

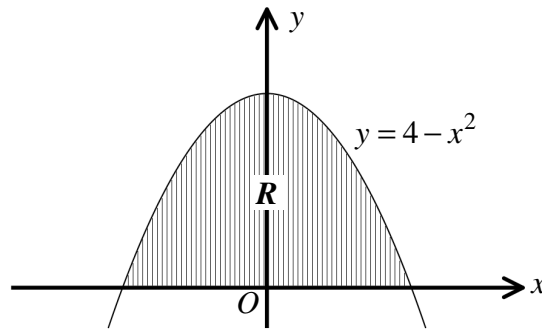
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INTEGRATION

VOLUME OF REVOLUTION

Created by T. Madas

Question 1 (**)



The figure above shows the graph of the curve with equation

$$y = 4 - x^2.$$

The shaded region R , is bounded by the curve and the x axis.

The region R is rotated through 2π radians about the x axis to form a solid of revolution.

Show that the volume of the solid is $\frac{256\pi}{15}$.

, proof

As the quadratic is symmetrical around the y-axis by 2π we double the area (or the entire area by π)

$$y = 4 - x^2 = (2-x)(2+x)$$

$$\Rightarrow V = \pi \int_{-2}^2 (4-x^2)^2 dx$$

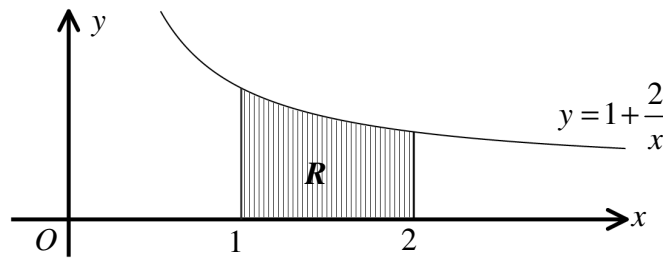
$$\Rightarrow V = \pi \int_{-2}^2 (4-2x^2+x^4) dx$$

$$\Rightarrow V = \pi \left[4x - \frac{2}{3}x^3 + \frac{1}{5}x^5 \right]_{-2}^2$$

$$\Rightarrow V = \pi \left[\left(8 - \frac{16}{3} + \frac{32}{5} \right) - 0 \right]$$

$$\therefore V = \frac{256\pi}{15}$$

Question 2 (**)



The figure above shows part of the graph of the curve with equation

$$y = 1 + \frac{2}{x}, \quad x \neq 0.$$

The region R , shown shaded in the figure above, is bounded by the curve, the straight lines with equations $x = 1$ and $x = 2$, and the x axis.

The region R is rotated through 360° about the x axis to form a solid of revolution.

Show that the volume of the solid is

$$\pi(3 + 4\ln 2).$$

, proof

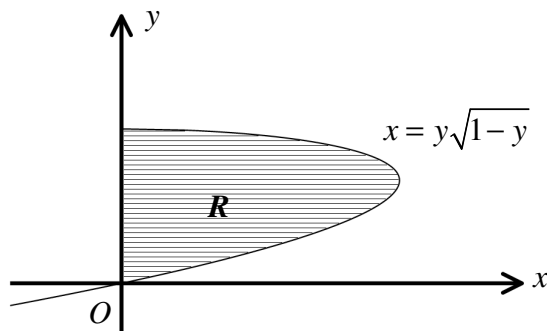
$$V = \pi \int_1^2 \left(1 + \frac{2}{x}\right)^2 dx = \pi \int_1^2 \left(1 + \frac{4}{x} + \frac{4}{x^2}\right) dx$$

$$= \pi \left[x + 4 \ln|x| - \frac{4}{x} \right]_1^2$$

$$= \pi \left[(2 + 4 \ln 2 - 2) - (1 + 4 \ln 1 - 4) \right]$$

$$= \pi(3 + 4 \ln 2)$$

Question 3 (**)



The figure above shows part of the graph of the curve with equation

$$x = y\sqrt{1-y}, \quad y \leq 1.$$

The shaded region R , bounded by the curve and the y axis is rotated through 2π radians about the y axis to form a solid of revolution.

Show that the volume of the solid is $\frac{\pi}{12}$.

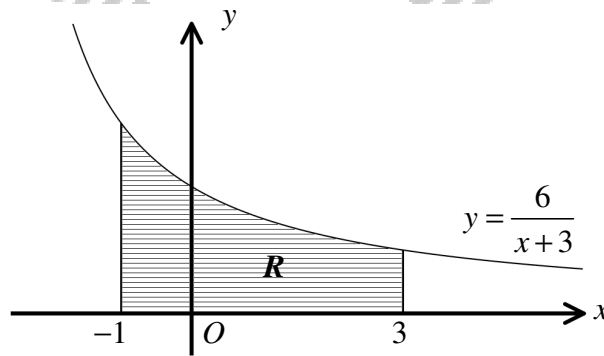
proof

The handwritten proof shows the following steps:

$$\begin{aligned}
 & \text{When } x=0 \\
 & 0 = y\sqrt{1-y} \\
 & y = 1
 \end{aligned}$$

$$\begin{aligned}
 V &= \pi \int_0^1 x^2 dy \\
 V &= \pi \int_0^1 y^2(1-y) dy \\
 V &= \pi \int_0^1 (y^2 - y^3) dy \\
 V &= \pi \left[\frac{1}{3}y^3 - \frac{1}{4}y^4 \right]_0^1 \\
 V &= \pi \left[\left(\frac{1}{3} - \frac{1}{4} \right) - 0 \right] \\
 V &= \frac{\pi}{12}
 \end{aligned}$$

Question 4 (**+)



The diagram above shows the graph of the curve with equation

$$y = \frac{6}{x+3}, \quad x \neq -3.$$

The region R , shown shaded in the figure above, is bounded by the curve, the coordinate axes and the straight lines with equations $x = -1$ and $x = 3$.

- a) Show that the area of R is exactly $6 \ln 3$.

The region R is rotated by 360° about the x axis to form a solid of revolution.

- b) Show that the volume of the solid generated is 12π .

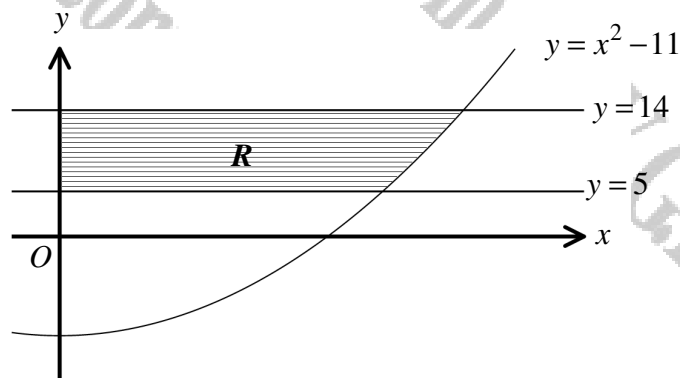
, proof

The handwritten solution shows the following steps:

(a) $A = \int_{-1}^3 \frac{6}{x+3} dx = [6 \ln|x+3|]_{-1}^3$
 $= 6 \ln 6 - 6 \ln 2 = 6(\ln 6 - \ln 2)$
 $= 6 \ln \left(\frac{6}{2}\right) = 6 \ln 3$

(b) $V = \pi \int_{-1}^3 \left(\frac{6}{x+3}\right)^2 dx = \pi \int_{-1}^3 \frac{36}{(x+3)^2} dx$
 $= \pi \int_{-1}^3 36(x+3)^{-2} dx = \pi \left[-36(x+3)^{-1} \right]_{-1}^3 = \pi \left[\frac{-36}{x+3} \right]_{-1}^3$
 $= \pi \left[\frac{-36}{6} - \frac{-36}{2} \right] = \pi(18 - 6) = 12\pi$

Question 5 (**+)



The figure above shows the parabola with equation

$$y = x^2 - 11.$$

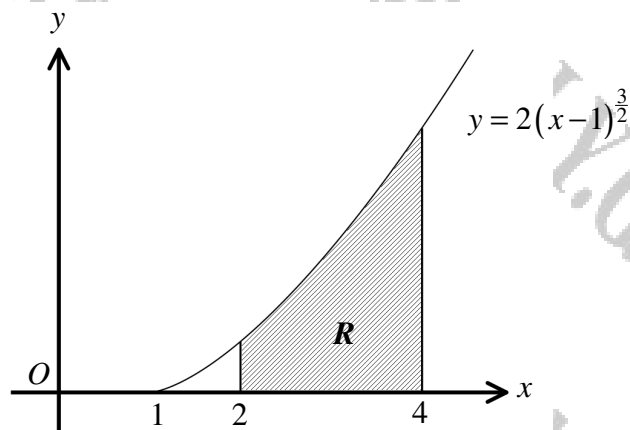
The shaded region R , is bounded by the curve, the y axis and the horizontal lines with equations $y = 5$ and $y = 14$.

This region R is rotated through 360° about the y axis to form a solid of revolution.

Show that the volume of the solid generated is $\frac{369\pi}{2}$.

proof

Question 6 (**)



The figure above shows part of the curve with equation

$$y = 2(x-1)^{\frac{3}{2}}.$$

The shaded region, labelled as R , bounded by the curve, the x axis and the straight lines with equations $x=2$ and $x=4$.

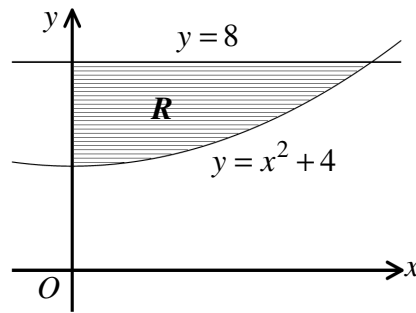
This region is rotated by 2π radians in the x axis, to form a solid of revolution S .

Show that the volume of S is 80π .

, proof

The handwritten solution shows the volume calculation using the disk method. It starts with the volume formula $V = \pi \int_a^b y^2 dx$ and substitutes the curve equation $y = 2(x-1)^{\frac{3}{2}}$. The integral is evaluated from $x=2$ to $x=4$, resulting in $V = \pi \int_2^4 4(x-1)^3 dx = 4\pi \int_2^4 (x-1)^3 dx = 4\pi \left[\frac{(x-1)^4}{4} \right]_2^4 = \pi [3^4 - 1^4] = 80\pi$.

Question 7 (**+)



The figure above shows the graph of the curve C with equation

$$y = x^2 + 4,$$

intersected by the straight line L with equation

$$y = 8.$$

The shaded region R , is bounded by C , the y axis and L .

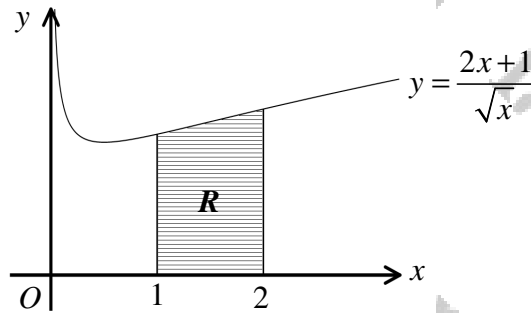
Show that when R is rotated through 2π radians about the y axis it will generate a volume of 8π cubic units.

proof

The handwritten proof shows the following steps:

$$\begin{aligned}
 V &= \pi \int_{y_1}^{y_2} x^2 dy = \pi \int_4^8 (y-4) dy \\
 &= \pi \left[\frac{1}{2}y^2 - 4y \right]_4^8 \\
 &= \pi [(32-32) - (8-16)] \\
 &= 8\pi
 \end{aligned}$$

Question 8 (**+)



The figure above shows the graph of the curve with equation

$$y = \frac{2x+1}{\sqrt{x}}, \quad x > 0.$$

The shaded region R is bounded by the curve, the x axis and the straight lines with equations $x=1$ and $x=2$.

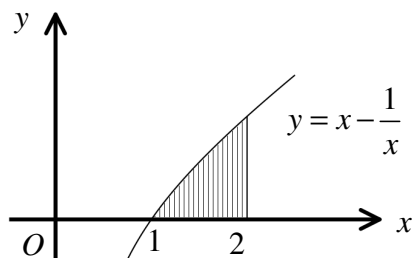
Find the volume that will be generated when R is rotated through 360° in the x axis.

Give the answer in the form $\pi(a+b\ln 2)$, where a and b are integers.

$$\pi(10 + \ln 2)$$

$$\begin{aligned}
 & y = \frac{2x+1}{\sqrt{x}} \\
 & \Rightarrow V = \pi \int_1^2 y^2 dx \\
 & \Rightarrow V = \pi \int_1^2 \left(\frac{2x+1}{\sqrt{x}} \right)^2 dx \\
 & \Rightarrow V = \pi \int_1^2 \frac{4x^2 + 4x + 1}{x} dx \\
 & 4x^2 \div x = 4x + \frac{1}{x} \\
 & \Rightarrow V = \pi \int_1^2 \left(4x + \frac{1}{x} \right) dx = \pi \left[2x^2 + \ln|x| \right]_1^2 \\
 & \Rightarrow V = \pi \left[(8 + \ln 2) - (2 + \ln 1) \right] \\
 & \Rightarrow V = \pi \left[6 + \ln 2 \right]
 \end{aligned}$$

Question 9 (***)



The figure above shows part of the curve with equation

$$y = x - \frac{1}{x}, \quad x \neq 0.$$

The shaded region bounded by the curve and the straight line with equation $x = 2$ is rotated by 360° about the x axis to form a solid of revolution.

Show that this volume is $\frac{5\pi}{6}$.

proof

$$\begin{aligned}
 y &= (x - \frac{1}{x})^2 = x^2 - 2 + \frac{1}{x^2} \\
 \therefore y^2 &= x^2 - 2 + \frac{1}{x^2} \\
 V &= \pi \int_{x_1}^{x_2} y^2 dx = \pi \int_1^2 (x^2 - 2 + \frac{1}{x^2}) dx = \pi \left[\frac{1}{3}x^3 - 2x - \frac{1}{x} \right]_1^2 \\
 &= \pi \left[\left(\frac{8}{3} - 4 - \frac{1}{2} \right) - \left(\frac{1}{3} - 2 - 1 \right) \right] = \frac{5\pi}{6} //
 \end{aligned}$$

Question 10 (***)

The curve C has equation

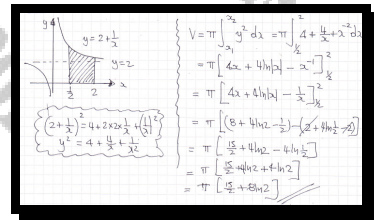
$$y = 2 + \frac{1}{x}, \quad x > 0.$$

The region bounded by C , the x axis and the lines $x = \frac{1}{2}$, $x = 2$ is rotated through 360° about the x axis.

Show that the volume of the solid formed is

$$\pi \left(\frac{15}{2} + 8 \ln 2 \right).$$

proof



Question 11 (***)

The curve C has equation

$$y = \sqrt{x} + \frac{4}{\sqrt{x}}, \quad x > 0.$$

The region bounded by C , the x axis and the lines $x=1$, $x=4$ is rotated through 360° about the x axis.

Show that the volume of the solid formed is

$$\frac{\pi}{2}(63 + 64 \ln 2).$$

proof

$$y^2 = (\sqrt{x} + \frac{4}{\sqrt{x}})^2 = (\sqrt{x})^2 + 2\sqrt{x}(\frac{4}{\sqrt{x}}) + (\frac{4}{\sqrt{x}})^2 = x + 8 + \frac{16}{x}$$

$$\therefore V = \pi \int_1^4 y^2 dx = \pi \int_1^4 (x + 8 + \frac{16}{x}) dx = \pi [\frac{1}{2}x^2 + 8x + 16 \ln x]_1^4$$

$$= \pi [(8 + 32 + 16 \ln 4) - (\frac{1}{2} + 8 + 16 \ln 1)] = \pi [\frac{53}{2} + 16 \ln 4]$$

$$= \pi [\frac{53}{2} + 32 \ln 2] = \frac{\pi}{2} [63 + 64 \ln 2]$$

Question 12 (***)

The curve C has equation

$$y = x^2 - 3x.$$

The region bounded by C and the x axis is rotated through 2π radians in the x axis.

Find the exact volume of the solid formed.

$\frac{81\pi}{10}$

$$V = \pi \int_0^3 (y)^2 dx = \pi \int_0^3 (x^2 - 3x)^2 dx$$

$$\Rightarrow V = \pi \int_0^3 (x^4 - 6x^3 + 9x^2) dx$$

$$\Rightarrow V = \pi [\frac{1}{5}x^5 - \frac{3}{2}x^4 + 3x^3]_0^3$$

$$\Rightarrow V = \pi [(243 - \frac{27}{2} \cdot 81 + 27 \cdot 27) - 0]$$

$$\Rightarrow V = \frac{81\pi}{10}$$

Question 13 (***)

The curve C has equation

$$y = x^{\frac{3}{2}} \sqrt{\ln x}, \quad x > 0.$$

The region bounded by C , the x axis and the straight lines with equations $x=1$ and $x=e$ is rotated through 360° about the x axis.

Use integration by parts to show that the volume of the solid formed is

$$\frac{1}{16} \pi (3e^4 + 1).$$

, proof

Handwritten solution showing the calculation of the volume of the solid of revolution. The solution uses integration by parts to evaluate the volume integral.

$$y = (x^{\frac{3}{2}} \ln x)^{\frac{1}{2}} = x^{\frac{3}{4}} \ln x \quad \left\{ \begin{array}{l} \text{BY PARTS} \\ \ln x \rightarrow \frac{1}{x} \\ \frac{3}{4}x^{\frac{3}{4}} \rightarrow \frac{3}{4}x^{-\frac{1}{4}} \end{array} \right.$$

$$V = \pi \int_1^e y^2 dx = \pi \int_1^e x^{\frac{3}{2}} \ln x dx$$

• INTEGRATE BY PARTS

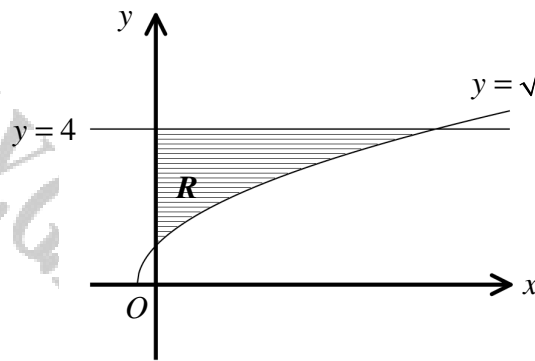
$$\frac{3}{2} x^{\frac{3}{2}} \ln x - \int \frac{3}{2} x^{\frac{3}{2}} \cdot \frac{1}{x} dx = \frac{3}{2} x^{\frac{3}{2}} \ln x - \int \frac{3}{2} x^{\frac{1}{2}} dx = \frac{3}{2} x^{\frac{3}{2}} \ln x - \frac{3}{2} \cdot \frac{2}{3} x^{\frac{3}{2}} + C$$

• HENCE

$$V = \pi \left[\frac{3}{2} x^{\frac{3}{2}} \ln x - x^{\frac{3}{2}} \right]_1^e = \pi \left[\left(\frac{3}{2} e^{\frac{3}{2}} \ln e - e^{\frac{3}{2}} \right) - \left(\frac{3}{2} \cdot 1^{\frac{3}{2}} - 1^{\frac{3}{2}} \right) \right]$$

$$= \pi \left[\frac{3}{2} e^{\frac{3}{2}} - e^{\frac{3}{2}} + 1 \right] = \pi \left[\frac{1}{2} e^{\frac{3}{2}} + 1 \right] = \frac{1}{16} \pi (3e^4 + 1)$$

Question 14 (***)



The curve C has equation

$$y = \sqrt{x+1}, \quad x > -1.$$

The region R is bounded by C , the y axis and the straight line with equation $y = 4$ is rotated through 360° about the y axis to form a solid of revolution.

Show that the volume of the solid is $\frac{828\pi}{5}$.

