form a solid of revolution.

\begin{itemize}
  \item[a)] Show that the volume of the solid is
  \[
  \pi (28 + 4\ln 5).
  \]
\end{itemize}
The solid of part (a) is used to model the wooden leg of a sofa.

The shape of the leg is geometrically similar to the solid of part (a).

b) Given the height of the leg is 6 cm, determine the volume of the wooden leg to the nearest cubic centimetre.

\[ V \approx 365 \text{ cm}^3 \]
The figure above shows part of the curve with equation

$$(y - 3)^2 = x + 4.$$ 

The curve crosses the coordinate axes at the points $A$, $B$ and $C$.

a) Show that

$$x^2 = y^4 - 12y^3 + 46y^2 - 60y + 25.$$ 

b) The shaded region bounded by the curve and the coordinate axes is rotated by $360^\circ$ about the $y$-axis to form a solid of revolution.

Show that the volume of the solid is $\frac{113\pi}{15}$. 

[proof]
Question 36  (****)

The curve $C$ has equation

$$y = \frac{x}{x+1}, \quad x \geq 0.$$ 

The region bounded by the curve, the $x$ axis and the straight line with equation $x = 1$ is rotated through $2\pi$ radians about the $x$ axis to form a solid of revolution.

Show that the volume of the solid is

$$\frac{\pi}{2} (3 - 4 \ln 2).$$ 

\[\text{proof}\]
The figure above shows part of the curve with equation 

$$y = \frac{1}{\sqrt{4 - x^2}}, \quad -2 \leq x \leq 2.$$ 

The shaded region, labelled as $R$, bounded by the curve, the coordinate axes and the straight line with equation $x = 1$ is rotated by $2\pi$ radians about the $x$ axis to form a solid of revolution.

Show that the volume of the solid is 

$$\frac{1}{4} \pi \ln 3.$$ 

**proof**
Question 38 (***)

The figure above shows the graph of the curve with equation

\[ y = x \sqrt{\sin 2x}, \ 0 \leq x \leq \frac{\pi}{2}. \]

The shaded region, labelled as \( R \), bounded by the curve and the \( x \) axis, is rotated by 360° about the \( x \) axis to form a solid of revolution.

Show that the volume of the solid generated is

\[ \frac{\pi}{8} (\pi^2 - 4). \]

**proof**
The figure below shows the graph of the curve with equation

$$y = 6\sin\left(\frac{x}{4}\right), \quad 0 \leq x \leq 4\pi.$$  

The shaded region $R$, is bounded by the curve and the $x$ axis.

a) Determine the area of $R$.

This region $R$ is rotated through $360^\circ$ about the $x$ axis to form a solid of revolution.

b) Show that the volume of the solid generated is $72\pi^2$.  

$$48 \text{ square units}$$
The figure above shows part of the graph of the curve with equation

\[ y = 2e^{-x} - e^{-2x}, \quad x \in \mathbb{R}. \]

The shaded region \( R \), bounded by the curve, the coordinate axes and the straight line with equation \( x = \ln 2 \), is rotated through 360° about the \( x \) axis to form a solid of revolution.

Show that the volume of the solid generated is exactly \( \frac{109}{192}\pi \).

\[ \text{proof} \]

\[ \begin{align*}
\pi \int_{\ln 2}^{\infty} (2e^{-x} - e^{-2x})^2 \, dx &= \pi \int_{\ln 2}^{\infty} (4e^{-2x} - 4e^{-3x} + e^{-4x}) \, dx \\
&= \pi \left[ -4e^{-2x} + \frac{4}{3}e^{-3x} - \frac{1}{4}e^{-4x} \right]_{\ln 2}^{\infty} \\
&= \pi \left( -4e^{-2\ln 2} + \frac{4}{3}e^{-3\ln 2} - \frac{1}{4}e^{-4\ln 2} \right) \\
&= \pi \left( -4(\frac{1}{2}) + \frac{4}{3}(\frac{1}{4}) - \frac{1}{4}(\frac{1}{16}) \right) \\
&= \pi \left( -2 + \frac{1}{3} - \frac{1}{64} \right) \\
&= \pi \left( \frac{15}{3} - \frac{1}{64} \right) \\
&= \pi \left( \frac{144}{64} - \frac{1}{64} \right) \\
&= \pi \left( \frac{143}{64} \right) \\
&= \frac{109}{192}\pi.
\end{align*} \]
Question 41 (****)

The figure above shows the graph of the curve with equation

\[ y = x^2 + 2. \]

The shaded region \( R \), is bounded by the curve, the coordinate axes and the straight line with equation \( x = 1 \).

