

INTEGRATION STRUCTURED EXAM QUESTIONS PART I

Question 1 (**)

Evaluate each of the following integrals, giving the answers in exact form.

a) $\int_0^2 \frac{1}{\sqrt{4x+1}} dx.$

b) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos 3x dx.$

$\boxed{}, \boxed{1}, \boxed{-\frac{1}{3}}$

$$\begin{aligned} \text{(a)} \int_0^2 \frac{1}{\sqrt{4x+1}} dx &= \int_0^2 (4x+1)^{-\frac{1}{2}} dx = \left[\frac{1}{2} (4x+1)^{\frac{1}{2}} \right]_0^2 = \left[\frac{1}{2} \sqrt{4x+1} \right]_0^2 \\ &= \frac{1}{2} \times 3 - \frac{1}{2} = \frac{3}{2} - \frac{1}{2} = 1 \\ \text{(b)} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos 3x dx &= \left[\frac{1}{3} \sin 3x \right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \frac{1}{3} \sin \pi - \frac{1}{3} \sin \frac{\pi}{2} = -\frac{1}{3} \end{aligned}$$

Question 2 (**)By using the substitution $u = 4 + 3x^2$, or otherwise, find

$$\int \frac{2x}{(4+3x^2)^2} dx.$$

$\boxed{\frac{1}{3}}, \boxed{-\frac{1}{3(4+3x^2)} + C}$

$$\begin{aligned} \int \frac{2x}{(4+3x^2)^2} dx &= \int \frac{2x}{u^2} \frac{du}{6x} \\ &= \int \frac{1}{3} u^{-2} du = -\frac{1}{3} u^{-1} + C \\ &= -\frac{1}{3} (4+3x^2)^{-1} + C = -\frac{1}{3(4+3x^2)} + C \end{aligned}$$

$$\begin{aligned} u &= 4+3x^2 \\ \frac{du}{dx} &= 6x \\ du &= 6x dx \\ dx &= \frac{du}{6x} \end{aligned}$$

Question 3 (**)

Show clearly that

$$\int_0^{\frac{1}{3}} x e^{3x} dx = \frac{1}{9}.$$

[4], proof

Handwritten solution for Question 3:

$$\begin{aligned} \int_0^{\frac{1}{3}} x e^{3x} dx &= \dots \text{by parts \& ignoring limits} \\ &= \frac{1}{3} x e^{3x} - \int \frac{1}{3} e^{3x} dx \\ &= \frac{1}{3} x e^{3x} - \frac{1}{9} e^{3x} + C \\ &\dots \text{Limits} \dots \\ &= \left[\frac{1}{3} x e^{3x} - \frac{1}{9} e^{3x} \right]_0^{\frac{1}{3}} = \left(\frac{1}{3} \cdot \frac{1}{3} e^1 - \frac{1}{9} e^1 \right) - \left(0 - \frac{1}{9} \right) = \frac{1}{9} \end{aligned}$$

The diagram shows the area under the curve $y = x e^{3x}$ from $x=0$ to $x=\frac{1}{3}$, with the curve and axes labeled.

Question 4 (**)

$$\frac{3x-5}{x-1} \equiv A + \frac{B}{x-1}.$$

- a) Determine the value of each of the constants A and B .
- b) Hence find

$$\int \frac{3x-5}{x-1} dx.$$

$$[A=3], [B=-2], [3x-2\ln|x-1|+C]$$

Handwritten solution for Question 4:

(a) $\frac{3x-5}{x-1} \equiv A + \frac{B}{x-1}$

Method 1: $\frac{3x-5}{x-1} \equiv \frac{A(x-1)+B}{x-1}$

• If $x=1$, $-2=B$

• If $x=0$, $-5=A+B$

$A=B+5$

$A=3$

Method 2: $\frac{3x-5}{x-1} = \frac{3(x-1)-2}{x-1} = 3 - \frac{2}{x-1}$

$\therefore A=3, B=-2$

(b) $\int \frac{3x-5}{x-1} dx = \int 3 - \frac{2}{x-1} dx = 3x - 2\ln|x-1| + C$

Question 5 (**)

Evaluate each of the following integrals, giving the answers in exact form.

a) $\int_0^4 e^{\frac{1}{2}x} dx.$

b) $\int_0^{\frac{\pi}{4}} \cos\left(3x + \frac{\pi}{4}\right) dx.$

$$\boxed{2}, \boxed{2(e^2 - 1)}, \boxed{-\frac{\sqrt{2}}{6}}$$

$$\begin{aligned} \text{a)} \quad \int_0^4 e^{\frac{1}{2}x} dx &= \left[2e^{\frac{1}{2}x} \right]_0^4 = 2e^2 - 2 = 2(e^2 - 1) \\ \text{b)} \quad \int_0^{\frac{\pi}{4}} \cos\left(3x + \frac{\pi}{4}\right) dx &= \left[\frac{1}{3} \sin\left(3x + \frac{\pi}{4}\right) \right]_0^{\frac{\pi}{4}} = \frac{1}{3} \sin\left(\frac{3\pi}{4} + \frac{\pi}{4}\right) - \frac{1}{3} \sin\left(\frac{\pi}{4}\right) \\ &= \frac{1}{3} \times \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{6} \end{aligned}$$

Question 6 (**)

$$\frac{5x+13}{(2x+1)(x+4)} \equiv \frac{A}{2x+1} + \frac{B}{x+4}.$$

a) Determine the value of each of the constants A and B .

b) Evaluate

$$\int_0^4 \frac{5x+13}{(2x+1)(x+4)} dx,$$

giving the answer as a single simplified natural logarithm.

$$\boxed{}, \boxed{A=3}, \boxed{B=1}, \boxed{\ln 54}$$

Question 7 (**)

By using the substitution $u^2 = 1 - x^2$, or otherwise, show that

$$\int_0^1 5x(1-x^2)^{\frac{3}{2}} dx = 1.$$

$$\boxed{}, \boxed{\text{proof}}$$

Question 8 (**)

Use integration by parts to find the value of

$$\int_0^{\frac{\pi}{4}} 4x \cos 4x \, dx.$$

$$\boxed{\frac{1}{4}}, \boxed{-\frac{1}{2}}$$

Handwritten solution for Question 8:

$$\begin{aligned} \int_0^{\frac{\pi}{4}} 4x \cos 4x \, dx &= \dots \text{by parts \& ignoring limits} \\ &= 2x \sin 4x - \int \sin 4x \, dx \\ &= 2x \sin 4x + \frac{1}{4} \cos 4x + C \\ &\dots \text{limits} \dots \\ &= \left[2x \sin 4x + \frac{1}{4} \cos 4x \right]_0^{\frac{\pi}{4}} \\ &= \left(\frac{\pi}{2} \sin \pi + \frac{1}{4} \cos \pi \right) - \left(0 + \frac{1}{4} \cos 0 \right) \\ &= -\frac{1}{4} - \left(\frac{1}{4} \right) = -\frac{1}{2} \end{aligned}$$

Diagram showing the choice of $u = 4x$ and $v = \cos 4x$.

Question 9 (**)By using the substitution $u = 1 - x^2$, or otherwise, find

$$\int \frac{12x}{(1-x^2)^{\frac{3}{2}}} \, dx.$$

$$\boxed{\frac{12}{\sqrt{1-x^2}}} + C$$

Handwritten solution for Question 9:

$$\begin{aligned} \int \frac{12x}{(1-x^2)^{\frac{3}{2}}} \, dx &= \int \frac{12x}{u^{\frac{3}{2}}} \cdot \frac{du}{-2x} \\ &= \int \frac{-6}{u^{\frac{3}{2}}} \, du = \int -6u^{-\frac{3}{2}} \, du \\ &= 12u^{-\frac{1}{2}} + C = \frac{12}{\sqrt{1-x^2}} + C \end{aligned}$$

Diagram showing the substitution $u = 1 - x^2$, $\frac{du}{dx} = -2x$, and $\frac{du}{dx} = \frac{du}{dx}$.

Question 10 (**)

Evaluate each of the following integrals, giving the answers in exact form.

a) $\int_0^3 \frac{4}{2x+3} dx.$

b) $\int_0^{\frac{\pi}{6}} \sin\left(4x + \frac{\pi}{6}\right) dx.$

$\boxed{}, \boxed{\ln 9}, \boxed{\frac{\sqrt{3}}{4}}$

$$\begin{aligned} \text{(a)} \quad \int_0^3 \frac{4}{2x+3} dx &= \left[2 \ln|2x+3| \right]_0^3 = 2 \ln 9 - 2 \ln 3 \\ &= 2 \ln 9 - \ln 9 = \ln 9 \quad (\text{or } 2 \ln 3) \\ \text{(b)} \quad \int_0^{\frac{\pi}{6}} \sin\left(4x + \frac{\pi}{6}\right) dx &= \left[-\frac{1}{4} \cos\left(4x + \frac{\pi}{6}\right) \right]_0^{\frac{\pi}{6}} = \frac{1}{4} \left[\cos\left(4x + \frac{\pi}{6}\right) \right]_0^{\frac{\pi}{6}} \\ &= \frac{1}{4} \left[\cos \frac{\pi}{2} - \cos \frac{\pi}{6} \right] = \frac{1}{4} \left[\frac{\sqrt{3}}{2} - \left(\frac{\sqrt{3}}{2}\right) \right] \\ &= \frac{1}{4} \times \sqrt{3} = \frac{\sqrt{3}}{4} \end{aligned}$$

Question 11 (**)

Use a trigonometric identity to integrate

$$\int \frac{1}{1 + \cos 2x} dx.$$

$\boxed{\frac{1}{2}}, \boxed{\frac{1}{2} \tan x + C}$

$$\begin{aligned} \int \frac{1}{1 + \cos 2x} dx &= \int \frac{1}{1 + (2\cos^2 x - 1)} dx \\ &= \int \frac{1}{2\cos^2 x} dx \\ &= \int \frac{1}{2} \sec^2 x dx \\ &= \frac{1}{2} \tan x + C \end{aligned}$$

Question 12 (**)

$$\frac{30}{(x+3)(9-2x)} \equiv \frac{A}{x+3} + \frac{B}{9-2x}.$$

- a) Determine the value of each of the constants A and B .
- b) Evaluate

$$\int_1^4 \frac{30}{(x+3)(9-2x)} dx,$$

giving the answer as a single simplified natural logarithm.

$$\boxed{40}, \boxed{A=2}, \boxed{B=4}, \boxed{4 \ln\left(\frac{7}{2}\right) = \ln\left(\frac{2401}{16}\right)}$$

$$\frac{30}{(x+3)(9-2x)} \equiv \frac{A}{x+3} + \frac{B}{9-2x}$$

$$\frac{30}{30} \equiv \frac{A(9-2x)}{A(9-2x)} + \frac{B(x+3)}{B(x+3)}$$

$$30 \equiv A(9-2x) + B(x+3)$$

$$\bullet \text{ let } x = -3, \quad 30 = 15A \Rightarrow A = 2$$

$$\bullet \text{ let } x = 4.5, \quad 30 = 9A + 3B$$

$$30 = 18 + 3B$$

$$3B = 12$$

$$B = 4$$

$$\text{(b)} \int_1^4 \frac{30}{(x+3)(9-2x)} dx = \int_1^4 \left(\frac{2}{x+3} + \frac{4}{9-2x} \right) dx = \left[2 \ln|x+3| - 2 \ln|9-2x| \right]_1^4$$

$$= (2 \ln 7 - 2 \ln 1) - (2 \ln 4 - 2 \ln 7)$$

$$= 2 \ln 7 - 2 \ln 4 + 2 \ln 7 = 4 \ln 7 - 2 \ln 4$$

$$= 4 \ln 7 - 4 \ln 2 = 4 (\ln 7 - \ln 2)$$

$$= 4 \ln\left(\frac{7}{2}\right)$$

Question 13 (**)

Evaluate each of the following integrals, giving the answers in exact form.

a) $\int_0^{\frac{1}{3}} e^{-3x} dx$.

b) $\int_0^{\frac{\pi}{4}} \sin\left(2x + \frac{\pi}{4}\right) dx$.

$$\frac{1}{3}(1 - e^{-1}), \quad \frac{\sqrt{2}}{2}$$

Handwritten solution for Question 13:

(a) $\int_0^{\frac{1}{3}} e^{-3x} dx = \left[-\frac{1}{3}e^{-3x}\right]_0^{\frac{1}{3}} = -\frac{1}{3}\left[e^{-1}\right]_0^{\frac{1}{3}} = -\frac{1}{3}\left(\frac{1}{e} - 1\right) = \frac{1}{3}\left(1 - \frac{1}{e}\right)$

(b) $\int_0^{\frac{\pi}{4}} \sin\left(2x + \frac{\pi}{4}\right) dx = \left[-\frac{1}{2}\cos\left(2x + \frac{\pi}{4}\right)\right]_0^{\frac{\pi}{4}} = -\frac{1}{2}\left[\cos\left(2\left(\frac{\pi}{4}\right) + \frac{\pi}{4}\right) - \cos\left(2(0) + \frac{\pi}{4}\right)\right]$
 $= -\frac{1}{2}\left[\cos\left(\frac{3\pi}{4}\right) - \cos\left(\frac{\pi}{4}\right)\right] = -\frac{1}{2}\left[-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right] = -\frac{1}{2}\left[-\sqrt{2}\right] = \frac{\sqrt{2}}{2}$

Question 14 (**+)

By using the substitution $u^2 = 16 - 7x^2$, or otherwise, show that

$$\int_0^1 \frac{x}{\sqrt{16 - 7x^2}} dx = \frac{1}{7}.$$

, proof

Handwritten solution for Question 14:

$\int_0^1 \frac{x}{\sqrt{16 - 7x^2}} dx = \int_4^3 \frac{-\frac{1}{2} du}{\sqrt{u^2}} = \int_4^3 -\frac{1}{2} \frac{1}{u} du = \left[-\frac{1}{2} \ln u\right]_4^3 = -\frac{1}{2} \ln 3 + \frac{1}{2} \ln 4 = \frac{1}{2} \ln \frac{4}{3}$

Wait, this is not the correct substitution. The correct substitution is $u^2 = 16 - 7x^2$. Let's re-evaluate:

$u^2 = 16 - 7x^2$
 $2u \frac{du}{dx} = -14x$
 $\frac{du}{dx} = -\frac{7x}{u}$
 $dx = -\frac{u}{7x} du$
 $\int_0^1 \frac{x}{\sqrt{16 - 7x^2}} dx = \int_4^3 \frac{x}{u} \left(-\frac{u}{7x}\right) du = \int_4^3 -\frac{1}{7} du = \left[-\frac{1}{7} u\right]_4^3 = -\frac{1}{7}(3 - 4) = \frac{1}{7}$

Question 15 (**+)Determine the value of the positive constant k given further that

$$\int_k^8 \frac{4}{2x-1} dx = 1.90038.$$

Give the value of k to an appropriate degree of accuracy.

$$\boxed{}, \boxed{k \approx 3.4}$$

INTEGRATE FIRST

$$\int_k^8 \frac{4}{2x-1} dx = \left[4 \ln|2x-1| \times \frac{1}{2} \right]_k^8 = \left[2 \ln(2x-1) \right]_k^8$$

$$= 2 \ln 15 - 2 \ln(2k-1)$$

NOW SOLVE THE EQUATION

$$\Rightarrow \int_k^8 \frac{4}{2x-1} dx = 1.90038$$

$$\Rightarrow 2 \ln 15 - 2 \ln(2k-1) = 1.90038$$

$$\Rightarrow 2 \ln 15 - 1.90038 = 2 \ln(2k-1)$$

$$\Rightarrow \ln(2k-1) = 1.75786 \dots$$

$$\Rightarrow 2k-1 = e^{1.75786 \dots}$$

$$\Rightarrow 2k-1 = 5.80003 \dots$$

$$\Rightarrow k = 3.40001 \dots$$

$\therefore k \approx 3.4$

Question 16 (**+)By using the substitution $u = 1 + 4 \ln x$, or otherwise, find

$$\int \frac{4}{x(1+4 \ln x)^2} dx.$$

$$\boxed{}, \boxed{-\frac{1}{1+4 \ln x} + C}$$

$$\int \frac{4}{x(1+4 \ln x)^2} dx = \int \frac{4}{x u^2} \frac{x}{4} du$$

$$= \int \frac{1}{u^2} du = \int u^{-2} du = -u^{-1} + C$$

$$= -\frac{1}{u} + C = -\frac{1}{1+4 \ln x} + C$$

$u = 1 + 4 \ln x$
 $\frac{du}{dx} = \frac{4}{x}$
 $4 dx = x du$
 $\frac{dx}{x} = \frac{1}{4} du$

Question 17 (**+)

$$\frac{8x}{4x-3} \equiv A + \frac{B}{4x-3}.$$

- a) Determine the value of each of the constants A and B .
- b) Hence, or otherwise, evaluate

$$\int_1^3 \frac{8x}{4x-3} dx,$$

giving the answer in terms of natural logarithms.

$$A=2, B=6, 4+3\ln 3$$

(a) $\frac{8x}{4x-3} \equiv A + \frac{B}{4x-3}$
 $\frac{8x}{4x-3} \equiv \frac{A(4x-3) + B}{4x-3}$
 $8x \equiv 4Ax - 3A + B$
 $\therefore 4A = 8 \quad -3A + B = 0$
 $A = 2 \quad B = 6$

(b) $\int_1^3 \frac{8x}{4x-3} dx = \int_1^3 \left(2 + \frac{6}{4x-3} \right) dx = \left[2x + \frac{3}{2} \ln|4x-3| \right]_1^3$
 $= \left[6 + \frac{3}{2} \ln 9 \right] - \left[2 + \frac{3}{2} \ln 1 \right] = 4 + 3 \ln 3$
 (Note: that the substitution $u = 4x-3$ would also work)

Question 18 (**+)

Use an appropriate integration method to find

$$\int (x+1)e^{x+1} dx.$$

$$\boxed{}, \boxed{xe^{x+1} + C}$$

$\int (x+1)e^{x+1} dx = \dots$ BY PARTS ...

$= (x+1)e^{x+1} - \int e^{x+1} dx$
 $= (x+1)e^{x+1} - e^{x+1} + C$
 $= e^{x+1}(x+1-1) + C$
 $= xe^{x+1} + C$

Question 19 (**+)

$$f(x) = 4xe^{2x}.$$

a) Use integration by parts to find $\int f(x) dx$.

b) Find an exact value for $\int_0^{\ln 2} f(x) dx$.

$$\boxed{}, \boxed{2xe^{2x} - e^{2x} + C}, \boxed{-3 + 8\ln 2}$$

(a) $\int 4xe^{2x} dx = \dots$ by parts
 $= (4x)(\frac{e^{2x}}{2}) - \int 2e^{2x} dx$
 $= 2xe^{2x} - e^{2x} + C$

(b) $\int_0^{\ln 2} 4xe^{2x} dx = [2xe^{2x} - e^{2x}]_0^{\ln 2}$
 $= (2\ln 2)e^{2\ln 2} - e^{2\ln 2} - [0 - 1]$
 $= 8\ln 2 - 4 + 1$
 $= -3 + 8\ln 2$

Question 20 (**+)

Evaluate each of the following integrals, giving the answers in exact form.

a) $\int_0^1 \frac{9}{(2x+1)^2} dx.$

b) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin\left(4x + \frac{\pi}{6}\right) dx.$

$$\boxed{3}, \boxed{-\frac{\sqrt{3}}{8}}$$

(a) $\int_0^1 \frac{9}{(2x+1)^2} dx = \int_0^1 9(2x+1)^{-2} dx = \left[-\frac{9}{2}(2x+1)^{-1} \right]_0^1 = \left[-\frac{9}{2(2x+1)} \right]_0^1$
 $= \frac{9}{2} \left[\frac{1}{2x+1} \right]_0^1 = \frac{9}{2} \left(\frac{1}{3} - \frac{1}{1} \right) = \frac{9}{2} \times \frac{-2}{3} = -3$

(b) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin\left(4x + \frac{\pi}{6}\right) dx = \left[-\frac{1}{4} \cos\left(4x + \frac{\pi}{6}\right) \right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \frac{1}{4} \left[\cos\left(4x + \frac{\pi}{6}\right) \right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$
 $= \frac{1}{4} \left[\cos\left(\frac{4\pi}{3} + \frac{\pi}{6}\right) - \cos\left(\frac{4\pi}{6} + \frac{\pi}{6}\right) \right] = \frac{1}{4} \left[\cos\left(\frac{3\pi}{2}\right) - \cos\left(\frac{5\pi}{6}\right) \right]$

Question 21 (**+)

By using the substitution $u = \ln x$, or otherwise, find an exact value for

$$\int_e^3 \frac{1}{x \ln x} dx.$$

$$\boxed{\ln(\ln 3)}$$

Handwritten solution showing the substitution $u = \ln x$, $\frac{du}{dx} = \frac{1}{x}$, and the integral evaluation:

$$\int_e^3 \frac{1}{x \ln x} dx = \int_1^{\ln 3} \frac{1}{u} du = \left[\ln |u| \right]_1^{\ln 3} = \ln(\ln 3) - \ln(1) = \ln(\ln 3)$$

Additional notes in the box: $u = \ln 3$, $\frac{du}{dx} = \frac{1}{x}$, $dx = x du$, $x = e$, $u = 1$, $x = 3$, $u = \ln 3$.

Question 22 (**+)

$$f(x) \equiv \frac{x-5}{x^2+5x+4}.$$

- a) Express $f(x)$ in partial fractions.
- b) Find the value of

$$\int_0^2 f(x) \, dx \, ,$$

giving the answer as a single simplified logarithm.

$$\boxed{}, \quad \boxed{f(x) \equiv \frac{3}{x+4} - \frac{2}{x+1}}, \quad \boxed{\int_0^2 f(x) \, dx = \ln\left(\frac{3}{8}\right)}$$

[illegible]

Question 23 (**+)

By using the substitution $u^2 = 4\cos x - 1$, or otherwise, find

$$\int \frac{\sin x}{\sqrt{4 \cos x - 1}} dx.$$

$$-\frac{1}{2}\sqrt{4\cos x-1}+C$$

$$\begin{aligned} \int \frac{\sin x}{\sqrt{4\cos x - 1}} dx &= \int \frac{\sin x}{\sqrt{x}} \left(\frac{-dx}{2\sin x} \right) du \\ &= \int -\frac{1}{2} du = -\frac{1}{2}u + C \\ &= -\frac{1}{2}\sqrt{4\cos x - 1} + C \end{aligned}$$

Question 24 (**+)

Use the substitution $u = \sqrt{2x-7}$ to find

$$\int_4^8 \frac{6x}{\sqrt{2x-7}} dx.$$

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Handwritten solution for Question 24:

$$\int_4^8 \frac{6x}{\sqrt{2x-7}} dx \quad \dots \text{by substitution}$$

$$= \int_1^3 \frac{6x}{u} \cdot u du = \int_1^3 3(2u) du$$

$$= \int_1^3 3(u^2+7) du = \int_1^3 3u^2 + 21 du$$

$$= \left[u^3 + 21u \right]_1^3 = (27+63) - (1+21)$$

$$= 90 - 22 = 68$$

Side notes:

$$u = \sqrt{2x-7}$$

$$u^2 = 2x-7$$

$$2u \frac{du}{dx} = 2$$

$$u \frac{du}{dx} = 1$$

$$x=4, u=1$$

$$x=8, u=3$$

$$2x = u^2 + 7$$

Question 25 (**+)

$$y = 2x \sin 2x + \cos 2x, \quad x \in \mathbb{R}$$

- a) Find an expression for $\frac{dy}{dx}$.
- b) Hence show that

$$\int_0^{\frac{\pi}{4}} x \cos 2x \, dx = \frac{1}{8}(\pi - 2).$$

$$\frac{dy}{dx} = 4x \cos 2x$$

Handwritten solution for Question 25:

(a) $\frac{d}{dx}(2x \sin 2x + \cos 2x) = 2x \sin 2x + 2x \cdot 2 \cos 2x - 2 \sin 2x$

$$= 2x \sin 2x + 4x \cos 2x - 2 \sin 2x$$

$$= 4x \cos 2x$$

(b) $\int_0^{\frac{\pi}{4}} 2x \cos 2x \, dx = \frac{1}{2} \int_0^{\frac{\pi}{4}} 4x \cos 2x \, dx = \frac{1}{2} \int_0^{\frac{\pi}{4}} (2x \sin 2x + \cos 2x) \, dx$

$$= \frac{1}{2} \left[\left(\frac{2x \sin 2x}{2} + \frac{\cos 2x}{2} \right) - (0 + 1) \right] = \frac{1}{2} \left(\frac{\pi}{2} - 1 \right)$$

$$= \frac{1}{8}(\pi - 2)$$

as required

Question 26 (**+)

Determine the value of the positive constant k given further that

$$\int_k^{\frac{1}{2}} \frac{6}{e^{2-3x}} dx = 0.1998.$$

Give the value of k to an appropriate degree of accuracy.

$$\boxed{}, \boxed{k \approx 0.44}$$

CHECK OUT THE INTEGRATION FIRST

$$\int_k^{\frac{1}{2}} \frac{6}{e^{2-3x}} dx = \int_k^{\frac{1}{2}} 6x e^{-(2-3x)} dx = \int_k^{\frac{1}{2}} 6e^{3x-2} dx$$

$$= \left[2e^{3x-2} \right]_k^{\frac{1}{2}} = 2e^{\frac{3}{2}-2} - 2e^{3k-2}$$

NOW SETTING UP AN EQUATION

$$\Rightarrow \int_k^{\frac{1}{2}} \frac{6}{e^{2-3x}} dx = 0.1998$$

$$\Rightarrow 2(e^{\frac{3}{2}-2} - e^{3k-2}) = 0.1998$$

$$\Rightarrow \frac{1}{e^{\frac{1}{2}}} - e^{3k-2} = 0.0999$$

$$\Rightarrow \frac{1}{\sqrt{e}} - 0.0999 = e^{3k-2}$$

$$\Rightarrow e^{3k-2} = 0.50630077 \dots$$

$$\Rightarrow 3k-2 = \ln(0.50630077 \dots)$$

$$\Rightarrow k = 0.440089924 \dots$$

$\therefore k \approx 0.44$

Question 27 (***)

$$y = \frac{3x}{2+x-x^2}$$

- a) Calculate the three missing values of y in the following table.

x	0	0.25	0.5	0.75	1
y	0				1.5

- b) Use the trapezium rule with all the values from the completed table of part (a) to find an estimate for

$$\int_0^1 \frac{3x}{2+x-x^2} dx$$

- c) Use a suitable method to find the exact value of

$$\int_0^1 \frac{3x}{2+x-x^2} dx$$

, 0.3429 , 0.6667 , 1.0286 , $\ln 2$

a)

x	0	0.25	0.5	0.75	1
y	0	0.3429	0.6667	1.0286	1.5

b)

$$\int_0^1 \frac{3x}{2+x-x^2} dx \approx \frac{1}{2} \left[\frac{3x}{2+x-x^2} \right]_0^1 + \frac{1}{2} \left[\frac{3x}{2+x-x^2} \right]_0^1$$

$$\approx \frac{1}{2} \left[0 + 1.5 + 2(0.3429 + 0.6667 + 1.0286) \right]$$

$$\approx 0.697$$

c)

$$\int_0^1 \frac{3x}{2+x-x^2} dx = \int_0^1 \frac{3x}{(x-2)(x+1)} dx$$

$$= \int_0^1 \frac{3x}{(x-2)(x+1)} dx$$

PROCEED BY PARTIAL FRACTIONS

$$\frac{3x}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}$$

$$3x = A(x+1) + B(x-2)$$

- IF $x = -1 \Rightarrow -3 = -3B \Rightarrow B = 1$
- IF $x = 2 \Rightarrow 6 = 2A \Rightarrow A = 3$

RETURNING TO THE INTEGRAL

$$= \int_0^1 \left(\frac{3}{x-2} + \frac{1}{x+1} \right) dx$$

$$= \left[3 \ln|x-2| + \ln|x+1| \right]_0^1$$

$$= \left[3 \ln|-1| + \ln|1| \right] - \left[3 \ln|-2| + \ln|1| \right]$$

$$= 3 \ln 2 - \ln 2$$

$$= \ln 2 \approx 0.693$$

Question 28 (***)

Use a suitable substitution to find

$$\int \frac{30x}{\sqrt{1-2x}} dx.$$

$$5(1-2x)^{\frac{3}{2}} - 15(1-2x)^{\frac{1}{2}} + C$$

Handwritten solution for the integral problem:

Method 1 (Substitution):

$$\int \frac{30x}{\sqrt{1-2x}} dx = \dots \text{ by substitution}$$

$$= \int \frac{30x}{\sqrt{u}} (-\frac{1}{2} du) = \int -15x \frac{du}{\sqrt{u}}$$

$$= \int -15(1-u) \frac{du}{\sqrt{u}} = \int -15 \left(\frac{1}{\sqrt{u}} - \sqrt{u} \right) du$$

$$= \int -15u^{-\frac{1}{2}} + 15u^{\frac{1}{2}} du = -15 \cdot 2u^{\frac{1}{2}} + 15 \cdot \frac{2}{3} u^{\frac{3}{2}} + C$$

$$= -30u^{\frac{1}{2}} + 10u^{\frac{3}{2}} + C$$

$$= -30(1-2x)^{\frac{1}{2}} + 10(1-2x)^{\frac{3}{2}} + C$$

$$= 10(1-2x)^{\frac{3}{2}} - 30(1-2x)^{\frac{1}{2}} + C$$

Method 2 (Alternative Substitution):

$$\int \frac{30x}{\sqrt{1-2x}} dx = \dots \int \frac{30x}{\sqrt{u}} \left(-\frac{du}{2} \right)$$

$$= \int -15x \frac{du}{\sqrt{u}} = \int -15 \left(\frac{1-u}{2} \right) \frac{du}{\sqrt{u}}$$

$$= \int -\frac{15}{2} \left(\frac{1}{\sqrt{u}} - \sqrt{u} \right) du = -\frac{15}{2} \left(2u^{\frac{1}{2}} - \frac{2}{3} u^{\frac{3}{2}} \right) + C$$

$$= -15u^{\frac{1}{2}} + 5u^{\frac{3}{2}} + C$$

$$= -15(1-2x)^{\frac{1}{2}} + 5(1-2x)^{\frac{3}{2}} + C$$

Both methods yield the same result: $5(1-2x)^{\frac{3}{2}} - 15(1-2x)^{\frac{1}{2}} + C$

Question 29 (***)

$$\frac{x^2+3}{x-1} \equiv Ax+B+\frac{C}{x-1}.$$

- a) Determine the value of each of the constants A , B and C .
- b) Hence, or otherwise, evaluate

$$\int_2^4 \frac{x^2+3}{x-1} dx,$$

giving the answer in terms of natural logarithms.

$$\boxed{}, \boxed{A=1}, \boxed{B=1}, \boxed{C=4}, \boxed{8+4\ln 3}$$

a) BY LONG DIVISION OR MULTIPLICATION

$$\frac{x^2+3}{x-1} = \frac{x(x-1) + (x-1) + 4}{x-1} = x+1 + \frac{4}{x-1}$$

$A=1$
 $B=1$
 $C=4$

ALTERNATIVE BY COMPARING COEFFICIENTS

$$\frac{x^2+3}{x-1} \equiv Ax+B+\frac{C}{x-1}$$

$$\frac{x^2+3}{x-1} \equiv \frac{Ax(x-1)+B(x-1)+C}{x-1}$$

$$x^2+3 \equiv Ax^2+(B-A)x+(C-B)$$

$A=1$	$B-A=0$	$C-B=3$
	$B-1=0$	$C-1=3$
	$B=1$	$C=4$

b) USING PART (a) WE HAVE

$$\int_2^4 \frac{x^2+3}{x-1} dx = \int_2^4 \left(x+1 + \frac{4}{x-1} \right) dx = \left[\frac{1}{2}x^2 + x + 4\ln|x-1| \right]_2^4$$

$$= (8+4+4\ln 3) - (2+2+4\ln 1)$$

$$= 8+4\ln 3$$

Question 30 (*)**Use the substitution $u = 1 + 2\cos x$ to find

$$\int_0^{\frac{\pi}{2}} (1 + 2\cos x)^3 \sin x \, dx.$$

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Handwritten solution for Question 30:

$$\int_0^{\frac{\pi}{2}} (1 + 2\cos x)^3 \sin x \, dx \dots \text{By the substitution } u = 1 + 2\cos x$$

$$\therefore = \int_3^1 u^3 \sin x \left(-\frac{du}{2\sin x} \right)$$

$$= \int_1^3 \frac{1}{2} u^3 \, du$$

$$= \left[\frac{1}{8} u^4 \right]_1^3 = \frac{81}{8} - \frac{1}{8} = 10$$

Boxed notes:

$$u = 1 + 2\cos x$$

$$\frac{du}{dx} = -2\sin x$$

$$dx = -\frac{du}{2\sin x}$$

$$\begin{array}{l} x=0 \rightarrow u=3 \\ x=\frac{\pi}{2} \rightarrow u=1 \end{array}$$

Question 31 (*)**By using the substitution $u = 3x + 1$, or otherwise, find

$$\int_0^5 x\sqrt{3x+1} \, dx.$$

$$\frac{204}{5} = 40.8$$

Handwritten solution for Question 31:

$$\int_0^5 x\sqrt{3x+1} \, dx = \int_1^{16} x u^{\frac{1}{2}} \frac{du}{3}$$

$$= \int_1^{16} \frac{1}{3} x u^{\frac{1}{2}} \, du = \int_1^{16} \frac{1}{9} (3u-1) u^{\frac{1}{2}} \, du$$

$$= \int_1^{16} \left(\frac{1}{3} u^{\frac{3}{2}} - \frac{1}{9} u^{\frac{1}{2}} \right) \, du$$

$$= \left[\frac{2}{15} u^{\frac{5}{2}} - \frac{2}{27} u^{\frac{3}{2}} \right]_1^{16} = \left(\frac{2 \cdot 16^{\frac{5}{2}}}{15} - \frac{2 \cdot 16^{\frac{3}{2}}}{27} \right) - \left(\frac{2}{15} - \frac{2}{27} \right)$$

$$= \frac{204}{5}$$

Boxed notes:

$$u = 3x + 1$$

$$\frac{du}{dx} = 3$$

$$dx = \frac{du}{3}$$

$$\begin{array}{l} x=0 \rightarrow u=1 \\ x=5 \rightarrow u=16 \\ u=3x+1 \end{array}$$

Question 32 (***)

Use an appropriate integration method to find an exact value for

$$\int_0^{\frac{\pi}{3}} 6x \sin 3x \, dx.$$

$$\boxed{}, \quad \boxed{\frac{2\pi}{3}}$$

Handwritten solution for Question 32:

$$\begin{aligned} \int_0^{\frac{\pi}{3}} 6x \sin 3x \, dx &= \dots \text{by parts: } \begin{matrix} u = 6x \\ dv = \sin 3x \end{matrix} \\ &= 6x \left(-\frac{1}{3} \cos 3x\right) - \int -2 \cos 3x \, dx \\ &= -2x \cos 3x + \int 2 \cos 3x \, dx \\ &= -2x \cos 3x + \frac{2}{3} \sin 3x + C \\ &\text{Introducing limits back} \\ &= \left[-2x \cos 3x + \frac{2}{3} \sin 3x\right]_0^{\frac{\pi}{3}} \\ &= \left[-\frac{2\pi}{3} \cos \pi + \frac{2}{3} \sin \pi\right] - \left[0 + 0\right] \\ &= -\frac{2\pi}{3}(-1) + 0 \\ &= \frac{2\pi}{3} \end{aligned}$$

Question 33 (***)

By using the substitution $u = \sec x$, or otherwise, find

$$\int \tan x \sec^4 x \, dx.$$

$$\boxed{\frac{1}{4} \sec^4 x + C}$$

Handwritten solution for Question 33:

$$\begin{aligned} \int \tan x \sec^4 x \, dx &= \dots \text{by substitution } \begin{matrix} u = \sec x \\ \frac{du}{dx} = \sec x \tan x \\ dx = \frac{du}{\sec x \tan x} \end{matrix} \\ &= \int \tan x \cdot u^4 \cdot \frac{du}{\sec x \tan x} = \int u^4 \cdot \frac{du}{u} \\ &= \int u^3 \, du = \frac{1}{4} u^4 + C = \frac{1}{4} \sec^4 x + C \end{aligned}$$

Question 34 (***)

$$\frac{3x^3 + 2x^2 - 3x + 8}{x+2} \equiv Ax^2 + Bx + C + \frac{D}{x+2}.$$

a) Find the value of each of the constants A , B , C and D .

b) Hence find

$$\int \frac{3x^3 + 2x^2 - 3x + 8}{x+2} dx.$$

$$\boxed{A=3}, \boxed{B=-4}, \boxed{C=5}, \boxed{D=-2}, \boxed{x^3 - 2x^2 + 5x - 2 \ln|x+2| + C}$$

(a)
$$\frac{3x^3 + 2x^2 - 3x + 8}{x+2} \equiv \frac{Ax^2 + Bx + C + \frac{D}{x+2}}{x+2}$$

$$\equiv \frac{Ax^2(x+2) + Bx(x+2) + C(x+2) + D}{x+2}$$

$$\equiv \frac{Ax^3 + 2Ax^2 + Bx^2 + 2Bx + Cx + 2C + D}{x+2}$$

$$\equiv \frac{Ax^3 + (2A+B)x^2 + (2B+C)x + (2C+D)}{x+2}$$

$A=3$ $2A+B=2$ $2B+C=-3$ $2C+D=8$
 $6+B=2$ $-8+C=-3$ $10+D=8$
 $B=-4$ $C=5$ $D=-2$

Alternative by long division

$$\begin{array}{r} 3x^2 - 4x + 5 \\ x+2 \overline{) 3x^3 + 2x^2 - 3x + 8} \\ \underline{3x^3 + 6x^2 + 6x + 12} \\ -4x^2 - 9x - 4 \\ \underline{-4x^2 - 8x - 8} \\ -x + 0 \\ \underline{-x - 2} \\ 2 \end{array}$$

(b)
$$\int \frac{3x^3 + 2x^2 - 3x + 8}{x+2} dx = \int 3x^2 - 4x + 5 - \frac{2}{x+2} dx$$

$$= x^3 - 2x^2 + 5x - 2 \ln|x+2| + C$$

Question 35 (***)

$$f(x) \equiv \frac{5}{3x^2 - 5x}$$

- a) Express $f(x)$ in partial fractions.
- b) Find the value of

$$\int_3^5 f(x) \, dx,$$

giving the answer as a single simplified logarithm.

$$\boxed{}, \quad f(x) \equiv \frac{3}{3x-5} - \frac{1}{x}, \quad \ln\left(\frac{3}{2}\right)$$

(a) $f(x) = \frac{5}{3x^2 - 5x} = \frac{5}{x(3x-5)} \Rightarrow \frac{A}{x} + \frac{B}{3x-5}$

$5 \equiv A(3x-5) + Bx$

\bullet If $x=0$, $5 = -5A$ \bullet If $x = \frac{5}{3}$

$A = -1$ $S = \frac{5}{3}$

$B = 3$

$\therefore f(x) = \frac{3}{3x-5} - \frac{1}{x}$

(b) $\int_3^5 f(x) \, dx = \int_3^5 \left(\frac{3}{3x-5} - \frac{1}{x} \right) dx = \left[\ln|3x-5| - \ln|x| \right]_3^5$

$= (\ln 10 - \ln 5) - (\ln 4 - \ln 3) = \ln 2 - \ln \frac{5}{3}$

$= \ln 2 + \ln \frac{3}{5} = \ln\left(\frac{3}{2}\right)$

Question 36 (*)**

By using the substitution $u = e^x$, or otherwise, show clearly that

$$\int_{-1}^1 \frac{e^x}{e^x + 1} dx = 1.$$

proof

$$\int_{-1}^1 \frac{e^x}{e^x + 1} dx = \int_{e^{-1}}^e \frac{1}{u+1} du = \left[\ln(u+1) \right]_{e^{-1}}^e$$

$$= \ln(e+1) - \ln\left(\frac{1}{e} + 1\right) = \ln(e+1) - \ln\left(\frac{1+e}{e}\right)$$

$$= \ln(e+1) - \ln(1+e) + \ln(e) = \ln(e) = 1$$

NOTE: This is a special case of the integral $\int \frac{f(x)}{f(x)+1} dx$ by noting that it is of the form $\int \frac{f'(x)}{f(x)+1} dx$.

Question 37 (*)**

Find an expression for the integral

$$\int \frac{3x-10}{x^2+5x-6} dx.$$

$$4 \ln|x+6| - \ln|x-1| + C$$

$$\int \frac{3x-10}{x^2+5x-6} dx = \int \frac{3x-10}{(x-1)(x+6)} dx = \dots \text{by partial fractions}$$

$$\frac{3x-10}{(x-1)(x+6)} = \frac{A}{x-1} + \frac{B}{x+6}$$

$$3x-10 = A(x+6) + B(x-1)$$

$$\begin{cases} 3(-1)-10 = 7A \Rightarrow A = -1 \\ 3(6)-10 = 7B \Rightarrow B = 4 \end{cases}$$

$$\therefore \int \frac{4}{x+6} - \frac{1}{x-1} dx = 4 \ln|x+6| - \ln|x-1| + C$$

Question 38 (*)**

By using the substitution $u = 1 + x^2$, or otherwise, find

$$\int \frac{x^3}{\sqrt{1+x^2}} dx.$$

$$\frac{1}{3}(1+x^2)^{\frac{3}{2}} - (1+x^2)^{\frac{1}{2}} + C$$

Handwritten solution for Question 38:

$$\begin{aligned} \int \frac{x^3}{\sqrt{1+x^2}} dx &= \int \frac{x^2}{\sqrt{u}} \cdot \frac{du}{2x} = \int \frac{x}{2\sqrt{u}} du \\ &= \int \frac{u-1}{2\sqrt{u}} du = \int \frac{u}{2\sqrt{u}} - \frac{1}{2\sqrt{u}} du \\ &= \int \frac{1}{2} u^{\frac{1}{2}} - \frac{1}{2} u^{-\frac{1}{2}} du = \frac{1}{3} u^{\frac{3}{2}} - u^{\frac{1}{2}} + C \\ &= \frac{1}{3} (1+x^2)^{\frac{3}{2}} - (1+x^2)^{\frac{1}{2}} + C \end{aligned}$$

Side notes: $u = 1+x^2$, $\frac{du}{dx} = 2x$, $dx = \frac{du}{2x}$, $u^2 = u-1$

Question 39 (*)**

Use integration by parts to find the value of

$$\int_1^e \ln x \, dx.$$

$$\boxed{1}, \boxed{1}$$

Handwritten solution for Question 39:

$$\begin{aligned} \int_1^e \ln x \, dx &= \text{by parts \& ignoring limits} \\ &= \int 1 \times \ln x \, dx = x \ln x - \int 1 \, dx \\ &= x \ln x - x + C \\ &\text{REINSTATE LIMITS...} \\ &= [x \ln x - x]_1^e \\ &= (e \ln e - e) - (1 \ln 1 - 1) \\ &= (e - e) + 1 \\ &= 1 \end{aligned}$$

Side note: Table for integration by parts: $\ln x$ and $\frac{1}{x}$ in the top row, x and 1 in the bottom row.

Question 40 (*)**Find the value of the constant k given that

$$\int_0^1 k(e^{2x} + 4x) dx = e^2 + 3.$$

$$k = 2$$

$$\begin{aligned} \int_0^1 k(e^{2x} + 4x) dx &= e^2 + 3 & \Rightarrow \frac{1}{2}k(e^2 + 3) &= e^2 + 3 \\ \Rightarrow \left[k\left(\frac{1}{2}e^{2x} + 2x\right) \right]_0^1 &= e^2 + 3 & \Rightarrow \frac{1}{2}k &= 1 \\ \Rightarrow k\left[\left(\frac{1}{2}e^2 + 2\right) - \left(\frac{1}{2} + 0\right)\right] &= e^2 + 3 & \Rightarrow k &= 2 \end{aligned}$$

Question 41 (*)**

Evaluate each of the following integrals, giving the answers in exact form.

a) $\int_0^{\ln 2} (e^x + 2e^{-x})^2 dx.$

b) $\int_0^{\frac{\pi}{4}} 1 - \sin 4x dx.$

$$3 + 4 \ln 2, \quad \frac{1}{4}(\pi - 2)$$

$$\begin{aligned} \text{a) } \int_0^{\ln 2} (e^x + 2e^{-x})^2 dx &= \int_0^{\ln 2} (e^{2x} + 2e^0 + 4e^{-2x}) dx = \int_0^{\ln 2} e^{2x} + 4 + 4e^{-2x} dx \\ &= \left[\frac{1}{2}e^{2x} + 4x - 2e^{-2x} \right]_0^{\ln 2} = \left(2 + 4 \ln 2 - \frac{1}{2} \right) - \left(\frac{1}{2} + 0 - 2 \right) \\ &= \frac{3}{2} + 4 \ln 2 - \frac{1}{2} + 2 = 3 + 4 \ln 2 \\ \text{b) } \int_0^{\frac{\pi}{4}} 1 - \sin 4x dx &= \left[x + \frac{1}{4} \cos 4x \right]_0^{\frac{\pi}{4}} = \left(\frac{\pi}{4} + \frac{1}{4} \cos \pi \right) - \left(0 + \frac{1}{4} \cos 0 \right) \\ &= \left(\frac{\pi}{4} - \frac{1}{4} \right) - \left(\frac{1}{4} \right) = \frac{\pi}{4} - \frac{1}{2} = \frac{1}{4}(\pi - 2) \end{aligned}$$

Question 42 (***)

By using the substitution $u = \tan x$ and the trigonometric identity $1 + \tan^2 x = \sec^2 x$, show clearly that

$$\int_0^{\frac{\pi}{3}} \sec^4 x \, dx = 2\sqrt{3}.$$

10, proof

Handwritten solution for Question 42:

$$\begin{aligned} \int_0^{\frac{\pi}{3}} \sec^4 x \, dx &= \int_0^{\frac{\pi}{3}} \sec^2 x \cdot \sec^2 x \, dx = \int_0^{\frac{\pi}{3}} \sec^2 x \cdot \frac{du}{dx} \, dx \\ &= \int_0^{\frac{\pi}{3}} (1 + \tan^2 x) \, dx = \int_0^{\frac{\pi}{3}} (1 + u^2) \, du \\ &= \left[u + \frac{1}{3}u^3 \right]_0^{\frac{\pi}{3}} = \left(\sqrt{3} + \frac{1}{3}(\sqrt{3})^3 \right) - (0 + 0) = 2\sqrt{3} \end{aligned}$$

Side notes in the solution:

- $u = \tan x$
- $\frac{du}{dx} = \sec^2 x$
- $dx = \frac{du}{\sec^2 x}$
- $x=0 \Rightarrow u=0$
- $x=\frac{\pi}{3} \Rightarrow u=\sqrt{3}$

Question 43 (***)

$$\frac{8x-1}{(2x-1)^2} \equiv \frac{A}{2x-1} + \frac{B}{(2x-1)^2}.$$

- a) Determine the value of each of the constants A and B .
- b) Hence find the exact value of

$$\int_1^{1.5} \frac{8x-1}{(2x-1)^2} \, dx.$$

$A=4$, $B=3$, $\frac{3}{4} + 2\ln 2$

Handwritten solution for Question 43:

(a) $\frac{8x-1}{(2x-1)^2} \equiv \frac{A}{2x-1} + \frac{B}{(2x-1)^2}$

Let $2x-1 = 0 \Rightarrow x = \frac{1}{2}$

Let $x=0 \Rightarrow -1 = -A + B$

Let $x=1 \Rightarrow 7 = 2A + B$

$A=4, B=3$

(b) $\int_1^{1.5} \frac{8x-1}{(2x-1)^2} \, dx = \int_1^{1.5} \frac{4}{2x-1} + \frac{3}{(2x-1)^2} \, dx$

$= \left[2\ln|2x-1| - \frac{3}{2(2x-1)} \right]_1^{1.5}$

$= \left(2\ln 2 - \frac{3}{2} \right) - \left(2\ln 1 - \frac{3}{2} \right) = 2\ln 2 - \frac{3}{2} + \frac{3}{2} = 2\ln 2$

Question 44 (***)

By using the substitution $u = x^2$, or otherwise, find

$$\int \frac{1}{4x^{\frac{1}{2}}\sqrt{x^2-1}} dx.$$

$$\boxed{}, \left(x^2-1\right)^{\frac{1}{2}} + C$$

USING THE SUBSTITUTION GIVEN

$$\begin{aligned} \Rightarrow u &= x^2 \\ \Rightarrow u^2 &= x \\ \Rightarrow x &= u^2 \\ \Rightarrow \frac{dx}{du} &= 2u \\ \Rightarrow dx &= 2u du \end{aligned}$$

FINDING THE INTEGRAL WE OBTAIN

$$\begin{aligned} \int \frac{1}{4x^{\frac{1}{2}}\sqrt{x^2-1}} dx &= \int \frac{1}{4u^{\frac{1}{2}}\sqrt{u-1}} (2u du) \\ &= \int \frac{1}{2} (u-1)^{-\frac{1}{2}} du \\ &= \frac{1}{2} (u-1)^{\frac{1}{2}} + C \\ &= \frac{1}{2} (x^2-1)^{\frac{1}{2}} + C \\ &= \sqrt{x^2-1} + C \end{aligned}$$

Question 44 (***)

- a) Use integration by parts to find

$$\int x \cos\left(\frac{1}{2}x\right) dx.$$

- b) Hence determine

$$\int x^2 \sin\left(\frac{1}{2}x\right) dx.$$

$$\boxed{}, \quad \boxed{2x \sin\left(\frac{1}{2}x\right) + 4 \cos\left(\frac{1}{2}x\right) + C},$$

$$\boxed{-2x^2 \cos\left(\frac{1}{2}x\right) + 8x \sin\left(\frac{1}{2}x\right) + 16 \cos\left(\frac{1}{2}x\right) + C}$$

a) SETTING UP INTEGRATION BY PARTS

$$\int x \cos\left(\frac{1}{2}x\right) dx = \dots$$

$$= \underline{2x \sin\left(\frac{1}{2}x\right)} - \int 2 \sin\left(\frac{1}{2}x\right) dx$$

$$= \underline{2x \sin\left(\frac{1}{2}x\right)} + 4 \cos\left(\frac{1}{2}x\right) + C$$

b) USING INTEGRATION BY PARTS a) over (a)

$$\int x^2 \sin\left(\frac{1}{2}x\right) dx = \dots$$

$$= \underline{-2x^2 \cos\left(\frac{1}{2}x\right)} - \int -4x \cos\left(\frac{1}{2}x\right) dx$$

$$= -2x^2 \cos\left(\frac{1}{2}x\right) + 4 \int x \cos\left(\frac{1}{2}x\right) dx$$

$$= -2x^2 \cos\left(\frac{1}{2}x\right) + 4 \left[2x \sin\left(\frac{1}{2}x\right) + 4 \cos\left(\frac{1}{2}x\right) \right] + C$$

$$= \underline{-2x^2 \cos\left(\frac{1}{2}x\right) + 8x \sin\left(\frac{1}{2}x\right) + 16 \cos\left(\frac{1}{2}x\right) + C}$$

Question 45 (*)**By using the substitution $u = x^2 + 6x$, or otherwise find

$$\int \frac{x+3}{(x^2+6x)^{\frac{1}{3}}} dx.$$

$$\boxed{\frac{1}{2}(x^2+6x)^{\frac{2}{3}} + C}$$

Handwritten solution for Question 45:

$$\int \frac{x+3}{(x^2+6x)^{\frac{1}{3}}} dx = \dots \text{by substitution or} \\ \text{Reverse Chain Rule (roughly)} \quad \begin{cases} u = x^2 + 6x \\ \frac{du}{dx} = 2x + 6 \\ dx = \frac{du}{2x+6} \end{cases}$$

$$= \int \frac{x+3}{u^{\frac{1}{3}}} \cdot \frac{du}{2x+6} = \int \frac{2x+6}{u^{\frac{1}{3}}} \times \frac{1}{2} \cdot \frac{du}{2x+6}$$

$$= \int \frac{1}{2} u^{-\frac{1}{3}} du = \frac{1}{2} \cdot \frac{3}{2} u^{\frac{2}{3}} + C = \frac{3}{4} u^{\frac{2}{3}} + C$$

$$= \frac{3}{4} (x^2+6x)^{\frac{2}{3}} + C$$

Question 46 (*)**Use the substitution $u = 10\cos x - 1$ to find

$$\int_0^{\frac{\pi}{3}} 15(10\cos x - 1)^{\frac{1}{2}} \sin x \, dx.$$

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Handwritten solution for Question 46:

$$\int_0^{\frac{\pi}{3}} 15(10\cos x - 1)^{\frac{1}{2}} \sin x \, dx = \dots \text{by substitution} \dots$$

$$= \int_9^4 15 u^{\frac{1}{2}} \sin x \left(\frac{du}{-10\sin x} \right)$$

$$= \int_9^4 \frac{3}{2} u^{\frac{1}{2}} du$$

$$= \left[u^{\frac{3}{2}} \right]_9^4 = 27 - 8 = 19$$

Side note:

$$\begin{cases} u = 10\cos x - 1 \\ \frac{du}{dx} = -10\sin x \\ du = -10\sin x \, dx \\ x=0, u=9 \\ x=\frac{\pi}{3}, u=4 \end{cases}$$

Question 47 (***)

$$\frac{2x^2 - x + 6}{x^2(3-2x)} \equiv \frac{A}{x} + \frac{B}{x^2} + \frac{C}{3-2x}.$$

- a) Determine the value of each of the constants A , B and C .
- b) Evaluate

$$\int_2^3 \frac{2x^2 - x + 6}{x^2(3-2x)} dx,$$

giving the answer in the form $p - \ln q$, where p and q are constants.

$$\boxed{A=1}, \boxed{B=2}, \boxed{C=4}, \boxed{\frac{1}{3} - \ln 6}$$

(a) $\frac{2x^2 - x + 6}{x^2(3-2x)} \equiv \frac{A}{x} + \frac{B}{x^2} + \frac{C}{3-2x}$
 $\frac{2x^2 - x + 6}{x^2(3-2x)} \equiv \frac{A(3-2x) + B(3-2x) + Cx^2}{x^2(3-2x)}$

- If $x = \frac{1}{2}$, $\frac{2(\frac{1}{2})^2 - \frac{1}{2} + 6}{(\frac{1}{2})^2(3-2(\frac{1}{2}))} = \frac{A}{\frac{1}{2}} + \frac{B}{(\frac{1}{2})^2} + \frac{C}{3-2(\frac{1}{2})}$
 $9 = \frac{A}{\frac{1}{2}} + \frac{B}{\frac{1}{4}} + \frac{C}{2}$
 $C = 4$
- If $x = 1$, $\frac{2(1)^2 - 1 + 6}{1^2(3-2(1))} = \frac{A}{1} + \frac{B}{1^2} + \frac{C}{3-2(1)}$
 $7 = A + B + C$
 $A = 1$
- If $x = 0$, $\frac{2(0)^2 - 0 + 6}{0^2(3-2(0))} = \frac{A}{0} + \frac{B}{0^2} + \frac{C}{3-2(0)}$
 $6 = 3B$
 $B = 2$

(b) $\int_2^3 \frac{2x^2 - x + 6}{x^2(3-2x)} dx = \int_2^3 \left(\frac{1}{x} + \frac{2}{x^2} + \frac{4}{3-2x} \right) dx$
 $= \left[\ln|x| - \frac{2}{x} - 2\ln|3-2x| \right]_2^3 = \left(\ln 3 - \frac{2}{3} - 2\ln 3 \right) - \left(\ln 2 - 1 - 2\ln 2 \right)$
 $= \ln 3 - 2\ln 3 - \ln 2 + \frac{2}{3} + 1 = \frac{1}{3} - \ln 3 - \ln 2$
 $= \frac{1}{3} - (\ln 3 + \ln 2) = \frac{1}{3} - \ln 6$

Question 48 (***)

x	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$
y	0	0.1309	0.4534	0.7854		0.6545

The table above shows tabulated values for the equation

$$y = x \sin 2x, \quad 0 \leq x \leq \frac{5\pi}{12}.$$

- a) Complete the missing value in the table.
- b) Use the trapezium rule with all the values from the table to find an approximate value for

$$\int_0^{\frac{5\pi}{12}} x \sin 2x \, dx.$$

- c) Use integration by parts to find an exact value for $\int_0^{\frac{5\pi}{12}} x \sin 2x \, dx.$

$$\boxed{}, \boxed{0.9069}, \boxed{0.682}, \boxed{\frac{5\pi\sqrt{3}}{48} + \frac{1}{8}}$$

Handwritten solution for part (c):

1) $y = \frac{5\pi}{12} \times \sin\left(2 \times \frac{5\pi}{12}\right) = \frac{5\pi}{12} \times \sin\frac{5\pi}{6} \approx 0.9069$

2) $\int_0^{\frac{5\pi}{12}} x \sin 2x \, dx \approx \frac{\text{TRAPEZIUM}}{2} \left[\text{FIRST} + (\text{LAST} + 2 \times \text{REST}) \right]$

$\approx \frac{\frac{5\pi}{12}}{2} \left[0 + 0.6545 + 2(0.1309 + 0.4534 + 0.7854 + 0.6545) \right]$

≈ 0.682

3) $\int_0^{\frac{5\pi}{12}} x \sin 2x \, dx = \dots$ INTEGRATION BY PARTS & INTRODUCING LIMITS

$= -\frac{1}{2} x \cos 2x - \int -\frac{1}{2} \cos 2x \, dx$

$= -\frac{1}{2} x \cos 2x + \int \frac{1}{4} \cos 2x \, dx$

$= -\frac{1}{2} x \cos 2x + \frac{1}{4} \sin 2x + C$

2) INTRODUCING LIMITS

$= \left[-\frac{1}{2} x \cos 2x + \frac{1}{4} \sin 2x \right]_0^{\frac{5\pi}{12}}$

$= \left(-\frac{1}{2} \times \frac{5\pi}{12} \times \cos\frac{5\pi}{6} + \frac{1}{4} \times \frac{1}{2} \right) - (0 + 0)$

$= \frac{5\pi\sqrt{3}}{48} + \frac{1}{8}$

Question 49 (***)

By using the substitution $u = e^x + 1$, or otherwise, find

$$\int \frac{e^{2x} - 2e^x}{e^x + 1} dx.$$

$$\boxed{}, \boxed{e^x - 3 \ln(e^x + 1) + C}$$

Handwritten solution for Question 49:

$$\begin{aligned} \int \frac{e^{2x} - 2e^x}{e^x + 1} dx &= \dots \text{by substitution} \\ u &= e^x + 1 \\ \frac{du}{dx} &= e^x \\ dx &= \frac{du}{e^x} \\ e^x &= u - 1 \end{aligned}$$

$$\begin{aligned} \int \frac{e^{2x} - 2e^x}{e^x + 1} dx &= \int \frac{u^2 - 2u}{u} \frac{du}{u-1} \\ &= \int \frac{(u-2)}{u} du = \int \frac{u-3}{u} du \\ &= \int \left(1 - \frac{3}{u}\right) du = u - 3 \ln|u| + C \\ &= (e^x + 1) - 3 \ln(e^x + 1) + C = e^x - 3 \ln(e^x + 1) + (1 + C) \\ &= e^x - 3 \ln(e^x + 1) + C \end{aligned}$$

Question 50 (***)

By using the substitution $u = 1 - \cos x$, or otherwise, find

$$\int \frac{\sin x \cos x}{1 - \cos x} dx.$$

$$\boxed{\cos x + \ln(1 - \cos x) + C}$$

Handwritten solution for Question 50:

$$\begin{aligned} \int \frac{\sin x \cos x}{1 - \cos x} dx &= \dots \text{by substitution} \\ u &= 1 - \cos x \\ \frac{du}{dx} &= \sin x \\ dx &= \frac{du}{\sin x} \\ \cos x &= 1 - u \end{aligned}$$

$$\begin{aligned} \int \frac{\sin x \cos x}{1 - \cos x} dx &= \int \frac{(1-u)u}{u} \frac{du}{1-u} \\ &= \int \frac{u}{1-u} du = \int \frac{1-u}{1-u} du \\ &= \int 1 du = u + C = 1 - \cos x + C \\ &= \cos x + \ln(1 - \cos x) + C \end{aligned}$$

Question 51 (***)

By using the substitution $u^2 = e^x - 1$, or otherwise, find

$$\int_{\ln 2}^{\ln 5} \frac{3e^{2x}}{\sqrt{e^x - 1}} dx.$$

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USING THE SUBSTITUTION GIVEN

$$u^2 = e^x - 1 \quad (\text{IN FACT THE SUBSTITUTION IS } u = \sqrt{e^x - 1})$$

$$2u \frac{du}{dx} = e^x$$

$$2u du = e^x dx$$

$$dx = \frac{2u}{e^x} du$$

LIMITS

$$x = \ln 5 \mapsto u = \sqrt{e^{\ln 5} - 1} = 2$$

$$x = \ln 2 \mapsto u = \sqrt{e^{\ln 2} - 1} = 1$$

TRANSFORMING THE INTEGRAL

$$\int_{\ln 2}^{\ln 5} \frac{3e^{2x}}{\sqrt{e^x - 1}} dx = \int_1^2 \frac{3e^{2x}}{e^x} \left(\frac{2u}{e^x} du \right)$$

$$= \int_1^2 6e^x du \quad \left(u^2 = e^x - 1 \right)$$

$$= \int_1^2 6(u^2 + 1) du$$

$$= \int_1^2 (6u^2 + 6) du$$

$$= \left[2u^3 + 6u \right]_1^2$$

$$= (16 + 12) - (2 + 6)$$

$$= 20$$

Question 52 (***)

Use trigonometric identities to find

$$\int \sec^2 x (1 + \cot^2 x) dx.$$

$\tan x - \cot x + C$

$$\int \sec^2 x (1 + \cot^2 x) dx = \int \sec^2 x + \sec^2 x \cot^2 x dx$$

$$= \int \sec^2 x + \frac{\sec^2 x \cot^2 x}{\cot^2 x} dx$$

$$= \int \sec^2 x + \csc^2 x dx \quad \left(\frac{d}{dx}(\tan x) = \sec^2 x \right)$$

$$= \tan x - \cot x + C \quad \left(\frac{d}{dx}(\cot x) = -\csc^2 x \right)$$

Question 53 (***)

By using the substitution $u = e^x + 1$, or otherwise, find the exact value of

$$\int_0^1 \frac{e^{2x}}{e^x + 1} dx.$$

$$e - 1 + \ln\left(\frac{2}{1+e}\right)$$

Handwritten solution for Question 53:

$$\int_0^1 \frac{e^{2x}}{e^x + 1} dx = \int_2^{e+1} \frac{u-1}{u} du = \int_2^{e+1} \left(1 - \frac{1}{u}\right) du = \left[u - \ln|u|\right]_2^{e+1} = (e+1) - \ln(e+1) - (2 - \ln 2) = e - 1 + \ln\left(\frac{2}{1+e}\right)$$

Substitution used: $u = e^x + 1$
 $\frac{du}{dx} = e^x$
 $dx = \frac{du}{e^x}$
 $x=0 \rightarrow u=2$
 $x=1 \rightarrow u=e+1$
 $e^x = u-1$

Question 54 (***)

By using the substitution $u = \sin x$, or otherwise, find

$$\int \cos^3 x \, dx.$$

$$\boxed{\sin x - \frac{1}{3} \sin^3 x + C}$$

Handwritten solution for Question 54:

$$\int \cos^3 x \, dx = \int \cos x \frac{du}{dx} \, dx = \int \cos x \, du = \int (1 - u^2) \, du = u - \frac{1}{3} u^3 + C = \sin x - \frac{1}{3} \sin^3 x + C$$

Substitution used: $u = \sin x$
 $\frac{du}{dx} = \cos x$
 $dx = \frac{du}{\cos x}$

Question 55 (***)

$$I = \int (x-1)(4-x)^{\frac{1}{2}} dx, \quad x \in \mathbb{R}, \quad x \leq 4.$$

- a) Use the substitution $u = (4-x)^{\frac{1}{2}}$ to find an expression for I .
- b) Show that the answer of part (a) can be written as

$$I = -\frac{2}{5}(x+1)(4-x)^{\frac{3}{2}} + C.$$

- c) Use integration by parts to verify the answer of part (b).

$$\boxed{C=1}, \quad I = \frac{2}{5}(4-x)^{\frac{5}{2}} - 2(4-x)^{\frac{3}{2}} + C$$

(a) $\int (x-1)(4-x)^{\frac{1}{2}} dx = \int (4-u^2)u(-2u du)$ $u = (4-x)^{\frac{1}{2}}$
 $u^2 = 4-x$
 $x = 4-u^2$
 $\frac{dx}{du} = -2u$
 $dx = -2u du$

$= \int (4-u^2)(-2u^2) du = \int -2u^4 + 2u^6 du$

$= -\frac{2}{5}u^5 + \frac{2}{7}u^7 + C = -\frac{2}{5}(4-x)^{\frac{5}{2}} + \frac{2}{7}(4-x)^{\frac{7}{2}} + C$

(b) $\therefore = \frac{2}{5}(4-x)^{\frac{5}{2}} - \frac{2}{7}(4-x)^{\frac{7}{2}} + C$

$= \frac{2}{5}(4-x)^{\frac{5}{2}} - \frac{2}{7}(4-x)^{\frac{5}{2}}(4-x) + C = -\frac{2}{35}(4-x)^{\frac{5}{2}}(4-x) + \frac{2}{5}(4-x)^{\frac{5}{2}} + C$

$= \frac{2}{35}(4-x)^{\frac{5}{2}}[4(4-x) + 7(4-x)] + C = \frac{2}{35}(4-x)^{\frac{5}{2}}(16-4x+28-7x) + C$

$= \frac{2}{35}(4-x)^{\frac{5}{2}}(44-11x) + C = -\frac{2}{5}(4-x)^{\frac{5}{2}}(x-4) + C$

(c) $\int (x-1)(4-x)^{\frac{1}{2}} dx = \dots$ by parts...