, proof

THEORY BY INSPECTING THE CURVE WHEN THE y AXIS AT $(0,1)$

$y = \sqrt{x+1}$
 $y^2 = x+1$
 $x = y^2 - 1$
 $x^2 = (y^2 - 1)^2$
 $x^2 = y^4 - 2y^2 + 1$

SETTING UP A DOUBLE INTEGRAL ABOUT THE y AXIS

$$\text{Volume} = \pi \int_1^4 (x^2) dy = \pi \int_1^4 (y^4 - 2y^2 + 1) dy$$

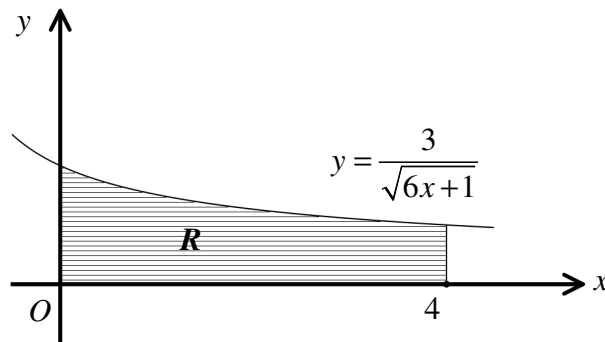
$$= \pi \left[\frac{y^5}{5} - \frac{2y^3}{3} + y \right]_1^4 = \pi \left(\frac{1024}{5} - \frac{128}{3} + 4 \right) - \pi \left(\frac{1}{5} - \frac{2}{3} + 1 \right)$$

$$= \frac{2448}{5}\pi - \frac{8}{15}\pi$$

$$= \frac{828}{5}\pi$$

As required

Question 15 (***)



The graph below shows the curve with equation

$$y = \frac{3}{\sqrt{6x+1}}, \quad x \neq -\frac{1}{6}.$$

The region R , shown in the figure shaded, is bounded by the curve, the coordinate axes and the straight line with equation $x = 4$.

- a) Show that the area of R is 4 square units.

The shaded region R is rotated by 2π radians about the x axis to form a solid of revolution.

- b) Show that the volume of the solid generated is $3\pi \ln 5$.

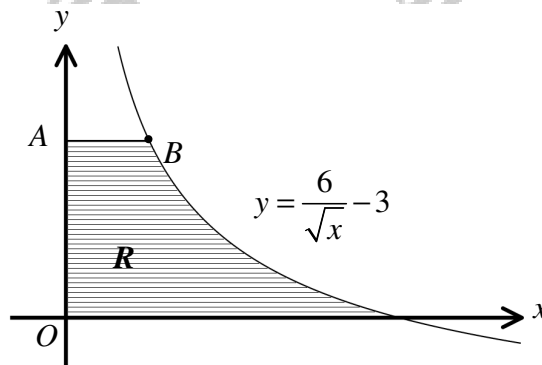
, proof

The handwritten solution shows the following steps:

(a) $\int_0^4 \frac{3}{\sqrt{6x+1}} dx = \int_1^25 \frac{3}{2\sqrt{u}} du = \frac{3}{2} \cdot 2\sqrt{u} \Big|_1^{25} = 3(\sqrt{25} - \sqrt{1}) = 3(5 - 1) = 12$

(b) $V = \pi \int_0^4 \left(\frac{3}{\sqrt{6x+1}}\right)^2 dx = \pi \int_0^4 \frac{9}{6x+1} dx = \pi \left[\frac{3}{2} \ln|6x+1| \right]_0^4 = \frac{3\pi}{2} (\ln 25 - \ln 1) = \frac{3\pi}{2} (2\ln 5) = 3\pi \ln 5$

Question 16 (***)



The figure above shows part of the graph of the curve with equation

$$y = \frac{6}{\sqrt{x}} - 3, \quad x > 0.$$

The point B lies on the curve where $x = 1$.

The shaded region R is bounded by the curve, the coordinate axes and a straight line segment AB , where AB is parallel to the x axis. The region R is rotated through 2π radians in the y axis to form a solid of revolution.

Show that the volume of this solid is 14π .

, proof

AS THE QUESTION IS ABOUT THE y AXIS, REARRANGE THE EQUATION FOR x .

$$\Rightarrow y = \frac{6}{\sqrt{x}} - 3$$

$$\Rightarrow y + 3 = \frac{6}{\sqrt{x}}$$

$$\Rightarrow (y+3)^2 = \frac{36}{x}$$

$$\Rightarrow x = \frac{36}{(y+3)^2}$$

$$\Rightarrow x^2 = \frac{1296}{(y+3)^4}$$

BY INSPECTING THE y COORDINATE OF B IS 3

THUS WE HAVE

$$V = \pi \int_{y=0}^{y=3} [x(y)]^2 dy = \pi \int_0^3 \frac{1296}{(y+3)^4} dy$$

$$= \pi \int_0^3 1296(y+3)^{-4} dy = \pi \left[\frac{1296}{-3} (y+3)^{-3} \right]_0^3$$

$$= \pi \left[\frac{432}{(y+3)^3} \right]_0^3 = 432\pi \left[\frac{1}{(y+3)^3} \right]_0^3$$

$$= 432\pi \left[\frac{1}{27} - \frac{1}{216} \right] = 432\pi \times \frac{7}{216} = 14\pi$$

AS REQUIRED

Question 17 (***)

The curve C has equation

$$y = x + \frac{1}{x^2}, \quad x > 0.$$

The region bounded by C , the x axis and the lines $x=1$, $x=2$ is rotated through 360° about the x axis.

Show that the volume of the solid formed is

$$\pi \left(\frac{21}{8} + 2 \ln 2 \right).$$

proof

The handwritten proof shows the following steps:

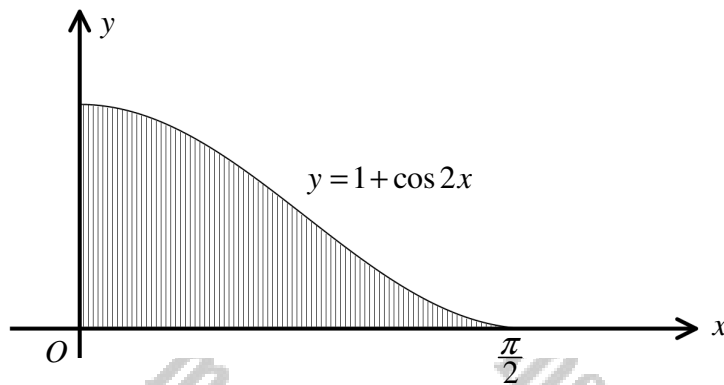
$$y = x + \frac{1}{x^2} = x^2 + 2x\left(\frac{1}{x}\right) + \left(\frac{1}{x}\right)^2 = x^2 + \frac{2}{x} + \frac{1}{x^2}$$

$$V = \pi \int_1^2 y^2 dx = \pi \int_1^2 \left(x^2 + \frac{2}{x} + x^{-2} \right)^2 dx = \pi \int_1^2 \left(x^4 + 2x + 2x^{-1} + \frac{4}{x^2} + 2x^{-2} + x^{-4} \right) dx$$

$$= \pi \left[\frac{1}{5}x^5 + 2x \ln|x| - \frac{1}{3x^3} \right]_1^2 = \pi \left[\left(\frac{32}{5} + 2 \ln 2 - \frac{1}{6} \right) - \left(\frac{1}{5} + 2 \ln 1 - \frac{1}{3} \right) \right]$$

$$= \pi \left[\frac{21}{8} + 2 \ln 2 \right]$$

Question 18 (***)



The figure above shows the graph of the curve with equation

$$y = 1 + \cos 2x, \quad 0 \leq x \leq \frac{\pi}{2}.$$

a) Show clearly that

$$(1 + \cos 2x)^2 \equiv \frac{3}{2} + 2 \cos 2x + \frac{1}{2} \cos 4x.$$

The shaded region bounded by the curve and the coordinate axes is rotated by 2π radians about the x axis to form a solid of revolution.

b) Show that the volume of the solid is

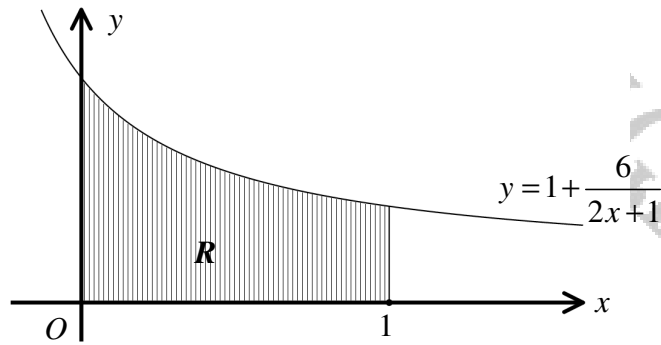
$$\frac{3}{4} \pi^2.$$

Q.E.D., proof

(a) $(1 + \cos 2x)^2 = 1 + 2\cos 2x + \cos^2 2x$
 $= 1 + 2\cos 2x + \left(\frac{1}{2} + \frac{1}{2}\cos 4x\right)$ ↖ $\cos^2 \theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$
 $= \frac{3}{2} + 2\cos 2x + \frac{1}{2}\cos 4x$ ↖ $\cos^2(\theta) = \frac{1}{2} + \frac{1}{2}\cos 2\theta$

(b) $V = \pi \int_0^{\pi/2} (1 + \cos 2x)^2 dx = \pi \int_0^{\pi/2} \left(\frac{3}{2} + 2\cos 2x + \frac{1}{2}\cos 4x\right) dx$
 $V = \pi \left[\frac{3}{2}x + \sin 2x + \frac{1}{8}\sin 4x \right]_0^{\pi/2}$
 $V = \pi \left[\left(\frac{3\pi}{4} + 0 + 0\right) - (0) \right]$
 $V = \frac{3}{4}\pi^2$

Question 19 (***)



The figure above shows the graph of the curve with equation

$$y = 1 + \frac{6}{2x+1}, \quad x \neq -\frac{1}{2}.$$

a) Show that

$$\left(1 + \frac{6}{2x+1}\right)^2 \equiv 1 + \frac{A}{2x+1} + \frac{B}{(2x+1)^2},$$

where A and B are constants to be found.

The shaded region, labelled as R , bounded by the curve, the coordinate axes and the line $x=1$ is rotated by 2π radians in the x axis to form a solid of revolution.

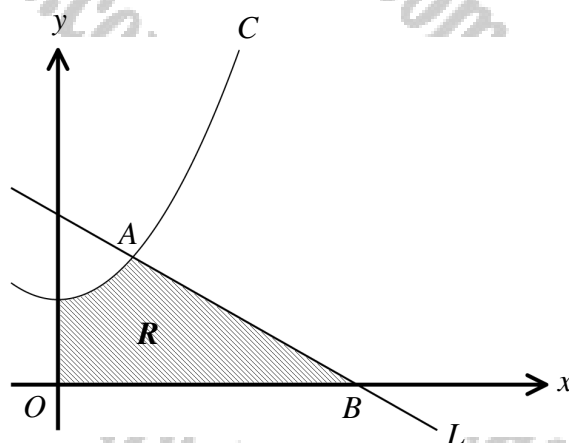
b) Show further that the volume generated is

$$\pi(13 + 6\ln 3).$$

, ,

(a) $\left(1 + \frac{6}{2x+1}\right)^2 = 1^2 + 2 \times 1 \times \frac{6}{2x+1} + \left(\frac{6}{2x+1}\right)^2 = 1 + \frac{12}{2x+1} + \frac{36}{(2x+1)^2}$
 (b) $V = \pi \int_{x=0}^{x=1} y^2 dx = \pi \int_0^1 \left(1 + \frac{12}{2x+1} + \frac{36}{(2x+1)^2}\right) dx$
 $= \pi \left[x + 6 \ln|2x+1| - \frac{18}{2x+1} \right]_0^1 = \pi \left[1 + 6 \ln 3 - \frac{18}{3} - \left(0 + 6 \ln 1 - \frac{18}{1}\right) \right] = \pi [13 + 6 \ln 3]$

Question 20 (***)



The figure above shows the graph of the curve C with equation

$$y = x^2 + 2,$$

intersected by the straight line L with equation

$$x + y = 4.$$

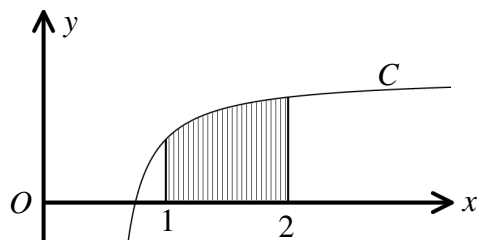
The point A is the intersection of C and L . The point B is the point where L meets the x axis.

The region R , shown shaded in the figure above, is bounded by C , L and the coordinate axes. This region is rotated by 360° in the x axis, forming a solid of revolution S .

Find an exact value for the volume of S .

$$\frac{218}{15} \pi$$

Question 21 (***)



The figure above shows part of the graph of the curve C with equation

$$y = 2 - \frac{1}{2x-1}, \quad x \neq \frac{1}{2}.$$

The shaded region bounded by C and the straight lines with equations $x=1$ and $x=2$, is rotated by 360° about the x axis, forming a solid of revolution.

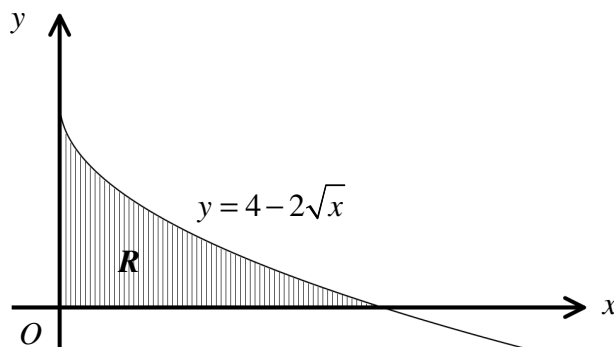
Show that the volume of the solid is

$$\pi \left(\frac{13}{3} - 2 \ln 3 \right).$$

proof

$y = 2 - \frac{1}{2x-1}$
 $V = \pi \int_1^2 \left(2 - \frac{1}{2x-1} \right)^2 dx$
 $= \pi \int_1^2 \left(4 - 2 \left(2x-1 \right)^{-1} + \left(2x-1 \right)^{-2} \right) dx$
 $= \pi \left[4x - 2 \ln |2x-1| - \frac{1}{2(2x-1)} \right]_1^2$
 $= \pi \left[\left(8 - 2 \ln 3 - \frac{1}{2} \right) - \left(4 - 2 \ln 1 - \frac{1}{2} \right) \right]$
 $= \pi \left[4 - 2 \ln 3 + \frac{1}{2} - 4 + \frac{1}{2} \right]$
 $= \pi \left[\frac{13}{3} - 2 \ln 3 \right]$

Question 22 (***)



The figure above shows the graph of the equation

$$y = 4 - 2\sqrt{x} \quad x \geq 0.$$

The shaded region R , bounded by the curve and the coordinate axes, is rotated through 4 right angles about the y axis to form a solid of revolution.

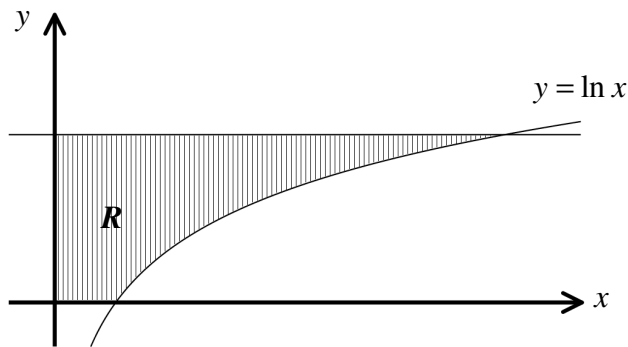
Show that the volume generated is $\frac{64\pi}{5}$.

proof

The handwritten proof shows the following steps:

- Graph of $y = 4 - 2\sqrt{x}$ is shown.
- When $y=0$, $x=4$.
- Volume calculation using the shell method: $V = \pi \int_0^4 x^2 dy = \pi \int_0^4 \frac{(4-y)^2}{4} dy$.
- Integration: $= \frac{\pi}{4} \int_0^4 (4-y)^2 dy = \frac{\pi}{4} \left[-\frac{1}{3}(4-y)^3 \right]_0^4 = \frac{\pi}{12} \left[(4-0)^3 - 0 \right] = \frac{\pi}{12} (64) = \frac{16\pi}{3}$.
- Final result: $\frac{64\pi}{5}$.

Question 23 (***)



The figure below shows the graph of the curve C with equation

$$y = \ln x, \quad x > 0,$$

intersected by the horizontal straight line L with equation

$$y = 2.$$

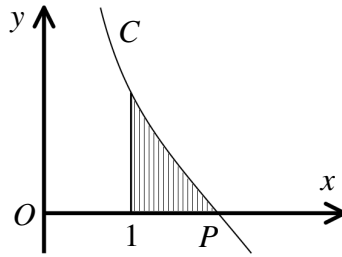
The shaded region R , bounded by C , L and the coordinate axes, is rotated through 2π radians in the y axis to form a solid of revolution.

Show that the volume of the solid is

$$\frac{1}{2}\pi(e^4 - 1).$$

proof

Question 24 (***)



The figure above shows part of the curve C with equation

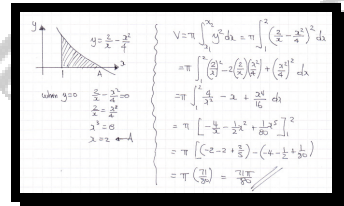
$$y = \frac{2}{x} - \frac{x^2}{4}, \quad x > 0.$$

The curve crosses the x axis at the point P .

The shaded region bounded by the curve, the straight line with equation $x = 1$ and the x axis is rotated by 360° about the x axis to form a solid of revolution.

Show that the volume of the solid is $\frac{71\pi}{80}$.

proof



Question 25 (***)

The curve C lies entirely above the x axis and has equation

$$y = 1 + \frac{1}{2\sqrt{x}}, \quad x \geq 0.$$

a) Show that

$$y^2 = 1 + \frac{1}{\sqrt{x}} + \frac{1}{4x}.$$

The region R is bounded by the curve, the x axis and the straight lines with equations $x=1$ and $x=4$.

b) Show that when R is rotated by 360° about the x axis, the solid generated has a volume

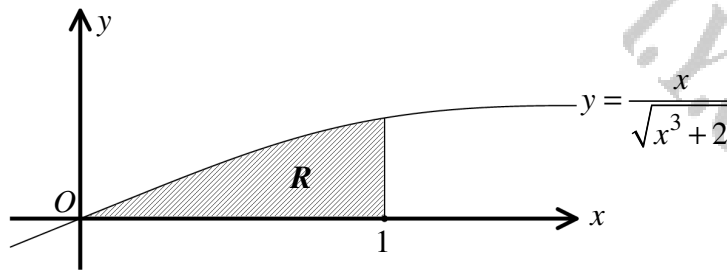
$$\pi(5 + \ln\sqrt{2}).$$

proof

Handwritten mathematical proof for Question 25b:

$$\begin{aligned} \text{(a)} \quad y^2 &= \left(1 + \frac{1}{2\sqrt{x}}\right)^2 = 1^2 + 2 \times 1 \times \frac{1}{2\sqrt{x}} + \left(\frac{1}{2\sqrt{x}}\right)^2 \\ &= 1 + \frac{2}{2\sqrt{x}} + \frac{1}{4x} = 1 + \frac{1}{\sqrt{x}} + \frac{1}{4x} \\ \text{(b)} \quad V &= \pi \int_1^4 y^2 dx = \pi \int_1^4 \left[1 + \frac{1}{\sqrt{x}} + \frac{1}{4x}\right] dx \\ &= \pi \left[x + 2\sqrt{x} + \frac{1}{4} \ln|x| \right]_1^4 = \pi \left[(4 + 2 + \frac{1}{4} \ln 4) - (1 + 2 + \frac{1}{4} \ln 1) \right] \\ &= \pi \left(5 + \frac{1}{2} \ln 4 \right) = \pi \left(5 + \frac{1}{2} \ln 2^2 \right) = \pi(5 + \ln\sqrt{2}) \end{aligned}$$

Question 26 (***)



The figure above shows part of the curve with equation

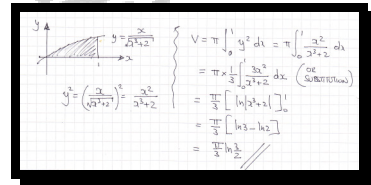
$$y = \frac{x}{\sqrt{x^3 + 2}}, \quad x^3 > -2.$$

The shaded region R , bounded by the curve, the x axis and the straight line with equation $x = 1$, is rotated by 360° about the x axis to form a solid of revolution.

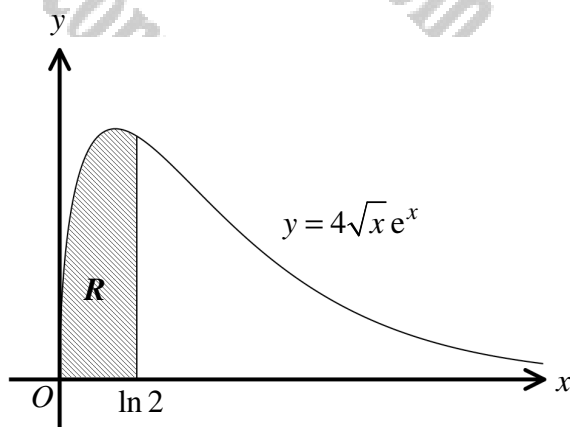
Show that the solid has a volume of

$$\frac{\pi}{3} \ln\left(\frac{3}{2}\right).$$

, proof



Question 27 (***)



The figure above shows the graph of the curve C with equation

$$y = 4\sqrt{x}e^x, \quad x \geq 0.$$

The shaded region R bounded by the curve, the x axis and the vertical straight line with equation $x = \ln 2$, is rotated by 2π radians in the x axis, forming a solid of revolution S .