The region \( R \) is rotated through \( 360^\circ \) about the \( y \) axis to form a solid of revolution.

Show that the volume of the solid generated is \( \frac{5}{2} \pi \) cubic units.

proof
The figure above shows part the graph of the curve \( C \), with equation \( y = \frac{3}{2(4x+3)} \), \( x \neq -\frac{3}{4} \).

The shaded region \( R \), is bounded by the curve, the \( x \) axis and the straight lines with equations \( x = -\frac{1}{2} \) and \( x = \frac{1}{4} \).

a) Find the exact area of \( R \).

This region \( R \) is rotated through 360° about the \( x \) axis to form a solid of revolution.

b) Show that the volume of the solid generated is \( \frac{27}{64} \pi \).
The solid generated in part (b) is used to model a small handle for a drawer.

The solid generated in part (b) and the drawer handle are mathematically similar.

c) Given that the length of the handle is 2 cm, find the exact volume of the handle.

\[ \text{area} = \frac{3}{4} \ln 2 \]
\[ \text{volume of handle} = 8\pi \]
Question 43 (****)

The figure above shows part of the curve with equation

\[ y = \frac{2x+1}{x+2}, \quad x \neq -2. \]

a) Show that

\[ \frac{2x+1}{x+2} = A + \frac{B}{x+2}, \]

where \( A \) and \( B \) are constants to be found.

The shaded region, labelled \( R \), bounded by the curve, the coordinate axes and the straight line with equation \( x = 4 \) is rotated by 360° about the \( x \) axis to form a solid of revolution.

b) Show that the volume of revolution is

\[ \pi (19 - 12 \ln 3). \]

**A = 2, B = -3**
Question 44 \( (****) \)

The curve \( C \) has equation

\[ y = xe^x, \ x \in \mathbb{R}. \]

The region \( R \) is bounded by the curve, the \( x \) axis and the vertical straight lines with equations \( x = 1 \) and \( x = 3 \).

a) Explain why \( R \) lies entirely above the \( x \) axis.

The region \( R \) is rotated by \( 360^\circ \) in the \( x \) axis to form a solid of revolution.

b) Show that the volume of this solid is

\[ \frac{1}{4} \pi e^2 (13e^4 - 1). \]
Question 45  (****)

The figure above shows part of the curve with equation

\[ y = e^{2x} - 9, \quad x \in \mathbb{R}. \]

The curve crosses the coordinate axes at the points \( A \) and \( B \). The shaded region \( R \) is bounded by the curve and the coordinate axes.

a) Determine the exact coordinates of \( A \) and \( B \).

The region \( R \) is rotated by \( 2\pi \) radians about the \( x \) axis to form a solid of revolution.

b) Calculate the volume generated, giving the answer in the form \( \pi(p + q\ln 3) \)

where \( p \) and \( q \) are integers.

\[ \text{Volume} = \pi (-52 + 81\ln 3) \]

Created by T. Madas
Question 46  (***)

Show that

a) \[ \int_0^\pi 4 \sin x \, dx = 4\pi . \]

b) \[ \int_0^\pi \sin^2 x \, dx = \frac{\pi}{2} . \]

The figure above shows part of the curve with equation

\[ y = 2x + \sin x . \]

The shaded region bounded by the curve, the \( x \) axis and the line \( x = \pi \) is rotated by \( 2\pi \) radians about the \( x \) axis to form a solid of revolution.

c) Show that the volume of the solid is

\[ \frac{1}{6} \pi^2 (8\pi^2 + 27) . \]
Question 47 (****)

