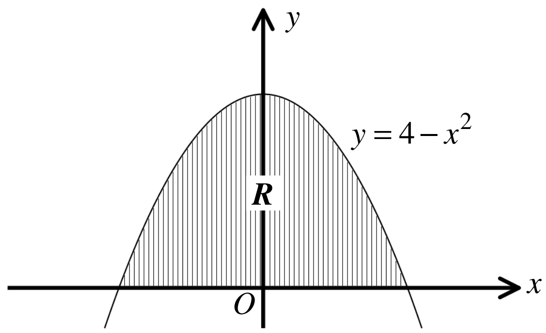
INTEGRATION

VOLUME OF REVOLUTION

Question 1 (**)



The figure above shows the graph of the curve with equation

$$y = 4 - x^2 .$$

The shaded region R, is bounded by the curve and the x axis.

The region R is rotated through 2π radians about the x axis to form a solid of revolution.

Show that the volume of the solid is $\dfrac{256\pi}{15}$.

$\boxed{}$, $\boxed{\text{proof}}$

AS THE QUADRATIC IS SYMMETRICAL CONSIDER THE REVOLUTION BY 2π OF HALF THE AREA (OR THE ENTIRE AREA BY π)

$y = 4 - x^2 = (2-x)(x+2)$

$\Rightarrow V = \pi \int_{x_1}^{x_2} [y(x)]^2 \, dx$

$\Rightarrow V = \pi \int_0^2 (4-x^2)^2 \, dx$

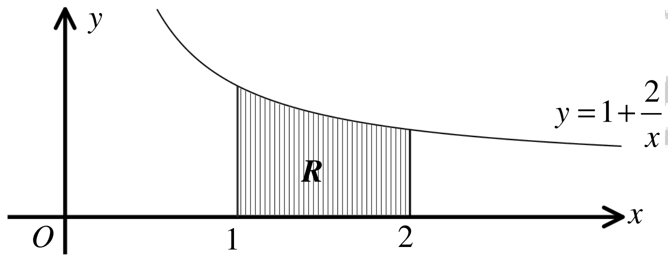
$\Rightarrow V = \pi \int_0^2 16 - 8x^2 + x^4 \, dx$

$\Rightarrow V = \pi \left[16x - \frac{8}{3}x^3 + \frac{1}{5}x^5 \right]_0^2$

$\Rightarrow V = \pi \left[\left(32 - \frac{64}{3} + \frac{32}{5}\right) - 0 \right]$

$\therefore V = \frac{256}{15}\pi$

Question 2 (**)



The figure above shows part of the graph of the curve with equation

$$y = 1 + \frac{2}{x}, \ x \neq 0 .$$

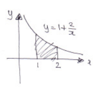
The region R, shown shaded in the figure above, is bounded by the curve, the straight lines with equations $x = 1$ and $x = 2$, and the x axis.

The region R is rotated through $360°$ about the x axis to form a solid of revolution.

Show that the volume of the solid is

$$\pi(3 + 4\ln 2).$$

$\boxed{}$, $\boxed{\text{proof}}$



$$V = \pi\int_{x_1}^{x_2} (y(x))^2 \, dx = \pi\int_1^2 \left(1+\frac{2}{x}\right)^2 dx$$

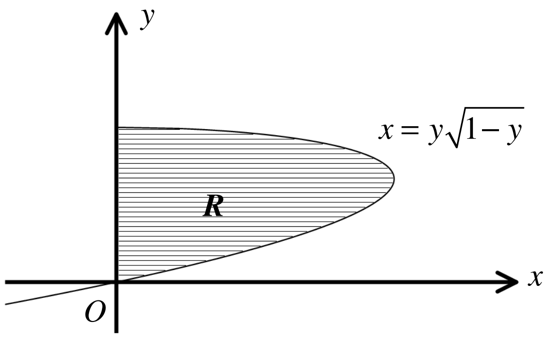
$$\therefore V = \pi\int_1^2 1 + \frac{4}{x} + \frac{4}{x^2} \, dx$$

$$V = \pi\left[x + 4\ln|x| - \frac{4}{x}\right]_1^2$$

$$V = \pi\left[(\cancel{2} + 4\ln 2 - \cancel{2}) - (1 + \cancel{4\ln 1} - 4)\right]$$

$$V = \pi(3 + 4\ln 2)$$

Question 3 (**)



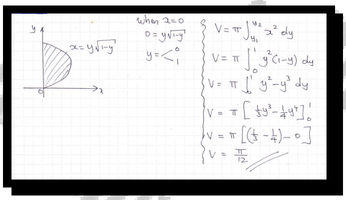
The figure above shows part of the graph of the curve with equation

$$x = y\sqrt{1-y}, \ \ y \le 1.$$

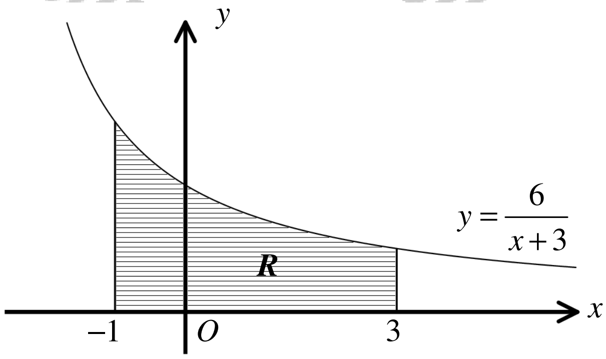
The shaded region R, bounded by the curve and the y axis is rotated through 2π radians about the y axis to form a solid of revolution.

Show that the volume of the solid is $\dfrac{\pi}{12}$.

proof



Question 4 (**+)



The diagram above shows the graph of the curve with equation

$$y = \frac{6}{x+3}, \; x \neq -3.$$

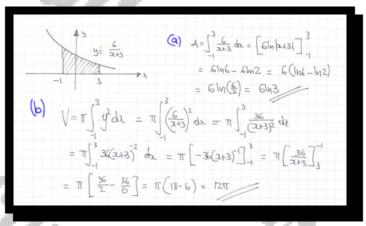
The region R, shown shaded in the figure above, is bounded by the curve, the coordinate axes and the straight lines with equations $x = -1$ and $x = 3$.

a) Show that the area of R is exactly $6\ln 3$.

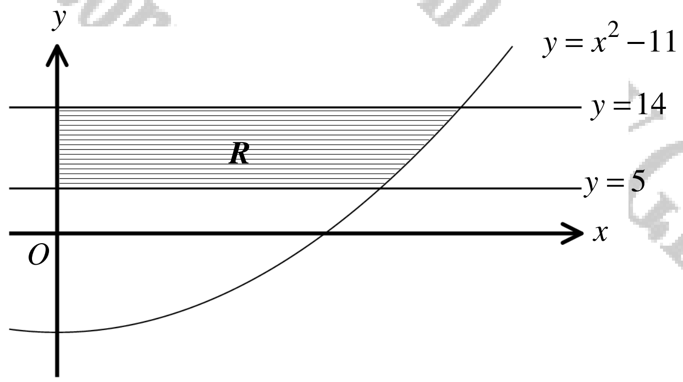
The region R is rotated by $360°$ about the x axis to form a solid of revolution.

b) Show that the volume of the solid generated is 12π.

$\boxed{\quad}$, $\boxed{\text{proof}}$



Question 5 (**+)



The figure above shows the parabola with equation

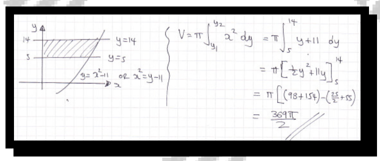
$$y = x^2 - 11.$$

The shaded region R, is bounded by the curve, the y axis and the horizontal lines with equations $y = 5$ and $y = 14$.

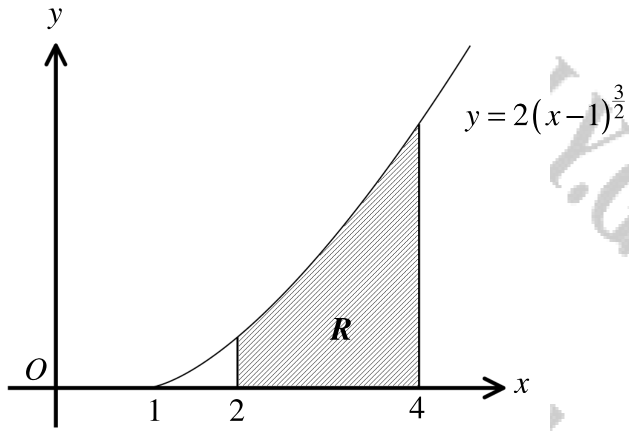
This region R is rotated through $360°$ about the y axis to form a solid of revolution.

Show that the volume of the solid generated is $\dfrac{369\pi}{2}$.

$\boxed{\text{proof}}$



Question 6 (**)



The figure above shows part of the curve with equation

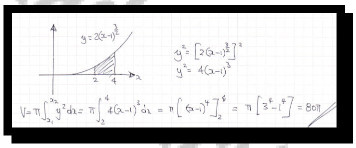
$$y = 2(x-1)^{\frac{3}{2}}.$$

The shaded region, labelled as R, bounded by the curve, the x axis and the straight lines with equations $x = 2$ and $x = 4$.

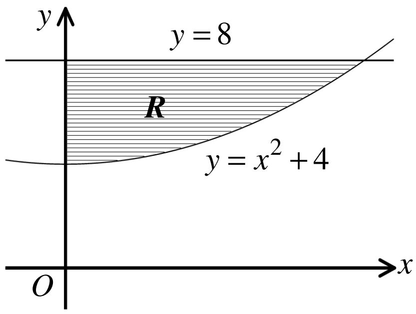
This region is rotated by 2π radians in the x axis, to form a solid of revolution S.

Show that the volume of S is 80π.

$\boxed{}$, $\boxed{\text{proof}}$



Question 7 (+)**



The figure above shows the graph of the curve C with equation

$$y = x^2 + 4,$$

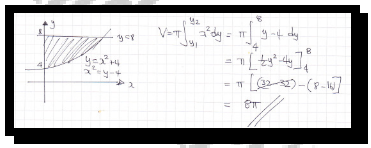
intersected by the straight line L with equation

$$y = 8.$$

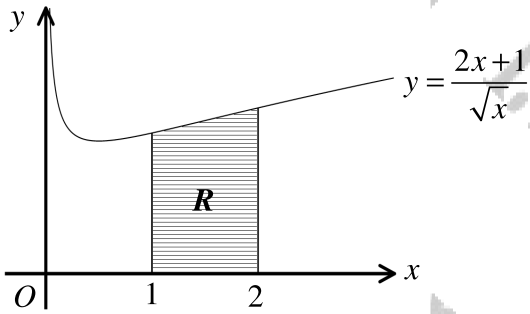
The shaded region R, is bounded by C, the y axis and L.

Show that when R is rotated through 2π radians about the y axis it will generate a volume of 8π cubic units.

proof



Question 8 (+)**



The figure above shows the graph of the curve with equation

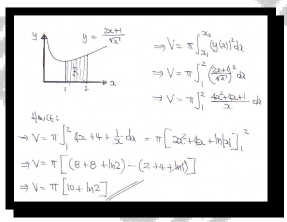
$$y = \frac{2x+1}{\sqrt{x}}, \quad x > 0 .$$

The shaded region R is bounded by the curve, the x axis and the straight lines with equations $x = 1$ and $x = 2$.

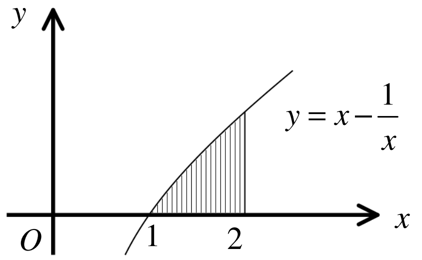
Find the volume that will be generated when R is rotated through $360°$ in the x axis.

Give the answer in the form $\pi(a + b\ln 2)$, where a and b are integers.

$\boxed{\pi(10 + \ln 2)}$



Question 9 (***)



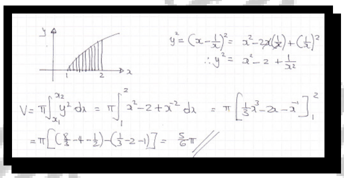
The figure above shows part of the curve with equation

$$y = x - \frac{1}{x}, \ x \neq 0 .$$

The shaded region bounded by the curve and the straight line with equation $x = 2$ is rotated by $360°$ about the x axis to form a solid of revolution.

Show that this volume is $\frac{5\pi}{6}$.

proof



Question 10 (***)

The curve C has equation

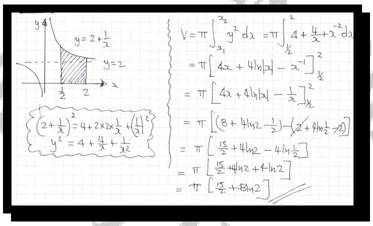
$$y = 2 + \frac{1}{x}, \; x > 0.$$

The region bounded by C, the x axis and the lines $x = \frac{1}{2}$, $x = 2$ is rotated through $360°$ about the x axis.

Show that the volume of the solid formed is

$$\pi\left(\frac{15}{2} + 8\ln 2\right).$$

$\boxed{\text{proof}}$



Question 11 **(***)**

The curve C has equation

$$y = \sqrt{x} + \frac{4}{\sqrt{x}}, \quad x > 0 .$$

The region bounded by C, the x axis and the lines $x = 1$, $x = 4$ is rotated through $360°$ about the x axis.

Show that the volume of the solid formed is

$$\frac{\pi}{2}\left(63 + 64\ln 2\right).$$

$\boxed{\text{proof}}$

$$y^2 = \left(\sqrt{x} + \frac{4}{\sqrt{x}}\right)^2 = (\sqrt{x})^2 + 2\cancel{\sqrt{x}}\left(\frac{4}{\cancel{\sqrt{x}}}\right) + \left(\frac{4}{\sqrt{x}}\right)^2 = x + 8 + \frac{16}{x}$$

$$\therefore V = \pi\int_{x_1}^{x_2} y^2\,dx = \pi\int_1^4 x + 8 + \frac{16}{x}\,dx = \pi\left[\frac{1}{2}x^2 + 8x + 16\ln x\right]_1^4$$

$$= \pi\left[(8 + 32 + 16\ln 4) - \left(\frac{1}{2} + 8 + \cancel{\ln 1}\right)\right] = \pi\left[\frac{63}{2} + 16\ln 4\right]$$

$$= \pi\left[\frac{63}{2} + 32\ln 2\right] = \frac{\pi}{2}\left[63 + 64\ln 2\right]$$

Question 12 **(***)**

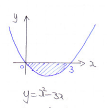
The curve C has equation

$$y = x^2 - 3x .$$

The region bounded by C and the x axis is rotated through 2π radians in the x axis.

Find the exact volume of the solid formed.

$\boxed{\dfrac{81\pi}{10}}$



$y = x^2 - 3x$

$y = x(x-3)$

$$V = \pi\int_{x_1}^{x_2} \left(y(x)\right)^2 dx = \pi\int_0^3 \left(x^2 - 3x\right)^2 dx$$

$$\Rightarrow V = \pi\int_0^3 x^4 - 6x^3 + 9x^2\,dx$$

$$\Rightarrow V = \pi\left[\frac{1}{5}x^5 - \frac{3}{2}x^4 + 3x^3\right]_0^3$$

$$\Rightarrow V = \pi\left[\left(\frac{243}{5} - \frac{243}{2} + 81\right) - (0)\right]$$

$$\Rightarrow V = \frac{81\pi}{10}$$

Question 13 (***)

The curve C has equation

$$y = x^{\frac{3}{2}}\sqrt{\ln x}\,, \quad x > 0\,.$$

The region bounded by C, the x axis and the straight lines with equations $x = 1$ and $x = \mathrm{e}$ is rotated through $360°$ about the x axis.

Use integration by parts to show that the volume of the solid formed is

$$\frac{1}{16}\pi\left(3\mathrm{e}^4 + 1\right).$$

$\boxed{}$, $\boxed{\text{proof}}$

$$y^2 = \left(x^{\frac{3}{2}}\sqrt{\ln x}\right)^2 = x^3 \ln x$$

$$V = \pi\int_{x_1}^{x_2} y^2\,dx = \pi\int_1^{\mathrm{e}} x^3 \ln x$$

BY PARTS

$$\left\{\begin{array}{l} \ln x \longrightarrow \frac{1}{x} \\ \frac{1}{4}x^4 \longrightarrow x^3 \end{array}\right.$$

- IGNORING π & LIMITS

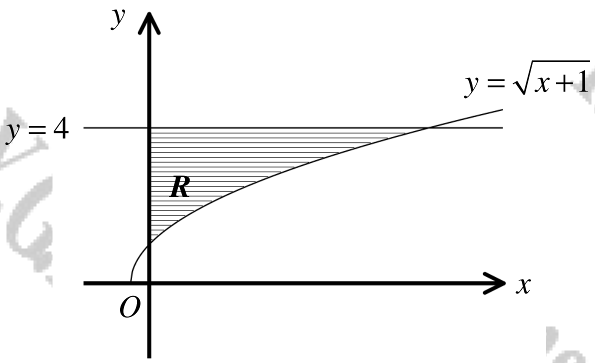
$$\frac{1}{4}x^4\ln x - \int \frac{1}{4}x^4\left(\frac{1}{x}\right)dx = \frac{1}{4}x^4\ln x - \int \frac{1}{4}x^3\,dx = \frac{1}{4}x^4\ln x - \frac{1}{16}x^4 + C$$

- HENCE

$$V = \pi\left[\frac{1}{4}x^4\ln x - \frac{1}{16}x^4\right]_1^{\mathrm{e}} = \pi\left[\left(\frac{1}{4}\mathrm{e}^4\ln \mathrm{e} - \frac{1}{16}\mathrm{e}^4\right) - \left(\cancel{\frac{1}{4}\ln 1} - \frac{1}{16}\right)\right]$$

$$= \pi\left[\frac{1}{4}\mathrm{e}^4 - \frac{1}{16}\mathrm{e}^4 + \frac{1}{16}\right] = \pi\left[\frac{3}{16}\mathrm{e}^4 + \frac{1}{16}\right] = \frac{1}{16}\pi\left(3\mathrm{e}^4 + 1\right)$$

Question 14 (***)



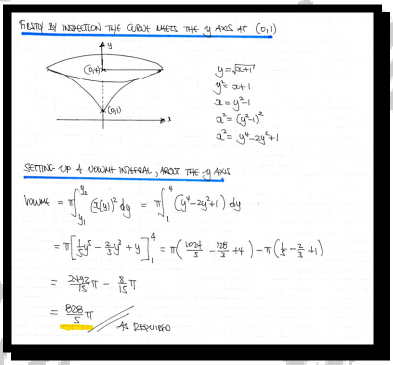
The curve C has equation

$$y = \sqrt{x+1}, \quad x > -1.$$

The region R is bounded by C, the y axis and the straight line with equation $y = 4$ is rotated through $360°$ about the y axis to form a solid of revolution.

Show that the volume of the solid is $\dfrac{828\pi}{5}$.

proof


FIRSTLY BY INSPECTION THE CURVE MEETS THE y AXIS AT $(0,1)$

$(0,4)$ $(0,1)$

$y = \sqrt{x+1}$
$y^2 = x+1$
$x = y^2 - 1$
$x^2 = (y^2-1)^2$
$x^2 = y^4 - 2y^2 + 1$

SETTING UP A VOLUME INTEGRAL, ABOUT THE y AXIS

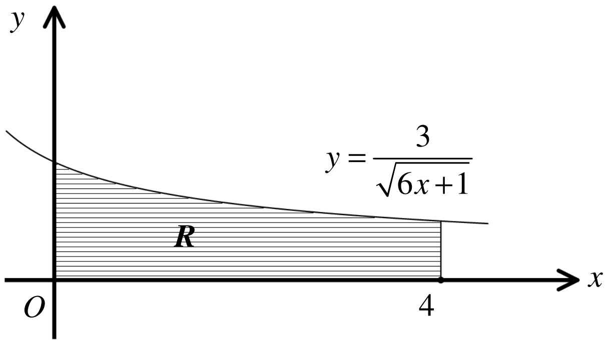
$$\text{VOLUME} = \pi \int_{y_1}^{y_2} (x(y))^2 \, dy = \pi \int_1^4 (y^4 - 2y^2 + 1) \, dy$$

$$= \pi \left[\frac{1}{5}y^5 - \frac{2}{3}y^3 + y \right]_1^4 = \pi\left(\frac{1024}{5} - \frac{128}{3} + 4 \right) - \pi\left(\frac{1}{5} - \frac{2}{3} + 1 \right)$$

$$= \frac{2492}{15}\pi - \frac{8}{15}\pi$$

$$= \frac{828}{5}\pi$$ AS REQUIRED

Question 15 (***)



The graph below shows the curve with equation

$$y = \frac{3}{\sqrt{6x+1}}, \quad x \neq -\frac{1}{6}.$$

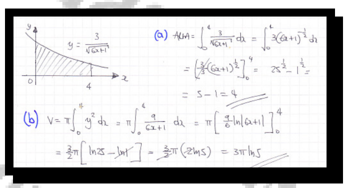
The region R, shown in the figure shaded, is bounded by the curve, the coordinate axes and the straight line with equation $x = 4$.

a) Show that the area of R is 4 square units.