$= \frac{2}{5}(4-x)^{\frac{5}{2}} - \frac{2}{7}(4-x)^{\frac{7}{2}} + C$

$= \frac{2}{5}(4-x)^{\frac{5}{2}} - \frac{2}{7}(4-x)^{\frac{5}{2}}(4-x) + C = -\frac{2}{35}(4-x)^{\frac{5}{2}}(4-x) + \frac{2}{5}(4-x)^{\frac{5}{2}} + C$

$= \frac{2}{35}(4-x)^{\frac{5}{2}}[4(4-x) + 7(4-x)] + C = \frac{2}{35}(4-x)^{\frac{5}{2}}(16-4x+28-7x) + C$

$= \frac{2}{35}(4-x)^{\frac{5}{2}}(44-11x) + C = -\frac{2}{5}(4-x)^{\frac{5}{2}}(x-4) + C$

Question 56 (*)**Find the value of the constant k given that

$$\int_0^{\ln 4} (4k-1)e^{2.5x} + k e^{-0.5x} dx = 190.$$

$$k = 4$$

Handwritten solution for Question 56:

$$\int_0^{\ln 4} (4k-1)e^{2.5x} + k e^{-0.5x} dx = 190$$

$$\Rightarrow \left[\frac{4k-1}{2.5} e^{2.5x} - \frac{k}{0.5} e^{-0.5x} \right]_0^{\ln 4} = 190$$

$$\Rightarrow \left[\frac{4k-1}{2.5} e^{2.5 \ln 4} - \frac{k}{0.5} e^{-0.5 \ln 4} \right] - \left[\frac{4k-1}{2.5} - \frac{k}{0.5} \right] = 190$$

$$\Rightarrow \frac{4k-1}{2.5} \cdot 2^5 - \frac{k}{0.5} \cdot \frac{1}{2} - \frac{4k-1}{2.5} + \frac{k}{0.5} = 190$$

$$\Rightarrow \frac{4k-1}{2.5} (32 - 1) - \frac{k}{1} + \frac{k}{0.5} = 190$$

$$\Rightarrow \frac{4k-1}{2.5} \cdot 31 - k + 2k = 190$$

$$\Rightarrow \frac{4k-1}{2.5} \cdot 31 + k = 190$$

$$\Rightarrow 4k-1 + 5k = 950$$

$$\Rightarrow 9k = 951$$

$$\Rightarrow k = 105.666...$$

Question 57 (*)**By using the substitution $u = 2x-1$, or otherwise, show that

$$\int \frac{2x}{\sqrt{2x-1}} dx = \frac{2}{3}(x+1)\sqrt{2x-1} + C.$$

proof

Handwritten solution for Question 57:

$$\int \frac{2x}{\sqrt{2x-1}} dx = \int \frac{2x}{\sqrt{u}} \cdot \frac{du}{2}$$

$$= \int \frac{u+1}{\sqrt{u}} \cdot \frac{du}{2} = \int \frac{u+1}{2\sqrt{u}} du = \int \frac{u}{2\sqrt{u}} + \frac{1}{2\sqrt{u}} du$$

$$= \int \frac{1}{2} u^{\frac{1}{2}} + \frac{1}{2} u^{-\frac{1}{2}} du = \frac{1}{2} \cdot \frac{2}{3} u^{\frac{3}{2}} + \frac{1}{2} \cdot 2 u^{\frac{1}{2}} + C$$

$$= \frac{1}{3} (2x-1)^{\frac{3}{2}} + (2x-1)^{\frac{1}{2}} + C = \frac{1}{3} (2x-1)^{\frac{1}{2}} [(2x-1) + 3] + C$$

$$= \frac{1}{3} (2x-1)^{\frac{1}{2}} (2x+2) + C = \frac{2}{3} (2x-1)^{\frac{1}{2}} (x+1) + C$$

Question 58 (***)

$$f(x) \equiv \frac{70}{x(x-2)(x+5)}$$

a) Express $f(x)$ in partial fractions.

b) Show that $\int_3^4 f(x) dx$ can be written in the form $p \ln 3 + q \ln 2$, where p and q are integers to be found.

$$f(x) \equiv \frac{2}{x+5} + \frac{5}{x-2} - \frac{7}{x}, \quad \int_3^4 f(x) dx = 11 \ln 3 - 15 \ln 2$$

$$f(x) = \frac{70}{x(x-2)(x+5)} = \frac{A}{x-2} + \frac{B}{x+5} + \frac{C}{x}$$

$$70 = A(x-2)(x+5) + Bx(x-2) + Cx(x-2)$$

- If $x=2$, $70 = 14B \Rightarrow B=5$
- If $x=0$, $70 = -10A \Rightarrow A=-7$
- If $x=-5$, $70 = 35C \Rightarrow C=2$

$$f(x) = \frac{5}{x-2} - \frac{7}{x+5} + \frac{2}{x}$$

$$\int_3^4 f(x) dx = \int_3^4 \left(\frac{5}{x-2} - \frac{7}{x+5} + \frac{2}{x} \right) dx = \left[5 \ln|x-2| - 7 \ln|x+5| + 2 \ln|x| \right]_3^4$$

$$= (5 \ln 2 - 7 \ln 7 + 2 \ln 4) - (5 \ln 3 - 7 \ln 8 + 2 \ln 3)$$

$$= 5 \ln 2 - 14 \ln 7 + 4 \ln 4 - 5 \ln 3 + 14 \ln 8 - 4 \ln 3 = 11 \ln 3 - 15 \ln 2$$

Question 59 (***)

Use trigonometric identities to find

$$\int \frac{1}{\cos^2 x \tan^2 x} dx$$

$$-\cot x + C$$

$$\int \frac{1}{\cos^2 x \tan^2 x} dx = \int \frac{\sec^2 x}{\tan^2 x} dx = \int \frac{\sec^2 x}{\tan^2 x} dx$$

$$= \int \sec^2 x \cot^2 x dx = -\cot x + C$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\int \frac{1}{\cos^2 x \tan^2 x} dx = \int \sec^2 x \cot^2 x dx = \dots$$

$$\text{since } \frac{d}{dx}(\tan x) = \sec^2 x$$

$$= -\frac{1}{\tan x} + C = -\cot x + C$$

Question 60 (***)

$$f(x) \equiv \frac{2x+2}{(1-x)(1-2x)}$$

a) Express $f(x)$ in partial fractions.

b) Show that $\int_{1.5}^2 f(x) dx$ can be written in the form $p \ln 2 + q \ln 3$, where p and q are integers to be found.

$$f(x) \equiv \frac{6}{1-2x} - \frac{4}{1-x}, \quad \int_{1.5}^2 f(x) dx = 7 \ln 2 - 3 \ln 3$$

a) $f(x) = \frac{2x+2}{(1-x)(1-2x)} \equiv \frac{A}{1-x} + \frac{B}{1-2x}$
 $\frac{2x+2}{(1-x)(1-2x)} \equiv \frac{A(1-2x) + B(1-x)}{(1-x)(1-2x)}$
 $2x+2 \equiv A(1-2x) + B(1-x)$
 $2x+2 \equiv A - 2Ax + B - Bx$
 $2x+2 \equiv (A+B) - (2A+B)x$
 $\begin{cases} A+B = 2 \\ -2A-B = 2 \end{cases} \Rightarrow \begin{matrix} A = -4 \\ B = 6 \end{matrix}$
 $\therefore f(x) = \frac{6}{1-2x} - \frac{4}{1-x}$

b) $\int_{1.5}^2 f(x) dx = \int_{1.5}^2 \left(\frac{6}{1-2x} - \frac{4}{1-x} \right) dx$
 $= \left[-3 \ln |1-2x| + 4 \ln |1-x| \right]_{1.5}^2$
 $= (-3 \ln 3 + 4 \ln 1) - (-3 \ln 2 + 4 \ln 1.5)$
 $= -3 \ln 3 + 4 \ln 1 + 3 \ln 2 - 4 \ln 1.5$
 $= -3 \ln 3 + 3 \ln 2 - 4 \ln 1.5$
 $= 7 \ln 2 - 3 \ln 3$

Question 61 (***)

Use a suitable method to find

$$\int \ln\left(\frac{x}{2}\right) dx.$$

$$x \ln\left(\frac{x}{2}\right) - x + C$$

$\int \ln\left(\frac{x}{2}\right) dx = \int \ln(x) - \ln(2) dx$
 $= x \ln(x) - \int \frac{1}{x} dx - \ln(2) \cdot x$
 $= x \ln(x) - \ln(x) - x \ln(2)$
 $= x \ln(x) - x \ln(2) - \ln(x)$
 $= x \ln\left(\frac{x}{2}\right) - x + C$

Question 62 (***)

$$f(x) \equiv \frac{32-17x}{(x+1)(3x-4)^2}.$$

- a)** Express $f(x)$ in partial fractions.
- b)** Show that

$$\int_0^1 f(x) \, dx,$$

can be evaluated in the form $p + \ln q$, where p and q are integers to be found.

$$f(x) \equiv \frac{4}{(3x-4)^2} - \frac{3}{(3x-4)} + \frac{1}{(x+1)},$$

$$\int_0^1 f(x) \, dx = 1 + \ln 8$$

(a) $\frac{32-17x}{(x+1)(3x-4)^2} = \frac{A}{x+1} + \frac{B}{3x-4} + \frac{C}{3x-4}$

$32-17x = A(3x-4)^2 + B(x+1) + C(x+1)(3x-4)$

• If $x = -1$, $49 = 49A \Rightarrow A = 1$

• If $x = \frac{4}{3}$, $32 - \frac{68}{3} = \frac{7}{3}B \Rightarrow B = 4$

• If $x = 0$, $32 = 16A + B - 4C \Rightarrow C = -3$

$\therefore f(x) = \frac{1}{x+1} + \frac{4}{3x-4} - \frac{3}{3x-4}$

(b) $\int_0^1 f(x) dx = \int_0^1 \frac{1}{x+1} + \frac{4}{3x-4} - \frac{3}{3x-4} dx$

$= [\ln|x+1| - \frac{1}{3} \ln|3x-4| - \ln|3x-4|]_0^1$

$= (\ln 2 + \frac{4}{3} - \ln 1) - (\ln 1 + \frac{1}{3} - \ln 1)$

$= \ln 2 + \ln 4 + \frac{4}{3} - \frac{1}{3} = 1 + \ln 8$

Question 63 (***)

Use a trigonometric identity to find the exact value of

$$\int_0^{\frac{\pi}{4}} \sin^2 2x \, dx.$$

$$\frac{\pi}{8}$$

Handwritten solution for Question 63:

$$\begin{aligned} \int_0^{\frac{\pi}{4}} \sin^2 2x \, dx &= \int_0^{\frac{\pi}{4}} \left(\frac{1}{2} - \frac{1}{2} \cos 4x \right) dx \\ &= \left[\frac{1}{2}x - \frac{1}{8} \sin 4x \right]_0^{\frac{\pi}{4}} \\ &= \left(\frac{\pi}{8} - \frac{1}{8} \sin \pi \right) - (0 - 0) = \frac{\pi}{8} \end{aligned}$$

Trigonometric identity used: $\sin^2 2x = \frac{1}{2} - \frac{1}{2} \cos 4x$

Question 64 (***)Use integration by parts **twice** to find an exact value for

$$\int_0^{\frac{\pi}{2}} 4x^2 \cos x \, dx.$$

$$\pi^2 - 8$$

Handwritten solution for Question 64:

$$\begin{aligned} \int_0^{\frac{\pi}{2}} 4x^2 \cos x \, dx &= \text{by parts 1st time} \\ &= 4x^2 \sin x - \int 8x \sin x \, dx \\ &= 4x^2 \sin x - \left[-8x \cos x - \int -8 \cos x \, dx \right] \\ &= 4x^2 \sin x + 8x \cos x - \int 8 \cos x \, dx \\ &\quad \dots \text{rearrange units} \dots \\ &= \left[4x^2 \sin x + 8x \cos x - 8 \sin x \right]_0^{\frac{\pi}{2}} \\ &= \left(4 \left(\frac{\pi}{2} \right)^2 \sin \frac{\pi}{2} + 8 \left(\frac{\pi}{2} \right) \cos \frac{\pi}{2} - 8 \sin \frac{\pi}{2} \right) - (0 + 0 - 0) \\ &= \pi^2 - 8 \end{aligned}$$

Integration by parts tables shown:

$4x^2$	$\cos x$
$\sin x$	$\cos x$

$8x$	$\sin x$
$\cos x$	$\sin x$

Question 65 (***)

$$\frac{2}{(u-2)(u+2)} \equiv \frac{A}{u-2} + \frac{B}{u+2}.$$

- a) Find the value of A and B in the above identity.
- b) By using the substitution $u = \sqrt{x}$, or otherwise, find

$$\int \frac{1}{\sqrt{x}(x-4)} dx.$$

$$\boxed{A = \frac{1}{2}}, \quad \boxed{B = -\frac{1}{2}}, \quad \boxed{\frac{1}{2} \ln \left| \frac{\sqrt{x}-2}{\sqrt{x}+2} \right| + C}$$

Handwritten solution for Question 65:

(a) $\frac{2}{(u-2)(u+2)} \equiv \frac{A}{u-2} + \frac{B}{u+2}$
 $\frac{2}{2} \equiv \frac{A(u+2)}{(u+2)} + \frac{B(u-2)}{(u-2)}$
 $2 \equiv A(u+2) + B(u-2)$
 $2 \equiv Au + 2A + Bu - 2B$
 $2 \equiv (A+B)u + (2A-2B)$
 $\begin{cases} A+B=0 \\ 2A-2B=2 \end{cases} \Rightarrow \begin{cases} B=-A \\ 2A-2(-A)=2 \end{cases} \Rightarrow \begin{cases} B=-A \\ 4A=2 \end{cases} \Rightarrow \begin{cases} A=\frac{1}{2} \\ B=-\frac{1}{2} \end{cases}$

(b) $\int \frac{1}{\sqrt{x}(x-4)} dx = \int \frac{1}{u(u^2-4)} \cdot 2u du$
 $= \int \frac{2}{u^2-4} du = \int \frac{2}{(u-2)(u+2)} du$
 $= \int \left(\frac{1/2}{u-2} - \frac{1/2}{u+2} \right) du$
 $= \frac{1}{2} \ln|u-2| - \frac{1}{2} \ln|u+2| + C = \frac{1}{2} \ln \left| \frac{u-2}{u+2} \right| + C$
 $= \frac{1}{2} \ln \left| \frac{\sqrt{x}-2}{\sqrt{x}+2} \right| + C$

Question 66 (***)

By using the substitution $u = \cos x$, or otherwise, find

$$\int \frac{1+\cos x}{\sin x} dx.$$

$$\boxed{\ln |\cos x - 1| + C}$$

Handwritten solution for Question 66:

$\int \frac{1+\cos x}{\sin x} dx = \int \frac{1+u}{- \sin x} \left(\frac{du}{- \sin x} \right)$
 $= \int \frac{1+u}{- \sin^2 x} du = \int \frac{1+u}{1-u^2} du$
 $= \int \frac{1+u}{(1-u)(1+u)} du = \int \frac{1}{1-u} du$
 $= \int \frac{1}{1-u} du = - \ln|1-u| + C = - \ln|1-\cos x| + C$
 $= \ln|\cos x - 1| + C$

Question 67 (***)

Use integration by parts to show that

$$\int \frac{4 \ln x}{x^3} dx = -\frac{1 + 2 \ln x}{x^2} + C.$$

$$\sin x - \frac{1}{3} \sin^3 x + C$$

$$\begin{aligned} \int \frac{4 \ln x}{x^3} dx &= \int (4 \ln x) x^{-3} dx = \text{by parts} \quad \left\{ \begin{array}{l} 4 \ln x \rightarrow \frac{4}{x} \\ x^{-3} \rightarrow x^{-2} \end{array} \right\} \\ &= -2x^{-2} \ln x - \int -2x^{-2} \frac{1}{x} dx \\ &= -\frac{4 \ln x}{x^2} + \int 2x^{-3} dx \\ &= -\frac{4 \ln x}{x^2} - x^{-2} + C \\ &= -\frac{4 \ln x}{x^2} - \frac{1}{x^2} + C \\ &= -\frac{1}{x^2} [4 \ln x + 1] + C \quad \text{square 40} \end{aligned}$$

Question 68 (***)

By considering the differentiation of a product of two appropriate functions, find

$$\int e^x (\tan x + \sec^2 x) dx.$$

$$e^x \tan x + C$$

$$\begin{aligned} \int e^x (\tan x + \sec^2 x) dx &= \dots \text{by inspection since } \frac{d}{dx}(\tan x) = \sec^2 x \\ &= \dots \frac{d}{dx}(e^x \tan x) = e^x \tan x + e^x \sec^2 x \\ \therefore \int e^x (\tan x + \sec^2 x) dx &= e^x \tan x + C \end{aligned}$$

Question 69 (***)

$$\frac{2x^2-3}{(x-1)^2} \equiv A + \frac{B}{x-1} + \frac{C}{(x-1)^2}.$$

- a) Determine the value of each of the constants A , B and C .
- b) Evaluate

$$\int_2^3 \frac{2x^2-3}{(x-1)^2} dx,$$

giving the answer in the form $p + \ln q$, where p and q are constants.

$$\boxed{A=2}, \boxed{B=4}, \boxed{C=-1}, \boxed{\frac{3}{2} + \ln 16}$$

(a) $\frac{2x^2-3}{(x-1)^2} \equiv A + \frac{B}{x-1} + \frac{C}{(x-1)^2}$

$\frac{2x^2-3}{(x-1)^2} \equiv \frac{A(x-1)^2 + B(x-1) + C}{(x-1)^2}$

$2x^2-3 \equiv A(x-1)^2 + B(x-1) + C$

• If $x=1$, $-1=C$

• If $x=0$, $-3=A-B+C$

$-3=A-B-1$

$A-B=-2$

• If $x=2$, $5=A+B+C$

$5=A+B-1$

$A+B=6$

$A+B=6$

$A-B=-2$

$2A=4$

$A=2$

$B=4$

$C=-1$

(b) $\int_2^3 \frac{2x^2-3}{(x-1)^2} dx = \int_2^3 \left(2 + \frac{4}{x-1} - (x-1)^{-2} \right) dx$

$= \left[2x + 4\ln|x-1| + (x-1)^{-1} \right]_2^3$

$= \left(2 \times 3 + 4\ln|3-1| + \frac{1}{3-1} \right) - \left(2 \times 2 + 4\ln|2-1| + \frac{1}{2-1} \right)$

$= \frac{3}{2} + 4\ln 2$

Question 70 (***)By using the substitution $u = 2x - 1$, or otherwise, find

$$\int \frac{16x^2}{2x-1} dx.$$

$$(2x-1)^2 + 4(2x-1) + 2\ln|2x-1| + C = 4x^2 + 4x + 2\ln|2x-1| + C$$

Handwritten solution for Question 70 using substitution $u = 2x - 1$.

$\int \frac{16x^2}{2x-1} dx = \dots$ by substitution...
 $u = 2x - 1$
 $\frac{du}{dx} = 2$
 $dx = \frac{du}{2}$
 $2x = u + 1$
 $4x^2 = u^2 + 2u + 1$

$= \int \frac{16x^2}{u} \cdot \frac{du}{2} = \int \frac{8x^2}{u} du = \int \frac{2(u^2 + 2u + 1)}{u} du$
 $= \int \frac{2(u^2 + 2u + 1)}{u} du = \int (2u + 4 + \frac{2}{u}) du$
 $= u^2 + 4u + 2\ln|u| + C = (2x-1)^2 + 4(2x-1) + 2\ln|2x-1| + C$
 $= 4x^2 - 4x + 1 + 8x - 4 + 2\ln|2x-1| + C$
 $= 4x^2 + 4x - 3 + 2\ln|2x-1| + C$
 $= 4x^2 + 4x + 2\ln|2x-1| + C$

Alternatively without substitution:
 $\int \frac{16x^2}{2x-1} dx = \int \frac{8x(2x-1) + 4(2x-1) + 4}{2x-1} dx$
 $= \int (8x + 4 + \frac{4}{2x-1}) dx$
 $= 4x^2 + 4x + 2\ln|2x-1| + C$

Question 71 (***)

Use integration by parts to show that

$$\int_0^{\frac{\pi}{4}} 4x \sec^2 x \, dx = \pi - 2\ln 2.$$

proof

Handwritten solution for Question 71 using integration by parts.

$\int_0^{\frac{\pi}{4}} 4x \sec^2 x \, dx = \dots$ by parts & knowing units
 $= 4x \tan x - \int 4 \tan x$
 $= 4x \tan x - 4 \ln|\sec x| + C$
 $= \dots$ units...
 $= [4x \tan x - 4 \ln|\sec x|]_0^{\frac{\pi}{4}}$
 $= (\pi - 4 \ln 2) - (0 - 4 \ln 1)$
 $= \pi - 4 \ln 2$
 $= \pi - 2 \ln 2$

(Note: The handwritten solution shows a boxed result of $\pi - 2 \ln 2$ and a note "as required".)

Question 72 (***)

$$\frac{18}{(3u-1)(3u+1)} \equiv \frac{A}{3u-1} + \frac{B}{3u+1}.$$

- Find the value of A and B in the above identity.
- By using the substitution $x = u^2$, or otherwise, find

$$\int \frac{9}{\sqrt{x}(9x-1)} dx.$$

$$\boxed{A=9}, \boxed{B=-9}, \left[3\ln\left|\frac{3\sqrt{x}-1}{3\sqrt{x}+1}\right| + C \right]$$

$$\begin{aligned} \text{(a)} \quad \frac{18}{(3u-1)(3u+1)} &\equiv \frac{A}{3u-1} + \frac{B}{3u+1} \\ \frac{18}{(3u-1)(3u+1)} &\equiv \frac{A(3u+1) + B(3u-1)}{(3u-1)(3u+1)} \\ \text{If } u = \frac{1}{3} &\Rightarrow 18 = 2A \Rightarrow A = 9 \\ \text{If } u = -\frac{1}{3} &\Rightarrow 18 = -2B \Rightarrow B = -9 \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \int \frac{9}{\sqrt{x}(2x-1)} dx &= \int \frac{9}{\sqrt{u}(2u-1)} (2x du) \\ &= \int \frac{9}{\sqrt{u} \cdot 2} du = \int \frac{18}{\sqrt{u}(2u-1)} du \\ &= \int \frac{9}{3u-1} - \frac{9}{3u+1} du = 3 \ln|3u-1| - 3 \ln|3u+1| + C \\ &= 3 \ln \left| \frac{3u-1}{3u+1} \right| + C \\ &= 3 \ln \left| \frac{3\sqrt{x}-1}{3\sqrt{x}+1} \right| + C \end{aligned}$$

$$\begin{aligned} x &= u^2 \\ \frac{dx}{du} &= 2u \\ \frac{dx}{du} &= 2u \quad du \end{aligned}$$

Question 73 (***)

Use an appropriate integration method to find an exact value for each of the following integrals

a) $\int_0^{\frac{\pi}{4}} \cos^2 x - \sin^2 x \, dx.$

b) $\int_1^e 4x \ln x \, dx.$

$$\frac{1}{2}, e^2 + 1$$

(a) $\int_0^{\frac{\pi}{4}} \cos^2 x - \sin^2 x \, dx = \int_0^{\frac{\pi}{4}} \cos 2x \, dx = \left[\frac{1}{2} \sin 2x \right]_0^{\frac{\pi}{4}} = \frac{1}{2} \sin \frac{\pi}{2} - \frac{1}{2} \sin 0 = \frac{1}{2}$

(b) $\int_1^e 4x \ln x \, dx = \dots$ by parts & ignoring limits

$u = \ln x$	$\frac{1}{x}$
$\frac{du}{dx} = \frac{1}{x}$	dx

$= 2x^2 \ln x - \int 2x^2 \left(\frac{1}{x} \right) dx$
 $= 2x^2 \ln x - \int 2x \, dx$
 $= 2x^2 \ln x - x^2 + C$
 evaluate limits...
 $= \left[2x^2 \ln x - x^2 \right]_1^e$
 $= (2e^2 \ln e - e^2) - (2 \ln 1 - 1)$
 $= e^2 + 1$

Question 74 (***)

Use the substitution $u = x^2$, followed by integration by parts to find

$$\int x^3 e^{x^2} \, dx.$$

$$\frac{1}{2} e^{x^2} (x^2 - 1) + C$$

$\int x^3 e^{x^2} \, dx = \int x^2 e^{x^2} \frac{du}{dx} \, dx = \int \frac{1}{2} u e^u \, du$

by parts

$u = u$	$\frac{1}{2} e^u$
$\frac{du}{du} = 1$	du

$= \frac{1}{2} u e^u - \int \frac{1}{2} e^u \, du$
 $= \frac{1}{2} u e^u - \frac{1}{2} e^u + C$
 $= \frac{1}{2} e^u (u - 1) + C$
 $= \frac{1}{2} e^{x^2} (x^2 - 1) + C$

Question 75 (***)

Use integration by parts to find the exact value of

$$\int_{\sqrt{e}}^e 16x^3 \ln x \, dx.$$

$$3e^2(e^2 - 1)$$

Handwritten solution for Question 75:

$$\begin{aligned} \int_{\sqrt{e}}^e 16x^3 \ln x \, dx &= \dots \text{ by parts \& ignoring limits} \\ &= 4x^4 \ln x - \int 4x^3 \, dx \\ &= 4x^4 \ln x - x^4 + C \\ &\dots \text{ REINTEGRATE UNITS} \dots \\ &= [4x^4 \ln x - x^4]_{\sqrt{e}}^e = (4e^4 \ln e - e^4) - (4e^2 \ln \sqrt{e} - e^2) \\ &= (4e^4 \ln e - e^4) - (2e^2 \ln e - e^2) = (4e^4 - e^4) - (2e^2 - e^2) \\ &= 3e^4 - e^2 = 3e^2(e^2 - 1) \end{aligned}$$

Question 76 (***)By using the substitution $u = \sqrt{2x-3}$, or otherwise, find an expression for

$$\int (2x-1)\sqrt{2x-3} \, dx.$$

$$\frac{1}{5}(2x-3)^{\frac{5}{2}} + \frac{2}{3}(2x-3)^{\frac{3}{2}} + C$$

Handwritten solution for Question 76:

$$\begin{aligned} \int (2x-1)\sqrt{2x-3} \, dx &= \dots \text{ by substitution} \dots \\ &= \int (2x-1) u \, (u \, du) = \int (2x-1) u^2 \, du \\ &= \int (u^2+2) u^2 \, du = \int u^4 + 2u^3 \, du \\ &= \frac{1}{5}u^5 + \frac{2}{4}u^4 = \frac{1}{5}(2x-3)^{\frac{5}{2}} + \frac{2}{3}(2x-3)^{\frac{3}{2}} + C \end{aligned}$$

Substitution details:

$$\begin{aligned} u &= \sqrt{2x-3} \\ u^2 &= 2x-3 \\ \frac{du}{dx} &= 2 \\ dx &= u \, du \\ 2x-3 &= u^2 \\ 2x-1 &= u^2+2 \end{aligned}$$

Question 77 (***)

$$\frac{3x^2 - 2x + 1}{2x(x-1)^2} \equiv \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}.$$

- a) Determine the value of each of the constants A , B and C .
- b) Evaluate

$$\int_4^9 \frac{3x^2 - 2x + 1}{2x(x-1)^2} dx,$$

giving the answer in the form $p + q \ln 4$, where p and q are constants.

$$\boxed{A = \frac{1}{2}}, \quad \boxed{B = 1}, \quad \boxed{C = 1}, \quad \boxed{\frac{5}{24} + \ln 4}$$

(a) $\frac{3x^2 - 2x + 1}{2x(x-1)^2} \equiv \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$

$$\frac{1}{2}(3x^2 - 2x + 1) \equiv A(x-1)^2 + Bx(x-1) + Cx$$

- If $x=1 \Rightarrow 1 = C$
- If $x=0 \Rightarrow \frac{1}{2} = A$
- If $x=2 \Rightarrow \frac{3}{2} = A + 2B + 2C$
 $\frac{3}{2} = \frac{1}{2} + 2B + 2$
 $2B = 2$
 $B = 1$

$\therefore A = \frac{1}{2}$
 $B = 1$
 $C = 1$

(b) $\int_4^9 \frac{3x^2 - 2x + 1}{2x(x-1)^2} dx = \int_4^9 \left[\frac{\frac{1}{2}}{x} + \frac{1}{x-1} + \frac{1}{(x-1)^2} \right] dx$

$$= \left[\frac{1}{2} \ln|x| + \ln|x-1| - \frac{1}{x-1} \right]_4^9$$

$$= \left(\frac{1}{2} \ln 9 + \ln 8 - \frac{1}{8} \right) - \left(\frac{1}{2} \ln 4 + \ln 3 - \frac{1}{2} \right)$$

$$= \ln 3 + \ln 8 - \frac{1}{8} - \ln 2 - \ln 3 + \frac{1}{2}$$

$$= \ln 4 + \frac{5}{24}$$

Question 78 (***)

It is given that

$$\sin 3x \equiv 3\sin x - 4\sin^3 x.$$

- a) Prove the above trigonometric identity, by writing $\sin 3x$ as $\sin(2x+x)$.
- b) Hence, or otherwise, find the exact value of

$$\int_0^{\frac{\pi}{2}} \sin^3 x \, dx.$$

$$\frac{2}{3}$$

(a) $\sin 3x = \sin(2x+x) = \sin 2x \cos x + \cos 2x \sin x$
 $= (2\sin x \cos x) \cos x + (1-2\sin^2 x) \sin x$
 $= 2\sin x \cos^2 x + \sin x - 2\sin^3 x$
 $= 2\sin x (1-\sin^2 x) + \sin x - 2\sin^3 x$
 $= 2\sin x - 2\sin^3 x + \sin x - 2\sin^3 x$
 $= 3\sin x - 4\sin^3 x$
 $= \text{RHS}$

(b) $\int_0^{\frac{\pi}{2}} \sin^3 x \, dx = \dots$
 $\sin^3 x = 3\sin x - 4\sin^3 x$
 $4\sin^3 x = 3\sin x - \sin^3 x$
 $\sin^3 x = \frac{3}{4}\sin x - \frac{1}{4}\sin^3 x$
 $= \dots \int_0^{\frac{\pi}{2}} \left(\frac{3}{4}\sin x - \frac{1}{4}\sin^3 x \right) dx$
 $= \left[-\frac{3}{4}\cos x + \frac{1}{4}\cos^3 x \right]_0^{\frac{\pi}{2}} =$
 $= \left[-\frac{3}{4}\cos \frac{\pi}{2} + \frac{1}{4}\cos^3 \frac{\pi}{2} \right] - \left[-\frac{3}{4}\cos 0 + \frac{1}{4}\cos^3 0 \right]$
 $= -\left(-\frac{3}{4} + \frac{1}{4} \right) = \frac{1}{2}$

Question 79 (***)

Use integration by parts to find an exact value for

$$\int_1^{\frac{\pi}{3}} 2 \sin x \ln(\sec x) \, dx.$$

$$1 - \ln 2$$

Handwritten solution for the integral:

$$\begin{aligned} \int_1^{\frac{\pi}{3}} 2 \sin x \ln(\sec x) \, dx &= \dots \text{by parts \& ignoring limits} \\ &= -(2 \cos x) \ln(\sec x) - \int -2 \cos x \tan x \, dx \\ &= -(2 \cos x) \ln(\sec x) + \int 2 \sin x \, dx \\ &= -2 \cos x \ln(\sec x) - 2 \cos x + C \\ &= \dots \text{limits} \\ &= \left[-2 \cos x \ln(\sec x) - 2 \cos x \right]_1^{\frac{\pi}{3}} = \left[2 \cos x \ln(\sec x) + 2 \cos x \right]_{\frac{\pi}{3}}^1 \\ &= [0 + 2] - [\ln 2 + 1] = 1 - \ln 2 \end{aligned}$$

Question 80 (***)

x	0	$\frac{\pi}{18}$	$\frac{\pi}{9}$	$\frac{\pi}{6}$
y	0	0.1632		0.2500

The table above shows tabulated values for the equation

$$y = \sin x \cos 2x, \quad 0 \leq x \leq \frac{\pi}{6}.$$

- a) Complete the missing value in the table. (1)
- b) Use the trapezium rule with all the values from the table to find an approximate value for

$$\int_0^{\frac{\pi}{6}} \sin x \cos 2x \, dx. \quad (3)$$

- c) By using the substitution $u = \cos x$, or otherwise, find an exact value for (6)

$$\int_0^{\frac{\pi}{6}} \sin x \cos 2x \, dx.$$

$$\boxed{}, \boxed{0.2620}, \boxed{0.096}, \boxed{\frac{1}{12}(3\sqrt{3}-4)}$$

a) $y = \sin x \cos 2x \approx 0.9397$

b) $\int_0^{\frac{\pi}{6}} \sin x \cos 2x \, dx \approx \frac{h}{3} [y_0 + 4y_1 + 2y_2 + y_3]$
 $= \frac{\pi/18}{3} [0 + 4(0.1632) + 2(0.2500) + 0]$
 $= 0.09602 \dots$
 ≈ 0.096

c) $\int_0^{\frac{\pi}{6}} \sin x \cos 2x \, dx$
 $= \int_{\cos 0}^{\cos \frac{\pi}{6}} \sin x \cos 2x \, dx$
 $= \int_{1}^{\frac{\sqrt{3}}{2}} -\cos 2x \, dx = \int_{\frac{\sqrt{3}}{2}}^1 \cos 2x \, dx$
 $= \left[\frac{1}{2} \sin 2x \right]_{\frac{\sqrt{3}}{2}}^1 = \left(\frac{1}{2} \sin 2 - \frac{1}{2} \sin \sqrt{3} \right)$
 $= \frac{1}{2} \left(\sin 2 - \sin \sqrt{3} \right) \approx \frac{1}{2} (0.9093 - 0.8660) \approx 0.0216$
 $\approx \frac{1}{12}(3\sqrt{3}-4)$

Question 81 (***)

$$\frac{12x^2 + x + 3}{(6x+1)(2x^2+1)} \equiv \frac{A}{6x+1} + \frac{Bx+C}{2x^2+1}.$$

- a) Determine the value of each of the constants A , B and C .
- b) Evaluate

$$\int_0^2 \frac{12x^2 + x + 3}{(6x+1)(2x^2+1)} dx,$$

giving the answer in the form $p \ln q$, where p and q are constants.

$$\boxed{A=3}, \boxed{B=1}, \boxed{C=0}, \boxed{\frac{1}{2} \ln 39}$$

(a) $\frac{12x^2+x+3}{(6x+1)(2x^2+1)} \equiv \frac{A}{6x+1} + \frac{Bx+C}{2x^2+1}$

$12x^2+x+3 \equiv A(2x^2+1) + (6x+1)(Bx+C)$

- If $x = -\frac{1}{6}$, $\frac{19}{18} = \frac{19}{18}A \Rightarrow A=3$
- If $x=0$, $3 = A+C \Rightarrow C=0$
- If $x=1$, $16 = 34 + 7(B+C)$
 $16 = 7 + 7B$
 $B=1$

(b) $\int_0^2 \frac{12x^2+x+3}{(6x+1)(2x^2+1)} dx = \int_0^2 \frac{3}{6x+1} + \frac{x}{2x^2+1} dx$

$= \int_0^2 \frac{3}{6x+1} + \frac{1}{2} \left(\frac{1}{2x^2+1} \right) dx$

$= \left(\frac{1}{2} \ln|6x+1| + \frac{1}{4} \ln(2x^2+1) \right) \Big|_0^2 = \left(\frac{1}{2} \ln 13 + \frac{1}{4} \ln 5 \right) - \left(\frac{1}{2} \ln 1 + \frac{1}{4} \ln 1 \right)$

$= \frac{1}{2} \ln 13 + \frac{1}{4} \ln 5 = \frac{1}{2} (\ln 13 + \ln 5) = \frac{1}{2} \ln 39$

Question 82 (***)

By using the trigonometric identity

$$\cos 2\theta \equiv 2\cos^2 \theta - 1$$

and the fact that $\frac{d}{dx}(\tan x) = \sec^2 x$, show clearly that

$$\int_0^{\frac{\pi}{3}} \frac{1}{1 + \cos x} dx = \frac{\sqrt{3}}{3}.$$

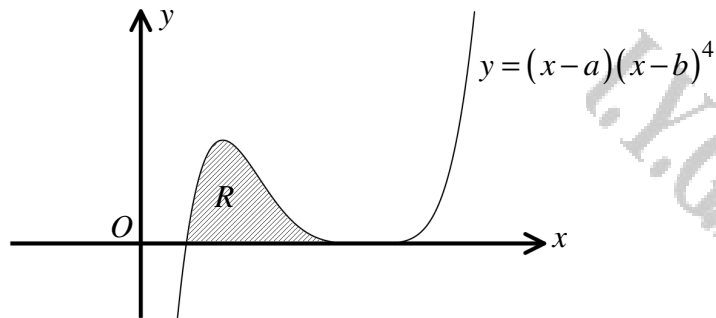
proof

Handwritten proof showing the integration of $\frac{1}{1 + \cos x}$ from 0 to $\frac{\pi}{3}$. The steps are as follows:

$$\begin{aligned} \int_0^{\frac{\pi}{3}} \frac{1}{1 + \cos x} dx &= \dots \text{ (using the identity } \cos 2\theta = 2\cos^2 \theta - 1 \text{)} \\ &= \int_0^{\frac{\pi}{3}} \frac{1}{2\cos^2 \frac{x}{2}} dx = \int_0^{\frac{\pi}{3}} \frac{1}{2} \sec^2 \frac{x}{2} dx \\ &= \left[\tan \frac{x}{2} \right]_0^{\frac{\pi}{3}} = \tan \frac{\pi}{6} - \tan 0 = \frac{\sqrt{3}}{3} \end{aligned}$$

The final result is $\frac{\sqrt{3}}{3}$, which is the required value.

Question 83 (***)



The figure above shows the graph of the curve with equation

$$y = (x-a)(x-b)^4,$$

where a and b are positive constants.

The shaded region R is bounded by the curve and the x axis.

By using integration by parts, or otherwise, show that the area of the shaded region is

$$\frac{1}{30}(a-b)^6.$$

proof

$$\begin{aligned} R &= \int_a^b (x-a)(x-b)^4 dx \\ &\Rightarrow R = \left[\frac{1}{2}(x-a)(x-b)^4 - \int \frac{1}{2}(x-b)^4 dx \right]_a^b \\ &\Rightarrow R = \left[\frac{1}{2}(x-a)(x-b)^4 - \frac{1}{20}(x-b)^5 \right]_a^b \\ &\Rightarrow R = \left(0 - 0 \right) - \left(0 - \frac{1}{20}(a-b)^5 \right) \\ &\Rightarrow R = \frac{1}{20}(a-b)^5 \end{aligned}$$

By using
 $\frac{d}{dx}(x-a) = 1$
 $\frac{d}{dx}(x-b)^5 = 5(x-b)^4$
 $\frac{1}{5} \frac{d}{dx}(x-b)^5 = (x-b)^4$

As required

Question 84 (***)

By using the substitution $u = \sqrt{x}$, or otherwise, show that

$$\int_0^{36} \frac{1}{\sqrt{x}(\sqrt{x}+2)} dx = \ln 16.$$

proof

Handwritten solution for Question 84:

Method 1: Direct integration

$$\int_0^{36} \frac{dx}{\sqrt{x}(\sqrt{x}+2)} = \int_0^{36} \frac{x^{-1/2}}{x^{1/2}+2} dx$$

Let $u = \sqrt{x} \Rightarrow \frac{du}{dx} = \frac{1}{2\sqrt{x}} \Rightarrow dx = 2\sqrt{x} du = 2u du$

When $x=0, u=0$; when $x=36, u=6$

$$= \int_0^6 \frac{2u du}{u(u+2)} = 2 \int_0^6 \frac{du}{u+2} = 2 [\ln|u+2|]_0^6 = 2(\ln 8 - \ln 2) = 2 \ln 4 = \ln 16$$

Method 2: Substitution

$$\int_0^{36} \frac{1}{\sqrt{x}(\sqrt{x}+2)} dx = \int_0^6 \frac{1}{u(u+2)} \cdot 2u du$$

Let $u = \sqrt{x} \Rightarrow \frac{du}{dx} = \frac{1}{2\sqrt{x}} \Rightarrow dx = 2\sqrt{x} du = 2u du$

When $x=0, u=0$; when $x=36, u=6$

$$= \int_0^6 \frac{2u du}{u(u+2)} = 2 \int_0^6 \frac{du}{u+2} = 2 [\ln|u+2|]_0^6 = 2(\ln 8 - \ln 2) = 2 \ln 4 = \ln 16$$

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Question 85 (***)

By using the substitution $u = 3x-1$, or otherwise, find

$$\int \frac{9x^2}{3x-1} dx.$$

$$\frac{1}{6}(3x-1)^2 + \frac{2}{3}(3x-1) + \frac{1}{3} \ln|3x-1| + C$$

Handwritten solution for Question 85:

Let $u = 3x-1 \Rightarrow \frac{du}{dx} = 3 \Rightarrow dx = \frac{du}{3}$

$$\int \frac{9x^2}{3x-1} dx = \int \frac{9x^2}{u} \cdot \frac{du}{3} = 3 \int \frac{x^2}{u} du$$

Let $u = 3x-1 \Rightarrow x = \frac{u+1}{3} \Rightarrow x^2 = \frac{(u+1)^2}{9}$

$$= 3 \int \frac{\frac{(u+1)^2}{9}}{u} du = \int \frac{(u+1)^2}{u} du = \int \frac{u^2 + 2u + 1}{u} du$$

$$= \int \left(u + \frac{2}{1} + \frac{1}{u} \right) du = \frac{1}{2}u^2 + 2u + \ln|u| + C$$

Substitute back $u = 3x-1$

$$= \frac{1}{2}(3x-1)^2 + 2(3x-1) + \ln|3x-1| + C$$

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Question 86 (*)**

By using the substitution $u = 2x + 3$, or otherwise, show clearly that

$$\int_{-1}^0 6 \ln(2x+3) \, dx = 9 \ln 3 - 6.$$

proof

Handwritten proof for Question 86:

$$\begin{aligned} \int_{-1}^0 6 \ln(2x+3) \, dx &= \dots \text{using the substitution } u=2x+3 \\ &= \int_1^3 6 \ln u \times \frac{du}{2} \\ &= \int_1^3 3 \ln u \, du \\ &= \dots \text{by parts, ignoring limits} \\ &= 3u \ln u - \int 3u \left(\frac{1}{u}\right) du \\ &= 3u \ln u - \int 3 \, du \\ &= 3u \ln u - 3u + C \\ &= \dots \text{limits} \dots \\ &= \left[3u \ln u - 3u \right]_1^3 \\ &= (9 \ln 3 - 9) - (3 \ln 1 - 3) \\ &= 9 \ln 3 - 6 \end{aligned}$$

Side notes in the proof:

- $u = 2x + 3$
- $\frac{du}{dx} = 2$
- $dx = \frac{du}{2}$
- $x = -1, u = 1$
- $x = 0, u = 3$
- Table for integration by parts:

$\ln u$	$\frac{1}{u}$
$3u$	3

Question 87 (*)**

By using the substitution $u = \tan x$, or otherwise, find

$$\int \sec^4 x \, dx.$$

$$\tan x + \frac{1}{3} \tan^3 x + C$$

Handwritten proof for Question 87:

$$\begin{aligned} \int \sec^4 x \, dx &= \int \sec^2 x \frac{du}{\sec^2 x} = \int \sec^2 x \, du \\ &\text{but } 1 + \tan^2 x = \sec^2 x \\ &= \int (1 + \tan^2 x) \, du = \int (1 + u^2) \, du \\ &= u + \frac{1}{3} u^3 + C = \tan x + \frac{1}{3} \tan^3 x + C \end{aligned}$$

Side notes in the proof:

- $u = \tan x$
- $\frac{du}{dx} = \sec^2 x$
- $dx = \frac{du}{\sec^2 x}$

Question 88 (***)

$$f(t) \equiv \frac{2}{(t-1)(t+1)} \equiv \frac{A}{(t-1)} + \frac{B}{(t+1)}.$$

- a) Find the value of each of the constants A and B in the above identity.
- b) Use the substitution $x = t^2 - 2$, $t > 0$ to show that

$$\int_4^9 \frac{1}{(x+1)\sqrt{x+2}} dx = \ln \frac{3}{2}.$$

$$\boxed{A=2}, \boxed{B=2}$$

a) $f(t) = \frac{2}{(t-1)(t+1)} = \frac{A}{t-1} + \frac{B}{t+1}$
 $2 = A(t+1) + B(t-1)$
 If $t=1 \Rightarrow 2 = 2B \Rightarrow B=1$
 If $t=-1 \Rightarrow 2 = -2A \Rightarrow A=-1$

b) Let $x = t^2 - 2$, $t > 0$
 $\frac{dx}{dt} = 2t$
 $dx = 2t dt$
 $x=4 \Rightarrow t=2$
 $x=9 \Rightarrow t=3$

$\int_4^9 \frac{1}{(x+1)\sqrt{x+2}} dx = \dots$ APPLY THE SUBSTITUTION
 $= \int_2^3 \frac{1}{(t^2-1)\sqrt{t^2}} (2t dt)$
 $= \int_2^3 \frac{2t}{(t^2-1)t} dt = \int_2^3 \frac{2}{t^2-1} dt$
 $= \int_2^3 \left(\frac{1}{t-1} - \frac{1}{t+1} \right) dt = \left[\ln|t-1| - \ln|t+1| \right]_2^3$
 $= (\ln 2 - \ln 4) - (\ln 1 - \ln 3) = \ln 2 - \ln 4 + \ln 3 = \ln \frac{3}{2}$

Question 89 (***)

x	0	$\frac{2\pi}{5}$	$\frac{4\pi}{5}$	$\frac{6\pi}{5}$	$\frac{8\pi}{5}$	2π
y	0	0.2031	0.8602			0

The table above shows some tabulated values for the equation

$$y = \sin^3\left(\frac{1}{2}x\right), \quad 0 \leq x \leq 2\pi.$$

- a) Complete the missing values in the table.
- b) Use the trapezium rule with all the values from the table to find an approximate value for

$$\int_0^{2\pi} \sin^3\left(\frac{1}{2}x\right) dx.$$

- c) By using the substitution $u = \cos\left(\frac{1}{2}x\right)$, or otherwise, find the value of the integral of part (b).

$$\boxed{}, \boxed{0.8602, 0.2031}, \boxed{2.672}, \boxed{\frac{8}{3}}$$

Handwritten solution for Question 89:

a) $y = \sin^3\left(\frac{\pi}{5}\right) = 0.8602$
 $y = \sin^3\left(\frac{3\pi}{5}\right) = 0.2031$

b) $\int_0^{2\pi} \sin^3\left(\frac{1}{2}x\right) dx = \frac{\text{Trapezium}}{2} \left[\text{First} + \text{Last} + 2 \times \text{Rest} \right]$
 $= \frac{2\pi \times 2}{2} [0 + 0 + 2(0.2031 + 0.8602 + \dots + 0.2031)]$
 ≈ 2.672

c) $\int_0^{2\pi} \sin^3\left(\frac{1}{2}x\right) dx$
 $= \int_{-1}^1 \sin^2\left(\frac{1}{2}x\right) \times \frac{du}{-\frac{1}{2}\cos\left(\frac{1}{2}x\right)}$
 $= \int_{-1}^1 -2\sin^2\left(\frac{1}{2}x\right) du$
 $= \int_{-1}^1 2\sin^2\left(\frac{1}{2}x\right) du$
 $= \int_{-1}^1 2(1 - \cos^2\left(\frac{1}{2}x\right)) du$
 $= \int_{-1}^1 2(1 - u^2) du$
 $= \int_{-1}^1 (2 - 2u^2) du$
 $= \left[2u - \frac{2}{3}u^3 \right]_{-1}^1$
 $= \left(2 - \frac{2}{3} \right) - \left(-2 + \frac{2}{3} \right)$
 $= \frac{4}{3} - \left(-\frac{4}{3} \right) = \frac{8}{3}$

Question 90 (***)

Use trigonometric identities to integrate

$$\int \frac{\cos 2x}{1 - \cos^2 2x} dx.$$

, $-\frac{1}{2} \operatorname{cosec} 2x + C$

$$\int \frac{\cos 2x}{1 - \cos^2 2x} dx = \int \frac{\cos 2x}{\sin^2 2x} dx = \int \frac{\cos 2x}{\sin 2x} \cdot \frac{1}{\sin 2x} dx$$

$$\int \cot 2x \operatorname{cosec} 2x dx = -\frac{1}{2} \operatorname{cosec} 2x + C //$$

Question 91 (****)By using the substitution $u = 2x^{\frac{5}{2}} + 1$, or otherwise, find an exact simplified value for

$$\int_0^1 \frac{10x^4}{2x^{\frac{5}{2}} + 1} dx.$$

, $2 - \ln 3$

$$\int_0^1 \frac{10x^4}{2x^{\frac{5}{2}} + 1} dx = \dots \text{by substitution}$$

$$= \int_1^3 \frac{10u^{\frac{4}{5}}}{u} \frac{du}{5u^{\frac{3}{5}}} = \int_1^3 \frac{2u^{\frac{4}{5}}}{u} du$$

$$= \int_1^3 \frac{u - 1}{u} du = \int_1^3 \left(1 - \frac{1}{u}\right) du$$

$$= \left[u - \ln|u|\right]_1^3 = (3 - \ln 3) - (1 - \ln 1)$$

$$= 2 - \ln 3 //$$

$u = 2x^{\frac{5}{2}} + 1$
 $\frac{du}{dx} = 5x^{\frac{3}{2}}$
 $\frac{du}{dx} = \frac{du}{5x^{\frac{3}{2}}}$
 $2 \times 0 \rightarrow u = 1$
 $2 \times 1 \rightarrow u = 3$
 $2x^{\frac{5}{2}} = u - 1$

Question 92 (****)

$$\frac{2u^2}{u-1} \equiv Au + B + \frac{C}{u-1}.$$

- a) Find the value of each of the constants A , B and C in the above identity.
- b) Use the substitution $u = \sqrt{x}$ to show

$$\int_4^9 \frac{\sqrt{x}}{\sqrt{x}-1} dx = 7 + 2 \ln 2.$$

$$A=2, B=2, C=2$$

(a) $\frac{2u^2}{u-1} \equiv Au + B + \frac{C}{u-1}$

$2u^2 \equiv Au(u-1) + B(u-1) + C$

$2u^2 \equiv Au^2 - Au + Bu - B + C$

$2u^2 \equiv Au^2 + (B-A)u + (C-B)$

$\bullet A=2$
 $\bullet B-A=0 \Rightarrow B=2$
 $\bullet C-B=0 \Rightarrow C=2$

(b) $\int_4^9 \frac{\sqrt{x}}{\sqrt{x}-1} dx = \dots$ by substitution

$u = \sqrt{x}$
 $\frac{du}{dx} = \frac{1}{2\sqrt{x}} \Rightarrow 2u \frac{du}{dx} = 1$

$= \int_2^3 \frac{2u^2}{u-1} du = \int_2^3 \frac{2u^2}{u-1} du = \int_2^3 \frac{2u^2}{u-1} du$

$= \left[u^2 + 2u + 2 \ln|u-1| \right]_2^3$

$= (9 + 6 + 2 \ln 2) - (4 + 4 + 2 \ln 1) = 7 + 2 \ln 2$

Question 93 (****)

$$\int_1^9 \frac{1}{2x(1+\sqrt{x})} dx.$$

- a) Show that the substitution $u = \sqrt{x}$ transforms the above integral to

$$\int_{x_1}^{x_2} \frac{1}{u(u+1)} du,$$

where x_1 and x_2 are constants to be found.

- b) Hence find an exact value for the original integral.

$$\ln\left(\frac{3}{2}\right)$$

(a) $\int_1^9 \frac{1}{2x(1+\sqrt{x})} dx \dots$ USE THE SUBSTITUTION $u = \sqrt{x}$
 $u^2 = x$
 $2u \frac{du}{dx} = 1$
 $2u du = dx$
 $u=1, x=1$
 $u=3, x=9$

(b) BY PARTIAL FRACTIONS $\frac{1}{u(u+1)} = \frac{A}{u} + \frac{B}{u+1}$ If $u=0, 1=A$
 $1 = A(u+1) + Bu$ If $u=-1, 1=-B$
 $\therefore \int_1^9 \frac{1}{u} - \frac{1}{u+1} du = [\ln|u| - \ln|u+1|]_1^9$
 $= (\ln 9 - \ln 10) - (\ln 1 - \ln 2) = \ln 9 - \ln 10 + \ln 2$
 $= \ln \frac{3 \times 2}{4} = \ln \frac{3}{2}$

Question 94 (****)

$$f(x) \equiv \frac{x^3}{x^2 - 4}, \quad x \neq \pm 2.$$

- a) Use a suitable substitution to show that

$$\int_{\sqrt{6}}^{\sqrt{8}} f(x) \, dx = 1 + \ln 4.$$

- b) Express $f(x)$ in the form

$$Ax + B + \frac{C}{x-2} + \frac{D}{x+2},$$

where A , B , C and D are constants to be found.

- c) Use the result part (b) to verify the result of part (a).

$$\boxed{A=1}, \quad \boxed{B=0}, \quad \boxed{C=2}, \quad \boxed{D=2}$$

(a) $\int_{\sqrt{6}}^{\sqrt{8}} \frac{x^3}{x^2-4} \, dx = \dots$ substitution first...