Find an exact value for the volume of S , giving the answer in the form $\pi(a + b \ln 2)$ where a and b are integers.

,

SOLVE THE STANDARD FORMULA FOR VOLUME OF REVOLUTION 4/2020

The 2nd part

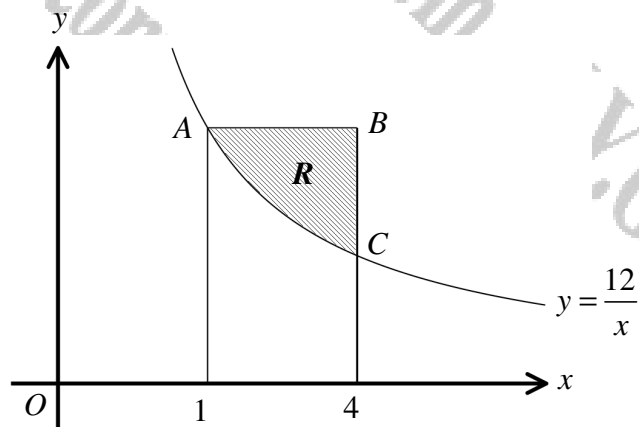
$$V = \pi \int_0^{\ln 2} [4\sqrt{x}]^2 dx$$

$$V = \pi \int_0^{\ln 2} (4\sqrt{x})^2 dx = \pi \int_0^{\ln 2} 16x dx$$

PROCEED BY INTEGRATION BY PARTS

$V = \pi \int_0^{\ln 2} (8x)(2e^x) dx$	$\frac{8x}{2x} \mid \frac{\pi}{2e^x}$
$V = \pi \left[(8x^2) \cdot \frac{1}{2} - \int_0^{\ln 2} 8e^{2x} dx \right]$	
$V = \pi \left[8x^2 - 4e^{2x} \right]_0^{\ln 2}$	
$V = \pi \left[(8 \ln^2 2 - 4e^{2 \ln 2}) - (0 - 4) \right]$	
$V = \pi \left[32 \ln^2 2 - 16 + 4 \right]$	
$V = \pi \left[32 \ln^2 2 - 12 \right]$	
<u>OR</u> $V = 4\pi \left[-3 + 8 \ln 2 \right]$	

Question 28 (***)



The figure above shows part of the graph of the curve with equation

$$y = \frac{12}{x}, \quad x \neq 0.$$

The points A and C lie on the curve where $x=1$ and $x=4$, respectively. The point B is such that AB is parallel to the x axis and BC is parallel to the y axis.

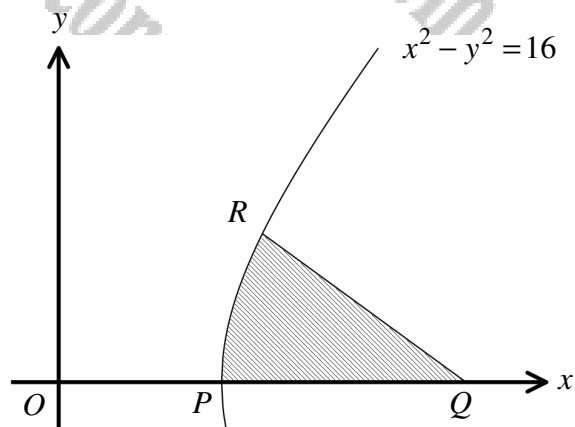
The region R , shown shaded in the figure above, is bounded by the curve and the straight line segments AB and BC . This region is rotated by 2π radians in the x axis, forming a solid of revolution S .

Find the exact value for the volume of S .

324 π

$V = \pi \int_1^4 \left(\frac{12}{x} \right)^2 dx - \pi \int_1^4 \left(\frac{12}{4} \right)^2 dx$
 $\Rightarrow V = \pi \int_1^4 \left[\frac{144}{x^2} - 9 \right] dx = \pi \left[\frac{144}{-x} - 9x \right]_1^4$
 $\Rightarrow V = \pi (144 - 36) = 108\pi$
 $\bullet V_{\text{ext}} = \pi r^2 h$
 $V_{\text{ext}} = \pi (12)^2 (3)$
 $V_{\text{ext}} = 432\pi$
 $\therefore \text{Required volume} = 432\pi - 108\pi = 324\pi$

Question 29 (***)



The figure above shows part of the graph of the hyperbola C with equation

$$x^2 - y^2 = 16.$$

The hyperbola crosses the x axis at $P(4,0)$, the point $R(5,3)$ lies on C and the point $Q(11,0)$ lies on the x axis.

The shaded region bounded by the curve, the x axis and the straight line segment RQ is rotated by 2π radians in the x axis, forming a solid of revolution S .

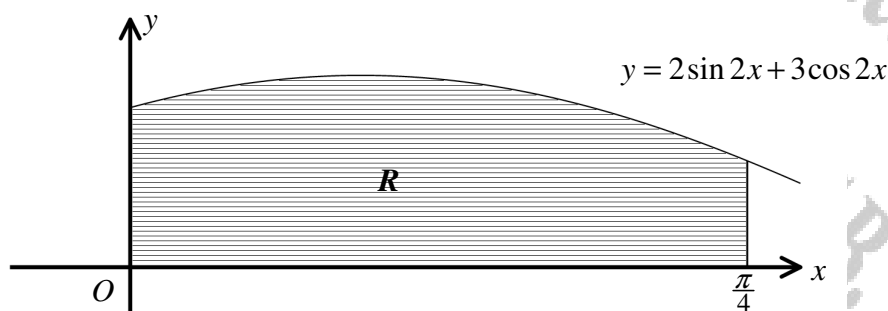
Find an exact value for the volume of S .

$$\frac{67}{3}\pi$$

\bullet Volume of cone = $\frac{1}{3}\pi r^2 h$
 $= \frac{1}{3}\pi \times 3^2 \times 6 = 18\pi$

$V = \pi \int_4^{11} (y^2) dx = \pi \int_4^{11} (x^2 - 16) dx = \pi \left[\frac{1}{3}x^3 - 16x \right]_4^{11}$
 $\therefore V = \pi \left[\left(\frac{1331}{3} - 176 \right) - \left(\frac{64}{3} - 64 \right) \right] = \pi \left[\frac{1331}{3} - \frac{64}{3} + 16 \right]$
 $\therefore V = \frac{67}{3}\pi + 16\pi = \frac{83}{3}\pi$

Question 30 (***)



The figure above shows part of the curve C , with equation

$$y = 2 \sin 2x + 3 \cos 2x.$$

a) Show that

$$y^2 = A + B \cos 4x + C \sin 4x,$$

where A , B and C are constants.

The shaded region R is bounded by the curve, the line $x = \frac{\pi}{4}$ and the coordinate axes.

b) Find the area of R .

The region R is rotated by 2π radians in the x axis forming a solid of revolution S .

c) Show that the volume of S is

$$\frac{\pi}{8} (13\pi + 24).$$

, $A = \frac{13}{2}, B = \frac{5}{2}, C = 6$, $\text{area} = 2.5$

(a) $y^2 = (2\sin 2x + 3\cos 2x)^2 = 4\sin^2 2x + 12\sin 2x \cos 2x + 9\cos^2 2x$
 $= 4(\frac{1}{2} - \frac{1}{2}\cos 4x) + 6(2\sin 2x \cos 2x) + 9(\frac{1}{2} + \frac{1}{2}\cos 4x)$
 $= 2 - 2\cos 4x + 6\sin 4x + \frac{9}{2} + \frac{9}{2}\cos 4x$
 $= \frac{13}{2} + 3\sin 4x + 2\cos 4x$

(b) $\text{Area} = \int_0^{\pi/4} (2\sin 2x + 3\cos 2x) dx = [-\cos 2x + \frac{3}{2}\sin 2x]_0^{\pi/4}$
 $= [(-\frac{1}{\sqrt{2}} + \frac{3}{2}) - (-1 + 0)] = \frac{1}{\sqrt{2}} + \frac{1}{2} = \frac{1 + \sqrt{2}}{2}$

(c) $V = \pi \int_0^{\pi/4} (y^2) dx = \pi \int_0^{\pi/4} (\frac{13}{2} + 3\sin 4x + 2\cos 4x) dx$
 $= \pi [\frac{13x}{2} - \frac{3}{4}\cos 4x + \frac{1}{2}\sin 4x]_0^{\pi/4}$
 $= \pi [\frac{13\pi}{8} - \frac{3}{4}(\frac{1}{\sqrt{2}} + \frac{3}{2}) + \frac{1}{2}(0 - \frac{3}{2})]$
 $= \pi [\frac{13\pi}{8} + 3] = \frac{\pi}{8} (13\pi + 24)$

Question 31 (***)

The point P lies on the curve with equation

$$y = x^2, \quad x \geq 0.$$

The straight line L_1 is parallel to the x axis and passes through P . The finite region R_1 is bounded by the curve, L_1 and the y axis.

The straight line L_2 is parallel to the y axis and passes through P . The finite region R_2 is bounded by the curve, L_2 and the x axis.

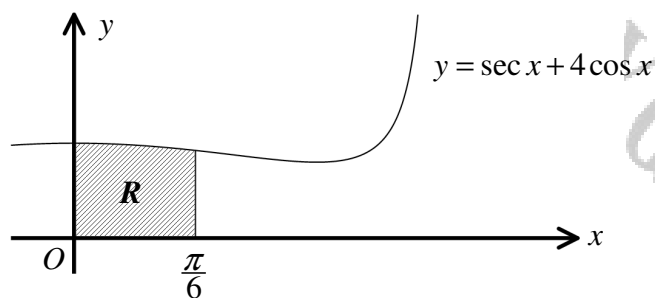
When R_1 is fully revolved about the y axis the volume of the solid formed is equal to the volume of the solid formed when R_2 is fully revolved about the x axis.

Determine the x coordinate of P .

, $x = \frac{5}{2}$

• LET P HAVE CO-ORDINATES (a, a^2) , $a > 0$
 • VOLUME OF REVOLUTION ABOUT THE x AXIS
 $V_x = \pi \int_{y_1}^{y_2} y^2 dy$
 $V_x = \pi \int_0^{a^2} (x^2)^2 dx = \pi \int_0^{a^2} x^4 dx = \frac{1}{5} \pi [x^5]_0^{a^2} = \frac{1}{5} \pi a^{10}$
 • VOLUME OF REVOLUTION ABOUT THE y AXIS
 $V_y = \pi \int_{x_1}^{x_2} x^2 dx$
 $V_y = \pi \int_0^a y^2 dy = \frac{1}{3} \pi [y^3]_0^{a^2} = \frac{1}{3} \pi a^6$
 • NOW $V_x = V_y$
 $\frac{1}{5} \pi a^{10} = \frac{1}{3} \pi a^6$
 $a^4 = \frac{5}{3} a^2$
 $a = \frac{\sqrt{15}}{3}$ ($a \neq 0$)

Question 32 (****)



The figure above shows part of the curve with equation

$$y = \sec x + 4 \cos x$$

The shaded region, labelled R , bounded by the curve, the coordinate axes and the straight line with equation $x = \frac{\pi}{6}$ is rotated by 2π radians in the x axis to form a solid of revolution.

Show that the solid has a volume of

$$\frac{\pi}{3} (8\pi + 7\sqrt{3}).$$

, proof

$y = \sec x + 4 \cos x$
 $V = \pi \int_0^{\pi/6} (y(x))^2 dx$
 $V = \pi \int_0^{\pi/6} (\sec x + 4 \cos x)^2 dx$
 $V = \pi \int_0^{\pi/6} \sec^2 x + 8 \sec x \cos x + 16 \cos^2 x dx$
 $V = \pi \int_0^{\pi/6} \sec^2 x + 8 + 16 \left(\frac{x}{2} + \frac{\sin 2x}{4} \right) dx$
 $V = \pi \left[\tan x + 8x + 4 \sin 2x \right]_0^{\pi/6}$
 $\Rightarrow V = \pi \left[\tan \frac{\pi}{6} + 8 \left(\frac{\pi}{6} \right) + 4 \sin \frac{\pi}{3} \right]$
 $\Rightarrow V = \pi \left[\frac{1}{\sqrt{3}} + \frac{4\pi}{3} + 4 \left(\frac{\sqrt{3}}{2} \right) \right]$
 $\Rightarrow V = \frac{\pi}{3} (8\pi + 7\sqrt{3})$

Question 33 (***)

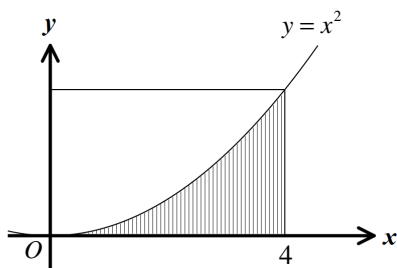


Figure 1

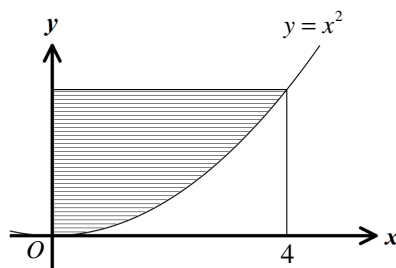


Figure 2

The figures above show part of the parabola with equation

$$y = x^2.$$

The shaded region, shown in Figure 1, is bounded by the curve, the x axis and the line $x = 4$. This region is revolved by 2π radians about the x axis, to form a solid of revolution.

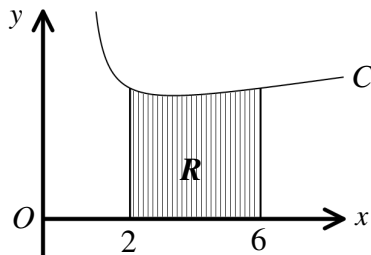
- a) Show that the solid has a volume of $\frac{1024\pi}{5}$.

The shaded region, shown in Figure 2, is bounded by the curve, the y axis and a horizontal line originating from a point on the parabola where $x = 4$. This region is revolved by 2π radians about the y axis, to form a solid of revolution.

- b) Show that the solid has a volume of 128π .
- c) Hence find the value of the volume generated when the region shown in figure 1 is revolved by 2π radians about the y axis.

128 π

Question 34 (****)



The figure above shows part of the curve C with equation

$$y = \frac{x+1}{\sqrt{x-1}}, \quad x \geq 1.$$

The shaded region R is bounded by the curve, the x axis and the straight lines with equations $x=2$ and $x=6$. The region R is rotated by 360° about the x axis to form a solid of revolution.

- a) Show that the volume of the solid is

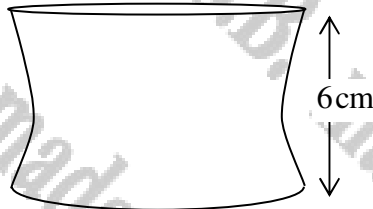
$$\pi(28 + 4\ln 5).$$

[continues overleaf]

[continued from overleaf]

The solid of part (a) is used to model the wooden leg of a sofa.

The shape of the leg is geometrically similar to the solid of part (a).



- b) Given the height of the leg is 6 cm, determine the volume of the wooden leg to the nearest cubic centimetre.

, $\approx 365 \text{ cm}^3$

a) VOLUME OF REVOLUTION IN Cartesian, about the z axis

$$V = \pi \int_{x_1}^{x_2} (y(x))^2 dx = \pi \int_2^6 \left(\frac{2x-1}{x-1}\right)^2 dx = \pi \int_2^6 \frac{(2x-1)^2}{(x-1)^2} dx$$

BY SUBSTITUTION or MANIPULATION

• $u = 2x - 1$ or $x = u + 1$
 $\frac{du}{dx} = 2$
 $du = 2dx$
 $x = 2 \rightarrow u = 3$
 $x = 6 \rightarrow u = 11$

$\Rightarrow V = \pi \int_3^{11} \frac{u^2}{\left(\frac{u}{2}\right)^2} \frac{du}{2}$

$\Rightarrow V = \pi \int_3^{11} \frac{u^2}{\frac{u^2}{4}} \frac{du}{2}$

$\Rightarrow V = \pi \int_3^{11} \frac{u^2 \cdot 4}{u^2} \frac{du}{2}$

$\Rightarrow V = \pi \int_3^{11} 2 du$

$\Rightarrow V = \pi [2u]_3^{11}$

$\Rightarrow V = \pi [2(11) - 2(3)]$

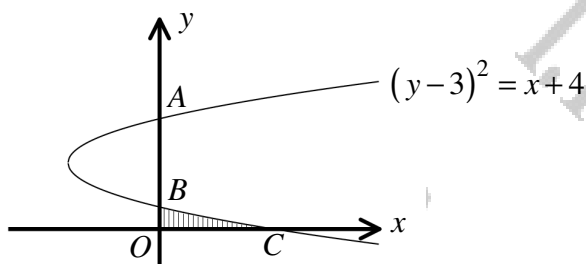
$\Rightarrow V = \pi [20 + 4hs]$

b) WORKS AT THE SIMILAR STAGES

$V = \pi (20 + 4hs)$ $V = ?$

$V = v \times (\text{scale factor})^3$
 $V = \pi (20 + 4hs) \times (6)^3$
 $V \approx 365$

Question 35 (****)



The figure above shows part of the curve with equation

$$(y-3)^2 = x+4.$$

The curve crosses the coordinate axes at the points A , B and C .

a) Show that

$$x^2 = y^4 - 12y^3 + 46y^2 - 60y + 25.$$

b) The shaded region bounded by the curve and the coordinate axes is rotated by 360° about the y axis to form a solid of revolution.

Show that the volume of the solid is $\frac{113\pi}{15}$.

proof

$(y-3)^2 = x+4$
 $(y-3)^2 - 4 = x$
 $y^2 - 6y + 9 - 4 = x$
 $y^2 - 6y + 5 = x$
 $x^2 = (y^2 - 6y + 5)^2$
 $x^2 = (y^2 - 6y + 5)(y^2 - 6y + 5)$
 $x^2 = y^4 - 6y^3 + 5y^2 - 6y^3 + 36y^2 - 30y + 5y^2 - 30y + 25$
 $x^2 = y^4 - 12y^3 + 41y^2 - 60y + 25$

(b) when $x=0$ $(y-3)^2 = 4$
 $y-3 = \pm 2$
 $y = 1$ or 5
 $\therefore V = \pi \int_1^5 x^2 dy = \pi \int_1^5 (y^2 - 6y + 5)^2 dy$
 $V = \pi \int_1^5 (y^4 - 12y^3 + 26y^2 - 60y + 25) dy$
 $V = \pi \left[\frac{1}{5}y^5 - 3y^4 + 26 \cdot \frac{1}{3}y^3 - 30y^2 + 25y \right]_1^5$
 $V = \pi \left[\frac{1}{5}(5^5 - 1^5) - 3(5^4 - 1^4) + \frac{26}{3}(5^3 - 1^3) - 30(5^2 - 1^2) + 25(5 - 1) \right]$
 $V = \pi \times \frac{113}{15}$
 $V = \frac{113\pi}{15}$

Question 36 (***)

The curve C has equation

$$y = \frac{x}{x+1}, \quad x \geq 0.$$

The region bounded by the curve, the x axis and the straight line with equation $x = 1$ is rotated through 2π radians about the x axis to form a solid of revolution.

Show that the volume of the solid is

$$\frac{\pi}{2}(3 - 4 \ln 2).$$

proof

$$V = \pi \int_0^1 \left(\frac{x}{x+1}\right)^2 dx = \pi \int_0^1 \left(\frac{x^2}{(x+1)^2}\right) dx$$

$$= \pi \int_0^1 \frac{x^2}{(x+1)^2} dx = \pi \int_0^1 \frac{x^2 - 2x + 1 + 2x - 1}{(x+1)^2} dx$$

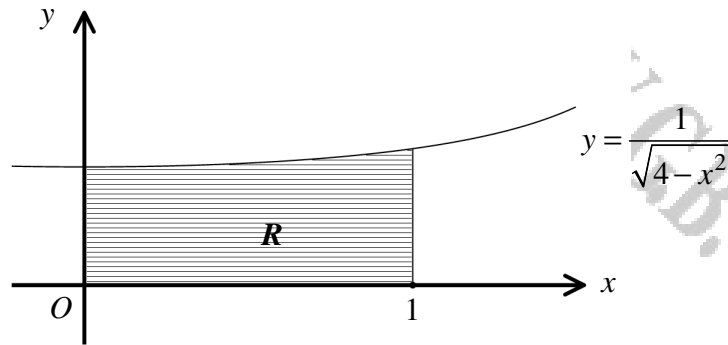
$$= \pi \int_0^1 \left(1 - \frac{2}{x+1} + \frac{1}{(x+1)^2}\right) dx = \pi \left[x - 2 \ln|x+1| - \frac{1}{x+1} \right]_0^1$$

$$= \pi \left[\left(1 - 2 \ln 2 - \frac{1}{2}\right) - \left(0 - 2 \ln 1 - 1\right) \right]$$

$$= \pi \left(\frac{1}{2} - 2 \ln 2 \right) = \frac{\pi}{2} (3 - 4 \ln 2)$$

Diagram: A coordinate system showing the region bounded by the curve $y = \frac{x}{x+1}$, the x -axis, and the vertical line $x = 1$. The region is shaded and labeled as the area to be rotated.