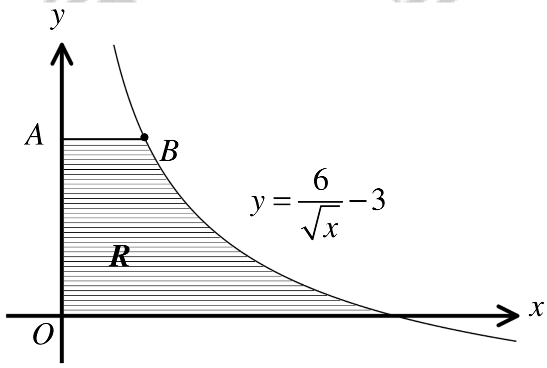
The shaded region R is rotated by 2π radians about the x axis to form a solid of revolution.

b) Show that the volume of the solid generated is $3\pi \ln 5$.

$\boxed{}$, $\boxed{\text{proof}}$



Question 16 (***)



The figure above shows part of the graph of the curve with equation

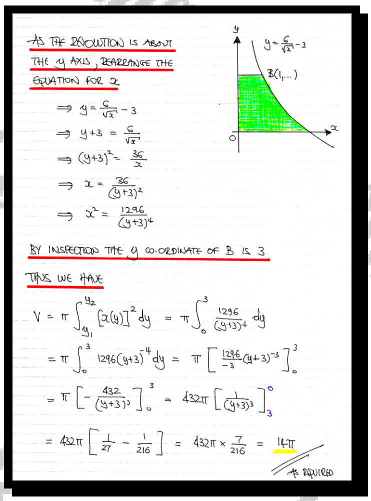
$$y = \frac{6}{\sqrt{x}} - 3, \quad x > 0.$$

The point B lies on the curve where $x = 1$.

The shaded region R is bounded by the curve, the coordinate axes and a straight line segment AB, where AB is parallel to the x axis. The region R is rotated through 2π radians in the y axis to form a solid of revolution.

Show that the volume of this solid is 14π.

$\boxed{}$, $\boxed{\text{proof}}$


AS THE REVOLUTION IS ABOUT THE y AXIS, REARRANGE THE EQUATION FOR x

$$\Rightarrow y = \frac{6}{\sqrt{x}} - 3$$
$$\Rightarrow y + 3 = \frac{6}{\sqrt{x}}$$
$$\Rightarrow (y+3)^2 = \frac{36}{x}$$
$$\Rightarrow x = \frac{36}{(y+3)^2}$$
$$\Rightarrow x^2 = \frac{1296}{(y+3)^4}$$

BY INSPECTION THE y CO-ORDINATE OF B IS 3

THUS WE HAVE

$$V = \pi \int_{y_1}^{y_2} [x(y)]^2 \, dy = \pi \int_0^3 \frac{1296}{(y+3)^4} \, dy$$
$$= \pi \int_0^3 1296(y+3)^{-4} \, dy = \pi \left[\frac{1296}{-3}(y+3)^{-3} \right]_0^3$$
$$= \pi \left[-\frac{432}{(y+3)^3} \right]_0^3 = 432\pi \left[\frac{1}{(y+3)^3} \right]_3^0$$
$$= 432\pi \left[\frac{1}{27} - \frac{1}{216} \right] = 432\pi \times \frac{7}{216} = 14\pi$$

AS REQUIRED

Question 17 (***)

The curve C has equation

$$y = x + \frac{1}{x^2}, \ x > 0.$$

The region bounded by C, the x axis and the lines $x = 1$, $x = 2$ is rotated through $360°$ about the x axis.

Show that the volume of the solid formed is

$$\pi\left(\frac{21}{8} + 2\ln 2\right).$$

proof

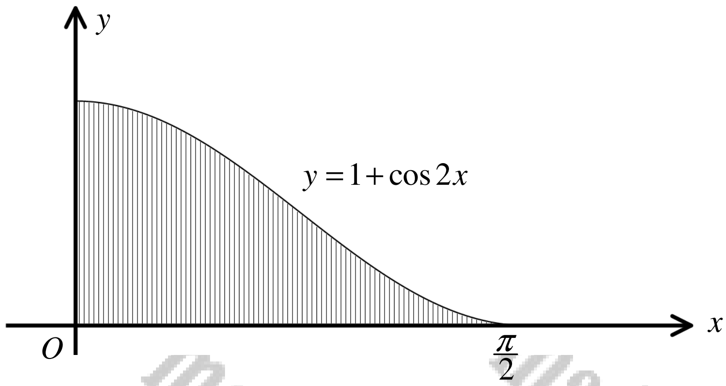
$$y^2 = \left(x + \frac{1}{x^2}\right)^2 = x^2 + 2x\left(\frac{1}{x^2}\right) + \left(\frac{1}{x^2}\right)^2 = x^2 + \frac{2}{x} + \frac{1}{x^4}$$

$$V = \pi\int_{x_1}^{x_2} y^2 \, dx = \pi\int_1^2 x^2 + \frac{2}{x} + x^{-4} \, dx = \pi\left[\frac{1}{3}x^3 + 2\ln|x| - \frac{1}{3}x^{-3}\right]_1^2$$

$$= \pi\left[\frac{1}{3}x^3 + 2\ln|x| - \frac{1}{3x^3}\right]_1^2 = \pi\left[\left(\frac{8}{3} + 2\ln 2 - \frac{1}{24}\right) - \left(\cancel{\frac{1}{3}} \cancel{\ln 1} \cancel{- \frac{1}{3}}\right)\right]$$

$$= \pi\left[\frac{21}{8} - 2\ln 2\right]$$

Question 18 (***)



The figure above shows the graph of the curve with equation

$$y = 1 + \cos 2x, \ 0 \le x \le \frac{\pi}{2}.$$

a) Show clearly that

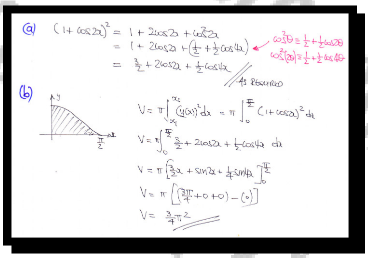
$$(1 + \cos 2x)^2 \equiv \frac{3}{2} + 2\cos 2x + \frac{1}{2}\cos 4x.$$

The shaded region bounded by the curve and the coordinate axes is rotated by 2π radians about the x axis to form a solid of revolution.

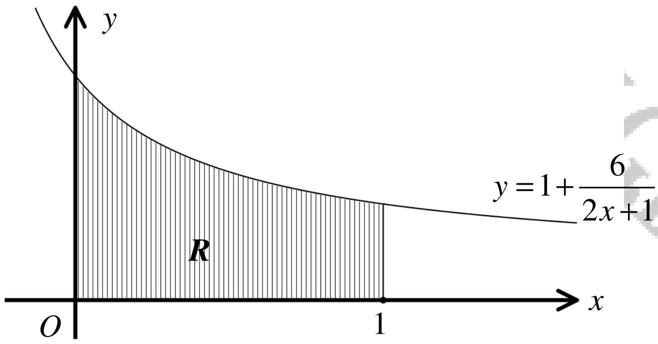
b) Show that the volume of the solid is

$$\frac{3}{4}\pi^2.$$

C4A, proof



Question 19 (***+)



The figure above shows the graph of the curve with equation

$$y = 1 + \frac{6}{2x+1}, \quad x \neq -\frac{1}{2}.$$

a) Show that

$$\left(1 + \frac{6}{2x+1}\right)^2 \equiv 1 + \frac{A}{2x+1} + \frac{B}{(2x+1)^2},$$

where A and B are constants to be found.

The shaded region, labelled as R, bounded by the curve, the coordinate axes and the line $x = 1$ is rotated by 2π radians in the x axis to form a solid of revolution.

b) Show further that the volume generated is

$$\pi(13 + 6\ln 3).$$

$\boxed{}$, $\boxed{A = 12}$, $\boxed{B = 36}$

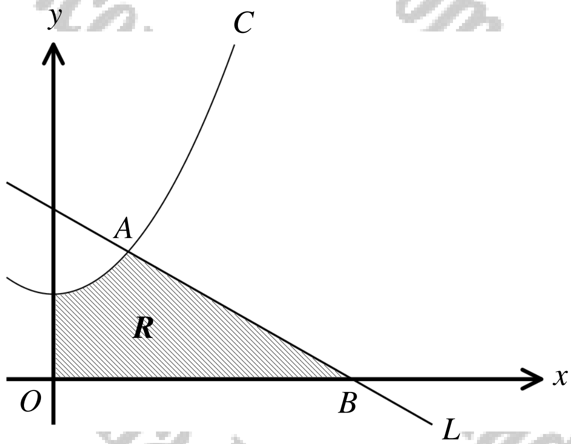
(a) $\left(1+\frac{6}{2x+1}\right)^2 = 1^2 + 2\times1\times\frac{6}{2x+1} + \left(\frac{6}{2x+1}\right)^2 = 1 + \frac{12}{2x+1} + \frac{36}{(2x+1)^2}$

(b) $V = \pi\int_{x_1}^{x_2} y^2\,dx = \pi\int_0^1 1 + \frac{12}{2x+1} + 36(2x+1)^{-2}\,dx =$

$= \pi\left[x + 6\ln|2x+1| - 18(2x+1)^{-1}\right]_0^1 = \pi\left[x + 6\ln|2x+1| - \frac{18}{2x+1}\right]_0^1$

$= \pi\left[(1 + 6\ln 3 - 6) - (0 + \cancel{6\ln 1} - 18)\right] = \pi\left[13 + 6\ln 3\right]$

Question 20 (***+)



The figure above shows the graph of the curve C with equation

$$y = x^2 + 2,$$

intersected by the straight line L with equation

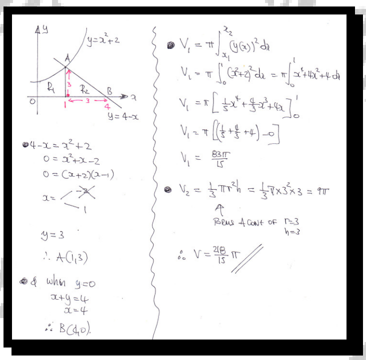
$$x + y = 4.$$

The point A is the intersection of C and L. The point B is the point where L meets the x axis.

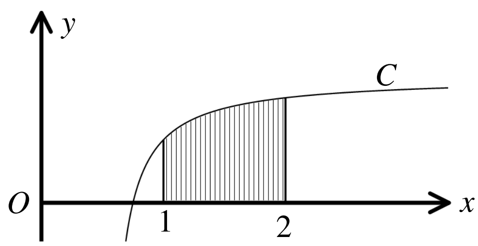
The region R, shown shaded in the figure above, is bounded by C, L and the coordinate axes. This region is rotated by 360° in the x axis, forming a solid of revolution S.

Find an exact value for the volume of S.

$$\boxed{\frac{218}{15}\pi}$$



Question 21 (***+)



The figure above shows part of the graph of the curve C with equation

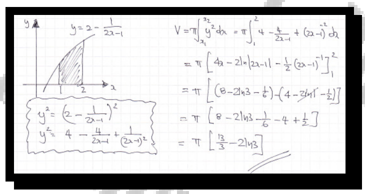
$$y = 2 - \frac{1}{2x-1}, \quad x \neq \frac{1}{2}.$$

The shaded region bounded by C and the straight lines with equations $x=1$ and $x=2$, is rotated by $360°$ about the x axis, forming a solid of revolution.

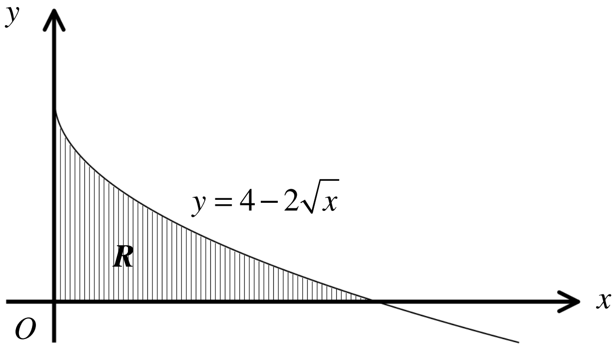
Show that the volume of the solid is

$$\pi\left(\frac{13}{3} - 2\ln 3\right).$$

proof



Question 22 (***+)



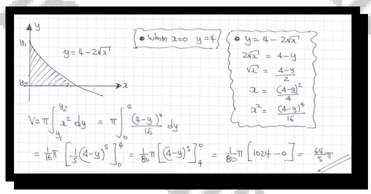
The figure above shows the graph of the equation

$$y = 4 - 2\sqrt{x}, \quad x \geq 0.$$

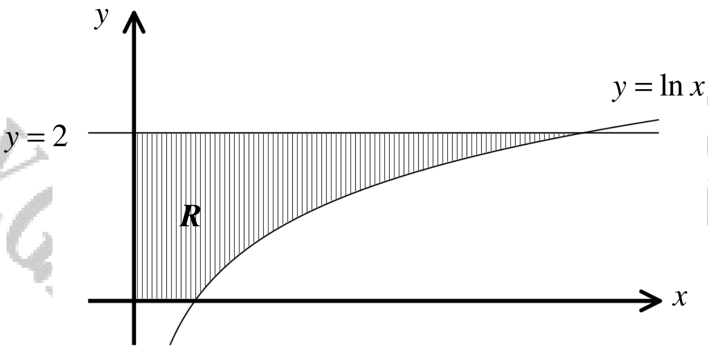
The shaded region R, bounded by the curve and the coordinate axes, is rotated through 4 right angles about the y axis to form a solid of revolution.

Show that the volume generated is $\frac{64\pi}{5}$.

proof



Question 23 (***+)



The figure below shows the graph of the curve C with equation

$$y = \ln x\,,\ \ x > 0\,,$$

intersected by the horizontal straight line L with equation

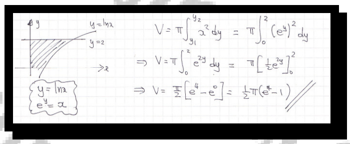
$$y = 2\,.$$

The shaded region R, bounded by C, L and the coordinate axes, is rotated through 2π radians in the y axis to form a solid of revolution.

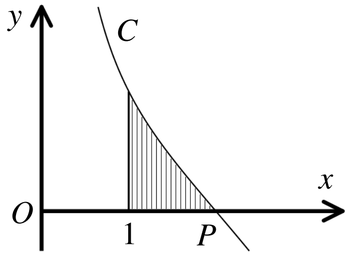
Show that the volume of the solid is

$$\tfrac{1}{2}\pi\left(\mathrm{e}^4 - 1\right).$$

proof



Question 24 (***+)



The figure above shows part of the curve C with equation

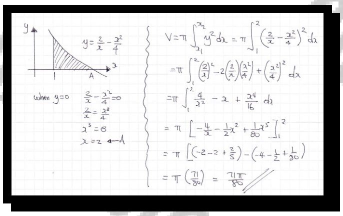
$$y = \frac{2}{x} - \frac{x^2}{4}, \ x > 0 .$$

The curve crosses the x axis at the point P.

The shaded region bounded by the curve, the straight line with equation $x = 1$ and the x axis is rotated by $360°$ about the x axis to form a solid of revolution.

Show that the volume of the solid is $\dfrac{71\pi}{80}$.

proof



Question 25 (***+)

The curve C lies entirely above the x axis and has equation

$$y = 1 + \frac{1}{2\sqrt{x}}, \quad x \geq 0.$$

a) Show that

$$y^2 = 1 + \frac{1}{\sqrt{x}} + \frac{1}{4x}.$$

The region R is bounded by the curve, the x axis and the straight lines with equations $x = 1$ and $x = 4$.

b) Show that when R is rotated by $360°$ about the x axis, the solid generated has a volume

$$\pi\left(5 + \ln\sqrt{2}\right).$$

$\boxed{\text{proof}}$

(a) $y^2 = \left(1 + \frac{1}{2\sqrt{x}}\right)^2 = 1^2 + 2 \times 1 \times \frac{1}{2\sqrt{x}} + \left(\frac{1}{2\sqrt{x}}\right)^2$

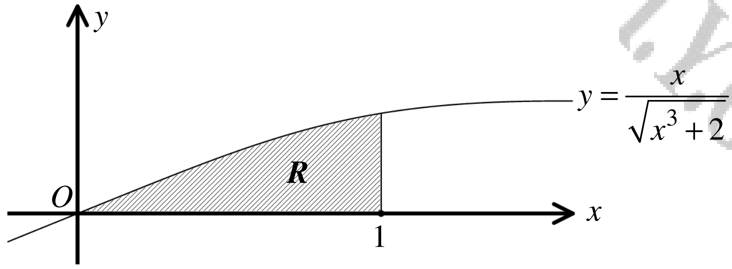
$= 1 + \frac{\cancel{2}}{\cancel{2}\sqrt{x}} + \frac{1}{4x} = 1 + \frac{1}{\sqrt{x}} + \frac{1}{4x}$

(b) $V = \pi \int_1^4 y^2 \, dx = \pi \int_1^4 1 + x^{-\frac{1}{2}} + \frac{1}{4} \times \frac{1}{x} \, dx$

$= \pi \left[x + 2x^{\frac{1}{2}} + \frac{1}{4}\ln|x| \right]_1^4 = \pi \left[\left(4 + 4 + \frac{1}{4}\ln 4\right) - \left(1 + 2 + \cancel{\frac{1}{4}\ln 1}\right) \right]$

$= \pi\left(5 + \frac{1}{4}\ln 4\right) = \pi\left(5 + \frac{1}{2}\ln 2\right) = \pi\left(5 + \ln\sqrt{2}\right)$

Question 26 (***+)



The figure above shows part of the curve with equation

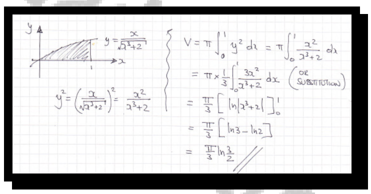
$$y = \frac{x}{\sqrt{x^3+2}}, \quad x^3 > -2 .$$

The shaded region R, bounded by the curve, the x axis and the straight line with equation $x = 1$, is rotated by $360°$ about the x axis to form a solid of revolution.

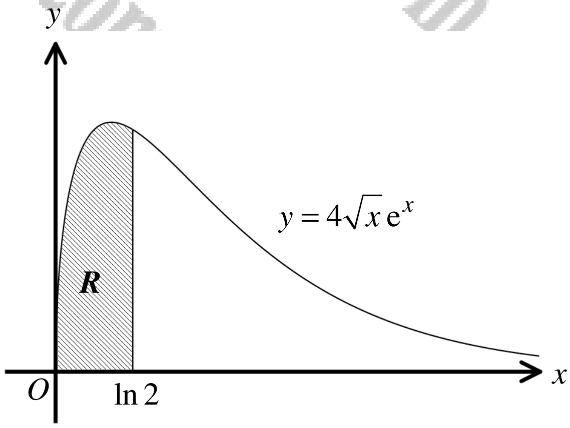
Show that the solid has a volume of

$$\frac{\pi}{3}\ln\left(\frac{3}{2}\right).$$

$\boxed{}$, $\boxed{\text{proof}}$



Question 27 (***+)



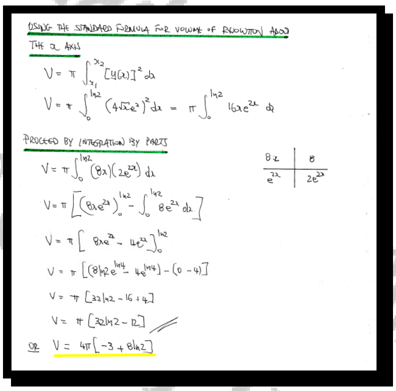
The figure above shows the graph of the curve C with equation

$$y = 4\sqrt{x}\,e^{x}, \quad x \ge 0.$$

The shaded region R bounded by the curve, the x axis and the vertical straight line with equation $x = \ln 2$, is rotated by 2π radians in the x axis, forming a solid of revolution S.

Find an exact value for the volume of S, giving the answer in the form $\pi(a + b\ln 2)$ where a and b are integers.