$u = x^2 - 4$
 $\frac{du}{dx} = 2x$
 $dx = \frac{du}{2x}$
 $2x \sqrt{u}, u=2$
 $2x^2 = u+4$

$= \int_2^4 \frac{x^3}{x^2-4} \cdot \frac{du}{2x} = \int_2^4 \frac{x^2}{2(x^2-4)} du = \int_2^4 \frac{u+4}{2(u+4)} du$
 $= \int_2^4 \frac{1}{2} + \frac{2}{u+4} du = \left[\frac{1}{2}u + 2 \ln|u+4| \right]_2^4$
 $= \left(\frac{1}{2}(4) + 2 \ln 8 \right) - \left(\frac{1}{2}(2) + 2 \ln 6 \right) = 1 + 2 \ln 4 - 2 \ln 2$
 $= 1 + 2(\ln 4 - \ln 2) = 1 + 2 \ln 2 = 1 + \ln 4$

(b) $\frac{x^3}{x^2-4} = \frac{x^3}{(x-2)(x+2)} = Ax + B + \frac{C}{x-2} + \frac{D}{x+2}$
 $\frac{x^3}{(x-2)(x+2)} = \frac{Ax+B}{(x-2)(x+2)} + \frac{C}{x-2} + \frac{D}{x+2}$
 $\frac{x^3}{(x-2)(x+2)} = \frac{(Ax+B)(x-2)(x+2) + C(x+2) + D(x-2)}{(x-2)(x+2)}$
 $x^3 = (Ax+B)(x^2-4) + C(x+2) + D(x-2)$
 $x^3 = Ax^3 + Bx^2 - 4Ax - 4B + Cx + 2C + Dx - 2D$
 $x^3 = Ax^3 + Bx^2 + (-4A+C+D)x + (-4B+2C-2D)$
 $\bullet \text{ If } x^3: A=1$
 $\bullet \text{ If } x^2: B=0$
 $\bullet \text{ If } x: -4A+C+D=0 \Rightarrow -4(1)+C+D=0 \Rightarrow C+D=4$
 $\bullet \text{ If } 1: -4B+2C-2D=0 \Rightarrow 0+2C-2D=0 \Rightarrow C=D$
 $C+D=4$
 $2C=4$
 $C=2$
 $D=2$

(c) $\int_{\sqrt{6}}^{\sqrt{8}} \frac{x^3}{x^2-4} \, dx = \int_{\sqrt{6}}^{\sqrt{8}} \left(x + \frac{2}{x-2} + \frac{2}{x+2} \right) dx$
 $= \left[\frac{1}{2}x^2 + 2 \ln|x-2| + 2 \ln|x+2| \right]_{\sqrt{6}}^{\sqrt{8}}$
 $= \left(\frac{1}{2}(4) + 2 \ln 2 + 2 \ln 4 \right) - \left(\frac{1}{2}(6) + 2 \ln 4 + 2 \ln 2 \right)$
 $= (2 + 2 \ln 4) - (3 + 2 \ln 2) = 1 + 2 \ln 4 - 2 \ln 2$
 $= 1 + \ln 4 - \ln 2 = 1 + \ln 4$

Question 95 (**)**

By using the cosine double angle identities and the fact that $\frac{d}{dx}(\tan x) = \sec^2 x$, show clearly that

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1 - \cos 2x}{1 + \cos 2x} dx = \frac{1}{6}(4\sqrt{3} - \pi).$$

proof

Handwritten proof for Question 95:

$$\begin{aligned} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1 - \cos 2x}{1 + \cos 2x} dx &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1 - (1 - 2\sin^2 x)}{1 + (2\cos^2 x - 1)} dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{2\sin^2 x}{2\cos^2 x} dx \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \tan^2 x dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (\sec^2 x - 1) dx = \left[\tan x - x \right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} \\ &= \left[\tan \frac{\pi}{3} - \frac{\pi}{3} \right] - \left[\tan \frac{\pi}{6} - \frac{\pi}{6} \right] = \left(\sqrt{3} - \frac{\pi}{3} \right) - \left(\frac{\sqrt{3}}{3} - \frac{\pi}{6} \right) \\ &= \frac{2\sqrt{3}}{3} - \frac{\pi}{6} = \frac{1}{6}(4\sqrt{3} - \pi) \quad \text{A.S. 21/0/00} \end{aligned}$$

Question 96 (**)**

Use the substitution $x = \sin \theta$ to find the exact value of

$$\int_0^{\frac{1}{2}} \frac{1}{(1-x^2)^{\frac{3}{2}}} dx.$$

 $\frac{\sqrt{3}}{3}$

Handwritten proof for Question 96:

$$\begin{aligned} \int_0^{\frac{1}{2}} \frac{1}{(1-x^2)^{\frac{3}{2}}} dx &= \int_0^{\frac{\pi}{6}} \frac{1}{(1-\sin^2 \theta)^{\frac{3}{2}}} \cos \theta d\theta \\ &= \int_0^{\frac{\pi}{6}} \frac{\cos \theta}{(\cos^2 \theta)^{\frac{3}{2}}} d\theta = \int_0^{\frac{\pi}{6}} \frac{\cos \theta}{\cos^3 \theta} d\theta \\ &= \int_0^{\frac{\pi}{6}} \frac{1}{\cos^2 \theta} d\theta = \int_0^{\frac{\pi}{6}} \sec^2 \theta d\theta \\ &= \left[\tan \theta \right]_0^{\frac{\pi}{6}} = \tan \frac{\pi}{6} - \tan 0 = \frac{\sqrt{3}}{3} \end{aligned}$$

Substitution details:

- $x = \sin \theta$
- $\frac{dx}{d\theta} = \cos \theta$
- $x=0, \theta=0$
- $x=\frac{1}{2}, \theta=\frac{\pi}{6}$

Question 97 (****)

$$f(x) \equiv \frac{1}{x(x^2+1)}, \quad x \neq 0.$$

- a) Use the substitution $x = \tan \theta$ to find

$$\int f(x) dx.$$

- b) Find the value of each of the constants A , B and C , so that

$$f(x) \equiv \frac{A}{x} + \frac{Bx+C}{x^2+1}.$$

- c) Use the result of part (b) to independently verify the answer of part (a).

$$\ln\left(\frac{x}{\sqrt{x^2+1}}\right), \quad \boxed{A=1}, \quad \boxed{B=-1}, \quad \boxed{C=0}$$

(a) $\int \frac{1}{x(x^2+1)} dx$ by substitution
 $= \int \frac{1}{\tan \theta (\tan^2 \theta + 1)} \sec^2 \theta d\theta = \int \frac{\sec \theta}{\tan \theta \sec^2 \theta} d\theta$
 $= \int \cos \theta d\theta = \ln|\sin \theta| + C$
 $= \ln\left|\frac{x}{\sqrt{x^2+1}}\right| + C$

(b) $\frac{1}{x(x^2+1)} \equiv \frac{A}{x} + \frac{Bx+C}{x^2+1}$
 $1 \equiv A(x^2+1) + x(Bx+C)$
 $1 \equiv Ax^2 + A + Bx^2 + Cx$
 $1 \equiv (A+B)x^2 + Cx + A$
 $\therefore A=1, C=0, B=-1$

(c) $\int \frac{1}{x(x^2+1)} dx = \int \frac{1}{x} - \frac{x}{x^2+1} dx = \int \frac{1}{x} - \frac{1}{2} \left(\frac{2x}{x^2+1} \right) dx$
 $= \ln|x| - \frac{1}{2} \ln|x^2+1| + C$
 $= \ln|x| - \ln\sqrt{x^2+1} + C$
 $= \ln\left|\frac{x}{\sqrt{x^2+1}}\right| + C$

Question 98 (****)

$$\frac{4x^3 - x^2 + 20x - 4}{x^2 + 4} \equiv Ax + B + \frac{Cx + D}{x^2 + 4}.$$

a) Determine the value of each of the constants A , B , C and D .

b) Hence show that

$$\int_0^2 \frac{4x^3 - x^2 + 20x - 4}{x^2 + 4} dx = 6 + 2\ln 2,$$

$$A=4, B=-1, C=4, D=0$$

(a) $\frac{4x^3 - x^2 + 20x - 4}{x^2 + 4} \equiv Ax + B + \frac{Cx + D}{x^2 + 4}$
 $4x^3 - x^2 + 20x - 4 \equiv (Ax + B)(x^2 + 4) + Cx + D$
 $\equiv Ax^3 + Bx^2 + 4Ax + 4B + Cx + D$
 $\equiv Ax^3 + Bx^2 + (4A + C)x + (4B + D)$
 $\therefore A=4, B=-1, 4A+C=20, 4B+D=-4$
 $C=4, D=0$

(b) $\int_0^2 \frac{4x^3 - x^2 + 20x - 4}{x^2 + 4} dx = \int_0^2 (4x - 1 + \frac{4x}{x^2 + 4}) dx$
 $= \int_0^2 4x - 1 + 2\left(\frac{2x}{x^2 + 4}\right) dx$
 $= [2x^2 - x + 2\ln(x^2 + 4)]_0^2$
 $= (8 - 2 + 2\ln 8) - (0 + 2\ln 4)$
 $= 6 + 2\ln 8 - 2\ln 4 = 6 + 2\ln 2$

Question 99 (****)

Use a cosine double identity and integration by parts to find

$$\int 4x \cos^2 x dx.$$

$$x^2 + x \sin 2x + \frac{1}{2} \cos 2x + C$$

$\int 4x \cos^2 x dx = \int 4x \left(\frac{1 + \cos 2x}{2}\right) dx$
 $= \int 2x + 2x \cos 2x dx$
 $= \int 2x dx + \int 2x \cos 2x dx$
 $= x^2 + \int 2x \cos 2x dx$
 $= x^2 + 2x \sin 2x - \int 2 \sin 2x dx$
 $= x^2 + 2x \sin 2x + \cos 2x + C$

Question 100 (**)**

By using the substitution $u = 2x^2 - 8x + 3$, or otherwise, find the exact value

$$\int_4^6 \frac{x-2}{2x^2-8x+3} dx.$$

$$\boxed{\frac{1}{2} \ln 3}$$

Question 101 (**)**

$$\frac{6u}{(u-2)(u+1)} \equiv \frac{A}{u-2} + \frac{B}{u+1}.$$

a) Find the value of each of the constants A and B in the above identity.

b) By using the substitution $u = \sqrt{x}$, or otherwise, show that

$$\int_0^1 \frac{3}{(\sqrt{x}-2)(\sqrt{x}+1)} dx = -\ln 4.$$

$$\boxed{A=4}, \boxed{B=2}$$

Question 102 (****)

$$\frac{4t^2}{t-1} \equiv At + B + \frac{C}{t-1}.$$

a) Determine the value of each of the constants A , B and C .

b) Use the substitution $t = x^{\frac{1}{4}}$ to show

$$\int_{16}^{81} \frac{1}{x^{\frac{1}{2}} - x^{\frac{1}{4}}} dx = 14 + 4 \ln 2.$$

$$\boxed{}, \boxed{A=4}, \boxed{B=4}, \boxed{C=4}$$

a) By Division

$$\frac{4t^2}{t-1} = \frac{4t(t-1) + 4(t-1) + 4}{t-1} = 4t + 4 + \frac{4}{t-1}$$

$\therefore A=B=C=4$

ALTERNATIVE BY ALGEBRAIC DIVISION

$$\begin{array}{r} 4t+4 \\ t-1 \overline{) 4t^2} \\ \underline{-4t^2+4t} \\ 4t \\ \underline{-4t+4} \\ 4 \end{array}$$

$\therefore \frac{4t^2}{t-1} = 4t + 4 + \frac{4}{t-1}$

$\therefore A=B=C=4$

ALTERNATIVE BY COMPARING

$$\Rightarrow \frac{4t^2}{t-1} \equiv \frac{At^2+Bt+C}{t-1}$$

$$\Rightarrow \frac{4t^2}{t-1} \equiv \frac{At(t-1) + B(t-1) + C}{t-1}$$

$$\Rightarrow 4t^2 \equiv At^2 - At + Bt - B + C$$

$$\Rightarrow 4t^2 \equiv At^2 + (B-A)t + (C-B)$$

$\therefore A=4 \quad B-A=0 \quad C-B=0$
 $A=4 \quad B=4 \quad C=4$

b) USING THE SUBSTITUTION GIVEN

$$\int_{16}^{81} \frac{1}{x^{\frac{1}{2}} - x^{\frac{1}{4}}} dx = \dots$$

$$= \int_2^3 \frac{1}{t^2 - t} (4t^3 dt) = \int_2^3 \frac{4t^3}{t(t-1)} dt$$

$$= \int_2^3 \frac{4t^2}{t-1} dt$$

USING PART (a)

$$= \int_2^3 4t + 4 + \frac{4}{t-1} dt$$

$$= [2t^2 + 4t + 4 \ln|t-1|]_2^3$$

$$= (18 + 12 + 4 \ln 2) - (8 + 8 + 4 \ln 1)$$

$$= 14 + 4 \ln 2$$

\therefore Proved

$t = x^{\frac{1}{4}}$
 $t^4 = x$
 $4t^3 = \frac{dx}{dt}$
 $dx = 4t^3 dt$
 $x=16 \rightarrow t=2$
 $x=81 \rightarrow t=3$

Question 103 (****)

It is given that

$$\int_k^{2k} \frac{3x-5}{x(x-1)} dx = \ln 72,$$

determine the value of k , $0 < k < 1$.

$$\boxed{}, \quad k = \frac{1}{4}$$

SINCE BY PARTIAL FRACTIONS

$$\frac{3x-5}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$$

$$\frac{3x-5}{x(x-1)} = \frac{A(x-1) + Bx}{x(x-1)}$$

If $2x \Rightarrow -2 = B$
 $\Rightarrow B = -2$

If $x=0 \Rightarrow -5 = -A$
 $\Rightarrow A = 5$

HENCE THE INTEGRAL BECOMES

$$\Rightarrow \int_k^{2k} \frac{3x-5}{x(x-1)} dx = \ln 72$$

$$\Rightarrow \int_k^{2k} \left(\frac{5}{x} - \frac{2}{x-1} \right) dx = \ln 72$$

$$\Rightarrow \left[5 \ln|x| - 2 \ln|x-1| \right]_k^{2k} = \ln 72$$

$$\Rightarrow \left[5 \ln|2k| - 2 \ln|2k-1| \right] - \left[5 \ln|k| - 2 \ln|k-1| \right] = \ln 72$$

$$\Rightarrow 5 \ln|2k| - 2 \ln|2k-1| - 5 \ln|k| + 2 \ln|k-1| = \ln 72$$

$$\Rightarrow 5 \ln \left| \frac{2k}{k} \right| + 2 \ln \left| \frac{k-1}{2k-1} \right| = \ln 72$$

$$\Rightarrow 5 \ln 2 + 2 \ln \left| \frac{k-1}{2k-1} \right| = \ln 72$$

$$\Rightarrow 2 \ln \left| \frac{k-1}{2k-1} \right| = \ln 72 - 5 \ln 2$$

$$\Rightarrow 2 \ln \left| \frac{k-1}{2k-1} \right| = \ln 72 - \ln 32$$

$$\Rightarrow 2 \ln \left| \frac{k-1}{2k-1} \right| = \ln \frac{72}{32}$$

$$\Rightarrow 2 \ln \left| \frac{k-1}{2k-1} \right| = \ln \left(\frac{9}{4} \right)$$

$$\Rightarrow \ln \left| \frac{k-1}{2k-1} \right| = \ln \frac{3}{2}$$

$$\Rightarrow \frac{k-1}{2k-1} = \frac{3}{2}$$

$$\Rightarrow 2k-1 = 2k-2$$

$$\Rightarrow 4k = 1$$

$$\Rightarrow k = \frac{1}{4}$$

Question 104 (****)

By using the substitution $u = 2x^{\frac{3}{2}} - 1$, or otherwise, find an expression for the integral

$$\int \frac{6x^2}{2x^{\frac{3}{2}} - 1} dx.$$

$$\boxed{}, \quad 2x^{\frac{1}{2}} + \ln \left| 2x^{\frac{3}{2}} - 1 \right| + C$$

Handwritten solution for the integral $\int \frac{6x^2}{2x^{\frac{3}{2}} - 1} dx$.

USING THE SUBSTITUTION GIVEN

$$\begin{aligned} \int \frac{6x^2}{2x^{\frac{3}{2}} - 1} dx &= \int \frac{6x^2}{u} \left(\frac{du}{3x^{\frac{1}{2}}} \right) \\ &= \int \frac{6x^2}{3x^{\frac{1}{2}}u} du = \int \frac{2x^{\frac{3}{2}}}{u} du = \int \frac{u+1}{u} du \\ &= \int 1 + \frac{1}{u} du = u + \ln|u| + C \\ &= (2x^{\frac{3}{2}} - 1) + \ln|2x^{\frac{3}{2}} - 1| + C \\ &= 2x^{\frac{3}{2}} + \ln|2x^{\frac{3}{2}} - 1| + C \end{aligned}$$

Boxed notes for substitution:

$$\begin{aligned} u &= 2x^{\frac{3}{2}} - 1 \\ \frac{du}{dx} &= 3x^{\frac{1}{2}} \\ du &= 3x^{\frac{1}{2}} dx \\ dx &= \frac{du}{3x^{\frac{1}{2}}} \\ 2x^{\frac{3}{2}} &= u+1 \end{aligned}$$

ALTERNATIVE BY MANIPULATION (DIVISION) OR RECOGNITION

$$\begin{aligned} \int \frac{6x^2}{2x^{\frac{3}{2}} - 1} dx &= \int \frac{3x^{\frac{3}{2}}(2x^{\frac{3}{2}} - 1) + 3x^{\frac{3}{2}}}{2x^{\frac{3}{2}} - 1} dx \\ &= \int 3x^{\frac{3}{2}} + \frac{3x^{\frac{3}{2}}}{2x^{\frac{3}{2}} - 1} dx \\ &= \int 3x^{\frac{3}{2}} dx + \int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C \\ &= \frac{3}{2} x^{\frac{5}{2}} + \ln|2x^{\frac{3}{2}} - 1| + C \end{aligned}$$

At the bottom, it says "At the bottom" with a diagonal line.

Question 105 (****)

Use the substitution $t = \sqrt{1-x^3}$ to show that

$$\int x^5 \sqrt{1-x^3} \, dx = -\frac{2}{45} (3x^3 + 2) (1-x^3)^{\frac{3}{2}} + C.$$

☐ , proof

USING THE GIVEN SUBSTITUTION

$$\begin{aligned} & \int x^5 \sqrt{1-x^3} \, dx \\ &= \int x^5 t \left(\frac{2t}{-3x^2} \right) dt \\ &= \int -\frac{2}{3} t^2 x^3 \, dt \\ &= \int -\frac{2}{3} t^2 (1-t^2) \, dt \\ &= -\frac{2}{3} \int t^2 - t^4 \, dt \\ &= -\frac{2}{3} \left[\frac{1}{3} t^3 - \frac{1}{5} t^5 \right] + C \\ &= -\frac{2}{3} \times \frac{1}{15} [5t^3 - 3t^5] + C \\ &= -\frac{2}{45} [5t^3 - 3t^5] + C \\ &= -\frac{2}{45} t^3 (5 - 3t^2) + C \\ &= -\frac{2}{45} (1-x^3)^{\frac{3}{2}} [5 - 3(1-x^3)] + C \\ &= -\frac{2}{45} (1-x^3)^{\frac{3}{2}} (2 + 3x^3) + C \\ &= -\frac{2}{45} (3x^3 + 2) (1-x^3)^{\frac{3}{2}} + C \end{aligned}$$

AK BILAL

Question 106 (****)

Use an appropriate substitution, followed by partial fractions, to show that

$$\int_{e^3}^{e^5} \frac{5}{2x[(\ln x)^2 + \ln x - 6]} dx = \ln\left(\frac{3}{2}\right).$$

You may assume that the integral converges.

, proof

BY SUBSTITUTION

$$u = \ln x \quad \frac{du}{dx} = \frac{1}{x} \quad dx = x du$$

When $x = e^3$, $u = 3$
When $x = e^5$, $u = 5$

TRANSFORMING THE INTEGRAL

$$\int_{e^3}^{e^5} \frac{5}{2x[(\ln x)^2 + \ln x - 6]} dx = \int_3^5 \frac{5}{2(u^2 + u - 6)} du$$

BY PARTIAL FRACTIONS

$$\frac{1}{(u+3)(u-2)} = \frac{A}{u+3} + \frac{B}{u-2}$$

$$1 = A(u-2) + B(u+3)$$

If $u = 2$, $1 = 5B \Rightarrow B = \frac{1}{5}$
If $u = -3$, $1 = -5A \Rightarrow A = -\frac{1}{5}$

PUTTING THE H.A.F.

$$\int_3^5 \frac{1}{2(u+3)(u-2)} du = \frac{1}{2} \int_3^5 \left(\frac{-1/5}{u+3} + \frac{1/5}{u-2} \right) du$$

$$= \frac{1}{2} \left[-\frac{1}{5} \ln|u+3| + \frac{1}{5} \ln|u-2| \right]_3^5$$

$$= \frac{1}{2} \left[-\frac{1}{5} \ln 8 + \frac{1}{5} \ln 3 + \frac{1}{5} \ln 5 - \frac{1}{5} \ln 1 \right]$$

$$= \frac{1}{2} \ln \frac{3}{2} = \ln \left(\frac{3}{2} \right)^{\frac{1}{2}}$$

Question 107 (****)

$$\frac{2u^3}{u+1} \equiv Au^2 + Bu + C + \frac{D}{u+1}.$$

- a) Find the value of each of the constants A , B and C in the above identity.
- b) Use the substitution $u = \sqrt{x}$ to show

$$\int_0^1 \frac{x}{1+\sqrt{x}} dx = \frac{5}{3} - 2 \ln 2.$$

$$\boxed{A=2}, \boxed{B=-2}, \boxed{C=2}, \boxed{D=-2}$$

(a) $\frac{2u^3}{u+1} \equiv Au^2 + Bu + C + \frac{D}{u+1}$
 $2u^3 \equiv Au^3 + Bu^2 + Cu + D$
 $2u^3 \equiv Au^3 + Au^2 + Bu^2 + Bu + Cu + C + D$
 $2u^3 \equiv Au^3 + (A+B)u^2 + (B+C)u + (C+D)$
 $A+2=0$ $A+B=0$ $B+C=0$ $C+D=0$
 $2+2=0$ $-2+C=0$ $-2+C=0$ $2+D=0$
 $B=-2$ $C=2$ $D=-2$

(b) $\int_0^1 \frac{x}{1+\sqrt{x}} dx = \dots$ by substitution
 $u = \sqrt{x}$
 $\frac{du}{dx} = \frac{1}{2\sqrt{x}} \Rightarrow 2u \frac{du}{dx} = 1$
 $2u du = dx$
 $2 \cdot 0 = 0$ $2 \cdot 1 = 2$
 $u=0$ $u=1$
 $= \left[\frac{2}{3}u^3 - u^2 + 2u - 2\ln|u+1| \right]_0^1$
 $= \left(\frac{2}{3} - 1 + 2 - 2\ln 2 \right) - (0 - 2\ln 1) = \frac{5}{3} - 2\ln 2$

Question 109 (****)

$$\int \frac{\cos x}{1 - \cos x} dx.$$

- a) Show by multiplying the numerator and denominator of the integrand by $(1 + \cos x)$, that the above integral can eventually be written as

$$\int \cot x \operatorname{cosec} x + \cot^2 x dx.$$

- b) Hence show further that

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos x}{1 - \cos x} dx = \frac{1}{4} (4\sqrt{2} - \pi).$$

☐ , ☐ proof

(a) $\int \frac{\cos x}{1 - \cos x} dx = \int \frac{\cos x (1 + \cos x)}{(1 - \cos x)(1 + \cos x)} dx = \int \frac{\cos x (1 + \cos x)}{1 - \cos^2 x} dx$
 $= \int \frac{\cos x + \cos^2 x}{\sin^2 x} dx = \int \frac{\cos x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x} dx = \int \frac{\cos x}{\sin^2 x} + \cot^2 x dx$
 $= \int \cot x \operatorname{cosec} x + \cot^2 x dx$

(b) Hence
 $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos x}{1 - \cos x} dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot x \operatorname{cosec} x + \cot^2 x dx = \left[\cot x \operatorname{cosec} x + \cot^2 x - \cot x \right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$
 $= \left[-\cot x - \cot^2 x - \cot x \right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \left[-2\cot x - \cot^2 x \right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$
 $= \left(\frac{\pi}{2} + \sqrt{2} + 1 \right) - \left(\frac{\pi}{4} + 1 + 0 \right) = -\frac{\pi}{4} + \sqrt{2} = \frac{1}{4} (4\sqrt{2} - \pi)$

Question 109 (***)

By using the substitution $x = 9 \sin^2 \theta$, or otherwise, find the exact value of

$$\int_0^{\frac{9}{4}} \frac{1}{\sqrt{x(9-x)}} dx.$$

$$\frac{\pi}{3}$$

Question 110 (****)

It is given that

$$\cos^4 \theta \equiv \frac{3}{8} + \frac{1}{2} \cos 2\theta + \frac{1}{8} \cos 4\theta.$$

- Prove the validity of the above trigonometric identity.
- Use the substitution $u = \sin \theta$ to show

$$\int_0^1 \sqrt{(1-x^2)^3} dx = \frac{3\pi}{16}.$$

proof

Question 111 (****)

$$y = \frac{x^2 + 2x - 2}{x^2 - 2x + 2}.$$

- a) Find the value of each of the constants A , B and C , so that

$$y \equiv A + \frac{Bx + C}{x^2 - 2x + 2}.$$

- b) Hence, or otherwise, show that

$$\int_{\frac{1}{2}}^{\frac{3}{2}} y \, dx = 1.$$

$$\boxed{A=1}, \boxed{B=4}, \boxed{C=-4}$$

(a) $\frac{x^2 + 2x - 2}{x^2 - 2x + 2} \equiv A + \frac{Bx + C}{x^2 - 2x + 2}$
 $\frac{x^2 + 2x - 2}{x^2 - 2x + 2} \equiv \frac{A(x^2 - 2x + 2) + Bx + C}{x^2 - 2x + 2}$
 $x^2 + 2x - 2 \equiv A(x^2 - 2x + 2) + Bx + C$
 $x^2 + 2x - 2 \equiv Ax^2 + (-2A + B)x + (2A + C)$
 $A = 1$
 $-2A + B = 2 \Rightarrow B = 4$
 $2A + C = -2 \Rightarrow C = -4$

(b) $\int_{\frac{1}{2}}^{\frac{3}{2}} \frac{x^2 + 2x - 2}{x^2 - 2x + 2} dx = \int_{\frac{1}{2}}^{\frac{3}{2}} 1 + \frac{4x - 4}{x^2 - 2x + 2} dx$
 $= \int_{\frac{1}{2}}^{\frac{3}{2}} 1 + 2 \frac{2x - 2}{x^2 - 2x + 2} dx$
 $= \left[x + 2 \ln|x^2 - 2x + 2| \right]_{\frac{1}{2}}^{\frac{3}{2}}$
 $= \left(\frac{3}{2} + 2 \ln \frac{5}{4} \right) - \left(\frac{1}{2} + 2 \ln \frac{3}{4} \right)$
 $= 1$

Question 112 (****)

$$f(x) \equiv \frac{2}{x + \sqrt{2x-1}}, \quad x \geq \frac{1}{2}.$$

- a) Use the substitution $u = \sqrt{2x-1}$ transforms to show

$$\int_1^5 f(x) \, dx \equiv \int_{u_1}^{u_2} \frac{4u}{(u+1)^2} \, du,$$

where u_1 and u_2 are constants to be found.

- b) By using another suitable substitution, or otherwise, show that

$$\int_1^5 f(x) \, dx = -1 + \ln 16.$$

 , proof

a) USING THE SUBSTITUTION GIVEN

$$\begin{aligned} \int_1^5 f(x) \, dx &= \int_1^5 \frac{2}{x + \sqrt{2x-1}} \, dx \\ &= \int_1^5 \frac{2u}{u^2 + u} \, du = \int_1^5 \frac{2u}{u(u+1)} \, du \\ &= \int_1^5 \frac{2}{u+1} \, du = \int_1^5 \frac{2}{u+1} \, du \\ &= \int_1^5 \frac{2}{u+1} \, du \end{aligned}$$

As expected

ALTERNATIVE SUBSTITUTION

$$\begin{aligned} \dots &= \int_2^4 \frac{4}{v^2} \, dv = \int_2^4 \frac{4}{v^2} \, dv \\ &= 4 \int_2^4 \frac{1}{v^2} \, dv = 4 \int_2^4 v^{-2} \, dv \\ &= 4 \left[-\frac{1}{v} \right]_2^4 = 4 \left[-\frac{1}{4} + \frac{1}{2} \right] \\ &= 4 \left[\frac{1}{2} - \frac{1}{4} \right] = 4 \left[\frac{2}{4} - \frac{1}{4} \right] = 4 \left[\frac{1}{4} \right] = 1 \end{aligned}$$

or $-1 + \ln 16$

Question 113 (**)**Use integration by parts find an exact value in terms of e , for the integral

$$\int_1^e (\ln x)^2 dx.$$

$$\boxed{e-2}$$

Handwritten solution for Question 113:

$$\begin{aligned} \int_1^e (\ln x)^2 dx &= \text{by parts \& ignoring limits} \\ &= \int 1 \times (\ln x)^2 dx \\ &= x(\ln x)^2 - \int 2 \ln x dx \\ &= \text{by part again} \\ &= x(\ln x)^2 - [2x \ln x - \int 2 dx] \\ &= x(\ln x)^2 - 2x \ln x + 2x \\ &\dots \text{limits} \\ &= [x(\ln x)^2 - 2x \ln x + 2x]_1^e \\ &= (e - 2e + 2e) - (0 - 0 + 2) = e - 2 \end{aligned}$$

Question 114 (**)**Use the fact that $\frac{d}{dx}(\sec x) = \sec x \tan x$, to find

$$\int \sin x (1 + \sec^2 x) dx.$$

$$\boxed{\sec x - \cos x + C}$$

Handwritten solution for Question 114:

Since $\frac{d}{dx}(\sec x) = \sec x \tan x$ then

$$\begin{aligned} \int \sin x (\sec^2 x + 1) dx &= \int \sin x \sec^2 x + \sin x dx \\ &= \int \sin x \frac{1}{\cos^2 x} + \sin x dx \\ &= \int \tan x \sec x + \sin x dx \\ &= \sec x - \cos x + C \end{aligned}$$

Question 115 (****)

$$\frac{1}{u(u+1)} \equiv \frac{A}{u} + \frac{B}{u+1}.$$

- a) Find the value of A and B in the above identity.
- b) By using the substitution $u = e^x$, or otherwise, show that

$$\int_0^{\ln 2} \frac{1}{1+e^x} dx = \ln\left(\frac{4}{3}\right).$$

$$\boxed{A=1}, \boxed{B=-1}$$

Question 116 (****)

By completing the square in the expression $4x^2 + 4x$, or otherwise, show that

$$\int \frac{4x^2 + 4x}{\sqrt{2x+1}} dx = A(2x+1)^{\frac{5}{2}} + B(2x+1)^{\frac{1}{2}} + C,$$

where A and B are constants to be found and C is the arbitrary constant of the integration.

$$\boxed{A=\frac{1}{5}}, \boxed{B=-1}$$

Question 117 (****)

$$u^3 + 1 \equiv (u+1)(u^2 + Au + 1).$$

- a) Determine the value of A in the above identity.
- b) Use the substitution $u = e^x$ to show

$$\int_0^{\ln 2} \frac{e^{3x} + 1}{e^x + 1} dx = \frac{1}{2} + \ln 2.$$

$$\boxed{A=1}, \quad \boxed{B=\frac{1}{2}}, \quad \boxed{C=-\frac{1}{2}}$$

Question 118 (****)

Use the substitution $u = x^{\frac{1}{4}}$ to find

$$\int_1^{16} \frac{2x^{\frac{1}{4}} + 1}{4x^{\frac{5}{4}} + 4x} dx,$$

giving the final answer as an exact simplified natural logarithm.

$$\boxed{\ln 3}$$

Question 119 (****)

$$\frac{x^3}{x^2+1} \equiv Ax + B + \frac{Cx + D}{x^2+1}.$$

- a) Determine the value of each of the constants A , B , C and D .
- b) Use integration by parts to show that

$$\int x \ln(x^2+1) dx = \frac{1}{2}(x^2+1) \ln(x^2+1) - \frac{1}{2}x^2 + C.$$

$$\boxed{A=1}, \boxed{B=0}, \boxed{C=-1}, \boxed{D=0}$$

(a) $\frac{x^3}{x^2+1} \equiv Ax + B + \frac{Cx+D}{x^2+1}$
 $x^3 \equiv (Ax+B)(x^2+1) + (Cx+D)$
 $x^3 \equiv Ax^3 + Bx^2 + Cx + D$
 $x^3 \equiv Ax^3 + Bx^2 + (C+D)x + D$
 $\begin{cases} A=1 \\ B=0 \\ C+D=0 \\ D=0 \end{cases} \Rightarrow \begin{cases} A=1 \\ B=0 \\ C=-1 \\ D=0 \end{cases}$

(b) $\int x \ln(x^2+1) dx \dots$ INTEGRATION BY PARTS
 $= \frac{1}{2}x^2 \ln(x^2+1) - \int \frac{x^2}{x^2+1} dx$
 $= \frac{1}{2}x^2 \ln(x^2+1) - \int x^2 \left(\frac{x^2+1}{x^2+1} \right) dx$
 $= \frac{1}{2}x^2 \ln(x^2+1) - \int x^2 \left(1 + \frac{-1}{x^2+1} \right) dx$
 $= \frac{1}{2}x^2 \ln(x^2+1) - \left[\frac{1}{2}x^2 - \frac{1}{2} \ln(x^2+1) \right] + C$
 $= \frac{1}{2}x^2 \ln(x^2+1) + \frac{1}{2} \ln(x^2+1) - \frac{1}{2}x^2 + C$
 $= \frac{1}{2}(x^2+1) \ln(x^2+1) - \frac{1}{2}x^2 + C$ as required

Question 120 (****)

By writing $\sec x$ as the fraction $\frac{\sec x}{1}$ and multiplying the numerator and the denominator by $(\sec x + \tan x)$, find

$$\int \sec x dx.$$

$$\boxed{\ln|\sec x + \tan x| + C}$$

$\int \sec x dx = \int \frac{\sec x}{1} dx = \int \frac{\sec x (\sec x + \tan x)}{\sec x + \tan x} dx$
 $= \int \frac{\sec^2 x + \sec x \tan x}{\sec^2 x + \tan^2 x} dx$
 $\text{or } \frac{d}{dx}(\tan x) = \sec^2 x$
 $\frac{d}{dx}(\sec x) = \sec x \tan x$
 $\left. \begin{matrix} \frac{d}{dx}(\tan x) = \sec^2 x \\ \frac{d}{dx}(\sec x) = \sec x \tan x \end{matrix} \right\} \text{I.E. OF THE FORM } \int \frac{f'(x)}{f(x)} dx$
 $= \ln|\sec x + \tan x| + C$

Question 121 (****)

$$\frac{4}{(1-u)^2(1+u)} \equiv \frac{A}{(1-u)^2} + \frac{B}{1-u} + \frac{C}{1+u}.$$

- a) Find the value of A , B and C in the above identity.
- b) Hence by using a suitable substitution find the exact value of

$$\int_0^{\frac{\pi}{6}} \frac{4}{\cos x(1-\sin x)} dx.$$

$$\boxed{}, \boxed{A=2}, \boxed{B=1}, \boxed{C=1}, \boxed{2+\ln 3}$$

(a) $\frac{4}{(1-u)^2(1+u)} = \frac{A}{(1-u)^2} + \frac{B}{1-u} + \frac{C}{1+u}$
 $\frac{4}{(1-u)^2(1+u)} \equiv \frac{A}{(1-u)^2} + \frac{B}{1-u} + \frac{C}{1+u}$
 \bullet If $u=1$; $4=2A \Rightarrow A=2$
 \bullet If $u=-1$; $4=4C \Rightarrow C=1$
 \bullet If $u=0$; $4=A+B+C \Rightarrow 4=2+B+1 \Rightarrow B=1$

(b) $\int_0^{\frac{\pi}{6}} \frac{4}{\cos x(1-\sin x)} dx = \dots$ by substitution $u = \sin x$
 $\frac{du}{dx} = \cos x$
 $dx = \frac{du}{\cos x}$
 $\frac{4}{\cos x(1-\sin x)} \cdot \frac{du}{\cos x} = \frac{4}{\cos^2 x(1-\sin x)} du$
 $\frac{4}{\cos^2 x(1-\sin x)} = \frac{4}{(1-\sin^2 x)(1-\sin x)} = \frac{4}{(1-\sin x)(1+\sin x)(1-\sin x)} = \frac{4}{(1-\sin x)^2(1+\sin x)}$
 $= \int_0^{\frac{\pi}{6}} \frac{4}{(1-\sin x)^2(1+\sin x)} dx = \int_0^{\frac{\pi}{6}} \left(\frac{2}{(1-\sin x)^2} + \frac{1}{1-\sin x} + \frac{1}{1+\sin x} \right) dx$
 $= \left[\frac{2}{1-\sin x} - \ln|1-\sin x| + \ln|1+\sin x| \right]_0^{\frac{\pi}{6}}$
 $= \left(4 - \ln \frac{1}{2} + \ln \frac{3}{2} \right) - (2 + \ln 1 + \ln 1) = 2 + \ln 3$

Question 122 (****)

Use the substitution $u = \ln x$, followed by integration by parts to find

$$\int \frac{1 - \ln x}{x^2} dx.$$

$$\frac{\ln x}{x} + C$$

Handwritten solution for Question 122:

$$\begin{aligned} \int \frac{1 - \ln x}{x^2} dx &= \text{using the substitution given } u = \ln x \\ &= \int \frac{1 - u}{x^2} \cdot x \, du = \int \frac{1 - u}{x} du = \int \frac{1 - u}{e^u} du \\ &= \int (1 - u) e^{-u} du \\ &\quad \text{by parts} \\ &= -(1 - u) e^{-u} - \int e^{-u} du \\ &= (u - 1) e^{-u} + e^{-u} + C \\ &= u e^{-u} - e^{-u} + e^{-u} + C \\ &= \frac{u}{e^u} + C = \frac{\ln x}{x} + C \end{aligned}$$

Question 123 (****)

$$y = \arctan x, \quad x \in \mathbb{R}.$$

- a) By writing the above equation as $x = \tan y$, show clearly that

$$\frac{dy}{dx} = \frac{1}{1 + x^2}.$$

- b) Use integration by parts and the result of part (a) to find

$$\int 2 \arctan x \, dx.$$

$$2x \arctan x - \ln(1 + x^2) + C$$

Handwritten solution for Question 123:

a) $y = \arctan x$
 $\Rightarrow \tan y = x$
 $\Rightarrow \frac{dy}{dx} = \frac{1}{1 + x^2}$

b) $\int 2 \arctan x \, dx = \dots$ by parts
 $= 2x \arctan x - \int \frac{2x}{1 + x^2} dx$
 $= 2x \arctan x - \ln(1 + x^2) + C$

Question 124 (****)

$$f(x) \equiv \frac{4x^2 - 23x + 21}{x^2 - 4x + 3}, \quad x \neq 1, \quad x \neq 3.$$

a) Express $f(x)$ in partial fractions.

b) Hence find an exact value for

$$\int_2^{2.5} f(x) \, dx.$$

$$f(x) \equiv 4 - \frac{1}{x-1} - \frac{6}{x-3}, \quad \left[2 - \ln\left(\frac{3}{128}\right) \right]$$

a) $\frac{4x^2 - 23x + 21}{x^2 - 4x + 3} \equiv \frac{4x^2 - 23x + 21}{(x-1)(x-3)} \equiv A + \frac{B}{x-1} + \frac{C}{x-3}$
 "Heuristic" $A=4$ by inspection
 $4x^2 - 23x + 21 \equiv A(x-1)(x-3) + B(x-3) + C(x-1)$
 • If $x=1 \Rightarrow 4 - 23 + 21 \equiv -2B$
 $-2B = -2$
 $B = 1$
 • If $x=3 \Rightarrow 36 - 69 + 21 \equiv 2C$
 $-12 = 2C$
 $C = -6$
 • If $x=0 \Rightarrow 21 \equiv 3A - 3B - C$
 $21 \equiv 3A + 3 + 6$
 $12 \equiv 3A$
 $A = 4$
 $\therefore f(x) = 4 - \frac{1}{x-1} - \frac{6}{x-3}$
 b) $\int_2^{2.5} f(x) \, dx = \int_2^{2.5} \left(4 - \frac{1}{x-1} - \frac{6}{x-3} \right) dx = \left[4x - \ln|x-1| - 6\ln|x-3| \right]_2^{2.5}$
 $= \left[10 - \ln\frac{3}{2} - 6\ln\frac{1}{2} \right] - \left[8 - \ln|2-1| - 6\ln|2-3| \right]$
 $= 2 - \ln\frac{3}{2} - 6\ln\frac{1}{2} = 2 - \left[\ln\frac{3}{2} + 6\ln\frac{1}{2} \right]$
 $= 2 - \ln\frac{3}{128}$

Question 125 (****)

By using the substitution $u = (2x+1)^{\frac{1}{2}}$, or otherwise, show that

$$\int_0^4 e^{\sqrt{2x+1}} dx = 2e^3.$$

proof

Handwritten proof for Question 125:

$$\int_0^4 e^{\sqrt{2x+1}} dx = \int_1^3 e^u (u du) = \int_1^3 u e^u du \dots$$

By parts & reversing limits...

$$= u e^u - \int e^u du = [u e^u - e^u]_1^3$$

$$= (3e^3 - e^3) - (e^1 - e^1) = 2e^3$$

Boxed notes:

- $u = (2x+1)^{\frac{1}{2}}$
- $\frac{du}{dx} = \frac{1}{\sqrt{2}}$
- $\frac{dx}{du} = \sqrt{2}$
- $x=0 \rightarrow u=1$
- $x=4 \rightarrow u=3$

Question 126 (****)

Use the substitution $u = x^2 + 2$, followed by integration by parts to show that

$$\int_2^4 2x^3 \ln(x^2 + 2) dx = 1 + 2 \ln 2.$$

proof

Handwritten proof for Question 126:

$$\int_2^4 2x^3 \ln(x^2+2) dx = \int_2^4 2x^3 \ln u \frac{du}{2x} = \int_2^4 x^2 \ln u du = \int_2^4 (u-2) \ln u du$$

By parts & reversing limits...

$$= \left(\frac{1}{2} u^2 - 2u \right) \ln u - \int \frac{1}{2} u - 2 du$$

$$= \left(\frac{1}{2} u^2 - 2u \right) \ln u - \frac{1}{4} u^2 + 2u + C$$

By parts

$$\ln u \quad \frac{1}{u}$$

$$\frac{1}{2} u^2 - 2u \quad u-2$$

Reversing limits...

$$= \left(\frac{1}{2} u^2 - 2u \right) \ln u - \frac{1}{4} u^2 + 2u \Big|_2^4$$

$$= (0 - 4 + 8) - (-2 \ln 2 - 1 + 4) = 4 - (3 - 2 \ln 2) = 1 + 2 \ln 2$$

Question 127 (****)

By considering the trigonometric expansions of $\sin(5x+3x)$ and $\sin(5x-3x)$, show clearly that

$$\int_0^{\frac{\pi}{4}} \cos 3x \sin 5x \, dx = \frac{1}{4}.$$

proof

$$\begin{aligned} \sin(5x+3x) &= \sin 5x \cos 3x + \cos 5x \sin 3x \\ \sin(5x-3x) &= \sin 5x \cos 3x - \cos 5x \sin 3x \\ \hline \sin 5x \cos 3x + \sin 3x \cos 5x &= 2 \sin 5x \cos 3x \\ \sin 5x \cos 3x &= \frac{1}{2} \sin 5x \cos 3x + \frac{1}{2} \sin 3x \cos 5x \\ \int_0^{\frac{\pi}{4}} \cos 3x \sin 5x \, dx &= \int_0^{\frac{\pi}{4}} \frac{1}{2} \sin 5x + \frac{1}{2} \sin 3x \, dx \\ &= \left[-\frac{1}{10} \cos 5x - \frac{1}{6} \cos 3x \right]_0^{\frac{\pi}{4}} = \left[\frac{1}{10} \cos 5x + \frac{1}{6} \cos 3x \right]_0^{\frac{\pi}{4}} \\ &= \left[\frac{1}{10} \cos 0 + \frac{1}{6} \cos 0 \right] - \left[\frac{1}{10} \cos \frac{5\pi}{4} + \frac{1}{6} \cos \frac{3\pi}{4} \right] \\ &= \left(\frac{1}{10} + \frac{1}{6} \right) - \left(\frac{1}{10} + 0 \right) = \frac{1}{6} \quad \text{As required} \end{aligned}$$

Question 128 (****)

By using the substitution $x = 2 + (u-1)^2$, or otherwise, find

$$\int \frac{1}{1 + \sqrt{x-2}} \, dx.$$

$$2\sqrt{x-2} + 2\ln(1 + \sqrt{x-2}) + C$$

$$\begin{aligned} \int \frac{1}{1 + \sqrt{x-2}} \, dx &= \int \frac{1}{1 + \sqrt{(u-1)^2}} \cdot 2(u-1) \, du \\ &= \int \frac{2(u-1)}{1 + |u-1|} \, du \\ &= \int \frac{2(u-1)}{1 + (u-1)} \, du = \int \frac{2(u-1)}{u} \, du \\ &= \int \frac{2u-2}{u} \, du = \int 2 - \frac{2}{u} \, du = 2u - 2\ln|u| + C \\ &= 2(1 + \sqrt{x-2}) - 2\ln(1 + \sqrt{x-2}) + C \\ &= 2 + 2\sqrt{x-2} - 2\ln(1 + \sqrt{x-2}) + C \\ &= 2\sqrt{x-2} - 2\ln(1 + \sqrt{x-2}) + C \end{aligned}$$

Question 129 (****)

$$\frac{6t^3}{t+1} \equiv At^2 + Bt + C + \frac{D}{t+1}.$$

a) Determine the value of each of the constants A , B , C and D .

b) Use the substitution $t = x^{\frac{1}{6}}$ to show

$$\int_1^{64} \frac{1}{\sqrt{x} + \sqrt[3]{x}} dx = 11 + 6 \ln\left(\frac{2}{3}\right).$$

$$A = 6, \quad B = -6, \quad C = 6, \quad D = -6$$

(a) $\frac{6t^3}{t+1} \equiv At^2 + Bt + C + \frac{D}{t+1}$
 $6t^3 \equiv (At^2 + Bt + C)(t+1) + D$
 $6t^3 \equiv At^3 + Bt^2 + Ct + At^2 + Bt + C + D$
 $6t^3 \equiv At^3 + (A+B)t^2 + (B+C)t + (C+D)$
 $\begin{cases} A=6 \\ A+B=0 \\ B+C=0 \\ C+D=0 \end{cases} \Rightarrow \begin{cases} A=6 \\ B=-6 \\ C=6 \\ D=-6 \end{cases}$

(b) $\int_1^{64} \frac{1}{\sqrt{x} + \sqrt[3]{x}} dx = \dots$ by substitution $t = x^{\frac{1}{6}}$
 $t^6 = x \Rightarrow \frac{dt}{dx} = \frac{1}{6x^{\frac{5}{6}}} \Rightarrow dx = 6t^5 dt$
 $\int_1^2 \frac{1}{t^3 + t^2} (6t^5 dt) = \int_1^2 \frac{6t^3}{t^2 + t} dt$
 $= \int_1^2 \frac{6t^3}{t(t+1)} dt = \dots$ from part (a)
 $= \int_1^2 (6t - 6t + 6 - \frac{6}{t+1}) dt$
 $= [3t^2 - 3t^2 + 6t - 6 \ln|t+1|]_1^2$
 $= (16 - 3 \times 2^2 - 6 \ln 3) - (2 - 3 - 6 \ln 2)$
 $= 11 + 6 \ln 2 - 6 \ln 3 = 11 + 6 \ln \frac{2}{3}$
 As required

Question 130 (****)

$$f(x) = 8x \ln(2x+1), \quad x > -\frac{1}{2}.$$

- a) Use the substitution $u = 2x+1$ to show that

$$\int f(x) dx = \int (2u-2) \ln u \, du.$$

- b) Use integration by parts to show that

$$\int f(x) dx = (4x^2 - 1) \ln(2x+1) - 2x^2 + 2x + C.$$

proof

$$\begin{aligned} \text{a) } \int 8x \ln(2x+1) dx &= \dots \text{ by substitution} \\ &= \int 8x \ln u \cdot \frac{du}{2} = \int 4x \ln u \, du = \int 2(2x) \ln u \, du \\ &= \int 2(u-1) \ln u \, du = \int (2u-2) \ln u \, du \\ \text{b) } &= \dots \text{ by parts } \dots \\ &= (u^2-2u) \ln u - \int (u^2-2u) \, du \\ &= (u^2-2u) \ln u - \left(\frac{1}{3}u^3 - 2u \right) + C \\ &= \left[(2x+1)^2 - 2(2x+1) \right] \ln(2x+1) - \left(\frac{1}{3}(2x+1)^3 + 2(2x+1) \right) + C \\ &= (4x^2+4x+1-4x-2) \ln(2x+1) - \left(\frac{1}{3}(8x^3+12x^2+6x+1) + 4x+2 \right) + C \\ &= (4x^2-1) \ln(2x+1) - 2x^2-2x + C \\ &= (4x^2-1) \ln(2x+1) - 2x^2+2x + C \end{aligned}$$

$\ln u$	$\frac{1}{u}$
u^2-2u	$2u-2$

Question 131 (****)

$$y = \frac{(1 + \sin x)^2}{\cos^2 x}$$

- a) Calculate the two missing values of the following table.

x	$\frac{\pi}{6}$		$\frac{\pi}{4}$	$\frac{7\pi}{24}$	$\frac{\pi}{3}$
y	3		5.8284	8.6784	13.9282

- b) Use the trapezium rule with all the values from the completed table of part (a) to find an estimate for

$$\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{(1 + \sin x)^2}{\cos^2 x} dx$$

- c) Use trigonometric identities to find the exact value of

$$\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{(1 + \sin x)^2}{\cos^2 x} dx$$

$$\boxed{}, \quad x = \frac{5\pi}{24}, \quad \boxed{4.112}, \quad \boxed{4 - \frac{1}{6}\pi}$$

a) Fill in the table

x	$\frac{\pi}{6}$	$\frac{5\pi}{24}$	$\frac{\pi}{4}$	$\frac{7\pi}{24}$	$\frac{\pi}{3}$
y	3	4.112	5.8284	8.6784	13.9282

By the trapezium rule

$$\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{(1 + \sin x)^2}{\cos^2 x} dx \approx \frac{1}{2} \left[\frac{1}{6}\pi (3 + 4.112) + 2 \left(4.112 + 5.8284 + 8.6784 + 13.9282 \right) + \frac{1}{6}\pi (5.8284 + 13.9282) \right]$$

$$\approx 3.515$$

b) Proceed by direct integration

$$\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{(1 + \sin x)^2}{\cos^2 x} dx = \int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{1 + 2\sin x + \sin^2 x}{\cos^2 x} dx$$

$$= \int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \left(\frac{1}{\cos^2 x} + \frac{2\sin x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} \right) dx = \int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \left(\sec^2 x + \frac{2\sin x}{\cos^2 x} + \frac{1 - \cos^2 x}{\cos^2 x} \right) dx$$

$$= \int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \left(\sec^2 x + 2\sec x \tan x + \sec^2 x - 1 \right) dx$$

$$= \int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} (2\sec x + 2\sec x \tan x - 1) dx$$

$$= 2\sec x + 2\sec x \tan x - x \Big|_{\frac{1}{6}\pi}^{\frac{1}{3}\pi}$$

$$= 2\sec\left(\frac{1}{3}\pi\right) + 2\sec\left(\frac{1}{3}\pi\right)\tan\left(\frac{1}{3}\pi\right) - \frac{1}{3}\pi - \left(2\sec\left(\frac{1}{6}\pi\right) + 2\sec\left(\frac{1}{6}\pi\right)\tan\left(\frac{1}{6}\pi\right) - \frac{1}{6}\pi \right)$$

$$= 2\sqrt{3} + 4 - \frac{\pi}{3} - \left(2 + \frac{4}{3} - \frac{\pi}{6} \right)$$

$$= 2\sqrt{3} + 4 - \frac{\pi}{3} - \frac{8}{3} + \frac{\pi}{6}$$

$$= 2\sqrt{3} + 4 - \frac{\pi}{6}$$

Now use what that

$$\frac{d}{dx}(\tan x) = \sec^2 x \quad \& \quad \frac{d}{dx}(\sec x) = \sec x \tan x$$

Therefore the previous have

$$= \left(2\sec x + 2\sec x \tan x - x \right) \Big|_{\frac{1}{6}\pi}^{\frac{1}{3}\pi}$$

$$= \left(2\sec\left(\frac{1}{3}\pi\right) + 2\sec\left(\frac{1}{3}\pi\right)\tan\left(\frac{1}{3}\pi\right) - \frac{1}{3}\pi \right) - \left(2\sec\left(\frac{1}{6}\pi\right) + 2\sec\left(\frac{1}{6}\pi\right)\tan\left(\frac{1}{6}\pi\right) - \frac{1}{6}\pi \right)$$

$$= \left(2\sqrt{3} + 4 - \frac{\pi}{3} \right) - \left(2 + \frac{4}{3} - \frac{\pi}{6} \right)$$

$$= 2\sqrt{3} + 4 - \frac{\pi}{3} - \frac{8}{3} + \frac{\pi}{6}$$

$$= 2\sqrt{3} + 4 - \frac{\pi}{6}$$

Question 132 (****)

$$y = \ln(\sec x + \tan x).$$

- a) Express $\frac{dy}{dx}$ as a single trigonometric function.
- b) Hence find

$$\int x \sec x \tan x \, dx.$$

$$x \sec x - \ln|\sec x + \tan x| + C$$

a) $y = \ln(\sec x + \tan x)$
 $\frac{dy}{dx} = \frac{1}{\sec x + \tan x} \times (\sec x \tan x + \sec^2 x)$
 $= \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} = \frac{\sec x (\tan x + \sec x)}{\sec x + \tan x} = \sec x$

b) $\int x \sec x \tan x \, dx = \dots$ by part ...
 $= x \sec x - \int \sec x \, dx$
 $= x \sec x - \ln|\sec x + \tan x| + C$

Question 133 (****)

Use the substitution $x = 2 \sin \theta$ to find the exact value of

$$\int_0^1 \frac{12}{(4-x^2)^{\frac{3}{2}}} \, dx.$$

$$\sqrt{3}$$

$\int_0^1 \frac{12}{(4-x^2)^{\frac{3}{2}}} \, dx = \dots$ using the substitution given

$x = 2 \sin \theta$
 $\frac{dx}{d\theta} = 2 \cos \theta$
 $dx = 2 \cos \theta \, d\theta$
 $x=0, 0=2 \sin \theta$
 $\sin \theta = 0$
 $\theta = 0$
 $x=1, 1=2 \sin \theta$
 $\sin \theta = \frac{1}{2}$
 $\theta = \frac{\pi}{6}$

$\int_0^1 \frac{12}{(4-x^2)^{\frac{3}{2}}} \, dx = \int_0^{\frac{\pi}{6}} \frac{12}{(4-4 \sin^2 \theta)^{\frac{3}{2}}} (2 \cos \theta \, d\theta)$
 $= \int_0^{\frac{\pi}{6}} \frac{24 \cos \theta}{(4 \cos^2 \theta)^{\frac{3}{2}}} \, d\theta = \int_0^{\frac{\pi}{6}} \frac{24 \cos \theta}{8 \cos^3 \theta} \, d\theta = \int_0^{\frac{\pi}{6}} \frac{3}{\cos^2 \theta} \, d\theta$
 $= \int_0^{\frac{\pi}{6}} 3 \sec^2 \theta \, d\theta = [3 \tan \theta]_0^{\frac{\pi}{6}} = 3 \tan \frac{\pi}{6} - 3 \tan 0$
 $= 3 \times \frac{1}{\sqrt{3}} = \sqrt{3}$

Question 134 (****)

$$\frac{u^2}{u^2-9} \equiv A + \frac{B}{u-3} + \frac{C}{u+3}.$$

a) Find the value of A , B and C in the above identity.

b) By using the substitution $u = \sqrt{x^2+9}$, or otherwise, find

$$\int \frac{\sqrt{x^2+9}}{x} dx.$$

$$\boxed{A=1}, \quad \boxed{B=\frac{3}{2}}, \quad \boxed{C=-\frac{3}{2}}, \quad \boxed{\sqrt{x^2+9} + \frac{3}{2} \ln \left| \frac{\sqrt{x^2+9}-3}{\sqrt{x^2+9}+3} \right| + C}$$

(a) $\frac{u^2}{u^2-9} = \frac{u^2}{(u-3)(u+3)} \equiv A + \frac{B}{u-3} + \frac{C}{u+3}$

$\boxed{u^2 \equiv A(u-3)(u+3) + B(u+3) + C(u-3)}$

Let $u=3$, $9 = 6B \Rightarrow B = \frac{3}{2}$
 Let $u=-3$, $9 = -6C \Rightarrow C = -\frac{3}{2}$
 Let $u=0$, $0 = -9A + 3B - 3C$
 $9A = \frac{3}{2} - (-\frac{3}{2})$
 $9A = 3$
 $A = \frac{1}{3}$

(b) $u = \sqrt{x^2+9}$
 $u^2 = x^2+9$
 $x \frac{du}{dx} = 2x$
 $\frac{dx}{x} = \frac{du}{u}$
 $x^2 = u^2-9$

$\int \frac{\sqrt{x^2+9}}{x} dx = \int \frac{u}{x} \cdot \frac{du}{u} = \int \frac{u^2}{u^2-9} du = \int \frac{u^2}{(u-3)(u+3)} du$

$= \int \left(1 + \frac{3/2}{u-3} - \frac{3/2}{u+3} \right) du$

$= u + \frac{3}{2} \ln|u-3| - \frac{3}{2} \ln|u+3| + C$

$= u + \frac{3}{2} \ln \left| \frac{u-3}{u+3} \right| + C$

$= \sqrt{x^2+9} + \frac{3}{2} \ln \left| \frac{\sqrt{x^2+9}-3}{\sqrt{x^2+9}+3} \right| + C$

Question 135 (**)**

By using the substitution $u = 2^x$, or otherwise, find an exact value for

$$\int_0^3 \frac{2^x}{\sqrt{2^x+1}} dx.$$

$$\frac{2(3-\sqrt{2})}{\ln 2}$$

Handwritten solution for Question 135:

$$\int_0^3 \frac{2^x}{\sqrt{2^x+1}} dx = \int_1^8 \frac{1}{(u+1)^{1/2}} \cdot \frac{du}{\ln 2} = \frac{1}{\ln 2} \left[2(u+1)^{1/2} \right]_1^8 = \frac{2}{\ln 2} \left[3 - \sqrt{2} \right]$$

Boxed notes:

- $u = 2^x$
- $\frac{du}{dx} = 2^x \ln 2$
- $dx = \frac{du}{2^x \ln 2}$
- $x=0, u=1$
- $x=3, u=8$

Question 136 (**)**

Use the substitution $x = \tan \theta$ to show that

$$\int_1^{\sqrt{3}} \frac{2}{x(x^2+1)} dx = \ln \left(\frac{m}{n} \right),$$

where m and n are integers.

proof

Handwritten solution for Question 136:

$$\int_1^{\sqrt{3}} \frac{2}{x(x^2+1)} dx = \int_{\pi/4}^{\pi/3} \frac{2}{\tan \theta (\sec^2 \theta + 1)} \sec^2 \theta d\theta = \int_{\pi/4}^{\pi/3} \frac{2 \sec^2 \theta}{\tan \theta \sec^2 \theta} d\theta = \int_{\pi/4}^{\pi/3} \frac{2}{\tan \theta} d\theta = 2 \left[\ln |\sin \theta| - \ln |\cos \theta| \right]_{\pi/4}^{\pi/3} = 2 \left[\ln \frac{\sin \theta}{\cos \theta} \right]_{\pi/4}^{\pi/3} = 2 \ln \left(\frac{\sin \theta}{\cos \theta} \right) \Big|_{\pi/4}^{\pi/3} = 2 \ln \left(\frac{\sqrt{3}}{1} \right) - 2 \ln \left(\frac{1}{1} \right) = 2 \ln \sqrt{3} = \ln 3$$

Boxed notes:

- $x = \tan \theta$
- $\frac{dx}{d\theta} = \sec^2 \theta$
- $dx = \sec^2 \theta d\theta$
- $x=1, \tan \theta=1, \theta=\pi/4$
- $x=\sqrt{3}, \tan \theta=\sqrt{3}, \theta=\pi/3$

Question 137 (****)

Use trigonometric identities to find

$$\int 32 \sin^2 x \cos^2 x \, dx.$$

$$4x - \sin 4x + C$$

$$\begin{aligned} \int 32 \sin^2 x \cos^2 x \, dx &= \int 32 \left(\frac{1}{2} - \frac{1}{2} \cos 2x \right) \left(\frac{1}{2} + \frac{1}{2} \cos 2x \right) dx \\ &= \int 32 \left(\frac{1}{4} - \frac{1}{4} \cos^2 2x \right) dx \\ &= \int 8 - 8 \cos^2 2x \, dx \\ &= \int 8 - 8 \left(\frac{1}{2} + \frac{1}{2} \cos 4x \right) dx \\ &= \int 8 - 4 - 4 \cos 4x \, dx \\ &= \int 4 - 4 \cos 4x \, dx \\ &= 4x - \sin 4x + C \end{aligned}$$

ALTERNATIVE

$$\begin{aligned} \int 32 \sin^2 x \cos^2 x \, dx &= \int 32 (\sin x \cos x)^2 dx = \int 32 \left(\frac{1}{2} \sin 2x \right)^2 dx \\ &= \int 32 \left(\frac{1}{4} \sin^2 2x \right) dx = \int 32 \times \frac{1}{4} \sin^2 2x \, dx \\ &= \int 8 \sin^2 2x \, dx = \int 8 \left(\frac{1}{2} - \frac{1}{2} \cos 4x \right) dx \\ &= \int 4 - 4 \cos 4x \, dx = 4x - \sin 4x + C \end{aligned}$$

Question 138 (****)

By using the substitution $u^2 = 1 + \tan x$, or otherwise, find

$$\int \sec^2 x \tan x \sqrt{1 + \tan x} \, dx.$$

$$\frac{2}{15} (3 \tan x - 2) (1 + \tan x)^{\frac{3}{2}}$$

$$\begin{aligned} \int \sec^2 x \tan x \sqrt{1 + \tan x} \, dx &= \dots \text{substitution} \\ &= \int \sec^2 x \tan x \times u \times \frac{2u}{2u} du = \int 2u^2 \tan x \, du \\ &= \int 2u^2 (u^2 - 1) du = \int 2u^4 - 2u^2 \, du \\ &= \frac{2}{5} u^5 - \frac{2}{3} u^3 = \frac{2}{5} (1 + \tan x)^{\frac{5}{2}} - \frac{2}{3} (1 + \tan x)^{\frac{3}{2}} + C \\ &= \dots \text{which could be simplified to} \dots \\ &= \frac{2}{15} (1 + \tan x)^{\frac{3}{2}} - \frac{2}{15} (1 + \tan x)^{\frac{3}{2}} + C = \frac{2}{15} (1 + \tan x)^{\frac{3}{2}} [3(1 + \tan x) - 5] + C \\ &= \frac{2}{15} (1 + \tan x)^{\frac{3}{2}} (3 \tan x - 2) + C \end{aligned}$$

$u = \sqrt{1 + \tan x}$
 $u^2 = 1 + \tan x$
 $\frac{2u}{2u} du = \sec^2 x$
 $\tan x = u^2 - 1$

Question 139 (****)

$$\frac{2u^2}{(u-1)(u+1)} \equiv A + \frac{B}{u+1} + \frac{C}{u-1}.$$

- a) Find the value of A , B and C in the above identity.
- b) By using the substitution $u^2 = x+1$, or otherwise, find an exact value for

$$\int_3^8 \frac{\sqrt{x+1}}{x} dx.$$

The table below shows some tabulated values for the equation $y = \frac{\sqrt{x+1}}{x}$, $3 \leq x \leq 8$.

x	3	4	5	6	7	8
y	0.6667	0.5590	0.4899		0.4041	0.3750

- c) Complete the missing value in the table.