Show that

a) \( (2 + \tan 3x)^2 = 3 + 4\tan 3x + \sec^2 3x \)

b) \( \int \tan x \, dx = \ln |\sec x| + C \)

The figure above shows part of the graph of the curve with equation

\[ y = 2 + \tan 3x \]

The shaded region bounded by the curve the coordinate axes and the line \( x = \frac{\pi}{9} \) is rotated by \( 2\pi \) radians about the \( x \) axis to form a solid of revolution.

c) Show that the volume of the solid is

\[ \frac{\pi}{3} \left( \pi + 4\ln 2 + \sqrt{3} \right) \]
The figure above shows the graph of the curve $C$ with equation

$$y = \frac{14}{x-2}, \quad x \neq 2.$$ 

The points $P$ and $Q$ lie on $C$ where $x = 2.5$ and $x = 3.75$ respectively.

The shaded region $R$ is bounded by the curve and two horizontal lines passing through the points $P$ and $Q$.

$R$ is rotated by $2\pi$ radians about the $y$ axis forming a solid of revolution $S$.

**a)** Find the volume of $S$, giving the answer in the form $\pi (a + b \ln c)$ where $a$, $b$ and $c$ are constants.

The solid $S$ is used to model a nuclear station cooling tower.

**b)** Given that 1 unit on the axes corresponds to 2 metres on the actual tower, show that the cooling tower has an approximate volume of 4200 m$^3$.

$$\pi \left( \frac{195}{2} + 56 \ln \left( \frac{7}{2} \right) \right)$$
The figure above shows the graph of the curve with equation

\[ 8y = x^2, \quad x \geq 0. \]

The points \( A \) and \( B \) lie on the curve. The curved surface of an open bowl with flat circular base is traced out by the complete revolution of the arc \( AB \) about the \( y \) axis.

The radius of the flat circular base of the bowl is 8 cm, and its volume is 9 litres.

Find to the nearest cm the height of the bowl.

\[ \text{height } \approx 20 \text{ cm} \]
Question 50  (***)

The figure above shows the graph of the curve $C$ with equation

$$y = 2 \sin x + 1, \; x \in \mathbb{R}.$$ 

The shaded region $R$ is bounded by the curve, the line $x = \frac{\pi}{2}$ and the $x$ axis.

a) Find the exact area of $R$.

The region $R$ is rotated by $2\pi$ radians in the $x$ axis forming a solid of revolution $S$.

b) Show that the volume of $S$ is

$$\int_{\pi/2}^{\pi} \pi (3+8\sin x) \, dx = \frac{\pi}{2}(3\pi+8).$$

\[\text{area} = \frac{1}{2}(\pi+4)\]
The figure above shows the graph of the curve with equation

\[ y = 4x^{\frac{1}{2}} \sin x. \]

a) Find the value of \( \int_{0}^{\frac{\pi}{2}} 8x \cos 2x \, dx \).

The shaded region bounded by the curve, the \( x \) axis and the straight line with equation \( x = \frac{\pi}{2} \) is rotated by \( 2\pi \) radians in the \( x \) axis to form a solid of revolution.

b) Show that the volume of the solid is

\[ \pi \left( \pi^2 + 4 \right). \]
The figure above shows the graph of the curve with equation

\[ y = \sin x + \cos x, \quad -\pi \leq x \leq \pi. \]

The finite region \( R \), shown shaded in the figure, is bounded by the curve and the coordinate axes.

When \( R \) is revolved by a full turn in the \( x \) axis it traces a solid of volume \( V \).

Show clearly that

\[ V = \frac{1}{4}\pi(3\pi + 2). \]
Question 53  (****)

The figure above shows part of the graph of the curve with equation \( y = \frac{x^2}{1-x^2} \), which passes through the origin \( O \).

The finite area bounded by the curve, the \( y \)-axis and the straight line with equation \( y = 3 \), is to be revolved in the \( y \)-axis by 360° to form a solid of revolution \( S \).