$\boxed{}$, $\boxed{\pi(-12 + 32\ln 2)}$


USING THE STANDARD FORMULA FOR VOLUME OF REVOLUTION ABOUT THE x AXIS

$V = \pi \int_{x_1}^{x_2} [y(x)]^2 \, dx$

$V = \pi \int_0^{\ln 2} (4\sqrt{x}e^{x})^2 \, dx = \pi \int_0^{\ln 2} 16xe^{2x} \, dx$

PROCEED BY INTEGRATION BY PARTS

$8x$	8
e^{2x}	$2e^{2x}$

$V = \pi \int_0^{\ln 2} (8x)(2e^{2x}) \, dx$

$V = \pi \left[\left(8xe^{2x}\right)_0^{\ln 2} - \int_0^{\ln 2} 8e^{2x} \, dx \right]$

$V = \pi \left[8xe^{2x} - 4e^{2x} \right]_0^{\ln 2}$

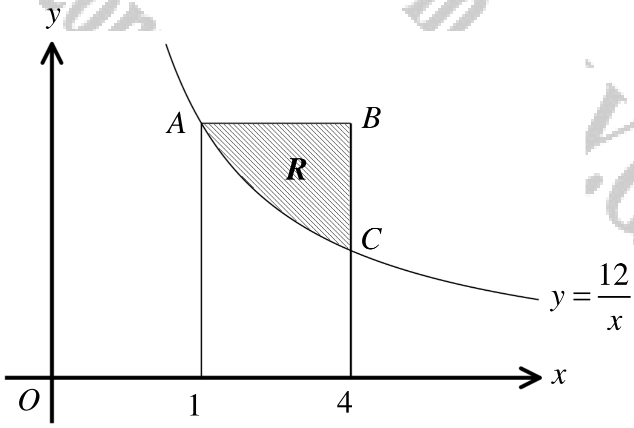
$V = \pi \left[(8\ln 2\, e^{\ln 4} - 4e^{\ln 4}) - (0 - 4) \right]$

$V = \pi [32\ln 2 - 16 + 4]$

$V = \pi [32\ln 2 - 12]$

OR $V = 4\pi[-3 + 8\ln 2]$

Question 28 (***+)



The figure above shows part of the graph of the curve with equation

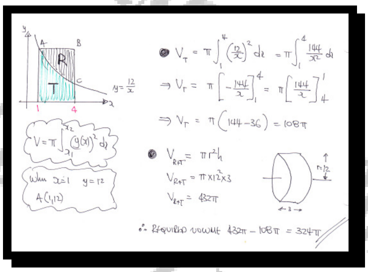
$$y = \frac{12}{x}, \ x \neq 0 .$$

The points A and C lie on the curve where $x = 1$ and $x = 4$, respectively. The point B is such so that AB is parallel to the x axis and BC is parallel to the y axis.

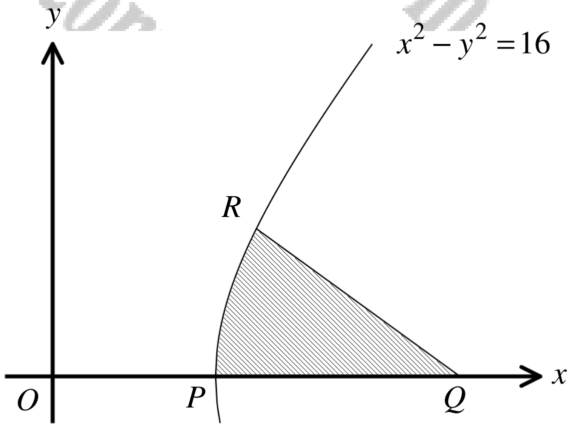
The region R, shown shaded in the figure above, is bounded by the curve and the straight line segments AB and BC. This region is rotated by 2π radians in the x axis, forming a solid of revolution S.

Find the exact value for the volume of S.

$\boxed{324\pi}$



Question 29 (***+)



The figure above shows part of the graph of the hyperbola C with equation

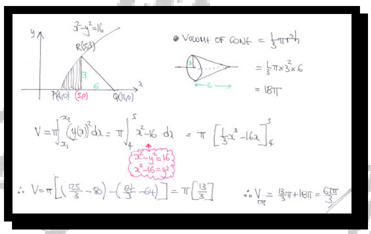
$$x^2 - y^2 = 16 .$$

The hyperbola crosses the x axis at $P(4,0)$, the point $R(5,3)$ lies on C and the point $Q(11,0)$ lies on the x axis.

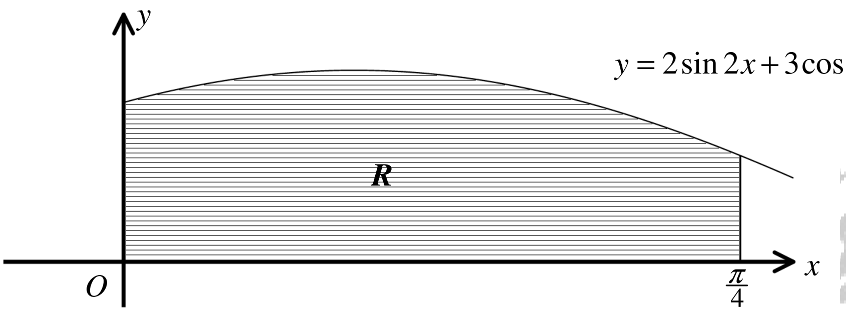
The shaded region bounded by the curve, the x axis and the straight line segment RQ is rotated by 2π radians in the x axis, forming a solid of revolution S.

Find an exact value for the volume of S.

$$\boxed{\frac{67}{3}\pi}$$



Question 30 (***+)



The figure above shows part of the curve C, with equation

$$y = 2\sin 2x + 3\cos 2x .$$

a) Show that

$$y^2 = A + B\cos 4x + C\sin 4x ,$$

where A, B and C are constants.

The shaded region R is bounded by the curve, the line $x = \frac{\pi}{4}$ and the coordinate axes.

b) Find the area of R.

The region R is rotated by 2π radians in the x axis forming a solid of revolution S.

c) Show that the volume of S is

$$\frac{\pi}{8}(13\pi + 24) .$$

$\boxed{}$, $\boxed{A = \frac{13}{2}, B = \frac{5}{2}, C = 6}$, $\boxed{\text{area} = 2.5}$

(a) $y^2 = (2\sin 2x + 3\cos 2x)^2 = 4\sin^2 2x + 12\sin 2x\cos 2x + 9\cos^2 2x$
$= 4\left(\frac{1}{2} - \frac{1}{2}\cos 4x\right) + 6(2\sin 2x\cos 2x) + 9\left(\frac{1}{2} + \frac{1}{2}\cos 4x\right)$
$= 2 - 2\cos 4x + 6\sin 4x + \frac{9}{2} + \frac{9}{2}\cos 4x$
$= \frac{13}{2} + \frac{5}{2}\cos 4x + 6\sin 4x$

(b) AREA $= \int_0^{\frac{\pi}{4}} 2\sin 2x + 3\cos 2x \, dx = \left[-\cos 2x + \frac{3}{2}\sin 2x\right]_0^{\frac{\pi}{4}}$
$= \left[\left(0 + \frac{3}{2}\right) - (-1 + 0)\right] = \frac{3}{2} + 1 = \frac{5}{2}$

(c) $V = \pi \int_{x_1}^{x_2} (y(x))^2 \, dx = \pi \int_0^{\frac{\pi}{4}} \frac{13}{2} + \frac{5}{2}\cos 4x + 6\sin 4x \, dx$
$= \left[\frac{13}{2}x + \frac{5}{8}\sin 4x - \frac{3}{2}\cos 4x\right]_0^{\frac{\pi}{4}}$
$= \pi\left(\frac{13\pi}{8} + 0 + \frac{3}{2}\right) - \left(0 + 0 - \frac{3}{2}\right)$
$= \pi\left[\frac{13\pi}{8} + 3\right] = \frac{\pi}{8}\left[13\pi + 24\right]$ AS REQUIRED

Question 31 (***+)

The point P lies on the curve with equation

$$y = x^2, \quad x \geq 0.$$

The straight line L_1 is parallel to the x axis and passes through P. The finite region R_1 is bounded by the curve, L_1 and the y axis.

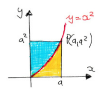
The straight line L_2 is parallel to the y axis and passes through P. The finite region R_2 is bounded by the curve, L_2 and the x axis.

When R_1 is fully revolved about the y axis the volume of the solid formed is equal to the volume of the solid formed when R_2 is fully revolved about the x axis.

Determine the x coordinate of P.

$\boxed{}$, $\boxed{x = \frac{5}{2}}$

- LET P HAVE CO-ORDINATES (a, a^2), $a > 0$



- VOLUME OF REVOLUTION ABOUT THE x AXIS

$$V_x = \pi \int_{x_1}^{x_2} y^2 \, dx$$

$$V_x = \pi \int_0^a (x^2)^2 \, dx = \pi \int_0^a x^4 \, dx = \frac{1}{5}\pi \left[x^5 \right]_0^a = \frac{1}{5}\pi a^5$$

- VOLUME OF REVOLUTION ABOUT THE y AXIS

$$V_y = \pi \int_{y_1}^{y_2} x^2 \, dy$$

$$V_y = \pi \int_0^{a^2} y \, dy = \frac{1}{2}\pi \left[y^2 \right]_0^{a^2} = \frac{1}{2}\pi a^4$$

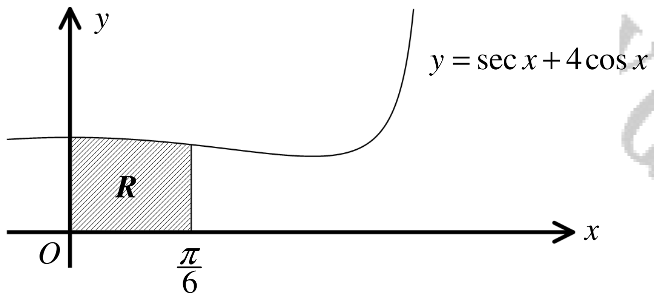
- NOW $V_x = V_y$

$$\frac{1}{5}\cancel{\pi} a^5 = \frac{1}{2}\cancel{\pi} a^4$$

$$a^5 = \frac{5}{2}a^4$$

$$a = \frac{5}{2} \qquad (a \neq 0)$$

Question 32 (****)



The figure above shows part of the curve with equation

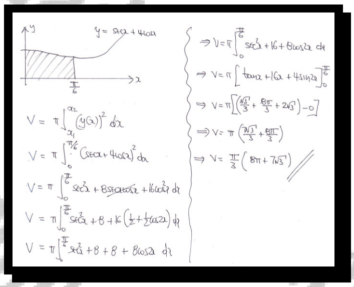
$$y = \sec x + 4\cos x .$$

The shaded region, labelled R, bounded by the curve, the coordinate axes and the straight line with equation $x = \frac{\pi}{6}$ is rotated by 2π radians in the x axis to form a solid of revolution.

Show that the solid has a volume of

$$\frac{\pi}{3}\left(8\pi + 7\sqrt{3}\right).$$

$\boxed{}$, $\boxed{\text{proof}}$



Question 33 (****)

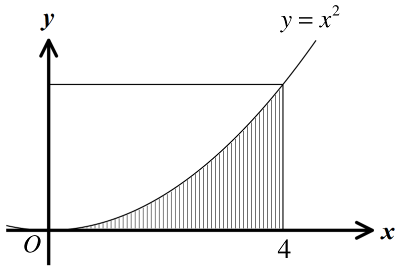


Figure 1

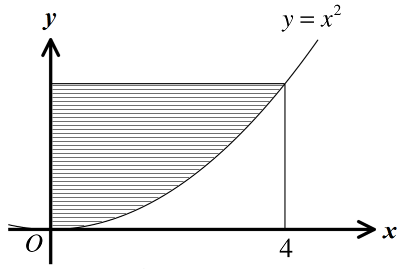


Figure 2

The figures above show part of the parabola with equation

$$y = x^2.$$

The shaded region, shown in Figure 1, is bounded by the curve, the x axis and the line $x = 4$. This region is revolved by 2π radians about the x axis, to form a solid of revolution.

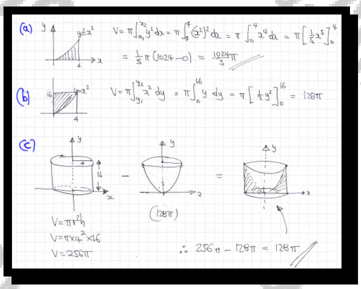
a) Show that the solid has a volume of $\frac{1024\pi}{5}$.

The shaded region, shown in Figure 2, is bounded by the curve, the y axis and a horizontal line originating from a point on the parabola where $x = 4$. This region is revolved by 2π radians about the y axis, to form a solid of revolution.

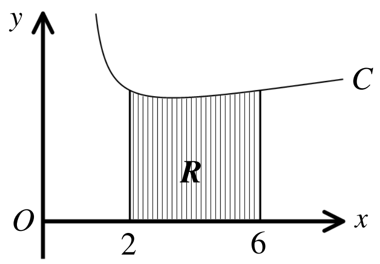
b) Show that the solid has a volume of 128π.

c) Hence find the value of the volume generated when the region shown in figure 1 is revolved by 2π radians about the y axis.

$\boxed{128\pi}$



Question 34 (****)



The figure above shows part of the curve C with equation

$$y = \frac{x+1}{\sqrt{x-1}}, \ x \geq 1.$$

The shaded region R is bounded by the curve, the x axis and the straight lines with equations $x = 2$ and $x = 6$. The region R is rotated by 360° about the x axis to form a solid of revolution.

a) Show that the volume of the solid is

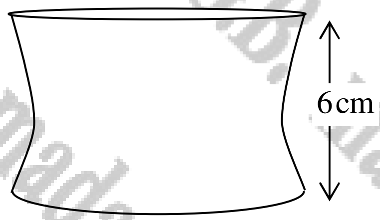
$$\pi\left(28 + 4\ln 5\right).$$

[continues overleaf]

[continued from overleaf]

The solid of part **(a)** is used to model the wooden leg of a sofa.

The shape of the leg is geometrically similar to the solid of part **(a)**.



b) Given the height of the leg is 6 cm, determine the volume of the wooden leg to the nearest cubic centimetre.

$\boxed{}$, $\boxed{\approx 365 \text{ cm}^3}$

a) VOLUME OF REVOLUTION IN CARTESIAN, ABOUT THE x AXIS

$$V = \pi\int_{x_1}^{x_2} (y(x))^2\,dx = \pi\int_2^6 \left(\frac{x+1}{\sqrt{x-1}}\right)^2 dx = \pi\int_2^6 \frac{(x+1)^2}{x-1}\,dx$$

BY SUBSTITUTION OR MANIPULATION

BY SUBSTITUTION

- $u = x-1$ or $x = u+1$
  $\frac{du}{dx} = 1$
  $du = dx$
- $x=2 \mapsto u=1$
  $x=6 \mapsto u=5$

$\Rightarrow V = \pi\int_1^5 \frac{(x+1)^2}{u}\,du$

$\Rightarrow V = \pi\int_1^5 \frac{(u+1+1)^2}{u}\,du$

$\Rightarrow V = \pi\int_1^5 \frac{(u+2)^2}{u}\,du$

$\Rightarrow V = \pi\int_1^5 \frac{u^2+4u+4}{u}\,du$

$\Rightarrow V = \pi\int_1^5 u + 4 + \frac{4}{u}\,du$

$\Rightarrow V = \pi\left[\frac{1}{2}u^2 + 4u + 4\ln|u|\right]_1^5$

$\Rightarrow V = \pi\left[\left(\frac{25}{2} + 20 + 4\ln 5\right) - \left(\frac{1}{2} + 4 + \cancel{4\ln 1}\right)\right]$

$\Rightarrow V = \pi\left[28 + 4\ln 5\right]$

MANIPULATION

$\Rightarrow V = \pi\int_2^6 \frac{[(x-1)+2]^2}{x-1}\,dx$

$\Rightarrow V = \pi\int_2^6 \frac{(x-1)^2 + 4(x-1) + 4}{x-1}\,dx$

$\Rightarrow V = \pi\int_2^6 \frac{(x-1)^2}{x-1} + \frac{4(x-1)}{x-1} + \frac{4}{x-1}\,dx$

$\Rightarrow V = \pi\int_2^6 x - 1 + 4 + \frac{4}{x-1}\,dx$

$\Rightarrow V = \pi\int_2^6 x + 3 + \frac{4}{x-1}\,dx$

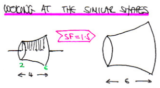
$\Rightarrow V = \pi\left[\frac{1}{2}x^2 + 3x + 4\ln|x-1|\right]_2^6$

$\Rightarrow V = \pi\left[(18 + 18 + 4\ln 5) - (2 + 6 + \cancel{\ln 1})\right]$

$\Rightarrow V = \pi\left[28 + 4\ln 5\right]$

AS OPPOSITE

b) LOOKING AT THE SIMILAR SHAPES



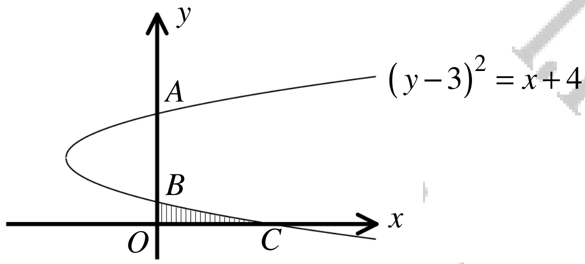
$v = \pi(28 + 4\ln 5)$ $\qquad V = ?$

$V = v \times (\text{SCALE FACTOR})^3$

$V = \pi(28 + 4\ln 5) \times (1.5)^3$

$V \simeq 365$

Question 35 (****)



The figure above shows part of the curve with equation

$$(y-3)^2 = x+4.$$

The curve crosses the coordinate axes at the points A, B and C.

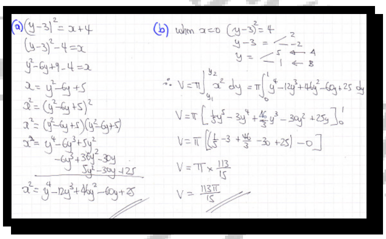
a) Show that

$$x^2 = y^4 - 12y^3 + 46y^2 - 60y + 25.$$

b) The shaded region bounded by the curve and the coordinate axes is rotated by $360°$ about the y axis to form a solid of revolution.

Show that the volume of the solid is $\dfrac{113\pi}{15}$.

proof



(a) $(y-3)^2 = x+4$

$(y-3)^2 - 4 = x$

$y^2 - 6y + 9 - 4 = x$

$x = y^2 - 6y + 5$

$x^2 = (y^2 - 6y + 5)^2$

$x^2 = (y^2 - 6y + 5)(y^2 - 6y + 5)$

$x^2 = y^4 - 6y^3 + 5y^2$
$\quad - 6y^3 + 36y^2 - 30y$
$\quad + 5y^2 - 30y + 25$

$x^2 = y^4 - 12y^3 + 46y^2 - 60y + 25$

(b) when $x=0$ $(y-3)^2 = 4$

$y - 3 = 2, -2$

$y = 5 \leftarrow A$, $1 \leftarrow B$

$\therefore V = \pi \int_{y_1}^{y_2} x^2 \, dy = \pi \int_0^1 y^4 - 12y^3 + 46y^2 - 60y + 25 \, dy$

$V = \pi \left[\frac{1}{5}y^5 - 3y^4 + \frac{46}{3}y^3 - 30y^2 + 25y \right]_0^1$

$V = \pi \left[\left(\frac{1}{5} - 3 + \frac{46}{3} - 30 + 25 \right) - 0 \right]$

$V = \pi \times \frac{113}{15}$

$V = \frac{113\pi}{15}$

Question 36 **(****)**

The curve C has equation

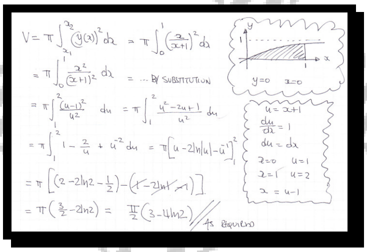
$$y = \frac{x}{x+1}, \quad x \geq 0 .$$

The region bounded by the curve, the x axis and the straight line with equation $x = 1$ is rotated through 2π radians about the x axis to form a solid of revolution.