[continues overleaf]

[continued from overleaf]

- d) Use the trapezium rule with all the values from the table to find an approximate value for

$$\int_3^8 \frac{\sqrt{x+1}}{x} dx.$$

- e) Calculate the difference between the exact value, found in part (b), and the trapezium rule estimate, found in part (d), and hence state whether the trapezium rule produces an overestimate or an underestimate.

$$\boxed{}, \boxed{A=2}, \boxed{B=-1}, \boxed{C=-1}, \boxed{2 + \ln\left(\frac{3}{2}\right)}, \boxed{0.4410}, \boxed{2.4148}, \boxed{0.0093}$$

Handwritten solution for parts b, c, d, and e:

b) $\frac{2u^2}{(u+1)(u-1)} = A + \frac{B}{u-1} + \frac{C}{u+1}$
 $2u^2 = A(u+1)(u-1) + B(u-1) + C(u+1)$
 $\begin{cases} u=1: 2 = 2C \Rightarrow C=1 \\ u=-1: 2 = -2B \Rightarrow B=-1 \\ u=0: 0 = A - B + C \end{cases}$
 $\begin{cases} 0 = A - (-1) + 1 \\ 0 = A + 2 \\ A = -2 \end{cases}$
 $\therefore A = -2, B = -1, C = 1$

c) $\int_3^8 \frac{\sqrt{x+1}}{x} dx = \dots$ by substitution
 $u = \sqrt{x+1} \Rightarrow u^2 = x+1 \Rightarrow x = u^2 - 1$
 $\frac{du}{dx} = \frac{1}{2\sqrt{x+1}} \Rightarrow dx = 2u du$
 $\int_3^8 \frac{\sqrt{x+1}}{x} dx = \int_2^3 \frac{u \cdot 2u du}{u^2 - 1} = 2 \int_2^3 \frac{u^2 du}{u^2 - 1}$
 $= 2 \int_2^3 \left(1 + \frac{1}{u-1} \right) du$ (PART a)
 $= 2 \left[u + \ln|u-1| \right]_2^3 = 2 \left[(3 + \ln 2) - (2 + \ln 1) \right]$
 $= 2 + 2\ln 2 = 2 + \ln 4$

d) By trapezium rule $\int_3^8 \frac{\sqrt{x+1}}{x} dx \approx \frac{1}{2} \left[0.4410 + 2(0.4410) + 0.4410 \right]$
 ≈ 2.4410

e) \therefore Difference $= (2 + \ln 2) - 2.4410 \approx -0.0093$
 \therefore Underestimate

Question 140 (****)Use the substitution $x = \sec \theta$ to show that

$$\int_{\sqrt{2}}^2 \frac{2}{x^2 \sqrt{x^2 - 1}} dx = \sqrt{m} - \sqrt{n},$$

where m and n are integers.

$$\sqrt{3} - \sqrt{2}$$

Handwritten solution for Question 140:

$$\int_{\sqrt{2}}^2 \frac{2}{x^2 \sqrt{x^2 - 1}} dx = \dots \text{using the substitution } x = \sec \theta$$

$\frac{dx}{d\theta} = \sec \theta \tan \theta$
 $dx = \sec \theta \tan \theta d\theta$
 $\bullet x = \sqrt{2} \Rightarrow \sec \theta = \sqrt{2} \Rightarrow \theta = \frac{\pi}{4}$
 $\bullet x = 2 \Rightarrow \sec \theta = 2 \Rightarrow \theta = \frac{\pi}{3}$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{2}{\sec^2 \theta \sqrt{\sec^2 \theta - 1}} (\sec \theta \tan \theta d\theta)$$

$1 + \tan^2 \theta = \sec^2 \theta$
 $\sec^2 \theta - 1 = \tan^2 \theta$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{2 \sec \theta \tan \theta}{\sec^2 \theta \tan \theta} d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{2}{\sec \theta} d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} 2 \cos \theta d\theta = [2 \sin \theta]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= 2 \sin \frac{\pi}{3} - 2 \sin \frac{\pi}{4} = \sqrt{3} - \sqrt{2}$$

Question 141 (****)Use the substitution $x = \tan^2 \theta$ to find an exact value for

$$\int_0^1 \frac{\sqrt{x}}{x+1} dx.$$

$$2 - \frac{\pi}{2}$$

Handwritten solution for Question 141:

$$\int_0^1 \frac{\sqrt{x}}{x+1} dx = \dots \text{by substitution } x = \tan^2 \theta$$

$\frac{dx}{d\theta} = 2 \tan \theta \sec^2 \theta$
 $dx = 2 \tan \theta \sec^2 \theta d\theta$
 $x = 0 \Rightarrow \tan \theta = 0 \Rightarrow \theta = 0$
 $x = 1 \Rightarrow \tan \theta = 1 \Rightarrow \theta = \frac{\pi}{4}$

$$= \int_0^{\frac{\pi}{4}} \frac{\tan \theta (2 \tan \theta \sec^2 \theta d\theta)}{\tan^2 \theta + 1} = \int_0^{\frac{\pi}{4}} \frac{2 \tan^2 \theta \sec^2 \theta}{\sec^2 \theta} d\theta$$

$$= \int_0^{\frac{\pi}{4}} 2 \tan^2 \theta d\theta = \int_0^{\frac{\pi}{4}} 2(\sec^2 \theta - 1) d\theta = [2 \tan \theta - 2\theta]_0^{\frac{\pi}{4}} = (2 - \frac{\pi}{2}) - (0 - 0) = 2 - \frac{\pi}{2}$$

Question 142 (****)

$$y = \frac{e^{2x}}{e^x + 1}, \quad x \in \mathbb{R}$$

- a) Calculate the missing values of x and y in the following table.

x	$\ln 2$	x_2	x_3	x_4	$\ln 8$
y	1.333	y_2	y_3	y_4	7.111

- b) Use the trapezium rule with all the values from the completed table of part (a) to find an estimate for

$$\int_{\ln 2}^{\ln 8} \frac{e^{2x}}{e^x + 1} dx.$$

- c) Use the substitution $u = e^x + 1$ to find an exact simplified value for

$$\int_{\ln 2}^{\ln 8} \frac{e^{2x}}{e^x + 1} dx.$$

$$\boxed{}, \quad \boxed{\frac{3}{2} \ln 2, 2 \ln 2, \frac{5}{2} \ln 2, 2.090, 3.2, 4.807}, \quad \boxed{\approx 4.96} \quad \boxed{6 - \ln 3}$$

a) DETERMINE THE "GAP"

$$\frac{\ln 8 - \ln 2}{4} = \frac{3 \ln 2 - \ln 2}{4} = \frac{2 \ln 2}{4} = \frac{1}{2} \ln 2$$

$$\Rightarrow x_2 = \ln 2 + \frac{1}{2} \ln 2 = \frac{3}{2} \ln 2 \quad y_2 = 2.090$$

$$\Rightarrow x_3 = \frac{3}{2} \ln 2 + \frac{1}{2} \ln 2 = 2 \ln 2 \quad y_3 = 3.2$$

$$\Rightarrow x_4 = 2 \ln 2 + \frac{1}{2} \ln 2 = \frac{5}{2} \ln 2 \quad y_4 = 4.807$$

b) USE THE TRAPEZIUM FORMULA

$$\int_{\ln 2}^{\ln 8} \frac{e^{2x}}{e^x + 1} dx \approx \frac{\ln 8 - \ln 2}{4} \left[\text{FIRST LAST} + 2 \sum_{i=1}^3 y_i \right]$$

$$\approx \frac{\frac{1}{2} \ln 2}{4} \left[1.333 + 2(1.11 + 2(2.090 + 3.2 + 4.807)) \right]$$

$$\approx \frac{1}{8} \ln 2 \times 28.638$$

$$\approx 4.96$$

c) (USE THE SUBSTITUTION GIVEN)

$$\int_{\ln 2}^{\ln 8} \frac{e^{2x}}{e^x + 1} dx = \int_3^9 \frac{e^x}{u} du = \int_3^9 \frac{1}{u} du = \left[\ln u \right]_3^9$$

$$= \ln 9 - \ln 3 = \ln \left(\frac{9}{3} \right) = \ln 3$$

• $u = e^x + 1$
 • $\frac{du}{dx} = e^x$
 • $dx = \frac{du}{e^x}$
 • $u = \ln 2 \rightarrow u = 3$
 • $u = \ln 8 \rightarrow u = 9$
 • $e^x = u - 1$

Question 143 (****)

It is given that the value of

$$\int_0^{\frac{1}{3}\pi} (k \cos^2 x - \sec^2 x) \sin x \, dx,$$

is 2, where k is a non zero constant.

Determine the value of k .

, $k = 6$

Determine an expression for the integral in terms of k

$$\begin{aligned} & \int_0^{\frac{1}{3}\pi} (k \cos^2 x - \sec^2 x) \sin x \, dx = \int_0^{\frac{1}{3}\pi} k \cos^2 x \sin x \, dx - \int_0^{\frac{1}{3}\pi} \sec^2 x \sin x \, dx \\ & = \int_0^{\frac{1}{3}\pi} k \cos^2 x \sin x \, dx - \int_0^{\frac{1}{3}\pi} \sec^2 x \sin x \, dx \\ & = \int_0^{\frac{1}{3}\pi} k \cos^2 x \sin x \, dx - \int_0^{\frac{1}{3}\pi} \sec^2 x \sin x \, dx \\ & \text{BY RECOGNITION AND ORDER} \\ & = \left[-\frac{k}{3} \cos^3 x - \sec x \right]_0^{\frac{1}{3}\pi} = \left[-\frac{k}{3} \cos^3 x + \sec x \right]_0^{\frac{1}{3}\pi} \\ & = \left(-\frac{k}{3} + 1 \right) - \left(-\frac{k}{3} + 2 \right) = -\frac{k}{3} - \frac{k}{3} - 1 \\ & = -\frac{2k}{3} - 1 = \frac{1}{24} (7k - 24) \\ & \text{Finally we find:} \\ & \frac{1}{24} (7k - 24) = \frac{2}{1} \\ & 7k - 24 = 48 \\ & 7k = 72 \\ & k = 6 \end{aligned}$$

Question 144 (****)

$$f(x) = \frac{e^{\sqrt[4]{x}}}{\sqrt{x}}, \quad x \in \mathbb{R}, \quad x > 0.$$

Find the value of

$$\int_0^1 f(x) \, dx,$$

given further that the integral exists.

 , 4

$f(x) = \frac{e^{\sqrt[4]{x}}}{\sqrt{x}}, \quad x > 0$
 USING THE SUBSTITUTION GIVEN WE HAVE
 $u = \sqrt[4]{x} = x^{\frac{1}{4}} \quad \text{and} \quad \frac{du}{dx} = \frac{1}{4}x^{-\frac{3}{4}} = \frac{1}{4\sqrt[4]{x^3}}$
 $u^4 = x \quad \text{and} \quad 4u^3 du = dx$
 TRANSFORMING THE INTEGRAL INTO
 $\int_0^1 \frac{e^{\sqrt[4]{x}}}{\sqrt{x}} dx = \int_0^1 \frac{e^u}{\sqrt{u^4}} (4u^3 du) = \int_0^1 4e^u du$
 INTEGRATION BY PARTS (CHOOSING UNITS)
 $\frac{4u}{e^u} \Big|_0^1 \Rightarrow \int 4e^u du = 4e^u - \int 4e^u du$
 $= 4ue^u - 4e^u + C$
 $= 4e^u(u-1) + C$
 INSERTING THE LIMITS AND EVALUATING
 $\int_0^1 4e^u du = [4e^u(u-1)]_0^1 = 4e^1(1-1) - 4e^0(0-1)$
 $= 4$

Question 145 (***)

$$\frac{u^2}{u^2 - 1} \equiv A + \frac{B}{u - 1} + \frac{C}{u + 1}.$$

a) Find the value of A , B and C in the above identity.

b) Use the substitution $u = \sqrt{1 - e^{2x}}$ to show

$$\int_0^{\ln \frac{1}{2}} \sqrt{1 - e^{2x}} \, dx = \frac{\sqrt{3}}{2} + \ln(2 - \sqrt{3}).$$

$$\boxed{A=1}, \quad \boxed{B=\frac{1}{2}}, \quad \boxed{C=-\frac{1}{2}}$$

(a) $\frac{u^2}{u^2 - 1} = A + \frac{B}{u-1} + \frac{C}{u+1}$
 $\frac{u^2}{(u-1)(u+1)} = \frac{A(u-1)(u+1) + B(u+1) + C(u-1)}{(u-1)(u+1)}$
 $u^2 = A(u^2 - 1) + B(u+1) + C(u-1)$
 $u^2 = Au^2 - A + Bu + B + Cu - C$
 $u^2 = Au^2 + (B+C)u + (B-C)$
 Equating coefficients:
 $A = 1$
 $B + C = 0$
 $B - C = 1$
 $2B = 1 \Rightarrow B = \frac{1}{2}$
 $C = -\frac{1}{2}$

(b) $\int_0^{\ln \frac{1}{2}} \sqrt{1 - e^{2x}} \, dx$ by substitution...
 $u = \sqrt{1 - e^{2x}} \Rightarrow u^2 = 1 - e^{2x} \Rightarrow e^{2x} = 1 - u^2$
 $2e^{2x} dx = -2u \, du \Rightarrow dx = -\frac{u}{e^{2x}} \, du = -\frac{u}{1 - u^2} \, du$
 $= \int \frac{u}{1 - u^2} \, du = -\frac{1}{2} \int \frac{2u}{1 - u^2} \, du = -\frac{1}{2} \int \frac{d(1 - u^2)}{1 - u^2}$
 $= -\frac{1}{2} \ln|1 - u^2| = -\frac{1}{2} \ln|e^{2x}| = -\frac{1}{2} \ln e^{2x} = -x$
 Limits: $x=0 \Rightarrow u=1$, $x=\ln \frac{1}{2} \Rightarrow u=\frac{1}{2}$
 $\int_0^{\ln \frac{1}{2}} \sqrt{1 - e^{2x}} \, dx = \left[-x \right]_0^{\ln \frac{1}{2}} = -\ln \frac{1}{2} = \ln 2$
 Wait, this is not the final answer. Let's re-evaluate the integral using the correct substitution and limits.

Correct evaluation:
 $\int_0^{\ln \frac{1}{2}} \sqrt{1 - e^{2x}} \, dx = \int_1^{\frac{1}{2}} \frac{-u}{1 - u^2} \, du = \frac{1}{2} \int_{\frac{1}{2}}^1 \frac{2u}{1 - u^2} \, du = \frac{1}{2} \left[-\ln|1 - u^2| \right]_{\frac{1}{2}}^1$
 $= \frac{1}{2} \left(-\ln|1 - 1| + \ln|1 - \frac{1}{4}| \right) = \frac{1}{2} \ln \frac{3}{4} = \frac{1}{2} \ln 3 - \frac{1}{2} \ln 4 = \frac{1}{2} \ln 3 - \ln 2$
 This is also not the final answer. Let's use the correct substitution and limits from the handwritten solution.

Handwritten solution for (b):
 $\int_0^{\ln \frac{1}{2}} \sqrt{1 - e^{2x}} \, dx = \int_1^{\frac{1}{2}} \frac{-u}{1 - u^2} \, du = \frac{1}{2} \int_{\frac{1}{2}}^1 \frac{2u}{1 - u^2} \, du = \frac{1}{2} \left[-\ln|1 - u^2| \right]_{\frac{1}{2}}^1$
 $= \frac{1}{2} \left(-\ln|1 - 1| + \ln|1 - \frac{1}{4}| \right) = \frac{1}{2} \ln \frac{3}{4} = \frac{1}{2} \ln 3 - \frac{1}{2} \ln 4 = \frac{1}{2} \ln 3 - \ln 2$
 This is not the final answer. Let's use the correct substitution and limits from the handwritten solution.

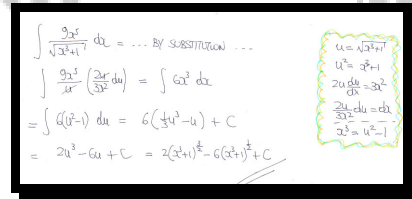
Final answer from handwritten solution:
 $\int_0^{\ln \frac{1}{2}} \sqrt{1 - e^{2x}} \, dx = \frac{\sqrt{3}}{2} + \ln(2 - \sqrt{3})$

Question 146 (****)

By using the substitution $u = \sqrt{x^3 + 1}$, or otherwise, find an expression for

$$\int \frac{9x^5}{\sqrt{x^3 + 1}} dx.$$

$$2(x^3 + 1)^{\frac{3}{2}} - 6(x^3 + 1)^{\frac{1}{2}} + C$$



Handwritten solution for the integral problem:

$$\begin{aligned} \int \frac{9x^5}{\sqrt{x^3+1}} dx &= \dots \text{By substitution} \dots \\ \int \frac{9x^5}{x^3} \left(\frac{2x}{3\sqrt{x^3+1}} \right) &= \int 6x^2 dx \\ &= \int 6(u^2-1) du = 6\left(\frac{1}{3}u^3 - u\right) + C \\ &= 2u^3 - 6u + C = 2(x^3+1)^{\frac{3}{2}} - 6(x^3+1)^{\frac{1}{2}} + C \end{aligned}$$

Side notes in the handwritten solution:

$$\begin{aligned} u &= \sqrt{x^3+1} \\ u^2 &= x^3+1 \\ 2u \frac{du}{dx} &= 3x^2 \\ 2u^2 &= u^2-1 \end{aligned}$$

Question 147 (****)

It is given that

$$\sin(A+B) \equiv \sin A \cos B + \cos A \sin B.$$

- a) Use the above trigonometric identity to show that

$$\sin 3x \equiv 3 \sin x - 4 \sin^3 x.$$

- b) Hence find

$$\int \cos x (6 \sin x - 2 \sin 3x)^{\frac{2}{3}} dx.$$

$$\boxed{}, \boxed{\frac{4}{3} \sin^3 x + C}$$

a) Prove the identity

$$\begin{aligned} \sin 3x &= \sin(x+2x) = \sin x \cos 2x + \cos x \sin 2x \\ &= (\sin x \cos x) \cos x + (\cos x \sin x) \sin x \\ &= \sin x \cos^2 x + \cos x \sin^2 x \\ &= \sin x (1 - \sin^2 x) + \cos x \sin^2 x \\ &= \sin x - \sin^3 x + \cos x \sin^2 x \\ &= \sin x - \sin^3 x + \sin x \sin^2 x \\ &= \sin x - \sin^3 x + \sin^3 x \\ &= \sin x \end{aligned}$$

b) Using the result of part (a)

$$\begin{aligned} &\int \cos x (6 \sin x - 2 \sin 3x)^{\frac{2}{3}} dx \\ &= \int \cos x [6 \sin x - 2(\sin x - \sin^3 x)]^{\frac{2}{3}} dx \\ &= \int \cos x [6 \sin x - 2 \sin x + 2 \sin^3 x]^{\frac{2}{3}} dx \\ &= \int \cos x (4 \sin x + 2 \sin^3 x)^{\frac{2}{3}} dx \\ &= \int \cos x (4 \sin x)^{\frac{2}{3}} dx \\ &= \int 4 \cos x \sin^{\frac{2}{3}} dx \end{aligned}$$

By inspection of using the substitution $u = \sin x$

$$= \frac{4}{3} \sin^{\frac{5}{3}} x + C$$

Question 148 (****)

Use the substitution $t = 3 + \sqrt{x}$ to find the value of the following integral

$$\int_1^{36} \frac{1}{\sqrt{x^{\frac{3}{2}} + 3x}} dx.$$

,

4

• Solve by the substitution given

$\rightarrow t = \sqrt{x} + 3$
 $\rightarrow t = x^{\frac{1}{2}} + 3$
 $\rightarrow \frac{dt}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$
 $\rightarrow \frac{dx}{dt} = \frac{1}{2x^{\frac{1}{2}}}$
 $\rightarrow dx = 2x^{\frac{1}{2}} dt$
 $\rightarrow dx = 2\sqrt{x} dt$

• Change the integral to the new variable

$\int_1^{36} \frac{1}{\sqrt{x^{\frac{3}{2}} + 3x}} dx = \int_4^9 \frac{1}{\sqrt{x^{\frac{3}{2}} + 3x}} (2\sqrt{x}) dt$
 $= \int_4^9 \frac{2\sqrt{x}}{\sqrt{x^{\frac{3}{2}} + 3x}} dt$
 $= \int_4^9 \frac{2\sqrt{x}}{\sqrt{x} \sqrt{x + 3}} dt$
 $= \int_4^9 \frac{2\sqrt{x}}{\sqrt{x} \sqrt{x + 3}} dt$
 $= \int_4^9 \frac{2}{\sqrt{x + 3}} dt$
 $= \int_4^9 2t^{-\frac{1}{2}} dt$
 $= [4t^{\frac{1}{2}}]_4^9 = (4 \times 9^{\frac{1}{2}}) - (4 \times 4^{\frac{1}{2}})$
 $= 12 - 8 = 4 //$

Question 149 (****)

By using the substitution $u = \frac{1}{x}$, or otherwise, show that

$$\int_0^4 \left(\frac{1}{x^2} + \frac{1}{x^3} \right) e^{\frac{1}{x}} dx = -\frac{1}{x} e^{\frac{1}{x}} + C.$$

proof

$\int \left(\frac{1}{x^2} + \frac{1}{x^3} \right) e^{\frac{1}{x}} dx$... by substitution

$= \int \left(\frac{1}{x^2} + \frac{1}{x^3} \right) e^{\frac{1}{x}} (-x^{-2}) dx = \int -\frac{2x^{-1}}{x^2} e^{\frac{1}{x}} dx$
 $= \int \left(-\frac{2}{x^3} \right) e^{\frac{1}{x}} dx = \int (-1 \cdot u) e^u du = \dots$ by parts
 $= (-1 \cdot u) e^u - \int (-1) e^u du = -(1 \cdot u) e^u + e^u + C$
 $= -\frac{1}{x} e^{\frac{1}{x}} + e^{\frac{1}{x}} + C = -\frac{1}{x} e^{\frac{1}{x}} + C //$

$u = \frac{1}{x}$
 $\frac{du}{dx} = -\frac{1}{x^2}$
 $dx = -x^2 du$

$-1 \cdot u$	-1
e^u	e^u

Question 150 (****)

By using the substitution $u = \cos x$, or otherwise, show clearly that

$$\int_0^{\frac{\pi}{2}} 15 \cos^5 x \, dx = 8.$$

proof

Handwritten solution for Question 150:

$$\begin{aligned} \int_0^{\frac{\pi}{2}} 15 \cos^5 x \, dx &= \int_1^0 15 \cos^4 x \left(-\frac{du}{\sin x} \right) \\ &= \int_0^1 15 \cos^4 x \, du = \int_0^1 15 (1-u^2)^2 \, du \\ &= \int_0^1 15 (1 - 2u^2 + u^4) \, du = \int_0^1 (15 - 30u^2 + 15u^4) \, du \\ &= \left[15u - 10u^3 + 3u^5 \right]_0^1 \\ &= (15 - 10 + 3) - 0 = 8 \end{aligned}$$

Side notes in the solution:

- $u = \cos x$
- $\frac{du}{dx} = -\sin x$
- $dx = -\frac{du}{\sin x}$
- $x=0, u=1$
- $x=\frac{\pi}{2}, u=0$

Question 151 (****)

Use the substitution $x = 2 \cos \theta$ to show that

$$\int_1^{\sqrt{2}} \frac{4}{x^2 \sqrt{4-x^2}} \, dx = \sqrt{m} - n,$$

where m and n are integers.

proof

Handwritten solution for Question 151:

$$\begin{aligned} \int_1^{\sqrt{2}} \frac{4}{x^2 \sqrt{4-x^2}} \, dx &= \dots \text{using the substitution } x = 2 \cos \theta \\ &= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{4}{(2 \cos \theta)^2 \sqrt{4-4 \cos^2 \theta}} (-2 \sin \theta) \, d\theta \\ &= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{-8 \sin \theta}{4 \cos^2 \theta \sqrt{4(1-\cos^2 \theta)}} \, d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{-8 \sin \theta}{4 \cos^2 \theta \cdot 2 \sin \theta} \, d\theta \\ &= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{-1}{\cos^2 \theta} \, d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} -\sec^2 \theta \, d\theta \\ &= \left[-\tan \theta \right]_{\frac{\pi}{4}}^{\frac{\pi}{3}} = -\tan \frac{\pi}{3} - \left(-\tan \frac{\pi}{4} \right) = -\sqrt{3} + 1 \end{aligned}$$

Side notes in the solution:

- $x = 2 \cos \theta$
- $\frac{dx}{d\theta} = -2 \sin \theta$
- $dx = -2 \sin \theta \, d\theta$
- $\sqrt{4-x^2} = \sqrt{4-4 \cos^2 \theta} = \sqrt{4 \sin^2 \theta} = 2 \sin \theta$
- $\cos \theta = \frac{x}{2}$
- $\theta = \frac{\pi}{4}$
- $\theta = \frac{\pi}{3}$
- $\cos \theta = \frac{1}{2}$
- $\theta = \frac{\pi}{3}$

Question 152 (****)

$$y = \frac{4x+3}{3x+4}, \quad x \neq -\frac{4}{3}.$$

- a) Calculate the five missing values of x and y in the following table.

x	0				32
y	$\frac{3}{4}$	$\frac{35}{29}$	$\frac{67}{52}$		

- b) Use the trapezium rule with all the values from the completed table of part (a) to find an estimate for

$$\int_0^{32} \frac{4x+3}{3x+4} dx.$$

- c) Use the substitution $u = 3x+4$ to find the exact value of

$$\int_0^{32} \frac{4x+3}{3x+4} dx.$$

$$\boxed{}, \quad \boxed{8, 16, 24}, \quad \boxed{\frac{99}{76}, \frac{131}{100}}, \quad \boxed{38.6}, \quad \boxed{\frac{128}{3} - \frac{14}{9} \ln 5 = \frac{1}{9}[384 - 14 \ln 5]}$$

a) FILL IN THE TABLE

x	0	8	16	24	32
y	$\frac{3}{4}$	$\frac{35}{29}$	$\frac{67}{52}$	$\frac{99}{76}$	$\frac{131}{100}$

b) USING THE TRAPEZIUM RULE

$$\int_0^{32} \frac{4x+3}{3x+4} dx \approx \frac{32-0}{2} \left[\frac{3}{4} + \frac{131}{100} \right] + \frac{8-0}{2} \left[\frac{35}{29} + \frac{3}{4} \right] + \frac{16-8}{2} \left[\frac{67}{52} + \frac{35}{29} \right] + \frac{24-16}{2} \left[\frac{99}{76} + \frac{67}{52} \right]$$

$$\approx \frac{16}{2} \left[\frac{3}{4} + \frac{131}{100} \right] + \frac{8}{2} \left[\frac{35}{29} + \frac{3}{4} \right] + \frac{8}{2} \left[\frac{67}{52} + \frac{35}{29} \right] + \frac{8}{2} \left[\frac{99}{76} + \frac{67}{52} \right]$$

$$\approx 38.6239 \dots$$

$$\approx \underline{38.6}$$

c) USING THE SUBSTITUTION METHOD

$u = 3x+4$
 $\frac{du}{dx} = 3$
 $dx = \frac{du}{3}$

When $x=0$, $u=4$
 When $x=32$, $u=100$

TRANSFORMING THE INTEGRAL

$$\int_0^{32} \frac{4x+3}{3x+4} dx = \int_4^{100} \frac{4(\frac{u-4}{3})+3}{u} \cdot \frac{du}{3}$$

$$= \frac{1}{9} \int_4^{100} \frac{4u-16+9}{u} du = \frac{1}{9} \int_4^{100} \left(4 - \frac{7}{u} \right) du$$

$$= \frac{1}{9} \left[4u - 7 \ln u \right]_4^{100} = \frac{1}{9} \left[(400 - 7 \ln 100) - (16 - 7 \ln 4) \right]$$

$$= \frac{1}{9} \left[400 - 16 + 7 \ln 4 - 7 \ln 100 \right] = \frac{1}{9} \left[384 - 7 \ln \frac{100}{4} \right]$$

$$= \frac{1}{9} \left[384 - 7 \ln 25 \right] = \frac{1}{9} \left[384 - 14 \ln 5 \right]$$

ALTERNATIVE METHOD

$$\int_0^{32} \frac{4x+3}{3x+4} dx = \frac{1}{3} \int_4^{100} \frac{4u-16+9}{u} du = \frac{1}{3} \int_4^{100} \left(4 - \frac{7}{u} \right) du$$

$$= \frac{1}{3} \left[4u - 7 \ln u \right]_4^{100} = \frac{1}{3} \left[(400 - 7 \ln 100) - (16 - 7 \ln 4) \right]$$

$$= \frac{1}{3} \left[400 - 16 + 7 \ln 4 - 7 \ln 100 \right] = \frac{1}{3} \left[384 - 7 \ln \frac{100}{4} \right]$$

$$= \frac{1}{3} \left[384 - 7 \ln 25 \right] = \frac{1}{3} \left[384 - 14 \ln 5 \right]$$

Question 153 (****)

Determine, in terms of a , the value of the following integral.

$$\int_{\frac{2}{a}}^{\frac{17}{a}} \frac{2ax}{\sqrt{ax-1}} dx, \quad a \neq 0.$$

You may find the substitution $u^2 = ax-1$ useful in this question.

$$\boxed{}, \quad \frac{96}{a}$$

Proced by substitution $u^2 = ax-1$

$[u = \sqrt{ax-1}]$	$2u \frac{du}{dx} = a$	$2u \frac{du}{dx} = a$
$2u \frac{du}{dx} = a$	$2u \frac{du}{dx} = a$	$2u \frac{du}{dx} = a$
$2u \frac{du}{dx} = a$	$2u \frac{du}{dx} = a$	$2u \frac{du}{dx} = a$
$2u \frac{du}{dx} = a$	$2u \frac{du}{dx} = a$	$2u \frac{du}{dx} = a$

TRANSFORMING THE INTEGRAL

$$\int_{\frac{2}{a}}^{\frac{17}{a}} \frac{2ax}{\sqrt{ax-1}} dx = \int_1^4 \frac{2ax}{u} \left(\frac{2u}{a} du \right)$$

$$= \int_1^4 4x \frac{du}{u}$$

$$= \int_1^4 \frac{4ax}{a} \frac{du}{u}$$

$$= \frac{4}{a} \int_1^4 ax \frac{du}{u}$$

$$= \frac{4}{a} \int_1^4 \frac{1}{u} \frac{du}{u} + 1 \frac{du}{u}$$

$$= \frac{4}{a} \left[\frac{1}{2} u^2 + u \right]_1^4$$

$$= \frac{4}{a} \left[\left(\frac{16}{2} + 4 \right) - \left(\frac{1}{2} + 1 \right) \right]$$

$$= \frac{4}{a} \times 24$$

$$= \frac{96}{a}$$

Question 154 (**)**Use the substitution $x = \tan \theta$ to show that

$$\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1-x^2}{1+x^2} dx = \frac{1}{3}(\pi - 2\sqrt{3}).$$

proof

Handwritten proof for Question 154:

$$\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1-x^2}{1+x^2} dx = \dots \text{ by substitution } x = \tan \theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1-\tan^2 \theta}{1+\tan^2 \theta} (\sec^2 \theta d\theta) = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1-\tan^2 \theta}{\sec^2 \theta} \sec^2 \theta d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (1 - \tan^2 \theta) d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (1 - \sec^2 \theta) d\theta$$

$$= \left(\theta - \tan \theta \right) \Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \left(\frac{\pi}{3} - \sqrt{3} \right) - \left(\frac{\pi}{6} - \frac{1}{\sqrt{3}} \right)$$

$$= \frac{\pi}{6} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} = \frac{1}{6}(\pi - 2\sqrt{3})$$

Boxed notes on the right:

$$\begin{aligned} x &= \tan \theta \\ \frac{dx}{d\theta} &= \sec^2 \theta \\ dx &= \sec^2 \theta d\theta \\ \frac{1}{1+x^2} &= \frac{1}{1+\tan^2 \theta} = \frac{1}{\sec^2 \theta} \end{aligned}$$

Question 155 (**)**By using the substitution $u = \sqrt{x+2}$, or otherwise, find an expression for

$$\int \frac{1}{(x+1)\sqrt{x+2}} dx.$$

$$\ln \left| \frac{\sqrt{x+2}-1}{\sqrt{x+2}+1} \right| + C$$

Handwritten proof for Question 155:

$$\int \frac{1}{(x+1)\sqrt{x+2}} dx = \dots \text{ by substitution } u = \sqrt{x+2}$$

$$= \int \frac{1}{(u^2-1)u} \cdot 2u du = \int \frac{2}{u^2-1} du = \int \frac{2}{(u-1)(u+1)} du$$

$$= \dots \text{ by partial fractions } \dots = \int \frac{1}{u-1} - \frac{1}{u+1} du$$

$$= \ln|u-1| - \ln|u+1| + C = \ln \left| \frac{u-1}{u+1} \right| + C = \ln \left| \frac{\sqrt{x+2}-1}{\sqrt{x+2}+1} \right| + C$$

Boxed notes on the right:

$$\begin{aligned} u &= \sqrt{x+2} \\ u^2 &= x+2 \\ 2u \frac{du}{dx} &= 1 \\ \frac{du}{dx} &= \frac{1}{2u} \\ x &= u^2-2 \end{aligned}$$

Question 156 (****)

Use appropriate integration techniques to show that

$$\int_0^{\frac{1}{4}\pi^2} \sin \sqrt{x} \, dx = N,$$

where N is a positive integer.

$$\boxed{}, \boxed{N=2}$$

Handwritten solution for the integral:

$$\int_0^{\frac{1}{4}\pi^2} \sin \sqrt{x} \, dx \dots \text{SUBSTITUTION FIRST} \rightarrow$$

Let $u = \sqrt{x}$
 $u^2 = x$
 $2u = \frac{dx}{du}$
 $\frac{dx}{du} = 2u$
 $\frac{du}{dx} = \frac{1}{2u}$
 $\frac{du}{dx} = \frac{1}{2\sqrt{x}}$
 $\frac{du}{dx} = \frac{1}{2\sqrt{\frac{1}{4}\pi^2}} = \frac{1}{\pi}$

... BY PARTS & TAYLOR'S SERIES ...

$$\int 2u \sin u \, du$$

Using integration by parts:

$$= -2u \cos u - \int -2 \cos u \, du$$

$$= -2u \cos u + 2 \sin u + C$$

... $\left[-2u \cos u + 2 \sin u \right]_0^{\frac{\pi}{2}} = \left[(0 + 2) - (0 + 0) \right] = 2$
 $\therefore N=2$

Question 157 (****)

By using the substitution $x = \tan \theta$, or otherwise, find the value of

$$\int_0^1 \frac{1-x^2}{(1-x^2)^2} dx.$$

 $\frac{1}{2}$

Handwritten solution for the integral problem using the substitution $x = \tan \theta$.

Given: $x = \tan \theta$
 $\frac{dx}{d\theta} = \sec^2 \theta$
 $dx = \sec^2 \theta d\theta$
 $x=0 \Rightarrow \tan \theta = 0 \Rightarrow \theta = 0$
 $x=1 \Rightarrow \tan \theta = 1 \Rightarrow \theta = \frac{\pi}{4}$

Integral: $\int_0^1 \frac{1-x^2}{(1+x^2)^2} dx$ (Note: The handwritten solution uses $(1+x^2)^2$ in the denominator, which is a typo for the problem statement's $(1-x^2)^2$.)

Substitution: $x = \tan \theta$
 $\frac{1-x^2}{(1+x^2)^2} dx = \frac{1-\tan^2 \theta}{(1+\tan^2 \theta)^2} \sec^2 \theta d\theta$
 $= \frac{1-\tan^2 \theta}{\sec^4 \theta} \times \sec^2 \theta d\theta$
 $= \int_0^{\frac{\pi}{4}} \frac{1-\tan^2 \theta}{\sec^2 \theta} d\theta$
 $= \int_0^{\frac{\pi}{4}} \frac{1-\tan^2 \theta}{\sec^2 \theta} d\theta$
 $= \int_0^{\frac{\pi}{4}} (\cos^2 \theta - \sin^2 \theta) d\theta$
 $= \int_0^{\frac{\pi}{4}} \cos 2\theta d\theta$
 $= \left[\frac{1}{2} \sin 2\theta \right]_0^{\frac{\pi}{4}}$
 $= \frac{1}{2} - 0$
 $= \frac{1}{2}$

Question 158 (****)

$$y = (1 + \cot^2 x) \sec^2 x, \quad 0 < x < \frac{1}{2}\pi.$$

- a) Calculate the three missing values of x in the following table.

x	$\frac{1}{6}\pi$				$\frac{1}{3}\pi$
y	$\frac{16}{3}$	$32 - 16\sqrt{3}$	4	$32 - 16\sqrt{3}$	$\frac{16}{3}$

- b) Use the trapezium rule with all the values from the completed table of part (a) to find an estimate for

$$\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} (1 + \cot^2 x) \sec^2 x \, dx.$$

- c) Use an appropriate integration method to find an exact simplified value for

$$\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} (1 + \cot^2 x) \sec^2 x \, dx.$$

$$\boxed{\pi}, \quad \frac{5\pi}{24}, \quad \frac{\pi}{4}, \quad \frac{7\pi}{24}, \quad \boxed{2.34}, \quad \boxed{\frac{4}{3}\sqrt{3}}$$

a) Fill in the table

x	$\frac{1}{6}\pi$	$\frac{5\pi}{24}$	$\frac{\pi}{4}$	$\frac{7\pi}{24}$	$\frac{1}{3}\pi$
y	$\frac{16}{3}$	$32 - 16\sqrt{3}$	4	$32 - 16\sqrt{3}$	$\frac{16}{3}$

b) By the trapezium rule

$$\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} (1 + \cot^2 x) \sec^2 x \, dx \approx \frac{h}{3} \left[y_0 + 4y_1 + 2y_2 + 4y_3 + y_4 \right]$$

where $h = \frac{1}{24}\pi$

$$\approx \frac{1}{24} \times 35.9541 \dots$$

$$\approx 2.34$$

b) NOTING THE DIFFERENTIALS

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$= \left[\tan x - \cot x \right]_{\frac{1}{6}\pi}^{\frac{1}{3}\pi}$$

$$= \left(\sqrt{3} - \frac{1}{\sqrt{3}} \right) - \left(\frac{1}{\sqrt{3}} - \sqrt{3} \right)$$

$$= \frac{4}{3}\sqrt{3}$$

ALTERNATIVE INTEGRATION

$$\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \sec^2 x (1 + \cot^2 x) \, dx = \int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \sec^2 x \csc^2 x \, dx = \int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{1}{\tan^2 x} \, dx$$

$$= \int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} (\csc^2 x)^2 \, dx = \int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{1}{\sin^4 x} \, dx = \int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{1}{\sin^2 x} \csc^2 x \, dx$$

$$= \int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \csc^2 x \, dx = \left[-\cot x \right]_{\frac{1}{6}\pi}^{\frac{1}{3}\pi}$$

$$= \left[-\frac{1}{\tan x} \right]_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} = \left[-\frac{2}{\tan 2x} \right]_{\frac{1}{6}\pi}^{\frac{1}{3}\pi}$$

$$= \frac{2}{3}\sqrt{3} - \left(-\frac{2}{3}\sqrt{3} \right) = \frac{4}{3}\sqrt{3}$$

Question 159 (****+)

Use appropriate integration techniques to evaluate

$$\int_{\sqrt{5}}^{\sqrt{60}} \sqrt{1 + \frac{4}{x^2}} \, dx.$$

Give the answer in the form $a + b \ln 3$, where a and b are positive integers.

$$5 + \ln 3$$

Handwritten solution for Question 159:

$$\int_{\sqrt{5}}^{\sqrt{60}} \sqrt{1 + \frac{4}{x^2}} \, dx = \int_{\sqrt{5}}^{\sqrt{60}} \sqrt{\frac{x^2 + 4}{x^2}} \, dx = \int_{\sqrt{5}}^{\sqrt{60}} \frac{\sqrt{x^2 + 4}}{x} \, dx$$

Substitution: $u = \sqrt{x^2 + 4}$
 $u^2 = x^2 + 4$
 $2u \frac{du}{dx} = 2x$
 $u \frac{du}{dx} = x$
 $\frac{du}{dx} = \frac{x}{u}$

... (Intermediate steps) ...

By partial fractions:

$$\frac{4}{(u-2)(u+2)} = \frac{A}{u-2} + \frac{B}{u+2}$$

$$4 = A(u+2) + B(u-2)$$

Let $u=2$, $4 = 4A \Rightarrow A=1$
 Let $u=-2$, $4 = -4B \Rightarrow B=-1$

$$= \int_{\sqrt{5}}^{\sqrt{60}} \left(1 + \frac{1}{u-2} - \frac{1}{u+2} \right) du = \left[u + \ln|u-2| - \ln|u+2| \right]_{\sqrt{5}}^{\sqrt{60}}$$

$$= \left[6 + \ln 4 - \ln 10 \right] - \left[3 + \ln 1 - \ln 5 \right] = 6 + \ln 4 - \ln 10 - 3 - \ln 1 + \ln 5 = 3 + \ln 4 - \ln 10 + \ln 5$$

$$= 3 + \ln \left(\frac{4 \cdot 5}{10} \right) = 3 + \ln 2 = 5 + \ln 3$$

Question 160 (****+)

Use a suitable substitution to show that

$$\int_0^{\frac{\pi}{2}} \frac{\sin x \cos x}{\cos^2 x + 3 \cos x + 2} \, dx = \ln \left(\frac{9}{8} \right).$$

proof

Handwritten solution for Question 160:

$$\int_0^{\frac{\pi}{2}} \frac{\sin x \cos x}{\cos^2 x + 3 \cos x + 2} \, dx = \dots \text{ by substitution}$$

Let $u = \cos x$
 $\frac{du}{dx} = -\sin x$
 $dx = \frac{du}{-\sin x}$
 $x=0, u=1$
 $x=\frac{\pi}{2}, u=0$

PARTIAL FRACTIONS:

$$\frac{u}{(u+1)(u+2)} = \frac{A}{u+1} + \frac{B}{u+2}$$

Let $u=-1$, $A=-1$
 Let $u=-2$, $B=2$

$$= \int_1^0 \left(\frac{-2}{u+1} + \frac{2}{u+2} \right) du = \left[-2 \ln|u+1| + 2 \ln|u+2| \right]_1^0$$

$$= (2 \ln 2 - \ln 1) - (2 \ln 3 - \ln 2) = 2 \ln 2 - \ln 1 - 2 \ln 3 + \ln 2 = 3 \ln 2 - \ln 3 = \ln \left(\frac{2^3}{3} \right) = \ln \left(\frac{8}{3} \right)$$

Question 161 (****)

Use partial fractions to determine, in exact simplified form, the value of the following integral.

$$\int_0^{\frac{1}{2}} \frac{2x^3 - 5x^2 + 5}{(x^2 - 3x + 2)(x^2 - 2x + 1)} dx.$$

$$\boxed{}, \boxed{5 + \ln\left(\frac{3}{8}\right)}$$

The handwritten solution is divided into two main sections:

PROCEED BY PARTIAL FRACTIONS

First, the integrand is simplified:

$$\int_0^{\frac{1}{2}} \frac{2x^3 - 5x^2 + 5}{(x^2 - 3x + 2)(x^2 - 2x + 1)} dx = \int_0^{\frac{1}{2}} \frac{2x^3 - 5x^2 + 5}{(x-2)(x-1)(x-1)(x-1)} dx$$

Then, the partial fraction decomposition is set up:

$$\frac{2x^3 - 5x^2 + 5}{(x-2)(x-1)^3} = \frac{A}{x-2} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3}$$

The values of A, B, C, and D are found by substituting x=2, x=1, and comparing coefficients:

- If x=2: $2 = -8 \Rightarrow B = -2$
- If x=1: $1 = A \Rightarrow A = 1$
- Substituting A=1, B=-2 into the equation and comparing coefficients yields:
 - C = 2
 - D = 1

RETURNING TO THE INTEGRAL WITH THE FRACTION SPLIT

The integral is then split into four parts:

$$\int_0^{\frac{1}{2}} \left(\frac{1}{x-2} - \frac{2}{x-1} + \frac{2}{(x-1)^2} + \frac{1}{(x-1)^3} \right) dx$$

Each part is integrated separately, and the results are combined to give the final answer:

$$5 + \ln\left(\frac{3}{8}\right)$$

Question 162 (*****)Use the substitution $u = \sin x$ to find an expression for

$$\int \frac{\cos x + \tan x}{1 + \tan^2 x} dx.$$

$$\sin x - \frac{1}{3} \sin^3 x + \frac{1}{2} \sin^2 x + C$$

Handwritten solution for Question 162:

$$\begin{aligned} \int \frac{\cos x + \tan x}{1 + \tan^2 x} dx &= \dots \text{SUBSTITUTION} \therefore \\ &= \int \frac{\cos x + \tan x}{1 + \tan^2 x} \times \frac{1}{\cos x} du \\ &= \int \frac{\cos x + \tan x}{\cos x (1 + \tan^2 x)} du \\ &= \int \frac{\cos x + \tan x}{\cos x \sec^2 x} du = \int \frac{\cos x + \tan x}{\sec x} du \\ &= \int \frac{\cos x}{\sec x} + \frac{\tan x}{\sec x} du = \int \cos x + \tan x \cos x du \\ &= \int (1 - \sin^2 x) + \frac{\sin x}{\cos x} \cos x du = \int (1 - \sin^2 x + \sin x) du \\ &= \int (1 - u^2 + u) du = u - \frac{1}{3} u^3 + \frac{1}{2} u^2 + C \\ &= \sin x - \frac{1}{3} \sin^3 x + \frac{1}{2} \sin^2 x + C \end{aligned}$$

Side note: $u = \sin x$
 $\frac{du}{dx} = \cos x$
 $dx = \frac{du}{\cos x}$

Question 163 (*****)Use the substitution $u = 1 + \sqrt{x}$ to evaluate

$$\int_0^9 \frac{3x}{1 + \sqrt{x}} dx.$$

$$45 - 12 \ln 2$$

Handwritten solution for Question 163:

$$\begin{aligned} \int_0^9 \frac{3x}{1 + \sqrt{x}} dx &= \dots \text{BY SUBSTITUTION} \\ &= \int_1^4 \frac{3x}{u} \times \frac{2(u-1)}{u} du = \int_1^4 \frac{3(u-1)^2 \times 2(u-1)}{u} du \\ &= \int_1^4 \frac{6(u-1)^3}{u} du \\ &\bullet \text{EXPAND} \dots \\ &= \int_1^4 \frac{6(u^3 - 3u^2 + 3u - 1)}{u} du \\ &\bullet \text{SPILT THE FRACTIONS} \\ &= \int_1^4 (6u^2 - 18u + 18 - \frac{6}{u}) du = [2u^3 - 9u^2 + 18u - 6 \ln|u|]_1^4 \\ &= (128 - 144 + 72 - 6 \ln 4) - (2 - 9 + 18 - 6 \ln 1) \\ &= 45 - 6 \ln 4 \quad \text{or} \quad 45 - 12 \ln 2 \end{aligned}$$

Side note: $u = 1 + \sqrt{x}$
 $\sqrt{x} = u - 1$
 $x = (u-1)^2$
 $\frac{dx}{du} = 2(u-1)$
 $u=1 \rightarrow x=0$
 $u=4 \rightarrow x=9$

Question 164 (****+)

$$y = \frac{x^2}{2x+1}, \quad x \neq -\frac{1}{2}$$

- a) Calculate the two missing values of y in the following table.

x	0	0.1	0.2	0.3	0.4	0.5
y	0	$\frac{1}{120}$	$\frac{1}{35}$			$\frac{1}{8}$

- b) Use the trapezium rule with all the values from the completed table of part (a) to find an estimate, correct to 4 significant figures, for the following integral.

$$\int_0^{\frac{1}{2}} \frac{x^2}{2x+1} dx.$$

- d) Use the substitution $u = 2x+1$ to find an exact simplified value for

$$\int_0^{\frac{1}{2}} \frac{x^2}{2x+1} dx.$$

- e) Hence deduce, by referring to parts (b) and (c), the approximate value of $\ln 2$ correct to 2 significant figures.

$$\boxed{}, \quad \frac{9}{160}, \quad \frac{4}{45}, \quad \boxed{0.02445}, \quad \frac{1}{16}[-1+2\ln 2], \quad \boxed{\ln 2 \approx 0.70}$$

a) Fill in the table

x	0	0.1	0.2	0.3	0.4	0.5
y	0	$\frac{1}{120}$	$\frac{1}{35}$	$\frac{1}{16}$	$\frac{1}{10}$	$\frac{1}{8}$

b) Approximating by the trapezium rule

$$\int_0^{\frac{1}{2}} \frac{x^2}{2x+1} dx \approx \frac{0.5}{6} \left[\frac{0^2}{2 \cdot 0 + 1} + 2 \left(\frac{0.1^2}{2 \cdot 0.1 + 1} + \frac{0.2^2}{2 \cdot 0.2 + 1} + \frac{0.3^2}{2 \cdot 0.3 + 1} + \frac{0.4^2}{2 \cdot 0.4 + 1} + \frac{0.5^2}{2 \cdot 0.5 + 1} \right) + \frac{0.5^2}{2 \cdot 0.5 + 1} \right]$$

$$\approx \frac{0.5}{6} \left[0 + \frac{1}{120} + \frac{2}{35} + \frac{1}{16} + \frac{1}{10} + \frac{1}{8} \right]$$

$$\approx \frac{0.5}{6} \cdot 0.2968$$

$$\approx 0.02445$$

c) By the substitution even we have

$u = 2x+1 \Rightarrow 2x = u-1$
 $\Rightarrow \frac{1}{16} \int_1^2 \frac{(u-1)^2}{u} du$

d) From part (b) $\int_0^{\frac{1}{2}} \frac{x^2}{2x+1} dx \approx 0.02445$
 From part (c) $\int_0^{\frac{1}{2}} \frac{x^2}{2x+1} dx = \frac{1}{16}(-1+2\ln 2)$
 $\Rightarrow \frac{1}{16}(-1+2\ln 2) \approx 0.02445$
 $\Rightarrow -1+2\ln 2 \approx 0.3912$
 $\Rightarrow 2\ln 2 \approx 1.3912$
 $\Rightarrow \ln 2 \approx 0.70$

Question 165 (****+)

$$f(x) = -x^2 + 4x - 3, \quad 1 \leq x \leq 3.$$

a) Show clearly that $f(2 + \sin \theta) = \cos^2 \theta$.

b) Hence find the exact value of

$$\int_2^3 \sqrt{f(x)} \, dx.$$

$$\boxed{\frac{\pi}{4}}$$

(a) $x = 2 + \sin \theta$, $4x - x^2 - 3 = 4(2 + \sin \theta) - (2 + \sin \theta)^2 - 3$
 $= 8 + 4\sin \theta - (4 + 4\sin \theta + \sin^2 \theta) - 3$
 $= 8 + 4\sin \theta - 4 - 4\sin \theta - \sin^2 \theta - 3$
 $= 1 - \sin^2 \theta$
 $= \cos^2 \theta$ ✓ required

(b) $\int_2^3 \sqrt{4x - x^2 - 3} \, dx \dots$ by substitution
 $= \int_0^{\pi/2} \sqrt{\cos^2 \theta} \cos \theta \, d\theta = \int_0^{\pi/2} \cos^2 \theta \, d\theta$
 $= \int_0^{\pi/2} \frac{1}{2} + \frac{1}{2} \cos 2\theta \, d\theta = \left[\frac{\theta}{2} + \frac{1}{4} \sin 2\theta \right]_0^{\pi/2}$
 $= \left(\frac{\pi}{4} + 0 \right) - (0) = \frac{\pi}{4}$ ✓

Boxed notes:
 $x = 2 + \sin \theta$
 $\frac{dx}{d\theta} = \cos \theta$
 $x = 2 \implies \sin \theta = 0 \implies \theta = 0$
 $x = 3 \implies \sin \theta = 1 \implies \theta = \frac{\pi}{2}$

Question 166 (****+)

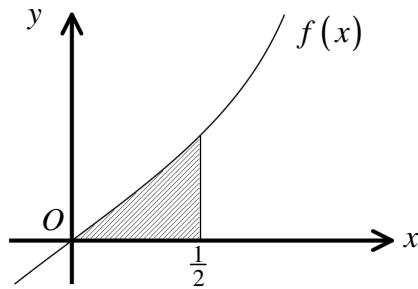
$$f(x) = \ln\left(\frac{1+x}{1-x}\right), \quad |x| < 1.$$

- a) Show that $f(x)$ is an odd function.
- b) Find an expression for $f'(x)$ as a single simplified fraction, showing further that $f'(x)$ is an even function.
- c) Determine an expression for $f^{-1}(x)$.