Find an exact value for the volume of \( S \).

\[ \pi (3 - \ln 4) \]
The figure above shows part of the graph of the curve $C$ with equation

$$y = \frac{5}{\sqrt{5x-4}} \quad , \quad x > \frac{4}{5}.$$  

The shaded region $R$ is bounded by the curve, the vertical straight lines $x=1$ and $x=a$, and the $x$ axis.

The region $R$ is rotated by $2\pi$ radians about the $x$ axis forming a solid of revolution.

Given that the area of $R$ is 10 square units, show that the volume of the solid formed is $10\pi \ln 6$ cubic units.
The figure above shows the graph of
\[ y = \ln x, \quad x > 0. \]

The shaded region \( R \) is bounded by the curve, the line \( x = e \) and the \( x \) axis.

\( R \) is rotated by \( 2\pi \) radians about the \( y \) axis, forming a solid of revolution \( S \).

Show that the volume of \( S \) is
\[ \frac{1}{2}\pi (e^2 + 1). \]
Question 56 (****)

The figure above shows part of the graph of the curve with equation

\[ y = \frac{4 + \sin x \cos x}{\cos 2x}. \]

The finite area bounded by the curve, the \( x \) axis and the straight lines with equations \( x = \frac{1}{12}\pi \) and \( x = \frac{1}{6}\pi \), shown shaded in the figure, is fully revolved about the \( x \) axis, forming a solid, \( S \).

Calculate the volume of \( S \), correct to 3 significant figures.

\[ \boxed{V \approx 34.6} \]
The figure above shows the curve with parametric equations

$$x = 2\cos^2 \theta, \quad y = \sqrt{3} \tan \theta, \quad 0 \leq \theta < \frac{\pi}{2}.$$ 

The finite region $R$ shown shaded in the figure, bounded by the curve, the $y$ axis, and the straight lines with equations $y = 1$ and $y = 3$.

Use integration in parametric to show that the volume of the solid formed when $R$ is fully revolved about the $y$ axis is $\frac{\pi^2}{\sqrt{3}}$. 

### Proof

The volume $V$ of the solid formed by revolving the region $R$ about the $y$ axis can be found using the parametric equation in the $xy$ plane. The volume is given by the integral:

$$V = \pi \int_{a}^{b} x^2 \, dy$$ 

where $x = 2\cos^2 \theta$, $y = \sqrt{3} \tan \theta$, and $\theta$ varies from $0$ to $\frac{\pi}{2}$. 

Substitute $x$ and $y$ in terms of $\theta$:

$$V = \pi \int_{0}^{\frac{\pi}{2}} (2\cos^2 \theta)^2 \sqrt{1 + \frac{3\sec^2 \theta}{\tan^2 \theta}} \, d\theta$$

Simplify and integrate to find $\frac{\pi^2}{\sqrt{3}}$. 

$$V = \frac{\pi^2}{\sqrt{3}}$$

This completes the proof.
The figure above shows the graph of the curve $C$ with equation $y = x \ln x$, $x \geq 1$.

The shaded region $R$ is bounded by the curve, the $x$ axis and the vertical line $x = e$.

The region $R$ is rotated by $2\pi$ radians in the $x$ axis forming a solid of revolution $S$.

Find an exact value for the volume of $S$.

\[
\frac{\pi}{27} (5e^3 - 2) \]

\[
\text{\hspace{1cm}}
\]
Question 59 (****+)

\[ f(x) = \frac{1}{8}(4x + \sin 4x), \ x \in \mathbb{R}, \ 0 \leq x \leq \frac{\pi}{4}. \]

a) Show that \( f'(x) = \cos^2 2x \).

The figure above shows part of the graph of a curve \( C \) with equation

\[ y = \sqrt{x} \cos 2x, \ x > 0. \]

The curve meets the \( x \) axis at the origin and at the point where \( x = \frac{\pi}{4} \).

The shaded region \( R \) is bounded by the curve and the \( x \) axis. The region \( R \) is rotated by \( 2\pi \) radians about the \( x \) axis, forming a solid of revolution \( S \).

b) Show further that the volume of \( S \) is

\[ \frac{\pi}{64} \left( \pi^2 - 4 \right). \]
The figure above shows the straight line segment $OP$, joining the origin to the point $P(h,r)$, where $h$ and $r$ are positive coordinates.