Show that the volume of the solid is

$$\frac{\pi}{2}\left(3 - 4\ln 2\right).$$

$\boxed{\text{proof}}$



$$V = \pi \int_{x_1}^{x_2} (y(x))^2 \, dx = \pi \int_0^1 \left(\frac{x}{x+1}\right)^2 dx$$

$$= \pi \int_0^1 \frac{x^2}{(x+1)^2} \, dx = \ldots \text{ BY SUBSTITUTION}$$

$$= \pi \int_1^2 \frac{(u-1)^2}{u^2} \, du = \pi \int_1^2 \frac{u^2 - 2u + 1}{u^2} \, du$$

$$= \pi \int_1^2 1 - \frac{2}{u} + u^{-2} \, du = \pi \left[u - 2\ln|u| - u^{-1} \right]_1^2$$

$$= \pi \left[\left(2 - 2\ln 2 - \frac{1}{2}\right) - \left(\cancel{1} - \cancel{2\ln 1} - \cancel{1}\right) \right]$$

$$= \pi \left(\frac{3}{2} - 2\ln 2 \right) = \frac{\pi}{2}\left(3 - 4\ln 2\right)$$

AS REQUIRED

$u = x+1$

$\frac{du}{dx} = 1$

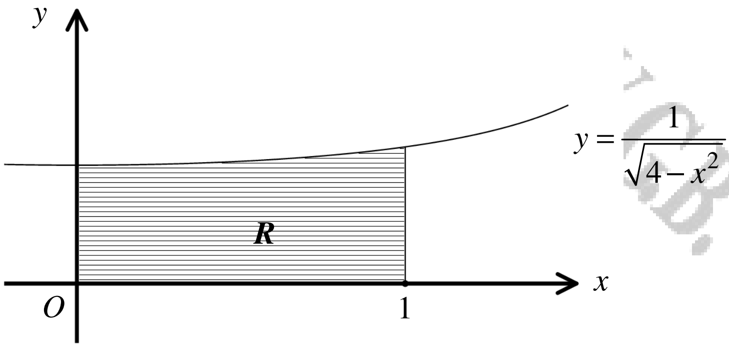
$du = dx$

$x=0 \quad u=1$

$x=1 \quad u=2$

$x = u-1$

Question 37 (****)



The figure above shows part of the curve with equation

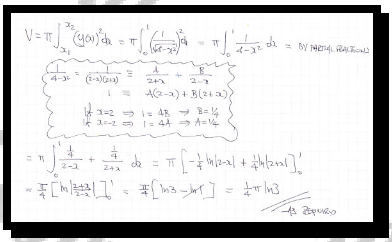
$$y = \frac{1}{\sqrt{4-x^2}}, \quad -2 \le x \le 2.$$

The shaded region, labelled as R, bounded by the curve, the coordinate axes and the straight line with equation $x = 1$ is rotated by 2π radians about the x axis to form a solid of revolution.

Show that the volume of the solid is

$$\frac{1}{4}\pi \ln 3.$$

proof



$V = \pi \int_{x_1}^{x_2} (y(x))^2 \, dx = \pi \int_0^1 \left(\frac{1}{\sqrt{4-x^2}}\right)^2 dx = \pi \int_0^1 \frac{1}{4-x^2} \, dx =$ BY PARTIAL FRACTIONS

$\frac{1}{4-x^2} = \frac{1}{(2-x)(2+x)} \equiv \frac{A}{2+x} + \frac{B}{2-x}$

$1 \equiv A(2-x) + B(2+x)$

If $x=2 \Rightarrow 1 = 4B \Rightarrow B = \frac{1}{4}$

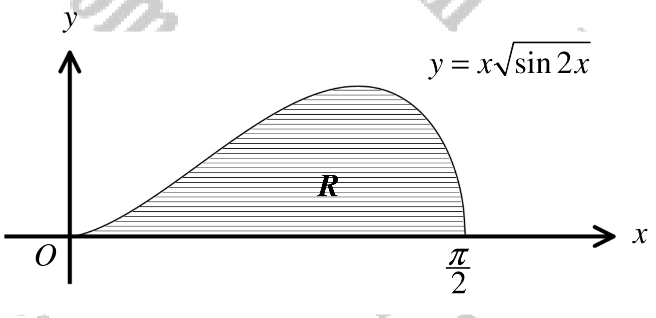
If $x=-2 \Rightarrow 1 = 4A \Rightarrow A = \frac{1}{4}$

$= \pi \int_0^1 \frac{\frac{1}{4}}{2-x} + \frac{\frac{1}{4}}{2+x} \, dx = \pi \left[-\frac{1}{4}\ln|2-x| + \frac{1}{4}\ln|2+x| \right]_0^1$

$= \frac{\pi}{4}\left[\ln\left|\frac{2+x}{2-x}\right| \right]_0^1 = \frac{\pi}{4}\left[\ln 3 - \cancel{\ln 1} \right] = \frac{1}{4}\pi \ln 3$

AS REQUIRED

Question 38 (****)



The figure above shows the graph of the curve with equation

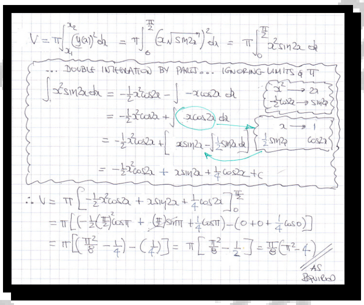
$$y = x\sqrt{\sin 2x}\,,\ 0 \le x \le \frac{\pi}{2}\,.$$

The shaded region, labelled as R, bounded by the curve and the x axis, is rotated by $360°$ about the x axis to form a solid of revolution.

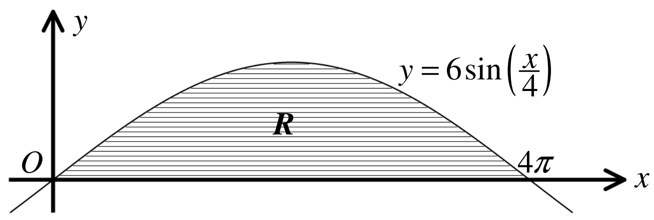
Show that the volume of the solid generated is

$$\frac{\pi}{8}\left(\pi^2 - 4\right).$$

$\boxed{\text{proof}}$



Question 39 (****)



The figure below shows the graph of the curve with equation

$$y = 6\sin\left(\frac{x}{4}\right),\ 0 \le x \le 4\pi .$$

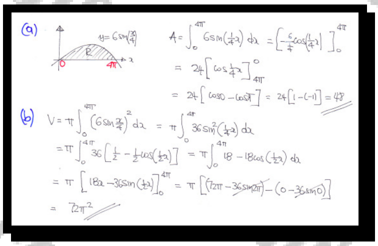
The shaded region R, is bounded by the curve and the x axis.

a) Determine the area of R.

This region R is rotated through $360°$ about the x axis to form a solid of revolution.

b) Show that the volume of the solid generated is $72\pi^2$.

$\boxed{}$, $\boxed{48 \text{ square units}}$



(a) $A = \int_0^{4\pi} 6\sin\left(\frac{1}{4}x\right)\,dx = \left[-\frac{6}{\frac{1}{4}}\cos\left(\frac{1}{4}x\right)\right]_0^{4\pi}$

$= 24\left[\cos\frac{1}{4}x\right]_{4\pi}^{0}$

$= 24\left[\cos 0 - \cos\pi\right] = 24\left[1-(-1)\right] = 48$

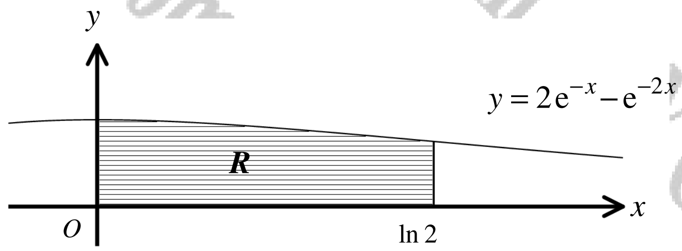
(b) $V = \pi\int_0^{4\pi}\left(6\sin\frac{x}{4}\right)^2 dx = \pi\int_0^{4\pi} 36\sin^2\left(\frac{1}{4}x\right)dx$

$= \pi\int_0^{4\pi} 36\left[\frac{1}{2} - \frac{1}{2}\cos\left(\frac{1}{2}x\right)\right] = \pi\int_0^{4\pi} 18 - 18\cos\left(\frac{1}{2}x\right)dx$

$= \pi\left[18x - 36\sin\left(\frac{1}{2}x\right)\right]_0^{4\pi} = \pi\left[\left(72\pi - \cancel{36\sin 2\pi}\right) - \left(0 - \cancel{36\sin 0}\right)\right]$

$= 72\pi^2$

Question 40 (****)



The figure above shows part of the graph of the curve with equation

$$y = 2e^{-x} - e^{-2x} \quad x \in \mathbb{R}.$$

The shaded region R, bounded by the curve, the coordinate axes and the straight line with equation $x = \ln 2$, is rotated through 360° about the x axis to form a solid of revolution.

Show that the volume of the solid generated is exactly $\frac{109}{192}\pi$.

$\boxed{\text{proof}}$

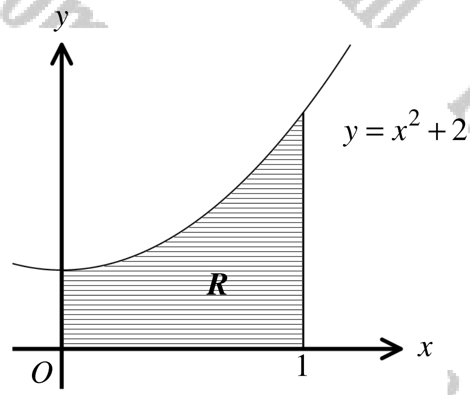
$$V = \pi\int_{x_1}^{x_2} y^2\,dx = \pi\int_0^{\ln 2} \left(2e^{-x} - e^{-2x}\right)^2 dx = \pi\int_0^{\ln 2} 4e^{-2x} - 4e^{-3x} + e^{-4x}\,dx$$

$$= \pi\left[-2e^{-2x} + \frac{4}{3}e^{-3x} - \frac{1}{4}e^{-4x}\right]_0^{\ln 2} = \pi\left[2e^{-2x} - \frac{4}{3}e^{-3x} + \frac{1}{4}e^{-4x}\right]_{\ln 2}^{0}$$

$$= \pi\left[\left(2 - \frac{4}{3} + \frac{1}{4}\right) - \left(\frac{1}{2} - \frac{1}{6} + \frac{1}{64}\right)\right] = \frac{109}{192}\pi$$

AS REQUIRED

Question 41 (****)



The figure babove shows the graph of the curve with equation

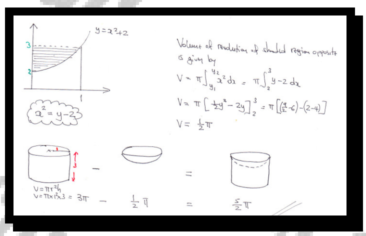
$$y = x^2 + 2 .$$

The shaded region R, is bounded by the curve, the coordinate axes and the straight line with equation $x = 1$.

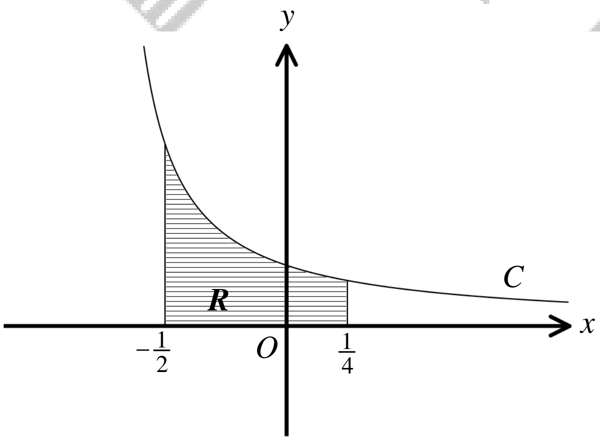
The region R is rotated through $360°$ about the $\boldsymbol{y}$ **axis** to form a solid of revolution.

Show that the volume of the solid generated is $\frac{5}{2}\pi$ cubic units.

$\boxed{\text{proof}}$



Question 42 (****)



The figure above shows part the graph of the curve C, with equation

$$y = \frac{3}{2(4x+3)}, \quad x \neq -\frac{3}{4}.$$

The shaded region R, is bounded by the curve, the x axis and the straight lines with equations $x = -\frac{1}{2}$ and $x = \frac{1}{4}$.

a) Find the exact area of R.

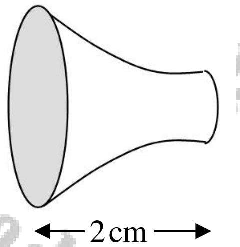
This region R is rotated through $360°$ about the x axis to form a solid of revolution.

b) Show that the volume of the solid generated is $\frac{27}{64}\pi$.

[continues overleaf]

[continues from overleaf]

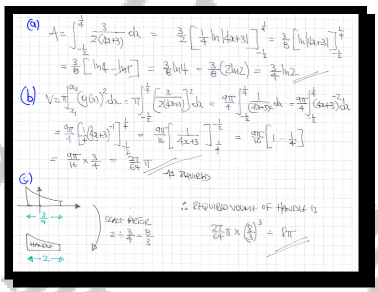
The solid generated in part (b) is used to model a small handle for a drawer.



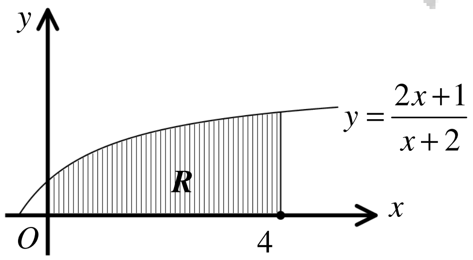
The solid generated in part (b) and the drawer handle are mathematically similar.

c) Given that the length of the handle is 2 cm, find the exact volume of the handle.

$\boxed{\text{area} = \frac{3}{4}\ln 2}$, $\boxed{\text{volume of handle} = 8\pi}$



Question 43 (****)



The figure above shows part of the curve with equation

$$y = \frac{2x+1}{x+2}, \quad x \neq -2 .$$

a) Show that

$$\frac{2x+1}{x+2} = A + \frac{B}{x+2},$$

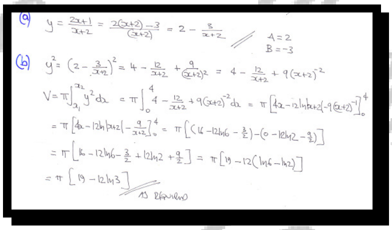
where A are B are constants to be found.

The shaded region, labelled R, bounded by the curve, the coordinate axes and the straight line with equation $x = 4$ is rotated by 360° about the x axis to form a solid of revolution.

b) Show that the volume of revolution is

$$\pi(19 - 12\ln 3).$$

$\boxed{A = 2, B = -3}$



(a) $y = \frac{2x+1}{x+2} = \frac{2(x+2)-3}{(x+2)} = 2 - \frac{3}{x+2}$ // $A = 2$, $B = -3$

(b) $y^2 = \left(2 - \frac{3}{x+2}\right)^2 = 4 - \frac{12}{x+2} + \frac{9}{(x+2)^2} = 4 - \frac{12}{x+2} + 9(x+2)^{-2}$

$V = \pi\int_{x_1}^{x_2} y^2\,dx = \pi\int_0^4 4 - \frac{12}{x+2} + 9(x+2)^{-2}\,dx = \pi\left[4x - 12\ln|x+2| - 9(x+2)^{-1}\right]_0^4$

$= \pi\left[4x - 12\ln|x+2| - \frac{9}{x+2}\right]_0^4 = \pi\left[(16 - 12\ln 6 - \frac{3}{2}) - (0 - 12\ln 2 - \frac{9}{2})\right]$

$= \pi\left[16 - 12\ln 6 - \frac{3}{2} + 12\ln 2 + \frac{9}{2}\right] = \pi\left[19 - 12(\ln 6 - \ln 2)\right]$

$= \pi\left[19 - 12\ln 3\right]$ // AS REQUIRED

Question 44 (****)

The curve C has equation

$$y = x\mathrm{e}^{x}, \quad x \in \mathbb{R}.$$

The region R is bounded by the curve, the x axis and the vertical straight lines with equations $x = 1$ and $x = 3$.

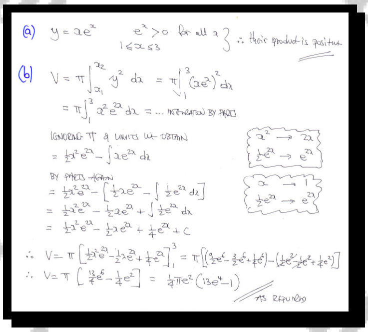
a) Explain why R lies entirely above the x axis.

The region R is rotated by $360°$ in the x axis to form a solid of revolution.

b) Show that the volume of this solid is

$$\frac{1}{4}\pi \mathrm{e}^{2}\left(13\mathrm{e}^{4} - 1\right).$$

proof


(a) $y = xe^{x}$ $\quad$ $e^{x} > 0$ for all x, $1 \leq x \leq 3$ $\Big\}$ $\therefore$ their product is positive

(b) $V = \pi \int_{x_1}^{x_2} y^2 \, dx = \pi \int_{1}^{3} (xe^{x})^2 \, dx$

$= \pi \int_{1}^{3} x^2 e^{2x} \, dx = \dots$ INTEGRATION BY PARTS

IGNORING π & LIMITS WE OBTAIN

$= \frac{1}{2}x^2e^{2x} - \int xe^{2x} \, dx$

$x^2 \rightarrow 2x$

$\frac{1}{2}e^{2x} \rightarrow e^{2x}$

BY PARTS AGAIN

$= \frac{1}{2}x^2e^{2x} - \left[\frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x} \, dx\right]$

$x \rightarrow 1$

$\frac{1}{2}e^{2x} \rightarrow e^{2x}$

$= \frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \int \frac{1}{2}e^{2x} \, dx$

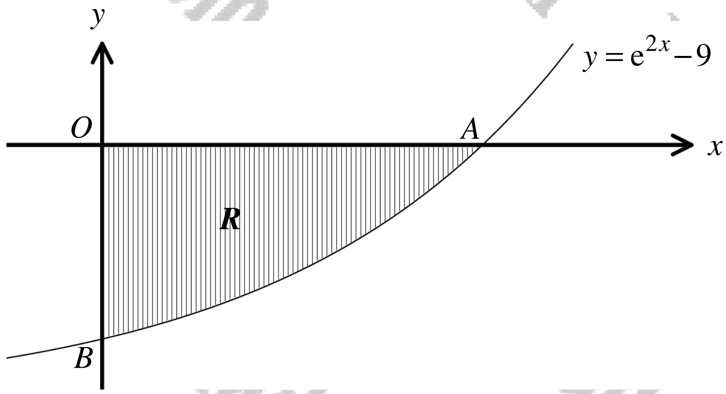
$= \frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + C$

$\therefore V = \pi\left[\frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x}\right]_1^3 = \pi\left[\left(\frac{9}{2}e^6 - \frac{3}{2}e^6 + \frac{1}{4}e^6\right) - \left(\frac{1}{2}e^2 - \frac{1}{2}e^2 + \frac{1}{4}e^2\right)\right]$

$\therefore V = \pi\left[\frac{13}{4}e^6 - \frac{1}{4}e^2\right] = \frac{1}{4}\pi e^2(13e^4 - 1)$

AS REQUIRED

Question 45 (****)



The figure above shows part of the curve with equation

$$y = e^{2x} - 9, \ x \in \mathbb{R}.$$

The curve crosses the coordinate axes at the points A and B. The shaded region R is bounded by the curve and the coordinate axes.

a) Determine the exact coordinates of A and B.

The region R is rotated by 2π radians about the x axis to form a solid of revolution.

b) Calculate the volume generated, giving the answer in the form $\pi\left(p + q\ln 3\right)$ where p and q are integers.

$\boxed{(\ln 3, 0)}$, $\boxed{(0, -8)}$, $\boxed{V = \pi\left(-52 + 81\ln 3\right)}$

(a) $x=0$, $y=-8$, $B(0,-8)$

$y=0$, $e^{2x}=9$, $2x=\ln 9$, $x=\ln 3$, $A(\ln 3, 0)$

(b) $V = \pi\int_{x_1}^{x_2} y^2\,dx = \pi\int_0^{\ln 3}\left(e^{2x}-9\right)^2 dx$

$V = \pi\int_0^{\ln 3} e^{4x} - 18e^{2x} + 81\,dx$

$V = \pi\left[\frac{1}{4}e^{4x} - 9e^{2x} + 81x\right]_0^{\ln 3}$

$V = \pi\left[\left(\frac{81}{4} - 81 + 81\ln 3\right) - \left(\frac{1}{4} - 9\right)\right]$

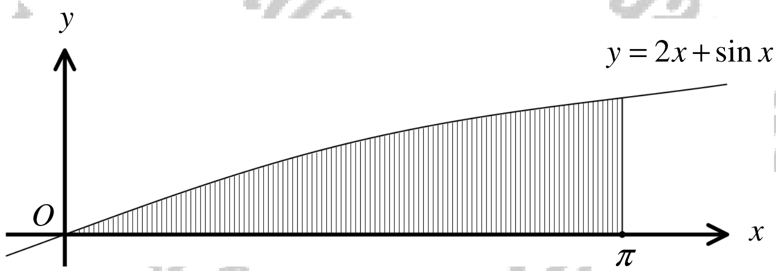
$V = \pi\left[-52 + 81\ln 3\right]$

Question 46 (****)

Show that

a) $\displaystyle\int_0^{\pi} 4x\sin x \; dx = 4\pi$.

b) $\displaystyle\int_0^{\pi} \sin^2 x \; dx = \frac{\pi}{2}$.