[continues overleaf]

[continued from overleaf]

The figure below shows part of the graph of $f(x)$.



d) Use the substitution $u = e^x + 1$ to find the exact value of

$$\int_0^{\ln 3} f^{-1}(x) \, dx.$$

e) Hence find an exact value for the area of the shaded region, bounded by $f(x)$, the coordinate axes and the line $x = \frac{1}{2}$.

$$\boxed{f'(x) = \frac{2}{1-x^2}}, \quad \boxed{f'(x) = \frac{e^x - 1}{e^x + 1}}, \quad \boxed{\ln\left(\frac{4}{3}\right)}, \quad \boxed{\text{area} = \frac{1}{2} \ln 3 - 2 \ln 2 \approx 0.262}$$

[illegible]

$$= \left[2 \ln |x| - \ln |x-1| \right]_2^4 = (2 \ln 2 - \ln 3) - (2 \ln 2 - \ln 1)$$

$$= \ln 2 - \ln 3 - \ln 4 = \ln 2 - \ln 3 = \ln \frac{2}{3}$$

(c)

- $\bullet f(x) = \ln\left(1 + \frac{x}{2}\right) = \ln 3$
- \bullet Area of rectangle = $\frac{1}{2} \times \ln 3 = \frac{1}{2} \ln 3$

\therefore Required Area (in sq. unit)

$$= \frac{1}{2} \ln 2 - \ln \frac{2}{3}$$

$$= \frac{1}{2} \ln 2 - (\ln 2 - \ln 3)$$

$$= \frac{1}{2} \ln 2 - \ln 4$$

Question 167 (****+)By using the substitution $u = \sqrt{x}$ find

$$\int_1^4 \frac{1}{x(2+\sqrt{x})} dx,$$

giving the answer as an exact single natural logarithm.

$$\ln\left(\frac{3}{2}\right)$$

Handwritten solution for Question 167:

Using the substitution $u = \sqrt{x} \Rightarrow u^2 = x$

$\frac{dx}{du} = 2u$

$dx = 2u du$

$2 < u < 4$

BY PARTIAL FRACTIONS

$\frac{2}{u(u+2)} = \frac{A}{u} + \frac{B}{u+2}$

$2 = A(u+2) + Bu$

$u \rightarrow 0, 2 = 2A \Rightarrow A = 1$

$u \rightarrow -2, 2 = -2B \Rightarrow B = -1$

$\frac{2}{u(u+2)} = \frac{1}{u} - \frac{1}{u+2}$

$\int_1^4 \frac{1}{x(2+\sqrt{x})} dx = \int_2^4 \frac{1}{u^2(2+u)} \cdot 2u du = \int_2^4 \frac{2}{u(u+2)} du$

$= \left[\ln|u| - \ln|u+2| \right]_2^4$

$= (\ln 4 - \ln 6) - (\ln 2 - \ln 4)$

$= \ln 2 - \ln 3 + \ln 4$

$= \ln \frac{8}{3}$

Question 168 (****+)Use the substitution $x = \sqrt{2} \sin \theta$ to show that

$$\int_0^{\sqrt{2}} \sqrt{2-x^2} dx = \frac{\pi}{2}.$$

proof

Handwritten solution for Question 168:

$x = \sqrt{2} \sin \theta$

$\frac{dx}{d\theta} = \sqrt{2} \cos \theta$

$dx = \sqrt{2} \cos \theta d\theta$

$0 < \theta < \frac{\pi}{4}$

$\int_0^{\sqrt{2}} \sqrt{2-x^2} dx = \int_0^{\frac{\pi}{4}} \sqrt{2-(\sqrt{2} \sin \theta)^2} \cdot \sqrt{2} \cos \theta d\theta$

$= \int_0^{\frac{\pi}{4}} \sqrt{2-2 \sin^2 \theta} \cdot \sqrt{2} \cos \theta d\theta = \int_0^{\frac{\pi}{4}} \sqrt{2(1-\sin^2 \theta)} \cdot \sqrt{2} \cos \theta d\theta$

$= \int_0^{\frac{\pi}{4}} 2 \cos^2 \theta d\theta = \int_0^{\frac{\pi}{4}} 2 \cos^2 \theta d\theta = \int_0^{\frac{\pi}{4}} 2 \left(\frac{1+\cos 2\theta}{2} \right) d\theta$

$= \int_0^{\frac{\pi}{4}} (1+\cos 2\theta) d\theta = \left[\theta + \frac{1}{2} \sin 2\theta \right]_0^{\frac{\pi}{4}}$

$= \left(\frac{\pi}{4} + \frac{1}{2} \sin \frac{\pi}{2} \right) - \left(0 + \frac{1}{2} \sin 0 \right) = \frac{\pi}{4} + \frac{1}{2}$

$= \frac{\pi}{2}$

Question 169 (****+)

$$\frac{1}{x(x^2+1)} \equiv \frac{A}{x} + \frac{Bx+C}{x^2+1}.$$

- a) Find the value of each of the constants A , B and C .
- b) Use the substitution $x = \cos \theta$ to show

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{2 \sin \theta}{\cos \theta + \cos^3 \theta} d\theta = \ln\left(\frac{5}{3}\right).$$

$$\boxed{A=1}, \boxed{B=-1}, \boxed{C=0}$$

(a) $\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$
 $1 = A(x^2+1) + x(Bx+C)$
 $1 = Ax^2 + A + Bx^2 + Cx$
 $1 = (A+B)x^2 + Cx + A$
 $\begin{cases} A+B=0 \\ C=0 \\ A=1 \end{cases} \Rightarrow \begin{cases} B=-1 \\ C=0 \\ A=1 \end{cases}$

(b) $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{2 \sin \theta}{\cos \theta + \cos^3 \theta} d\theta = \dots$ by substitution $x = \cos \theta$
 $\frac{dx}{d\theta} = -\sin \theta$
 $d\theta = \frac{dx}{-\sin \theta}$
 $\theta = \frac{\pi}{3} \Rightarrow x = \frac{1}{2}$
 $\theta = \frac{\pi}{4} \Rightarrow x = \frac{\sqrt{2}}{2}$

$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{2 \sin \theta}{\cos \theta + \cos^3 \theta} d\theta = \int_{\frac{\sqrt{2}}{2}}^{\frac{1}{2}} \frac{2}{x(1+x^2)} \cdot \frac{dx}{-\sin \theta}$
 $= -2 \int_{\frac{\sqrt{2}}{2}}^{\frac{1}{2}} \frac{1}{x(1+x^2)} dx = -2 \int_{\frac{\sqrt{2}}{2}}^{\frac{1}{2}} \left(\frac{1}{x} - \frac{x}{1+x^2} \right) dx$
 $= -2 \left[\ln|x| - \frac{1}{2} \ln|1+x^2| \right]_{\frac{\sqrt{2}}{2}}^{\frac{1}{2}}$
 $= -2 \left[\ln\left(\frac{1}{2}\right) - \frac{1}{2} \ln\left(1 + \frac{1}{4}\right) \right] - \left[\ln\left(\frac{\sqrt{2}}{2}\right) - \frac{1}{2} \ln\left(1 + \frac{2}{4}\right) \right]$
 $= -\ln\left(\frac{1}{2}\right) + \ln\left(\frac{5}{4}\right) - \ln\left(\frac{\sqrt{2}}{2}\right) + \ln\left(\frac{3}{4}\right) = \ln\left(\frac{5}{3}\right)$

Question 170 (****+)

By using the substitution $u = 1 - \tan^2 x$, or otherwise, find the exact value of

$$\int_0^{\frac{\pi}{6}} \tan x \sec 2x \, dx.$$

$$\boxed{\frac{1}{2} \ln\left(\frac{3}{2}\right)}$$

Handwritten solution for the integral problem:

$$\int_0^{\frac{\pi}{6}} \tan x \sec 2x \, dx = \dots \text{ BY SUBSTITUTION}$$

$$u = 1 - \tan^2 x$$

$$\frac{du}{dx} = -2 \tan x \sec^2 x$$

$$dx = -\frac{du}{2 \tan x \sec^2 x}$$

$$x=0 \rightarrow u=1$$

$$x=\frac{\pi}{6} \rightarrow u=\frac{1}{2}$$

$$= \int_1^{\frac{1}{2}} \left(\frac{\tan x \sec 2x}{\sec^2 x} \right) \left(-\frac{du}{2 \tan x \sec^2 x} \right)$$

$$= \frac{1}{2} \int_{\frac{1}{2}}^1 \frac{\sec 2x}{\sec^2 x} du$$

$$= \frac{1}{2} \int_{\frac{1}{2}}^1 \frac{1}{\cos 2x \sec^2 x} du$$

$$= \frac{1}{2} \int_{\frac{1}{2}}^1 \frac{1}{(\cos^2 x - \sin^2 x) \sec^2 x} du = \frac{1}{2} \int_{\frac{1}{2}}^1 \frac{1}{1 - \sin^2 x \sec^2 x} du$$

$$= \frac{1}{2} \int_{\frac{1}{2}}^1 \frac{1}{1 - \frac{\sin^2 x}{\cos^2 x}} du = \frac{1}{2} \int_{\frac{1}{2}}^1 \frac{1}{1 - \tan^2 x} du = \frac{1}{2} \int_{\frac{1}{2}}^1 \frac{1}{u} du$$

$$= \left[\frac{1}{2} \ln |u| \right]_{\frac{1}{2}}^1 = \frac{1}{2} \ln 1 - \frac{1}{2} \ln \frac{1}{2} = -\frac{1}{2} \ln \frac{1}{2} = \frac{1}{2} \ln \frac{2}{1} = \frac{1}{2} \ln 2$$

Alternatively from ...

$$\dots = \frac{1}{2} \int_{\frac{1}{2}}^1 \frac{\sec 2x}{\sec^2 x} du = \frac{1}{2} \int_{\frac{1}{2}}^1 \frac{\cos 2x}{\cos^2 x} du = \frac{1}{2} \int_{\frac{1}{2}}^1 \frac{\cos 2x}{\cos^2 x - \sin^2 x} du$$

Divide top & bottom of the integrand by $\cos^2 x$ w/ certain

$$= \frac{1}{2} \int_{\frac{1}{2}}^1 \frac{1 - \sin^2 x}{1 - \tan^2 x} du = \frac{1}{2} \int_{\frac{1}{2}}^1 \frac{1}{1 - \tan^2 x} du = \frac{1}{2} \int_{\frac{1}{2}}^1 \frac{1}{u} du$$

Question 171 (****+)

a) Show clearly that $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$.

b) Use trigonometric identities to find

$$\int \frac{1}{\sin^2 x \cos^2 x} dx.$$

$$-2 \cot 2x + C$$

(a) $\frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) = \frac{\sin x(-\sin x) - \cos x(\cos x)}{\sin^2 x}$
 $= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\frac{\sin^2 x + \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x}$
 $= -\operatorname{cosec}^2 x$

(b) $\int \frac{1}{\sin^2 x \cos^2 x} dx = \int \frac{1}{\left(\frac{1}{2} \cos 2x\right) \left(\frac{1}{2} \cos 2x\right)} dx$
 $= \int \frac{1}{\frac{1}{4} \cos^2 2x} dx = \int \frac{4}{\cos^2 2x} dx$ (Reciprocal of cosine is secant)
 $= \int \frac{4}{1 - \cos^2 2x} dx = \int \frac{4}{\sin^2 2x} dx$
 $= \int 4 \operatorname{cosec}^2 2x dx = \dots$ by part (a)
 $= -2 \cot 2x + C$

Alternative
 $\int \frac{1}{\sin^2 x \cos^2 x} dx = \int \frac{1}{(\sin x \cos x)^2} dx = \int \frac{1}{\left(\frac{1}{2} \sin 2x\right)^2} dx$
 $= \int \frac{1}{\left(\frac{1}{4} \sin^2 2x\right)} dx = \int \frac{4}{\sin^2 2x} dx$
 $= \int \frac{4}{\sin^2 2x} dx = \int 4 \operatorname{cosec}^2 2x dx$
 $= \dots$ by part (a) $\dots = -2 \cot 2x + C$

Alternative
 $\int \frac{1}{\sin^2 x \cos^2 x} dx = \int \frac{\sec^2 x + \tan^2 x}{\cos^2 x \sin^2 x} dx = \int \frac{\sec^2 x}{\cos^2 x \sin^2 x} + \frac{\tan^2 x}{\cos^2 x \sin^2 x} dx$
 $= \int \sec^2 x \csc^2 x + \sec^2 x \tan^2 x dx = \int \tan^2 x \sec^2 x + \sec^2 x \tan^2 x dx$
 $= \int \tan^2 x \sec^2 x dx = \int \tan^2 x d(\tan x) = \int (u^2) du = \frac{1}{3} u^3 + C = \frac{1}{3} \tan^3 x + C$

Question 172 (****+)

Use the substitution $x = \operatorname{cosec} \theta$ to find the exact value of

$$\int_{\sqrt{2}}^2 \frac{\sqrt{x^2 - 1}}{x} dx.$$

$$\sqrt{3} - 1 - \frac{\pi}{12}$$

$\int_{\sqrt{2}}^2 \frac{\sqrt{x^2 - 1}}{x} dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\sqrt{\operatorname{cosec}^2 \theta - 1}}{\operatorname{cosec} \theta} (\operatorname{cosec} \theta d\theta)$
 $= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\sqrt{\sec^2 \theta}}{\sec \theta} (\sec \theta d\theta) = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sec \theta d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1}{\cos \theta} d\theta$
 $= \left[\ln |\sec \theta + \tan \theta| \right]_{\frac{\pi}{6}}^{\frac{\pi}{4}} = \left(\ln \left(\sqrt{2} + 1 \right) \right) - \left(\ln \left(\sqrt{3} + 1 \right) \right)$
 $= \ln \left(\frac{\sqrt{2} + 1}{\sqrt{3} + 1} \right) = \ln \left(\frac{(\sqrt{2} + 1)(\sqrt{3} - 1)}{(\sqrt{3} + 1)(\sqrt{3} - 1)} \right) = \ln \left(\frac{(\sqrt{2} + 1)(\sqrt{3} - 1)}{2} \right)$
 $= \ln \left(\frac{\sqrt{6} - \sqrt{2} + \sqrt{3} - 1}{2} \right)$

Alternative
 $\int_{\sqrt{2}}^2 \frac{\sqrt{x^2 - 1}}{x} dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\sqrt{\sec^2 \theta - 1}}{\sec \theta} (\sec \theta d\theta) = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sec \theta d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1}{\cos \theta} d\theta$
 $= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1}{\cos \theta} d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\cos \theta}{\cos^2 \theta} d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\cos \theta}{1 - \sin^2 \theta} d\theta$
 $= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\cos \theta}{1 - \sin^2 \theta} d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\cos \theta}{(1 - \sin \theta)(1 + \sin \theta)} d\theta$
 $= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\cos \theta}{1 - \sin^2 \theta} d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\cos \theta}{1 - \sin^2 \theta} d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\cos \theta}{1 - \sin^2 \theta} d\theta$
 $= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\cos \theta}{1 - \sin^2 \theta} d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\cos \theta}{1 - \sin^2 \theta} d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\cos \theta}{1 - \sin^2 \theta} d\theta$

Question 173 (****+)

a) Write down an expression for $\frac{d}{dx}(e^{\cos x})$.

b) By using integration by parts, or otherwise, show that

$$\int e^{\cos x} \cos x \sin x \, dx = e^x (1 - \cos x) + \text{constant}.$$

$$\frac{d}{dx}(e^{\cos x}) = -e^{\cos x} \sin x$$

(a) $\frac{d}{dx}(e^{\cos x}) = \frac{d}{dx}(e^u) = e^u \frac{du}{dx} = e^{\cos x} (-\sin x) = -e^{\cos x} \sin x$

(b) $\int e^{\cos x} \cos x \sin x \, dx$
 Let $u = \cos x$, then $\frac{du}{dx} = -\sin x$
 $\therefore \sin x \, dx = -du$
 $\int e^u u (-du) = -\int u e^u \, du$
 Using integration by parts: $\int u e^u \, du = u e^u - \int e^u \, du = u e^u - e^u + C$
 $\therefore -\int u e^u \, du = -u e^u + e^u + C = e^u (1 - u) + C$
 $= e^{\cos x} (1 - \cos x) + C$

Diagram: A right-angled triangle with hypotenuse $\sqrt{1-x^2}$, one side x , and the other side $\sqrt{1-x^2}$. The angle is labeled θ . The text "By Pythagoras" is written below the triangle.

Question 174 (****+)

By using trigonometric identities, show that

$$\int_0^{\frac{\pi}{4}} \sin^4 x + \cos^4 x \, dx = \frac{3\pi}{16}.$$

proof

$$\begin{aligned} \int_0^{\frac{\pi}{4}} \sin^2 x + \cos^2 x \, dx &= \int_0^{\frac{\pi}{4}} (\sin^2 x + \cos^2 x) + 2\sin^2 x \cos^2 x - 2\sin^2 x \cos^2 x \, dx \\ &= \int_0^{\frac{\pi}{4}} (\sin^2 x + \cos^2 x) - \frac{1}{2}(4\sin^2 x \cos^2 x) \, dx = \int_0^{\frac{\pi}{4}} 1 - \frac{1}{2}(\sin 2x)^2 \, dx \\ &= \int_0^{\frac{\pi}{4}} 1 - \frac{1}{2}\sin^2 2x \, dx = \int_0^{\frac{\pi}{4}} 1 - \frac{1}{2}\left(\frac{1}{2} - \frac{1}{2}\cos 4x\right) \, dx \\ &= \int_0^{\frac{\pi}{4}} \frac{3}{4} + \frac{1}{4}\cos 4x \, dx = \left[\frac{3}{4}x + \frac{1}{16}\sin 4x\right]_0^{\frac{\pi}{4}} \\ &= \left(\frac{3\pi}{16} - 0\right) - (0 - 0) = \frac{3\pi}{16} \end{aligned}$$

ALTERNATE VARIATION

$$\begin{aligned} \int_0^{\frac{\pi}{4}} \sin^2 x + \cos^2 x \, dx &= \int_0^{\frac{\pi}{4}} (\sin^2 x)^2 + (\cos^2 x)^2 \, dx \\ &= \int_0^{\frac{\pi}{4}} \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right)^2 + \left(\frac{1}{2} + \frac{1}{2}\cos 2x\right)^2 \, dx \\ &= \int_0^{\frac{\pi}{4}} \frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{4}\cos^2 2x + \frac{1}{4} + \frac{1}{2}\cos 2x + \frac{1}{4}\cos^2 2x \, dx \\ &= \int_0^{\frac{\pi}{4}} \frac{1}{2} + \frac{1}{2}\cos^2 2x \, dx = \int_0^{\frac{\pi}{4}} \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2}\cos 4x\right) \, dx \\ &= \int_0^{\frac{\pi}{4}} \frac{3}{4} + \frac{1}{4}\cos 4x \, dx = \dots = \frac{3\pi}{16} \text{ as above} \end{aligned}$$

NOTE: $\cos^2 \theta \equiv \frac{1}{2} + \frac{1}{2}\cos 2\theta$
 $\sin^2 \theta \equiv \frac{1}{2} - \frac{1}{2}\cos 2\theta$

Question 175 (****+)

By using the substitution $u = \sin 2x$, or otherwise, find an exact simplified value for the following trigonometric integral.

$$\int_0^{\frac{1}{4}\pi} \frac{1 - \tan^2 x}{\sec^2 x + 2 \tan x} dx.$$

$$\boxed{}, \frac{1}{2} \ln 2$$

USING THE SUBSTITUTION (even)

$$\int_0^{\frac{1}{4}\pi} \frac{1 - \tan^2 x}{\sec^2 x + 2 \tan x} dx = \int_0^1 \frac{1 - \tan^2 x}{\sec^2 x + 2 \tan x} \left(\frac{dx}{\sin 2x} \right)$$

$$= \frac{1}{2} \int_0^1 \frac{1 - \tan^2 x}{(\sec^2 x + 2 \tan x) \sin 2x} du$$

SWITCH EVERYTHING INTO SINES & COSINES

$$= \frac{1}{2} \int_0^1 \frac{1 - \frac{\sin^2 x}{\cos^2 x}}{\left(\frac{1}{\cos^2 x} + \frac{2 \sin x}{\cos x} \right) (\sin x \cos x)} du$$

MULTIPLY TOP & BOTTOM OF THE DOUBLE FRACTION BY $\cos^2 x$

$$= \frac{1}{2} \int_0^1 \frac{(\cos^2 x - \sin^2 x)}{(1 + 2 \sin x \cos x)(\sin x \cos x)} du = \frac{1}{2} \int_0^1 \frac{1}{1 + 2 \sin x \cos x} du$$

$$= \frac{1}{2} \int_0^1 \frac{1}{1 + \sin 2x} du = \frac{1}{2} \int_0^1 \frac{1}{1 + u} du$$

$$= \frac{1}{2} [\ln |1+u|]_0^1 = \frac{1}{2} [\ln 2 - \ln 1] = \frac{1}{2} \ln 2 //$$

Question 176 (****+)

By using a suitable substitution, or otherwise, find the value of

$$\int_0^1 \frac{x \arcsin x}{\sqrt{1-x^2}} dx.$$

$$\boxed{1}$$

BY SUBSTITUTION

$$\int_0^1 \frac{x \arcsin x}{\sqrt{1-x^2}} dx = \dots$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin \theta \arcsin(\sin \theta)}{\sqrt{1-\sin^2 \theta}} (\cos \theta d\theta)$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin \theta \times \theta}{\sqrt{\cos^2 \theta}} \cos \theta d\theta$$

$$= \int_0^{\frac{\pi}{2}} \theta \sin \theta d\theta$$

BY PARTS

θ	1
$-\cos \theta$	$\sin \theta$

$$= \left[-\theta \cos \theta \right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos \theta d\theta$$

$$= \left[\sin \theta \right]_0^{\frac{\pi}{2}} = 1 //$$

Question 177 (****+)

- a) Use the substitution $u = 2x - 1$ to show that

$$\int_1^5 \frac{x+1}{(2x-1)^{\frac{3}{2}}} dx = 2.$$

- b) By using integration by parts and the result of part (a), find the value of

$$\int_1^5 \frac{(x+1)^2}{(2x-1)^{\frac{5}{2}}} dx.$$

20
9

(a) $\int_1^5 \frac{x+1}{(2x-1)^{\frac{3}{2}}} dx$... substitution $u = 2x-1$

$\frac{du}{dx} = 2 \Rightarrow \frac{dx}{du} = \frac{1}{2}$

$x=1, u=1$
 $x=5, u=9$

$\int_1^5 \frac{x+1}{(2x-1)^{\frac{3}{2}}} dx = \int_1^9 \frac{\frac{u+1}{2} \cdot \frac{1}{2}}{u^{\frac{3}{2}}} du = \int_1^9 \frac{u+1}{4u^{\frac{3}{2}}} du$

$= \int_1^9 \frac{u+2}{4u^{\frac{3}{2}}} du = \int_1^9 \frac{u+3}{4u^{\frac{3}{2}}} du$

$= \frac{1}{4} \int_1^9 \frac{u+3}{u^{\frac{3}{2}}} du = \frac{1}{4} \int_1^9 \left(\frac{u}{u^{\frac{3}{2}}} + \frac{3}{u^{\frac{3}{2}}} \right) du = \frac{1}{4} \int_1^9 \left(u^{-\frac{1}{2}} + 3u^{-\frac{3}{2}} \right) du$

$= \frac{1}{4} \left[2u^{\frac{1}{2}} - 6u^{\frac{1}{2}} \right]_1^9 = \frac{1}{4} \left[(6-2) - (2-6) \right] = \frac{1}{4} [4-2+6] = 2$

(b) $\int_1^5 \frac{(x+1)^2}{(2x-1)^{\frac{5}{2}}} dx = \int_1^5 \frac{(x+1)^2}{(2x-1)^{\frac{3}{2}}} \cdot \frac{1}{2x-1} dx$... by parts

$\frac{(x+1)^2}{(2x-1)^{\frac{3}{2}}} = \frac{(x+1)^2}{(2x-1)^{\frac{3}{2}}} \cdot \frac{1}{2x-1}$

$= \left[\frac{(x+1)^2}{2(2x-1)^{\frac{3}{2}}} \right]_1^5 - \int_1^5 \frac{2(x+1)}{(2x-1)^{\frac{3}{2}}} dx$

$= \left[\frac{(x+1)^2}{2(2x-1)^{\frac{3}{2}}} \right]_1^5 + \frac{2}{3} \int_1^5 \frac{x+1}{(2x-1)^{\frac{3}{2}}} dx$

$= \left(\frac{4}{3} - \frac{1}{3} \right) + \frac{2}{3} \times 2 = \frac{20}{9}$

Question 178 (****+)

Use the substitution $u = \ln x$ to show that

$$\int 3^{\ln x} dx = \frac{x(3^{\ln x})}{1 + \ln 3} + \text{constant}.$$

V, ☐, proof

START WITH AN OBVIOUS SUBSTITUTION

- $u = \ln x$
- $\frac{du}{dx} = \frac{1}{x}$
- $dx = x du$
- $a^u = a$
- $\frac{d}{dx} a^u = a^u \ln a$

$\int 3^{\ln x} dx = \int 3^u x du = \int (3e)^u du = \int a^u du$ where $a = 3e$

NOW WE KNOW THAT

$$\frac{d}{du} (a^u) = a^u \ln a \Rightarrow a^u = \int a^u \ln a du$$

$$\Rightarrow \frac{1}{\ln a} a^u = \int a^u du$$

RETURNING TO OUR ORIGINAL $\ln x$

$$\int a^u du = \frac{1}{\ln a} a^u + C = \frac{1}{\ln(3e)} (3e)^u + C$$

$$= \frac{3^u x}{\ln 3 + \ln e} + C = \frac{3^{\ln x} x}{\ln 3 + 1} + C$$

$$= \frac{3^{\ln x} x}{1 + \ln 3} + C = \frac{x(3^{\ln x})}{1 + \ln 3} + C$$

Question 179 (****+)

$$J = \int_{-1}^1 \frac{1}{1 + e^{-x}} \, dx.$$

- a)** Show that the substitution $u = 1 + e^{-x}$ transforms J into

$$\int_{1+e}^{1+e^{-1}} \frac{1}{u(1-u)} du.$$

- b) By expressing $\frac{1}{u(1-u)}$ into partial fractions show clearly that $J = 1$.

proof

[illegible]

Question 180 (****+)Use the substitution $x = \tan \theta$ to find the exact value of

$$\int_0^1 \frac{8}{(1+x^2)^2} dx.$$

$$\boxed{\pi}, \boxed{\pi+2}$$

Handwritten solution for Question 180:

$$\int_0^1 \frac{8}{(1+x^2)^2} dx = \dots \text{USE THE SUBSTITUTION (GIVEN)}$$

Let $x = \tan \theta$
 $\frac{dx}{d\theta} = \sec^2 \theta$
 $dx = \sec^2 \theta d\theta$
 $x=0, \tan \theta=0 \Rightarrow \theta=0$
 $x=1, \tan \theta=1 \Rightarrow \theta=\frac{\pi}{4}$

$$= \int_0^{\frac{\pi}{4}} \frac{8 \sec^2 \theta}{(1+\tan^2 \theta)^2} d\theta = \int_0^{\frac{\pi}{4}} \frac{8 \sec^2 \theta}{\sec^4 \theta} d\theta = \int_0^{\frac{\pi}{4}} 8 \cos^2 \theta d\theta$$

STANDARD INTEGRATION KNOWLEDGE...

$$= \int_0^{\frac{\pi}{4}} 8 \left(\frac{1+\cos 2\theta}{2} \right) d\theta = \int_0^{\frac{\pi}{4}} (4+4\cos 2\theta) d\theta$$

$$= \left[4\theta + 2\sin 2\theta \right]_0^{\frac{\pi}{4}} = \left(\pi + 2\sin \frac{\pi}{2} \right) - (0 + 2\sin 0) = \pi + 2$$

Question 181 (****+)Use the substitution $u = 400 - 20\sqrt{x}$ to show that

$$\int_0^{100} \frac{1}{400 - 20\sqrt{x}} dx = -1 + 2 \ln 2.$$

proof

Handwritten solution for Question 181:

$$\int_0^{100} \frac{1}{400 - 20\sqrt{x}} dx = \text{USE THE SUBSTITUTION}$$

Let $u = 400 - 20\sqrt{x}$
 $20\sqrt{x} = 400 - u$
 $\sqrt{x} = 20 - \frac{1}{20}u$
 $x = \left(20 - \frac{1}{20}u\right)^2$
 $\frac{dx}{du} = 2\left(20 - \frac{1}{20}u\right) \times \left(-\frac{1}{20}\right)$
 $dx = \left(-\frac{1}{10}\left(20 - \frac{1}{20}u\right)\right) du$
 $2=0 \rightarrow u=400$
 $2=100 \rightarrow u=200$

$$= \int_{200}^{400} \frac{1}{u} \times \left(-\frac{1}{10}\left(20 - \frac{1}{20}u\right)\right) du$$

USE KNOWLEDGE TO SPLIT UP THE UNIT & FIND OUT $\frac{1}{u}$

$$= \frac{1}{10} \int_{200}^{400} \left(\frac{20}{u} - \frac{1}{20} \right) du = \frac{1}{10} \left[20 \ln |u| - \frac{1}{20} u \right]_{200}^{400}$$

$$= \frac{1}{10} \left[\left(20 \ln 400 - 20 \right) - \left(20 \ln 200 - 10 \right) \right] = \frac{1}{10} \left[20 \ln 2 - 10 \right]$$

$$= \frac{1}{10} \left[20 \ln 2 - 10 \right] = 2 \ln 2 - 1$$

As 24+10/10

Question 182 (****+)

It is given that

$$\frac{(2x^2 - 10x + 7)(x^2 - 3x - 3)}{(x-4)^2} \equiv Ax^2 + Bx + C + \frac{D}{x-4} + \frac{E}{(x-4)^2}.$$

- a) Find the value of A , B , C , D and E in the above identity.
- b) Hence find the exact value of

$$\int_0^3 f(x) \, dx.$$

$$A=2, B=0, C=-1, D=1, E=-1, \frac{57}{4} - \ln 4$$

(a) $\frac{(2x^2 - 10x + 7)(x^2 - 3x - 3)}{(x-4)^2} \equiv Ax^2 + Bx + C + \frac{D}{x-4} + \frac{E}{(x-4)^2}$

$(2x^2 - 10x + 7)(x^2 - 3x - 3) \equiv (Ax^2 + Bx + C)(x-4)^2 + D(x-4) + E$

• If $x=4$ $(-1)(1) = E \Rightarrow E = -1$

Expand

$$2x^4 - 6x^3 - 4x^2 - 10x^3 + 30x^2 + 30x - 3x^3 + 9x^2 - 21x - 21 \equiv (Ax^2 + Bx + C)(x^2 - 8x + 16) + Dx - 4D - 1$$

$$\Rightarrow 2x^4 - 16x^3 + 31x^2 + 9x - 21 \equiv Ax^4 - 8Ax^3 + 16Ax^2 - 8Bx^3 + 64Bx^2 + 16Bx - 8Cx^2 + 64Cx + 16C + Dx - 4D - 1$$

$$\Rightarrow 2x^4 - 16x^3 + 31x^2 + 9x - 21 \equiv Ax^4 + (-8A)x^3 + (16A - 8B)x^2 + (16B - 8C + D)x + (16C - 4D - 1)$$

$\Rightarrow A=2$

$B - 8A = -16$	$16A - 8B + C = 31$	$16C - 4D = -20$
$B - 16 = -16$	$32 + C = 31$	$-16 - 4D = -20$
$B = 0$	$C = -1$	$-4D = 4$
		$D = -1$

(b) $\int_0^3 f(x) \, dx = \int_0^3 2x^2 - 1 + \frac{1}{x-4} - \frac{1}{(x-4)^2} \, dx$

$$= \left[\frac{2}{3}x^3 - x + \ln|x-4| + \frac{1}{x-4} \right]_0^3$$

$$= \left(\frac{2}{3} \cdot 27 - 3 + \ln|3-4| - 1 \right) - \left(0 - 0 + \ln|0-4| - \frac{1}{4} \right)$$

$$= 14 - \ln 4 + \frac{1}{4}$$

$$= \frac{57}{4} - \ln 4$$

Question 183 (*****)

$$x = \frac{1}{2}(-1 + 4 \tan \theta), \quad -\frac{1}{2}\pi < \theta < \frac{1}{2}\pi.$$

a) Use trigonometric identities to show that

$$4x^2 + 4x + 17 = 16 \sec^2 \theta.$$

b) Hence find the exact value of

$$\int_{-\frac{1}{2}}^{\frac{3}{2}} \frac{1}{4x^2 + 4x + 17} dx.$$

$$\boxed{\frac{1}{16}}, \quad \boxed{\frac{\pi}{32}}$$

a) SUBSTITUTE, EXPAND & TIDY

$$\begin{aligned}
 4x^2 + 4x + 17 &= 4 \left[\frac{1}{4}(-1 + 4 \tan \theta)^2 \right] + 4 \left[\frac{1}{2}(-1 + 4 \tan \theta) \right] + 17 \\
 &= 4 \times \frac{1}{4}(-1 + 4 \tan \theta)^2 + 2(-1 + 4 \tan \theta) + 17 \\
 &= 1 - 8 \tan \theta + 16 \tan^2 \theta - 2 + 8 \tan \theta + 17 \\
 &= 16 + 16 \tan^2 \theta \\
 &= 16(1 + \tan^2 \theta) \\
 &= 16 \sec^2 \theta \quad \text{as required}
 \end{aligned}$$

b) BY SUBSTITUTION FROM PART (a)

$$\begin{aligned}
 \Rightarrow x &= \frac{1}{2}(-1 + 4 \tan \theta) = -\frac{1}{2} + 2 \tan \theta \\
 \Rightarrow \frac{dx}{d\theta} &= 2 \sec^2 \theta \\
 \Rightarrow dx &= 2 \sec^2 \theta d\theta
 \end{aligned}$$

• when $x = -\frac{1}{2}$

$$\begin{aligned}
 -\frac{1}{2} &= -\frac{1}{2} + 2 \tan \theta \\
 0 &= 2 \tan \theta \\
 \theta &= 0
 \end{aligned}$$

• when $x = \frac{3}{2}$

$$\begin{aligned}
 \frac{3}{2} &= -\frac{1}{2} + 2 \tan \theta \\
 2 &= 2 \tan \theta \\
 \tan \theta &= 1 \\
 \theta &= \frac{\pi}{4}
 \end{aligned}$$

TRANSFORMING THE INTEGRAL VITECS

$$\begin{aligned}
 \int_{-\frac{1}{2}}^{\frac{3}{2}} \frac{1}{4x^2 + 4x + 17} dx &= \int_0^{\frac{\pi}{4}} \frac{1}{16 \sec^2 \theta} (2 \sec^2 \theta d\theta) \\
 &= \int_0^{\frac{\pi}{4}} \frac{1}{8} d\theta \\
 &= \left[\frac{\theta}{8} \right]_0^{\frac{\pi}{4}} \\
 &= \frac{1}{8} \left[\frac{\pi}{4} - 0 \right] \\
 &= \frac{\pi}{32}
 \end{aligned}$$

Question 184 (****+)

It is given that

$$\sin(A+B) \equiv \sin A \cos B + \cos A \sin B.$$

Use the above trigonometric identity to show that

$$\sin 3x \equiv 3 \sin x - 4 \sin^3 x,$$

and hence find

$$\int \sqrt[3]{3 \sin 2x - 2 \sin 3x \cos x} \, dx.$$

$$-\frac{3}{2} \sin^{\frac{4}{3}} x + C$$

$$\begin{aligned} \sin 3x &= \sin(2x+x) = \sin 2x \cos x + \cos 2x \sin x \\ &= (2 \sin x \cos x) \cos x + (1-2 \sin^2 x) \sin x \\ &= 2 \sin x \cos^2 x + \sin x - 2 \sin^3 x \\ &= 2 \sin x (1-\sin^2 x) + \sin x - 2 \sin^3 x \\ &= 2 \sin x - 2 \sin^3 x + \sin x - 2 \sin^3 x \\ &= 3 \sin x - 4 \sin^3 x \end{aligned}$$

$$\begin{aligned} \int (3 \sin x - 4 \sin^3 x)^{\frac{1}{3}} dx &= \int (3 \sin x - 4 \sin^3 x)^{\frac{1}{3}} dx \\ &= \int (3 \sin x - 4 \sin^3 x)^{\frac{1}{3}} dx \\ &= \int 3 \sin x (\cos^2 x)^{\frac{1}{3}} dx \\ &= \frac{3}{2} (\cos^2 x)^{\frac{2}{3}} + C \\ &= -\frac{3}{2} (\cos^2 x)^{\frac{2}{3}} + C = -\frac{3}{2} \cos^{\frac{4}{3}} x + C \end{aligned}$$

Question 185 (****+)

Use the substitution $u = x + \frac{\pi}{4}$ to show that

$$\int_{\frac{\pi}{4}}^{\frac{7\pi}{12}} \frac{\sin x}{\sin\left(x + \frac{\pi}{4}\right)} dx = \frac{\sqrt{2}}{6}(\pi + \ln 8).$$

proof

Handwritten proof of the integral using the substitution $u = x + \frac{\pi}{4}$.

The integral is transformed as follows:

$$\int_{\frac{\pi}{4}}^{\frac{7\pi}{12}} \frac{\sin x}{\sin\left(x + \frac{\pi}{4}\right)} dx = \int_{\frac{\pi}{4}}^{\frac{7\pi}{12}} \frac{\sin\left(u - \frac{\pi}{4}\right)}{\sin u} du$$

Using the identity $\sin\left(u - \frac{\pi}{4}\right) = \sin u \cos \frac{\pi}{4} - \cos u \sin \frac{\pi}{4}$, the integral becomes:

$$\int_{\frac{\pi}{4}}^{\frac{7\pi}{12}} \frac{\sin u \cos \frac{\pi}{4} - \cos u \sin \frac{\pi}{4}}{\sin u} du = \int_{\frac{\pi}{4}}^{\frac{7\pi}{12}} \left(\cos \frac{\pi}{4} - \cot u \sin \frac{\pi}{4} \right) du$$

Since $\cos \frac{\pi}{4} = \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$, the integral simplifies to:

$$\frac{\sqrt{2}}{2} \int_{\frac{\pi}{4}}^{\frac{7\pi}{12}} (1 - \cot u) du = \frac{\sqrt{2}}{2} \left[u - \ln |\sin u| \right]_{\frac{\pi}{4}}^{\frac{7\pi}{12}}$$

Evaluating the limits:

$$= \frac{\sqrt{2}}{2} \left[\left(\frac{7\pi}{12} - \ln \frac{1}{2} \right) - \left(\frac{\pi}{4} - \ln \frac{\sqrt{2}}{2} \right) \right] = \frac{\sqrt{2}}{2} \left[\frac{\pi}{3} - \ln \frac{1}{2} \right]$$

$$= \frac{\sqrt{2}}{2} \left[\frac{\pi}{3} + \ln 2 \right] = \frac{\sqrt{2}}{6} (\pi + 3 \ln 2) = \frac{\sqrt{2}}{6} (\pi + \ln 8)$$

Boxed notes on the right side of the page:

- $u = x + \frac{\pi}{4}$
- $x = u - \frac{\pi}{4}$
- $\frac{dx}{du} = 1$
- $dx = du$
- $x = \frac{\pi}{4} \Rightarrow u = \frac{\pi}{2}$
- $x = \frac{7\pi}{12} \Rightarrow u = \frac{7\pi}{12} + \frac{\pi}{4} = \frac{10\pi}{12} = \frac{5\pi}{6}$

Question 186 (****+)

By using the substitution $u = \sqrt[3]{x}$, or otherwise, show that

$$\int_0^{\sqrt{27}} \frac{2}{x + \sqrt[3]{x}} dx = 6 \ln 2.$$

☐ , proof

Handwritten solution for Question 186 using the substitution $u = \sqrt[3]{x}$.

USING THE SUBSTITUTION (6/10)

$$\int_0^{\sqrt{27}} \frac{2}{\sqrt[3]{x^3} + x} dx = \int_0^{\sqrt{27}} \frac{2}{u^3 + u^3} (3u^2 du)$$

$$= \int_0^{\sqrt{27}} \frac{6u^2}{u(1+u^2)} du = \int_0^{\sqrt{27}} \frac{6u}{u^2+1} du$$

$$= 3 \int_0^{\sqrt{27}} \frac{2u}{u^2+1} du$$

Using the above answer (or another substitution)

$$= 3 \left[\ln(u^2+1) \right]_0^{\sqrt{27}} = 3 \left[\ln 4 - \ln 1 \right]$$

$$= 3 \times 2 \ln 2$$

$$= 6 \ln 2$$

As Required

Substitution Table:

- $u = \sqrt[3]{x}$
- $u^3 = x$
- $x = u^3$
- $\frac{dx}{du} = 3u^2$
- $dx = 3u^2 du$
- $u = \sqrt[3]{27}$
- $27^{\frac{1}{3}} = u^3$
- $3^{\frac{1}{3}} = u$
- $u = 3$
- $u = 0, u = 0$

Question 187 (****+)

$$I = \int_0^1 \frac{3}{(1+8x^2)^{\frac{3}{2}}} dx$$

- a) Use the substitution $x = \frac{1}{\sqrt{8}} \tan \theta$ to show that

$$I = \frac{3}{\sqrt{8}} \sin(\arctan \sqrt{8}).$$

- b) Show, presenting detailed calculations, that $I = 1$.

proof

Handwritten solution for Question 187b:

(a) $\int_0^1 \frac{3}{(1+8x^2)^{\frac{3}{2}}} dx = \dots$

Substitution: $x = \frac{1}{\sqrt{8}} \tan \theta$, $dx = \frac{1}{\sqrt{8}} \sec^2 \theta d\theta$

When $x=0$, $\theta=0$. When $x=1$, $\theta = \arctan \sqrt{8}$.

Then $I = \int_0^{\arctan \sqrt{8}} \frac{3}{(1+8(\frac{1}{8} \tan^2 \theta))^{\frac{3}{2}}} \cdot \frac{1}{\sqrt{8}} \sec^2 \theta d\theta$

$= \int_0^{\arctan \sqrt{8}} \frac{3}{(1+\tan^2 \theta)^{\frac{3}{2}}} \cdot \frac{1}{\sqrt{8}} \sec^2 \theta d\theta$

$= \int_0^{\arctan \sqrt{8}} \frac{3}{\sec^3 \theta} \cdot \frac{1}{\sqrt{8}} \sec^2 \theta d\theta$

$= \frac{3}{\sqrt{8}} \int_0^{\arctan \sqrt{8}} \cos \theta d\theta$

$= \frac{3}{\sqrt{8}} [\sin \theta]_0^{\arctan \sqrt{8}} = \frac{3}{\sqrt{8}} (\sin(\arctan \sqrt{8}) - 0)$

$= \frac{3}{\sqrt{8}} \sin(\arctan \sqrt{8})$

(b) To show $I = 1$, we need to show $\sin(\arctan \sqrt{8}) = \frac{\sqrt{8}}{3}$.

Consider a right-angled triangle with angle $\theta = \arctan \sqrt{8}$. The opposite side is $\sqrt{8}$, the adjacent side is 1, and the hypotenuse is $\sqrt{1+8} = 3$.

Therefore, $\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{\sqrt{8}}{3}$.

$\therefore \frac{3}{\sqrt{8}} \sin(\arctan \sqrt{8}) = \frac{3}{\sqrt{8}} \cdot \frac{\sqrt{8}}{3} = 1$

Question 188 (****+)

$$I = \int_1^2 \frac{1}{x^2 - x\sqrt{x^2 - 1}} dx.$$

a) Show that the substitution $x = \sec \theta$ transforms I to

$$I = \int_0^{\frac{1}{3}\pi} \frac{\tan \theta}{\sec \theta - \tan \theta} d\theta.$$

b) Hence use trigonometric identities to show that

$$I = 1 + \sqrt{3} - \frac{1}{3}\pi.$$

☐ , proof

a) USING THE SUBSTITUTION GIVEN

$x = \sec \theta$ $x=1 \rightarrow \theta=0$
 $\frac{dx}{d\theta} = \sec \theta \tan \theta$ $x=2 \rightarrow \theta=\frac{\pi}{3}$
 $dx = \sec \theta \tan \theta d\theta$

TRANSFORMING THE INTEGRAL

$$\int_1^2 \frac{1}{x^2 - x\sqrt{x^2 - 1}} dx = \int_0^{\frac{\pi}{3}} \frac{\sec \theta \tan \theta}{\sec^2 \theta - \sec \theta \sqrt{\sec^2 \theta - 1}} d\theta$$

$$= \int_0^{\frac{\pi}{3}} \frac{\sec \theta \tan \theta}{\sec^2 \theta - \sec \theta \tan \theta} d\theta = \int_0^{\frac{\pi}{3}} \frac{\sec \theta \tan \theta}{\sec \theta (\sec \theta - \tan \theta)} d\theta$$

$$= \int_0^{\frac{\pi}{3}} \frac{\tan \theta}{\sec \theta - \tan \theta} d\theta$$

As required

b) USING $(\sec \theta - \tan \theta)(\sec \theta + \tan \theta) = \sec^2 \theta - \tan^2 \theta = 1$

$$= \int_0^{\frac{\pi}{3}} \frac{\tan \theta (\tan \theta + \sec \theta)}{(\sec \theta - \tan \theta)(\sec \theta + \tan \theta)} d\theta$$

$$= \int_0^{\frac{\pi}{3}} \frac{\tan^2 \theta + \tan \theta \sec \theta}{1} d\theta$$

$$= \int_0^{\frac{\pi}{3}} \tan^2 \theta + \tan \theta \sec \theta d\theta$$

$$= \int_0^{\frac{\pi}{3}} \sec^2 \theta - 1 + \tan \theta \sec \theta d\theta$$

$$= [\tan \theta - \theta + \sec \theta]_0^{\frac{\pi}{3}}$$

$$= \left(\sqrt{3} - \frac{\pi}{3} + 2\right) - (0 - 0 + 1)$$

$$= \sqrt{3} - \frac{\pi}{3} + 1$$

As required

Question 189 (****+)

Use integration by parts to find the value of

$$\int_0^{\frac{\pi}{2}} e^{\cos x} \sin 2x \, dx.$$

2

$$\int_0^{\frac{\pi}{2}} e^{\cos x} \sin 2x \, dx = \int_0^{\frac{\pi}{2}} 2e^{\cos x} \sin x \cos x \, dx \quad \text{By parts}$$

$$= \left[-2e^{\cos x} \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 2e^{\cos x} \sin x \, dx$$

$$= \left[-2e^{\cos x} \right]_0^{\frac{\pi}{2}} - \left[-2e^{\cos x} \right]_0^{\frac{\pi}{2}}$$

$$= \left[-2e^{\cos x} - 2e^{\cos x} \right]_0^{\frac{\pi}{2}}$$

$$= (-2e^0 - 2e^0) - (-2e^1 - 2e^1)$$

$$= 2$$

ALTERNATIVE

$$\int_0^{\frac{\pi}{2}} e^{\cos x} \sin 2x \, dx = \int_0^{\frac{\pi}{2}} 2e^{\cos x} \sin x \cos x \, dx \quad \text{Substitution}$$

$$= \int_1^0 2e^u \sin x \left(\frac{du}{dx} \right) dx = \int_1^0 2ue^u \, du$$

By parts

$$= \left[2ue^u \right]_1^0 - \int_1^0 2e^u \, du = \left[2ue^u - 2e^u \right]_1^0$$

$$= (2e^0 - 2e^0) - (0 - 2) = 2$$

Handwritten notes:
 For the first method, a table shows:

$$\begin{array}{c|c} 2\cos x & -2\sin x \\ \hline \cos x & e^{\cos x} \end{array}$$
 with the result $-2e^{\cos x}$ and a note "Recognition".
 For the second method, a box shows:

$$\begin{array}{l} u = \cos x \\ \frac{du}{dx} = -\sin x \\ dx = \frac{du}{-\sin x} \\ x=0 \quad u=1 \\ x=\frac{\pi}{2} \quad u=0 \end{array}$$

Question 190 (****+)

$$\sin 2x \equiv \frac{2 \tan x}{1 + \tan^2 x}$$

a) Prove the validity of the above trigonometric identity.

b) Express $\frac{8}{(3t+1)(t+3)}$ into partial fractions.

c) Hence use the substitution $t = \tan x$ to show that

$$\int_0^{\frac{\pi}{4}} \frac{8}{3+5\sin 2x} dx = \ln 3.$$

$$\frac{8}{(3t+1)(t+3)} = \frac{3}{3t+1} - \frac{1}{t+3}$$

a) STARTING FROM THE R.H.S.

$$\begin{aligned} \text{R.H.S.} &= \frac{2 \tan x}{1 + \tan^2 x} = \frac{2 \tan x}{\sec^2 x} = 2 \tan x \cos^2 x \\ &= 2 \times \frac{\sin x}{\cos x} \times \cos^2 x = 2 \sin x \cos x = \sin 2x \\ &= \text{L.H.S.} \end{aligned}$$

As required

b) THE PARTIAL FRACTIONS NEXT

$$\frac{8}{(3t+1)(t+3)} \equiv \frac{A}{3t+1} + \frac{B}{t+3}$$

$$8 \equiv A(t+3) + B(3t+1)$$

• If $t = -3$ • If $t = -\frac{1}{3}$

$$\begin{aligned} 8 &= -6B & 8 &= \frac{8}{3}A \\ B &= -\frac{4}{3} & A &= 3 \end{aligned}$$

$$\therefore \frac{8}{(3t+1)(t+3)} = \frac{3}{3t+1} - \frac{1}{t+3}$$

c) USING THE SUBSTITUTION GIVEN IN THE PREVIOUS PART

$$\begin{aligned} \Rightarrow t &= \tan x \\ \Rightarrow \frac{dt}{dx} &= \sec^2 x \end{aligned}$$

$\Rightarrow dx = \frac{dt}{\sec^2 x}$

$\Rightarrow dx = \frac{dt}{1+t^2}$

• WHEN $x=0$, $t=0$
• WHEN $x=\frac{\pi}{4}$, $t=1$

TRANSFORMING THE INTEGRAL

$$\begin{aligned} \int_0^{\frac{\pi}{4}} \frac{8}{3+5\sin 2x} dx &= \int_0^1 \frac{8}{3+5\left(\frac{2t}{1+t^2}\right)} \cdot \frac{dt}{1+t^2} \quad \leftarrow \text{BY PART (a)} \\ &= \int_0^1 \frac{8}{3+5\left(\frac{2t}{1+t^2}\right)} \cdot \left(\frac{dt}{1+t^2}\right) \quad \leftarrow \text{BY THE SUBSTITUTION} \\ &= \int_0^1 \frac{8}{3+5\left(\frac{2t}{1+t^2}\right)} \cdot \frac{dt}{1+t^2} = \int_0^1 \frac{8}{3t^2+10t+3} dt \\ &= \int_0^1 \frac{8}{(3t+1)(t+3)} dt = \int_0^1 \left(\frac{3}{3t+1} - \frac{1}{t+3} \right) dt \quad \leftarrow \text{BY (b)} \\ &= \left[\ln|3t+1| - \ln|t+3| \right]_0^1 = (\ln 4 - \ln 3) - (\ln 1 - \ln 3) \\ &= \ln 3 \end{aligned}$$

As required

Question 191 (****+)

It is given that for some constants A and B

$$6 \sin x \equiv A(\cos x + \sin x) + B(\cos x - \sin x).$$

- Find the value of A and the value of B .
- Hence find

$$\int \frac{6 \sin x}{\cos x + \sin x} dx.$$

$$\boxed{}, \boxed{A=3}, \boxed{B=-3}, \boxed{3x - 3 \ln |\cos x + \sin x| + C}$$

a) EXPAND & COMBINE

$$\Rightarrow 6 \sin x \equiv A(\cos x + \sin x) + B(\cos x - \sin x)$$

$$\Rightarrow 6 \sin x \equiv (A+B)\cos x + (A-B)\sin x$$

$$\rightarrow \begin{cases} A+B=0 \\ A-B=6 \end{cases} \quad \text{ADDING & SUBTRACTING GIVES}$$

$$\frac{-A=3}{B=-3}$$

b) USING PART (a)

$$\int \frac{6 \sin x}{\cos x + \sin x} dx = \int \frac{3(\cos x + \sin x) - 3(\cos x - \sin x)}{\cos x + \sin x} dx$$

$$= \int \frac{3(\cos x + \sin x)}{\cos x + \sin x} - \frac{3(\cos x - \sin x)}{\cos x + \sin x} dx$$

$$= \int 3 - 3 \left(\frac{\cos x - \sin x}{\cos x + \sin x} \right) dx$$

THIS IS OF THE FORM $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C$

$$= 3x - 3 \ln |\cos x + \sin x| + C$$

Question 192 (****+)

Use the substitution $x = 2 \sin \theta$ to show that

$$\int_0^2 \sqrt{4-x^2} dx = \pi.$$

proof

$\int_0^2 \sqrt{4-x^2} dx = \dots = \int_0^{\pi/2} \sqrt{4-4\sin^2 \theta} (2\cos \theta) d\theta$

$$= \int_0^{\pi/2} 2\cos \theta \sqrt{4(1-\sin^2 \theta)} d\theta = \int_0^{\pi/2} 2\cos \theta \sqrt{4\cos^2 \theta} d\theta$$

$$= \int_0^{\pi/2} 4\cos^2 \theta d\theta = \text{TRIG IDENTITY} \therefore \int_0^{\pi/2} 4 \left(\frac{1+\cos 2\theta}{2} \right) d\theta$$

$$= \int_0^{\pi/2} 2+2\cos 2\theta d\theta = \left(2\theta + \sin 2\theta \right) \Big|_0^{\pi/2}$$

$$= \left(\pi - \sin \pi \right) - (0 - \sin 0) = \pi$$

$x = 2\cos \theta$
 $\frac{dx}{d\theta} = -2\sin \theta$
 $2=0, 0=2\cos \theta$
 $\cos \theta = 0$
 $\theta = \frac{\pi}{2}$
 $x=2, 2=2\cos \theta$
 $\cos \theta = 1$
 $\theta = 0$

Question 193 (****+)

Use the substitution $u = 1 + x^2 e^{-3x}$ to find an expression for

$$\int \frac{x(2-3x)}{e^{3x} + x^2} dx.$$

$$\boxed{}, \ln(1 + x^2 e^{-3x}) + C$$

USING THE SUBSTITUTION GIVEN

$$\Rightarrow u = 1 + x^2 e^{-3x}$$

$$\Rightarrow \frac{du}{dx} = 2x e^{-3x} + x^2 (-3e^{-3x})$$

$$\Rightarrow \frac{du}{dx} = 2x e^{-3x} - 3x^2 e^{-3x}$$

$$\Rightarrow \frac{du}{dx} = x e^{-3x} (2 - 3x)$$

$$\Rightarrow dx = \frac{du}{x(2-3x)e^{-3x}}$$

TRANSFORMING THE INTEGRAL

$$\int \frac{x(2-3x)}{e^{3x} + x^2} dx = \int \frac{x(2-3x)}{x^2 + x^2} \left(\frac{du}{x(2-3x)e^{-3x}} \right)$$

$$= \int \frac{1}{e^3 + 1} \times \frac{1}{e^{3x}} du$$

$$= \int \frac{1}{1 + x^2 e^{-3x}} du$$

$$= \int \frac{1}{u} du$$

$$= \ln|u| + C$$

$$= \ln(1 + x^2 e^{-3x}) + C$$

Use the substitution $u = \frac{1}{x} + x e^x$ to find an expression for

$$\int \frac{x^3 + x^2 - e^{-x}}{x^3 + x e^{-x}} dx.$$

$$\boxed{}, \ln \left| \frac{1}{x} + x e^x \right| + C$$

$$\begin{aligned}
 u &= \frac{1}{x} + 2e^x \\
 \frac{du}{dx} &= -\frac{1}{x^2} + e^x + 2e^x \\
 dx &= \frac{-\frac{1}{x^2} + e^x + 2e^x}{\frac{1}{x^2} + 2e^x - 1} du \quad \leftarrow \text{MULTIPLY TOP BOTTOM BY } x^2 \\
 \int \frac{x^2 + 2x^2 e^x - e^x}{x^2 + 2x^2 e^x - 1} dx &= \int \frac{\frac{x^2 + 2x^2 e^x - e^x}{x^2 + e^x} \times \frac{x^2}{x^2 + 2x^2 e^x - 1}}{\frac{x^2 + 2x^2 e^x - e^x}{x^2 + 2x^2 e^x - 1} \times \frac{x^2}{x^2 + 2x^2 e^x - 1}} dx \\
 &= \int \frac{\frac{x^2 + 2x^2 e^x - e^x}{x^2 + 2x^2 e^x - 1}}{\frac{x^2 + 2x^2 e^x - e^x}{x^2 + 2x^2 e^x - 1}} dx \\
 &= \int \frac{1}{x} dx \\
 &= \ln|x| + C = \ln\left|\frac{1}{x} + 2e^x\right| + C
 \end{aligned}$$

Question 195 (****+)

$$\int \frac{1}{\sqrt{x^2 + x^n}} dx, \quad n \neq 2, \quad x \geq 0.$$

- a) Show that the substitution $u^2 = 1 + x^{n-2}$ transforms above integral into

$$\frac{1}{n-2} \int \frac{2}{(u-1)(u+1)} du.$$

- b) Use partial fractions to find, in terms of x and n , an integrated expression for the original integral.

$$\boxed{}, \quad \frac{1}{n-2} \ln \left| \frac{\sqrt{1+x^{n-2}}-1}{\sqrt{1+x^{n-2}}+1} \right| + C$$

a) USING THE SUBSTITUTION GIVEN $u = (1+x^{n-2})^{\frac{1}{2}}$

$$\Rightarrow u^2 = 1+x^{n-2} \iff x^{n-2} = u^2 - 1$$

$$\Rightarrow 2u \frac{du}{dx} = (u-1)2x^{n-3}$$

$$\Rightarrow 2u du = (u-1)2x^{n-3} dx$$

$$\Rightarrow dx = \frac{2u}{(n-2)x^3} du$$

TRANSFORMING THE INTEGRAL

$$\int \frac{1}{\sqrt{x^2+x^n}} dx = \int \frac{1}{|x|\sqrt{1-x^{n-2}}} dx \quad (x \geq 0)$$

$$= \int \frac{1}{x} \cdot \frac{2u}{(n-2)x^{n-3}} du = \int \frac{2}{(n-2)x^{n-2}} du$$

$$= \int \frac{2}{(n-2)(u^2-1)} du = \frac{1}{n-2} \int \frac{2}{(u-1)(u+1)} du \quad \text{As required}$$

b) PROCEED BY PARTIAL FRACTIONS

$$\frac{2}{(u-1)(u+1)} = \frac{A}{u-1} + \frac{B}{u+1}$$

$$2 = A(u+1) + B(u-1)$$

• If $u=1$ $2 = 2A$ $A = 1$	• If $u=-1$ $2 = -2B$ $B = -1$
-----------------------------------	--------------------------------------

...

$$= \frac{1}{n-2} \int \left(\frac{1}{u-1} - \frac{1}{u+1} \right) du$$

$$= \frac{1}{n-2} \left[\ln|u-1| - \ln|u+1| \right] + C$$

$$= \frac{1}{n-2} \ln \left| \frac{u-1}{u+1} \right| + C$$

$$= \frac{1}{n-2} \ln \left| \frac{\sqrt{1+x^{n-2}}-1}{\sqrt{1+x^{n-2}}+1} \right| + C$$

Question 196 (****+)

$$f(x) \equiv 2 - \sqrt{x-1}, \quad x \geq 1.$$

- a) Find a simplified expression for $g(x)$ so that $f(x)g(x) = 1$.
- b) Hence, or otherwise, find

$$\int \frac{5-x}{2-\sqrt{x-1}} dx.$$

$$\boxed{\frac{1}{f(x)}}, \quad g(x) \equiv \frac{2+\sqrt{x-1}}{5-x}, \quad \boxed{2x + \frac{1}{2}(x-1)^{\frac{3}{2}} + C}$$

a) $f(x) \equiv 2 - \sqrt{x-1}, \quad x \geq 1$
USING THE GIVEN
 $\Rightarrow f(x)g(x) = 1$
 $\Rightarrow g(x) = \frac{1}{f(x)} = \frac{1}{2 - \sqrt{x-1}}$
 $\Rightarrow \frac{1}{f(x)} = \frac{(2 + \sqrt{x-1})}{(2 - \sqrt{x-1})(2 + \sqrt{x-1})}$
 $\Rightarrow \frac{1}{f(x)} = \frac{2 + \sqrt{x-1}}{4 - (x-1)}$
 $\Rightarrow \frac{1}{f(x)} = \frac{2 + \sqrt{x-1}}{5-x}$

b) ALSO FROM (a)
 $\int \frac{5-x}{2-\sqrt{x-1}} dx = \int \frac{5-x}{f(x)} dx = \int (5-x) \times \frac{1}{f(x)} dx$
 $= \int (5-x) \frac{2+\sqrt{x-1}}{5-x} dx = \int 2 + \sqrt{x-1} dx$
 $= \int 2 + (x-1)^{\frac{1}{2}} dx$
 $= 2x + \frac{2}{3}(x-1)^{\frac{3}{2}} + C$

Question 197 (****+)

By using the substitution $u = 1 + \sin^2 x$, or otherwise, show clearly that

$$\int_0^{\frac{\pi}{4}} \frac{4 \tan x}{1 + \sin^2 x} dx = \ln 3.$$

proof

Handwritten solution for the integral problem:

$$\int_0^{\frac{\pi}{4}} \frac{4 \tan x}{1 + \sin^2 x} dx = \dots \text{ by substitution } \dots$$

$$= \int_1^{\frac{5}{4}} \frac{4 \tan x}{u} \frac{du}{2 \sin x \cos x}$$

$$= \int_1^{\frac{5}{4}} \frac{2 \tan x}{u \cos x} du = \int_1^{\frac{5}{4}} \frac{2}{u} \cdot \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} du$$

$$= \int_1^{\frac{5}{4}} \frac{2}{u \cos^2 x} du = \int_1^{\frac{5}{4}} \frac{2}{u(1 - \sin^2 x)} du$$

$$= \int_1^{\frac{5}{4}} \frac{2}{u(1 - (u-1))} du = \int_1^{\frac{5}{4}} \frac{2}{u(2-u)} du$$

By partial fractions

$$\frac{2}{u(2-u)} = \frac{A}{u} + \frac{B}{2-u}$$

$$2 = A(2-u) + Bu$$

$$14 \quad u=0 \Rightarrow 2A=2 \Rightarrow A=1$$

$$16 \quad u=2 \Rightarrow 2B=2 \Rightarrow B=1$$

$$= \int_1^{\frac{5}{4}} \frac{1}{u} + \frac{1}{2-u} du = [\ln|2-u| + \ln|u|]_1^{\frac{5}{4}} = [\ln|u| - \ln|2-u|]_1^{\frac{5}{4}}$$

$$= [\ln \frac{5}{4} - \ln \frac{1}{2}] - [\ln 1 - \ln 1] = \ln \frac{5}{4} + \ln 2 = \ln 3$$

Question 198 (****+)

$$\sec x \equiv \frac{\cos x}{1 - \sin^2 x}.$$

- a) Prove the validity of the above trigonometric identity.
- b) Use the substitution $u = \sin x$ to show that

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec x \, dx = \frac{1}{2} \ln \left(\frac{7 + 4\sqrt{3}}{3} \right).$$

- c) Show clearly that

$$\frac{1}{2} \ln \left(\frac{7 + 4\sqrt{3}}{3} \right) = \ln \left(1 + \frac{2}{3} \sqrt{3} \right)$$

proof

(a) LHS = $\sec x = \frac{1}{\cos x} = \frac{\cos x}{\cos^2 x} = \frac{\cos x}{1 - \sin^2 x}$ As Required

(b) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec x \, dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos x}{1 - \sin^2 x} \, dx$ substitution $u = \sin x$
 $\frac{du}{dx} = \cos x$
 $dx = \frac{du}{\cos x}$
 $x = \frac{\pi}{6} \Rightarrow u = \frac{1}{2}$
 $x = \frac{\pi}{3} \Rightarrow u = \frac{\sqrt{3}}{2}$

$= \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{\cos x}{1 - u^2} \cdot \frac{du}{\cos x} = \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{1}{(1-u)(1+u)} \, du$

$= \dots$ by partial fractions

$= \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \left[\frac{\frac{1}{2}}{1-u} + \frac{-\frac{1}{2}}{1+u} \right] \, du = \left[\frac{1}{2} \ln|1-u| - \frac{1}{2} \ln|1+u| \right]_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}$

$= \left[\frac{1}{2} \ln \left(\frac{1-u}{1+u} \right) \right]_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} = \frac{1}{2} \left[\ln \left(\frac{1 - \frac{\sqrt{3}}{2}}{1 + \frac{\sqrt{3}}{2}} \right) - \ln \left(\frac{1 - \frac{1}{2}}{1 + \frac{1}{2}} \right) \right]$

$= \frac{1}{2} \left[\ln \left(\frac{2 - \sqrt{3}}{2 + \sqrt{3}} \right) - \ln \left(\frac{1}{3} \right) \right] = \frac{1}{2} \left[\ln \left(\frac{2 - \sqrt{3}}{2 + \sqrt{3}} \right) + \ln 3 \right]$

$= \frac{1}{2} \ln \left(\frac{7 + 4\sqrt{3}}{3} \right)$ As Required

(c) $\frac{1}{2} \ln \left(\frac{7 + 4\sqrt{3}}{3} \right) = \frac{1}{2} \ln \left(\frac{21 + 12\sqrt{3}}{9} \right) = \frac{1}{2} \ln \left[\frac{9 + 2 \times 3 \times 2\sqrt{3} + 12}{9} \right]$

$= \frac{1}{2} \ln \left[\frac{3^2 + 2 \times 3 \times 2\sqrt{3} + (2\sqrt{3})^2}{9} \right]$

$= \frac{1}{2} \ln \left[\frac{(3 + 2\sqrt{3})^2}{9} \right] = \ln \left(\frac{3 + 2\sqrt{3}}{3} \right)$

$= \ln \left(1 + \frac{2}{3} \sqrt{3} \right)$ As Required

Question 199 (****+)

Use the substitution $x = 2\sec\theta$, to find the exact value of

$$\int_{\frac{4}{\sqrt{3}}}^4 \frac{6}{(x^2 - 4)^{\frac{3}{2}}} dx.$$

$$3 - \sqrt{3}$$

Handwritten solution for Question 199:

Given: $x = 2\sec\theta$
 $\frac{dx}{d\theta} = 2\sec\theta \tan\theta$
 $dx = 2\sec\theta \tan\theta d\theta$

Substitution steps:

- $x = \frac{4}{\sqrt{3}} \Rightarrow \frac{4}{\sqrt{3}} = 2\sec\theta \Rightarrow \sec\theta = \frac{2}{\sqrt{3}} \Rightarrow \cos\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{\pi}{6}$
- $x = 4 \Rightarrow 4 = 2\sec\theta \Rightarrow \sec\theta = 2 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$

Integration:

$$\int_{\frac{4}{\sqrt{3}}}^4 \frac{6}{(x^2 - 4)^{\frac{3}{2}}} dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{6}{(4\sec^2\theta - 4)^{\frac{3}{2}}} \cdot 2\sec\theta \tan\theta d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{12\sec\theta \tan\theta}{[4(\sec^2\theta - 1)]^{\frac{3}{2}}} d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{12\sec\theta \tan\theta}{(4\tan^2\theta)^{\frac{3}{2}}} d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{12\sec\theta \tan\theta}{8\tan^3\theta} d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{3\sec\theta}{2\tan^2\theta} d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{3}{2} \frac{\cos\theta}{\sin^2\theta} d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{3\cos\theta}{2\sin^2\theta} d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{3}{2} \cot\theta \csc\theta d\theta$$

$$= \frac{3}{2} [\csc\theta]_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \frac{3}{2} \left[2 - \frac{2}{\sqrt{3}} \right] = 3 - \sqrt{3}$$

Final answer: $3 - \sqrt{3}$

Question 200 (***)

Use the substitution $u = 1 + x e^{\sin x}$ to find an exact simplified value for the following definite integral.

$$\int_0^{\pi} \frac{1 + x \cos x}{x + e^{-\sin x}} dx.$$

$$\boxed{}, \ln(1 + \pi)$$

USING THE SUBSTITUTION GIVEN

$$\Rightarrow u = 1 + x e^{\sin x}$$

$$\Rightarrow \frac{du}{dx} = 1 + x e^{\sin x} + x \cdot e^{\sin x} \cos x$$

$$\Rightarrow \frac{du}{dx} = 1 + x e^{\sin x} (1 + \cos x)$$

$$\Rightarrow \frac{du}{dx} = e^{\sin x} (1 + x \cos x)$$

$$\Rightarrow dx = \frac{du}{e^{\sin x} (1 + x \cos x)}$$

CHANGING THE LIMITS

$$x=0 \rightarrow u=1$$

$$x=\pi \rightarrow u=1+\pi$$

TRANSFORMING THE INTEGRAL

$$\int_0^{\pi} \frac{1 + x \cos x}{x + e^{-\sin x}} dx = \int_1^{1+\pi} \frac{1 + x \cos x}{x + e^{-\sin x}} \times \frac{du}{e^{\sin x} (1 + x \cos x)}$$

$$= \int_1^{1+\pi} \frac{1}{x e^{\sin x}} du$$

$$= \int_1^{1+\pi} \frac{1}{u} du$$

$$= \left[\ln |u| \right]_1^{1+\pi}$$

$$= \ln(1+\pi) - \ln 1$$

$$= \ln(1+\pi)$$

Question 201 (****+)

$$I = \int \frac{1}{x^2 \sqrt{4-x^2}} dx.$$

- a)** Use the substitution $x = 2 \sin \theta$ to show clearly that

$$I = -\frac{\sqrt{4-x^2}}{4x} + C.$$

- b)** Verify the answer to part **(a)** by using the substitution $u = \frac{2}{x}$.

proof


(4) $\int \frac{1}{x^2 \sqrt{4-x^2}} dx = \dots$ by substitution

$x = 2 \sin(\theta)$
 $\frac{dx}{d\theta} = 2 \cos(\theta)$
 $dx = 2 \cos(\theta) d\theta$

$= \int \frac{2 \cos(\theta)}{4 \sin^2(\theta) \sqrt{4 - 4 \sin^2(\theta)}} d\theta = \int \frac{2 \cos(\theta)}{4 \sin^2(\theta) \sqrt{4 \cos^2(\theta)}} d\theta$

$= \int \frac{2 \cos(\theta)}{4 \sin^2(\theta) (2 \cos(\theta))} d\theta = \int \frac{1}{2 \sin^2(\theta)} d\theta = -\frac{1}{2} \cot(\theta) + C$

Now $x = 2 \sin(\theta)$

$\frac{x}{2} = \sin(\theta)$ $\sin(\theta) = \frac{3}{4}$
 $\therefore \cot(\theta) = \frac{4}{3}$
 $\therefore \cot(\theta) = \frac{4}{3}$ $= -\frac{\sqrt{4-x^2}}{4x} + C$

(5) $\int \frac{1}{x^2 \sqrt{4-x^2}} dx = \dots$ by substitution

$u = \frac{x}{2}$, $x = \frac{2}{u}$

$= \int \frac{1}{\frac{4}{u^2} \sqrt{4 - \frac{4}{u^2}}} \left(-\frac{2}{u^2} du \right) = \int \frac{u^2}{4 \sqrt{\frac{4u^2 - 4}{u^2}}} \left(-\frac{2}{u^2} du \right)$
 $= \int \frac{-1}{2 \sqrt{4u^2 - 4}} du = \int -\frac{1}{2 \sqrt{4u^2 - 4}} du$

By RECOGNITION or by using the SUBSTITUTION \dots
 (4) $\frac{1}{\sqrt{4u^2 - 4}} = \frac{1}{2 \sqrt{u^2 - 1}}$

$= -\frac{1}{2} u \sqrt{u^2 - 1} - \frac{1}{2} \ln \left| (u^2 - 1) \right| + C$

$= -\frac{1}{2} \sqrt{\frac{x^2}{4} - 1} + C = -\frac{1}{4} \sqrt{\frac{x^2 - 4}{x^2}} + C$

$= -\frac{1}{4} \frac{\sqrt{x^2 - 4}}{x} + C = -\frac{\sqrt{4 - x^2}}{4x} + C$

As before

Question 202 (***)

By using the substitution $x = -\frac{1}{2} + \frac{1}{2} \sin \theta$, or otherwise, find the exact value of

$$\int_{-\frac{1}{4}}^0 \frac{3}{\sqrt{-x(x+1)}} dx.$$

, π

● USING THE SUBSTITUTION GIVEN

$$x = -\frac{1}{2} + \frac{1}{2} \sin \theta$$

$$dx = \frac{1}{2} \cos \theta d\theta$$

● CHANGING THE LIMITS

$x = 0 \Rightarrow 0 = -\frac{1}{2} + \frac{1}{2} \sin \theta$ $\frac{1}{2} = \frac{1}{2} \sin \theta$ $\sin \theta = 1$ $\theta = \frac{\pi}{2}$	$x = -\frac{1}{4} \Rightarrow -\frac{1}{4} = -\frac{1}{2} + \frac{1}{2} \sin \theta$ $\frac{1}{4} = \frac{1}{2} \sin \theta$ $\sin \theta = \frac{1}{2}$ $\theta = \frac{\pi}{6}$
--	--

● TRANSFORMING THE DENOMINATOR OF THE INTEGRAND

$$\begin{aligned} \sqrt{-x(x+1)} &= \sqrt{-\left(-\frac{1}{2} + \frac{1}{2} \sin \theta\right)\left(-\frac{1}{2} + \frac{1}{2} \sin \theta + 1\right)} \\ &= \sqrt{\left(\frac{1}{2} - \frac{1}{2} \sin \theta\right)\left(\frac{1}{2} + \frac{1}{2} \sin \theta\right)} \\ &= \sqrt{\frac{1}{4} - \frac{1}{4} \sin^2 \theta} \\ &= \sqrt{\frac{1}{4}(1 - \sin^2 \theta)} \\ &= \sqrt{\frac{1}{4} \cos^2 \theta} \\ &= \frac{1}{2} \cos \theta \end{aligned}$$

● HENCE THE INTEGRAL NOW BECOMES

$$\begin{aligned} \int_{-\frac{1}{4}}^0 \frac{3}{\sqrt{-x(x+1)}} dx &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{3}{\frac{1}{2} \cos \theta} \left(\frac{1}{2} \cos \theta d\theta\right) \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 3 d\theta \\ &= \left[3\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} \\ &= \frac{3\pi}{2} - \frac{\pi}{2} \\ &= \pi \end{aligned}$$

Question 203 (****+)

By using multiplying the numerator and denominator of the integrand by $(\sec x + 1)$, and manipulating it further by various trigonometric identities, show clearly that

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{6}{\sec x - 1} dx = 12 - \pi.$$

proof

$$\begin{aligned} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{6}{\sec x - 1} dx &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{6(\sec x + 1)}{(\sec x - 1)(\sec x + 1)} dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{6(\sec x + 1)}{\sec^2 x - 1} dx \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{6(\sec x + 1)}{\tan^2 x} dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 6(\sec x + 1) \cot^2 x dx \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 6 \left(\frac{1}{\cos x} + 1 \right) \frac{\cos^2 x}{\sin^2 x} dx \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 6 \left(\frac{\cos x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x} \right) dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 6 \left(\frac{\cos x}{\sin^2 x} + \frac{1}{\tan^2 x} \right) dx \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 6 \left(\cot x \csc x + \operatorname{cosec}^2 x - 1 \right) dx \\ &= \left[-6 \operatorname{cosec} x - 6 \cot x - 6x \right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} \\ &= \left[-6 \operatorname{cosec} x + 6 \cot x + 6x \right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} \\ &= \left(-6 \operatorname{cosec} \frac{\pi}{3} + 6 \cot \frac{\pi}{3} + \pi \right) - \left(-6 \operatorname{cosec} \frac{\pi}{6} + 6 \cot \frac{\pi}{6} + 2\pi \right) \\ &= \left(-12 + 6\sqrt{3} + \pi \right) - \left(-12\sqrt{3} + 2\pi\sqrt{3} + 2\pi \right) \\ &= 12 - \pi \quad \text{As required} \end{aligned}$$

Question 204 (****+)

By changing the base of the logarithmic integrand into base e and further using integration by parts, show that

$$\int_1^e \log_{10} x \, dx = \frac{1}{\ln 10}.$$

 , proof

CHANGING INTO BASE e

$$\int_1^e \log_{10} x \, dx = \int_1^e \frac{\log_e x}{\log_e 10} \, dx \quad \boxed{\log_a b = \frac{\log_e b}{\log_e a}}$$

$$= \int_1^e \frac{\ln x}{\ln 10} \, dx$$

$$= \int_1^e \frac{1}{\ln 10} (\ln x) \, dx$$

INTEGRATION BY PARTS: $\frac{1}{\ln 10}$ IS A CONSTANT

$\ln x$	$\frac{1}{x}$
$\frac{1}{\ln 10}$	$\frac{1}{\ln 10}$

$$= \left[\frac{\ln x}{\ln 10} x \right]_1^e - \int_1^e \frac{1}{\ln 10} x \cdot \frac{1}{x} \, dx$$

$$= \left[\frac{x \ln x}{\ln 10} \right]_1^e - \int_1^e \frac{1}{\ln 10} \, dx$$

$$= \left[\frac{x \ln x}{\ln 10} - \frac{1}{\ln 10} x \right]_1^e$$

$$= \left(\frac{e \ln e}{\ln 10} - \frac{e}{\ln 10} \right) - \left(\frac{1 \ln 1}{\ln 10} - \frac{1}{\ln 10} \right)$$

$$= \left(\frac{e}{\ln 10} - \frac{e}{\ln 10} \right) + \frac{1}{\ln 10}$$

$$= \frac{1}{\ln 10}$$

* REQUIRED

Question 205 (****+)

Use trigonometric identities to find

$$\int \frac{1}{\operatorname{cosec} 2x - \cot 2x} \, dx,$$

giving the answer in the form $\ln |f(x)|$

$\ln |\sin x| + C$

$$\int \frac{1}{\operatorname{cosec} 2x - \cot 2x} \, dx = \int \frac{1}{\frac{1}{\sin 2x} - \frac{\cos 2x}{\sin 2x}} \, dx = \int \frac{1}{\frac{1 - \cos 2x}{\sin 2x}} \, dx$$

Use the trig identity $1 - \cos 2x = 2 \sin^2 x$

$$= \int \frac{\sin 2x}{1 - \cos 2x} \, dx = \int \frac{2 \sin x \cos x}{2 \sin^2 x} \, dx = \int \frac{\cos x}{\sin x} \, dx$$

$$= \int \frac{\cos x}{\sin x} \, dx = \dots \text{ of the type } \int \frac{f'(x)}{f(x)} \, dx = \ln |f(x)| + C$$

Question 206 (****+)

By using the substitution $u = \tan x$, or otherwise, find the exact value of

$$\int_0^{\frac{\pi}{4}} \frac{1}{(\cos x + 2 \sin x)^2} dx.$$

 $\frac{1}{3}$

Handwritten solution for the integral problem using the substitution $u = \tan x$.

Method 1: By Substitution

Let $u = \tan x$
 $\frac{du}{dx} = \sec^2 x$
 $dx = \frac{du}{\sec^2 x}$
 $x=0 \Rightarrow u=0$
 $x=\frac{\pi}{4} \Rightarrow u=1$

The integral becomes:

$$\int_0^1 \frac{1}{(\cos x + 2 \sin x)^2} \frac{du}{\sec^2 x} = \int_0^1 \frac{1}{(\cos x + 2 \sin x)^2 \sec^2 x} du$$

Since $\sec^2 x = 1 + \tan^2 x = 1 + u^2$, and $\cos x = \frac{1}{\sqrt{1+u^2}}$, $\sin x = \frac{u}{\sqrt{1+u^2}}$, we have:

$$= \int_0^1 \frac{1}{(1 + 2u)^2} du = \int_0^1 \frac{1}{(1+2u)^2} du$$

Let $v = 1+2u$, then $dv = 2 du$, so $du = \frac{dv}{2}$.

$$= \int_1^3 \frac{1}{v^2} \cdot \frac{dv}{2} = \left[-\frac{1}{2v} \right]_1^3 = -\frac{1}{2 \cdot 3} - \left(-\frac{1}{2 \cdot 1} \right) = -\frac{1}{6} + \frac{1}{2} = \frac{1}{3}$$

Method 2: By Recognising the Chain Rule

Let $u = \tan x$, then $\frac{du}{dx} = \sec^2 x$.

The integral becomes:

$$\int_0^{\frac{\pi}{4}} \frac{1}{(\cos x + 2 \sin x)^2} dx = \int_0^{\frac{\pi}{4}} \frac{1}{(\cos x (1 + 2 \tan x))^2} dx = \int_0^{\frac{\pi}{4}} \frac{1}{\cos^2 x (1 + 2 \tan x)^2} dx$$

Since $\frac{1}{\cos^2 x} = \sec^2 x = \frac{du}{dx}$, we have:

$$= \int_0^{\frac{\pi}{4}} \frac{1}{(1 + 2 \tan x)^2} \frac{du}{dx} dx = \int_0^1 \frac{1}{(1 + 2u)^2} du$$

Let $v = 1+2u$, then $dv = 2 du$, so $du = \frac{dv}{2}$.

$$= \int_1^3 \frac{1}{v^2} \cdot \frac{dv}{2} = \left[-\frac{1}{2v} \right]_1^3 = -\frac{1}{6} + \frac{1}{2} = \frac{1}{3}$$

Question 207 (***)

Use trigonometric identities and integration by parts to find an exact value for

$$\int_0^{\frac{\pi}{2}} 9x \sin x \sin 2x \, dx.$$

$$\boxed{}, \boxed{3\pi - 4}$$

START BY REWRITING THE DOUBLE ANGLE IN THE INTEGRAND

$$\begin{aligned} \int_0^{\frac{\pi}{2}} 9x \sin x \sin 2x \, dx &= \int_0^{\frac{\pi}{2}} 9x \sin x (2 \sin x \cos x) \, dx \\ &= \int_0^{\frac{\pi}{2}} 18x \sin^2 x \cos x \, dx \\ &= \int_0^{\frac{\pi}{2}} (6x) (3 \sin^2 x \cos x) \, dx \end{aligned}$$

NEXT INTEGRATION BY PARTS

$6x$	6	$\dots = [6x \sin^3 x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 6 \sin^3 x \, dx$
$\sin^2 x$	$3 \sin^2 x \cos x$	$= 3\pi - 0 - \int_0^{\frac{\pi}{2}} 6 \sin^3 x \, dx$

BY RECOGNITION

NOW MANIPULATE INTO "RECOGNISABLE" FORM OR USE THE SUBSTITUTION $u = \cos x$, WE OBTAIN

$$\begin{aligned} \dots &= 3\pi - \int_0^{\frac{\pi}{2}} 6 \sin^2 x \sin x \, dx \\ &= 3\pi - \int_0^{\frac{\pi}{2}} 6 \sin x (1 - \cos x) \, dx \\ &= 3\pi - \int_0^{\frac{\pi}{2}} 6 \sin x - 6 \sin x \cos x \, dx \end{aligned}$$

BY RECOGNITION

$$\begin{aligned} &= 3\pi - [-6 \cos x + 2 \cos^2 x]_0^{\frac{\pi}{2}} \\ &= 3\pi + [6 \cos x - 2 \cos^2 x]_0^{\frac{\pi}{2}} \\ &= 3\pi + [(0 - 0) - (6 - 2)] \\ &= 3\pi - 4 \end{aligned}$$

Question 208 (****+)

$$I \equiv \int \frac{1}{1 + \sin 2x} dx.$$

- a) Integrate I by multiplying the numerator and denominator of the integrand by $(1 - \sin 2x)$.
- b) Hence evaluate

$$\int_0^{\frac{\pi}{8}} \frac{1}{1 + \sin 2x} dx.$$

- c) Use the substitution $t = \tan x$ to integrate I .
- d) Hence evaluate

$$\int_0^{\frac{\pi}{4}} \frac{1}{1 + \sin 2x} dx.$$

$$\boxed{\frac{1}{2} \tan 2x - \frac{1}{2} \sec 2x + C}, \quad \boxed{\frac{1}{2}(2 - \sqrt{2})}, \quad \boxed{-\frac{1}{1 + \tan x} + C}, \quad \boxed{\frac{1}{2}}$$

Handwritten solution for Question 208:

(a) $I = \int \frac{1}{1 + \sin 2x} dx = \int \frac{1 - \sin 2x}{(1 + \sin 2x)(1 - \sin 2x)} dx = \int \frac{1 - \sin 2x}{1 - \sin^2 2x} dx = \int \frac{1 - \sin 2x}{\cos^2 2x} dx$
 $= \int \frac{1}{\cos^2 2x} - \frac{\sin 2x}{\cos^2 2x} dx = \int \frac{1}{\cos^2 2x} - \frac{1}{\cos 2x} dx$
 $= \int \sec^2 2x - \sec 2x dx = \frac{1}{2} \tan 2x - \frac{1}{2} \sec 2x + C$

(b) $\int_0^{\frac{\pi}{8}} \frac{1}{1 + \sin 2x} dx = \left[\frac{1}{2} \tan 2x - \frac{1}{2} \sec 2x \right]_0^{\frac{\pi}{8}} = \frac{1}{2} [(1 - \sqrt{2}) - (0 - 1)]$
 $= \frac{1}{2} (2 - \sqrt{2})$

(c) $I = \int \frac{1}{1 + \sin 2x} dx = \int \frac{1}{1 + 2 \tan x \sec x} dx = \int \frac{\sec^2 x}{\sec^2 x + 2 \tan x \sec x} dx$
 $= \int \frac{\sec^2 x}{\sec^2 x + 2 \tan x} dx = \int \frac{1 + \tan^2 x}{1 + \tan^2 x + 2 \tan x} dx$
 $= \int \frac{1 + \tan^2 x}{(1 + \tan x)^2} dx$
 Now let $t = \tan x$
 $\frac{dt}{dx} = \sec^2 x$
 $dx = \frac{dt}{1 + t^2}$
 $= \int \frac{1 + t^2}{(1 + t)^2} \cdot \frac{dt}{1 + t^2} = \int \frac{1}{(1 + t)^2} dt = -\frac{1}{1 + t} + C$
 $= -\frac{1}{1 + \tan x} + C$

(d) Similarly $\int_0^{\frac{\pi}{4}} \frac{1}{1 + \sin 2x} dx = \left[-\frac{1}{1 + \tan x} \right]_0^{\frac{\pi}{4}} = \left[-\frac{1}{1 + \tan x} \right]_{\frac{\pi}{4}}$
 $= 1 - \frac{1}{2} = \frac{1}{2}$

Question 209 (****+)

Show clearly that

$$\int_0^1 \sqrt{x^2 - x^4} \, dx = \frac{1}{3}.$$

, proof

PROCEED AS FOLLOWS

$$\begin{aligned} \int_0^1 \sqrt{x^2 - x^4} \, dx &= \int_0^1 \sqrt{x^2(1-x^2)} \, dx \\ &= \int_0^1 x \sqrt{1-x^2} \, dx \\ &= \int_0^1 x \sqrt{1-x^2} \, dx \end{aligned}$$

USING THE SUBSTITUTION $u = 1-x^2$

$u = 1-x^2$	$\frac{du}{dx} = -2x$	$x = 0 \rightarrow u = 1$	$x = 1 \rightarrow u = 0$
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TRANSFORMING THE INTEGRAL

$$= \int_1^0 \sqrt{u} \left(-\frac{1}{2}\right) du = -\frac{1}{2} \int_1^0 \sqrt{u} \, du$$

BY RECOGNITION

$$\left[-\frac{1}{2} \cdot \frac{2}{3} u^{3/2}\right]_1^0 = -\frac{1}{3} \left[u^{3/2}\right]_1^0 = -\frac{1}{3} [0 - 1] = \frac{1}{3}$$

As required

Question 210 (*****)

Use the substitution $u = 1 + x^2 \operatorname{cosec} x$ to find an expression for

$$\int \frac{2x - x^2 \cot x}{x^2 + \sin x} dx.$$

$$\boxed{}, \ln |1 + x^2 \operatorname{cosec} x| + C$$

Handwritten solution for Question 210:

USING THE SUBSTITUTION GIVEN

$$\begin{aligned} \Rightarrow u &= 1 + x^2 \operatorname{cosec} x \\ \Rightarrow \frac{du}{dx} &= 2x \operatorname{cosec} x + x^2 \operatorname{cosec} x \cot x \\ \Rightarrow \frac{du}{dx} &= x \operatorname{cosec} x (2 + x \cot x) \\ \Rightarrow du &= \frac{du}{(2 + x \cot x) \operatorname{cosec} x} \end{aligned}$$

SUBSTITUTE INTO THE INTEGRAL

$$\begin{aligned} \int \frac{2x - x^2 \cot x}{x^2 + \sin x} dx &= \int \frac{x(2 - x \cot x)}{x^2 + \sin x} \cdot \frac{du}{(2 + x \cot x) \operatorname{cosec} x} \\ &= \int \frac{1}{(2 + x \cot x) \operatorname{cosec} x} du \\ &= \int \frac{1}{3 \operatorname{cosec} x + 1} du \\ &= \int \frac{1}{u} du \\ &= \ln |u| + C \\ &= \ln |1 + x^2 \operatorname{cosec} x| + C \end{aligned}$$

Question 211 (****+)

Use a suitable trigonometric substitution to find an exact simplified value for

$$\int_0^a x^{\frac{1}{2}} \sqrt{a-x} \, dx,$$

where a is a positive constant.

$$\frac{1}{8} \pi a^2$$

Handwritten solution for the integral:

$$\begin{aligned} \int_0^a x^{\frac{1}{2}} (a-x)^{\frac{1}{2}} dx & \dots \text{By substitution} \\ &= \int_0^{\frac{\pi}{2}} \sqrt{a \sin \theta} (a - a \sin \theta)^{\frac{1}{2}} (a \cos \theta) d\theta \\ &= \int_0^{\frac{\pi}{2}} \sqrt{a} \sin^{\frac{1}{2}} \theta \sqrt{a(1 - \sin \theta)} \times a \cos \theta d\theta \\ &= \int_0^{\frac{\pi}{2}} a^2 \sin^{\frac{1}{2}} \theta \cos \theta d\theta \\ &= \int_0^{\frac{\pi}{2}} \frac{1}{2} a^2 (2 \sin \theta \cos \theta) d\theta \\ &= \int_0^{\frac{\pi}{2}} \frac{1}{2} a^2 (\sin 2\theta) d\theta = \frac{1}{2} a^2 \int_0^{\frac{\pi}{2}} \frac{1}{2} - \frac{1}{2} \cos 4\theta d\theta \\ &= \frac{1}{4} a^2 \int_0^{\frac{\pi}{2}} 1 - \cos 4\theta d\theta = \frac{1}{4} a^2 \left[\theta - \frac{1}{4} \sin 4\theta \right]_0^{\frac{\pi}{2}} \\ &= \frac{1}{4} a^2 \left[\left(\frac{\pi}{2} - 0 \right) - \left(0 - 0 \right) \right] = \frac{\pi a^2}{8} \end{aligned}$$

Boxed notes on the right side of the solution:

$$\begin{aligned} x &= a \sin \theta \\ dx &= a \cos \theta d\theta \\ \frac{x}{a} &= \sin \theta \end{aligned}$$

Question 212 (****+)

$$f(u) \equiv \frac{1}{u^2 + 5u + 6}.$$

a) Express $f(u)$ into partial fractions.

$$I = \int_{\arcsin \frac{3}{5}}^{\arccos \frac{3}{5}} \frac{1}{(\sin x + 2 \cos x)(\sin x + 3 \cos x)} dx.$$

b) Express I in the form

$$I = \int_{\arcsin \frac{3}{5}}^{\arccos \frac{3}{5}} \frac{\sec^2 x}{g(\tan x)} dx,$$

where g is a function to be found.

c) Hence show that

$$I = \ln\left(\frac{a}{b}\right),$$

where a and b are positive integers to be found.

$$\boxed{150}, \quad f(u) \equiv \frac{1}{u+2} - \frac{1}{u+3}, \quad g(\tan x) \equiv (2 + \tan x)(3 + \tan x), \quad I = \ln\left(\frac{150}{143}\right)$$

The handwritten solution is divided into three parts: a), b), and c).

a) Partial fraction decomposition of $f(u) = \frac{1}{u^2 + 5u + 6}$. The denominator is factored as $(u+2)(u+3)$. The decomposition is $\frac{1}{(u+2)(u+3)} = \frac{A}{u+2} + \frac{B}{u+3}$. Solving for A and B yields $A = -1$ and $B = 1$. Thus, $f(u) = \frac{1}{u+3} - \frac{1}{u+2}$.

b) Integration of $I = \int_{\arcsin \frac{3}{5}}^{\arccos \frac{3}{5}} \frac{1}{(\sin x + 2 \cos x)(\sin x + 3 \cos x)} dx$. The integrand is rewritten using the partial fractions from part a) as $\frac{1}{\sin x + 3 \cos x} - \frac{1}{\sin x + 2 \cos x}$. The integral is then evaluated using the substitution $u = \tan x$, which gives $du = \sec^2 x dx$. The limits of integration are $u = \frac{3}{4}$ at $x = \arcsin \frac{3}{5}$ and $u = \frac{3}{5}$ at $x = \arccos \frac{3}{5}$. The integral becomes $I = \int_{\frac{3}{4}}^{\frac{3}{5}} \left(\frac{1}{u+3} - \frac{1}{u+2} \right) du$. Evaluating this gives $I = \left[\ln|u+3| - \ln|u+2| \right]_{\frac{3}{4}}^{\frac{3}{5}} = \ln\left(\frac{150}{143}\right)$.

c) A diagram of a right-angled triangle with a vertical side of 3, a horizontal side of 4, and a hypotenuse of 5. The angle at the bottom-left vertex is x . The sine of x is $\frac{3}{5}$ and the cosine is $\frac{4}{5}$. The angle at the top-right vertex is $\frac{\pi}{2} - x$, and its sine is $\frac{4}{5}$ and cosine is $\frac{3}{5}$. This confirms the values of $\sin x$ and $\cos x$ at the limits of integration.

Question 213 (****+)

Find in exact simplified form an expression for

$$\int \frac{3x}{x - \sqrt{x^2 - 1}} dx.$$

$$\boxed{}, \quad \boxed{x^3 + (x^2 - 1)^{\frac{3}{2}} + C}$$

$$\begin{aligned} \int \frac{3x}{x - \sqrt{x^2 - 1}} dx &= \int \frac{3x(x + \sqrt{x^2 - 1})}{(x - \sqrt{x^2 - 1})(x + \sqrt{x^2 - 1})} dx \\ &= \int \frac{3x^2 + 3x\sqrt{x^2 - 1}}{x^2 - (x^2 - 1)} dx = \int \frac{3x^2 + 3x\sqrt{x^2 - 1}}{1} dx \\ &= \int 3x^2 + 3x\sqrt{x^2 - 1} dx = x^3 + (x^2 - 1)^{\frac{3}{2}} + C \end{aligned}$$

ALTERNATIVE BY TRIGONOMETRIC SUBSTITUTION

$$\begin{aligned} \int \frac{3x}{x - \sqrt{x^2 - 1}} dx &= \int \frac{3 \sec \theta (\sec \theta \tan \theta d\theta)}{\sec \theta - \sqrt{\sec^2 \theta - 1}} \quad \begin{matrix} x = \sec \theta \\ dx = \sec \theta \tan \theta d\theta \end{matrix} \\ &= \int \frac{3 \sec^2 \theta \tan \theta}{\sec \theta - \tan \theta} d\theta = \int \frac{3 \sec^2 \theta \tan \theta (\sec \theta + \tan \theta)}{(\sec \theta - \tan \theta)(\sec \theta + \tan \theta)} d\theta \\ &= \int \frac{3 \sec^3 \theta \tan \theta + 3 \sec^2 \theta \tan^2 \theta}{1 + \tan^2 \theta} d\theta = \int 3 \sec^3 \theta \tan \theta + 3 \sec^2 \theta \tan \theta d\theta \\ \text{NOW BY RECOGNITION: } \frac{d}{d\theta}(\sec \theta) &= 3 \sec \theta (\sec \theta \tan \theta) = 3 \sec^2 \theta \tan \theta \\ \frac{d}{d\theta}(\tan \theta) &= 3 \tan \theta \times \sec \theta = 3 \sec \theta \tan \theta \\ \therefore \sec^3 \theta + \tan^3 \theta + C &= \sec \theta + [\sec \theta - 1]^{\frac{3}{2}} + C \\ &= x^3 + (x^2 - 1)^{\frac{3}{2}} + C \end{aligned}$$

ALTERNATIVE BY HYPERBOLIC SUBSTITUTION

$$\begin{aligned} \int \frac{3x}{x - \sqrt{x^2 - 1}} dx &= \int \frac{3 \cosh \theta (\sinh \theta d\theta)}{\cosh \theta - \sqrt{\cosh^2 \theta - 1}} \\ &= \int \frac{3 \cosh \theta \sinh \theta}{\cosh \theta - \sinh \theta} d\theta \\ &= \int \frac{3 \cosh \theta \sinh \theta (\cosh \theta + \sinh \theta)}{(\cosh \theta - \sinh \theta)(\cosh \theta + \sinh \theta)} d\theta \\ &= \int \frac{3 \cosh^2 \theta \sinh \theta + 3 \cosh \theta \sinh^2 \theta}{\cosh^2 \theta - \sinh^2 \theta} d\theta \\ &= \int 3 \cosh^2 \theta \sinh \theta + 3 \sinh^2 \theta \cosh \theta d\theta \\ &= \cosh^3 \theta + \sinh^3 \theta + C \\ &= x^3 + (x^2 - 1)^{\frac{3}{2}} + C \end{aligned}$$

$\begin{matrix} x = \cosh \theta \\ \theta = \operatorname{arccosh} x \\ dx = \sinh \theta d\theta \end{matrix}$

Question 214 (*****)

Use the substitution $u = \sin x + x \tan x$ to find an expression for

$$\int \frac{2x + \sin 2x + 2\cos^3 x}{(x + \cos x)\sin 2x} dx.$$

$$\boxed{}, \ln|\sin x + x \tan x| + C$$

Using the substitution $u = \sin x + x \tan x$
 $u = \sin x + x \tan x$
 $\frac{du}{dx} = \cos x + \tan x + x \sec^2 x$
 $\frac{du}{dx} = \frac{\cos x + \tan x + x \sec^2 x}{\cos x + \tan x + x \sec^2 x} \frac{du}{dx}$
Then solve the given integral
 $\int \frac{2x + \sin 2x + 2\cos^3 x}{(x + \cos x)\sin 2x} dx$
 $= \int \frac{2x + 2\sin x \cos x + 2\cos^3 x}{(x + \cos x)(2\cos x \sin x)} \times \frac{1}{\cos x + \tan x + x \sec^2 x} du$
 $= \int \frac{2x + 2\sin x \cos x + 2\cos^3 x}{(x + \cos x)\sin 2x} \times \frac{1}{\cos x + \frac{\sin x}{\cos x} + \frac{x}{\cos^2 x}} du$
 $= \int \frac{2x + 2\sin x \cos x + 2\cos^3 x}{(x + \cos x)\sin 2x} \times \frac{\cos^2 x}{\cos^2 x + \sin x \cos x + x} du$
 $= \int \frac{2x + 2\sin x \cos x + 2\cos^3 x}{(x + \cos x)\sin 2x} \times \frac{\cos^2 x}{\cos^2 x + \sin x \cos x + x} du$
 $= \int \frac{\cos^2 x}{x \sin 2x + \sin^2 x} du$
 $= \int \frac{1}{2x \tan x + \sin x} du$
 $= \int \frac{1}{u} du$
 $= \ln|u| + C$
 $= \ln|\sin x + x \tan x| + C$

Question 215 (****+)

Find, in exact simplified form, the value of

$$\int_{\ln(\ln 2)}^{\ln[2\ln(e+1)]} e^{x+e^x} dx.$$

$$\boxed{}, \boxed{e^2 + 2e - 1}$$

START BY REWRITING THE INTEGRAL, USING THE RULES OF INDICES

$$\int_{\ln(\ln 2)}^{\ln[2\ln(e+1)]} e^{x+e^x} dx = \int_{\ln(\ln 2)}^{\ln[2\ln(e+1)]} e^x \cdot e^{e^x} dx$$

BY SUBSTITUTION (OR RECOGNITION)

$u = e^x$ $\frac{du}{dx} = e^x$ $\frac{du}{dx} = u$ $dx = \frac{du}{u}$	LIMITS TRANSFORM TO $a = \ln(\ln 2), u = e^{\ln(\ln 2)} = \ln 2$ $b = \ln[2\ln(e+1)], u = e^{\ln[2\ln(e+1)]} = [2\ln(e+1)]^2$
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TRANSFORMING THE INTEGRAL YIELDS

$$\begin{aligned}
 &= \int_{\ln 2}^{[2\ln(e+1)]^2} \frac{1}{u} \cdot u \cdot \left(\frac{du}{u}\right) = \int_{\ln 2}^{[2\ln(e+1)]^2} \frac{1}{u} du \\
 &= \left[\ln u \right]_{\ln 2}^{[2\ln(e+1)]^2} = \ln([2\ln(e+1)]^2) - \ln(\ln 2) \\
 &= (e+1)^2 - 2 \\
 &= e^2 + 2e + 1 - 2 \\
 &= e^2 + 2e - 1
 \end{aligned}$$

Question 216 (****)

By using the substitution $u = 1 + \cos^4 x$, or otherwise, find the exact value of

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{4 \cot^3 x}{1 + 2 \cot^2 x + 2 \cot^4 x} dx.$$

$$\boxed{}, \left[\ln\left(\frac{5}{4}\right) \right]$$

The image shows two pages of handwritten work. The left page is titled "USING THE SUBSTITUTION GIVEN" and shows the substitution $u = 1 + \cos^4 x$ with $du = -4\cos^3 x \sin x dx$. It then shows the transformation of the integral, simplifying the denominator to $1 + 2\cos^2 x + 2\cos^4 x$ and using the identity $\cos^2 x = 1 - \sin^2 x$ to simplify the denominator to $1 + \cos^2 x$. The right page shows the integral in terms of u , $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{1 + \cos^2 x} dx$, and then evaluates it as $\int_1^{\frac{5}{4}} \frac{1}{u} du = \ln\left(\frac{5}{4}\right)$.

Question 217 (****)

$$\frac{2}{u(u-2)} \equiv \frac{A}{u-2} + \frac{B}{u}.$$

- a)** Find the value of each of the constants A and B .
- b)** By using the substitution $u = 1 + \cos^2 x$, or otherwise, show clearly that

$$\int \frac{4 \cot x}{1 + \cos^2 x} dx = -\ln(\operatorname{cosec}^2 x + \cot^2 x) + C.$$

$$\boxed{A=1}, \quad \boxed{B=1}$$

(a) $\frac{2}{u(u+2)} \equiv \frac{A}{u-2} + \frac{B}{u}$ \bullet If $u=0$, $2 = -2B \Rightarrow B = -1$
 $2 \equiv A(u+2) + B(u-2)$ \bullet If $u=2$, $2 = 2A \Rightarrow A = 1$

(b) $\int \frac{4dx}{1+\cos^2 x} = \dots$ (ANSWER THE QUESTION) GIVEN $u = 1 + \cos^2 x$
 $\frac{du}{dx} = -2\cos x \sin x$
 $\frac{du}{dx} = -\frac{du}{2\cos x \sin x}$
 $\frac{dx}{\cos x \sin x} = -\frac{du}{u}$
 $\int -\frac{2}{u} \cdot \frac{\cos x}{\sin x} \times \frac{1}{\cos x \sin x} du = \int -\frac{2}{u} \times \frac{1}{\sin^2 x} du$
 $\int -\frac{2}{u} \times \frac{1}{1-\cos^2 x} du = \int -\frac{2}{u} \times \frac{1}{1-(u-1)} du$
 $\int \frac{-2}{u(2-u)} du = \int \frac{-2}{u(2-u)} du = \dots$ part a ...
 $\int \frac{1}{u-2} - \frac{1}{u} du = \ln|u-2| - \ln|u| = \ln \left| \frac{u-2}{u} \right| + C$
 $= \ln \left| \frac{1+\cos^2 x - 2}{1+\cos^2 x} \right| + C = \ln \left| \frac{1-\cos^2 x}{1+\cos^2 x} \right| + C = \ln \left| \frac{\sin^2 x}{1+\cos^2 x} \right| + C$
 $= \ln \left(\frac{\sin^2 x}{1+\cos^2 x} \right) + C = -\ln \left(\frac{1+\cos^2 x}{\sin^2 x} \right) + C$
 $= -\ln \left(\frac{1}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x} \right) + C = -\ln (\csc^2 x + \cot^2 x) + C$
 (10 marks)

Question 218 (****)

By using the substitution $\tan \theta = \sqrt{x^3 - 1}$, or otherwise, find an exact value for the following integral.

$$\int_1^{\sqrt[3]{2}} \frac{\sqrt{x^3 - 1}}{\frac{1}{6}x} dx.$$

 , $4 - \pi$

Handwritten solution for the integral:

$$\int_1^{\sqrt[3]{2}} \frac{\sqrt{x^3 - 1}}{\frac{1}{6}x} dx = \dots$$

By substitution $\tan \theta = \sqrt{x^3 - 1}$

$$\tan^2 \theta = x^3 - 1$$

$$x^3 = 1 + \tan^2 \theta$$

$$x^3 = \sec^2 \theta$$

$$3x^2 \frac{dx}{d\theta} = 2 \sec^2 \theta \tan \theta$$

$$dx = \frac{2 \sec^2 \theta \tan \theta}{3x^2} d\theta$$

$$x=1 \rightarrow \theta=0$$

$$x=\sqrt[3]{2} \rightarrow \theta=\frac{\pi}{4}$$

$$\int_0^{\frac{\pi}{4}} \frac{4 \sec^2 \theta \tan \theta}{1 + \tan^2 \theta} d\theta$$

$$\int_0^{\frac{\pi}{4}} \frac{4 \sec^2 \theta \tan \theta}{\sec^2 \theta} d\theta$$

$$\int_0^{\frac{\pi}{4}} 4 \tan \theta d\theta$$

$$= \int_0^{\frac{\pi}{4}} 4(x^3 - 1) d\theta$$

$$= [2(\tan^2 \theta - \theta)]_0^{\frac{\pi}{4}}$$

$$= 2(1 - \frac{\pi}{4}) = 2 - \frac{\pi}{2}$$

Question 219 (****)

Use appropriate integration methods to find an exact simplified value for

$$\int_0^{\frac{1}{2}} \cos(5 \arcsin x) \, dx.$$

$$\sqrt{\quad}, \quad \square, \quad \frac{\sqrt{3}}{16}$$

PROCEED BY A SUBSTITUTION

$\theta = \arcsin x$
 $\sin \theta = x$ & limits $x=0 \rightarrow \theta=0$
 $dx = \cos \theta \, d\theta$ $x=\frac{1}{2} \rightarrow \theta=\frac{\pi}{6}$

TRANSFORM THE INTEGRAL

$$\int_0^{\frac{1}{2}} \cos(5 \arcsin x) \, dx = \int_0^{\frac{\pi}{6}} \cos(5\theta) (\cos \theta \, d\theta) = \int_0^{\frac{\pi}{6}} \cos(5\theta) \cos \theta \, d\theta$$

GETTING SOME TRIG IDENTITIES (OR CONTINUE WITH DOUBLE INTEGRATION BY PARTS)

$$\begin{aligned} \cos(5\theta + \theta) &\equiv \cos(5\theta)\cos\theta - \sin(5\theta)\sin\theta \\ \cos(5\theta - \theta) &\equiv \cos(5\theta)\cos\theta + \sin(5\theta)\sin\theta \end{aligned} \quad \text{--- ADDING ---}$$

$$\begin{aligned} \Rightarrow \cos\theta + \cos(5\theta) &\equiv 2\cos(5\theta)\cos\theta \\ \Rightarrow \cos(5\theta)\cos\theta &\equiv \frac{1}{2}\cos\theta + \frac{1}{2}\cos(5\theta) \end{aligned}$$

RETURNING TO THE INTEGRAL

$$\begin{aligned} \therefore \int_0^{\frac{\pi}{6}} \frac{1}{2}\cos\theta + \frac{1}{2}\cos(5\theta) \, d\theta &= \left[\frac{1}{2}\sin\theta + \frac{1}{10}\sin(5\theta) \right]_0^{\frac{\pi}{6}} \\ &= \left(0 + \frac{\sqrt{3}}{16} \right) - (0 + 0) \\ &= \frac{\sqrt{3}}{16} \end{aligned}$$

Question 220 (****)

By using the substitution $e^x = \frac{1}{u}$, or otherwise, show clearly that

$$\int \frac{9}{e^x \sqrt{e^{2x} - 9}} dx = \frac{\sqrt{e^{2x} - 9}}{e^x} + C.$$

proof

$$\int \frac{q}{e^x \sqrt{e^x - 5}} dx = \dots \text{by substitution}$$

$$= \int \frac{q}{\frac{1}{u} \sqrt{\frac{1}{u^2} - 5}} \cdot \left(-\frac{du}{u^2} \right) = \int \frac{q \cancel{u}}{\sqrt{\frac{1}{u^2} - 5}} \cdot \left(-\frac{du}{u} \right)$$

$$= \int \frac{-q}{\frac{\sqrt{1 - 5u^2}}{u^2}} du = \int \frac{-q u^2}{\sqrt{1 - 5u^2}} du$$

$$= \int \frac{-\frac{q}{u}}{\sqrt{1 - 5u^2}} du = -\frac{1}{5} \int \frac{1}{\sqrt{1 - 5u^2}} du$$

BY FINANCING OTHER DUFF
OR AVERAGE SUBSTITUTION

$$= (-\frac{1}{5}) \arcsin \frac{1}{\sqrt{5}} + C$$

$$= (-\frac{1}{5}) \arcsin \frac{1}{\sqrt{5}} + C = \sqrt{\frac{e^x - 5}{e^x}} + C$$

$$= \sqrt{\frac{e^x - 5}{e^x}} + C$$

AS FINANCING

$$e^x = \frac{1}{u} \quad u = \frac{1}{e^x}$$

$$\frac{du}{dx} = -e^{-x}$$

$$\frac{du}{dx} = -\frac{1}{e^x}$$

$$dx = -e^x du$$

$$dx = -\frac{du}{u}$$

Question 221 (****)

By using a reciprocal substitution, or otherwise, find the value of the following integral.

$$\int_1^2 \frac{x^2 - 1}{x^3 \sqrt{2x^4 - 2x^2 + 1}} dx.$$

$$\square, \frac{1}{8}$$

$$\begin{aligned}
 & \int_{-1}^1 \frac{x^3 - 1}{x^3 \sqrt{2x^4 - 2x^2 + 1}} dx \quad \dots \text{SUBSTITUTION} \\
 &= \int_{-1}^1 \frac{\frac{1}{u} - 1}{\frac{1}{u^3} \sqrt{u^4 - 2u^2 + 1}} \left(-\frac{1}{u^2} du \right) \\
 &= \int_{-1}^1 \frac{1}{u^3} \frac{1 - u^3}{\sqrt{u^4 - 2u^2 + 1}} du \\
 &= \int_{-1}^1 \frac{1}{u^3} \frac{1 - u^3}{\sqrt{u^4 - 2u^2 + 1}} du \\
 &= \int_{-1}^1 \frac{\frac{1}{u^3} - \frac{1}{u}}{\sqrt{u^4 - 2u^2 + 1}} du \\
 &\text{MULTIPLY TOP A BITTEN OF THE INVERSE BY } u^5 \\
 &= \int_{-1}^1 \frac{u - u^3}{\sqrt{u^4 - 2u^2 + 1}} du \\
 &\text{NOW! NOTICE THAT} \\
 &\frac{d}{dx} (u^4 - 2u^2 + 1) = 4u^3 - 4u = 4(u - u^3)
 \end{aligned}$$

$$\begin{aligned}
 u &= \frac{1}{x} \\
 \frac{1}{u} &= \frac{1}{x} \\
 \frac{du}{dx} &= -\frac{1}{x^2} du \\
 x=1 &\mapsto u=1 \\
 x=-2 &\mapsto u=-\frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 &\text{BECOMES AS} \\
 &= \int_{\frac{1}{2}}^1 (u - u^3) (u^4 - 2u^2 + 1)^{-\frac{1}{2}} du \\
 &= -\frac{1}{4} \left[\frac{1}{2} - 4(u - u^3)(u^4 - 2u^2 + 1)^{-\frac{1}{2}} \right]_{\frac{1}{2}}^1 \\
 &\text{BY RECOGNITION} \\
 &= -\frac{1}{4} \left[2(u^4 - 2u^2 + 1)^{-\frac{1}{2}} \right]_{\frac{1}{2}}^1 \\
 &= -\frac{1}{2} \left[\frac{1}{\sqrt{u^4 - 2u^2 + 1}} \right]_{\frac{1}{2}}^1 \\
 &= \frac{1}{2} \sqrt{\frac{1}{\frac{1}{16} - 1 + 1}} - \frac{1}{2} \\
 &= \frac{5}{8} - \frac{1}{2} \\
 &= \frac{1}{8}
 \end{aligned}$$

Question 222 (**) non calculator**

Show clearly that

$$\int_0^{\frac{\pi}{2} - \arctan \frac{12}{5}} 5 \cos x - 12 \sin x \, dx = 1.$$

proof

Handwritten solution for Question 222:

$$\begin{aligned} \int_0^{\frac{\pi}{2} - \arctan \frac{12}{5}} 5 \cos x - 12 \sin x \, dx &= \left[5 \sin x + 12 \cos x \right]_0^{\frac{\pi}{2} - \arctan \frac{12}{5}} \\ &= \left[5 \sin \left(\frac{\pi}{2} - \arctan \frac{12}{5} \right) + 12 \cos \left(\frac{\pi}{2} - \arctan \frac{12}{5} \right) \right] - [0 + 12] \\ &= 5 \sin \left(\frac{\pi}{2} - \arctan \frac{12}{5} \right) + 12 \cos \left(\frac{\pi}{2} - \arctan \frac{12}{5} \right) - 12 \\ &= 5 \sin \left(\frac{\pi}{2} - \alpha \right) + 12 \cos \left(\frac{\pi}{2} - \alpha \right) - 12 \\ &= 5 \times 1 \times \frac{5}{13} + 12 \times 1 \times \frac{12}{13} - 12 \\ &= \frac{25}{13} + \frac{144}{13} - 12 \\ &= \frac{169}{13} - 12 \\ &= 13 - 12 \\ &= 1 \end{aligned}$$

Diagram: A right-angled triangle with angle α , opposite side 12, and adjacent side 5. The hypotenuse is 13. The angle is $\alpha = \arctan \frac{12}{5}$.

Question 223 (**)**

Use trigonometric identities to find

$$\int \frac{1}{\cos x \sin^2 x} \, dx.$$

$\ln |\sec x + \tan x| - \operatorname{cosec} x + C$

Handwritten solution for Question 223:

$$\begin{aligned} \int \frac{1}{\cos x \sin^2 x} \, dx &= \int \frac{\operatorname{cosec}^2 x}{\cos x} \, dx = \int \frac{1 + \cot^2 x}{\cos x} \, dx \\ &= \int \frac{1}{\cos x} + \frac{\cot^2 x}{\cos x} \, dx = \int \sec x + \cot^2 x \sec x \, dx \\ &= \int \sec x + \frac{\cos^2 x}{\sin^2 x} \times \frac{1}{\cos x} \, dx = \int \sec x + \frac{\cos x}{\sin^2 x} \, dx \\ &= \int \sec x + \frac{\cos x}{\sin x} \times \frac{1}{\sin x} \, dx = \int \sec x + \cot x \operatorname{cosec} x \, dx \\ &= \ln |\sec x + \tan x| - \operatorname{cosec} x + C \end{aligned}$$

Alternative method for the integral:

$$\frac{1}{\cos x \sin^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos x \sin^2 x} = \frac{\cos^2 x}{\cos x \sin^2 x} + \frac{\sin^2 x}{\cos x \sin^2 x} = \frac{\cos x}{\sin^2 x} + \frac{1}{\cos x} = \dots \cot x \operatorname{cosec} x + \sec x \dots \text{etc}$$

Question 224

$$I = \int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{\cot^3 x}{\operatorname{cosec} x} dx.$$

Use appropriate integration techniques to show that

$$I = \frac{1}{6} [a + b\sqrt{3}],$$

where a and b are integers to be found.

$$\boxed{\frac{1}{6}}, I = \frac{1}{6} [15 - 7\sqrt{3}]$$

7. First try write the integrand in terms of sines and cosines

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos 2x}{\cos x} dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos^2 x \times \cos x}{\sin^2 x} dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos^2 x}{\sin^2 x} dx$$

8. By substitution

$$\begin{aligned} u &= \sin x \\ \frac{du}{dx} &= \cos x \\ dx &= \frac{du}{\cos x} \\ \frac{2+\frac{1}{u}}{\frac{1}{u}} &\rightarrow u + \frac{1}{u} \\ \frac{2+\frac{1}{u}}{\frac{1}{u}} &\rightarrow u + \frac{1}{u} \end{aligned}$$

9. We now have

$$\begin{aligned} \dots &= \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{\cos^2 x}{u^2} \frac{du}{\cos x} = \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{\cos^2 x}{u^2} du = \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{1 - \sin^2 x}{u^2} du \\ &= \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{1 - u^2}{u^2} du = \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{1}{u^2} - 1 du = \left[-\frac{1}{u} - u \right]_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \\ &= \left(-\frac{1}{u} + u \right) \Big|_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} = \left[\frac{1-u^2}{u} \right]_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \\ &= \frac{1 + \frac{1}{4}}{\frac{1}{2}} - \frac{1 + \frac{3}{4}}{\frac{\sqrt{3}}{2}} = \frac{4+1}{2} - \frac{4+3}{2\sqrt{3}} \\ &= \frac{5}{2} - \frac{7}{2\sqrt{3}} = \frac{5}{2} - \frac{7\sqrt{3}}{6} \\ &= \frac{1}{6} [15 - 7\sqrt{3}] \end{aligned}$$

Question 225 (****)

$$I = \int_1^a \frac{1}{\left(x^{\frac{4}{3}} + 7x\right)^{\frac{2}{3}}} dx.$$

Given that $I = 9$, determine the value a .

$$\boxed{}, \boxed{a = 8000}$$

ISOLATING THE UNIT & THE EQUATION WE HAVE

$$\int \frac{1}{(x^{\frac{4}{3}} + 7x)^{\frac{2}{3}}} dx = \int \frac{1}{(x(x^{\frac{1}{3}} + 7))^{\frac{2}{3}}} dx$$

$$= \int \frac{1}{x^{\frac{2}{3}}(x^{\frac{1}{3}} + 7)^{\frac{2}{3}}} dx = \int x^{-\frac{2}{3}}(x^{\frac{1}{3}} + 7)^{-\frac{2}{3}} dx$$

BY SUBSTITUTION AS $\frac{d}{dx}(x^{\frac{1}{3}}) = \frac{1}{3}x^{-\frac{2}{3}}$ (WE OBTAIN)

$$\frac{d}{dx} \left[(x^{\frac{1}{3}} + 7)^{-\frac{2}{3}} \right] = \frac{1}{3}(x^{\frac{1}{3}} + 7)^{-\frac{2}{3}} \times \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{9}x^{-\frac{2}{3}}(x^{\frac{1}{3}} + 7)^{-\frac{2}{3}}$$

THUS WE HAVE

$$\int \frac{1}{(x^{\frac{4}{3}} + 7x)^{\frac{2}{3}}} dx = 9(x^{\frac{1}{3}} + 7)^{-\frac{2}{3}}$$

FINDING THE LIMITS & EQUATION

$$\Rightarrow \int_1^a \frac{1}{(x^{\frac{4}{3}} + 7x)^{\frac{2}{3}}} dx = 9$$

$$\Rightarrow \left[9(x^{\frac{1}{3}} + 7)^{-\frac{2}{3}} \right]_1^a = 9$$

$$\Rightarrow (a^{\frac{1}{3}} + 7)^{-\frac{2}{3}} - 8^{-\frac{2}{3}} = 1$$

$$\Rightarrow (a^{\frac{1}{3}} + 7)^{-\frac{2}{3}} = 3$$

$$\Rightarrow a^{\frac{1}{3}} + 7 = 27$$

$$\Rightarrow a^{\frac{1}{3}} = 20 \quad \therefore a = 20^3 = 8000$$

Question 226 (****)

By using a suitable trigonometric substitution, show clearly that

$$\int_0^{\frac{1}{2}} \sqrt{\frac{16x}{1-x}} dx = \pi - 2.$$

proof

$\int_0^{\frac{1}{2}} \sqrt{\frac{16x}{1-x}} dx$ by substitution

$$= \int_0^{\frac{\pi}{6}} \sqrt{\frac{16 \sin^2 \theta}{1 - \sin^2 \theta}} (2 \cos \theta d\theta)$$

$$= \int_0^{\frac{\pi}{6}} \frac{16 \sin^2 \theta}{\cos \theta} (2 \cos \theta d\theta)$$

$$= \int_0^{\frac{\pi}{6}} \frac{4 \sin^2 \theta}{\cos \theta} (2 \cos \theta d\theta) = \int_0^{\frac{\pi}{6}} 8 \sin^2 \theta d\theta = \int_0^{\frac{\pi}{6}} 8 \left(\frac{1}{2} - \frac{1}{2} \cos 2\theta \right) d\theta$$

$$= \int_0^{\frac{\pi}{6}} (4 - 4 \cos 2\theta) d\theta = \left[4\theta - 2 \sin 2\theta \right]_0^{\frac{\pi}{6}} = \left(\pi - 2 \right) - (0 - 0) = \pi - 2$$

$x = \sin^2 \theta$
 $\frac{dx}{d\theta} = 2 \sin \theta \cos \theta$
 $dx = 2 \sin \theta \cos \theta d\theta$
 $x = 0, \theta = 0$
 $x = \frac{1}{2}, \theta = \frac{\pi}{6}$

Question 227 (****)

Find an exact simplified value for

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{\sin x \sin 2x} dx.$$

$$\frac{1}{2} \ln \left[\frac{2 + \sqrt{3}}{\sqrt{3}} \right] + 1 - \frac{\sqrt{3}}{3}$$

Handwritten solution for the integral problem:

$$\begin{aligned} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{\sin x \sin 2x} dx &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos 2x + \sin 2x}{2 \sin x \cos x} dx \\ &= \frac{1}{2} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos 2x}{\sin x \cos x} + \sec x dx = \frac{1}{2} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos 2x}{\sin x} + \sec x dx \\ &= \frac{1}{2} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (\cot x \cos x + \sec x) dx = \frac{1}{2} \left[-\cos x + \ln |\sec x + \tan x| \right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} \\ &= \frac{1}{2} \left[\left(\ln(2 + \sqrt{3}) - \frac{2}{\sqrt{3}} \right) - \left(-2 + \ln \left(\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} \right) \right) \right] \\ &= \frac{1}{2} \left[\ln(2 + \sqrt{3}) - \frac{2}{\sqrt{3}} + 2 - \ln \sqrt{3} \right] \\ &= \frac{1}{2} \left[\ln \left(\frac{2 + \sqrt{3}}{\sqrt{3}} \right) + 2 \left(1 - \frac{1}{\sqrt{3}} \right) \right] \\ &= \frac{1}{2} \ln \left(\frac{2 + \sqrt{3}}{\sqrt{3}} \right) + 1 - \frac{\sqrt{3}}{3} \end{aligned}$$

Question 228 (****)

Evaluate the following definite integral.

$$\int_0^1 e^{\arccos x} dx.$$

Give the answer in exact simplified form.