Question 37 (****)



The figure above shows part of the curve with equation

$$y = \frac{1}{\sqrt{4-x^2}}, \quad -2 \leq x \leq 2.$$

The shaded region, labelled as R , bounded by the curve, the coordinate axes and the straight line with equation $x=1$ is rotated by 2π radians about the x axis to form a solid of revolution.

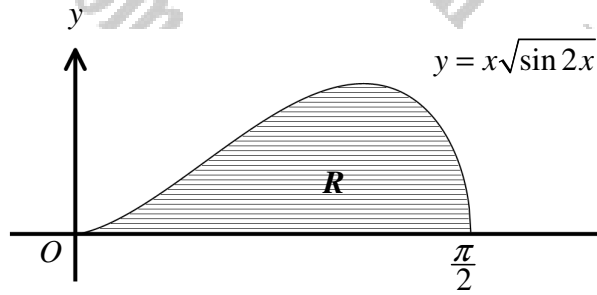
Show that the volume of the solid is

$$\frac{1}{4}\pi \ln 3.$$

proof

$$\begin{aligned}
 V &= \pi \int_0^1 (y(x))^2 dx = \pi \int_0^1 \left(\frac{1}{\sqrt{4-x^2}}\right)^2 dx = \pi \int_0^1 \frac{1}{4-x^2} dx = \pi \int_0^1 \frac{1}{(2-x)(2+x)} dx \\
 \frac{1}{(2-x)(2+x)} &= \frac{A}{2-x} + \frac{B}{2+x} \\
 1 &= A(2+x) + B(2-x) \\
 \frac{1}{2-x} &\Rightarrow 1 = 4A \Rightarrow A = \frac{1}{4} \\
 \frac{1}{2+x} &\Rightarrow 1 = 4B \Rightarrow B = \frac{1}{4} \\
 &= \pi \int_0^1 \left(\frac{1}{4} + \frac{1}{4}\right) dx = \pi \int_0^1 \left(-\frac{1}{4} \ln|2-x| + \frac{1}{4} \ln|2+x|\right) dx \\
 &= \frac{\pi}{4} \left[\ln\left|\frac{2+x}{2-x}\right| \right]_0^1 = \frac{\pi}{4} \left[\ln 3 - \ln 1 \right] = \frac{1}{4}\pi \ln 3
 \end{aligned}$$

Question 38 (***)



The figure above shows the graph of the curve with equation

$$y = x\sqrt{\sin 2x}, \quad 0 \leq x \leq \frac{\pi}{2}.$$

The shaded region, labelled as R , bounded by the curve and the x axis, is rotated by 360° about the x axis to form a solid of revolution.

Show that the volume of the solid generated is

$$\frac{\pi}{8}(\pi^2 - 4).$$

proof

The handwritten proof shows the following steps:

$$V = \pi \int_0^{\frac{\pi}{2}} (y^2) dx = \pi \int_0^{\frac{\pi}{2}} (x\sqrt{\sin 2x})^2 dx = \pi \int_0^{\frac{\pi}{2}} x^2 \sin 2x dx$$

... DOUBLE INTEGRATION BY PARTS ...

$$\int x^2 \sin 2x dx = \frac{1}{2} x^2 \cos 2x - \int x \cos 2x dx$$

$$= \frac{1}{2} x^2 \cos 2x + \int x \sin 2x dx$$

$$= \frac{1}{2} x^2 \cos 2x + \frac{1}{2} x^2 \sin 2x - \int \sin 2x dx$$

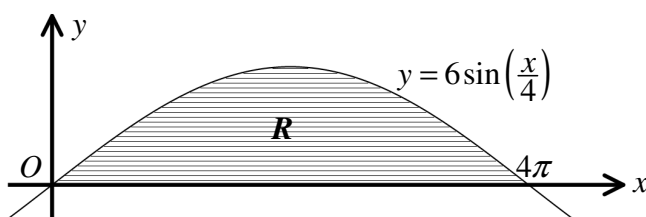
$$= \frac{1}{2} x^2 \cos 2x + \frac{1}{2} x^2 \sin 2x + \frac{1}{4} \cos 2x + C$$

$$\therefore V = \pi \left[\frac{1}{2} x^2 \cos 2x + \frac{1}{2} x^2 \sin 2x + \frac{1}{4} \cos 2x \right]_0^{\frac{\pi}{2}}$$

$$= \pi \left[\left(\frac{1}{2} \left(\frac{\pi}{2} \right)^2 \cos \pi + \left(\frac{\pi}{2} \right)^2 \sin \pi + \frac{1}{4} \cos \pi \right) - \left(0 + 0 + \frac{1}{4} \cos 0 \right) \right]$$

$$= \pi \left[\left(\frac{\pi^2}{8} - \frac{1}{4} \right) - \left(\frac{1}{4} \right) \right] = \pi \left[\frac{\pi^2}{8} - \frac{1}{2} \right] = \frac{\pi}{8} (\pi^2 - 4)$$

Question 39 (****)



The figure below shows the graph of the curve with equation

$$y = 6 \sin\left(\frac{x}{4}\right), \quad 0 \leq x \leq 4\pi.$$

The shaded region R , is bounded by the curve and the x axis.

- a) Determine the area of R .

This region R is rotated through 360° about the x axis to form a solid of revolution.

- b) Show that the volume of the solid generated is $72\pi^2$.

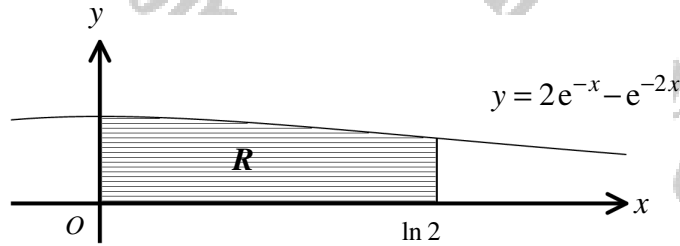
, 48 square units

The handwritten solution shows the following steps:

(a) $A = \int_0^{4\pi} 6 \sin\left(\frac{x}{4}\right) dx = \left[-\frac{24}{1} \cos\left(\frac{x}{4}\right)\right]_0^{4\pi}$
 $= 24 \left[\cos\left(\frac{4\pi}{4}\right) - \cos\left(\frac{0}{4}\right) \right] = 24 [1 - 1] = 0$ (Note: The handwritten work shows a result of 48, which is likely a typo for the area calculation, as the integral of a sine wave over a full period is zero. The correct calculation should be $24 [1 - (-1)] = 48$.)

(b) $V = \pi \int_0^{4\pi} \left(6 \sin\left(\frac{x}{4}\right)\right)^2 dx = \pi \int_0^{4\pi} 36 \sin^2\left(\frac{x}{4}\right) dx$
 $= \pi \int_0^{4\pi} 36 \left[\frac{1}{2} - \frac{1}{2} \cos\left(\frac{x}{2}\right) \right] dx = \pi \int_0^{4\pi} 18 - 18 \cos\left(\frac{x}{2}\right) dx$
 $= \pi \left[18x - 36 \sin\left(\frac{x}{2}\right) \right]_0^{4\pi} = \pi \left[(72\pi - 36 \sin(2\pi)) - (0 - 36 \sin(0)) \right]$
 $= 72\pi^2$

Question 40 (***)



The figure above shows part of the graph of the curve with equation

$$y = 2e^{-x} - e^{-2x} \quad x \in \mathbb{R}.$$

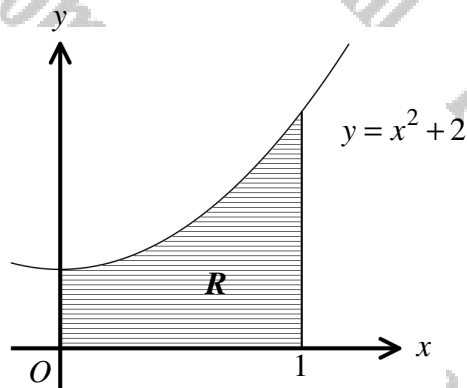
The shaded region R , bounded by the curve, the coordinate axes and the straight line with equation $x = \ln 2$, is rotated through 360° about the x axis to form a solid of revolution.

Show that the volume of the solid generated is exactly $\frac{109}{192}\pi$.

proof

$$\begin{aligned} V &= \pi \int_0^{\ln 2} y^2 dx = \pi \int_0^{\ln 2} (2e^{-x} - e^{-2x})^2 dx = \pi \int_0^{\ln 2} (4e^{-2x} - 4e^{-3x} + e^{-4x}) dx \\ &= \pi \left[-2e^{-2x} + \frac{4}{3}e^{-3x} - \frac{1}{4}e^{-4x} \right]_0^{\ln 2} = \pi \left[2e^{-2 \ln 2} - \frac{4}{3}e^{-3 \ln 2} + \frac{1}{4}e^{-4 \ln 2} \right] \\ &= \pi \left[2 \left(\frac{1}{2} \right)^2 - \frac{4}{3} \left(\frac{1}{2} \right)^3 + \frac{1}{4} \left(\frac{1}{2} \right)^4 \right] = \frac{109}{192} \pi \end{aligned}$$

Question 41 (****)



The figure above shows the graph of the curve with equation

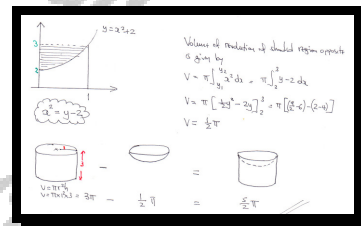
$$y = x^2 + 2.$$

The shaded region R , is bounded by the curve, the coordinate axes and the straight line with equation $x = 1$.

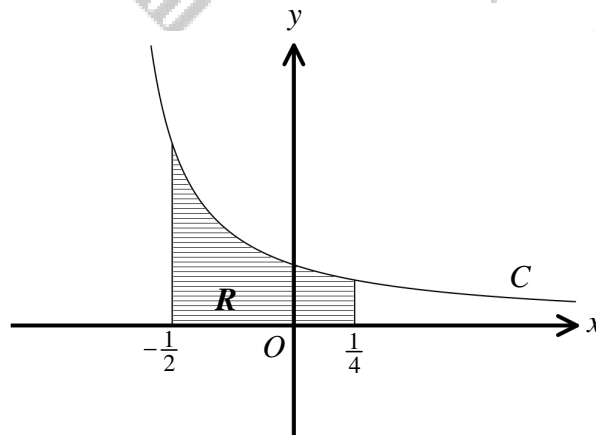
The region R is rotated through 360° about the y axis to form a solid of revolution.

Show that the volume of the solid generated is $\frac{5}{2}\pi$ cubic units.

proof



Question 42 (***)



The figure above shows part the graph of the curve C , with equation

$$y = \frac{3}{2(4x+3)}, \quad x \neq -\frac{3}{4}.$$

The shaded region R , is bounded by the curve, the x axis and the straight lines with equations $x = -\frac{1}{2}$ and $x = \frac{1}{4}$.

- a) Find the exact area of R .

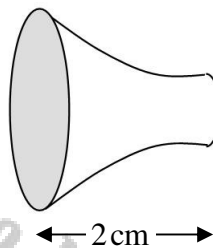
This region R is rotated through 360° about the x axis to form a solid of revolution.

- b) Show that the volume of the solid generated is $\frac{27}{64}\pi$.

[continues overleaf]

[continues from overleaf]

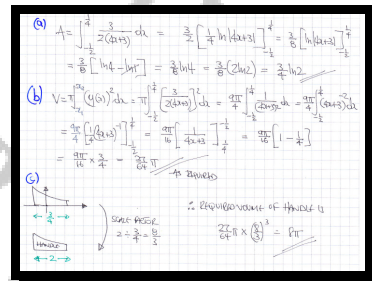
The solid generated in part (b) is used to model a small handle for a drawer.



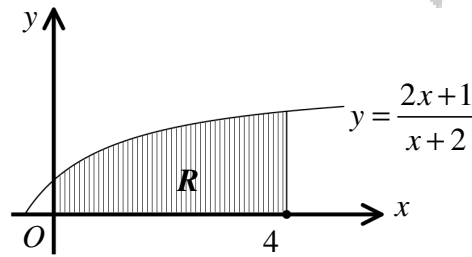
The solid generated in part (b) and the drawer handle are mathematically similar.

- c) Given that the length of the handle is 2 cm, find the exact volume of the handle.

area = $\frac{3}{4} \ln 2$, volume of handle = 8π



Question 43 (****)



The figure above shows part of the curve with equation

$$y = \frac{2x+1}{x+2}, \quad x \neq -2.$$

a) Show that

$$\frac{2x+1}{x+2} = A + \frac{B}{x+2},$$

where A and B are constants to be found.

The shaded region, labelled R , bounded by the curve, the coordinate axes and the straight line with equation $x = 4$ is rotated by 360° about the x axis to form a solid of revolution.

b) Show that the volume of revolution is

$$\pi(19 - 12\ln 3).$$

$$A = 2, B = -3$$

$$\begin{aligned} \text{(a)} \quad y &= \frac{2x+1}{x+2} = \frac{2(x+2)-3}{x+2} = 2 - \frac{3}{x+2} \quad \begin{matrix} A=2 \\ B=-3 \end{matrix} \\ \text{(b)} \quad y^2 &= \left(2 - \frac{3}{x+2}\right)^2 = 4 - \frac{12}{x+2} + \frac{9}{(x+2)^2} = 4 - \frac{12}{x+2} + 9(x+2)^{-2} \\ V &= \pi \int_0^4 y^2 dx = \pi \int_0^4 \left(4 - \frac{12}{x+2} + 9(x+2)^{-2}\right) dx = \pi \left[4x - 12 \ln|x+2| - \frac{9}{x+2}\right]_0^4 \\ &= \pi \left[4(4) - 12 \ln 6 - \frac{9}{6}\right] - \pi \left[4(0) - 12 \ln 2 - \frac{9}{2}\right] \\ &= \pi \left[16 - 12 \ln 6 - \frac{3}{2} + 12 \ln 2 + \frac{9}{2}\right] = \pi \left[19 - 12(\ln 6 - \ln 2)\right] \\ &= \pi \left[19 - 12 \ln 3\right] \quad \text{As required} \end{aligned}$$

Question 44 (****)

The curve C has equation

$$y = xe^x, \quad x \in \mathbb{R}.$$

The region R is bounded by the curve, the x axis and the vertical straight lines with equations $x=1$ and $x=3$.

- a) Explain why R lies entirely above the x axis.

The region R is rotated by 360° in the x axis to form a solid of revolution.

- b) Show that the volume of this solid is

$$\frac{1}{4}\pi e^2(13e^4 - 1).$$

proof

(a) $y = xe^x$ $e^x > 0$ for all x \therefore their product is positive
 $1 < x < 3$

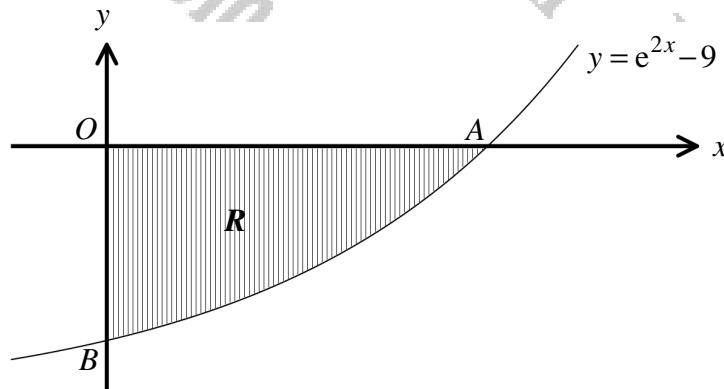
(b) $V = \pi \int_1^3 y^2 dx = \pi \int_1^3 (xe^x)^2 dx$
 $= \pi \int_1^3 x^2 e^{2x} dx$... INTEGRATE BY PARTS

Let $u = x^2$ $v = e^{2x}$
 $u' = 2x$ $v' = 2e^{2x}$

BY PARTS
 $= \frac{1}{2}x^2 e^{2x} - \int 2xe^{2x} dx$
 $= \frac{1}{2}x^2 e^{2x} - \int 2xe^{2x} dx$
 $= \frac{1}{2}x^2 e^{2x} - 2 \int xe^{2x} dx$
 $= \frac{1}{2}x^2 e^{2x} - 2 \left(\frac{1}{2}x^2 e^{2x} - \int 2xe^{2x} dx \right)$

$\therefore V = \pi \left[\frac{1}{2}x^2 e^{2x} - 2 \left(\frac{1}{2}x^2 e^{2x} - \int 2xe^{2x} dx \right) \right]$
 $\therefore V = \pi \left[\frac{1}{2}e^{2x} - 2e^{2x} \right] = \frac{1}{4}\pi e^2(13e^4 - 1)$

Question 45 (****)



The figure above shows part of the curve with equation

$$y = e^{2x} - 9, \quad x \in \mathbb{R}.$$

The curve crosses the coordinate axes at the points A and B. The shaded region R is bounded by the curve and the coordinate axes.

- a) Determine the exact coordinates of A and B.

The region R is rotated by 2π radians about the x axis to form a solid of revolution.

- b) Calculate the volume generated, giving the answer in the form $\pi(p + q \ln 3)$ where p and q are integers.