The figure above shows part of the curve with equation

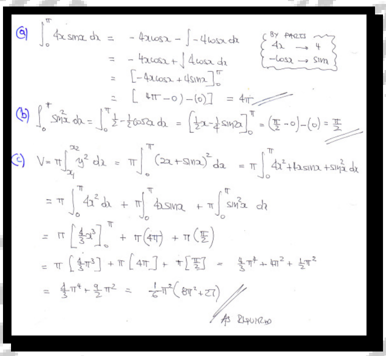
$$y = 2x + \sin x .$$

The shaded region bounded by the curve, the x axis and the line $x = \pi$ is rotated by 2π radians about the x axis to form a solid of revolution.

c) Show that the volume of the solid is

$$\frac{1}{6}\pi^2\left(8\pi^2 + 27\right).$$

C4P, proof



(a) $\int_0^{\pi} 4x\sin x \, dx = -4x\cos x - \int -4\cos x \, dx$

$= -4x\cos x + \int 4\cos x \, dx$

$= \left[-4x\cos x + 4\sin x\right]_0^{\pi}$

$= \left[4\pi - 0) - (0)\right] = 4\pi$

BY PARTS: $4x \to 4$, $-\cos x \to \sin x$

(b) $\int_0^{\pi} \sin^2 x \, dx = \int_0^{\pi} \frac{1}{2} - \frac{1}{2}\cos 2x \, dx = \left[\frac{1}{2}x - \frac{1}{4}\sin 2x\right]_0^{\pi} = \left(\frac{\pi}{2} - 0\right) - (0) = \frac{\pi}{2}$

(c) $V = \pi\int_{x_1}^{x_2} y^2 \, dx = \pi\int_0^{\pi} (2x + \sin x)^2 \, dx = \pi\int_0^{\pi} 4x^2 + 4x\sin x + \sin^2 x \, dx$

$= \pi\int_0^{\pi} 4x^2 \, dx + \pi\int_0^{\pi} 4x\sin x + \pi\int_0^{\pi} \sin^2 x \, dx$

$= \pi\left[\frac{4}{3}x^3\right]_0^{\pi} + \pi(4\pi) + \pi\left(\frac{\pi}{2}\right)$

$= \pi\left[\frac{4}{3}\pi^3\right] + \pi\left[4\pi\right] + \pi\left[\frac{\pi}{2}\right] = \frac{4}{3}\pi^4 + 4\pi^2 + \frac{1}{2}\pi^2$

$= \frac{4}{3}\pi^4 + \frac{9}{2}\pi^2 = \frac{1}{6}\pi^2(8\pi^2 + 27)$

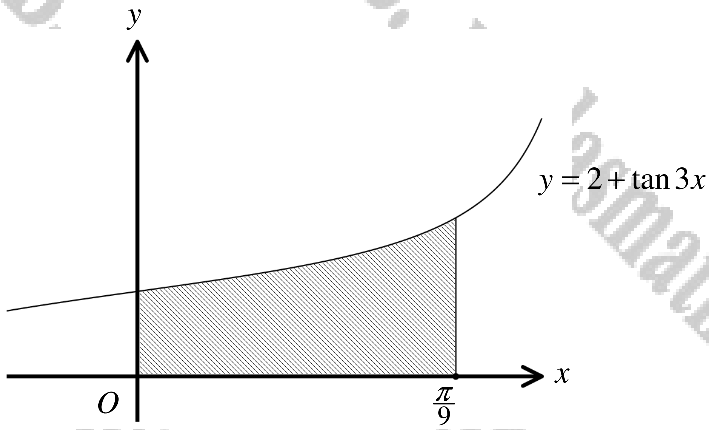
AS REQUIRED

Question 47 (****)

Show that

a) $(2+\tan 3x)^2 = 3 + 4\tan 3x + \sec^2 3x$

b) $\displaystyle\int \tan x \, dx = \ln|\sec x| + C$



The figure above shows part of the graph of the curve with equation

$$y = 2 + \tan 3x \, .$$

The shaded region bounded by the curve the coordinate axes and the line $x = \frac{\pi}{9}$ is rotated by 2π radians about the x axis to form a solid of revolution.

c) Show that the volume of the solid is

$$\frac{\pi}{3}\left(\pi + 4\ln 2 + \sqrt{3}\right).$$

$\boxed{}$, $\boxed{\text{proof}}$

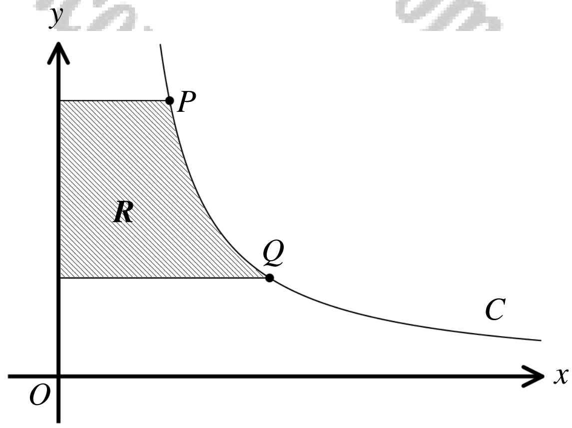
(a) $(2+\tan 3x)^2 = 4 + 4\tan 3x + \tan^2 3x = 4 + 4\tan 3x + (\sec^2 3x - 1)$
$= 3 + 4\tan 3x + \sec^2 3x$

(b) $\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx = -\int \frac{-\sin x}{\cos x} dx = -\ln|\cos x| + C$
$= \ln|\cos x|^{-1} + C = \ln\left|\frac{1}{\cos x}\right| + C = \ln|\sec x| + C$

(c) $V = \pi \int_{x_1}^{x_2} y^2 \, dx = \pi \int_0^{\frac{\pi}{9}} (2+\tan 3x)^2 \, dx = \pi \int_0^{\frac{\pi}{9}} 3 + 4\tan 3x + \sec^2 3x \, dx$
$= \pi \left[3x + \frac{4}{3}\ln|\sec 3x| + \frac{1}{3}\tan 3x \right]_0^{\frac{\pi}{9}}$
$= \pi \left[\left(\frac{\pi}{3} + \frac{4}{3}\ln 2 + \frac{\sqrt{3}}{3} \right) - \left(0 + \frac{4}{3}\ln 1 + 0 \right) \right]$
$= \frac{\pi}{3}\left(\pi + 4\ln 2 + \sqrt{3}\right)$

AS REQUIRED

Question 48 (****)



The figure above shows the graph of the curve C with equation

$$y = \frac{14}{x-2}, \quad x \neq 2.$$

The points P and Q lie on C where $x = 2.5$ and $x = 3.75$ respectively.

The shaded region R is bounded by the curve and two horizontal lines passing through the points P and Q.

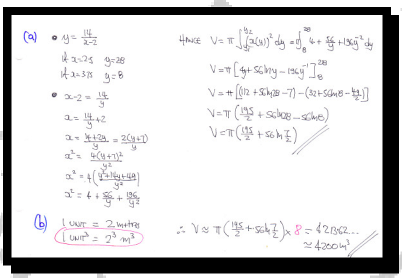
R is rotated by 2π radians about the y axis forming a solid of revolution S.

a) Find the volume of S, giving the answer in the form $\pi(a + b\ln c)$ where a, b and c are constants.

The solid S is used to model a nuclear station cooling tower.

b) Given that 1 unit on the axes corresponds to 2 metres on the actual tower, show that the cooling tower has an approximate volume of 4200 m^3.

$$\boxed{\pi\left(\frac{195}{2} + 56\ln\left(\frac{7}{2}\right)\right)}$$


(a) • $y = \frac{14}{x-2}$

If $x = 2.5$ $y = 28$

If $x = 3.75$ $y = 8$

• $x - 2 = \frac{14}{y}$

$x = \frac{14}{y} + 2$

$x = \frac{14+2y}{y} = \frac{2(y+7)}{y}$

$x^2 = \frac{4(y+7)^2}{y^2}$

$x^2 = 4\left(\frac{y^2+14y+49}{y^2}\right)$

$x^2 = 4 + \frac{56}{y} + \frac{196}{y^2}$

Hence $V = \pi\int_{y_1}^{y_2} (x(y))^2\,dy = \pi\int_8^{28} 4 + \frac{56}{y} + 196y^{-2}\,dy$

$V = \pi\left[4y + 56\ln y - 196y^{-1}\right]_8^{28}$

$V = \pi\left[(112 + 56\ln 28 - 7) - (32 + 56\ln 8 - \frac{49}{2})\right]$

$V = \pi\left(\frac{195}{2} + 56\ln 28 - 56\ln 8\right)$

$V = \pi\left(\frac{195}{2} + 56\ln\frac{7}{2}\right)$

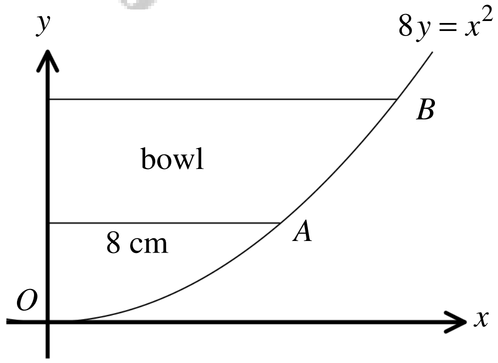
(b) 1 UNIT = 2 metres

1 UNIT^3 = 2^3 m^3

$\therefore V \approx \pi\left(\frac{195}{2} + 56\ln\frac{7}{2}\right) \times 8 = 4213.62\ldots$

$\simeq 4200 \text{ m}^3$

Question 49 (****)



The figure above shows the graph of the curve with equation

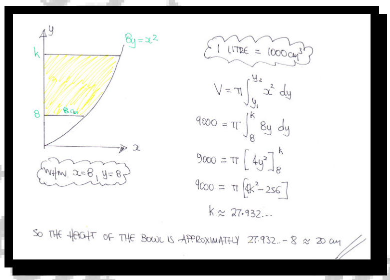
$$8y = x^2, \ x \geq 0.$$

The points A and B lie on the curve. The curved surface of an open bowl with flat circular base is traced out by the complete revolution of the arc AB about the y **axis**.

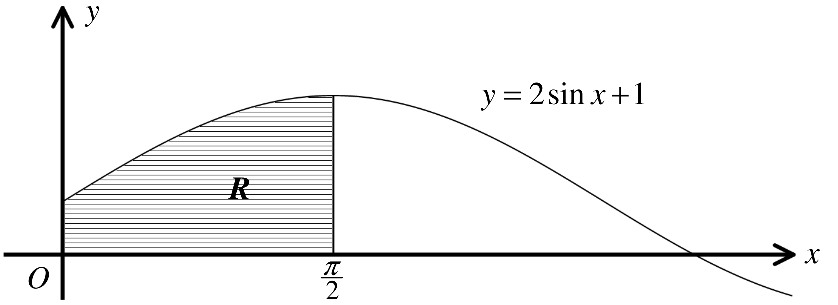
The radius of the flat circular base of the bowl is 8 cm, and its volume is 9 **litres**.

Find to the nearest cm the height of the bowl.

$$\boxed{\text{height} \approx 20\,\text{cm}}$$



Question 50 (****)



The figure above shows the graph of the curve C with equation

$$y = 2\sin x + 1, \quad x \in \mathbb{R}.$$

The shaded region R is bounded by the curve, the line $x = \dfrac{\pi}{2}$ and the x axis.

a) Find the exact area of R.

The region R is rotated by 2π radians in the x axis forming a solid of revolution S.

b) Show that the volume of S is

$$\frac{\pi}{2}(3\pi + 8).$$

$$\boxed{\text{area} = \tfrac{1}{2}(\pi + 4)}$$

(a) AREA $= \int_0^{\frac{\pi}{2}} 2\sin x + 1 \, dx = \left[-2\cos x + x\right]_0^{\frac{\pi}{2}} = \left(0 + \frac{\pi}{2}\right) - (-2 + 0)$

$= \frac{\pi}{2} + 2 = \frac{1}{2}(\pi + 4)$

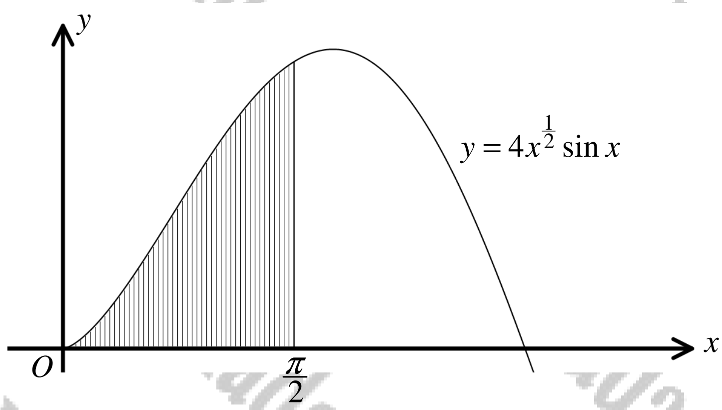
(b) VOLUME $= \pi \int_0^{\frac{\pi}{2}} (2\sin x + 1)^2 \, dx = \pi \int_0^{\frac{\pi}{2}} 4\sin^2 x + 4\sin x + 1 \, dx$

$= \pi \int_0^{\frac{\pi}{2}} 4\left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) + 4\sin x + 1 \, dx = \pi \int_0^{\frac{\pi}{2}} 3 + 4\sin x - 2\cos 2x \, dx$

$= \pi \left[3x - 4\cos x - \sin 2x\right]_0^{\frac{\pi}{2}} = \pi \left[\left(\frac{3\pi}{2} - 0 - 0\right) - (0 - 4 - 0)\right]$

$= \pi \left[\frac{3\pi}{2} + 4\right] = \frac{1}{2}\pi(3\pi + 8)$ AS REQUIRED

Question 51 (****)



The figure above shows the graph of the curve with equation

$$y = 4x^{\frac{1}{2}} \sin x .$$

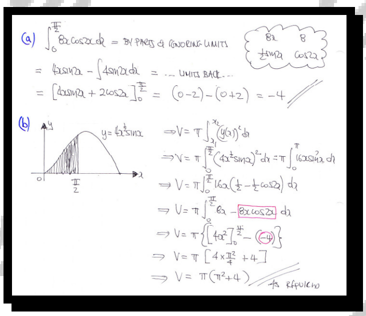
a) Find the value of $\displaystyle\int_0^{\frac{\pi}{2}} 8x \cos 2x \, dx$.

The shaded region bounded by the curve, the x axis and the straight line with equation $x = \frac{\pi}{2}$ is rotated by 2π radians in the x axis to form a solid of revolution.

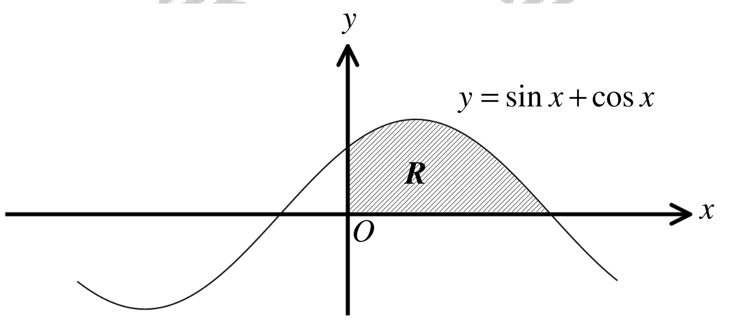
b) Show that the volume of the solid is

$$\pi\left(\pi^2 + 4\right).$$

$\boxed{-4}$



Question 52 (****)



The figure above shows the graph of the curve with equation

$$y = \sin x + \cos x \,, \; -\pi \le x \le \pi \,.$$

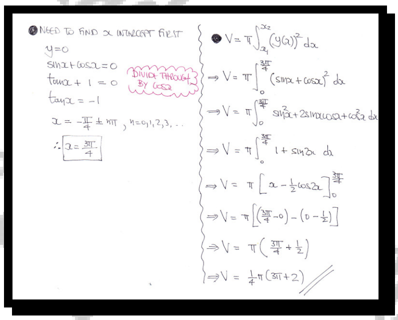
The finite region R, shown shaded in the figure, is bounded by the curve and the coordinate axes.

When R is revolved by a full turn in the x axis it traces a solid of volume V.

Show clearly that

$$V = \frac{1}{4}\pi\left(3\pi + 2\right).$$

$\boxed{}$, $\boxed{\text{proof}}$



NEED TO FIND x INTERCEPT FIRST

$y = 0$

$\sin x + \cos x = 0$

$\tan x + 1 = 0$ DIVIDE THROUGH BY $\cos x$

$\tan x = -1$

$x = -\frac{\pi}{4} \pm n\pi$, $n = 0, 1, 2, 3, \ldots$

$\therefore \boxed{x = \frac{3\pi}{4}}$

$V = \pi \int_{x_1}^{x_2} (y(x))^2 \, dx$

$\Rightarrow V = \pi \int_0^{\frac{3\pi}{4}} (\sin x + \cos x)^2 \, dx$

$\Rightarrow V = \pi \int_0^{\frac{3\pi}{4}} \sin^2 x + 2\sin x \cos x + \cos^2 x \, dx$

$\Rightarrow V = \pi \int_0^{\frac{3\pi}{4}} 1 + \sin 2x \, dx$

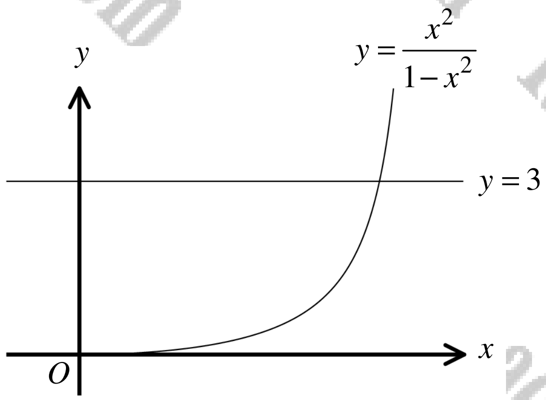
$\Rightarrow V = \pi \left[x - \frac{1}{2}\cos 2x \right]_0^{\frac{3\pi}{4}}$

$\Rightarrow V = \pi \left[\left(\frac{3\pi}{4} - 0\right) - \left(0 - \frac{1}{2}\right) \right]$

$\Rightarrow V = \pi \left(\frac{3\pi}{4} + \frac{1}{2} \right)$

$\Rightarrow V = \frac{1}{4}\pi(3\pi + 2)$

Question 53 (****)



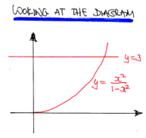
The figure above shows part of the graph of the curve with equation $y = \dfrac{x^2}{1-x^2}$, which passes through the origin O.

The finite area bounded by the curve, the y axis and the straight line with equation $y = 3$, is to be revolved in the **y axis** by $360°$ to form a solid of revolution S.

Find an exact value for the volume of S.

$\boxed{}$, $\boxed{\pi(3 - \ln 4)}$

LOOKING AT THE DIAGRAM



$y = \dfrac{x^2}{1-x^2}$

$y - x^2y = x^2$

$y = x^2 + x^2y$

$y = x^2(1+y)$

$x^2 = \dfrac{y}{1+y}$

SETTING UP A VOLUME INTEGRAL ABOUT y

$\Rightarrow V = \pi \int_{y_1}^{y_2} [x(y)]^2 \, dy$

$\Rightarrow V = \pi \int_0^3 \dfrac{y}{1+y} \, dy$

OR THE SUBSTITUTION $u = 1+y$

$\Rightarrow V = \pi \int_0^3 \dfrac{(1+y)-1}{1+y} \, dy$

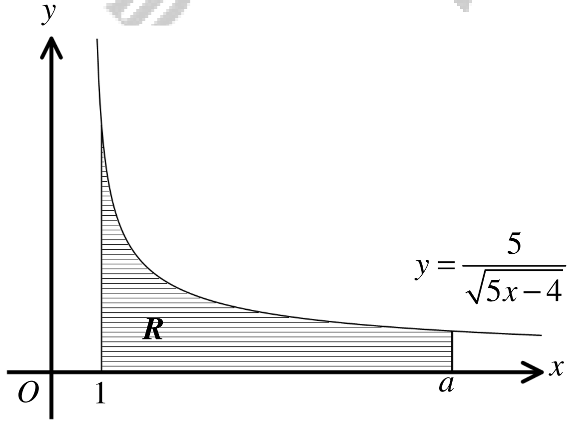
$\Rightarrow V = \pi \int_0^3 1 - \dfrac{1}{y+1} \, dy$

$\Rightarrow V = \pi \left[y - \ln|y+1| \right]_0^3$

$\Rightarrow V = \pi \left[(3 - \ln 4) - (0 - \cancel{\ln 1}) \right]$

$\Rightarrow V = \pi(3 - \ln 4)$

Question 54 (****)



The figure above shows part of the graph of the curve C with equation

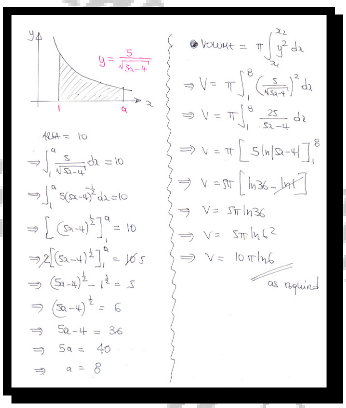
$$y = \frac{5}{\sqrt{5x-4}}, \quad x > \frac{4}{5}.$$

The shaded region R is bounded by the curve, the vertical straight lines $x = 1$ and $x = a$, and the x axis.