□, proof

START WITH A SUBSTITUTION

$$\int_0^1 e^{\arccos x} dx = \int_{\frac{\pi}{2}}^0 e^{\arccos x} (-\sin \theta d\theta)$$

$$= \int_{\frac{\pi}{2}}^0 e^{\arccos x} \sin \theta d\theta$$

$\theta = \arccos x$
 $x = \cos \theta$
 $dx = -\sin \theta d\theta$
 $x=0 \quad \theta = \frac{\pi}{2}$
 $x=1 \quad \theta = 0$

USE THE CHAIN RULE - CE BY DOUBLE

INTEGRATION BY PARTS - CE BY DOUBLE

$$\frac{1}{\sin \theta} \left[e^{\arccos x} \sin \theta \right] = e^{\arccos x}$$

$$e^{\arccos x} (\cos \theta) + e^{\arccos x} (-\sin \theta) = e^{\arccos x}$$

$$(A-B) \sin \theta + (A+B) \cos \theta = \sin \theta$$

$$A = \frac{1}{2}, B = -\frac{1}{2}$$

THIS IS THE ANSWER

$$= \left[e^{\arccos x} \left(\frac{1}{2} \sin \theta - \frac{1}{2} \cos \theta \right) \right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2} \left[e^{\arccos(1)} (1-0) - e^{\arccos(0)} (0-1) \right]$$

$$= \frac{1}{2} \left[e^0 + 1 \right]$$

Question 229 (****)

Use a suitable trigonometric substitution to find the exact value of

$$\int_{-1}^5 \sqrt{(1+x)(5-x)} \, dx.$$

$$\frac{9\pi}{2}$$

$$\int_{-1}^5 \sqrt{(1+x)(5-x)} \, dx = \int_{-1}^5 \sqrt{x^2 - 2x + 10} \, dx = \int_{-1}^5 \sqrt{x^2 - 2x + 1 + 9} \, dx$$

$$= \int_{-1}^5 \sqrt{(x-1)^2 + 9} \, dx = \int_{-1}^5 \sqrt{(x-1)^2 + 3^2} \, dx$$

Now by trigonometric substitution

$$x-1 = 3 \sin \theta$$

$$x = 1 + 3 \sin \theta$$

$$dx = 3 \cos \theta \, d\theta$$

$$x = -1 \Rightarrow \sin \theta = -\frac{2}{3} \Rightarrow \theta = -\frac{\pi}{6}$$

$$x = 5 \Rightarrow \sin \theta = \frac{4}{3} \Rightarrow \theta = \frac{\pi}{2}$$

$$= \int_{-\pi/6}^{\pi/2} \sqrt{9 \cos^2 \theta} (3 \cos \theta \, d\theta) = \int_{-\pi/6}^{\pi/2} 9 \cos^3 \theta \, d\theta$$

$$= \int_{-\pi/6}^{\pi/2} 9 \cos \theta (1 - \sin^2 \theta) \, d\theta = \int_{-\pi/6}^{\pi/2} (9 \cos \theta - 9 \cos \theta \sin^2 \theta) \, d\theta$$

$$= \int_{-\pi/6}^{\pi/2} 9 \cos \theta \, d\theta - \int_{-\pi/6}^{\pi/2} 9 \sin^2 \theta \cos \theta \, d\theta$$

$$= \left[9 \sin \theta - \frac{9}{3} \sin^3 \theta \right]_{-\pi/6}^{\pi/2}$$

$$= \left(\frac{9\pi}{2} + 0 \right) - \left(0 + 0 \right) = \frac{9\pi}{2}$$

Question 230 (****)

By using trigonometric identities, show that

$$\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cos^2 x} dx = \frac{1}{8}(16 - 3\pi).$$

, proof

PROCEED BY SPLITTING THE FRACTION

$$\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cos^2 x} dx = \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \frac{\sin^4 x}{\sin^2 x \cos^2 x} + \frac{\cos^4 x}{\sin^2 x \cos^2 x} dx$$

$$= \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\sin^2 x} dx = \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \frac{(1 - \cos^2 x)^2}{\cos^2 x} + \frac{(1 - \sin^2 x)^2}{\sin^2 x} dx$$

EXPANDING & SPLIT THE FRACTIONS AGAIN

$$= \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \frac{1 - 2\cos^2 x + \cos^4 x}{\cos^2 x} + \frac{1 - 2\sin^2 x + \sin^4 x}{\sin^2 x} dx$$

$$= \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \sec^2 x - 2 + \sec^4 x + \csc^2 x - 2 + \csc^4 x dx$$

$$= \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \sec^2 x + \csc^2 x + (\sec^4 x + \csc^4 x) - 4 dx$$

$$= \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \sec^2 x + \csc^2 x - 3 dx$$

$$= \left[\tan x - \cot x - 3x \right]_{\frac{\pi}{8}}^{\frac{\pi}{4}}$$

TRY BASIC EVALUATING

$$= \left[\tan x - \frac{1}{\tan x} - 3x \right]_{\frac{\pi}{8}}^{\frac{\pi}{4}}$$

$$= \left[\frac{\tan x - 1}{\tan x} - 3x \right]_{\frac{\pi}{8}}^{\frac{\pi}{4}}$$

$$= \left[3x + \frac{1 - \tan x}{\tan x} \right]_{\frac{\pi}{8}}^{\frac{\pi}{4}}$$

$$= \left[3x + 2\left(\frac{1 - \tan x}{2 \tan x}\right) \right]_{\frac{\pi}{8}}^{\frac{\pi}{4}}$$

$$= \left[3x + \frac{2}{\tan x} \right]_{\frac{\pi}{8}}^{\frac{\pi}{4}}$$

$$= \left(\frac{3\pi}{4} + \frac{2}{\tan \frac{\pi}{4}} \right) - \left(\frac{3\pi}{8} + \frac{2}{\tan \frac{\pi}{8}} \right)$$

$$= -\frac{3\pi}{8} + 2$$

$$= \frac{1}{8}(16 - 3\pi)$$

Question 231 (****)

Find, in exact simplified form, the value of the following integral.

$$\int_0^{\frac{1}{2}\pi} \sqrt{1 + 4\cos^2 2x - 4\cos 2x} \, dx.$$

$$\boxed{\frac{1}{6}\pi + \sqrt{3}}$$

MANIPULATING AS FOCUS

$$\int_0^{\frac{\pi}{2}} \sqrt{1 + 4\cos^2 2x - 4\cos 2x} \, dx = \int_0^{\frac{\pi}{2}} \sqrt{(2\cos 2x - 1)^2} \, dx$$

$$= \int_0^{\frac{\pi}{2}} |2\cos 2x - 1| \, dx$$

NOW THE CRITICAL VALUES FOR THE INTEGRAND

$$2\cos 2x - 1 = 0$$

$$\cos 2x = \frac{1}{2}$$

$$2x = \pm \frac{\pi}{3}, \pm \frac{5\pi}{3}, \dots$$

$$x = \pm \frac{\pi}{6}, \pm \frac{5\pi}{6}, \dots$$

$x = \frac{\pi}{6}$ IS A CRITICAL VALUE FOR THE INTEGRATION INTERVAL

$$\begin{array}{ll} 2\cos 2x - 1 > 0 & 0 < x < \frac{\pi}{6} \\ 2\cos 2x - 1 < 0 & \frac{\pi}{6} < x < \frac{\pi}{2} \end{array}$$

RETURNING TO THE INTEGRAL

$$\begin{aligned} \dots &= \int_0^{\frac{\pi}{6}} (2\cos 2x - 1) \, dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} -(2\cos 2x - 1) \, dx \\ &= \left[\sin 2x - x \right]_0^{\frac{\pi}{6}} + \left[-x - \sin 2x \right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} \\ &= \left(\frac{\sqrt{3}}{2} - \frac{\pi}{6} \right) - (0) + \left(-\frac{\pi}{2} - 0 \right) - \left(-\frac{\pi}{6} - \frac{\sqrt{3}}{2} \right) \\ &= -\frac{\sqrt{3}}{2} \times 2 + \frac{\pi}{3} - \frac{\pi}{2} = \sqrt{3} + \frac{\pi}{6} \end{aligned}$$

Question 232 (****)

By using a suitable cosine double angle trigonometric identity find

$$\int \frac{6}{(1 + \cos x)^2} \, dx.$$

$$\boxed{3 \tan \frac{x}{2} + \tan^3 \frac{x}{2} + C}$$

$$\begin{aligned} \int \frac{6}{(1 + \cos x)^2} \, dx &= \int \frac{6}{(1 + 2\cos^2 \frac{x}{2} - 1)^2} \, dx = \int \frac{6}{4\cos^4 \frac{x}{2}} \, dx \quad \cos 2\theta = 2\cos^2 \theta - 1 \\ &= \int \frac{3}{2\cos^4 \frac{x}{2}} \, dx = \int \frac{3}{2} \sec^4 \frac{x}{2} \, dx = \int \frac{3}{2} \sec^2 \frac{x}{2} (1 + \tan^2 \frac{x}{2}) \, dx \\ &= \int \frac{3}{2} \sec^2 \frac{x}{2} + \frac{3}{2} \sec^2 \frac{x}{2} \tan^2 \frac{x}{2} \, dx = \dots \text{by recognition} \\ &= 3 \tan \frac{x}{2} + \tan^3 \frac{x}{2} + C \end{aligned}$$

Question 233 (****)

By expressing the integrand in the form $\sec^2 x f(\tan x)$, or otherwise, find a simplified expression for the following integral.

$$\int \frac{3 \sin^2 x \cos^2 x}{(\cos^3 x - \sin^3 x)^2} dx.$$

$$\boxed{}, \quad \boxed{\frac{1}{1 - \tan^3 x} + C}$$

PROCEED AS BEFORE

$$\int \frac{3 \sin^2 x \cos^2 x}{(\cos^3 x - \sin^3 x)^2} dx = \int 3 \left(\frac{\sin x \cos x}{\cos^3 x - \sin^3 x} \right)^2 dx$$

$$= \int 3 \left[\frac{\sin x \cos x}{\cos^3 x - \sin^3 x} \right]^2 dx = \int \frac{3 (\tan x \sec x)^2}{(1 - \tan^3 x)^2} dx$$

$$= \int 3 \tan^2 x \sec^2 x (1 - \tan^3 x)^{-2} dx$$

BY REVERSE CHAIN RULE WE OBTAIN

$$\dots = (1 - \tan^3 x)^{-1} + C = \frac{1}{1 - \tan^3 x} + C$$

ALTERNATIVE BY SUBSTITUTION $u = \tan x$ OR $u = 1 - \tan^3 x$

$$\dots = \int 3 \tan^2 x \sec^2 x (1 - \tan^3 x)^{-2} dx$$

$$= \int 3 \tan^2 x \sec^2 x u^{-2} \left(\frac{du}{3 \sec^2 x \tan^2 x} \right)$$

$$= \int -u^{-2} du$$

$$= u^{-1} + C$$

$$= \frac{1}{1 - \tan^3 x} + C$$

AS BEFORE

$u = 1 - \tan^3 x$
 $\frac{du}{dx} = -3 \sec^2 x \tan^2 x$
 $dx = \frac{-1}{3 \sec^2 x \tan^2 x} du$

Question 234 (****)

$$\sec x \equiv \frac{1 + \tan^2\left(\frac{x}{2}\right)}{1 - \tan^2\left(\frac{x}{2}\right)}$$

a) Prove the validity of the above trigonometric identity.

b) Express $\frac{2}{1-t^2}$ into partial fractions.

c) Hence use the substitution $t = \tan\left(\frac{x}{2}\right)$ to show that

$$\int \sec x \, dx = \ln \left| \tan\left(\frac{x}{2} + \frac{\pi}{4}\right) \right| + C.$$

$$\frac{2}{1-t^2} = \frac{1}{1+t} + \frac{1}{1-t}$$

(a) $\sec x = \frac{1 + \tan^2\left(\frac{x}{2}\right)}{1 - \tan^2\left(\frac{x}{2}\right)} = \frac{1 + \frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}}{1 - \frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}} = \dots$ Multiply numerator & denominator by $\cos^2\frac{x}{2}$
 $= \frac{\cos^2\frac{x}{2} + \sin^2\frac{x}{2}}{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}} = \frac{1}{\cos(2 \times \frac{x}{2})} = \frac{1}{\cos x} = \sec x = \text{p.f.s}$
 $\cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \cos x$

(b) $\frac{2}{1-t^2} = \frac{2}{(1-t)(1+t)} = \frac{A}{1-t} + \frac{B}{1+t}$
 $\frac{2}{1-t^2} = \frac{A(1+t) + B(1-t)}{(1-t)(1+t)}$
 $2 = A(1+t) + B(1-t)$
 $\text{If } t=1, 2=2A, A=1$
 $\text{If } t=-1, 2=2B, B=1$
 $\text{Thus } \frac{2}{1-t^2} = \frac{1}{1-t} + \frac{1}{1+t}$

(c) $\int \sec x \, dx = \dots$ By part (a)
 $= \int \frac{1 + \tan^2\frac{x}{2}}{1 - \tan^2\frac{x}{2}} \, dx = \dots$ By substitution...
 $= \int \frac{1+t^2}{1-t^2} \left(\frac{2}{1+t^2} \, dt\right) = \int \frac{2}{1-t^2} \, dt$
 $= \int \frac{1}{1-t} + \frac{1}{1+t} \, dt = \ln|1-t| - \ln|1+t| + C$
 $= \ln\left|\frac{1-t}{1+t}\right| + C = \ln\left|\frac{1 - \tan\frac{x}{2}}{1 + \tan\frac{x}{2}}\right| + C$
 $\text{But } \tan\frac{x}{2} = t$
 $= \ln\left|\frac{1 - \tan\frac{x}{2}}{1 + \tan\frac{x}{2}}\right| + C = \ln\left|\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right| + C$

Side note: $t = \tan\frac{x}{2}$
 $\frac{dx}{dt} = \frac{1}{1+t^2}$
 $dx = \frac{2}{1+t^2} \, dt$
 $dx = \frac{2}{1+t^2} \, dt$

Question 235 (*****)

By using the substitution $e^x = \frac{1}{t}$, or otherwise, show clearly that

$$\int \frac{4}{e^x \sqrt{e^{2x} + 4}} dx = -\frac{\sqrt{e^{2x} + 4}}{e^x} + C.$$

proof

Handwritten solution for Question 235:

By substitution $e^x = \frac{1}{t}$

$$\int \frac{4}{e^x \sqrt{e^{2x} + 4}} dx = \int \frac{4}{\frac{1}{t} \sqrt{\frac{1}{t^2} + 4}} \left(-\frac{dt}{t^2}\right) = \int \frac{4t}{\sqrt{\frac{1}{t^2} + 4}} \left(-\frac{dt}{t^2}\right)$$

$$= \int \frac{-4}{\sqrt{1 + 4t^2}} dt = \int \frac{-4}{\sqrt{1 + 4t^2}} dt$$

$$= \int \frac{-4t}{\sqrt{1 + 4t^2}} dt = \int -4t (1 + 4t^2)^{-\frac{1}{2}} dt$$

By reverse chain rule or another substitution

$$= -(1 + 4t^2)^{\frac{1}{2}} + C = -\left(1 + \frac{4}{e^{2x}}\right)^{\frac{1}{2}} + C$$

$$= -\left(\frac{e^{2x} + 4}{e^{2x}}\right)^{\frac{1}{2}} + C = -\frac{\sqrt{e^{2x} + 4}}{e^x} + C \quad \text{As required}$$

Side note: $e^x = \frac{1}{t}$
 $\frac{d}{dx} \frac{1}{t} = -\frac{1}{t^2} \frac{dt}{dx}$
 $\frac{d}{dx} \frac{1}{t} = -\frac{1}{t^2} \frac{dt}{dx}$
 $\frac{dx}{dt} = -\frac{1}{t}$
 $\frac{dx}{dt} = -\frac{1}{t}$
 $\frac{dx}{dt} = -\frac{1}{t}$

Question 236 (*****)

Use the fact that $\sin A \equiv \cos\left(\frac{\pi}{2} - A\right)$ and other trigonometric identities to show that

$$\int_{\frac{5\pi}{12}}^{\frac{\pi}{4}} \frac{4}{\sqrt{1 - \sin 2x}} dx = 2 \ln 3.$$

proof

Handwritten solution for Question 236:

$$\int_{\frac{5\pi}{12}}^{\frac{\pi}{4}} \frac{4}{\sqrt{1 - \sin 2x}} dx = \int_{\frac{5\pi}{12}}^{\frac{\pi}{4}} \frac{4}{\sqrt{1 - \cos(\frac{\pi}{2} - 2x)}} dx$$

Now $\cos 2A = 1 - 2\sin^2 A$

$$\dots = \int_{\frac{5\pi}{12}}^{\frac{\pi}{4}} \frac{4}{\sqrt{1 - 2\sin^2(\frac{\pi}{4} - x)}} dx$$

$$= \int_{\frac{5\pi}{12}}^{\frac{\pi}{4}} \frac{4}{\sqrt{2\cos^2(\frac{\pi}{4} - x)}} dx = \frac{4}{\sqrt{2}} \int_{\frac{5\pi}{12}}^{\frac{\pi}{4}} \frac{1}{\cos(\frac{\pi}{4} - x)} dx$$

Standard result $\int \sec u du = \ln|\sec u| + C$

$$= 2\sqrt{2} \int_{\frac{5\pi}{12}}^{\frac{\pi}{4}} \sec(\frac{\pi}{4} - x) dx = \dots$$

$$= 2\sqrt{2} \left[-\ln|\tan(\frac{\pi}{4} - x)| \right]_{\frac{5\pi}{12}}^{\frac{\pi}{4}} = -2\sqrt{2} \left[\ln|\tan(\frac{\pi}{4} - \frac{\pi}{4})| - \ln|\tan(\frac{\pi}{4} - \frac{5\pi}{12})| \right]$$

$$= -2\sqrt{2} \left[\ln|1| - \ln|\sqrt{3}| \right] = -2\sqrt{2} \left[0 - \ln\sqrt{3} \right] = 2\sqrt{2} \ln\sqrt{3} = 2\sqrt{2} \times \frac{1}{2} \ln 3$$

$$= 2\ln 3$$

Question 237 (*****)

Use the substitution $x = \frac{1}{u}$ to find the value of

$$\int_{\frac{1}{2}}^2 \frac{\ln x}{1+x^2} dx.$$

0

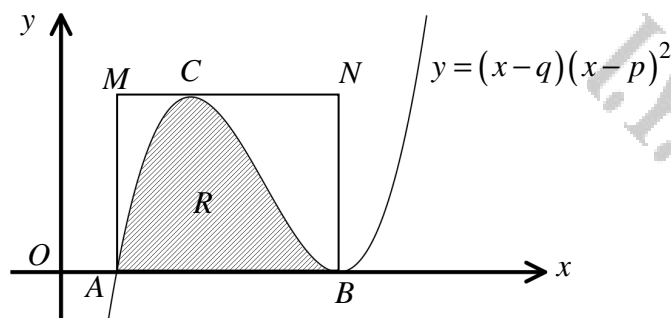
Question 238 (*****)

Use the substitution $x = \frac{1}{u^2+1}$ to show that

$$\int_{0.2}^{0.5} \frac{\sqrt{x-x^2}}{x^4} dx = \frac{256}{15}.$$

proof

Question 239 (****)



The figure above shows the graph of the curve with equation

$$y = (x-q)(x-p)^2,$$

where p and q are positive constants.

The curve meets the x axis at the points A and B . The region R , shown shaded in the figure, is bounded by the curve and the x axis.

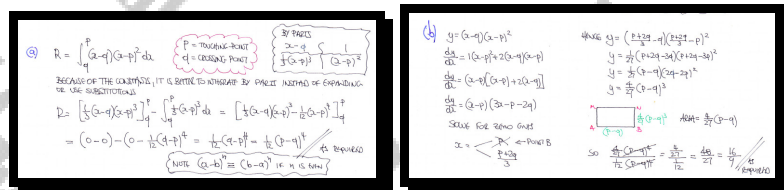
- a) Show that the area of the shaded region is

$$\frac{1}{12}(p-q)^4.$$

The point C is the local maximum of the curve. The rectangle $AMCN$ is such so that MCN is parallel to the x axis and both AM and BN are parallel to the y axis.

- b) Show that the area of the rectangle $AMCN$ is $\frac{16}{9}$ times as large as the area of R , regardless of the values of p and q .

proof



Question 240 (****)

$$f(x) = \frac{\sin 3x}{(\cos 7x + \cos x)^2 + (\sin 7x + \sin x)^2}, \quad x \in \mathbb{R}.$$

Use trigonometric identities to find the exact value of

$$\int_{\frac{1}{12}\pi}^{\frac{1}{9}\pi} f(x) \, dx.$$

$$\boxed{}, \quad \frac{2 - \sqrt{2}}{12}$$

EXPAND THE DENOMINATOR & TIDY UP

$$\int_{\frac{1}{12}\pi}^{\frac{1}{9}\pi} \frac{\sin 3x}{(\cos 7x + \cos x)^2 + (\sin 7x + \sin x)^2} \, dx$$

$$= \int_{\frac{1}{12}\pi}^{\frac{1}{9}\pi} \frac{\sin 3x}{\cos^2 7x + 2\cos 7x \cos x + \cos^2 x + \sin^2 7x + 2\sin 7x \sin x + \sin^2 x} \, dx$$

$$= \int_{\frac{1}{12}\pi}^{\frac{1}{9}\pi} \frac{\sin 3x}{1 + 1 + 2(\cos 7x \cos x + \sin 7x \sin x)} \, dx$$

$$= \int_{\frac{1}{12}\pi}^{\frac{1}{9}\pi} \frac{\sin 3x}{2 + 2\cos(7x - x)} \, dx$$

$$= \int_{\frac{1}{12}\pi}^{\frac{1}{9}\pi} \frac{\sin 3x}{2 + 2\cos 6x} \, dx$$

$$= \int_{\frac{1}{12}\pi}^{\frac{1}{9}\pi} \frac{\sin 3x}{4\cos^2 3x} \, dx$$

$$= \int_{\frac{1}{12}\pi}^{\frac{1}{9}\pi} \frac{\sin 3x}{4} \cdot \frac{1}{\cos^2 3x} \, dx$$

$$= \int_{\frac{1}{12}\pi}^{\frac{1}{9}\pi} \frac{1}{4} \sec 3x \, dx$$

$$= \left[\frac{1}{4} \sec 3x \right]_{\frac{1}{12}\pi}^{\frac{1}{9}\pi}$$

$$= \frac{1}{4} \left[\frac{1}{\cos 3x} \right]_{\frac{1}{12}\pi}^{\frac{1}{9}\pi}$$

$$= \frac{1}{4} \left[\frac{1}{\frac{1}{2}} - \frac{1}{\frac{\sqrt{3}}{2}} \right] = \frac{2 - \sqrt{3}}{4}$$

cos 2θ = 2cos² θ - 1
cos 6x = 2cos² 3x - 1
2cos² 3x = 4cos² 3x - 2
2 + 2cos 6x = 4cos² 3x

$\frac{d}{dx}(\sec) = \sec \tan$

$\frac{2 - \sqrt{2}}{12}$

Question 241 (****)

Use a suitable trigonometric manipulation to find an exact simplified answer for the following integral.

$$\int_0^{\frac{\pi}{3}} \frac{1}{(\cos x + \sqrt{3} \sin x)^2} dx.$$

$$\boxed{}, \frac{1}{4}\sqrt{3}$$

START WITH THE DENOMINATOR, BY AN ID-TRANSFORMATION
 OR A MANIPULATION

$$\begin{aligned} \cos x + \sqrt{3} \sin x &= 2 \left[\frac{1}{2} \cos x + \frac{\sqrt{3}}{2} \sin x \right] \\ &= 2 \left[\cos \frac{\pi}{6} \cos x + \sin \frac{\pi}{6} \sin x \right] \\ &= 2 \cos \left(\frac{\pi}{6} - x \right) \\ &= 2 \cos \left(x - \frac{\pi}{6} \right) \end{aligned}$$

HENCE THE INTEGRAL BECOMES

$$\begin{aligned} \int_0^{\frac{\pi}{3}} \frac{1}{(\cos x + \sqrt{3} \sin x)^2} dx &= \int_0^{\frac{\pi}{3}} \frac{1}{[2 \cos(x - \frac{\pi}{6})]^2} dx \\ &= \int_0^{\frac{\pi}{3}} \frac{1}{4 \cos^2(x - \frac{\pi}{6})} dx = \int_0^{\frac{\pi}{3}} \frac{1}{4} \sec^2(x - \frac{\pi}{6}) dx \\ &= \left[\frac{1}{4} \tan(x - \frac{\pi}{6}) \right]_0^{\frac{\pi}{3}} = \frac{1}{4} \left[\tan 0 - \tan \left(-\frac{\pi}{6} \right) \right] \\ &= \frac{1}{4} \left(0 + \tan \frac{\pi}{6} \right) = \frac{1}{4} \sqrt{3} \end{aligned}$$

Question 242 (****)

$$f(x) = 3\sin x - \cos x + 3, \quad x \in \mathbb{R}.$$

$$g(x) = \sin x + \cos x, \quad x \in \mathbb{R}.$$

- a) Express $f(x)$ in the form

$$A \times g(x) + B \times g'(x) + 3,$$

where A and B are constants.

- b) Express $g(x)$ in the form

$$R \cos(x - \phi),$$

where R and ϕ are positive constants.

- c) Hence find a simplified expression for

$$\int \frac{f(x)}{g(x)} dx.$$

$$\boxed{}, \boxed{A=1}, \boxed{B=-2}, \boxed{R=\sqrt{2}}, \boxed{\phi=\frac{1}{4}\pi},$$

$$x - 2 \ln |\sin x + \cos x| + \frac{3}{2} \sqrt{2} \ln \left| \sec\left(x - \frac{1}{4}\pi\right) + \tan\left(x - \frac{1}{4}\pi\right) \right| + C$$

Handwritten solution for Question 242:

a) DIFFERENTIATE $g(x)$
 $g(x) = \sin x + \cos x$
 $g'(x) = \cos x - \sin x$
 EQUATE & COMPARE COEFFICIENTS
 $\Rightarrow f(x) = A \cdot g(x) + B \cdot g'(x) + 3$
 $\Rightarrow 3\sin x - \cos x + 3 = A(\sin x + \cos x) + B(\cos x - \sin x) + 3$
 $\Rightarrow 3\sin x - \cos x = (A - B)\sin x + (A + B)\cos x$
 $\begin{cases} A - B = 3 \\ A + B = -1 \end{cases} \Rightarrow \begin{matrix} 2A = 2 \\ A = 1 \end{matrix} \quad \begin{matrix} B = -2 \end{matrix}$
 $\Rightarrow f(x) = g(x) - 2g'(x) + 3$

b) $g(x) = \sin x + \cos x$
 $\Rightarrow g(x) = \sqrt{2} \left(\frac{1}{\sqrt{2}} \sin x + \frac{1}{\sqrt{2}} \cos x \right)$
 $\Rightarrow g(x) = \sqrt{2} \left(\sin\left(x - \frac{\pi}{4}\right) + \cos\left(x - \frac{\pi}{4}\right) \right)$
 $\Rightarrow g(x) = \sqrt{2} \cos\left(x - \frac{\pi}{4}\right)$
 OR BE DONE ALSO BY COMPARING
 $\sin x + \cos x = R \cos\left(x - \frac{\pi}{4}\right)$

c) $\int \frac{f(x)}{g(x)} dx = \int \frac{g(x) - 2g'(x) + 3}{g(x)} dx$
 $= \int 1 + \frac{2g'(x)}{g(x)} + \frac{3}{g(x)} dx$
 $= \int 1 dx + 2 \int \frac{g'(x)}{g(x)} dx + \int \frac{3}{g(x)} dx$
 $= x + 2 \ln |g(x)| + \int \frac{3}{\sqrt{2} \cos\left(x - \frac{\pi}{4}\right)} dx$
 $= x + 2 \ln |g(x)| + \frac{3}{\sqrt{2}} \int \sec\left(x - \frac{\pi}{4}\right) dx$
 NOTING THAT $\int \sec u du = \ln |\sec u + \tan u| + C$
 $\Rightarrow \int \frac{f(x)}{g(x)} dx = x + 2 \ln |\sin x + \cos x| + \frac{3}{\sqrt{2}} \ln \left| \sec\left(x - \frac{\pi}{4}\right) + \tan\left(x - \frac{\pi}{4}\right) \right| + C$

Question 243 (****)

Use the substitution $x = 2\sec\theta$, to find a simplified expression for

$$\int \frac{6}{(x^2 - 4)^{\frac{3}{2}}} dx.$$

$$-\frac{x}{\sqrt{x^2 - 4}} + C$$

Handwritten solution for the integral using the substitution $x = 2\sec\theta$.

Given: $x = 2\sec\theta$

Then: $\frac{dx}{d\theta} = 2\sec\theta \tan\theta$
 $dx = 2\sec\theta \tan\theta d\theta$

Substitution into the integral:

$$\int \frac{6}{(x^2 - 4)^{\frac{3}{2}}} dx = \int \frac{6}{(4\sec^2\theta - 4)^{\frac{3}{2}}} (2\sec\theta \tan\theta d\theta)$$

$$= \int \frac{12\sec\theta \tan\theta}{[4(\sec^2\theta - 1)]^{\frac{3}{2}}} d\theta$$

$$= \int \frac{12\sec\theta \tan\theta}{(4\tan^2\theta)^{\frac{3}{2}}} d\theta$$

$$= \int \frac{12\sec\theta \tan\theta}{8\tan^3\theta} d\theta$$

$$= \int \frac{3\sec\theta}{2\tan^2\theta} d\theta$$

$$= \int \frac{3\sec\theta}{2\frac{\sin^2\theta}{\cos^2\theta}} d\theta$$

$$= \int \frac{3\cos^3\theta}{2\sin^2\theta} d\theta$$

$$= \int \frac{3\cos\theta}{2} d\theta \quad (\text{by recognizing } \frac{\cos^3\theta}{\sin^2\theta} = \cos\theta)$$

$$= \frac{3}{2} \int \cos\theta d\theta$$

$$= \frac{3}{2} \sin\theta + C$$

Alternative method:

$$\int \frac{3\cos\theta}{2} d\theta = \frac{3}{2} \int \cos\theta d\theta = \frac{3}{2} \sin\theta + C$$

Using the triangle:

Right-angled triangle with hypotenuse x , adjacent side 2 , and opposite side $\sqrt{x^2 - 4}$.

Then: $\sec\theta = \frac{x}{2}$
 $\sin\theta = \frac{\sqrt{x^2 - 4}}{x}$

Final answer: $-\frac{x}{\sqrt{x^2 - 4}} + C$

Question 244 (*****)

Find an exact simplified value for

$$\int_0^1 \frac{x^2 - 3x + 1}{\sqrt{1-x^2}} dx.$$

$$\boxed{\frac{3}{4}[\pi - 4]}$$

Handwritten solution for Question 244:

$$\begin{aligned} \int_0^1 \frac{x^2 - 3x + 1}{\sqrt{1-x^2}} dx &= \dots \text{ BY SUBSTITUTION} \\ &= \int_0^{\frac{\pi}{2}} \frac{\sin^2 \theta - 3\sin \theta + 1}{\sqrt{1-\sin^2 \theta}} (\cos \theta d\theta) \\ &= \int_0^{\frac{\pi}{2}} \frac{\sin^2 \theta - 3\sin \theta + 1}{\cos \theta} \times \cos \theta d\theta \\ &= \int_0^{\frac{\pi}{2}} \frac{1}{2} (\cos 2\theta - 3\sin \theta + 1) d\theta \\ &= \int_0^{\frac{\pi}{2}} \frac{1}{2} + \frac{1}{2} \cos 2\theta - 3\sin \theta d\theta \\ &= \left[\frac{\theta}{2} + \frac{1}{4} \sin 2\theta + 3\cos \theta \right]_0^{\frac{\pi}{2}} \\ &= \left(\frac{\frac{\pi}{2}}{2} + 0 + 0 \right) - \left(0 + 0 + 3 \right) \\ &= \frac{\pi}{4} - 3 \\ &= \frac{3}{4}[\pi - 4] \end{aligned}$$

Side note: $x = \sin \theta$
 $dx = \cos \theta d\theta$
 $\theta = \arcsin x$
 $x=0 \rightarrow \theta=0$
 $x=1 \rightarrow \theta=\frac{\pi}{2}$

Question 245 (*****)

Given that a and b are integers, evaluate

$$\int_{-\pi}^{\pi} (\cos ax - \sin bx)^2 dx.$$

$$\boxed{2\pi}$$

Handwritten solution for Question 245:

$$\begin{aligned} \int_{-\pi}^{\pi} (\cos ax - \sin bx)^2 dx &= \int_{-\pi}^{\pi} \frac{\cos^2 ax - 2\cos ax \sin bx + \sin^2 bx}{1} dx \\ &= 2 \int_0^{\pi} \cos^2 ax + \sin^2 bx dx = 2 \int_0^{\pi} \frac{1}{2} + \frac{1}{2} \cos 2ax + \frac{1}{2} - \frac{1}{2} \cos 2bx dx \\ &= \int_0^{\pi} 1 + \cos 2ax - \cos 2bx dx = \left[x + \frac{1}{2a} \sin 2ax - \frac{1}{2b} \sin 2bx \right]_0^{\pi} \\ &= \left[\pi + \frac{1}{2a} \sin(2a\pi) - \frac{1}{2b} \sin(2b\pi) \right] - \left[0 + 0 + 0 \right] \\ &\quad \text{BUT IF } a, b \text{ ARE INTEGERS } \sin(2a\pi) = \sin(2b\pi) = 0 \\ &= 2\pi \end{aligned}$$

Question 246 (****)

$$I = \int_{1.5}^2 \frac{(x-2)(2x^2-5x-1)}{(x-1)(x-3)} dx.$$

Use appropriate integrations techniques to show that

$$I = \frac{5}{4} - \ln k,$$

where k is a positive integer.

$$\boxed{}, \quad k = \boxed{6}$$

$\int_{1.5}^2 \frac{(x-2)(2x^2-5x-1)}{(x-1)(x-3)} dx = \frac{5}{4} - \ln k$

• FIRSTLY THE INTEGRAND IS IMPROPER — SO WE DIVIDE IT IN ORDER TO DIVIDE IT OUT

$$\frac{(x-2)(2x^2-5x-1)}{(x-1)(x-3)} = \frac{2x^3-5x^2-2x+2}{2x^2-5x+2}$$

$$\frac{2x^3-5x^2-2x+2}{2x^2-5x+2} = 2x-1 + \frac{3x-4}{2x^2-5x+2}$$

• NOW WE HAVE TO FIND THE PARTIAL FRACTIONS

$$\frac{3x-4}{2x^2-5x+2} = \frac{A}{x-1} + \frac{B}{x-3}$$

$$\frac{3x-4}{(x-1)(x-3)} = \frac{A}{x-1} + \frac{B}{x-3}$$

$$3x-4 = A(x-3) + B(x-1)$$

$$3x-4 = Ax-3A+Bx-B$$

$$3x-4 = (A+B)x - 3A - B$$

$$\begin{cases} A+B=3 \\ -3A-B=-4 \end{cases}$$

$$4A=7 \Rightarrow A=\frac{7}{4}$$

$$B=3-A=3-\frac{7}{4}=\frac{5}{4}$$

• THEREFORE

$$\frac{3x-4}{2x^2-5x+2} = \frac{7}{4(x-1)} + \frac{5}{4(x-3)}$$

$$\frac{(x-2)(2x^2-5x-1)}{(x-1)(x-3)} = 2x-1 + \frac{7}{4(x-1)} + \frac{5}{4(x-3)}$$

• NOW WE CAN INTEGRATE

$$\int_{1.5}^2 \left(2x-1 + \frac{7}{4(x-1)} + \frac{5}{4(x-3)} \right) dx$$

$$= \left[x^2 - x + \frac{7}{4} \ln|x-1| + \frac{5}{4} \ln|x-3| \right]_{1.5}^2$$

$$= \left(4 - 2 + \frac{7}{4} \ln|2-1| + \frac{5}{4} \ln|2-3| \right) - \left(\frac{9}{4} - \frac{3}{2} + \frac{7}{4} \ln|1.5-1| + \frac{5}{4} \ln|1.5-3| \right)$$

$$= \left(2 - \frac{7}{4} + \frac{5}{4} \ln \frac{1}{2} \right) - \left(\frac{9}{4} - \frac{3}{2} + \frac{7}{4} \ln \frac{1}{2} + \frac{5}{4} \ln \frac{3}{2} \right)$$

$$= \frac{5}{4} - \ln \frac{3}{2}$$

• THEREFORE

$$I = \frac{5}{4} - \ln \frac{3}{2}$$

• WE CAN REWRITE THIS AS

$$I = \frac{5}{4} - \ln k$$

• THEREFORE $k = \frac{3}{2}$

Question 247 (****)

$$I = \int_0^1 \left(x^{\frac{7}{6}} + 4x^{\frac{2}{3}} \right)^{-\frac{3}{4}} dx.$$

Use appropriate integration techniques to show that

$$I = 8 \left[\sqrt[4]{5} - \sqrt{2} \right].$$

, proof

$$\int_0^1 \frac{1}{(x^{\frac{7}{6}} + 4x^{\frac{2}{3}})^{\frac{3}{4}}} dx = 8[\sqrt[4]{5} - \sqrt{2}]$$

• SIMPLY BY PROCEEDING OUT OF THE RADICAL IN THE DENOMINATOR

$$\int_0^1 \frac{1}{[x^{\frac{7}{6}}(x^{\frac{1}{6}} + 4)^{\frac{3}{4}}]} dx = \int_0^1 \frac{1}{(x^{\frac{7}{6}})^{\frac{3}{4}}(x^{\frac{1}{6}} + 4)^{\frac{3}{4}}} dx$$

$$= \int_0^1 \frac{1}{x^{\frac{7}{2}}(x^{\frac{1}{6}} + 4)^{\frac{3}{4}}} dx$$

• BY SUBSTITUTION (OR RECOGNITION AS $(x^{\frac{1}{6}})^{\frac{1}{6}} x^{\frac{1}{6}}$)

$$u = x^{\frac{1}{6}}$$

$$du = \frac{1}{6} x^{-\frac{5}{6}} dx$$

$$dx = 6u^5 du$$

$$\text{LIMITS ARE CONSIDERED}$$

$$= \int_0^1 \frac{1}{u^{\frac{7}{2}}(u + 4)^{\frac{3}{4}}} (6u^5 du) = \int_0^1 \frac{6}{(u + 4)^{\frac{3}{4}}} du$$

$$= \int_0^1 2(u + 4)^{-\frac{3}{4}} du = \left[8(u + 4)^{\frac{1}{4}} \right]_0^1$$

$$= 8 \left[\sqrt[4]{5} - \sqrt[4]{4} \right] = 8 \left[\sqrt[4]{5} - \sqrt{2} \right]$$

Question 248 (****)

$$I = \int_0^{\frac{1}{4}\pi} \frac{1}{9\cos^2 x - \sin^2 x} dx.$$

By using a tangent substitution, or otherwise, show that

$$I = \frac{1}{6} \ln 2.$$

, **proof**

$$\int_0^{\frac{\pi}{4}} \frac{1}{9\cos^2 x - \sin^2 x} dx = \int_0^{\frac{\pi}{4}} \frac{1}{\cos^2 x (9 - \frac{\sin^2 x}{\cos^2 x})} dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{9 - \tan^2 x} dx$$

NOW USE THE SUBSTITUTION $u = \tan x$
 $\frac{du}{dx} = \sec^2 x$
 $dx = \frac{du}{\sec^2 x}$
 $x=0 \rightarrow u=0$
 $x=\frac{\pi}{4} \rightarrow u=1$

THE INTEGRAL NOW TRANSFORMS TO:

$$\int_0^1 \frac{\sec^2 x}{9 - u^2} \cdot \frac{du}{\sec^2 x} = \int_0^1 \frac{1}{9 - u^2} du$$

$$= \int_0^1 \frac{1}{(3-u)(3+u)} du$$

BY PARTIAL FRACTIONS (LOOK UP OR FOR METHOD)

$$\dots = \int_0^1 \left(\frac{\frac{1}{6}}{3-u} + \frac{\frac{1}{6}}{3+u} \right) du = \frac{1}{6} \left[\ln|3+u| - \ln|3-u| \right]_0^1$$

$$= \frac{1}{6} \left[(\ln 4 - \ln 2) - (\ln 3 - \ln 3) \right] = \frac{1}{6} \ln 2$$

Question 249 (****)

Find an exact simplified value for the following definite integral.

$$\int_0^{\infty} \frac{e^{8x} - e^{2x}}{(e^{8x} + 3)(e^{2x} + 3)} dx.$$

You may assume without proof that the integral converges.

$$\boxed{V}, \boxed{}, \boxed{\frac{1}{4} \ln 2}$$

SPLIT BY PARTIAL FRACTIONS (BY INSPECTION)

$$\frac{e^{8x} - e^{2x}}{(e^{8x} + 3)(e^{2x} + 3)} = -\frac{1}{e^{8x} + 3} + \frac{1}{e^{2x} + 3}$$

THIS GIVE THIS

$$\int_0^{\infty} \frac{e^{8x} - e^{2x}}{(e^{8x} + 3)(e^{2x} + 3)} dx = \int_0^{\infty} \left(-\frac{1}{e^{8x} + 3} + \frac{1}{e^{2x} + 3} \right) dx$$

$$= \int_0^{\infty} \frac{1}{e^{2x} + 3} dx - \int_0^{\infty} \frac{1}{e^{8x} + 3} dx$$

INTEGRATE ONE OF THE TWO, AS THEY ARE IDENTICAL IN STRUCTURE

$$\int_0^{\infty} \frac{1}{e^{2x} + 3} dx = \int_0^{\infty} \frac{e^{-2x}}{1 + 3e^{-2x}} dx = -\frac{1}{2} \int_0^{\infty} \frac{e^{-2x}}{1 + 3e^{-2x}} dx$$

$$= \left[\frac{1}{2} \ln(1 + 3e^{-2x}) \right]_0^{\infty} = -\frac{1}{2} \ln 1 + \frac{1}{2} \ln 4 = \frac{1}{2} \ln 4$$

THE OTHER ONE CAN BE DONE LIKE

$$\int_0^{\infty} \frac{1}{e^{8x} + 3} dx = -\int_0^{\infty} \frac{e^{-8x}}{1 + 3e^{-8x}} dx = -\frac{1}{8} \int_0^{\infty} \frac{e^{-8x}}{1 + 3e^{-8x}} dx = \left[\frac{1}{8} \ln(1 + 3e^{-8x}) \right]_0^{\infty}$$

$$= -\frac{1}{8} \ln 1 + \frac{1}{8} \ln 4 = -\frac{1}{8} \ln 4$$

COMBINING RESULTS

$$\int_0^{\infty} \frac{e^{8x} - e^{2x}}{(e^{8x} + 3)(e^{2x} + 3)} dx = \frac{1}{2} \ln 4 - \frac{1}{8} \ln 4 = \frac{3}{8} \ln 4 = \frac{3}{4} \ln 2$$

Question 250 (****)

$$I = \int_{-\frac{5}{2}}^{\frac{7}{2}} \frac{4x+1}{\sqrt{35+4x-4x^2}} dx.$$

By writing $35+4x-4x^2$ in completed the square form, followed by a suitable trigonometric substitution, show that

$$I = \frac{3}{2} \pi.$$

 , proof

• START BY COMPLETING THE SQUARE IN THE DENOMINATOR

$$35+4x-4x^2 = -(4x^2-4x-35) = -(4x^2-4x+1-36) = -(4x^2-4x+1) + 36 = (2x-1)^2 - 36$$

\uparrow
 $6 \sin \theta$ (or $6 \cos \theta$)

• USE THE SUBSTITUTION $2x-1 = 6 \sin \theta$

$$\frac{2 dx}{d\theta} = 6 \cos \theta$$

$$dx = 3 \cos \theta d\theta$$

$$2x-1 = 6 \sin \theta \Rightarrow x = \frac{6 \sin \theta + 1}{2}$$

$$2x-1 = -6 \sin \theta \Rightarrow \theta = -\frac{\pi}{2}$$

• UNLESS WE HAVE TRANSFORMED THE INTEGRAL TO

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{4x+1}{\sqrt{35+4x-4x^2}} dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{12 \sin \theta + 3}{\sqrt{36-36 \sin^2 \theta}} \times (3 \cos \theta d\theta)$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(3+12 \sin \theta)(3 \cos \theta)}{6(1-\sin^2 \theta)^{\frac{1}{2}}} d\theta = \frac{3}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(1+4 \sin \theta) \cos \theta}{\cos \theta} d\theta$$

$$= \frac{3}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1+4 \sin \theta) d\theta = 2 \times \frac{3}{2} \int_0^{\frac{\pi}{2}} 1 d\theta$$

$$= 3 \times \left[\theta \right]_0^{\frac{\pi}{2}} = \frac{3\pi}{2}$$

Question 251 (****)

$$I = \int_{-1}^1 (x+3)\sqrt{7-6x-x^2} \, dx$$

- a) Use a suitable trigonometric substitution to show that $I = 8\sqrt{3}$.
- b) Verify the answer of part (a) by an alternative method.

, proof

a) $\int_{-1}^1 (x+3)\sqrt{7-6x-x^2} \, dx = \int_{-1}^1 (x+3)\sqrt{-(x^2+6x+9)+16} \, dx$
 $= \int_{-1}^1 (x+3)\sqrt{-(x+3)^2+16} \, dx = \int_{-1}^1 (x+3)\sqrt{16-(x+3)^2} \, dx$

MATCH WITH A TRIGONOMETRIC SUBSTITUTION $16[1 - (\frac{x+3}{4})^2]$

Then $\frac{x+3}{4} = \sin\theta$ 4 units
 $x+3 = 4\sin\theta$ $x=-1 \rightarrow \theta = \frac{\pi}{6}$
 $x = -3+4\sin\theta$ $x=1 \rightarrow \theta = \frac{\pi}{2}$
 $dx = 4\cos\theta \, d\theta$

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 4\sin\theta \sqrt{16-16\sin^2\theta} (4\cos\theta \, d\theta)$
 $= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 4\sin\theta \sqrt{16(1-\sin^2\theta)} \, 4\cos\theta \, d\theta$
 $= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 4\sin\theta \times 4\cos\theta \times 4\cos\theta \, d\theta$
 $= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 64\sin\theta\cos^2\theta \, d\theta$
 $= \left[-\frac{64}{3}\cos^3\theta \right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} = \frac{64}{3} \left[\cos^3\theta \right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$
 $= \frac{64}{3} \left[\left(\frac{1}{2}\right)^3 - 0^3 \right] = \frac{64}{3} \times \frac{1}{8} = 8\sqrt{3}$

As Required

b) BY AN ALGEBRAIC SUBSTITUTION

$\int_{-1}^1 (x+3)\sqrt{7-6x-x^2} \, dx = \dots$

$\dots = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (x+3) u \left(\frac{-u}{2x+3} \, dx \right)$
 $= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} -u^2 \, du$
 $= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} -u^2 \, du$
 $= \left[-\frac{1}{3}u^3 \right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$
 $= \frac{1}{3} \left(\sqrt{12} \right)^3 = \frac{1}{3} \times 12 \times \sqrt{12} = 4\sqrt{12} = 8\sqrt{3}$

As before

VERIFICATION WITHOUT SUBSTITUTION SINCE THE ARGUMENT OF THE RADICAL DIFFERENTIATES TO $-6-2x = -2(3+x)$

$\int_{-1}^1 (x+3)(7-6x-x^2)^{\frac{1}{2}} \, dx \dots$ BY RECOGNITION

$= \left[(7-6x-x^2)^{\frac{3}{2}} \times \left(-\frac{1}{3}\right) \right]_{-1}^1 = \frac{1}{3} \left[(7-6x-x^2)^{\frac{3}{2}} \right]_{-1}^1$
 $= \frac{1}{3} \left[12^{\frac{3}{2}} - 0^{\frac{3}{2}} \right] = \frac{1}{3} \times 12 \times 12^{\frac{1}{2}} = 8\sqrt{12}$

As above.

Question 252 (****)

Use trigonometric identities to find the value of

$$\int_0^{\frac{\pi}{3}} 32 \sin x \sin 2x \sin 3x \, dx.$$

MP, 9

LOOKS AT THE CONVENTIONS SINCE WE USE AS EVIDENCE

$$\cos(2a+x) = \cos 2a \cos x - \sin 2a \sin x$$

$$\cos(2a-x) = \cos 2a \cos x + \sin 2a \sin x$$

SUBTRACTING \Rightarrow $\cos(2a-x) - \cos(2a+x) = 2 \sin 2a \sin x$

$$\cos 2x - \cos 4x = 2 \sin 3x \sin x$$

RETURNING TO THE INTEGRAL

$$\int_0^{\frac{\pi}{3}} 32 \sin x \sin 2x \sin 3x \, dx = \int_0^{\frac{\pi}{3}} 16 \sin 2x [2 \sin 3x \sin x] \, dx$$

$$= \int_0^{\frac{\pi}{3}} 16 \sin 2x [\cos x - \cos 3x] \, dx = \int_0^{\frac{\pi}{3}} 16 \sin 2x \cos x - 16 \sin 2x \cos 3x \, dx$$

$$= \int_0^{\frac{\pi}{3}} 8 (\sin 2x \cos x) - 16 \sin 2x (\cos 3x - 1) \, dx$$

$$= \int_0^{\frac{\pi}{3}} 8 \cos x - 32 \sin 2x \cos 3x + 16 \sin 2x \, dx$$

$$= \left[8 \sin x - 32 \sin 2x \cos 3x + 16 \sin 2x \right]_0^{\frac{\pi}{3}}$$

$$= \left[-2(4) + \frac{16}{3} \left(\frac{1}{2} \right) - 8 \left(\frac{\sqrt{3}}{2} \right) \right] - \left[-2 + \frac{16}{3} - 0 \right]$$

$$= 1 - \frac{8}{3} + 4 + 2 - \frac{16}{3} + 0$$

$$= 9$$

Question 253 (****)

Use suitable integration techniques to show that

$$\int_0^1 \frac{x^2}{(x^2+1)^3} dx = \frac{\pi}{32}.$$



,

proof

START WITH A TANGENT SUBSTITUTION

$$\int_0^1 \frac{x^2}{(x^2+1)^3} dx = \dots$$

$$= \int_0^{\frac{\pi}{4}} \frac{\tan^2 \theta}{(\sec^2 \theta)^3} (\sec^2 \theta d\theta)$$

$$= \int_0^{\frac{\pi}{4}} \frac{\tan^2 \theta \sec^2 \theta}{(\sec^6 \theta)} d\theta = \int_0^{\frac{\pi}{4}} \frac{\tan^2 \theta \sec^2 \theta}{\sec^4 \theta} d\theta = \int_0^{\frac{\pi}{4}} \frac{\tan^2 \theta}{\sec^2 \theta} d\theta$$

SWITCH INTO SINUS & COSINES AND ALGEBRAIC SIMPLIFICATIONS

$$= \int_0^{\frac{\pi}{4}} \frac{\tan^2 \theta \cos^2 \theta}{\cos^2 \theta} d\theta = \int_0^{\frac{\pi}{4}} \tan^2 \theta d\theta$$

$$= \int_0^{\frac{\pi}{4}} \left(\frac{1}{\cos^2 \theta} - 1 \right) d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{\cos^2 \theta} d\theta - \int_0^{\frac{\pi}{4}} 1 d\theta$$

$$= \left[\frac{1}{\cos \theta} - \theta \right]_0^{\frac{\pi}{4}}$$

$$= \left(\frac{1}{\cos \frac{\pi}{4}} - \frac{\pi}{4} \right) - \left(\frac{1}{\cos 0} - 0 \right) = \frac{\pi}{4} - \frac{\pi}{4} = 0$$

$u = \tan \theta$
 $du = \sec^2 \theta d\theta$
 $u=0 \rightarrow \theta=0$
 $u=1 \rightarrow \theta=\frac{\pi}{4}$

$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$
 $\sin 2\theta = 2 \sin \theta \cos \theta$

Use suitable integration techniques to show that

$$\int_{-\frac{1}{6}\ln 3}^{\frac{1}{6}\ln 3} 6e^{-3x} \arctan(e^{3x}) dx = \ln 3 + \frac{\pi\sqrt{3}}{9}.$$

□, proof

$$\begin{aligned}
 & \text{SMPR BY A SUBSTITUTION-AS FOLLOWS} \\
 & \int_{\frac{1}{\sqrt{e}}}^{\frac{1}{\sqrt{3}}} 6e^{-3x} \operatorname{arctan}(e^{3x}) \, dx \\
 & \dots = \int_{\frac{1}{\sqrt{e}}}^{\frac{1}{\sqrt{3}}} 6e^{-3x}(\theta) \frac{e^{3\theta}}{3e^{3\theta}} \, d\theta \\
 & = \int_{\frac{1}{\sqrt{e}}}^{\frac{1}{\sqrt{3}}} 2e^{-3\theta} \theta \operatorname{arctan}(e^{3\theta}) \, d\theta \\
 & = \int_{\frac{1}{\sqrt{e}}}^{\frac{1}{\sqrt{3}}} \frac{2\theta \operatorname{arctan}(e^{3\theta})}{(e^{3\theta})^2} \, d\theta \\
 & = \int_{\frac{1}{\sqrt{e}}}^{\frac{1}{\sqrt{3}}} \frac{2\theta \operatorname{arctan}(e^{3\theta})}{\ln(e^{\theta})} \, d\theta \\
 & = \int_{\frac{1}{\sqrt{e}}}^{\frac{1}{\sqrt{3}}} 2\theta \times \ln(e^{3\theta}) \, d\theta \\
 & = \int_{\frac{1}{\sqrt{e}}}^{\frac{1}{\sqrt{3}}} 2\theta \times \frac{\ln(e^{3\theta})}{\ln(e^{\theta})} \times \frac{1}{e^{\theta}} \, d\theta \\
 & = \int_{\frac{1}{\sqrt{e}}}^{\frac{1}{\sqrt{3}}} 2\theta \operatorname{arctan}(e^{\theta}) \, d\theta \quad \leftarrow \text{INTEGRATED BY PARTS} \\
 & = \left[-2\theta \ln(\theta) \right]_{\frac{1}{\sqrt{e}}}^{\frac{1}{\sqrt{3}}} + \int_{\frac{1}{\sqrt{e}}}^{\frac{1}{\sqrt{3}}} 2 \ln(\sin \theta) \, d\theta \\
 & = \left[-2\theta \ln(\theta) \right]_{\frac{1}{\sqrt{e}}}^{\frac{1}{\sqrt{3}}} - \left[2\theta \ln(\sin \theta) - \frac{1}{2} \ln^2 \theta \right] \\
 & = 2\ln \sqrt{\frac{3}{2}} - \frac{2\sqrt{3}}{3} \sqrt{\frac{3}{2}} - 2\ln \frac{1}{\sqrt{2}} + \frac{1}{2} \times \sqrt{2} \\
 & = \ln \frac{3}{2} - \frac{2\sqrt{6}}{3} + \ln 4 + \frac{\sqrt{2}}{2} \\
 & = \ln \left(\frac{3}{2} \times 4 \right) + \left(\frac{2}{3} + \frac{1}{3} \right) \sqrt{6} \\
 & = \ln 3 + \frac{\sqrt{6}}{3}
 \end{aligned}$$

Question 255 (****)

Use suitable integration techniques to show that

$$\int_0^{\frac{1}{2}\pi} \frac{1 + \cos x + \sin x - \tan x}{1 + \tan x} dx = 1.$$

You may assume that the above integral converges.

, proof

SPLITTING INTO SINUS & COSINES

$$\begin{aligned} & \int_0^{\frac{1}{2}\pi} \frac{1 + \cos x + \sin x - \tan x}{1 + \tan x} dx \\ &= \int_0^{\frac{1}{2}\pi} \frac{1 + \cos x + \sin x - \frac{\sin x}{\cos x}}{1 + \frac{\sin x}{\cos x}} dx \quad \left\{ \text{MULTIPLY TOP & BOTTOM BY } \cos x \right\} \\ &= \int_0^{\frac{1}{2}\pi} \frac{\cos x + \cos^2 x + \sin x \cos x - \sin x}{\cos x + \sin x} dx \end{aligned}$$

RECOGNISING THE TRIANGLE

$$\begin{aligned} &= \int_0^{\frac{1}{2}\pi} \frac{\cos x - \sin x + \cos x \sin x + \sin x}{\cos x + \sin x} dx \\ &= \int_0^{\frac{1}{2}\pi} \frac{\cos x - \sin x}{\cos x + \sin x} + \frac{\cos^2 x + \sin^2 x}{\cos x + \sin x} dx \\ &= \int_0^{\frac{1}{2}\pi} \frac{\cos x - \sin x}{\cos x + \sin x} + \frac{\cos(\sin x + \cos x)}{\cos x + \sin x} dx \\ &\quad \uparrow \\ &\quad \text{or THE FORM } \int \frac{f(x)}{f(x)} dx = \ln|f(x)| + C \end{aligned}$$

INTERPRETING RESULTS

$$\begin{aligned} &= \left[\ln|\cos x + \sin x| + \sin x \right]_0^{\frac{1}{2}\pi} \\ &= \left[\ln(\cos \frac{1}{2}\pi + \sin \frac{1}{2}\pi) + 1 \right] - \left[\ln(\cos 0 + \sin 0) + 0 \right] \\ &= 1 \end{aligned}$$

Question 256 (****)

Use a suitable trigonometric substitution to find a simplified expression for

$$\int \sqrt{(1+x)(5-x)} \, dx.$$

$$\frac{9}{2} \arcsin\left(\frac{x-2}{3}\right) + \frac{1}{2}(x-2)\sqrt{(1+x)(5-x)} + C$$

Handwritten solution for the integral $\int \sqrt{(1+x)(5-x)} \, dx$ using trigonometric substitution.

Step 1: Rewrite the integrand as $\sqrt{5-x+5x-x^2} = \sqrt{5-(x-2)^2}$.

Step 2: Use the substitution $x-2 = 3\sin\theta$. Then $dx = 3\cos\theta \, d\theta$.

Step 3: The integral becomes $\int \sqrt{9-9\sin^2\theta} \cdot 3\cos\theta \, d\theta = \int 9\cos^2\theta \, d\theta$.

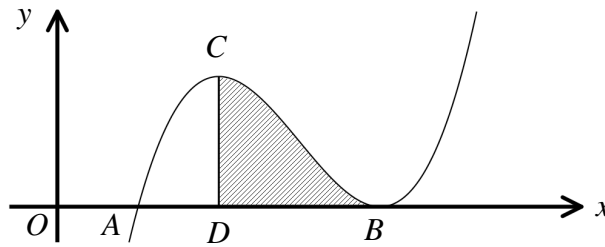
Step 4: Simplify the integral: $\int 9\cos^2\theta \, d\theta = \int 9 \cdot \frac{1+\cos(2\theta)}{2} \, d\theta = \frac{9}{2} \int (1+\cos(2\theta)) \, d\theta$.

Step 5: Integrate: $\frac{9}{2} \left(\theta + \frac{\sin(2\theta)}{2} \right) + C$.

Step 6: Substitute back $\theta = \arcsin\left(\frac{x-2}{3}\right)$ and $\sin(2\theta) = 2\sin\theta\cos\theta = \frac{2(x-2)\sqrt{9-(x-2)^2}}{9}$.

Final result: $\frac{9}{2} \arcsin\left(\frac{x-2}{3}\right) + \frac{1}{2}(x-2)\sqrt{(1+x)(5-x)} + C$.

Question 257 (****)



The figure above shows a cubic curve that crosses the x axis at $A(a,0)$ and touches the x axis at $B(b,0)$, where a and b are positive constants. The point C is a local maximum of the curve.

- a)** Find the x coordinate of C , in terms of a and b .

The point D lies on the x axis so that CD is parallel to the y axis.

- b) Show that $|AB| = 3|AD|$.

The region R is bounded by the curve, the line segment CD and the x axis.

- c) Use integration by parts to show that the area of R is $\frac{4}{81}(b-a)^4$.

$$x = \frac{1}{3}(2a + b)$$

(a) $\gamma = (a-b)(x-b)^2$ Solving for b ETO

$$\begin{aligned} \frac{d\gamma}{da} &= 1 \times (x-b)^2 + (x-a) \times 2(x-b) \\ &= (x-b)^2 + 2(x-b)(x-a) \\ &= (x-b)(x-b + 2(x-a)) \\ &= (x-b)(3x-b-2a) \end{aligned}$$

$\gamma = 0 \Rightarrow$

$$a < \frac{b}{3} \leftarrow C$$

(b) Now $|AB| = b-a$

$$|AB| = \frac{2a+b}{3} - a = \frac{2a+b-3a}{3} = \frac{b-a}{3}$$

So $3|AB| = 3 \times \frac{b-a}{3} = b-a = |AB|$ At $\frac{b-a}{3} = 0$

(c)
$$A(b) = \int_0^{2b} (x-a)(x-b)^2 dx \dots \text{by parts}$$

$$\frac{x-a}{\frac{1}{3}(x-b)^3} \quad \frac{1}{(x-b)^2}$$

$$\begin{aligned} &= \left[\frac{1}{3}(x-a)(x-b)^3 \right]_0^{2b} - \int_0^{2b} \frac{1}{3}(x-b)^3 dx \\ &= \left[\frac{1}{3}(x-a)(x-b)^3 - \frac{1}{12}(x-b)^4 \right]_0^{2b} \\ &= \frac{1}{12} \left[(x-b)^3 [4(x-a) - (x-b)] \right]_0^{2b} \\ &= \frac{1}{12} \left[(x-b)^3 (3x-4a+b) \right]_0^{2b} \\ &= \frac{1}{12} \left[0 - \left[\frac{(2a+b-b)^3}{3} (2(2a+b-b)+b) \right] \right] \\ &= -\frac{1}{12} \left[\frac{(2a+b-b)^3}{3} (2b-2a) \right] \\ &= -\frac{1}{12} \left[\frac{(2a-b)^3}{3} \times 2(b-a) \right] \\ &= -\frac{1}{12} \times \left(\frac{1}{3} \right) (b-a)^3 \times 2 \times (b-a) = \frac{1}{3!} (b-a)^4 \end{aligned}$$

\Rightarrow Required

Question 258 (*****)

By considering the derivatives of $e^x \sin x$ and $e^x \cos x$, find

$$\int e^x (2 \cos x - 3 \sin x) dx.$$

$$\boxed{\frac{1}{2} e^x (5 \cos x - \sin x) + C}$$

Handwritten solution for Question 258:

$$\begin{aligned} \frac{d}{dx}(e^x \sin x) &= e^x \sin x + e^x \cos x \\ \frac{d}{dx}(e^x \cos x) &= e^x \cos x - e^x \sin x \end{aligned}$$

Add & subtract gives

$$\begin{aligned} \frac{d}{dx}(e^x \sin x + e^x \cos x) &= 2e^x \cos x \\ \frac{d}{dx}(e^x \sin x - e^x \cos x) &= 2e^x \sin x \end{aligned}$$

Therefore

$$\begin{aligned} 2e^x \cos x - 3e^x \sin x &= 2 \frac{d}{dx} \left(\frac{1}{2} e^x (\sin x + \cos x) \right) - 3 \frac{d}{dx} \left(\frac{1}{2} e^x (\sin x - \cos x) \right) \\ 2e^x \cos x - 3e^x \sin x &= \frac{d}{dx} \left[e^x (\sin x + \cos x) - \frac{3}{2} e^x (\sin x - \cos x) \right] \\ 2e^x \cos x - 3e^x \sin x &= \frac{d}{dx} \left[\frac{1}{2} e^x (2 \cos x - \sin x) \right] \\ \therefore \int (2e^x \cos x - 3e^x \sin x) dx &= \frac{1}{2} e^x (2 \cos x - \sin x) + C \end{aligned}$$

Question 259 (*****)

Use integration by parts and suitable trigonometric identities to find

$$\int \sec^3 x dx.$$

$$\boxed{}, \boxed{\frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x| + C}$$

Handwritten solution for Question 259:

- $\int \sec^3 x dx = \int \sec x \cdot \sec^2 x dx \dots$

BY PARTS
$\sec x$ $\sec^2 x$
$\tan x$ $\sec x$
- $\begin{aligned} &= \sec x \tan x - \int \sec x (\sec^2 x - 1) dx \\ &= \sec x \tan x - \int \sec^3 x - \sec x dx \\ &= \sec x \tan x - \int \sec^3 x dx + \int \sec x dx \\ &= \sec x \tan x - \int \sec^3 x dx + \ln |\sec x + \tan x| \end{aligned}$
- COLLECTING THE RESULTS

$$\begin{aligned} \int \sec^3 x dx &= \sec x \tan x - \int \sec^3 x dx + \ln |\sec x + \tan x| \\ 2 \int \sec^3 x dx &= \sec x \tan x + \ln |\sec x + \tan x| + C \\ \int \sec^3 x dx &= \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x| + C \end{aligned}$$

Question 260 (****)

By using the substitution $\sqrt{x} = \tan \theta$, or otherwise, find a simplified expression for the following integral.

$$\int \frac{1-x}{\sqrt{x}(x+1)^2} dx.$$

$$\boxed{}, \quad \boxed{\frac{2\sqrt{x}}{x+1} + C}$$

Handwritten solution for the integral using the substitution $\sqrt{x} = \tan \theta$.