$(\ln 3, 0)$, $(0, -8)$, $V = \pi(-52 + 81 \ln 3)$

Handwritten solution for part (b):

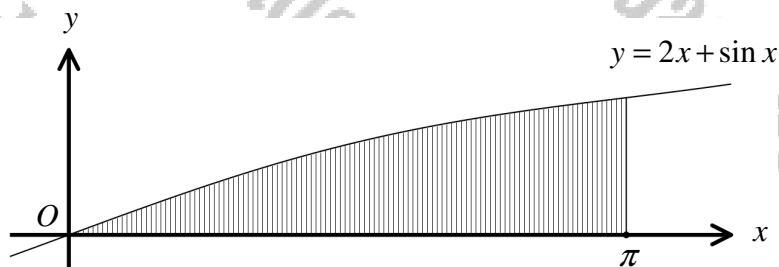
$$\begin{aligned}
 & \text{④ } \begin{matrix} x=0 & y=0 \\ y=8 & x=\ln 3 \\ B(0,-8) & \\ A(\ln 3, 0) & \end{matrix} \quad \begin{matrix} V = \pi \int_{\ln 3}^0 (e^{2x} - 9)^2 dx = \pi \int_{\ln 3}^0 (e^{4x} - 18e^{2x} + 81) dx \\ V = \pi \left[\frac{1}{4} e^{4x} - 9e^{2x} + 81x \right]_{\ln 3}^0 \\ V = \pi \left[\left(\frac{25}{4} - 9 + 81 \ln 3 \right) - \left(\frac{1}{4} - 9 \right) \right] \\ V = \pi [-2 + 81 \ln 3] \checkmark \end{matrix}
 \end{aligned}$$

Question 46 (****)

Show that

a) $\int_0^{\pi} 4x \sin x \, dx = 4\pi.$

b) $\int_0^{\pi} \sin^2 x \, dx = \frac{\pi}{2}.$



The figure above shows part of the curve with equation

$$y = 2x + \sin x.$$

The shaded region bounded by the curve, the x axis and the line $x = \pi$ is rotated by 2π radians about the x axis to form a solid of revolution.

c) Show that the volume of the solid is

$$\frac{1}{6} \pi^2 (8\pi^2 + 27).$$

Q46, proof

$\int_0^{\pi} 4x \sin x \, dx = -4x \cos x - \int -4 \cos x \, dx$
 $= -4x \cos x + \int 4 \cos x \, dx$
 $= [-4x \cos x + 4 \sin x]_0^{\pi}$
 $= [4\pi - 0 - 0] = 4\pi$

$\int_0^{\pi} \sin^2 x \, dx = \int_0^{\pi} \frac{1 - \cos 2x}{2} \, dx = \left[\frac{x}{2} - \frac{1}{4} \sin 2x \right]_0^{\pi} = \left(\frac{\pi}{2} - 0 \right) - 0 = \frac{\pi}{2}$

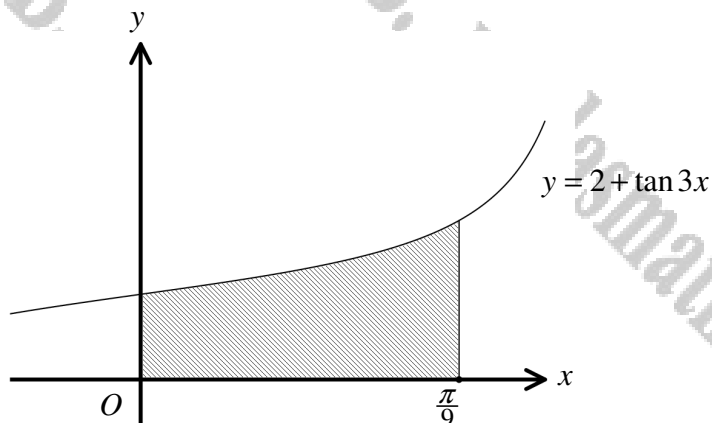
$V = \pi \int_0^{\pi} (2x + \sin x)^2 \, dx = \pi \int_0^{\pi} (4x^2 + 4x \sin x + \sin^2 x) \, dx$
 $= \pi \int_0^{\pi} 4x^2 \, dx + \pi \int_0^{\pi} 4x \sin x \, dx + \pi \int_0^{\pi} \sin^2 x \, dx$
 $= \pi \left[\frac{4}{3} x^3 \right]_0^{\pi} + \pi (4\pi) + \pi \left(\frac{\pi}{2} \right)$
 $= \pi \left[\frac{4}{3} \pi^3 \right] + \pi (4\pi) + \pi \left[\frac{\pi}{2} \right] = \frac{4}{3} \pi^4 + 4\pi^2 + \frac{1}{2} \pi^3$
 $= \frac{1}{6} \pi^2 (8\pi^2 + 27)$

Question 47 (****)

Show that

a) $(2 + \tan 3x)^2 = 3 + 4 \tan 3x + \sec^2 3x$

b) $\int \tan x \, dx = \ln|\sec x| + C$



The figure above shows part of the graph of the curve with equation

$$y = 2 + \tan 3x.$$

The shaded region bounded by the curve the coordinate axes and the line $x = \frac{\pi}{9}$ is rotated by 2π radians about the x axis to form a solid of revolution.

c) Show that the volume of the solid is

$$\frac{\pi}{3} (\pi + 4 \ln 2 + \sqrt{3}).$$

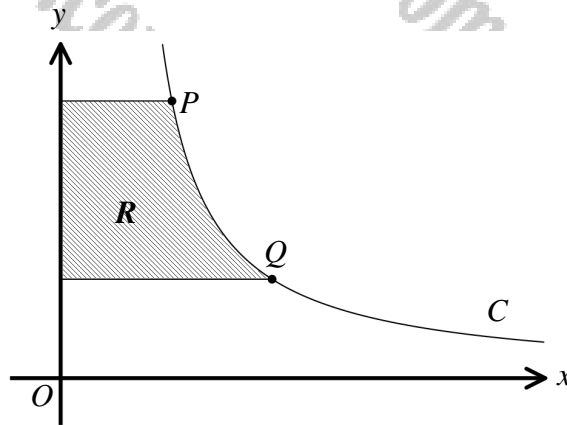
, proof

(a) $(2 + \tan 3x)^2 = 4 + 4 \tan 3x + \tan^2 3x = 4 + 4 \tan 3x + (\sec^2 3x - 1)$
 $= 3 + 4 \tan 3x + \sec^2 3x$

(b) $\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = - \int \frac{-\cos x}{\cos x} \, dx = - \ln|\cos x| + C$
 $= \ln|\sec x| + C = \ln|\sec x| + C$

(c) $V = \pi \int_0^{\pi/9} y^2 \, dx = \pi \int_0^{\pi/9} (2 + \tan 3x)^2 \, dx = \pi \int_0^{\pi/9} (3 + 4 \tan 3x + \sec^2 3x) \, dx$
 $= \pi \left[3x + \frac{4}{3} \ln|\sec 3x| + \frac{1}{3} \tan 3x \right]_0^{\pi/9}$
 $= \pi \left[\left(\frac{\pi}{3} + \frac{4}{3} \ln 2 + \frac{\sqrt{3}}{3} \right) - \left(0 + \frac{4}{3} \ln 1 + 0 \right) \right]$
 $= \frac{\pi}{3} (\pi + 4 \ln 2 + \sqrt{3})$

Question 48 (***)



The figure above shows the graph of the curve C with equation

$$y = \frac{14}{x-2}, \quad x \neq 2.$$

The points P and Q lie on C where $x = 2.5$ and $x = 3.75$ respectively.

The shaded region R is bounded by the curve and two horizontal lines passing through the points P and Q .

R is rotated by 2π radians about the y axis forming a solid of revolution S .

- a) Find the volume of S , giving the answer in the form $\pi(a + b \ln c)$ where a , b and c are constants.

The solid S is used to model a nuclear station cooling tower.

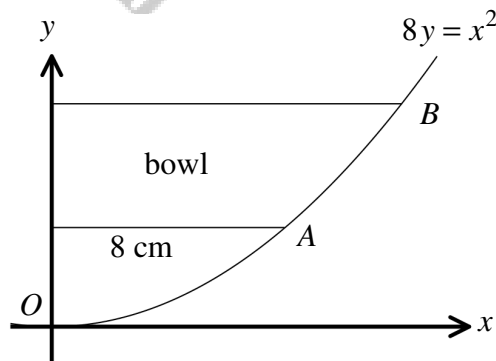
- b) Given that 1 unit on the axes corresponds to 2 metres on the actual tower, show that the cooling tower has an approximate volume of 4200 m^3 .

$$\pi \left(\frac{195}{2} + 56 \ln \left(\frac{7}{2} \right) \right)$$

(a) $y = \frac{14}{x-2}$
 $\int_2^{3.75} \frac{14}{x-2} dx = 14 \ln \left| \frac{3.75-2}{2-2} \right|$
 $\int_2^{3.75} \frac{14}{x-2} dx = 14 \ln \left| \frac{1.75}{0} \right|$
 $\int_2^{3.75} \frac{14}{x-2} dx = 14 \ln \left| \frac{1.75}{0} \right|$
 $\int_2^{3.75} \frac{14}{x-2} dx = 14 \ln \left| \frac{1.75}{0} \right|$

(b) $1 \text{ unit} = 2 \text{ metres}$
 $1 \text{ unit}^2 = 2^2 \text{ m}^2$
 $\therefore V = \pi \left(\frac{195}{2} + 56 \ln \left(\frac{7}{2} \right) \right) \times 2^3 = 4200 \text{ m}^3$

Question 49 (****)



The figure above shows the graph of the curve with equation

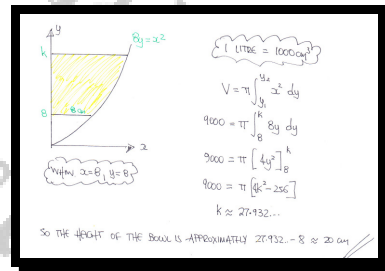
$$8y = x^2, \quad x \geq 0.$$

The points A and B lie on the curve. The curved surface of an open bowl with flat circular base is traced out by the complete revolution of the arc AB about the y axis.

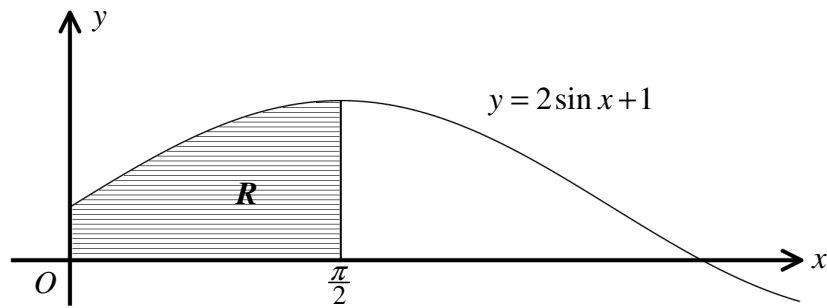
The radius of the flat circular base of the bowl is 8 cm, and its volume is 9 litres.

Find to the nearest cm the height of the bowl.

height ≈ 20 cm



Question 50 (****)



The figure above shows the graph of the curve C with equation

$$y = 2 \sin x + 1, \quad x \in \mathbb{R}.$$

The shaded region R is bounded by the curve, the line $x = \frac{\pi}{2}$ and the x axis.

- a) Find the exact area of R .

The region R is rotated by 2π radians in the x axis forming a solid of revolution S .

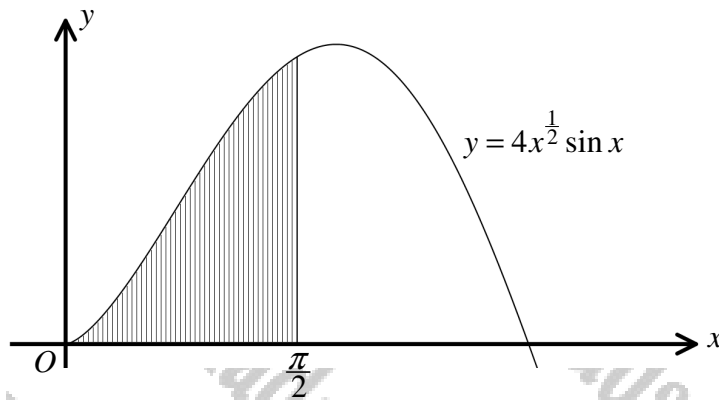
- b) Show that the volume of S is

$$\frac{\pi}{2}(3\pi + 8).$$

$$\text{area} = \frac{1}{2}(\pi + 4)$$

$$\begin{aligned} \text{(a)} \quad \text{Area} &= \int_0^{\frac{\pi}{2}} 2 \sin x + 1 \, dx = [-2 \cos x + x]_0^{\frac{\pi}{2}} = (0 + \frac{\pi}{2}) - (-2 + 0) \\ &= \frac{\pi}{2} + 2 = \frac{1}{2}(\pi + 4) \\ \text{(b)} \quad \text{Volume} &= \pi \int_0^{\frac{\pi}{2}} (2 \sin x + 1)^2 \, dx = \pi \int_0^{\frac{\pi}{2}} (4 \sin^2 x + 4 \sin x + 1) \, dx \\ &= \pi \int_0^{\frac{\pi}{2}} (2 - 2 \cos 2x) + 4 \sin x + 1 \, dx = \pi \int_0^{\frac{\pi}{2}} (3 + 4 \sin x - 2 \cos 2x) \, dx \\ &= \pi \left[3x - 4 \cos x - \sin 2x \right]_0^{\frac{\pi}{2}} = \pi [(3\frac{\pi}{2} - 0 - 0) - (0 - 4 - 0)] \\ &= \pi \left[\frac{3\pi}{2} + 4 \right] = \frac{1}{2} \pi (3\pi + 8) \end{aligned}$$

Question 51 (****)



The figure above shows the graph of the curve with equation

$$y = 4x^{\frac{1}{2}} \sin x.$$

- a) Find the value of $\int_0^{\frac{\pi}{2}} 8x \cos 2x \, dx$.

The shaded region bounded by the curve, the x axis and the straight line with equation $x = \frac{\pi}{2}$ is rotated by 2π radians in the x axis to form a solid of revolution.

- b) Show that the volume of the solid is

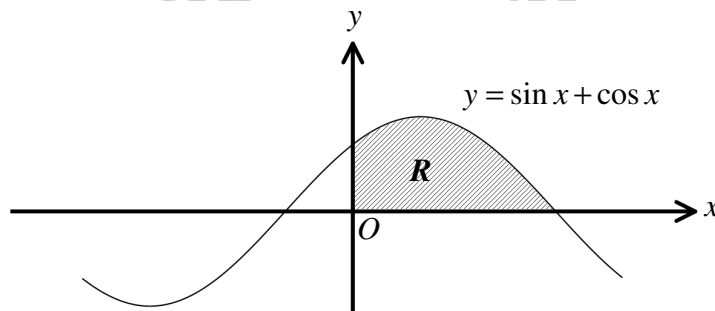
$$\pi(\pi^2 + 4).$$

4

(a) $\int_0^{\frac{\pi}{2}} 8x \cos 2x \, dx = 8 \int_0^{\frac{\pi}{2}} x \cos 2x \, dx$
 $= 8 \left[x \sin 2x - \int \sin 2x \, dx \right]_0^{\frac{\pi}{2}}$
 $= 8 \left[x \sin 2x + \frac{1}{2} \cos 2x \right]_0^{\frac{\pi}{2}}$
 $= 8 \left[\left(\frac{\pi}{2} \sin \pi + \frac{1}{2} \cos \pi \right) - \left(0 \sin 0 + \frac{1}{2} \cos 0 \right) \right]$
 $= 8 \left[\left(0 - \frac{1}{2} \right) - \left(0 + \frac{1}{2} \right) \right] = -4$

(b) $V = \pi \int_0^{\frac{\pi}{2}} (y^2) \, dx$
 $\Rightarrow V = \pi \int_0^{\frac{\pi}{2}} (4x^{\frac{1}{2}} \sin x)^2 \, dx = \pi \int_0^{\frac{\pi}{2}} 16x \sin^2 x \, dx$
 $\Rightarrow V = 16\pi \int_0^{\frac{\pi}{2}} x \sin^2 x \, dx$
 $\Rightarrow V = 16\pi \int_0^{\frac{\pi}{2}} x \left(\frac{1 - \cos 2x}{2} \right) \, dx$
 $\Rightarrow V = 8\pi \int_0^{\frac{\pi}{2}} (x - x \cos 2x) \, dx$
 $\Rightarrow V = 8\pi \left[\frac{x^2}{2} - \left(\frac{x \sin 2x}{2} - \int \sin 2x \, dx \right) \right]_0^{\frac{\pi}{2}}$
 $\Rightarrow V = 8\pi \left[\frac{x^2}{2} - \frac{x \sin 2x}{2} + \frac{1}{4} \cos 2x \right]_0^{\frac{\pi}{2}}$
 $\Rightarrow V = 8\pi \left[\left(\frac{\pi^2}{2} - \frac{\pi \sin \pi}{2} + \frac{1}{4} \cos \pi \right) - \left(0 - 0 + \frac{1}{4} \cos 0 \right) \right]$
 $\Rightarrow V = 8\pi \left[\frac{\pi^2}{2} - 0 - \frac{1}{4} - \frac{1}{4} \right]$
 $\Rightarrow V = 8\pi \left[\frac{\pi^2}{2} - \frac{1}{2} \right]$
 $\Rightarrow V = 4\pi(\pi^2 - 1)$

Question 52 (****)



The figure above shows the graph of the curve with equation

$$y = \sin x + \cos x, \quad -\pi \leq x \leq \pi.$$

The finite region R, shown shaded in the figure, is bounded by the curve and the coordinate axes.

When R is revolved by a full turn in the x axis it traces a solid of volume V.

Show clearly that

$$V = \frac{1}{4}\pi(3\pi + 2).$$

, proof

Handwritten solution showing the derivation of the volume V. It starts with the equation $y = \sin x + \cos x$ and finds the x-intercepts $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$. The volume is calculated using the disk method:

$$V = \pi \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin x + \cos x)^2 dx$$

$$= \pi \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin^2 x + 2\sin x \cos x + \cos^2 x) dx$$

$$= \pi \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (1 + \sin 2x) dx$$

$$= \pi \left[x - \frac{1}{2} \cos 2x \right]_{\frac{\pi}{4}}^{\frac{5\pi}{4}}$$

$$= \pi \left[\left(\frac{5\pi}{4} - 0 \right) - \left(\frac{\pi}{4} - 1 \right) \right]$$

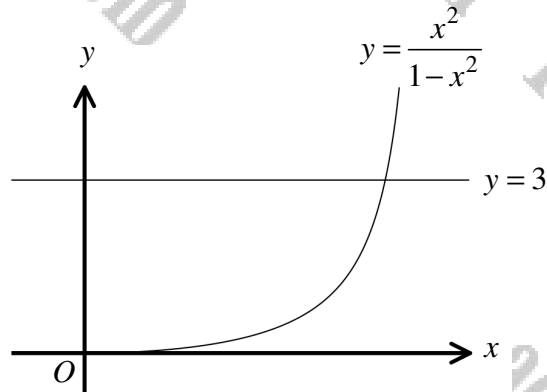
$$= \pi \left(\frac{5\pi}{4} + 1 - \frac{\pi}{4} \right)$$

$$= \pi \left(\frac{4\pi}{4} + 1 \right)$$

$$= \pi(\pi + 1)$$

The final result is $V = \frac{1}{4}\pi(3\pi + 2)$.

Question 53 (***)



The figure above shows part of the graph of the curve with equation $y = \frac{x^2}{1-x^2}$, which passes through the origin O .

The finite area bounded by the curve, the y axis and the straight line with equation $y = 3$, is to be revolved in the y axis by 360° to form a solid of revolution S .

Find an exact value for the volume of S .

, $\pi(3 - \ln 4)$

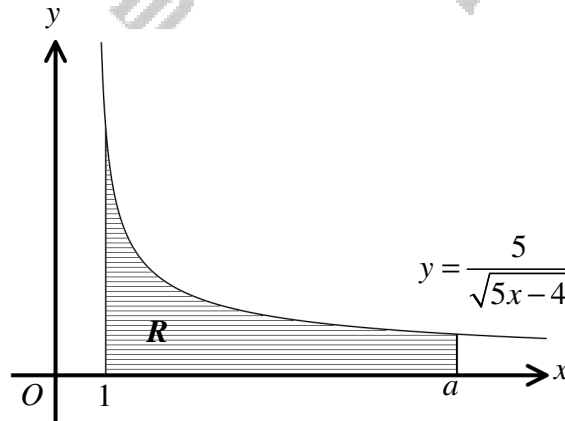
LOOKING AT THE QUESTION

$$\begin{aligned} y &= \frac{x^2}{1-x^2} \\ y - 3 &= -3 \\ y &= x^2 + 3 \\ x^2 &= y - 3 \end{aligned}$$

SETTING UP A VOLUME INTEGRAL ABOUT Y

$$\begin{aligned} \Rightarrow V &= \pi \int_0^3 [x(y)]^2 dy \\ \Rightarrow V &= \pi \int_0^3 \frac{y}{1-y} dy && \text{USE SUBSTITUTION } (x = \sqrt{y}) \\ \Rightarrow V &= \pi \int_0^3 \frac{(y+1) - 1}{1-y} dy \\ \Rightarrow V &= \pi \int_0^3 \left(1 - \frac{1}{1-y} \right) dy \\ \Rightarrow V &= \pi [y - \ln|1-y|]_0^3 \\ \Rightarrow V &= \pi [(3 - \ln 4) - (0 - 0)] \\ \Rightarrow V &= \pi(3 - \ln 4) \end{aligned}$$

Question 54 (****)



The figure above shows part of the graph of the curve C with equation

$$y = \frac{5}{\sqrt{5x-4}}, \quad x > \frac{4}{5}.$$

The shaded region R is bounded by the curve, the vertical straight lines $x=1$ and $x=a$, and the x axis.

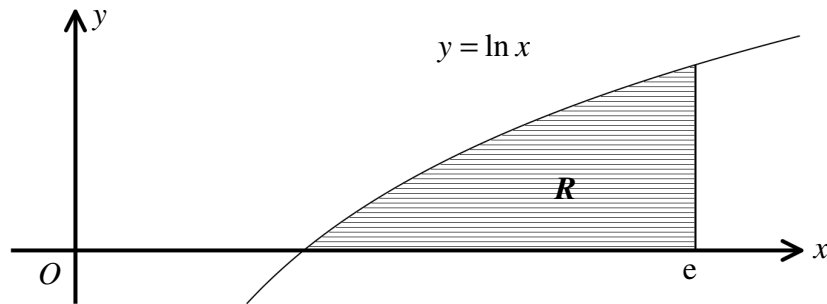
The region R is rotated by 2π radians about the x axis forming a solid of revolution.

Given that the area of R is 10 square units, show that the volume of the solid formed is $10\pi \ln 6$ cubic units.