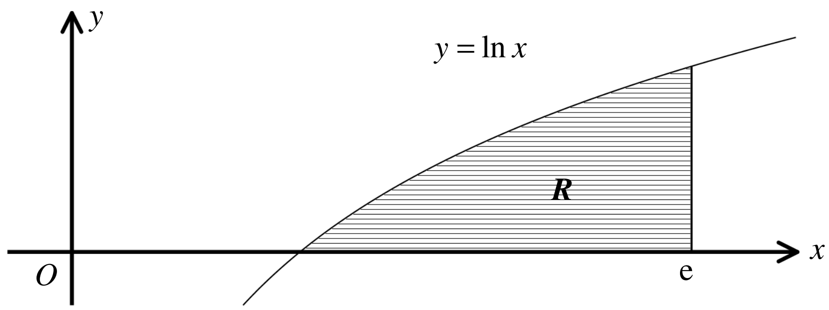
The region R is rotated by 2π radians about the x axis forming a solid of revolution.

Given that the area of R is 10 square units, show that the volume of the solid formed is $10\pi \ln 6$ cubic units.

$\boxed{}$, $\boxed{\text{proof}}$



Question 55 (****)



The figure above shows the graph of

$$y = \ln x, \quad x > 0.$$

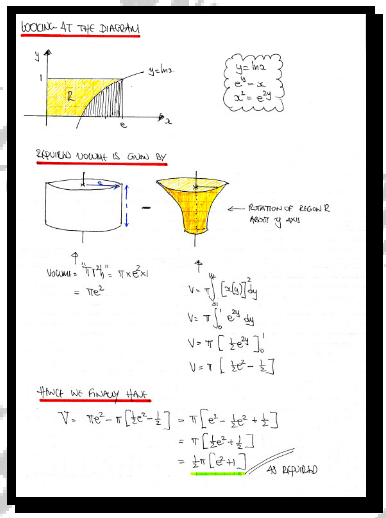
The shaded region R is bounded by the curve, the line $x = \mathrm{e}$ and the x axis.

R is rotated by 2π radians about the y axis, forming a solid of revolution S.

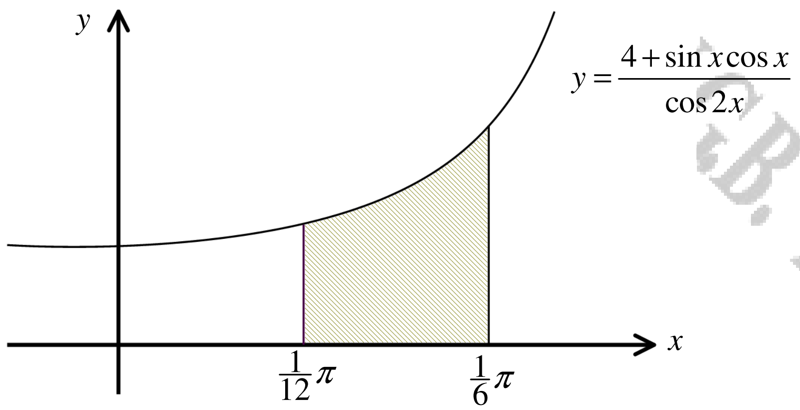
Show that the volume of S is

$$\frac{1}{2}\pi\left(\mathrm{e}^2+1\right).$$

$\boxed{}$, $\boxed{\text{proof}}$



Question 56 (****)



The figure above shows part of the graph of the curve with equation

$$y = \frac{4+\sin x\cos x}{\cos 2x}.$$

The finite area bounded by the curve, the x axis and the straight lines with equations $x = \frac{1}{12}\pi$ and $x = \frac{1}{6}\pi$, shown shaded in the figure, is fully revolved about the x axis, forming a solid, S.

Calculate the volume of S, correct to 3 significant figures.

$\boxed{}$, $\boxed{V \approx 34.6}$

START BY MANIPULATING THE FUNCTION

$$y = \frac{4+\sin x\cos x}{\cos 2x} = \frac{4+\frac{1}{2}(2\sin x\cos x)}{\cos 2x}$$

$$= \frac{4+\frac{1}{2}\sin 2x}{\cos 2x} = 4\sec 2x + \frac{1}{2}\tan 2x$$

HENCE THE VOLUME OF REVOLUTION IS GIVEN BY

$$\Longrightarrow V = \pi\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \left(4\sec 2x + \frac{1}{2}\tan 2x\right)^2 dx$$

$$\Longrightarrow V = \pi\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} 16\sec^2 2x + 4\sec 2x\tan 2x + \frac{1}{4}\tan^2 2x \, dx$$

$$\Longrightarrow V = \pi\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} 16\sec^2 2x + 4\sec 2x\tan 2x + \frac{1}{4}(\sec^2 2x - 1) \, dx$$

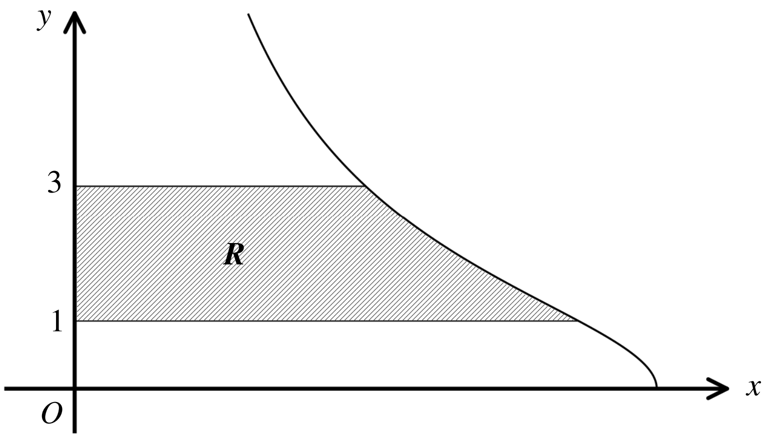
$$\Longrightarrow V = \pi\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \frac{65}{4}\sec^2 2x + 4\sec 2x\tan 2x - \frac{1}{4} \, dx$$

$$\Longrightarrow V = \pi\left[\frac{65}{8}\tan 2x + 2\sec 2x - \frac{1}{4}x\right]_{\frac{\pi}{12}}^{\frac{\pi}{6}}$$

$$\Longrightarrow V = \pi\left[\left(\frac{65}{8}\sqrt{3} + 4 - \frac{\pi}{24}\right) - \left(\frac{65}{24}\sqrt{3} + \frac{4\sqrt{3}}{3} - \frac{\pi}{48}\right)\right]$$

$$\Longrightarrow V = \pi\left[4 + \frac{49}{12}\sqrt{3} - \frac{\pi}{48}\right] \approx 34.6$$

Question 57 (****)



The figure above shows the curve with parametric equations

$$x = 2\cos^2\theta, \quad y = \sqrt{3}\tan\theta, \quad 0 \le \theta < \frac{\pi}{2}.$$

The finite region R shown shaded in the figure, bounded by the curve, the y axis, and the straight lines with equations $y = 1$ and $y = 3$.

Use integration in parametric to show that the volume of the solid formed when R is fully revolved about the y axis is $\dfrac{\pi^2}{\sqrt{3}}$.

$\boxed{}$, $\boxed{\text{proof}}$

WORKING IN PARAMETRIC

$$V = \pi\int_{y_1}^{y_2}[x(y)]^2\,dy = \pi\int_{\theta_1}^{\theta_2}[x(\theta)]^2\frac{dy}{d\theta}\,d\theta$$

TRANSFORMING THE INTEGRAL

$y = 1$, $\sqrt{3}\tan\theta = 1$, $\tan\theta = \frac{1}{\sqrt{3}}$, $\theta = \frac{\pi}{6}$

$y = 3$, $\sqrt{3}\tan\theta = 3$, $\tan\theta = \sqrt{3}$, $\theta = \frac{\pi}{3}$

HENCE WE HAVE

$$V = \pi\int_{\pi/6}^{\pi/3}\underbrace{(2\cos^2\theta)^2}_{x^2}\underbrace{(\sqrt{3}\sec^2\theta)}_{\frac{dy}{d\theta}}\,d\theta = 4\sqrt{3}\pi\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\cos^4\theta\sec^2\theta\,d\theta$$

$$= 4\sqrt{3}\pi\int_{\pi/6}^{\pi/3}\cos^2\theta\,d\theta = 4\sqrt{3}\pi\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{1}{2}+\frac{1}{2}\cos 2\theta\,d\theta$$

$$= 4\sqrt{3}\pi\left[\frac{1}{2}\theta+\frac{1}{4}\sin 2\theta\right]_{\pi/6}^{\pi/3}$$

$$= 4\sqrt{3}\pi\left[\left(\frac{\pi}{6}+\cancel{\frac{1}{4}\frac{\sqrt{3}}{2}}\right)-\left(\frac{\pi}{12}+\cancel{\frac{1}{4}\frac{\sqrt{3}}{2}}\right)\right]$$

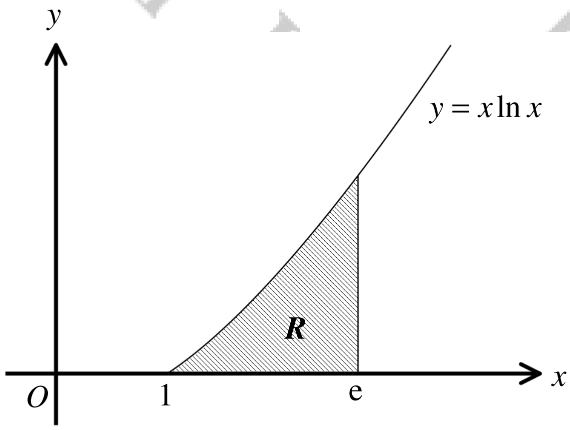
$$= 4\sqrt{3}\pi\times\frac{\pi}{12}$$

$$= \frac{\sqrt{3}\pi^2}{3} = \frac{\sqrt{3}\pi^2\sqrt{3}}{3\sqrt{3}} = \frac{3\pi^2}{3\sqrt{3}}$$

$$= \frac{\pi^2}{\sqrt{3}}$$

AS REQUIRED

Question 58 (****+)



The figure above shows the graph of the curve C with equation

$$y = x \ln x, \quad x \geq 1.$$

The shaded region R is bounded by the curve, the x axis and the vertical line $x = \mathrm{e}$.

The region R is rotated by 2π radians in the x axis forming a solid of revolution S.

Find an exact value for the volume of S.

$\boxed{}$, $\boxed{\dfrac{\pi}{27}\left(5\mathrm{e}^3 - 2\right)}$

USING THE STANDARD RESULT FOR VOLUME OF REVOLUTION IN x

$$\Rightarrow V = \pi \int_{x_1}^{x_2} [y(x)]^2 \, dx = \pi \int_1^e (x \ln x)^2 \, dx = \pi \int_1^e x^2 (\ln x)^2 \, dx$$

CONTINUE BY INTEGRATION BY PARTS & IGNORING π & LIMITS

$$\int x^2 (\ln x)^2 \, dx = \frac{1}{3}x^3 (\ln x)^2 - \int \frac{2}{3} x^2 \ln x \, dx$$

$(\ln x)^2$	$2\ln x \times \frac{1}{x}$
$\frac{1}{3}x^3$	x^2

BY PARTS AGAIN ON THIS INTEGRAL

$$\dots = \frac{1}{3}x^3(\ln x)^2 - \left[\frac{2}{9}x^3 \ln x - \int \frac{2}{9}x^2 \, dx\right]$$

$$\dots = \frac{1}{3}x^3(\ln x)^2 - \frac{2}{9}x^3 \ln x + \int \frac{2}{9}x^2 \, dx$$

$$\dots = \frac{1}{3}x^3(\ln x)^2 - \frac{2}{9}x^3 \ln x + \frac{2}{27}x^3 + C$$

$\ln x$	$\frac{1}{x}$
$\frac{2}{9}x^3$	$\frac{2}{3}x^2$

RETURNING TO THE MAIN LINE

$$V = \pi \left[\frac{1}{3}x^3(\ln x)^2 - \frac{2}{9}x^3 \ln x + \frac{2}{27}x^3\right]_1^e$$

$$V = \pi \left[\left(\frac{1}{3}e^3 - \frac{2}{9}e^3 + \frac{2}{27}e^3\right) - \left(0 - 0 + \frac{2}{27}\right)\right]$$

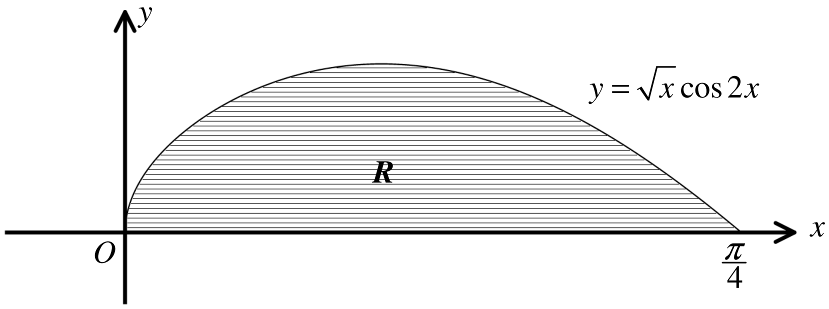
$$V = \pi \left[\frac{5}{27}e^3 - \frac{2}{27}\right]$$

$$V = \frac{\pi}{27}(5e^3 - 2)$$

Question 59 **(****+)**

$$f(x) = \frac{1}{8}(4x + \sin 4x), \ x \in \mathbb{R}, \ 0 \le x \le \frac{\pi}{4}.$$

a) Show that $f'(x) = \cos^2 2x$.



The figure above shows part of the graph of a curve C with equation

$$y = \sqrt{x}\cos 2x, \ x > 0.$$

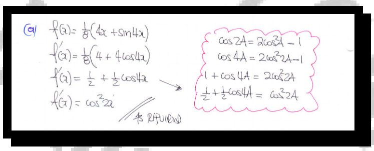
The curve meets the x axis at the origin and at the point where $x = \frac{\pi}{4}$.

The shaded region R is bounded by the curve and the x axis. The region R is rotated by 2π radians about the x axis, forming a solid of revolution S.

b) Show further that the volume of S is

$$\frac{\pi}{64}\left(\pi^2 - 4\right).$$

proof



(a) $f(x) = \frac{1}{8}(4x + \sin 4x)$

$f'(x) = \frac{1}{8}(4 + 4\cos 4x)$

$f'(x) = \frac{1}{2} + \frac{1}{2}\cos 4x$

$f'(x) = \cos^2 2x$

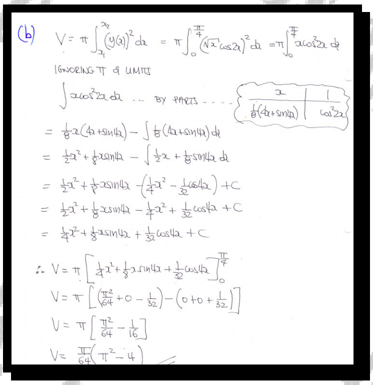
// AS REQUIRED

$\cos 2A = 2\cos^2 A - 1$

$\cos 4A = 2\cos^2 2A - 1$

$1 + \cos 4A = 2\cos^2 2A$

$\frac{1}{2} + \frac{1}{2}\cos 4A = \cos^2 2A$



(b) $V = \pi\int_{x_1}^{x_2}(y(x))^2\,dx = \pi\int_0^{\frac{\pi}{4}}(\sqrt{x}\cos 2x)^2\,dx = \pi\int_0^{\frac{\pi}{4}} x\cos^2 2x\,dx$

IGNORING π & LIMITS

$\int x\cos^2 2x\,dx$ ... BY PARTS ....

x	1
$\frac{1}{8}(4x+\sin 4x)$	$\cos^2 2x$

$= \frac{1}{8}x(4x + \sin 4x) - \int \frac{1}{8}(4x + \sin 4x)\,dx$

$= \frac{1}{2}x^2 + \frac{1}{8}x\sin 4x - \int \frac{1}{2}x + \frac{1}{8}\sin 4x\,dx$

$= \frac{1}{2}x^2 + \frac{1}{8}x\sin 4x - \left(\frac{1}{4}x^2 - \frac{1}{32}\cos 4x\right) + C$

$= \frac{1}{2}x^2 + \frac{1}{8}x\sin 4x - \frac{1}{4}x^2 + \frac{1}{32}\cos 4x + C$

$= \frac{1}{4}x^2 + \frac{1}{8}x\sin 4x + \frac{1}{32}\cos 4x + C$

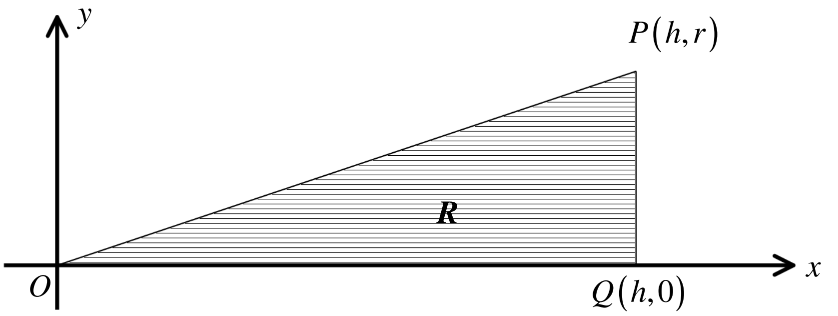
$\therefore V = \pi\left[\frac{1}{4}x^2 + \frac{1}{8}x\sin 4x + \frac{1}{32}\cos 4x\right]_0^{\frac{\pi}{4}}$

$V = \pi\left[\left(\frac{\pi^2}{64} + 0 - \frac{1}{32}\right) - \left(0 + 0 + \frac{1}{32}\right)\right]$

$V = \pi\left[\frac{\pi^2}{64} - \frac{1}{16}\right]$

$V = \frac{\pi}{64}\left(\pi^2 - 4\right)$

Question 60 (****+)



The figure above shows the straight line segment OP, joining the origin to the point $P(h,r)$, where h and r are positive coordinates.

The point $Q(h,0)$ lies on the x axis.

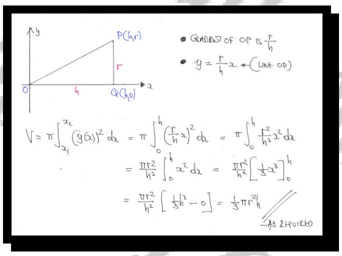
The shaded region R is bounded by the straight line segments OP, PQ and OQ.

The region R is rotated by 2π radians in the x axis to form a solid cone of height h and radius r.

Show by integration that the volume of the cone V is given by

$$V = \frac{1}{3}\pi r^2 h.$$

, proof



Question 61 (****+)

A finite region R is defined by the inequalities

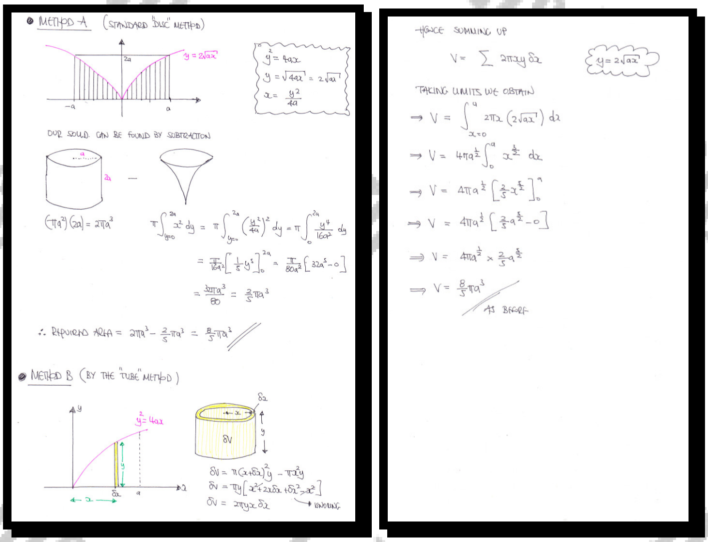
$$y^2 \le 4ax, \ 0 \le x \le a, \ y \ge 0,$$

where a is a positive constant.

The region R is rotated by 2π radians in the y axis forming a solid of revolution.