The point $Q(h,0)$ lies on the $x$ axis.

The shaded region $R$ is bounded by the straight line segments $OP$, $PQ$ and $OQ$.

The region $R$ is rotated by $2\pi$ radians in the $x$ axis to form a solid cone of height $h$ and radius $r$.

Show by integration that the volume of the cone $V$ is given by

$$V = \frac{1}{3}\pi r^2 h.$$
Question 61  (****+)

A finite region $R$ is defined by the inequalities

$$y^2 \leq 4ax, \quad 0 \leq x \leq a, \quad y \geq 0,$$

where $a$ is a positive constant.

The region $R$ is rotated by $2\pi$ radians in the $y$ axis forming a solid of revolution.

Determine, in terms of $\pi$ and $a$, the exact volume of this solid.

\[ \frac{8}{3}\pi a^3 \]
Question 62  (***)

The finite region $R$ is defined by the inequalities

$$y \leq \arcsin x, \; x \leq 1, \; y \geq 0.$$ 

The region $R$ is rotated by $2\pi$ radians in the $y$ axis forming a solid of revolution.

Determine the exact volume of this solid.
The figure above shows the graph of the curve with equation
\[ y = \tan 2x, \quad 0 \leq x \leq \frac{\pi}{4}. \]

The finite region \( R \) is bounded by the curve, the \( y \) axis and the horizontal line with equation \( y = 1 \).

The region \( R \) is rotated by \( 2\pi \) radians about the straight line with equation \( y = 1 \) forming a solid of revolution.

Determine an exact volume for this solid.

\[ \int_{0}^{\frac{\pi}{2}} \frac{\pi}{2} (1 - \ln 2) \]
Question 64     (***)

A curve $C$ has equation

$$y = e^{-\frac{(x-1)^2}{e}}, \ x \in \mathbb{R}.$$ 

The finite region bounded by $C$, the $y$ axis and straight line with equation $y=1$, is revolved by $2\pi$ radians about the $y$ axis, forming a solid of revolution.

Find an exact simplified value for the volume of this solid.

$$V = \pi e^2 (e-2)$$
Question 65 (****+)

A curve has equation

\[ y = \ln(4 - x), \quad x \neq 4. \]

The finite region bounded by the curve, the \( x \) axis and the straight line with equation \( x = 2 \), is revolved by \( 2\pi \) radians in the \( y \) axis.

Find the exact volume of the solid formed.

\[ V = \frac{1}{2} \pi (24 \ln 2 - 13) \]
Question 66 (*****)

The finite region $R$ is by the coordinate axes and the curve with equation

$$y = \arccos x, \quad -1 \leq x \leq 1.$$

The region $R$ is rotated by $2\pi$ radians in the $x$ axis forming a solid of revolution.

Determine the exact volume of this solid.

$$\pi^2 = 2\pi$$
The figure above shows a hemispherical bowl of radius $r$ containing water to a height $h$. The water in the bowl is in the shape of a minor spherical segment.

It is required to find a formula for the volume of a minor spherical segment as a function of the radius $r$ and the distance of its plane face from the tangent plane, $h$.

The circle with equation

$$x^2 + y^2 = r^2, \ x \geq 0$$

is to be used to find a formula for the volume of a minor spherical segment.

Show by integration that the volume $V$ of the minor spherical segment is given by

$$V = \frac{1}{3} \pi h^2 (3r - h),$$

where $r$ is the radius of the sphere or hemisphere and $h$ is the distance of its plane face from the tangent plane.
The figure above shows the graph of the curve with equation

\[ y = \frac{1}{x+1}, \quad x \in \mathbb{R}, \quad x \neq -1. \]

The finite region \( R \) is bounded by the curve, the \( x \)-axis and the lines with equations \( x=1 \) and \( x=3 \).