, proof

Handwritten solution showing the derivation of the volume of the solid of revolution. The solution starts with the area of region R being 10 square units, leading to the equation $\int_1^a \frac{5}{\sqrt{5x-4}} dx = 10$. This is simplified to $\int_1^a \frac{5}{2(5x-4)^{1/2}} dx = 10$, which further simplifies to $\int_1^a \frac{5}{(5x-4)^{1/2}} dx = 10$. The integral is evaluated to $2[(5x-4)^{1/2}]_1^a = 10$, leading to $(5a-4)^{1/2} - 1^2 = 5$, $(5a-4)^{1/2} = 6$, $5a-4 = 36$, $5a = 40$, and $a = 8$. The volume is then calculated as $V = \pi \int_1^8 y^2 dx = \pi \int_1^8 \frac{25}{5x-4} dx = \pi \int_1^8 \frac{25}{5x-4} dx = \pi [5 \ln|5x-4|]_1^8 = 5\pi [5 \ln 36 - 5 \ln 1] = 25\pi \ln 36 = 10\pi \ln 6$, as required.

Question 55 (***)



The figure above shows the graph of

$$y = \ln x, \quad x > 0.$$

The shaded region R is bounded by the curve, the line $x = e$ and the x axis.

R is rotated by 2π radians about the y axis, forming a solid of revolution S .

Show that the volume of S is

$$\frac{1}{2}\pi(e^2 + 1).$$

, proof

LOOK AT THE DIAGRAM

REQUIRED VOLUME IS GIVEN BY

VOLUME = $\pi r^2 h = \pi \times e^2 = \pi e^2$

$V = \int_0^1 \pi [2e^y]^2 dy$

$V = \pi \int_0^1 4e^{2y} dy$

$V = \pi [2e^{2y}]_0^1$

$V = \pi [2e^2 - 2]$

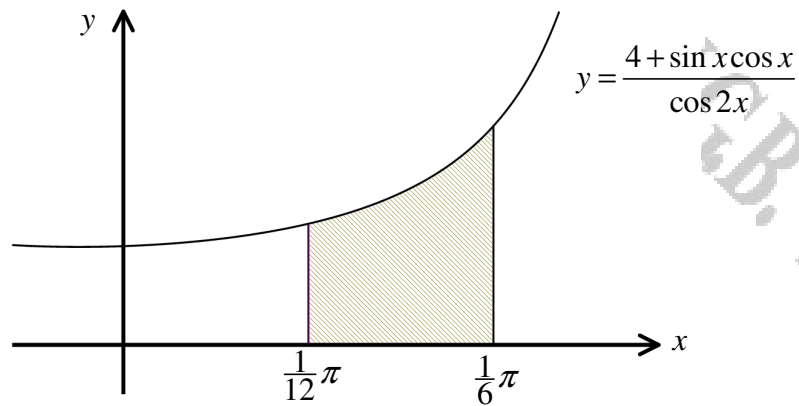
HENCE WE FINALLY HAVE

$V = \pi e^2 - \pi [2e^2 - 2] = \pi [e^2 - 2e^2 + 2]$

$= \pi [2 - e^2]$

$= \frac{1}{2}\pi [e^2 + 1]$ AS REQUIRED

Question 56 (****)



The figure above shows part of the graph of the curve with equation

$$y = \frac{4 + \sin x \cos x}{\cos 2x}$$

The finite area bounded by the curve, the x axis and the straight lines with equations $x = \frac{1}{12}\pi$ and $x = \frac{1}{6}\pi$, shown shaded in the figure, is fully revolved about the x axis, forming a solid, S .

Calculate the volume of S , correct to 3 significant figures.

, $V \approx 34.6$

START BY MANIPULATING THE FUNCTION

$$y = \frac{4 + \sin x \cos x}{\cos 2x} = 4 + \frac{1}{2} \frac{\sin 2x}{\cos 2x}$$

$$= 4 + \frac{1}{2} \tan 2x = 4 \sec 2x + \frac{1}{2} \tan 2x$$

HENCE THE VOLUME OF REVOLUTION IS GIVEN BY

$$\rightarrow V = \pi \int_{\frac{1}{12}\pi}^{\frac{1}{6}\pi} (4 \sec 2x + \frac{1}{2} \tan 2x)^2 dx$$

$$\rightarrow V = \pi \int_{\frac{1}{12}\pi}^{\frac{1}{6}\pi} 16 \sec^2 2x + 4 \sec 2x \tan 2x + \frac{1}{4} \tan^2 2x dx$$

$$\rightarrow V = \pi \int_{\frac{1}{12}\pi}^{\frac{1}{6}\pi} 16 \sec 2x + 4 \sec 2x \tan 2x + \frac{1}{4} (\sec^2 2x - 1) dx$$

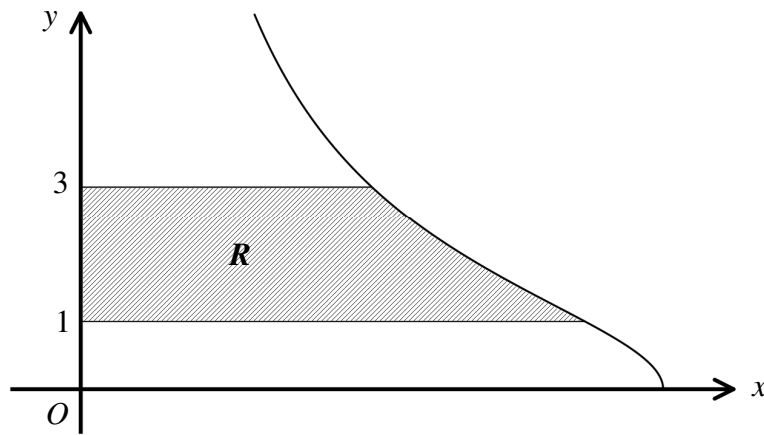
$$\rightarrow V = \pi \int_{\frac{1}{12}\pi}^{\frac{1}{6}\pi} \frac{65}{4} \sec 2x + 4 \sec 2x \tan 2x - \frac{1}{4} dx$$

$$\rightarrow V = \pi \left[\frac{65}{8} \ln |\sec 2x + \tan 2x| + 2 \sec 2x - \frac{1}{4} x \right]_{\frac{1}{12}\pi}^{\frac{1}{6}\pi}$$

$$\Rightarrow V = \pi \left[\left(\frac{65}{8} \sqrt{3} + 4 - \frac{\pi}{24} \right) - \left(\frac{65}{24} \sqrt{3} + \frac{4\sqrt{3}}{3} - \frac{\pi}{48} \right) \right]$$

$$\Rightarrow V = \pi \left[4 + \frac{4\sqrt{3}}{3} \sqrt{3} - \frac{\pi}{48} \right] \approx 34.6$$

Question 57 (****)



The figure above shows the curve with parametric equations

$$x = 2\cos^2 \theta, \quad y = \sqrt{3} \tan \theta, \quad 0 \leq \theta < \frac{\pi}{2}.$$

The finite region R shown shaded in the figure, bounded by the curve, the y axis, and the straight lines with equations $y=1$ and $y=3$.

Use integration in parametric to show that the volume of the solid formed when R is fully revolved about the y axis is $\frac{\pi^2}{\sqrt{3}}$.

, proof

WORKING IN PARAMETRIC

$$V = \pi \int_{y_1}^{y_2} [x(y)]^2 dy = \pi \int_{y_1}^{y_2} [2\cos^2 \theta]^2 \frac{dy}{d\theta} d\theta$$

TRANSFORMING THE INTEGRAL

$y=1$	$y=3$
$\sqrt{3} \tan \theta = 1$	$\sqrt{3} \tan \theta = 3$
$\tan \theta = \frac{1}{\sqrt{3}}$	$\tan \theta = \sqrt{3}$
$\theta = \frac{\pi}{6}$	$\theta = \frac{\pi}{3}$

THUS WE HAVE

$$V = \pi \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (2\cos^2 \theta)^2 (\sqrt{3} \sec^2 \theta) d\theta = 4\sqrt{3}\pi \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos^4 \theta \sec^2 \theta d\theta$$

$$= 4\sqrt{3}\pi \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos^2 \theta d\theta = 4\sqrt{3}\pi \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left(\frac{1}{2} + \frac{1}{2} \cos 2\theta \right) d\theta$$

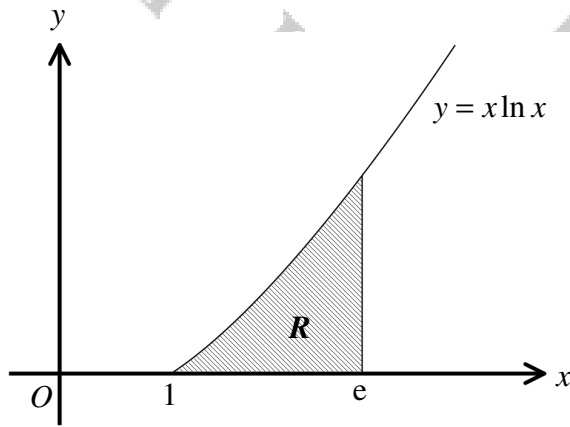
$$= 4\sqrt{3}\pi \left[\frac{\theta}{2} + \frac{1}{4} \sin 2\theta \right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= 4\sqrt{3}\pi \times \frac{\pi}{6}$$

$$= \frac{4\sqrt{3}\pi^2}{3} = \frac{4\sqrt{3} \times \sqrt{3}}{3} = \frac{4 \times 3}{3} = \frac{4\pi^2}{3}$$

As required

Question 58 (***)



The figure above shows the graph of the curve C with equation $y = x \ln x, x \geq 1$.

The shaded region R is bounded by the curve, the x axis and the vertical line $x = e$.

The region R is rotated by 2π radians in the x axis forming a solid of revolution S .

Find an exact value for the volume of S .

, $\frac{\pi}{27}(5e^3 - 2)$

USE THE STANDARD RESULT FOR VOLUME OF REVOLUTION IN 2

$$\Rightarrow V = \pi \int_1^e (x \ln x)^2 dx = \pi \int_1^e (x \ln x)^2 dx = \pi \int_1^e x^2 (\ln x)^2 dx$$

CONTINUE BY INTEGRATION BY PARTS & (WORKING IN 4 UNITS)

$$\int x^2 (\ln x)^2 dx = \frac{1}{3} x^3 (\ln x)^2 - \int \frac{2}{3} x^2 \ln x dx \quad \frac{(x \ln x)^2}{\frac{1}{3} x^3} \quad \frac{2x \ln x \cdot \frac{1}{3}}{x^2}$$

BY PARTS AGAIN ON THIS INTEGRAL

$$\dots = \frac{1}{3} x^3 (\ln x)^2 - \left[\frac{2}{3} x^2 \ln x - \int \frac{2}{3} x dx \right] \quad \frac{\ln x}{\frac{1}{3} x^2} \quad \frac{\frac{1}{3}}{\frac{2}{3} x^2}$$

$$\dots = \frac{1}{3} x^3 (\ln x)^2 - \frac{2}{3} x^2 \ln x + \frac{2}{9} x^2 + C$$

RETURNING TO THE ORIGINAL

$$V = \pi \left[\frac{1}{3} x^3 (\ln x)^2 - \frac{2}{3} x^2 \ln x + \frac{2}{9} x^2 \right]_1^e$$

$$V = \pi \left[\left(\frac{1}{3} e^3 - \frac{2}{3} e^2 + \frac{2}{9} e^2 \right) - \left(0 - 0 + \frac{2}{9} \right) \right]$$

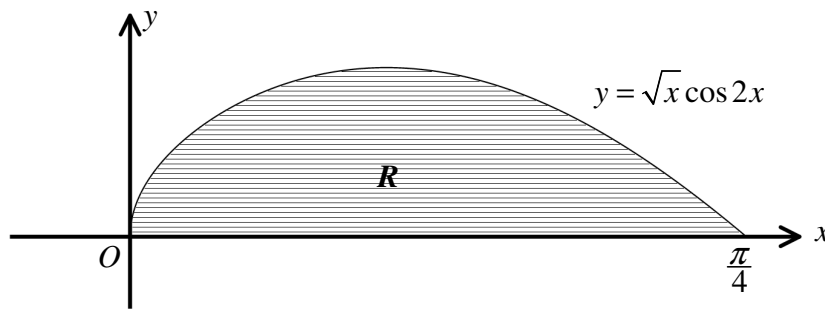
$$V = \pi \left[\frac{5e^3}{3} - \frac{2}{9} \right]$$

$$V = \frac{\pi}{27} (5e^3 - 2)$$

Question 59 (****+)

$$f(x) = \frac{1}{8}(4x + \sin 4x), \quad x \in \mathbb{R}, \quad 0 \leq x \leq \frac{\pi}{4}.$$

a) Show that $f'(x) = \cos^2 2x$.



The figure above shows part of the graph of a curve C with equation

$$y = \sqrt{x} \cos 2x, \quad x > 0.$$

The curve meets the x axis at the origin and at the point where $x = \frac{\pi}{4}$.

The shaded region R is bounded by the curve and the x axis. The region R is rotated by 2π radians about the x axis, forming a solid of revolution S .

b) Show further that the volume of S is

$$\frac{\pi}{64}(\pi^2 - 4).$$

, proof

(c)

$$\begin{aligned} f(x) &= \frac{1}{8}(4x + \sin 4x) \\ f'(x) &= \frac{1}{8}(4 + 4\cos 4x) \\ f'(x) &= \frac{1}{2} + \frac{1}{2}\cos 4x \\ f'(x) &= \cos^2 2x \end{aligned}$$

$\cos 2A = 2\cos^2 A - 1$
 $\cos 4A = 2\cos^2 2A - 1$
 $1 + \cos 4A = 2\cos^2 2A$
 $\frac{1}{2} + \frac{1}{2}\cos 4A = \cos^2 2A$

(b)

$$V = \pi \int_0^{\frac{\pi}{4}} (y(x))^2 dx = \pi \int_0^{\frac{\pi}{4}} (\sqrt{x} \cos 2x)^2 dx = \pi \int_0^{\frac{\pi}{4}} x \cos^2 2x dx$$

LOWERS π & LIMITS

$$\int x \cos^2 2x dx \dots \text{ by parts} \dots$$

x	$\cos^2 2x$
$\frac{1}{2}x^2 + \frac{1}{2}\sin 4x$	$-\int \frac{1}{2}x + \frac{1}{2}\sin 4x dx$

$$= \frac{1}{2}x^2 + \frac{1}{2}\sin 4x - \int \frac{1}{2}x + \frac{1}{2}\sin 4x dx$$

$$= \frac{1}{2}x^2 + \frac{1}{2}\sin 4x - \left(\frac{1}{4}x^2 - \frac{1}{8}\cos 4x \right) + C$$

$$= \frac{1}{4}x^2 + \frac{1}{2}\sin 4x - \frac{1}{4}x^2 + \frac{1}{8}\cos 4x + C$$

$$= \frac{1}{2}\sin 4x + \frac{1}{8}\cos 4x + C$$

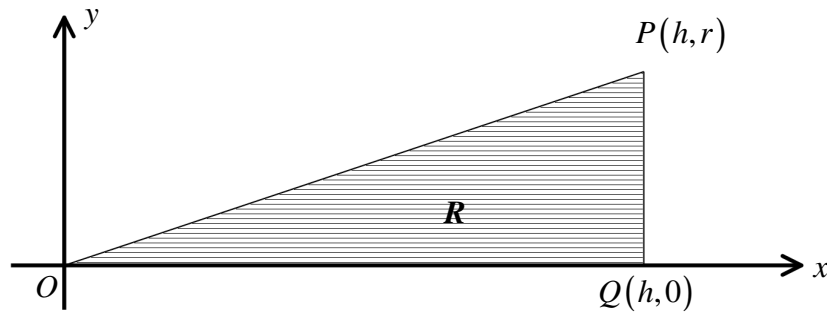
$$\therefore V = \pi \left[\frac{1}{2}\sin 4x + \frac{1}{8}\cos 4x \right]_0^{\frac{\pi}{4}}$$

$$V = \pi \left[\left(\frac{\pi^2}{4} + 0 - \frac{\pi^2}{4} \right) - \left(0 + 0 + \frac{1}{8} \right) \right]$$

$$V = \pi \left[\frac{\pi^2}{4} - \frac{1}{8} \right]$$

$$V = \frac{\pi}{64}(\pi^2 - 4)$$

Question 60 (****+)



The figure above shows the straight line segment OP , joining the origin to the point $P(h, r)$, where h and r are positive coordinates.

The point $Q(h, 0)$ lies on the x axis.

The shaded region R is bounded by the straight line segments OP , PQ and OQ .

The region R is rotated by 2π radians in the x axis to form a solid cone of height h and radius r .

Show by integration that the volume of the cone V is given by

$$V = \frac{1}{3} \pi r^2 h.$$

, proof

\bullet Gradient of $OP = \frac{r}{h}$
 \bullet $y = \frac{r}{h}x = (\text{slope} \times x)$

$$V = \pi \int_0^h (y(x))^2 dx = \pi \int_0^h \left(\frac{r}{h}x\right)^2 dx = \pi \int_0^h \frac{r^2}{h^2} x^2 dx$$

$$= \frac{\pi r^2}{h^2} \int_0^h x^2 dx = \frac{\pi r^2}{h^2} \left[\frac{1}{3}x^3\right]_0^h$$

$$= \frac{\pi r^2}{h^2} \left[\frac{1}{3}h^3 - 0\right] = \frac{1}{3} \pi r^2 h$$

Question 61 (***)

A finite region R is defined by the inequalities

$$y^2 \leq 4ax, \quad 0 \leq x \leq a, \quad y \geq 0,$$

where a is a positive constant.

The region R is rotated by 2π radians in the y axis forming a solid of revolution.

Determine, in terms of π and a , the exact volume of this solid.

, $\frac{8}{5}\pi a^3$

METHOD A (SPINDLE DISC METHOD)

Diagram shows the region R bounded by $y = \sqrt{4ax}$, $x = a$, and $y = 0$. The solid is formed by rotating this region around the y -axis. The volume is found by subtraction: a cylinder of radius a and height $2a$ minus a paraboloid of revolution.

$(\pi a^2)(2a) = 2\pi a^3$
 $\pi \int_0^a (2\sqrt{x})^2 dx = \pi \int_0^a 4x dx = 4\pi \left[\frac{x^2}{2} \right]_0^a = 2\pi a^2$
 $\therefore \text{Required Area} = 2\pi a^3 - 2\pi a^2 = \frac{8}{5}\pi a^3$

METHOD B (BY THE TUBE METHOD)

Diagram shows a vertical tube of width dy at position x . The height of the tube is $y = \sqrt{4ax}$. The volume element is $dV = \pi y^2 dx = \pi (4ax) dx = 4\pi ax dx$.

$dV = \pi (y^2) dy = \pi (4ax) dy$
 $dV = \pi (2\sqrt{x})^2 dx = 4\pi x dx$
 $dV = 2\pi y dy$

SHELL METHOD

Diagram shows a shell of radius x and height $y = \sqrt{4ax}$. The volume element is $dV = 2\pi x y dy = 2\pi x \sqrt{4ax} dy = 4\pi x^{3/2} a^{1/2} dy$.

$V = \int_0^a 2\pi x y dy$
 TRACING UNITS WE OBTAIN
 $\Rightarrow V = \int_0^a 2\pi x (2\sqrt{ax}) dx$
 $\Rightarrow V = 4\pi a^{1/2} \int_0^a x^{3/2} dx$
 $\Rightarrow V = 4\pi a^{1/2} \left[\frac{2}{5} x^{5/2} \right]_0^a$
 $\Rightarrow V = 4\pi a^{1/2} \times \frac{2}{5} a^{5/2}$
 $\Rightarrow V = \frac{8}{5}\pi a^3$

Question 62 (***)

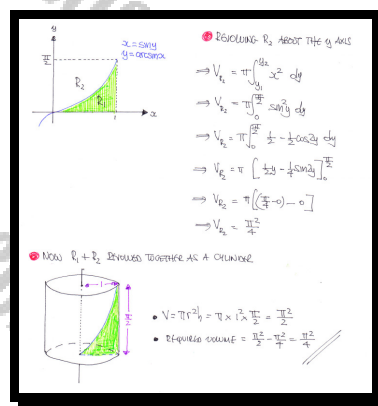
The finite region R is defined by the inequalities

$$y \leq \arcsin x, \quad x \leq 1, \quad y \geq 0.$$

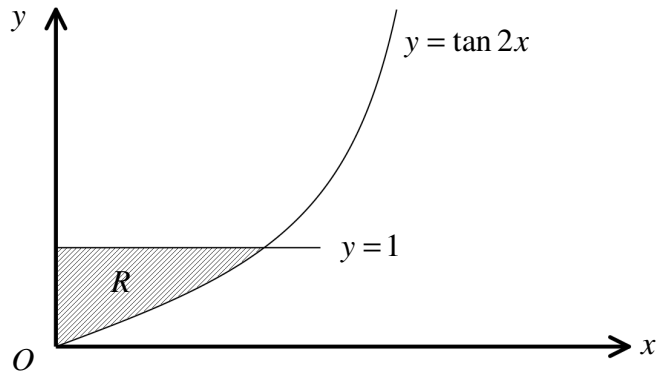
The region R is rotated by 2π radians in the y axis forming a solid of revolution.