Determine, in terms of π and a, the exact volume of this solid.

$\boxed{}$, $\boxed{\frac{8}{5}\pi a^3}$



Question 62 (****+)

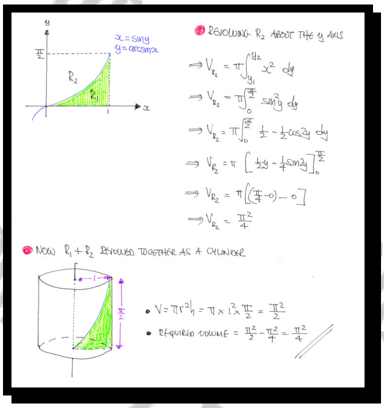
The finite region R is defined by the inequalities

$$y \le \arcsin x, \ x \le 1, \ y \ge 0.$$

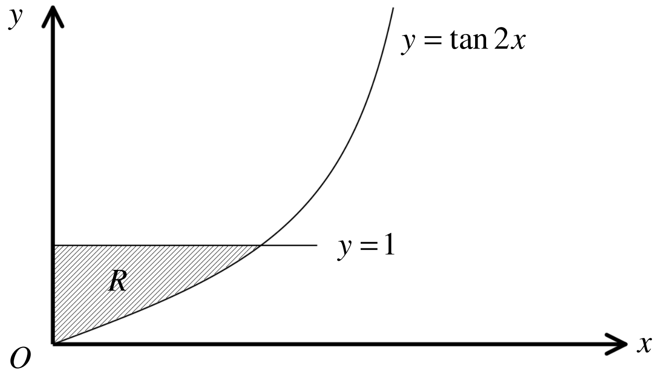
The region R is rotated by 2π radians in the y axis forming a solid of revolution.

Determine the exact volume of this solid.

$\boxed{}$, $\boxed{\frac{1}{4}\pi^2}$



Question 63 (****+)



The figure above shows the graph of the curve with equation

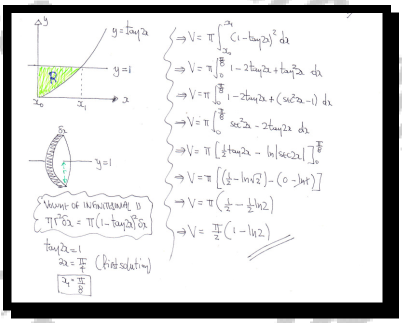
$$y = \tan 2x\,,\ 0 \le x \le \frac{\pi}{4}.$$

The finite region R is bounded by the curve, the y axis and the horizontal line with equation $y = 1$.

The region R is rotated by 2π radians about the straight line with equation $y = 1$ forming a solid of revolution.

Determine an exact volume for this solid.

$\boxed{\frac{\pi}{2}\left(1-\ln 2\right)}$



Question 64 (****+)

A curve C has equation

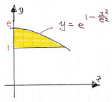
$$y = e^{1-\left(\frac{x}{e}\right)^2}, \quad x \in \mathbb{R},$$

The finite region bounded by C, the y axis and straight line with equation $y = 1$, is revolved by 2π radians about the y axis, forming a solid of revolution.

Find an exact simplified value for the volume of this solid.

$\boxed{\text{SP-C}}$, $\boxed{V = \pi e^2(e-2)}$

- AS THE REVOLUTION IS IN THE y AXIS, START BY REARRANGING THE FORMULA



$$\Rightarrow y = e^{1-\frac{x^2}{e^2}}$$
$$\Rightarrow \ln y = 1 - \frac{x^2}{e^2}$$
$$\Rightarrow \frac{x^2}{e^2} = 1 - \ln y$$
$$\Rightarrow x^2 = e^2(1-\ln y)$$

- NEXT THE VOLUME OF REVOLUTION ABOUT y IS GIVEN BY

$$\Rightarrow V = \pi\int_{y_1}^{y_2} (x(y))^2\,dy$$
$$\Rightarrow V = \pi\int_1^e e^2(1-\ln y)\,dy$$

- NEXT NOTE THAT $\int \ln x\,dx = x\ln|x| - x + C$, EITHER BY COMMON KNOWLEDGE, OR INTEGRATION BY PARTS

$$\Rightarrow V = \pi e^2\int_1^e 1 - \ln y\,dy$$
$$\Rightarrow V = \pi e^2\Big[y - (y\ln|y| - y)\Big]_1^e$$
$$\Rightarrow V = \pi e^2\Big[2y - y\ln y\Big]_1^e$$
$$\Rightarrow V = \pi e^2\Big[(2e - e\ln e) - (2 - \cancel{\ln 1})\Big]$$
$$\Rightarrow V = \pi e^2(2e - e - 2)$$
$$\Rightarrow V = \pi e^2(e-2)$$

Question 65 (****+)

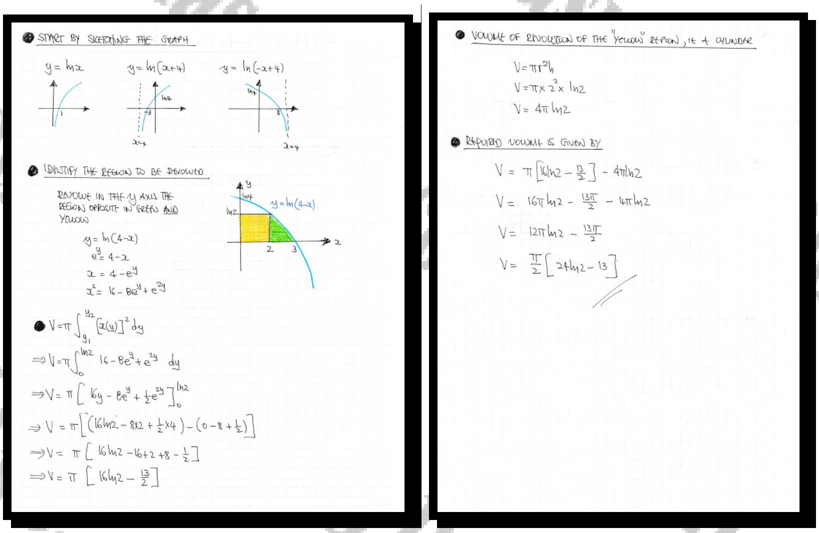
A curve has equation

$$y = \ln(4 - x), \quad x \in \mathbb{R}, \; x \neq 4.$$

The finite region bounded by the curve, the x axis and the straight line with equation $x = 2$, is revolved by 2π radians in the y axis.

Find the exact volume of the solid formed.

SP-I, $\boxed{V = \frac{1}{2}\pi(24\ln 2 - 13)}$



Question 66 (*****)

The finite region R is by the coordinate axes and the curve with equation

$$y = \arccos x\,,\ -1 \le x \le 1\,.$$

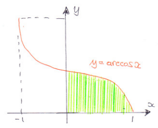
The region R is rotated by 2π radians in the x axis forming a solid of revolution.

Determine the exact volume of this solid.

$\boxed{}$, $\boxed{\pi^2 - 2\pi}$

$$\boxed{\text{Volume} = \pi \int_{x_1}^{x_2} [y(x)]^2\, dx}$$

$$\Rightarrow V = \pi \int_0^1 (\arccos x)^2\, dx$$



BY SUBSTITUTION

$$\theta = \arccos x$$
$$\cos\theta = x$$
$$dx = -\sin\theta\, d\theta$$
$$x = 0 \longmapsto \theta = \tfrac{\pi}{2}$$
$$x = 1 \longmapsto \theta = 0$$

$$\Rightarrow V = \pi \int_{\frac{\pi}{2}}^{0} \theta^2 (-\sin\theta\, d\theta)$$

$$\Rightarrow V = \pi \int_0^{\frac{\pi}{2}} \theta^2 \sin\theta\, d\theta$$

NOW BY PARTS (DIVIDE π ACROSS)

θ^2	2θ
$-\cos\theta$	$\sin\theta$

$$\Rightarrow \frac{1}{\pi} V = \cancel{\left[-\theta^2\cos\theta\right]_0^{\pi/2}} + \int_0^{\frac{\pi}{2}} 2\theta\cos\theta\, d\theta$$

BY PARTS AGAIN

2θ	2
$\sin\theta$	$\cos\theta$

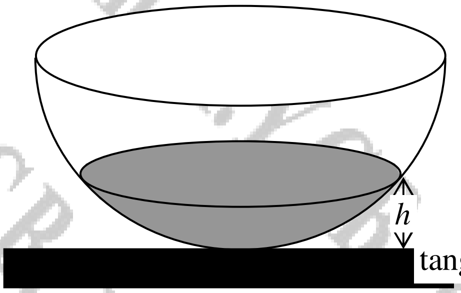
$$\Rightarrow \frac{1}{\pi} V = \left[2\theta\sin\theta\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 2\sin\theta\, d\theta$$

$$\Rightarrow \frac{1}{\pi} V = (\pi - 0) + \left[2\cos\theta\right]_0^{\frac{\pi}{2}}$$

$$\Rightarrow \frac{1}{\pi} V = \pi + (0 - 2)$$

$$\Rightarrow V = \pi(\pi - 2)$$

Question 67 (*****)



The figure above shows a hemispherical bowl of radius r containing water to a height h. The water in the bowl is in the shape of a minor spherical segment.

It is required to find a formula for the volume of a minor spherical segment as a function of the radius r and the distance of its plane face from the tangent plane, h.

The circle with equation

$$x^2 + y^2 = r^2, \; x \geq 0$$

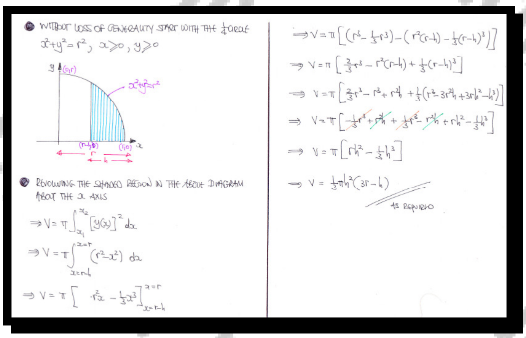
is to be used to find a formula for the volume of a minor spherical segment.

Show by integration that the volume V of the minor spherical segment is given by

$$V = \frac{1}{3}\pi h^2 (3r - h),$$

where r is the radius of the sphere or hemisphere and h is the distance of its plane face from the tangent plane.

proof



WITHOUT LOSS OF GENERALITY START WITH THE $\frac{1}{4}$ CIRCLE
$x^2+y^2=r^2, \; x \geq 0, \; y \geq 0$

REVOLVING THE SHADED REGION IN THE ABOVE DIAGRAM ABOUT THE x AXIS

$\Rightarrow V = \pi \int_{x_1}^{x_2} [y(x)]^2 \, dx$

$\Rightarrow V = \pi \int_{x=r-h}^{x=r} (r^2 - x^2) \, dx$

$\Rightarrow V = \pi \left[r^2x - \frac{1}{3}x^3 \right]_{x=r-h}^{x=r}$

$\Rightarrow V = \pi \left[\left(r^3 - \frac{1}{3}r^3\right) - \left(r^2(r-h) - \frac{1}{3}(r-h)^3\right) \right]$

$\Rightarrow V = \pi \left[\frac{2}{3}r^3 - r^2(r-h) + \frac{1}{3}(r-h)^3 \right]$

$\Rightarrow V = \pi \left[\frac{2}{3}r^3 - r^3 + r^2h + \frac{1}{3}(r^3 - 3r^2h + 3rh^2 - h^3) \right]$

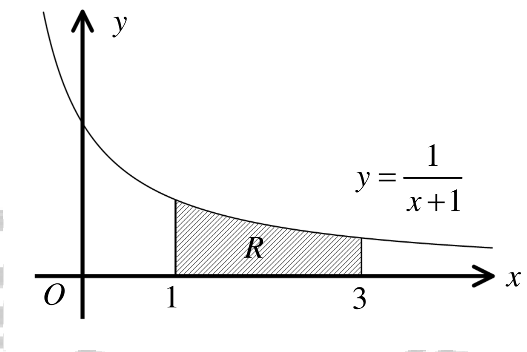
$\Rightarrow V = \pi \left[\cancel{-\frac{1}{3}r^3} + \cancel{r^2h} + \cancel{\frac{1}{3}r^3} - \cancel{r^2h} + rh^2 - \frac{1}{3}h^3 \right]$

$\Rightarrow V = \pi \left[rh^2 - \frac{1}{3}h^3 \right]$

$\Rightarrow V = \frac{1}{3}\pi h^2 (3r - h)$

AS REQUIRED

Question 68 (*****)



The figure above shows the graph of the curve with equation

$$y = \frac{1}{x+1}, \quad x \in \mathbb{R}, \quad x = -1 .$$

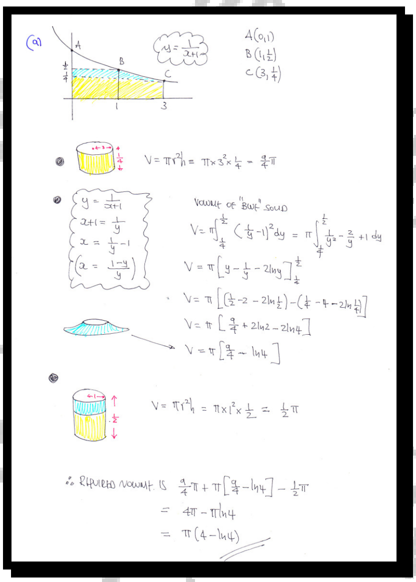
The finite region R is bounded by the curve, the x axis and the lines with equations $x = 1$ and $x = 3$.

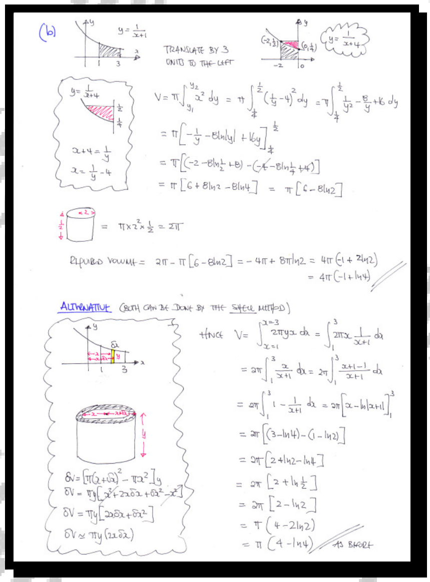
Determine the exact volume of the solid formed when the region R is revolved by 2π radians about …

a) … the y axis.

b) … the straight line with equation $x = 3$.

[illegible], $\boxed{\pi(4-\ln 4)}$, $\boxed{4\pi(-1+\ln 4)}$





Question 69 (*****)

The finite region R is bounded by the curve with equation

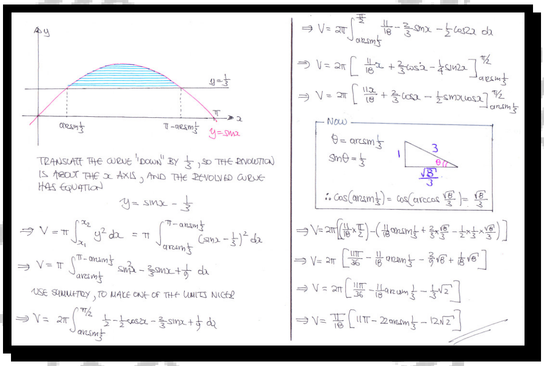
$$y = \sin x,\ 0 \le x \le \pi,$$

and the straight line with equation $y = \frac{1}{3}$.

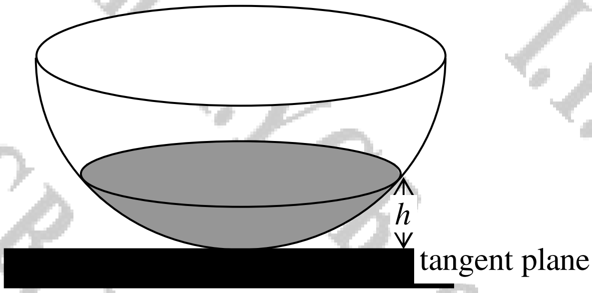
The region R is rotated by 2π radians in the straight line with equation $y = \frac{1}{3}$ forming a solid of revolution.

Determine the exact volume of this solid.

$\boxed{\text{SP-V}}$, $\boxed{V = \frac{\pi}{18}\left[11\pi - 22\arcsin\left(\frac{1}{3}\right) - 12\sqrt{2}\right]}$



Question 70 (*****)



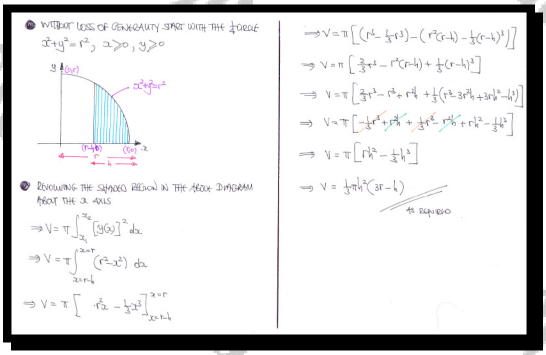
The figure above shows a hemispherical bowl of radius r containing water to a height h. The water in the bowl is in the shape of a minor spherical segment. It is required to find a formula for the volume of a minor spherical segment as a function of the radius r and the distance of its plane face from the tangent plane, h.

Show by integration that the volume V of the minor spherical segment is given by

$$V = \frac{1}{3}\pi h^2 (3r - h),$$

where r is the radius of the sphere or hemisphere and h is the distance of its plane face from the tangent plane.

$\boxed{}$, $\boxed{\text{proof}}$


WITHOUT LOSS OF GENERALITY START WITH THE $\frac{1}{4}$ CIRCLE
$x^2+y^2=r^2$, $x \geqslant 0$, $y \geqslant 0$

REVOLVING THE SHADED REGION IN THE ABOVE DIAGRAM ABOUT THE x AXIS

$\Rightarrow V = \pi \int_{x_1}^{x_2} [y(x)]^2 \, dx$

$\Rightarrow V = \pi \int_{x=r-h}^{x=r} (r^2 - x^2) \, dx$

$\Rightarrow V = \pi \left[r^2 x - \frac{1}{3}x^3 \right]_{x=r-h}^{x=r}$

$\Rightarrow V = \pi \left[\left(r^3 - \frac{1}{3}r^3\right) - \left(r^2(r-h) - \frac{1}{3}(r-h)^3\right) \right]$

$\Rightarrow V = \pi \left[\frac{2}{3}r^3 - r^2(r-h) + \frac{1}{3}(r-h)^3 \right]$

$\Rightarrow V = \pi \left[\frac{2}{3}r^3 - r^3 + r^2h + \frac{1}{3}(r^3 - 3r^2h + 3rh^2 - h^3) \right]$

$\Rightarrow V = \pi \left[\cancel{-\frac{1}{3}r^3} + \cancel{r^2h} + \cancel{\frac{1}{3}r^3} - \cancel{r^2h} + rh^2 - \frac{1}{3}h^3 \right]$

$\Rightarrow V = \pi \left[rh^2 - \frac{1}{3}h^3 \right]$

$\Rightarrow V = \frac{1}{3}\pi h^2 (3r - h)$

AS REQUIRED

Question 71 (*****)

A curve has equation

$$y = \frac{8}{x^2 - 4x + 8}, \quad x \in \mathbb{R}.$$

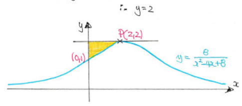
The finite region R is bounded by the curve, the y axis and the tangent to the curve at the stationary point of the curve.

Determine, in simplified exact form, the volume of the solid formed when R is fully revolved about the y axis.