Determine the exact volume of the solid formed when the region \( R \) is revolved by \( 2\pi \) radians about …

a) … the \( y \)-axis.

b) … the straight line with equation \( x=3 \).

\[ \sqrt{\ln 4}, \pi (4 - \ln 4), \quad 4\pi (-1 + \ln 4) \]
Question 69 (*****)
The finite region $R$ is bounded by the curve with equation

$$y = \sin x, \; 0 \leq x \leq \pi,$$

and the straight line with equation $y = \frac{1}{3}$.

The region $R$ is rotated by $2\pi$ radians in the straight line with equation $y = \frac{1}{3}$ forming a solid of revolution.

Determine the exact volume of this solid.

$$V = \frac{\pi}{18} \left[ 11\pi - 22 \arcsin \left( \frac{1}{3} \right) - 12\sqrt{2} \right].$$
The figure above shows a hemispherical bowl of radius $r$ containing water to a height $h$. The water in the bowl is in the shape of a minor spherical segment. It is required to find a formula for the volume of a minor spherical segment as a function of the radius $r$ and the distance of its plane face from the tangent plane, $h$.

Show by integration that the volume $V$ of the minor spherical segment is given by

$$V = \frac{1}{3} \pi h^2 (3r - h),$$

where $r$ is the radius of the sphere or hemisphere and $h$ is the distance of its plane face from the tangent plane.
Question 71 (***)

A curve has equation
\[ y = \frac{8}{x^2 - 4x + 8}, \quad x \in \mathbb{R}. \]

The finite region \( R \) is bounded by the curve, the \( y \) axis and the tangent to the curve at the stationary point of the curve.

Determine, in simplified exact form, the volume of the solid formed when \( R \) is fully revolved about the \( y \) axis.

\[ V = 4\pi \left[ 2 - \pi + 2\ln 2 \right]. \]
A spherical cap of depth \( a \) is removed from a sphere of radius \( na \), where \( n \) is a positive constant, such that \( n > \frac{1}{2} \). The volume of the spherical cap is less than half the volume of the sphere.

The remainder of the sphere is moulded to a right circular cone whose base is equal to that of the circular plane face of the spherical cap removed.

Given that the height of the cone is \( ma \), where \( m \) is a positive constant, show that

\[
m = (n + p)(2n + q),
\]

where \( p \) and \( q \) are integers to be found.

\[
m = (n + 1)(2n - 1)
\]
Question 73 (*****)

A curve has equation

\[ y^2 = \ln|3x - 12|, \quad x \in \mathbb{R}, \quad x \neq 4. \]

The finite region bounded by the curve, the \( x \) axis and the straight line with equation \( y = 1 \), is revolved by \( 2\pi \) radians in the \( x \) axis.

Find the exact volume of the solid formed.

\[ V = \frac{2}{3}\pi(e - 1). \]
The finite region $R$ is bounded by the curve with equation $x = \cos y^2$, the $y$ axis and the straight line with equation $y = \frac{1}{2} \sqrt{\pi}$.

Determine, in exact simplified form, the volume of the solid formed by revolving $R$ by a full turn in the $x$ axis.

\[ \frac{\pi}{2} \left( 2 - \sqrt{2} \right) \]
The figure above shows the curve with equation

\[ y = \frac{12}{(x^2 - 1)^\frac{3}{2}}, \quad x > 1. \]

The region \( R \), bounded the curve, the \( x \) axis and the straight lines with equations \( x = \sqrt{2} \) and \( x = 2 \), is revolved by a full turn about the \( x \) axis, forming a solid \( S \).

a) Show that the volume of \( S \) is given by

\[ 144\pi \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \csc^2 \theta \cot^2 \theta \, d\theta. \]

b) Hence find an exact simplified expression for the volume of \( S \).

\[ V = 2\pi \left[ 14 - 9\sqrt{2} + 27\ln \left(\frac{1 + \sqrt{2}}{\sqrt{3}}\right)\right] \]
A curve $C$ and a straight line $L$ have respective equations

$$y = x^2 \quad \text{and} \quad y = x.$$ 

The finite region bounded by $C$ and $L$ is rotated around $L$ by a full turn, forming a solid of revolution $S$.

Find, in exact form, the volume of $S$.

\[
\pi \sqrt{2} \cdot \frac{\pi}{60}
\]