Determine the exact volume of this solid.

, $\frac{1}{4}\pi^2$



Question 63 (***)



The figure above shows the graph of the curve with equation

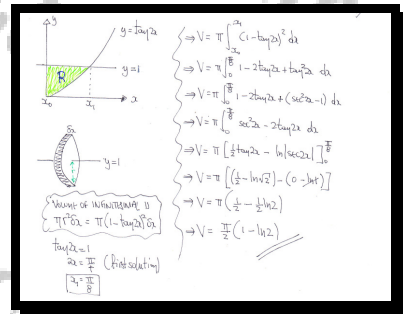
$$y = \tan 2x, \quad 0 \leq x \leq \frac{\pi}{4}.$$

The finite region R is bounded by the curve, the y axis and the horizontal line with equation $y = 1$.

The region R is rotated by 2π radians about the straight line with equation $y = 1$ forming a solid of revolution.

Determine an exact volume for this solid.

$$\frac{\pi}{2}(1 - \ln 2)$$



Question 64 (****+)A curve C has equation

$$y = e^{1 - \left(\frac{x}{e}\right)^2}, \quad x \in \mathbb{R},$$

The finite region bounded by C , the y axis and straight line with equation $y=1$, is revolved by 2π radians about the y axis, forming a solid of revolution.

Find an exact simplified value for the volume of this solid.

$$\boxed{}, \quad \boxed{V = \pi e^2 (e - 2)}$$

• AS THE REVOLUTION IS IN THE y AXIS, START BY REPRESENTING THE REGION
 $\Rightarrow y = e^{1 - \frac{x^2}{e^2}}$
 $\Rightarrow \ln y = 1 - \frac{x^2}{e^2}$
 $\Rightarrow \frac{x^2}{e^2} = 1 - \ln y$
 $\Rightarrow x^2 = e^2(1 - \ln y)$

• NEXT THE VOLUME OF REVOLUTION ABOUT y IS GIVEN BY
 $\Rightarrow V = \pi \int_0^1 (x(u))^2 dy$
 $\Rightarrow V = \pi \int_0^1 e^2(1 - \ln y) dy$

• NEXT NOTE THAT $\int \ln x dx = x \ln|x| - x + C$, EITHER BY COMMON KNOWLEDGE, OR INTEGRATION BY PARTS
 $\Rightarrow V = \pi e^2 \int_0^1 (1 - \ln y) dy$
 $\Rightarrow V = \pi e^2 \left[y - (y \ln|y| - y) \right]_0^1$
 $\Rightarrow V = \pi e^2 \left[2y - y \ln y \right]_0^1$
 $\Rightarrow V = \pi e^2 \left[(2e - e \ln e) - (2 - \ln 1) \right]$
 $\Rightarrow V = \pi e^2 (2e - e - 2)$
 $\Rightarrow V = \pi e^2 (e - 2)$

Question 65 (****+)

A curve has equation

$$y = \ln(4-x), \quad x \in \mathbb{R}, \quad x \neq 4.$$

The finite region bounded by the curve, the x axis and the straight line with equation $x = 2$, is revolved by 2π radians in the y axis.

Find the exact volume of the solid formed.

SP1, $V = \frac{1}{2}\pi(24\ln 2 - 13)$

● SKETCH BY SKETCHING THE GRAPH

$y = \ln x$ $y = \ln(x+4)$ $y = \ln(4-x)$

● IDENTIFY THE REGION TO BE REVOLVED

REVOLVE IN THE y -AXIS THE REGION OPPOSITE TO $y=0$ AND $y=0$ AND $x=2$

$y = \ln(4-x)$
 $\ln 2 = 4-x$
 $x = 4 - e^{\ln 2}$
 $x = 4 - 2e^{\ln 2}$

● $V = \pi \int_{y_1}^{y_2} [r(y)]^2 dy$
 $\Rightarrow V = \pi \int_0^{\ln 2} [4 - e^y]^2 dy$
 $\Rightarrow V = \pi \int_0^{\ln 2} [16 - 8e^y + \frac{1}{2}e^{2y}] dy$
 $\Rightarrow V = \pi [16y - 8e^y + \frac{1}{4}e^{2y}]_0^{\ln 2}$
 $\Rightarrow V = \pi [16\ln 2 - 6 + 2 + 8 - \frac{1}{2}]$
 $\Rightarrow V = \pi [16\ln 2 - \frac{13}{2}]$

● VOLUME OF REVOLUTION OF THE REGION OPPOSITE TO $y=0$ AND $x=2$

$V = \pi r^2 h$
 $V = \pi x^2 \times \ln 2$
 $V = 4\pi \ln 2$

● EXPAND ABOUT $x=2$ GIVEN BY

$V = \pi [4\ln 2 - \frac{13}{2}] - 4\pi \ln 2$
 $V = 16\pi \ln 2 - \frac{13\pi}{2} - 4\pi \ln 2$
 $V = 12\pi \ln 2 - \frac{13\pi}{2}$
 $V = \frac{\pi}{2} [24\ln 2 - 13]$

Question 66 (****)

The finite region R is by the coordinate axes and the curve with equation

$$y = \arccos x, \quad -1 \leq x \leq 1.$$

The region R is rotated by 2π radians in the x axis forming a solid of revolution.

Determine the exact volume of this solid.

, $\pi^2 - 2\pi$

$$V_{\text{solid}} = \pi \int_{-1}^1 (y(x))^2 dx$$

$$\Rightarrow V = \pi \int_{-1}^1 (\arccos x)^2 dx$$

By substitution

$$\Rightarrow V = \pi \int_{\frac{\pi}{2}}^0 \theta^2 (-\sin \theta d\theta)$$

$$\Rightarrow V = \pi \int_0^{\frac{\pi}{2}} \theta^2 \sin \theta d\theta$$

Now by PARTS (divide π arccos)

$$\Rightarrow \frac{1}{\pi} V = \left[-\theta^2 \cos \theta \right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} 2\theta \cos \theta d\theta$$

By PARTS again

$$\Rightarrow \frac{1}{\pi} V = \left[2\theta \sin \theta \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 2 \sin \theta d\theta$$

$$\Rightarrow \frac{1}{\pi} V = (\pi - 0) + \left[2 \cos \theta \right]_0^{\frac{\pi}{2}}$$

$$\Rightarrow \frac{1}{\pi} V = \pi + (0 - 2)$$

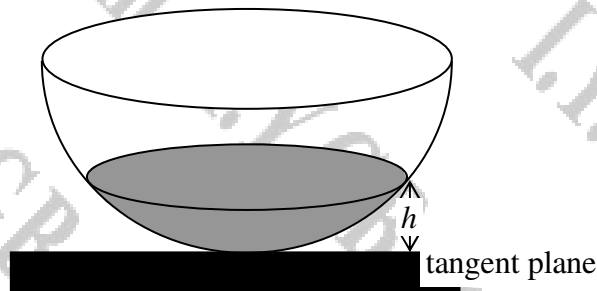
$$\Rightarrow V = \pi(\pi - 2)$$

$\theta = \arccos x$	$\theta = \frac{\pi}{2}$
$\cos \theta = x$	$x = 0 \rightarrow \theta = \frac{\pi}{2}$
$dx = -\sin \theta d\theta$	$x = 1 \rightarrow \theta = 0$

θ^2	2θ
$-\cos \theta$	$-\sin \theta$

2θ	2
$\sin \theta$	$\cos \theta$

Question 67 (*****)



The figure above shows a hemispherical bowl of radius r containing water to a height h . The water in the bowl is in the shape of a minor spherical segment.

It is required to find a formula for the volume of a minor spherical segment as a function of the radius r and the distance of its plane face from the tangent plane, h .

The circle with equation

$$x^2 + y^2 = r^2, \quad x \geq 0$$

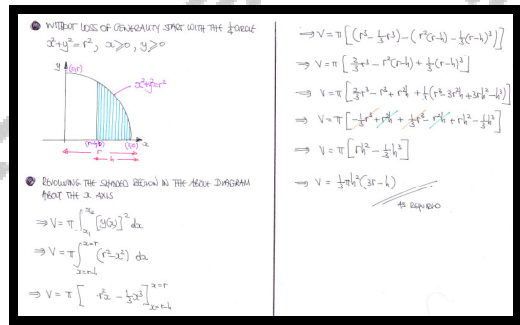
is to be used to find a formula for the volume of a minor spherical segment.

Show by integration that the volume V of the minor spherical segment is given by

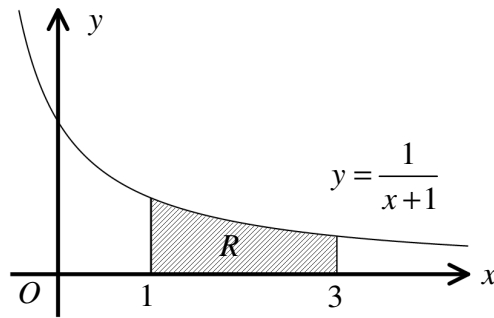
$$V = \frac{1}{3} \pi h^2 (3r - h),$$

where r is the radius of the sphere or hemisphere and h is the distance of its plane face from the tangent plane.

, proof



Question 68 (****)



The figure above shows the graph of the curve with equation

$$y = \frac{1}{x+1}, \quad x \in \mathbb{R}, \quad x > -1.$$

The finite region R is bounded by the curve, the x axis and the lines with equations $x=1$ and $x=3$.

Determine the exact volume of the solid formed when the region R is revolved by 2π radians about ...

- a) ... the y axis.
- b) ... the straight line with equation $x=3$.

, $\pi(4 - \ln 4)$, $4\pi(-1 + \ln 4)$

(a)

$A(0,1)$
 $B(1, \frac{1}{2})$
 $C(3, \frac{1}{4})$

$V = \pi r^2 h = \pi \times 3^2 \times \frac{1}{4} = \frac{9}{4}\pi$

$V = \pi \int_0^1 (y - \frac{1}{y})^2 dy = \pi \int_0^1 (y^2 - \frac{2}{y} + \frac{1}{y^2}) dy$
 $V = \pi [\frac{1}{3}y^3 - 2\ln y - \frac{1}{y}]_0^1$
 $V = \pi [\frac{1}{3} - 2\ln 1 - 1]$
 $V = \pi [\frac{1}{3} - 1]$
 $V = \pi [\frac{2}{3} - \ln 4]$

$V = \pi r^2 h = \pi \times 1^2 \times \frac{1}{2} = \frac{1}{2}\pi$

\therefore Required volume is $\frac{9}{4}\pi + \pi [\frac{2}{3} - \ln 4] - \frac{1}{2}\pi$
 $= 4\pi - \pi \ln 4$
 $= \pi(4 - \ln 4)$

(b)

$V = \pi \int_0^1 (3 - y)^2 dy = \pi \int_0^1 (9 - 6y + y^2) dy = \pi [9y - 3y^2 + \frac{1}{3}y^3]_0^1 = \pi [9 - 3 + \frac{1}{3}] = \pi [6 + \frac{1}{3}] = \frac{19}{3}\pi$

$V = \pi \int_0^1 (3 - \frac{1}{y})^2 dy = \pi \int_0^1 (9 - \frac{6}{y} + \frac{1}{y^2}) dy = \pi [9y - 6\ln y - \frac{1}{y}]_0^1 = \pi [9 - 6\ln 1 - 1] = 8\pi$

\therefore Required volume is $\frac{19}{3}\pi - 8\pi = \frac{19}{3}\pi - \frac{24}{3}\pi = -\frac{5}{3}\pi$

$V = \pi \int_0^1 (3 - \frac{1}{y})^2 dy = \pi [9y - 6\ln y - \frac{1}{y}]_0^1 = \pi [9 - 6\ln 1 - 1] = 8\pi$

$V = \pi \int_0^1 (3 - \frac{1}{y})^2 dy = \pi [9y - 6\ln y - \frac{1}{y}]_0^1 = \pi [9 - 6\ln 1 - 1] = 8\pi$

$V = \pi \int_0^1 (3 - \frac{1}{y})^2 dy = \pi [9y - 6\ln y - \frac{1}{y}]_0^1 = \pi [9 - 6\ln 1 - 1] = 8\pi$

Question 69 (****)

The finite region R is bounded by the curve with equation

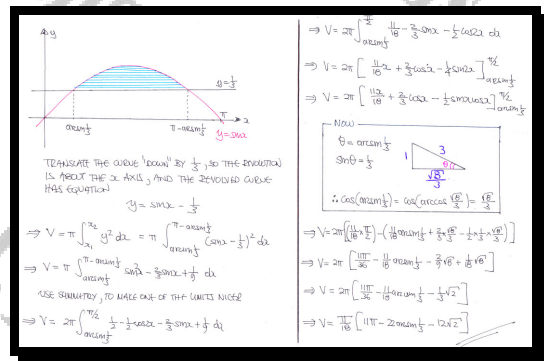
$$y = \sin x, \quad 0 \leq x \leq \pi,$$

and the straight line with equation $y = \frac{1}{3}$.

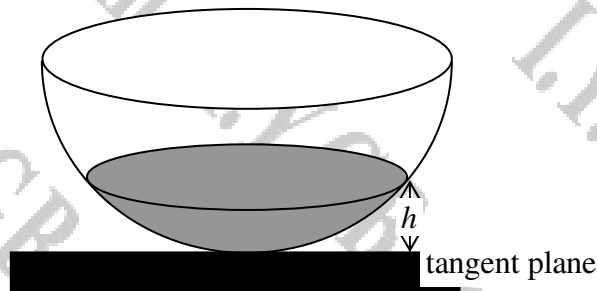
The region R is rotated by 2π radians in the straight line with equation $y = \frac{1}{3}$ forming a solid of revolution.

Determine the exact volume of this solid.

$$\boxed{V = \frac{\pi}{18} \left[11\pi - 22 \arcsin\left(\frac{1}{3}\right) - 12\sqrt{2} \right]}$$



Question 70 (****)



The figure above shows a hemispherical bowl of radius r containing water to a height h . The water in the bowl is in the shape of a minor spherical segment. It is required to find a formula for the volume of a minor spherical segment as a function of the radius r and the distance of its plane face from the tangent plane, h .

Show by integration that the volume V of the minor spherical segment is given by

$$V = \frac{1}{3} \pi h^2 (3r - h),$$

where r is the radius of the sphere or hemisphere and h is the distance of its plane face from the tangent plane.

, proof

$x^2 + y^2 = r^2, \quad x > 0, y > 0$
 $y = \sqrt{r^2 - x^2}$
 $V = \pi \int_{r-h}^r (\sqrt{r^2 - x^2})^2 dx$
 $V = \pi \int_{r-h}^r (r^2 - x^2) dx$
 $V = \pi \left[r^2 x - \frac{1}{3} x^3 \right]_{r-h}^r$
 $V = \pi \left[(r^3 - \frac{1}{3} r^3) - (r^2(r-h) - \frac{1}{3}(r-h)^3) \right]$
 $V = \pi \left[\frac{2}{3} r^3 - r^2(r-h) + \frac{1}{3}(r-h)^3 \right]$
 $V = \pi \left[\frac{2}{3} r^3 - r^3 + r^2 h + \frac{1}{3}(r^3 - 3r^2 h + 3r h^2 - h^3) \right]$
 $V = \pi \left[r^2 h - \frac{1}{3} h^3 \right]$
 $V = \frac{1}{3} \pi h^2 (3r - h)$
 as required.

Question 71 (*****)

A curve has equation

$$y = \frac{8}{x^2 - 4x + 8}, \quad x \in \mathbb{R}.$$

The finite region R is bounded by the curve, the y axis and the tangent to the curve at the stationary point of the curve.

Determine, in simplified exact form, the volume of the solid formed when R is fully revolved about the y axis.

, $V = 4\pi[2 - \pi + 2\ln 2]$

• START WITH A QUICK CHECK OF $y = \frac{8}{x^2 - 4x + 8} = \frac{8}{(x-2)^2 + 4}$

- As $x \rightarrow \pm\infty$ $y \rightarrow 0$ (HORIZONTAL ASYMPTOTE)
- $y > 0$ for all x
- $x=2$ IS A LINE OF SYMMETRY
- "HIGHER" VALUE of y occurs when $x=2$

• EVALUATE THE EQUATION FOR x^2

$$\Rightarrow y = \frac{8}{(x-2)^2 + 4}$$

$$\Rightarrow (x-2)^2 + 4 = \frac{8}{y}$$

$$\Rightarrow (x-2)^2 = \frac{8}{y} - 4$$

$$\Rightarrow x-2 = \pm \sqrt{\frac{8}{y} - 4}$$

$$\Rightarrow x = 2 \pm \sqrt{\frac{8}{y} - 4}$$

$$\Rightarrow x^2 = 4 \pm 8\sqrt{\frac{2}{y} - 1} + 4\left(\frac{2}{y} - 1\right)$$

$$\Rightarrow x^2 = 4 \pm 8\sqrt{\frac{2}{y} - 1} + \frac{8}{y} - 4$$

• NOW THE COORDINATES OF THE STATIONARY POINT

$$\Rightarrow V = \pi \int_{y_1}^{y_2} [x(y)]^2 dy = \pi \int_{\frac{1}{2}}^2 \left(\frac{8}{y} - 8\sqrt{\frac{2}{y} - 1}\right)^2 dy$$

$$\Rightarrow V = 8\pi \int_{\frac{1}{2}}^2 \left(\frac{1}{y} - \sqrt{\frac{2}{y} - 1}\right)^2 dy$$

$$\Rightarrow V = 8\pi \left[\ln|y| \right]_{\frac{1}{2}}^2 - 8\pi \int_{\frac{1}{2}}^2 \sqrt{\frac{2}{y} - 1} dy$$

$$\Rightarrow V = 8\pi (\ln 2 - \ln \frac{1}{2}) - 8\pi \int_{\frac{1}{2}}^2 \sqrt{\frac{2}{y} - 1} dy$$

$$y = 2\sin^2 \theta$$

$$\frac{dy}{d\theta} = 4\sin\theta \cos\theta$$

$$\sin^2 \theta = \frac{y}{2}$$

$$\theta = \arcsin\left(\sqrt{\frac{y}{2}}\right)$$

$$y = \frac{1}{2} \rightarrow \theta = \frac{\pi}{4}$$

$$y = 2 \rightarrow \theta = \frac{\pi}{2}$$

$$\Rightarrow V = 8\pi (\ln 2 - \ln \frac{1}{2}) - 8\pi \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sqrt{\frac{2 - 2\sin^2 \theta}{2\sin^2 \theta}} (4\sin\theta \cos\theta) d\theta$$

$$\Rightarrow V = 8\pi (\ln 2 - \ln \frac{1}{2}) - 8\pi \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1 - \sin^2 \theta}{\sin \theta} (4\sin\theta \cos\theta) d\theta$$

$$\Rightarrow V = 8\pi (\ln 2 - \ln \frac{1}{2}) - 8\pi \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos^2 \theta}{\sin \theta} (4\sin\theta \cos\theta) d\theta$$

$$= 8\pi \ln 2 - 8\pi \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 4\cos^2 \theta d\theta$$

• BY TRIGONOMETRIC IDENTITIES WE HAVE

$$\Rightarrow V = 8\pi \ln 2 - 8\pi \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 4\left(\frac{1}{2} + \frac{1}{2}\cos(2\theta)\right) d\theta$$

$$\Rightarrow V = 8\pi \ln 2 - 8\pi \left[\frac{\pi}{4} + 2\sin(2\theta) \right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$\Rightarrow V = 8\pi \ln 2 - 8\pi \left[\frac{\pi}{2} + 0 - \left(\frac{\pi}{4} + 1\right) \right]$$

$$\Rightarrow V = 8\pi \ln 2 - 8\pi \left[\frac{\pi}{4} - 1 \right]$$

$$\Rightarrow V = 8\pi \ln 2 - 4\pi^2 + 8\pi$$

$$\Rightarrow V = 4\pi [2\ln 2 - \pi + 2]$$

Question 72 (*****)

A spherical cap of depth a is removed from a sphere of radius na , where n is a positive constant, such that $n > \frac{1}{2}$. The volume of the spherical cap is less than half the volume of the sphere.

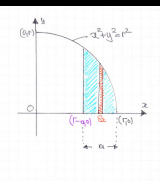
The remainder of the sphere is moulded to a right circular cone whose base is equal to that of the circular plane face of the spherical cap removed.