$\boxed{}$, $\boxed{V = 4\pi\left[2 - \pi + 2\ln 2\right]}$

START WITH A QUICK SKETCH OF $y = \frac{8}{x^2-4x+8} = \frac{8}{(x-2)^2+4}$

- AS $x \to \pm\infty$ $y \to 0$ (FROM ABOVE)
- $y > 0$ FOR ALL x
- $x = 2$ IS A LINE OF SYMMETRY
- "HIGHEST" VALUE OF y OCCURS WHEN $x = 2$ $\therefore y = 2$



REARRANGE THE EQUATION FOR x^2

$\Rightarrow y = \frac{8}{(x-2)^2+4}$

$\Rightarrow (x-2)^2 + 4 = \frac{8}{y}$

$\Rightarrow (x-2)^2 = \frac{8}{y} - 4$

$\Rightarrow x - 2 = -\sqrt{\frac{8-4y}{y}}$

$\Rightarrow x = 2 - 2\sqrt{\frac{2-y}{y}}$

$\Rightarrow x^2 = 4 - 8\sqrt{\frac{2-y}{y}} + 4\left(\frac{2-y}{y}\right)$

$\Rightarrow x^2 = 4 - 8\sqrt{\frac{2-y}{y}} + 4\left(\frac{2}{y} - 1\right)$

$\Rightarrow x^2 = \cancel{4} - 8\sqrt{\frac{2-y}{y}} + \frac{8}{y} \cancel{-4}$

NOW THE VOLUME CAN BE FOUND

$\Rightarrow V = \pi \int_{y_1}^{y_2} [x(y)]^2 \, dy = \pi \int_1^2 \frac{8}{y} - 8\sqrt{\frac{2-y}{y}} \, dy$

$\Rightarrow V = 8\pi \int_1^2 \frac{1}{y} - \sqrt{\frac{2-y}{y}} \, dy$

$\Rightarrow V = 8\pi \Big[\ln|y|\Big]_1^2 - 8\pi \int_1^2 \sqrt{\frac{2-y}{y}} \, dy$

$\Rightarrow V = 8\pi\left[\ln 2 - \cancel{\ln 1}\right] - 8\pi \int_1^2 \sqrt{\frac{2-y}{y}} \, dy$

$y = 2\sin^2\theta$
$dy = 4\sin\theta\cos\theta \, d\theta$
$\sin^2\theta = \frac{1}{2}y$
$\sin\theta = +\sqrt{\frac{y}{2}}$
$\theta = \arcsin\sqrt{\frac{y}{2}}$
$y = 1 \longmapsto \theta = \frac{\pi}{4}$
$y = 2 \longmapsto \theta = \frac{\pi}{2}$

$\Rightarrow V = 8\pi\ln 2 - 8\pi \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sqrt{\frac{2 - 2\sin^2\theta}{2\sin^2\theta}} \left(4\sin\theta\cos\theta \, d\theta\right)$

$\Rightarrow V = 8\pi\ln 2 - 8\pi \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sqrt{\frac{1-\sin^2\theta}{\sin^2\theta}} (4\sin\theta\cos\theta) \, d\theta$

$\Rightarrow V = 8\pi\ln 2 - 8\pi \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos\theta}{\cancel{\sin\theta}} (4\cancel{\sin\theta}\cos\theta) \, d\theta$

$= 8\pi\ln 2 - 8\pi \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 4\cos^2\theta \, d\theta$

BY TRIGONOMETRIC IDENTITIES WE HAVE

$\Rightarrow V = 8\pi\ln 2 - 8\pi \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 4\left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right) d\theta$

$\Rightarrow V = 8\pi\ln 2 - 8\pi \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 2 + 2\cos 2\theta \, d\theta$

$\Rightarrow V = 8\pi\ln 2 - 8\pi\Big[2\theta + \sin 2\theta\Big]_{\pi/4}^{\pi/2}$

$\Rightarrow V = 8\pi\ln 2 - 8\pi\left[(\pi + 0) - \left(\frac{\pi}{2} + 1\right)\right]$

$\Rightarrow V = 8\pi\ln 2 - 8\pi\left[\frac{\pi}{2} - 1\right]$

$\Rightarrow V = 8\pi\ln 2 - 4\pi^2 + 8\pi$

$\Rightarrow V = 4\pi\left[2\ln 2 - \pi + 2\right]$

Question 72 (*****)

A spherical cap of depth a is removed from a sphere of radius na, where n is a positive constant, such that $n > \frac{1}{2}$. The volume of the spherical cap is less than half the volume of the sphere.

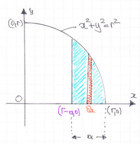
The remainder of the sphere is moulded to a right circular cone whose base is equal to that of the circular plane face of the spherical cap removed.

Given that the height of the cone is ma, where m is a positive constant, show that

$$m = (n+p)(2n+q),$$

where p and q are integers to be found.

$\boxed{}$, $\boxed{m = (n+1)(2n-1)}$



SET UP THE VOLUME OF THE SPHERICAL CAP, AS A VOLUME OF REVOLUTION OF THE CIRCLE $x^2+y^2=r^2$, $x>0$, $y>0$ REVOLVED ABOUT THE x AXIS

$$V_{CAP} = \pi \int_{x_1}^{x_2} [y(x)]^2 \, dx$$

$$V_{CAP} = \pi \int_{r-a}^{r} r^2 - x^2 \, dx$$

$$V_{CAP} = \pi \left[r^2 x - \tfrac{1}{3}x^3 \right]_{r-a}^{r}$$

TIDY THE EXPRESSION

$$V_{CAP} = \pi \left[r^3 - \tfrac{1}{3}r^3 - r^2(r-a) + \tfrac{1}{3}(r-a)^3 \right]$$

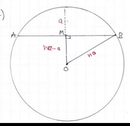
$$V_{CAP} = \pi \left[\cancel{r^3} - \tfrac{1}{3}r^3 - \cancel{r^3} + ar^2 + \tfrac{1}{3}(r^3 - 3r^2a + 3ra^2 - a^3) \right]$$

$$V_{CAP} = \pi \left[\cancel{-\tfrac{1}{3}r^3} + \cancel{ar^2} - \cancel{\tfrac{1}{3}r^3} - \cancel{r^2a} + ra^2 - \tfrac{1}{3}a^3 \right]$$

$$V_{CAP} = \pi \left[a^2 r - \tfrac{1}{3}a^3 \right]$$

$$V_{CAP} = \tfrac{1}{3}\pi a^2 [3r - a]$$

NOW REMOVING THE CAP (DIAGRAM OPPOSITE)



BY PYTHAGORAS

$$|MB|^2 = (na)^2 - (na-a)^2$$
$$= a^2[n^2 - (n-1)^2]$$
$$= a^2(n^2 - n^2 +$$

NOW THE VOLUME OF "OUR CAP"

$$V_{CAP} = \tfrac{1}{3}\pi a^2(3na - a) = \tfrac{1}{3}\pi a^3(3n-1)$$

$$V_{SPHERE} = \tfrac{4}{3}\pi r^3 = \tfrac{4}{3}\pi n^3 a^3$$

$$V_{CONE} = \tfrac{4}{3}\pi a^3 n^3 - \tfrac{1}{3}\pi a^3(3n-1)$$
$$= \tfrac{1}{3}\pi a^3 [4n^3 - (3n-1)]$$
$$= \tfrac{1}{3}\pi a^3 [4n^3 - 3n + 1]$$

FINALLY

$$\Rightarrow \tfrac{1}{3}\pi r^2 h = \tfrac{1}{3}\pi a^3 (4n^3 - 3n + 1)$$

$$\Rightarrow \tfrac{1}{3}\pi [a^2(2n-1)](ma) = \tfrac{1}{3}\pi a^3 (4n^3 - 3n + 1)$$

$$\Rightarrow \cancel{\tfrac{1}{3}\pi a^3}(m(2n-1)) = \cancel{\tfrac{1}{3}\pi a^3}(4n^3 - 3n + 1)$$

$$\Rightarrow m(2n-1) = 4n^3 - 3n + 1$$

$$\Rightarrow m = \frac{4n^3 - 3n + 1}{2n-1} \quad \Leftarrow \text{ BY INSPECTION } n=-1 \text{ IS A SOLUTION}$$

$$\Rightarrow m = \frac{4n^2(n+1) - 4n(n+1) + (n+1)}{2n-1}$$

$$\Rightarrow m = \frac{(n+1)(4n^2 - 4n + 1)}{2n-1}$$

$$\Rightarrow m = \frac{(n+1)(2n-1)^2}{2n-1}$$

$$\Rightarrow m = (n+1)(2n-1)$$

IE $p = 1$
$q = -1$

Question 73 (*****)

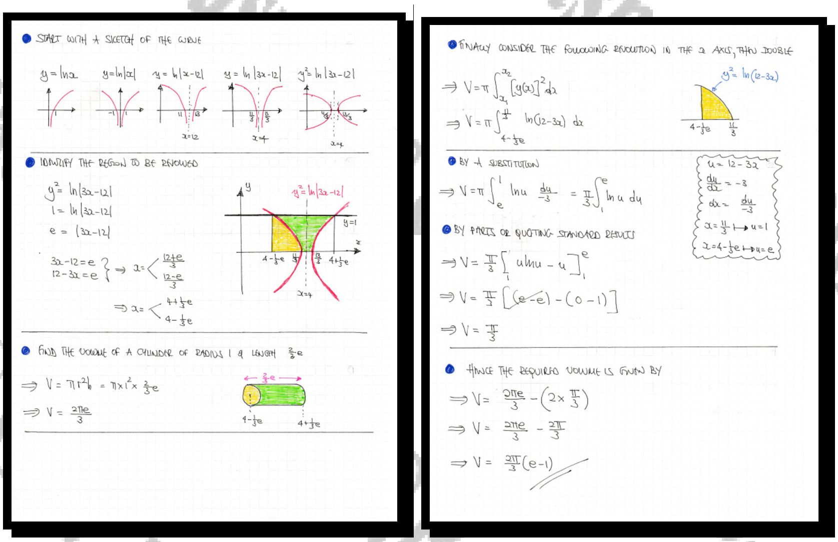
A curve has equation

$$y^2 = \ln|3x - 12|, \quad x \in \mathbb{R}, \; x \neq 4.$$

The finite region bounded by the curve, the x axis and the straight line with equation $y = 1$, is revolved by 2π radians in the x axis.

Find the exact volume of the solid formed.

$\boxed{V = \frac{2}{3}\pi(\mathrm{e} - 1)}$

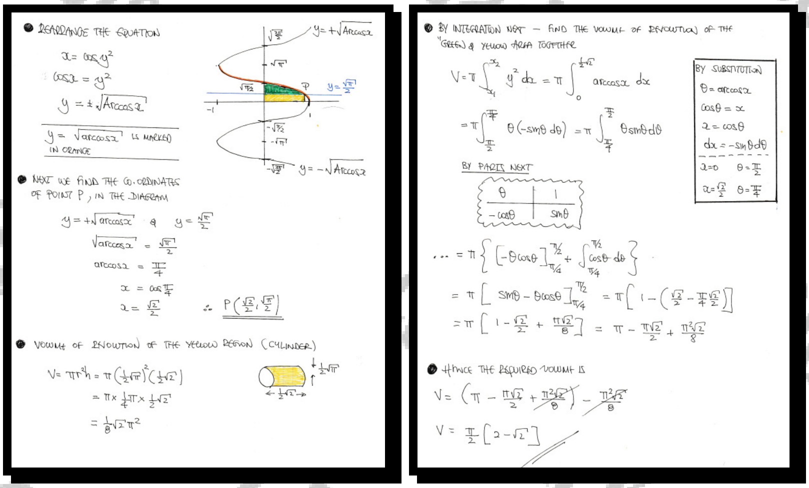


Question 74 (*****)

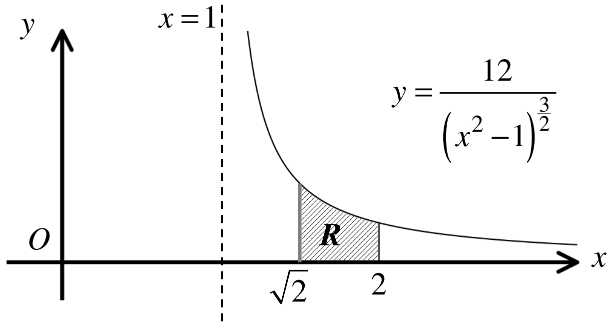
The finite region R is bounded by the curve with equation $x = \cos y^2$, the y axis and the straight line with equation $y = \frac{1}{2}\sqrt{\pi}$.

Determine, in exact simplified form, the volume of the solid formed by revolving R by a full turn in the x axis.

$\boxed{}$, $\boxed{\frac{\pi}{2}\left(2-\sqrt{2}\right)}$



Question 75 (*****)



The figure above shows the curve with equation

$$y = \frac{12}{\left(x^2-1\right)^{\frac{3}{2}}}, \; x > 1 .$$

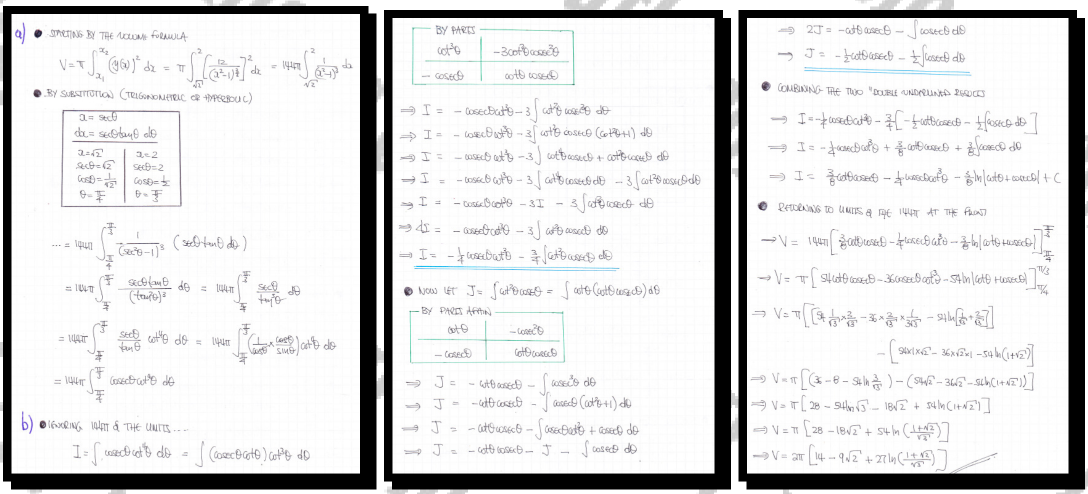
The region R, bounded the curve, the x axis and the straight lines with equations $x = \sqrt{2}$ and $x = 2$, is revolved by a full turn about the x axis, forming a solid S.

a) Show that the volume of S is given by

$$144\pi \int_{\frac{1}{4}\pi}^{\frac{1}{3}\pi} \operatorname{cosec}\theta \cot^4\theta \; d\theta .$$

b) Hence find an exact simplified expression for the volume of S.

$$\boxed{V = 2\pi\left[14 - 9\sqrt{2} + 27\ln\left(\frac{1+\sqrt{2}}{\sqrt{3}}\right)\right]}$$



Question 76 (*****)

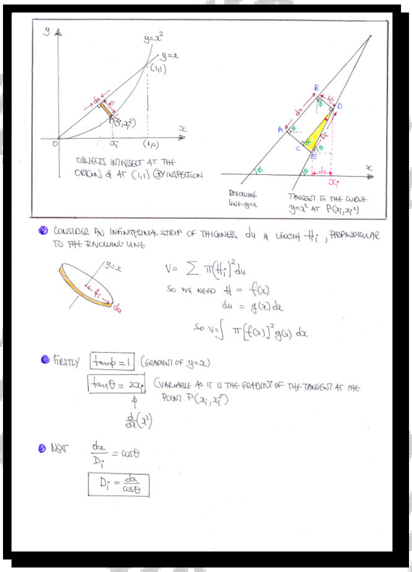
A curve C and a straight line L have respective equations

$$y = x^2 \quad \text{and} \quad y = x.$$

The finite region bounded by C and L is rotated around L by a full turn, forming a solid of revolution S.

Find, in exact form, the volume of S.

$\boxed{}$, $\boxed{\dfrac{\pi\sqrt{2}}{60}}$


OBJECTS INTERSECT AT THE ORIGIN & AT (1,1) (BY INSPECTION)

TANGENT TO THE CURVE $y=x^2$ AT $P(x_i, x_i^2)$

CONSIDER AN INFINITESIMAL STRIP OF THICKNESS du & LENGTH H_i, PERPENDICULAR TO THE REVOLVING LINE

$V = \sum \pi (H_i)^2 du$

SO WE NEED $H_i = f(x)$, $du = g(x)\,dx$

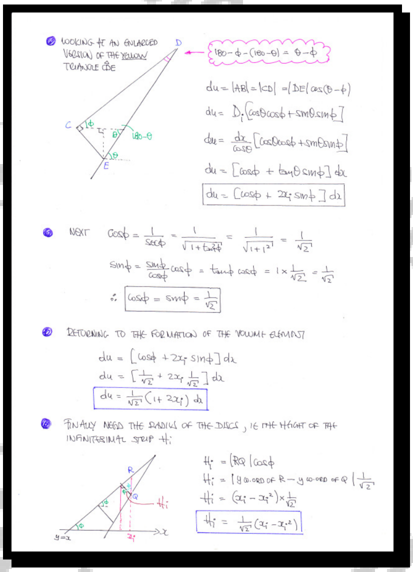
SO $V = \int \pi [f(x)]^2 g(x)\,dx$

FIRSTLY $\tan\phi = 1$ (GRADIENT OF $y=x$)

$\tan\theta = 2x_i$ (VARIABLE AS IT IS THE GRADIENT OF THE TANGENT AT THE POINT $P(x_i, x_i^2)$)

NEXT $\frac{dx}{D_i} = \cos\theta$

$\boxed{D_i = \frac{dx}{\cos\theta}}$


WORKING AT AN ENLARGED VERSION OF THE YELLOW TRIANGLE CDE

$180 - \phi - (180 - \theta) = \theta - \phi$

$du = |AB| = |CD| = |DE| \cos(\theta - \phi)$

$du = D_i [\cos\theta\cos\phi + \sin\theta\sin\phi]$

$du = \frac{dx}{\cos\theta}[\cos\theta\cos\phi + \sin\theta\sin\phi]$

$du = [\cos\phi + \tan\theta\sin\phi]\,dx$

$\boxed{du = [\cos\phi + 2x_i\sin\phi]\,dx}$

NEXT $\cos\phi = \frac{1}{\sec\phi} = \frac{1}{\sqrt{1+\tan^2\phi}} = \frac{1}{\sqrt{1+1^2}} = \frac{1}{\sqrt{2}}$

$\sin\phi = \frac{\sin\phi}{\cos\phi}\cos\phi = \tan\phi\cos\phi = 1 \times \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$

$\therefore \boxed{\cos\phi = \sin\phi = \frac{1}{\sqrt{2}}}$

RETURNING TO THE FORMATION OF THE VOLUME ELEMENT

$du = [\cos\phi + 2x_i\sin\phi]\,dx$

$du = [\frac{1}{\sqrt{2}} + 2x_i\frac{1}{\sqrt{2}}]\,dx$

$\boxed{du = \frac{1}{\sqrt{2}}(1 + 2x_i)\,dx}$

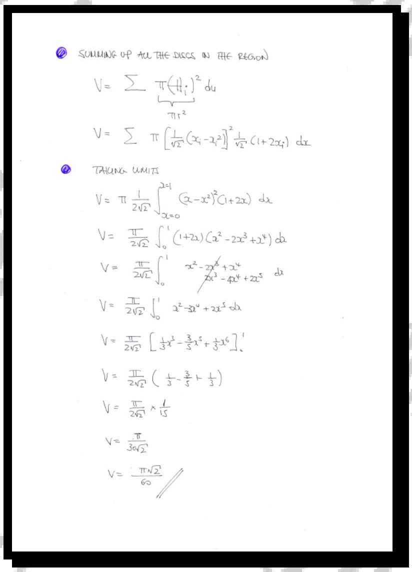
FINALLY NEED THE RADIUS OF THE DISCS, IE THE HEIGHT OF THE INFINITESIMAL STRIP H_i

$H_i = |RQ|\cos\phi$

$H_i = [y \text{ co-ord of } R - y \text{ co-ord of } Q]\frac{1}{\sqrt{2}}$

$H_i = (x_i - x_i^2) \times \frac{1}{\sqrt{2}}$

$\boxed{H_i = \frac{1}{\sqrt{2}}(x_i - x_i^2)}$


SUMMING UP ALL THE DISCS IN THE REGION

$V = \sum \underbrace{\pi (H_i)^2}_{\pi r^2} du$

$V = \sum \pi \left[\frac{1}{\sqrt{2}}(x_i - x_i^2)\right]^2 \frac{1}{\sqrt{2}}(1 + 2x_i)\,dx$

TAKING LIMITS

$V = \pi \frac{1}{2\sqrt{2}} \int_{x=0}^{x=1} (x - x^2)^2 (1 + 2x)\,dx$

$V = \frac{\pi}{2\sqrt{2}} \int_0^1 (1+2x)(x^2 - 2x^3 + x^4)\,dx$

$V = \frac{\pi}{2\sqrt{2}} \int_0^1 x^2 - 2x^3 + x^4 + 2x^3 - 4x^4 + 2x^5 \,dx$

$V = \frac{\pi}{2\sqrt{2}} \int_0^1 x^2 - 3x^4 + 2x^5\,dx$

$V = \frac{\pi}{2\sqrt{2}} \left[\frac{1}{3}x^3 - \frac{3}{5}x^5 + \frac{1}{3}x^6\right]_0^1$

$V = \frac{\pi}{2\sqrt{2}} \left(\frac{1}{3} - \frac{3}{5} + \frac{1}{3}\right)$

$V = \frac{\pi}{2\sqrt{2}} \times \frac{1}{15}$

$V = \frac{\pi}{30\sqrt{2}}$

$V = \frac{\pi\sqrt{2}}{60}$