Given that the height of the cone is ma , where m is a positive constant, show that

$$m = (n + p)(2n + q),$$

where p and q are integers to be found.

$$\boxed{}, \quad m = (n + 1)(2n - 1)$$



• SET UP THE VOLUME OF THE SPHERICAL CAP AS A VOLUME OF REVOLUTION OF THE CURVE $x^2 + y^2 = r^2, x > 0, y > 0$ REVOLVED ABOUT THE x -AXIS

$$V_{cap} = \pi \int_0^{na} (y(x))^2 dx$$

$$V_{cap} = \pi \int_0^{na} (r^2 - x^2) dx$$

$$V_{cap} = \pi \left[r^2x - \frac{1}{3}x^3 \right]_0^{na}$$

• TRY THE EXPRESSION

$$V_{cap} = \pi \left[r^2(na) - \frac{1}{3}(na)^3 \right]$$

$$V_{cap} = \pi \left[r^2na - \frac{1}{3}n^3a^3 \right]$$

$$V_{cap} = \pi \left[r^2na - \frac{1}{3}n^3a^3 \right]$$

$$V_{cap} = \pi \left[r^2na - \frac{1}{3}n^3a^3 \right]$$

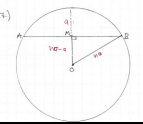
$$V_{cap} = \pi \left[r^2na - \frac{1}{3}n^3a^3 \right]$$

• NOW DETERMINE THE CONE (DIRECTLY WORKS)

BY PYTHAGORAS

$$MB^2 = (na)^2 - (ra)^2$$

$$= a^2(n^2 - r^2)$$

$$= a^2(n^2 - r^2)$$


• NOW THE VOLUME OF THE CONE

$$V_{cap} = \frac{1}{3}\pi r^2(3na - a) = \frac{1}{3}\pi r^2(3n - 1)a$$

$$V_{cone} = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2 ma$$

$$V_{cone} = \frac{1}{3}\pi r^2(3na - a) = \frac{1}{3}\pi r^2(3n - 1)a$$

$$= \frac{1}{3}\pi r^2 [4n^2 - (3n - 1)]$$

$$= \frac{1}{3}\pi r^2 [4n^2 - 3n + 1]$$

• FINALLY

$$\Rightarrow \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2 (4n^2 - 3n + 1)$$

$$\Rightarrow \frac{1}{3}\pi [r^2(3n - 1)](ma) = \frac{1}{3}\pi r^2 (4n^2 - 3n + 1)$$

$$\Rightarrow \frac{1}{3}\pi r^2 (3n - 1) = \frac{1}{3}\pi r^2 (4n^2 - 3n + 1)$$

$$\Rightarrow m(3n - 1) = 4n^2 - 3n + 1$$

$$\Rightarrow m = \frac{4n^2 - 3n + 1}{3n - 1}$$

← BY INSPECTION $m = 1$ IS A SOLUTION

$$\Rightarrow m = \frac{4n^2(1) - 3n(1) + 1}{3n - 1}$$

$$\Rightarrow m = \frac{(1)(1)(4n^2 - 3n + 1)}{3n - 1}$$

$$\Rightarrow m = \frac{(1)(1)(4n^2 - 3n + 1)}{3n - 1}$$

$$\Rightarrow m = \frac{(1)(1)(4n^2 - 3n + 1)}{3n - 1}$$

16 p=1
q=1

Question 73 (*****)

A curve has equation

$$y^2 = \ln|3x-12|, \quad x \in \mathbb{R}, \quad x \neq 4.$$

The finite region bounded by the curve, the x axis and the straight line with equation $y=1$, is revolved by 2π radians in the x axis.

Find the exact volume of the solid formed.

المسألة

$$V = \frac{2}{3}\pi(e-1)$$

• START WITH A SKETCH OF THE CURVE

• IDENTIFY THE REGION TO BE REVOLVED

$$y^2 = \ln|3x-12|$$

$$1 = \ln|3x-12|$$

$$e = |3x-12|$$

$$3x-12 = e \Rightarrow x < \frac{12+e}{3}$$

$$12-3x = e \Rightarrow x < \frac{12-e}{3}$$

$$\Rightarrow x < \frac{4+e}{3}$$

• FIND THE VOLUME OF A CYLINDER OF RADIUS r AND HEIGHT h

$$\Rightarrow V = \pi r^2 h = \pi \times \frac{1}{3} \times \frac{2}{3} e$$

$$\Rightarrow V = \frac{2\pi e}{9}$$

• SIMPLY CONSIDER THE FOLLOWING REVOLUTION IN THE x AXIS, THIS DOUBLE

$$\Rightarrow V = \pi \int_{4-\frac{1}{3}}^{4+\frac{1}{3}} [y(x)]^2 dx$$

$$\Rightarrow V = \pi \int_{4-\frac{1}{3}}^{4+\frac{1}{3}} \ln(2-3x) dx$$

• BY u SUBSTITUTION

$$\Rightarrow V = \pi \int_{-1}^1 \ln u \frac{du}{-3} = -\frac{\pi}{3} \int_{-1}^1 \ln u du$$

• BY FORMS OR QUOTIENT STANDARD RESULTS

$$\Rightarrow V = \frac{\pi}{3} \int_{-1}^1 [u \ln u - u] du$$

$$\Rightarrow V = \frac{\pi}{3} [(e-1) - (0-1)]$$

$$\Rightarrow V = \frac{2\pi}{3}$$

• FINISH THE REQUIRED VOLUME IS FOUND BY

$$\Rightarrow V = \frac{2\pi e}{9} - (2 \times \frac{2\pi}{3})$$

$$\Rightarrow V = \frac{2\pi e}{9} - \frac{4\pi}{3}$$

$$\Rightarrow V = \frac{2\pi}{9}(e-1)$$

Question 74 (*****)

The finite region R is bounded by the curve with equation $x = \cos y^2$, the y axis and the straight line with equation $y = \frac{1}{2}\sqrt{\pi}$.

Determine, in exact simplified form, the volume of the solid formed by revolving R by a full turn in the x axis.

, $\frac{\pi}{2}(2 - \sqrt{2})$

• REARRANGE THE EQUATION

$$x = \cos y^2$$

$$\cos^2 = x^2$$

$$y = \pm \sqrt{\arccos x}$$

$y = \sqrt{\arccos x}$ IS TAKEN IN CONSIDERATION

• NEXT WE FIND THE CO-ORDINATE OF POINT P, IN THE DIAGRAM

$$y = \sqrt{\arccos x} \quad \text{at } y = \frac{1}{2}\sqrt{\pi}$$

$$\sqrt{\arccos x} = \frac{1}{2}\sqrt{\pi}$$

$$\arccos x = \frac{\pi}{4}$$

$$x = \cos \frac{\pi}{4}$$

$$x = \frac{1}{\sqrt{2}}$$

$$\therefore P \left(\frac{1}{\sqrt{2}}, \frac{1}{2}\sqrt{\pi} \right)$$

• VOLUME OF REVOLUTION OF THE YELLOW REGION (CYLINDER)

$$V = \pi r^2 h = \pi \left(\frac{1}{2}\sqrt{\pi} \right)^2 \left(\frac{1}{\sqrt{2}} \right)$$

$$= \pi \times \frac{1}{4} \pi \times \frac{1}{\sqrt{2}}$$

$$= \frac{1}{8} \pi^2 \sqrt{2}$$

• BY INTEGRATION METHOD - FIND THE VOLUME OF REVOLUTION OF THE GREEN & YELLOW AREA TOGETHER

$$V = \pi \int_{-1}^{\frac{1}{\sqrt{2}}} y^2 dx = \pi \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \arccos x dx$$

$$= \pi \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \theta (-\sin \theta) d\theta = \pi \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \theta \sin \theta d\theta$$

BY PARTS NEXT

θ	1
$-\cos \theta$	$\sin \theta$

BY SUBSTITUTION

$$\theta = \arccos x$$

$$\cos \theta = x$$

$$x = \cos \theta$$

$$dx = -\sin \theta d\theta$$

$$x = \frac{1}{\sqrt{2}} \quad \theta = \frac{\pi}{4}$$

$$x = \frac{1}{2} \quad \theta = \frac{\pi}{3}$$

$$\dots = \pi \left\{ [-\theta \cos \theta]_{\frac{\pi}{4}}^{\frac{\pi}{2}} + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos \theta d\theta \right\}$$

$$= \pi \left[\sin \theta - \theta \cos \theta \right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \pi \left[1 - \left(\frac{\pi}{2} - \frac{\pi}{4} \frac{1}{\sqrt{2}} \right) \right]$$

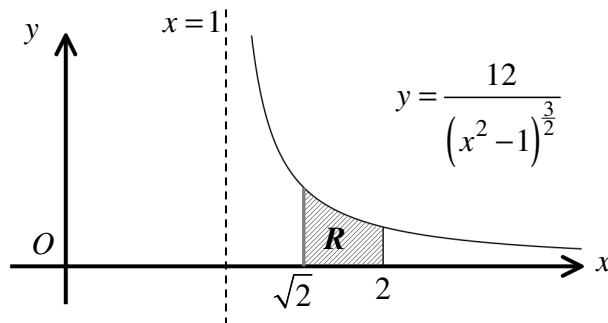
$$= \pi \left[1 - \frac{\pi}{2} + \frac{\pi \sqrt{2}}{8} \right] = \pi - \frac{\pi^2}{2} + \frac{\pi^2 \sqrt{2}}{8}$$

• HENCE THE REQUIRED VOLUME IS

$$V = \left(\pi - \frac{\pi^2}{2} + \frac{\pi^2 \sqrt{2}}{8} \right) - \frac{\pi^2 \sqrt{2}}{8}$$

$$V = \pi [2 - \sqrt{2}]$$

Question 75 (****)



The figure above shows the curve with equation

$$y = \frac{12}{(x^2 - 1)^{\frac{3}{2}}}, \quad x > 1.$$

The region R , bounded the curve, the x axis and the straight lines with equations $x = \sqrt{2}$ and $x = 2$, is revolved by a full turn about the x axis, forming a solid S .

a) Show that the volume of S is given by

$$144\pi \int_{\frac{1}{4}\pi}^{\frac{3}{4}\pi} \operatorname{cosec} \theta \cot^4 \theta \, d\theta.$$

b) Hence find an exact simplified expression for the volume of S .

$$\boxed{}, \quad V = 2\pi \left[14 - 9\sqrt{2} + 27 \ln \left(\frac{1 + \sqrt{2}}{\sqrt{3}} \right) \right]$$

a) **• SPLITTING BY THE COSEC PRODUCT**

$$V = \pi \int_{\sqrt{2}}^2 (y(x))^2 dx = \pi \int_{\sqrt{2}}^2 \left(\frac{12}{(x^2-1)^{3/2}} \right)^2 dx = 144\pi \int_{\sqrt{2}}^2 \frac{1}{(x^2-1)^3} dx$$

• BY SUBSTITUTION (TRIGONOMETRIC OR HYPERBOLIC)

$a = \operatorname{sech} b$	
$\frac{da}{dx} = \operatorname{sech} b \tanh b$	
$\frac{2a+2}{2a^2-2} = 2$	$\frac{2}{2a^2-2} = 2$
$\frac{2a-2}{2a^2-2} = \frac{1}{2}$	$\frac{2a-2}{2a^2-2} = \frac{1}{2}$
$b = \frac{1}{2}$	$b = \frac{1}{2}$

$$\dots = 144\pi \int_{\frac{1}{4}\pi}^{\frac{3}{4}\pi} \frac{1}{(\operatorname{sech} \theta)^3} (\operatorname{sech} \theta \tanh \theta) d\theta$$

$$= 144\pi \int_{\frac{1}{4}\pi}^{\frac{3}{4}\pi} \frac{\operatorname{sech}^2 \theta \tanh \theta}{(\tanh \theta)^3} d\theta = 144\pi \int_{\frac{1}{4}\pi}^{\frac{3}{4}\pi} \frac{\operatorname{sech}^2 \theta}{\tanh^2 \theta} d\theta$$

$$= 144\pi \int_{\frac{1}{4}\pi}^{\frac{3}{4}\pi} \frac{\operatorname{sech}^2 \theta}{\tanh^2 \theta} \operatorname{cosec}^2 \theta d\theta = 144\pi \int_{\frac{1}{4}\pi}^{\frac{3}{4}\pi} \frac{\operatorname{sech}^2 \theta}{\tanh^2 \theta} \operatorname{cosec}^2 \theta d\theta$$

b) **• FINDING WHAT IS THE UNITS ...**

$$I = \int \operatorname{sech}^2 \theta d\theta = \int (\operatorname{cosec}^2 \theta) \cot^4 \theta d\theta$$

• BY PARTS

$u^2 \theta$	$-3 \cot^2 \theta \operatorname{cosec}^2 \theta$
$-\operatorname{cosec}^2 \theta$	$\cot^2 \theta \operatorname{cosec}^2 \theta$

$$\Rightarrow I = -\operatorname{cosec}^2 \theta \cot^2 \theta - 3 \int \cot^2 \theta \operatorname{cosec}^2 \theta d\theta$$

$$\Rightarrow I = -\operatorname{cosec}^2 \theta \cot^2 \theta - 3 \int \cot^2 \theta \operatorname{cosec}^2 \theta d\theta$$

$$\Rightarrow I = -\operatorname{cosec}^2 \theta \cot^2 \theta - 3 \int \cot^2 \theta \operatorname{cosec}^2 \theta d\theta$$

$$\Rightarrow I = -\operatorname{cosec}^2 \theta \cot^2 \theta - 3 \int \cot^2 \theta \operatorname{cosec}^2 \theta d\theta$$

• NOW LET $J = \int \cot^2 \theta \operatorname{cosec}^2 \theta d\theta = \int \cot^2 \theta (\operatorname{cosec}^2 \theta) d\theta$

$\cot^2 \theta$	$-\operatorname{cosec} \theta$
$-\operatorname{cosec} \theta$	$\cot \theta \operatorname{cosec} \theta$

$$\Rightarrow J = -\cot \theta \operatorname{cosec} \theta - \int \operatorname{cosec} \theta d\theta$$

$$\Rightarrow J = -\cot \theta \operatorname{cosec} \theta - \int \operatorname{cosec} \theta d\theta$$

$$\Rightarrow J = -\cot \theta \operatorname{cosec} \theta - \int \operatorname{cosec} \theta d\theta$$

$$\Rightarrow J = -\cot \theta \operatorname{cosec} \theta - \int \operatorname{cosec} \theta d\theta$$

$$\Rightarrow 2I = -\cot \theta \operatorname{cosec} \theta - \int \operatorname{cosec} \theta d\theta$$

$$\Rightarrow I = -\frac{1}{2} \cot \theta \operatorname{cosec} \theta - \frac{1}{2} \int \operatorname{cosec} \theta d\theta$$

• COMBINING THE TWO TRIGONOMETRIC EXPRESSIONS

$$\Rightarrow I = -\frac{1}{2} \cot \theta \operatorname{cosec} \theta - \frac{1}{2} \int \operatorname{cosec} \theta d\theta$$

$$\Rightarrow I = -\frac{1}{2} \cot \theta \operatorname{cosec} \theta + \frac{1}{2} \ln |\operatorname{cosec} \theta - \cot \theta| + C$$

• RETURNING TO UNITS OF THE 144\pi AT THE FRONT

$$\Rightarrow V = 144\pi \left[-\frac{1}{2} \cot \theta \operatorname{cosec} \theta - \frac{1}{2} \int \operatorname{cosec} \theta d\theta \right]_{\frac{1}{4}\pi}^{\frac{3}{4}\pi}$$

$$\Rightarrow V = \pi \left[54 \cot \theta \operatorname{cosec} \theta - 27 \ln |\operatorname{cosec} \theta - \cot \theta| \right]_{\frac{1}{4}\pi}^{\frac{3}{4}\pi}$$

$$\Rightarrow V = \pi \left[\left(54 \cot \frac{3}{4}\pi \operatorname{cosec} \frac{3}{4}\pi - 27 \ln \left| \operatorname{cosec} \frac{3}{4}\pi - \cot \frac{3}{4}\pi \right| \right) - \left(54 \cot \frac{1}{4}\pi \operatorname{cosec} \frac{1}{4}\pi - 27 \ln \left| \operatorname{cosec} \frac{1}{4}\pi - \cot \frac{1}{4}\pi \right| \right) \right]$$

$$\Rightarrow V = \pi \left[27 \left(2\sqrt{2} - 18\sqrt{2} + 54 \ln \left(\frac{1+\sqrt{2}}{\sqrt{3}} \right) \right) \right]$$

$$\Rightarrow V = 27\pi \left[14 - 9\sqrt{2} + 27 \ln \left(\frac{1+\sqrt{2}}{\sqrt{3}} \right) \right]$$

Question 76 (****)

A curve C and a straight line L have respective equations

$$y = x^2 \quad \text{and} \quad y = x.$$

The finite region bounded by C and L is rotated around L by a full turn, forming a solid of revolution S .

Find, in exact form, the volume of S .

 , $\frac{\pi\sqrt{2}}{60}$

\bullet CONSIDER AN INFINITESIMAL SLICE OF THICKNESS du A LENGTH dx , PERPENDICULAR TO THE ROTATIONAL AXIS.
 $V = \int \pi(h_1)^2 dx$
 So we need $h_1 = f(x)$
 $du = g(x) dx$
 So $V = \int \pi(f(x))^2 g(x) dx$

\bullet FIRSTLY $\frac{dy}{dx} = 2x$ (Gradient of $y=x^2$)
 $\frac{dy}{dx} = 2x$ (SLOPE AS IT IS THE PERCENT OF THE TANGENT AT THE POINT $P(x_1, y_1)$)
 $\frac{dy}{dx}(x) = 2x$

\bullet NEXT $\frac{dx}{dy} = \frac{1}{2x}$
 $D_1 = \frac{dx}{dy}$

\bullet WORKING IF AN ENLARGED VERSION OF THE REGION TO MAKE LIFE EASIER
 $100 - 2 - (100 - 2) = 0 - 0$
 $du = |AB| = |CD| = |DE| (\cos(B) - \theta)$
 $du = D_1 [\cos(B)\cos(\theta) + \sin(B)\sin(\theta)]$
 $du = \frac{dx}{dy} [\cos(B)\cos(\theta) + \sin(B)\sin(\theta)]$
 $du = [\cos(\theta) + \tan(\theta)\sin(\theta)] dx$
 $du = [\cos(\theta) + 2x\sin(\theta)] dx$

\bullet NEXT $\cos(\theta) = \frac{1}{\sqrt{1+4x^2}} = \frac{1}{\sqrt{1+4x^2}}$
 $\sin(\theta) = \frac{2x}{\sqrt{1+4x^2}} = \frac{2x}{\sqrt{1+4x^2}}$
 $\therefore \cos(\theta) + \sin(\theta) = \frac{1+2x}{\sqrt{1+4x^2}}$

\bullet DETERMINING TO THE FORMATION OF THE VOLUME ELEMENT
 $du = [\cos(\theta) + 2x\sin(\theta)] dx$
 $du = \left[\frac{1}{\sqrt{1+4x^2}} + 2x \cdot \frac{2x}{\sqrt{1+4x^2}} \right] dx$
 $du = \frac{1+2x^2}{\sqrt{1+4x^2}} dx$

\bullet FINALLY NEED THE RADIUS OF THE DISCS, IE THE HEIGHT OF THE INFINITESIMAL SLICE h_1 .
 $h_1 = |BP| \cos(\theta)$
 $h_1 = |BQ \sin(\theta) + R - y| \cos(\theta) = | \frac{1}{\sqrt{2}} |$
 $h_1 = (x-2x^2) \times \frac{1}{\sqrt{2}}$
 $h_1 = \frac{1}{\sqrt{2}} (x-2x^2)$

\bullet SCALING UP ALL THE DISCS IN THE REGION
 $V = \int \pi \left(\frac{1}{\sqrt{2}} (x-2x^2) \right)^2 dx$
 $V = \int \pi \left(\frac{1}{2} (x-2x^2)^2 \right) \frac{1}{\sqrt{2}} (1+2x) dx$

\bullet TAKING LIMITS
 $V = \pi \frac{1}{2\sqrt{2}} \int_{x=0}^{x=1} (x-2x^2)^2 (1+2x) dx$
 $V = \frac{\pi}{2\sqrt{2}} \int_0^1 (1+2x)(x^2-2x^3)^2 dx$
 $V = \frac{\pi}{2\sqrt{2}} \int_0^1 (x^2-2x^3)^2 (1+2x) dx$
 $V = \frac{\pi}{2\sqrt{2}} \int_0^1 (x^4-2x^5+x^6) (1+2x) dx$
 $V = \frac{\pi}{2\sqrt{2}} \left[\frac{1}{5}x^5 - \frac{2}{6}x^6 + \frac{1}{7}x^7 \right]_0^1$
 $V = \frac{\pi}{2\sqrt{2}} \left(\frac{1}{5} - \frac{2}{6} + \frac{1}{7} \right)$
 $V = \frac{\pi}{2\sqrt{2}} \times \frac{1}{42}$
 $V = \frac{\pi}{84\sqrt{2}}$
 $V = \frac{\pi \times \sqrt{2}}{60}$