USING THE GIVEN SUBSTITUTION

$\sqrt{x} = \tan \theta$
 $x = \tan^2 \theta$
 $dx = 2 \tan \theta \sec^2 \theta d\theta$

... = $\int \frac{1 - \tan^2 \theta}{\tan \theta (\tan^2 \theta + 1)^2} (2 \tan \theta \sec^2 \theta d\theta)$
 $= \int \frac{2 \sec^2 \theta (1 - \tan^2 \theta)}{(\sec^2 \theta)^2} d\theta$
 $= \int \frac{2 \sec^2 \theta (1 - \tan^2 \theta)}{\sec^4 \theta} d\theta$
 $= \int \frac{2 (1 - \tan^2 \theta)}{\sec^2 \theta} d\theta$

CONVERT THE INTEGRAL INTO $\sec \theta$, SO IT MAY BE SPLIT

$= \int \frac{2 [1 - (\sec^2 \theta - 1)]}{\sec^2 \theta} d\theta$
 $= \int \frac{2 (2 - \sec^2 \theta)}{\sec^2 \theta} d\theta$
 $= \int 2 \left(\frac{2}{\sec^2 \theta} - \frac{\sec^2 \theta}{\sec^2 \theta} \right) d\theta$
 $= \int 2 [2 \cos^2 \theta - 1] d\theta$
 $= \int 2 \cos 2\theta d\theta$
 $= \sin 2\theta + C$

$\sin 2\theta = 2 \sin \theta \cos \theta + C$
 $= \frac{2 \sin \theta}{\cos \theta} \times \frac{1}{\sec \theta} + C$
 $= \frac{2 \tan \theta}{1 + \tan^2 \theta} + C$
 $= \frac{2 \sqrt{x}}{1 + x} + C$

Question 261 (****)

Find the value of the following definite integral.

$$\int_{\frac{1}{2}}^2 \frac{1}{x+x^4} dx$$

Give the answer in the form $\ln k$, where k is a positive integer.
 , $k=2$

THE INTEGRATION OF $\frac{1}{x+x^4} = \frac{1}{x(1+x^3)} = \frac{1}{x(1+x)(1-x+x^2)}$
 PARTIAL FRACTIONS (SPLIT INTO PARTIAL FRACTIONS (SPLIT INTO PARTIAL FRACTIONS))

$$\int_{\frac{1}{2}}^2 \frac{1}{x+x^4} dx = \int_{\frac{1}{2}}^2 \frac{1}{x(1+x^3)} dx = \int_{\frac{1}{2}}^2 \frac{A}{x} + \frac{B}{1+x} + \frac{C}{1-x+x^2} dx$$

BUT THIS IS OF THE FORM

$$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C$$

THUS IT CAN BE EVALUATED EASILY

$$= \int_{\frac{1}{2}}^2 \frac{1}{x^3+1} dx = \frac{1}{3} \int_{\frac{1}{2}}^2 \frac{3x^2}{x^3+1} dx = \left[\frac{1}{3} \ln|x^3+1| \right]_{\frac{1}{2}}^2$$

$$= \frac{1}{3} \left[\ln|2^3+1| \right]_{\frac{1}{2}}^2 = \frac{1}{3} \left[\ln|8+1| - \ln\left| \left(\frac{1}{2}\right)^3+1 \right| \right]$$

$$= \frac{1}{3} \left[\ln(9) - \ln\left(1+\frac{1}{8}\right) \right] = \frac{1}{3} \left[\ln(9) - \ln\left(\frac{9}{8}\right) \right]$$

$$= \frac{1}{3} \left[\ln(9) + \ln\left(\frac{8}{9}\right) \right] = \frac{1}{3} \ln(8) = \ln 2$$

Question 262 (****)

$$I = \int_{-2}^2 \frac{1}{\sqrt{1-ax+a^2}} dx, \quad a > 0, \quad a \neq 1.$$

Find the two possible values of I , giving the answer in terms of a where appropriate.

$$\boxed{}, \quad \begin{cases} I = 4 & 0 < a < 1 \\ I = \frac{4}{a} & a > 1 \end{cases}$$

$$I = \int_{-2}^2 \frac{1}{\sqrt{1-ax+a^2}} dx \quad a > 0, a \neq 1$$

BY INSPECTION (OR SUBSTITUTION)

$$\Rightarrow I = \int_{-2}^2 (1-ax+a^2)^{-\frac{1}{2}} dx = \int_{-2}^2 \frac{2(1-ax+a^2)^{-\frac{1}{2}}}{2} dx$$

$$\Rightarrow I = \frac{2}{a} \left[(1-ax+a^2)^{-\frac{1}{2}} \right]_{-2}^2 = \frac{2}{a} \left[(1+2a+a^2)^{-\frac{1}{2}} - (1-2a+a^2)^{-\frac{1}{2}} \right]$$

NOW THERE ARE TWO POSSIBILITIES

<p>IF $0 < a < 1$</p> $I = \frac{2}{a} \left[\sqrt{(a+1)^2} - \sqrt{(a-1)^2} \right]$ $I = \frac{2}{a} \left[(a+1) - (a-1) \right]$ $I = \frac{2}{a} \left[(a+1) - (a-1) \right]$ $I = \frac{2}{a} \times 2a$ $I = 4$	<p>IF $a > 1$</p> $I = \frac{2}{a} \left[\sqrt{(a+1)^2} - \sqrt{(a-1)^2} \right]$ $I = \frac{2}{a} \left[(a+1) - (a-1) \right]$ $I = \frac{2}{a} \left[(a+1) - (a-1) \right]$ $I = \frac{2}{a} \times 2$ $I = \frac{4}{a}$
---	---

Question 263 (****)

$$I = \int \frac{\cos^3 x}{(1 + \sin^2 x) \sin x} dx.$$

By using the substitution $u = \sin x + \operatorname{cosec} x$, or otherwise, show that

$$I = \ln \left| \frac{\sin x}{1 + \sin^2 x} \right| + \text{constant}$$

, proof

$$\begin{aligned}
 & \text{Using the substitution } u = \cos x \\
 & \frac{du}{dx} = -\sin x \Rightarrow dx = \frac{-1}{\sin x} du \\
 & \int \frac{\cos x}{\cos x - \sin x \cos x} dx = \int \frac{\cos x}{(1 + \sin^2 x) \sin x} \times \frac{-1}{\sin x} du \\
 & \text{Simplify to cancel and rewrite:} \\
 & = \int \frac{\cos x}{(1 + \sin^2 x) \sin x} \times \frac{-1}{\sin x} \times \frac{\cos x}{\cos x} du \\
 & = \int \frac{\cos x}{(1 + \sin^2 x) \sin x} \times \frac{1}{\cos x} \left(\frac{-1}{1 + \sin^2 x} \right) du \\
 & = \int \frac{\cos x}{(1 + \sin^2 x) \sin x} \times \frac{\sin x}{\cos x} du \quad \text{writing "top & bottom" of the fraction by } \sin x \\
 & = \int \frac{\cos x}{(1 + \sin^2 x) \sin x} \times \frac{\sin x}{\cos x} du \\
 & = \int \frac{-\sin x}{1 + \sin^2 x} du \\
 & = \int \frac{-\sin x \cos x}{\cos x + \sin^2 x \cos x} du
 \end{aligned}$$

Question 264 (****)

$$I = \int_0^{\frac{1}{2}\pi} x \cot x \, dx.$$

Use appropriate integration techniques to show that

$$I = \frac{1}{2}\pi \ln 2.$$

, proof

Start by integration by parts

$$\int_0^{\frac{\pi}{2}} x \cot x \, dx = \left[x \ln(\sin x) \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \ln(\sin x) \, dx$$

BECAUSE $x \rightarrow 0$ FACTOR THEN

$$\int_0^{\frac{\pi}{2}} x \cot x \, dx = - \int_0^{\frac{\pi}{2}} \ln(\sin x) \, dx$$

Now proceed as follows

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \ln(\sin x) \, dx \quad \dots \text{substitution}$$

$$\Rightarrow I = \int_{\frac{\pi}{2}}^0 \ln[\sin(\frac{\pi}{2} - x)] \, (-dx)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \ln[\sin(\frac{\pi}{2} - x)] \, dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \ln(\cos x) \, dx$$

THIS RESEMBLES THE SECOND PART OF Q, WE HAVE

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \ln(\sin x) \, dx = \int_0^{\frac{\pi}{2}} \ln(\cos x) \, dx$$

THEY ARE THE SAME

$$\Rightarrow I + I = \int_0^{\frac{\pi}{2}} \ln(\sin x) \, dx + \int_0^{\frac{\pi}{2}} \ln(\cos x) \, dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \ln(\sin x) + \ln(\cos x) \, dx = \int_0^{\frac{\pi}{2}} \ln(\sin x \cos x) \, dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \ln(\frac{1}{2} \sin 2x) \, dx = \int_0^{\frac{\pi}{2}} \ln(\frac{1}{2}) + \ln(\sin 2x) \, dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} -\ln 2 \, dx + \int_0^{\frac{\pi}{2}} \ln(\sin 2x) \, dx$$

$$\begin{aligned} u &= 2x \\ \frac{du}{dx} &= 2 \\ dx &= \frac{1}{2} du \\ 2x=0 &\rightarrow u=0 \\ 2x=\frac{\pi}{2} &\rightarrow u=\frac{\pi}{2} \end{aligned}$$

$$\Rightarrow 2I = -\ln 2 \left(\frac{\pi}{2} \right) + \int_0^{\frac{\pi}{2}} \ln(\sin u) \left(\frac{1}{2} du \right)$$

$$\Rightarrow 2I = -\frac{\pi}{2} \ln 2 + \frac{1}{2} \int_0^{\frac{\pi}{2}} \ln(\sin u) \, du$$

THE SAME FUNCTION IS GIVEN ABOUT $\frac{\pi}{2}$
SO WE CAN REUSE

$$\Rightarrow 2I = -\frac{\pi}{2} \ln 2 + \frac{1}{2} \times 2 \int_0^{\frac{\pi}{2}} \ln(\sin u) \, du$$

$$\Rightarrow 2I = -\frac{\pi}{2} \ln 2 + I$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \ln(\sin x) \, dx = -\frac{\pi}{2} \ln 2$$

THE FINALLY ANSWER THAT

$$\int_0^{\frac{\pi}{2}} x \cot x \, dx = - \int_0^{\frac{\pi}{2}} \ln(\sin x) \, dx = \frac{\pi}{2} \ln 2$$

Question 265 (*****)

Use the substitution $x = \frac{1}{u}$ to find the value of

$$\int_{\frac{1}{2}}^2 \frac{x^4 - 1}{x^2 \sqrt{x^4 + 1}} dx.$$

0

Handwritten solution for Question 265:

$$\int_{\frac{1}{2}}^2 \frac{x^4 - 1}{x^2 \sqrt{x^4 + 1}} dx = \dots$$

Substitution: $x = \frac{1}{u} \Rightarrow \frac{dx}{du} = -\frac{1}{u^2} \Rightarrow dx = -\frac{1}{u^2} du$

$$= \int_{\frac{1}{2}}^2 \frac{\frac{1}{u^4} - 1}{\frac{1}{u^2} \sqrt{\frac{1}{u^4} + 1}} \left(-\frac{1}{u^2} du\right) = \int_{\frac{1}{2}}^2 \frac{1 - u^4}{\sqrt{1 + u^4}} \left(-\frac{1}{u^2} du\right)$$

$$= \int_{\frac{1}{2}}^2 \frac{1 - u^4}{u^2 \sqrt{1 + u^4}} \left(-\frac{1}{u^2} du\right) = \int_{\frac{1}{2}}^2 \frac{1 - u^4}{u^4 \sqrt{1 + u^4}} du = - \int_{\frac{1}{2}}^2 \frac{1 - u^4}{u^4 \sqrt{1 + u^4}} du$$

Thus $\int_{\frac{1}{2}}^2 \frac{x^4 - 1}{x^2 \sqrt{x^4 + 1}} dx = - \int_{\frac{1}{2}}^2 \frac{1 - u^4}{u^4 \sqrt{1 + u^4}} du$

$\therefore \int_{\frac{1}{2}}^2 \frac{x^4 - 1}{x^2 \sqrt{x^4 + 1}} dx = 0$

Question 266 (*****)

Use a suitable substitution to find the value of

$$\int_2^4 \frac{\sqrt{\ln(9-x)}}{\sqrt{\ln(9-x)} + \sqrt{\ln(3+x)}} dx.$$

, 1

Handwritten solution for Question 266:

$$\text{Let } I = \int_2^4 \frac{\sqrt{\ln(9-x)}}{\sqrt{\ln(9-x)} + \sqrt{\ln(3+x)}} dx$$

Using the standard result:

$$\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$$

$$\Rightarrow I = \int_2^4 \frac{\sqrt{\ln(9-(6-x))}}{\sqrt{\ln(9-(6-x))} + \sqrt{\ln(3+(6-x))}} dx$$

$$\Rightarrow I = \int_2^4 \frac{\sqrt{\ln(3+x)}}{\sqrt{\ln(3+x)} + \sqrt{\ln(9-x)}} dx$$

Adding the expressions:

$$\Rightarrow 2I = \int_2^4 \frac{\sqrt{\ln(9-x)} + \sqrt{\ln(3+x)}}{\sqrt{\ln(9-x)} + \sqrt{\ln(3+x)}} dx = \int_2^4 1 dx$$

$$\Rightarrow 2I = 2$$

$$\Rightarrow I = 1$$

Question 267 (****)

Use an appropriate substitution followed by integration by parts to find a simplified expression for

$$\int \frac{[\ln(x^2+1) - 2\ln x] \sqrt{x^2+1}}{x^4} dx.$$

$$\boxed{}, \frac{2}{9x^3} (x^2+1)^{\frac{3}{2}} \left[1 - 3\ln\left(\frac{x^2+1}{x^2}\right) \right] + C$$

START BY MANIPULATING THE INTEGRAL TO FOCUS

$$\int \frac{[\ln(x^2+1) - 2\ln x] \sqrt{x^2+1}}{x^4} dx$$

$$= \int \ln\left(\frac{x^2+1}{x^2}\right) \times \frac{\sqrt{x^2+1}}{x^4} dx$$

$$= \int \ln\left(1 + \frac{1}{x^2}\right) \times \frac{\sqrt{x^2+1}}{x^4} dx$$

$$= \int \sqrt{\frac{x^2+1}{x^4}} \ln\left(1 + \frac{1}{x^2}\right) \times \frac{1}{x^2} dx$$

$$= \int \sqrt{1 + \frac{1}{x^2}} \ln\left(1 + \frac{1}{x^2}\right) \times \frac{1}{x^2} dx$$

Now use the "u substitution"

$$u = \sqrt{1 + \frac{1}{x^2}}$$

$$u^2 = 1 + \frac{1}{x^2}$$

$$2u \, du = -\frac{2}{x^3} dx \quad \therefore \frac{du}{x^3} = -u \, dx$$

TRANSFORM THE INTEGRAL

$$= \int u \ln u^2 (-u \, dx)$$

$$= \int -u^2 \ln u^2 \, du$$

$$= \int -2u^2 \ln u \, du$$

PROCEED BY INTEGRATION BY PARTS

$\ln u$	$\frac{1}{u}$
$-\frac{2}{3}u^3$	$-\frac{1}{2u^2}$

$$\dots = -\frac{2}{3}u^3 \ln u - \int \frac{2}{3}u^3 \left(\frac{1}{u}\right) du$$

$$= -\frac{2}{3}u^3 \ln u + \frac{2}{3} \int u^2 \, du$$

$$= -\frac{2}{3}u^3 \ln u + \frac{2}{9}u^3 + C$$

$$= \frac{2}{9}u^3 [1 - 3\ln u] + C$$

$$= \frac{2}{9}u^3 [1 - 3\ln u^2] + C$$

$$= \frac{2}{9} \left(\frac{x^2+1}{x^2}\right)^{\frac{3}{2}} [1 - 3\ln\left(\frac{x^2+1}{x^2}\right)] + C$$

$$= \frac{2}{9x^3} \sqrt{x^2+1}^3 \left[1 - 3\ln\left(\frac{x^2+1}{x^2}\right)\right] + C$$

Question 268 (****)

Use appropriate integration techniques to show that

$$\int_0^{\frac{1}{2}\pi} \frac{\sin^2 x}{\sin x + \cos x} dx = \frac{1}{\sqrt{2}} \ln(1 + \sqrt{2}).$$

□, proof

• LET $I = \int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sin x + \cos x} dx$
 BY SUBSTITUTION
 $\begin{cases} x = \frac{\pi}{2} - y \\ y = \frac{\pi}{2} - x \end{cases} \Rightarrow dx = -dy \quad \text{a.} \quad \begin{cases} x = \frac{\pi}{2} \mapsto y = 0 \\ x = 0 \mapsto y = \frac{\pi}{2} \end{cases}$
 $\Rightarrow I = \int_{\frac{\pi}{2}}^0 \frac{\sin^2(\frac{\pi}{2} - y)}{\sin(\frac{\pi}{2} - y) + \cos(\frac{\pi}{2} - y)} (-dy)$ $\sin(\frac{\pi}{2} - y) = \cos y$
 $\cos(\frac{\pi}{2} - y) = \sin y$
 $\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\cos^2 y}{\cos y + \sin y} dy$
 $\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sin x + \cos x} dx = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\sin x + \cos x} dx$
 • ADDING UP OBTAIN
 $\Rightarrow I + I = \int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sin x + \cos x} + \frac{\cos^2 x}{\sin x + \cos x} dx$
 $\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \frac{1}{\sin x + \cos x} dx$
 $\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{2} \left[\frac{1}{\sqrt{2}} \sin x + \frac{1}{\sqrt{2}} \cos x \right]} dx$
 $\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{2} (\cos x \cos \frac{\pi}{4} + \sin x \sin \frac{\pi}{4})} dx$

$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{2} \cos(x - \frac{\pi}{4})} dx$
 $\Rightarrow 2I = \frac{1}{\sqrt{2}} \int_0^{\frac{\pi}{2}} \sec(x - \frac{\pi}{4}) dx$
 • STANDARD RESULT
 $\int \sec \theta d\theta = \ln|\sec \theta + \tan \theta| + C$
 $\Rightarrow I = \frac{1}{2\sqrt{2}} \left[\ln \left| \sec(x - \frac{\pi}{4}) + \tan(x - \frac{\pi}{4}) \right| \right]_0^{\frac{\pi}{2}}$
 $\Rightarrow I = \frac{1}{2\sqrt{2}} \left[\ln \left| \sec \frac{\pi}{4} + \tan \frac{\pi}{4} \right| - \ln \left| \sec(-\frac{\pi}{4}) + \tan(-\frac{\pi}{4}) \right| \right]$
 $\Rightarrow I = \frac{1}{2\sqrt{2}} \left[\ln(\sqrt{2} + 1) - \ln(\sqrt{2} - 1) \right]$
 $\Rightarrow I = \frac{1}{2\sqrt{2}} \ln \left(\frac{\sqrt{2} + 1}{\sqrt{2} - 1} \right)$
 $\Rightarrow I = \frac{1}{2\sqrt{2}} \ln \left(\frac{(\sqrt{2} + 1)(\sqrt{2} + 1)}{(\sqrt{2} - 1)(\sqrt{2} + 1)} \right)$
 $\Rightarrow I = \frac{1}{2\sqrt{2}} \ln \left(\frac{(\sqrt{2} + 1)^2}{2 - 1} \right)$
 $\Rightarrow I = \frac{1}{2\sqrt{2}} \ln(\sqrt{2} + 1)^2$
 $\Rightarrow I = \frac{1}{\sqrt{2}} \ln(\sqrt{2} + 1)$
 As required

ALTERNATIVE FROM THE POINT WHERE THE INTEGRAL IS OBTAINED
 $\frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{1}{\cos x + \sin x} dx$
 BY LITTLE F IDIOTS
 • LET $t = \tan \frac{x}{2}$
 $\frac{dx}{dt} = \frac{2}{1+t^2}$
 $\frac{dx}{dt} = \frac{1}{\frac{1}{2}(1+t^2)}$
 $dx = \frac{2dt}{1+t^2}$
 $\cos x = \frac{1-t^2}{1+t^2}$
 $\sin x = \frac{2t}{1+t^2}$
 $\cos x + \sin x = \frac{1-t^2}{1+t^2} + \frac{2t}{1+t^2} = \frac{1-t^2+2t}{1+t^2} = \frac{(1-t)^2}{1+t^2}$
 • LIMITS
 $x=0 \mapsto t=0$
 $x=\frac{\pi}{2} \mapsto t=1$
 $= \frac{1}{2} \int_0^1 \frac{1}{\frac{(1-t)^2}{1+t^2}} \cdot \frac{2}{1+t^2} dt = \int_0^1 \frac{1}{(1-t)^2} dt$
 $= \int_0^1 \frac{1}{(1-t)^2} dt = \int_1^0 \frac{1}{(t-1)^2} dt$
 $= \int_1^0 \frac{1}{(t-1-\sqrt{2})(t-1+\sqrt{2})} dt$

NOW BY PARTIAL FRACTIONS (BY CHANCE)
 $= \int_1^0 \left[\frac{1+\sqrt{2}}{(t-1-\sqrt{2})} + \frac{1-\sqrt{2}}{(t-1+\sqrt{2})} \right] dt$
 $= \int_1^0 \left[\frac{2\sqrt{2}}{t-1-\sqrt{2}} - \frac{2\sqrt{2}}{t-1+\sqrt{2}} \right] dt$
 $= \frac{1}{2\sqrt{2}} \int_1^0 \left[\frac{1}{t-1-\sqrt{2}} - \frac{1}{t-1+\sqrt{2}} \right] dt$
 $= \frac{1}{2\sqrt{2}} \left[\ln \left| \frac{t-1-\sqrt{2}}{t-1+\sqrt{2}} \right| \right]_1^0$
 $= \frac{1}{2\sqrt{2}} \left[\ln \left| \frac{-1-\sqrt{2}}{-1+\sqrt{2}} \right| - \ln \left| \frac{-\sqrt{2}}{\sqrt{2}} \right| \right]$
 $= \frac{1}{2\sqrt{2}} \ln \left[\frac{-1-\sqrt{2}}{-1+\sqrt{2}} \times 1 \right]$
 $= \frac{1}{2\sqrt{2}} \ln \left[\frac{\sqrt{2}+1}{\sqrt{2}-1} \right]$
 $= \frac{1}{2\sqrt{2}} \ln \left(\frac{(\sqrt{2}+1)^2}{2-1} \right)$
 $= \frac{1}{2\sqrt{2}} \ln(\sqrt{2}+1)^2$
 $= \frac{1}{\sqrt{2}} \ln(\sqrt{2}+1)$
 As required

Question 269 (****)

Use appropriate integration techniques to show that

$$\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{1}{1 + \sqrt{\cot x}} dx = \frac{\pi}{12}.$$

proof

Handwritten solution for the integral problem:

Let $I = \int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{1}{1 + \sqrt{\cot x}} dx = \frac{\pi}{12}$.

Let $I = \int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{1}{1 + \sqrt{\cot x}} dx$... use the substitution

$u = \frac{\pi}{6} - x$
 $du = -dx$
 $\frac{\pi}{6} \rightarrow \frac{\pi}{3}$
 $\frac{1}{6}\pi \rightarrow \frac{1}{6}\pi$

$I = \int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{1}{1 + \sqrt{\cot(\frac{\pi}{6} - x)}} (-dx) = \int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{1}{1 + \sqrt{\tan x}} dx$

$2I = \int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{1}{1 + \sqrt{\cot x}} dx + \int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{1}{1 + \sqrt{\tan x}} dx$

$2I = \int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{1}{1 + \sqrt{\cot x}} + \frac{1}{1 + \sqrt{\tan x}} dx$

$2I = \int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{1}{1 + \frac{\cos x}{\sin x}} + \frac{1}{1 + \frac{\sin x}{\cos x}} dx$

Multiply top & bottom of the first fraction by $\sin x$ & top & bottom of the second fraction by $\cos x$

$2I = \int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{\sin x}{\sin x + \sqrt{\cos x}} + \frac{\cos x}{\cos x + \sqrt{\sin x}} dx$

$2I = \int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx = \int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} 1 dx = \frac{\pi}{6}$

$I = \frac{\pi}{12}$

As required

Question 270 (****)

Use appropriate integration techniques to find an exact answer for the following definite integral.

$$\int_{\frac{1}{2}\pi}^{2\pi - \arccos \frac{1}{8}} \sqrt[3]{3\sin 2x - 2\sin 3x \cos x} \, dx$$

$$\boxed{}, \boxed{\frac{3}{32}}$$

• START BY DRAWING AN EXPRESSION RE $\sin 2x$ BY IDENTITIES

$$\begin{aligned}\sin 3x &= \sin(2x+x) = \sin 2x \cos x + \cos 2x \sin x \\ &= (2\sin x \cos x) \cos x + (1-2\sin^2 x) \sin x \\ &= 2\sin x \cos^2 x + \sin x - 2\sin^3 x \\ &= 2\sin x(1-\sin^2 x) + \sin x - 2\sin^3 x \\ &= 3\sin x - 4\sin^3 x\end{aligned}$$

• HENCE THE INTEGRAL CAN NOW BE SIMPLIFIED

$$\begin{aligned}&\int_{\frac{1}{2}\pi}^{2\pi - \arccos \frac{1}{8}} \sqrt[3]{3\sin 2x - 2\sin 3x \cos x} \, dx \\ &= \int_{\frac{1}{2}\pi}^{2\pi - \arccos \frac{1}{8}} (6\sin x \cos^2 x - 2\cos x(3\sin x - 4\sin^3 x))^{\frac{1}{3}} \, dx \\ &= \int_{\frac{1}{2}\pi}^{2\pi - \arccos \frac{1}{8}} (6\sin x \cos^2 x - 6\sin x \cos x + 8\sin^3 x \cos x)^{\frac{1}{3}} \, dx \\ &= \int_{\frac{1}{2}\pi}^{2\pi - \arccos \frac{1}{8}} (8\sin x \cos x)^{\frac{1}{3}} \, dx \\ &= \int_{\frac{1}{2}\pi}^{2\pi - \arccos \frac{1}{8}} 2\sin x (\cos x)^{\frac{1}{3}} \, dx \\ &= -2 \times \frac{3}{4} \left[(\cos x)^{\frac{4}{3}} \right]_{\frac{1}{2}\pi}^{2\pi - \arccos \frac{1}{8}} \\ &= -\frac{3}{2} \left[(\cos(2\pi - \arccos \frac{1}{8}))^{\frac{4}{3}} - (\cos \frac{\pi}{2})^{\frac{4}{3}} \right] \\ &= -\frac{3}{2} \left[(\cos \arccos \frac{1}{8})^{\frac{4}{3}} + \sin^{\frac{4}{3}}(\arccos \frac{1}{8}) \right]^{\frac{1}{3}} \\ &= -\frac{3}{2} \left[1 \times \frac{1}{8} \right]^{\frac{1}{3}} \\ &= -\frac{3}{2} \times \frac{1}{16} \\ &= -\frac{3}{32}\end{aligned}$$

Question 271 (****)

- a) Use the compound angle identity $\cos(A+B)$ to show that

$$\cos\left(\frac{5\pi}{12}\right) = \frac{\sqrt{6}-\sqrt{2}}{4}$$

- b) Use a suitable trigonometric substitution to find the exact value of

$$\int_{\sqrt{2}}^{\sqrt{6+\sqrt{2}}} \frac{2}{x\sqrt{x^4-1}} dx.$$

$$\boxed{}, \quad \boxed{\frac{\pi}{24}}$$

a) $\cos \frac{5\pi}{12} = \cos\left(\frac{\pi}{4} + \frac{\pi}{6}\right) = \cos \frac{\pi}{4} \cos \frac{\pi}{6} - \sin \frac{\pi}{4} \sin \frac{\pi}{6}$
 $= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \frac{\sqrt{6}-\sqrt{2}}{4}$

b) $\int_{\sqrt{2}}^{\sqrt{6+\sqrt{2}}} \frac{1}{x\sqrt{x^4-1}} dx = \dots$

THE INTEGRAL TRANSFORMS TO

$$= \int_{\frac{\pi}{4}}^{\frac{5\pi}{12}} \frac{1}{2\sqrt{\sec^2\theta-1}} \times \frac{\sec\theta \tan\theta}{2\sec\theta} d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{5\pi}{12}} \frac{\sec\theta \tan\theta}{2\sec^2\theta \sqrt{\tan^2\theta}} d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{5\pi}{12}} \frac{\sec\theta \tan\theta}{2\sec\theta \tan\theta} d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{5\pi}{12}} \frac{1}{2} d\theta$$

$$= \left[\frac{1}{2} \theta \right]_{\frac{\pi}{4}}^{\frac{5\pi}{12}}$$

$$= \frac{1}{2} \times \frac{\pi}{12}$$

$$= \frac{\pi}{24}$$

BY SUBSTITUTION

- $x^2 = \sec\theta$
- $2x \frac{dx}{d\theta} = \sec\theta \tan\theta$
- $dx = \frac{\sec\theta \tan\theta}{2x} d\theta$
- $x = \sqrt{x^2}$
- $x^2 = 2$
- $2 = \sec\theta$
- $\theta = \frac{\pi}{3}$
- $x = \sqrt{x^2+1}$
- $x^2 = x^2+1$
- $\sec\theta = \sqrt{x^2+1}$
- $\cos\theta = \frac{1}{\sqrt{x^2+1}}$
- $\cos\theta = \frac{\sqrt{x^2-1}}{\sqrt{x^2+1}(\sqrt{x^2-1})}$
- $\cos\theta = \frac{\sqrt{x^2-1}}{x}$
- $\cos\theta = \frac{\sqrt{x^2-1}}{x}$
- $\theta = \frac{\pi}{12}$

Question 272 (*****)

It is given that the functions of x , $u(x)$ and $v(x)$ satisfy

$$\int u(x)v(x)dx = \left[\int u(x)dx \right] \times \left[\int v(x)dx \right], \text{ for } x \in \mathbb{R}, x \neq 0, x \neq 1.$$

a) Show clearly that

$$\frac{\int u(x)dx}{u(x)} + \frac{\int v(x)dx}{v(x)} = 1.$$

b) Given further that

$$\frac{\int u(x)dx}{u(x)} = \frac{1}{x},$$

show that

$$u(x) = A x e^{\frac{1}{2}x^2}, \text{ where } A \text{ is an arbitrary constant.}$$

c) Determine a similar expression for $v(x)$.

$$v(x) = B x e^x$$

(a) $\int uv dx = \int u dx \times \int v dx$
 Diff w.r.t x following the product rule
 $uv = u \int v dx + \int u dx \times v$
 Divide by uv
 $1 = \frac{\int v dx}{v} + \frac{\int u dx}{u}$
 As ZEPHRO

(b) $\frac{\int u dx}{u} = \frac{1}{x}$
 $\Rightarrow \int u dx = \frac{u}{x}$
 Diff w.r.t x
 $\Rightarrow u = \frac{d}{dx} \left(\frac{u}{x} \right)$
 $\Rightarrow u = \frac{x \frac{du}{dx} - u}{x^2}$
 $\Rightarrow x u = x \frac{du}{dx} - u$
 $\Rightarrow x u + u = x \frac{du}{dx}$
 $\Rightarrow u(x+1) = x \frac{du}{dx}$
 $\Rightarrow \frac{x+1}{x} dx = \frac{1}{u} du$
 $\Rightarrow \int \frac{x+1}{x} dx = \int \frac{1}{u} du$
 $\Rightarrow \frac{1}{2}x^2 + \ln|x| + C = \ln|u|$
 $\Rightarrow u = e^{\frac{1}{2}x^2 + \ln|x| + C}$
 $\Rightarrow u = e^{\frac{1}{2}x^2} \times e^{\ln|x|} \times e^C$
 $\Rightarrow u = A x e^{\frac{1}{2}x^2}$

(c) $1 + \frac{\int v dx}{v} + \frac{1}{x} = 1$
 $\Rightarrow \frac{\int v dx}{v} = 1 - \frac{1}{x}$
 $\Rightarrow \int v dx = v - \frac{v}{x}$
 Differentiate w.r.t
 $\Rightarrow v = \frac{dv}{dx} - \frac{x \frac{dv}{dx} - v}{x^2}$
 $\Rightarrow v x^2 = x^2 \frac{dv}{dx} - x \frac{dv}{dx} + v$
 $\Rightarrow v x^2 - v = (x^2 - x) \frac{dv}{dx}$
 $\Rightarrow v(x^2 - 1) = (x^2 - x) \frac{dv}{dx}$
 $\Rightarrow v(x+1)(x-1) = x(x-1) \frac{dv}{dx}$
 $\Rightarrow \frac{x+1}{x} dx = \frac{1}{v} dv$
 $\Rightarrow \int \frac{x+1}{x} dx = \int \frac{1}{v} dv$
 $\Rightarrow \ln|v| = x + \ln|x| + D$
 $\Rightarrow v = e^{x + \ln|x| + D}$
 $\Rightarrow v = e^x \times e^{\ln|x|} \times e^D$
 $\Rightarrow v = B x e^x$

Question 273 (****)

Use the substitution $\tan x = \frac{1}{2}(-1 + \sqrt{3} \tan \theta)$ to find the exact value of

$$\int_0^{\frac{\pi}{4}} \frac{\sqrt{3}}{2 + \sin 2x} dx.$$

$$\boxed{}, \frac{\pi}{6}$$

$$\int_0^{\frac{\pi}{4}} \frac{\sqrt{3}}{2+5\cos 2x} dx = \dots \text{ USING THE SUBSTITUTION}$$

$$\tan u = \frac{1}{\sqrt{3}} (\sqrt{3} \tan u - 1)$$

$$\sec^2 u \frac{du}{dx} = \frac{1}{\sqrt{3}} (\sec^2 u)$$

$$dx = \frac{\sqrt{3} \sec^2 u}{2+5\cos 2u} du$$

$$\bullet 2 \cdot 0 \quad 0 = \frac{1}{\sqrt{3}} (\sqrt{3} \tan u - 1)$$

$$\tan u = \frac{1}{\sqrt{3}}$$

$$u = \frac{\pi}{6}$$

$$\bullet 2 \cdot \frac{\pi}{4} \quad 1 = \frac{1}{\sqrt{3}} (\sqrt{3} \tan u - 1)$$

$$\tan u = \sqrt{3}$$

$$u = \frac{\pi}{3}$$

$$\int_0^{\frac{\pi}{4}} \frac{\sqrt{3}}{2+5\cos 2x} dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\frac{\sqrt{3}}{2} \sec^2 u}{1 + 5 \cos(2u)} du$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\frac{\sqrt{3}}{2} \sec^2 u}{\sec^2 u + \tan u} du$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\frac{\sqrt{3}}{2} \sec^2 u}{1 + \tan u + \tan^2 u} du$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\frac{\sqrt{3}}{2} \sec^2 u}{1 + \frac{1}{2}(\sqrt{3} \tan u - 1) + \frac{1}{4}(\sqrt{3} \tan u - 1)^2} du$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\frac{\sqrt{3}}{2} \sec^2 u}{1 + \frac{\sqrt{3}}{2} \tan u - \frac{1}{2} + \frac{3}{4} (3 \tan^2 u - 2\sqrt{3} \tan u + 1)} du$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\frac{\sqrt{3}}{2} \sec^2 u}{\frac{3}{4} + \frac{3}{4} \tan^2 u + \frac{3}{4} \tan u - \frac{\sqrt{3}}{2} \tan u + \frac{1}{4}} du$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\frac{\sqrt{3}}{2} \sec^2 u}{\frac{3}{4} + \frac{3}{4} \tan^2 u + \frac{3}{4} \tan u - \frac{\sqrt{3}}{2} \tan u + \frac{1}{4}} du = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\frac{\sqrt{3}}{2}}{\frac{3}{4} (\tan^2 u + \tan u + \frac{1}{3})} du = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\frac{\sqrt{3}}{2}}{\frac{3}{4} (\tan^2 u + \tan u + \frac{1}{3})} du$$

$$= \left[\frac{\pi}{4} - \frac{\pi}{6} \right] = \frac{\pi}{12}$$

Question 274 (****)

$$I = \int_{\arcsin \frac{3}{5}}^{\arccos \frac{2}{5}} \frac{1}{(\sin x + 2 \cos x)(\sin x + 3 \cos x)} dx.$$

Use appropriate integration techniques to show that

$$I = \ln \left(\frac{a}{b} \right),$$

where a and b are positive integers to be found.

$$\boxed{}, I = \ln \left(\frac{150}{143} \right)$$

The handwritten solution is divided into two main parts, each with a blue header.

Left Page:

- STRET MANIPULATING THE INTEGRAND AS BUCKS**: Shows the integrand $\frac{1}{(\sin x + 2 \cos x)(\sin x + 3 \cos x)}$ being rewritten as $\frac{1}{\sin^2 x + 5 \sin x \cos x + 6 \cos^2 x}$.
- DIVIDE "TOP & BOTTOM" BY $\cos^2 x$ TO CREATE TANGENTS**: The integrand becomes $\frac{\sec^2 x}{\tan^2 x + 5 \tan x + 6}$.
- BY SUBSTITUTION AS $\tan x$ DIFFERENTIATES TO $\sec^2 x$** :
 - Let $u = \tan x$, then $\frac{du}{dx} = \sec^2 x$.
 - The integral becomes $\int_{\frac{3}{4}}^{\frac{2}{3}} \frac{1}{u^2 + 5u + 6} du$.
 - A small diagram shows a right-angled triangle with sides 3, 4, 5, where $\tan \theta = \frac{3}{4}$ and $\sec \theta = \frac{5}{4}$.
 - It notes that $\arccos \frac{2}{5} = \arctan \frac{3}{4} \rightarrow u = \frac{3}{4}$ and $\arcsin \frac{3}{5} = \arctan \frac{3}{4} \rightarrow u = \frac{3}{4}$.

Right Page:

- BY PARTIAL FRACTIONS**: The integrand $\frac{1}{(u+2)(u+3)}$ is decomposed into $\frac{A}{u+2} + \frac{B}{u+3}$. Solving gives $A = 1$ and $B = -1$.
- RETURNING TO THE INTEGRAL**: The integral becomes $\int_{\frac{3}{4}}^{\frac{2}{3}} \left(\frac{1}{u+2} - \frac{1}{u+3} \right) du$.
- The final result is $\ln \frac{150}{143}$.

Question 275 (****)

Use appropriate integration techniques to show that

$$\int_0^1 \frac{1}{x + \sqrt{1-x^2}} dx = \frac{\pi}{4}.$$

□, proof

Using a Trigonometric Substitution

$\begin{aligned} x &= \sin \theta \\ dx &= \cos \theta \, d\theta \\ x=0 &\rightarrow \theta=0 \\ x=1 &\rightarrow \theta=\frac{\pi}{2} \end{aligned}$

TRANSFORMING THE INTEGRAL

$$\int_0^1 \frac{1}{x + \sqrt{1-x^2}} dx = \int_0^{\frac{\pi}{2}} \frac{1}{\sin \theta + \sqrt{1-\sin^2 \theta}} (\cos \theta \, d\theta)$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos \theta}{\sin \theta + \cos \theta} d\theta$$

PROCEED AS FOLLOWS

LET $I = \int_0^{\frac{\pi}{2}} \frac{\cos \theta}{\sin \theta + \cos \theta} d\theta$

using $\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\cos(\frac{\pi}{2}-\theta)}{\sin(\frac{\pi}{2}-\theta) + \cos(\frac{\pi}{2}-\theta)} d\theta$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sin \theta}{\cos \theta + \sin \theta} d\theta$$

THEN WE NOW HAVE

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \frac{\cos \theta}{\sin \theta + \cos \theta} d\theta + \int_0^{\frac{\pi}{2}} \frac{\sin \theta}{\cos \theta + \sin \theta} d\theta$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \frac{\cos \theta + \sin \theta}{\cos \theta + \sin \theta} d\theta$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 1 \, d\theta$$

$$\Rightarrow 2I = \frac{\pi}{2}$$

ALTERNATIVE USING THE TRIGONOMETRIC SUBSTITUTION

$$\Rightarrow I = \frac{\pi}{4}$$

$$\therefore \int_0^1 \frac{1}{x + \sqrt{1-x^2}} dx = \frac{\pi}{4}$$

ALTERNATIVE USING THE TRIGONOMETRIC SUBSTITUTION

$$\int_0^1 \frac{1}{x + \sqrt{1-x^2}} dx = \dots = \int_0^{\frac{\pi}{2}} \frac{\cos \theta}{\sin \theta + \cos \theta} d\theta$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{2 \cos \theta}{\sin \theta + \cos \theta} d\theta = \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{\cos \theta - \sin \theta + \sin \theta + \cos \theta}{\sin \theta + \cos \theta} d\theta$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{\cos \theta - \sin \theta}{\sin \theta + \cos \theta} d\theta + \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{\sin \theta + \cos \theta}{\sin \theta + \cos \theta} d\theta$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{\cos \theta - \sin \theta}{\sin \theta + \cos \theta} d\theta + \frac{1}{2} \int_0^{\frac{\pi}{2}} 1 \, d\theta$$

↑ This is of the form $\int \frac{f(x)}{f(x)} dx = \ln|f(x)| + C$

$$= \frac{1}{2} \left[\ln|\sin \theta + \cos \theta| + \theta \right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2} \left[\left(\ln 1 + \frac{\pi}{2} \right) - \left(\ln 1 + 0 \right) \right]$$

$$= \frac{\pi}{4}$$

As before

Question 276 (****)

Use polynomial division to find the exact value of

$$\int_0^1 \frac{x^4(1-x)^4}{x^2+1} dx.$$

You may assume that

$$\int \frac{1}{1+x^2} dx = \arctan x + \text{constant}.$$

$$\boxed{}, \quad \boxed{\frac{22}{7} - \pi}$$

• SIMP BY FULL EXPANDING THE NUMERATOR OF THE INTEGRAND

$$\int_0^1 \frac{x^4(1-x)^4}{x^2+1} dx = \int_0^1 \frac{x^4(1-4x+6x^2-4x^3+x^4)}{x^2+1} dx$$

$$= \int_0^1 \frac{x^6-4x^5+6x^4-4x^3+x^2}{x^2+1} dx$$

• BY LONG DIVISION NEXT

$$\begin{array}{r} x^4 - 4x^3 + 6x^2 - 4x + 4 \\ x^2 + 1 \overline{) x^6 - 4x^5 + 6x^4 - 4x^3 + x^2} \\ \underline{-x^6} \\ 4x^5 \\ \underline{-4x^5} \\ 6x^4 \\ \underline{-6x^4} \\ 4x^3 \\ \underline{-4x^3} \\ 4x^2 \\ \underline{-4x^2} \\ 4x \\ \underline{-4x} \\ 4 \end{array}$$

$$\therefore \frac{x^6-4x^5+6x^4-4x^3+x^2}{x^2+1} = x^4 - 4x^3 + 6x^2 - 4x + \frac{4}{x^2+1}$$

• RETURNING TO THE INTEGRAL

$$\begin{aligned} \dots &= \int_0^1 x^4 - 4x^3 + 6x^2 - 4x + \frac{4}{x^2+1} dx \\ &= \left[\frac{1}{5}x^5 - \frac{4}{4}x^4 + \frac{6}{3}x^3 - \frac{4}{2}x^2 + 4 \arctan x \right]_0^1 \\ &= \left(\frac{1}{5} - 1 + 2 - 2 + 4 \arctan 1 \right) - (0 - 0 + 0 - 0) \\ &= \frac{1}{5} - \frac{4}{5} + 5 - 4 \times \frac{\pi}{4} \\ &= \frac{1}{5} - \frac{4}{5} + 5 - \pi \\ &= 3 + \frac{1}{5} - \pi \\ &= \frac{22}{5} - \pi \end{aligned}$$

Question 277 (****)

Use integration by parts to find a simplified exact value for

$$\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} (\cos 2x + \sin 2x)(\ln \cos x + \ln \sin x) \, dx.$$

You may assume that

$$\int \operatorname{cosec} x \, dx = \ln \left| \tan \left(\frac{1}{2} x \right) \right| + \text{constant}.$$

$$\boxed{}, \quad \boxed{\frac{1}{2} \ln 2}$$

Handwritten solution for Question 277:

$$\begin{aligned} \int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} (\cos 2x + \sin 2x)(\ln \cos x + \ln \sin x) \, dx &= \int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} (\cos 2x + \sin 2x) \ln \left(\frac{1}{2} \sin 2x \right) \, dx \\ &= \left[\frac{1}{2} (\sin 2x - \cos 2x) \ln \left(\frac{1}{2} \sin 2x \right) \right]_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} - \int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} (\sin 2x - \cos 2x) \frac{\cos 2x}{\sin 2x} \, dx \\ &= \left[\frac{1}{2} (\sin 2x - \cos 2x) \ln \left(\frac{1}{2} \sin 2x \right) \right]_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} - \int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} \cos 2x - \frac{\cos^2 2x}{\sin 2x} \, dx \\ &= \left[\frac{1}{2} (\sin 2x - \cos 2x) \ln \left(\frac{1}{2} \sin 2x \right) \right]_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} - \int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} \cos 2x - \frac{1 - \sin^2 2x}{\sin 2x} \, dx \\ &= \left[\frac{1}{2} (\sin 2x - \cos 2x) \ln \left(\frac{1}{2} \sin 2x \right) \right]_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} - \int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} \cos 2x + \sin 2x - \operatorname{cosec} 2x \, dx \\ &= \left[\frac{1}{2} (\sin 2x - \cos 2x) \ln \left(\frac{1}{2} \sin 2x \right) - \frac{1}{2} \sin 2x + \frac{1}{2} \cos 2x + \frac{1}{2} \ln |\sin 2x| \right]_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} \\ &= \left[\frac{1}{2} \ln \left(\frac{1}{2} \sin \pi \right) - 0 - \frac{1}{2} + \frac{1}{2} \ln \left(\frac{1}{2} \sin \frac{\pi}{2} \right) \right] - \left[\frac{1}{2} \ln \frac{1}{2} - \frac{1}{2} + 0 + \frac{1}{2} \ln 1 \right] \\ &= \frac{1}{2} \ln \left(\sin \frac{\pi}{2} \cos \frac{\pi}{2} \right) - \frac{1}{2} + \frac{1}{2} \ln \left(\frac{\sin \pi}{\cos \pi} \right) - \frac{1}{2} \ln \frac{1}{2} + \frac{1}{2} \\ &= \frac{1}{2} \ln \left[\sin \frac{\pi}{2} \cos \frac{\pi}{2} \times \frac{\sin \frac{\pi}{2}}{\cos \frac{\pi}{2}} \right] + \frac{1}{2} \ln 2 \\ &= \frac{1}{2} \ln 1 + \frac{1}{2} \ln 2 \\ &= \frac{1}{2} \ln 2 \end{aligned}$$

Integration by parts table:

Integration by parts	
$u = \ln \left(\frac{1}{2} \sin 2x \right)$	$\frac{d}{dx} \ln \left(\frac{1}{2} \sin 2x \right) = \frac{\cos 2x}{\sin 2x}$
$v = \cos 2x - \cos 2x$	$\frac{d}{dx} (\cos 2x - \cos 2x) = -2 \sin 2x$

Final result:

$$\int \operatorname{cosec} 2x \, dx = \ln \left| \tan \left(\frac{1}{2} \sin 2x \right) \right| + C$$

$$\int \operatorname{cosec} 2x \, dx = \frac{1}{2} \ln |\sin 2x| + C$$

Question 278 (****)

It is given that

$$x^2 + x + 2 = (u - x)^2.$$

a) Show clearly that ...

i. ... $x = \frac{u^2 - 2}{2u + 1}.$

ii. ... $\frac{dx}{du} = \frac{2(u^2 + u + 2)}{(2u + 1)^2}.$

b) Find a simplified expression for

$$\int \frac{1}{x\sqrt{x^2 + x + 2}} dx.$$

$$\boxed{}, \frac{1}{\sqrt{2}} \ln \left| \frac{x + \sqrt{x^2 + x + 2} - \sqrt{2}}{x + \sqrt{x^2 + x + 2} + \sqrt{2}} \right| + C$$

$x^2 + x + 2 = (u - x)^2$
 $x^2 + x + 2 = u^2 - 2ux + x^2$
 $x + 2 = u^2 - 2ux$
 $2ux + x = u^2 - 2$
 $x(2u + 1) = u^2 - 2$
 $x = \frac{u^2 - 2}{2u + 1}$
 As required

$\frac{dx}{du} = \frac{(2u)(2u) - 2(u^2 - 2)}{(2u + 1)^2}$
 $\frac{dx}{du} = \frac{4u^2 + 4 - 2u^2 + 4}{(2u + 1)^2}$
 $\frac{dx}{du} = \frac{2u^2 + 8}{(2u + 1)^2}$
 $\frac{dx}{du} = \frac{2(u^2 + 4)}{(2u + 1)^2}$

(b) $\int \frac{1}{x\sqrt{x^2 + x + 2}} dx = \int \frac{1}{x(u - x)} \times \frac{2(u^2 + 4)}{(2u + 1)^2} du$
 $= \int \frac{\frac{u^2 - 2}{2u + 1} \left(u - \frac{u^2 - 2}{2u + 1} \right)}{\frac{u^2 - 2}{2u + 1} \times \frac{2(u^2 + 4)}{(2u + 1)^2}} du$
 $= \int \frac{\frac{u^2 - 2}{2u + 1} \times \frac{2u^2 + 4u + 2}{2u + 1}}{\frac{u^2 - 2}{2u + 1} \times \frac{2(u^2 + 4)}{(2u + 1)^2}} du = \int \frac{(2u^2 + 4u + 2)}{2(u^2 + 4)} du$
 $= \int \frac{u^2 + 2u + 1}{u^2 + 4} du = \int \frac{(u + 1)^2}{u^2 + 4} du$
 Now by partial fractions or long-divisional resor: $\frac{(u + 1)^2}{u^2 + 4} = \frac{1}{2} + \frac{u + 2}{u^2 + 4}$
 $= \frac{1}{2} \ln \left| \frac{u - 2}{u + 2} \right| + C$
 $= \frac{1}{\sqrt{2}} \ln \left| \frac{x + \sqrt{x^2 + x + 2} - \sqrt{2}}{x + \sqrt{x^2 + x + 2} + \sqrt{2}} \right| + C$
 Now $u - x = \sqrt{x^2 + x + 2}$
 $u = x + \sqrt{x^2 + x + 2}$

Question 279 (****)

It is given that a and b are distinct real constants and λ is a real parameter.

- a) Starting by the relationship between two functions of x , $f(x)$ and $g(x)$

$$[\lambda f(x) + g(x)]^2 \geq 0,$$

show clearly that

$$\lambda^2 \int_a^b [f(x)]^2 dx + 2\lambda \int_a^b f(x)g(x) dx + \int_a^b [g(x)]^2 dx \geq 0.$$

- b) Deduce the Cauchy Schwarz inequality for integrals

$$\left[\int_a^b f(x)g(x) dx \right]^2 \leq \left[\int_a^b [f(x)]^2 dx \right] \left[\int_a^b [g(x)]^2 dx \right].$$

[continues overleaf]

[continued from overleaf]

c) By letting $f(x) = \sqrt{\sin x}$ and $g(x) = 1$, show that

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin x} \, dx \leq \sqrt{\frac{\pi}{2}}.$$

d) By letting $f(x) = \sqrt{\sin x}$ and $g(x) = \cos x$, show that


$$\int_0^{\frac{\pi}{2}} \sqrt{\sin x} \, dx \geq \frac{64}{25\pi}.$$

proof

(a) $[f(x) + g(x)]^2 \geq 0$
 $\lambda^2 [f(x)]^2 + 2\lambda f(x)g(x) + [g(x)]^2 \geq 0$
 $\int_a^b \lambda^2 [f(x)]^2 dx + 2\lambda \int_a^b f(x)g(x) dx + \int_a^b [g(x)]^2 dx \geq [C]^b_a$
 $\lambda^2 \int_a^b [f(x)]^2 dx + 2\lambda \int_a^b f(x)g(x) dx + \int_a^b [g(x)]^2 dx \geq C - C$
 $\lambda^2 \int_a^b [f(x)]^2 dx + 2\lambda \int_a^b f(x)g(x) dx + \int_a^b [g(x)]^2 dx \geq 0$

(b) If the quadratic inequality for λ , since the definite integrals are constant numbers, the discriminant must be negative or zero.

This
 $4 \left[\int_a^b f(x)g(x) dx \right]^2 - 4 \left[\int_a^b [f(x)]^2 dx \right] \left[\int_a^b [g(x)]^2 dx \right] \leq 0$
 $\left[\int_a^b f(x)g(x) dx \right]^2 \leq \int_a^b [f(x)]^2 dx \int_a^b [g(x)]^2 dx$



(c) $\begin{cases} f(x) = 1 \\ g(x) = \sqrt{\sin x} = (\sin x)^{\frac{1}{2}} \end{cases}$

$$\left[\int_0^{\frac{\pi}{2}} (1 \cdot \sqrt{\sin x}) dx \right]^2 \leq \int_0^{\frac{\pi}{2}} 1^2 dx \int_0^{\frac{\pi}{2}} (\sqrt{\sin x})^2 dx$$

$$\left[\int_0^{\frac{\pi}{2}} \sqrt{\sin x} dx \right]^2 \leq \int_0^{\frac{\pi}{2}} 1 dx \int_0^{\frac{\pi}{2}} \sin x dx$$

$$\left[\int_0^{\frac{\pi}{2}} \sqrt{\sin x} dx \right]^2 \leq \frac{\pi}{2} \times [-\cos x]_0^{\frac{\pi}{2}}$$

$$\left[\int_0^{\frac{\pi}{2}} \sqrt{\sin x} dx \right]^2 \leq \frac{\pi}{2} \times 1$$

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin x} dx \leq \sqrt{\frac{\pi}{2}}$$

(d) $\begin{cases} f(x) = (\sin x)^{\frac{1}{2}} \\ g(x) = \cos x \end{cases}$

$$\left[\int_0^{\frac{\pi}{2}} (\sin x)^{\frac{1}{2}} \cos x dx \right]^2 \leq \int_0^{\frac{\pi}{2}} (\sin x)^{\frac{1}{2}} dx \int_0^{\frac{\pi}{2}} \cos^2 x dx$$

$$\left[\frac{2}{3} (\sin x)^{\frac{3}{2}} \right]_0^{\frac{\pi}{2}} \leq \int_0^{\frac{\pi}{2}} \sqrt{\sin x} dx \int_0^{\frac{\pi}{2}} \frac{1}{2} (1 + \cos 2x) dx$$

$$\left(\frac{2}{3} \right)^2 \leq \int_0^{\frac{\pi}{2}} \sqrt{\sin x} dx \times \left[\frac{1}{2} x + \frac{1}{4} \sin 2x \right]_0^{\frac{\pi}{2}}$$

$$\frac{16}{9} \leq \int_0^{\frac{\pi}{2}} \sqrt{\sin x} dx \times \frac{\pi}{4}$$

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin x} dx \geq \frac{64}{25\pi}$$

Question 280 (****)

By using the substitution $\sqrt{x} = \tan \theta$, or otherwise, find

$$\int \frac{(x+3)\sqrt{x}}{(x+1)^2} dx.$$

$$\boxed{}, \boxed{\frac{2x^{\frac{3}{2}}}{x+1} + C}$$

using the substitution given

$$\sqrt{x} = \tan \theta \quad [16 \quad \theta = \arcsin \sqrt{x}]$$

$$x = \tan^2 \theta$$

$$dx = 2 \sec^2 \theta \tan \theta d\theta$$

then substitute into the integrand

$$\int \frac{(x+3)\sqrt{x}}{(x+1)^2} dx = \int \frac{(3+\tan^2 \theta) \tan \theta}{(1+\tan^2 \theta)^2} \times 2 \sec^2 \theta \tan \theta d\theta$$

$$= \int \frac{2 \sec^2 \theta \tan^2 \theta (3+\tan^2 \theta)}{(\sec^2 \theta)^2} d\theta$$

$$= \int \frac{2 \tan^2 \theta (3+\tan^2 \theta)}{\sec^2 \theta} d\theta$$

substitute all into sec^2

$$= \int \frac{2(\sec^2 \theta - 1)(3 + \sec^2 \theta - 1)}{\sec^2 \theta} d\theta$$

$$= \int \frac{2(\sec^2 \theta - 1)(\sec^2 \theta + 2)}{\sec^2 \theta} d\theta$$

$$= \int \frac{2 \sec^2 \theta + 2 \sec^2 \theta - 4}{\sec^2 \theta} d\theta$$

$$= \int 2 \sec^2 \theta + 2 - \frac{4}{\sec^2 \theta} d\theta$$

$$= \int 2 \sec^2 \theta + 2 - 4 \cos^2 \theta d\theta$$

$$= \int 2 \sec^2 \theta + 2 - 2 \cos 2\theta d\theta$$

$$= 2 \tan \theta + 2\theta - \sin 2\theta + C$$

$$= 2 \tan \theta - 2 \sin \theta \cos \theta + C$$

$$= 2 \tan \theta - \frac{2 \sin \theta \cos \theta}{\frac{1}{\sec^2 \theta}} + C$$

$$= 2 \tan \theta - \frac{2 \sin \theta}{1 + \tan^2 \theta} + C$$

$$= 2\sqrt{x} - \frac{2\sqrt{x}}{1+x} + C$$

$$= 2\sqrt{x} \left[1 - \frac{1}{x+1} \right] + C$$

$$= 2\sqrt{x} \left[\frac{x+1-1}{x+1} \right] + C$$

$$= 2\sqrt{x} \left(\frac{x}{x+1} \right) + C$$

$$= \frac{2x^{\frac{3}{2}}}{x+1} + C$$

Question 281 (****)

By using the substitution $u = \sec x + \sqrt{\tan x}$, or otherwise, find

$$\int \frac{1+2\sin x\sqrt{\tan x}}{2\left[1+\cos x\sqrt{\tan x}\right]\cos x\sqrt{\tan x}} dx.$$

$$\boxed{}, \quad \ln|\sec x + \sqrt{\tan x}| + C$$

[illegible]

Question 282 (****)

It is given that

$$\int \frac{\cot x \operatorname{cosec} x + 2 \cot x}{1 + \operatorname{cosec} x} dx \equiv \ln \left| [1 + f(x)] f(x) \right| + \text{constant}.$$

Using integration techniques, determine an expression for $f(x)$.

, $f(x) = \sin x$

LOOKING AT THE INTEGRAND & NOTING THAT $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

$$\begin{aligned} \int \frac{\cot x \operatorname{cosec} x + 2 \cot x}{1 + \operatorname{cosec} x} dx &= \int \frac{\operatorname{cosec} x \cot x}{1 + \operatorname{cosec} x} dx + \int \frac{2 \cot x}{1 + \operatorname{cosec} x} dx \\ &= \int \frac{\operatorname{cosec} x \cot x}{1 + \operatorname{cosec} x} dx + \int \frac{2 \cot x}{1 + \operatorname{cosec} x} dx \\ &= -\ln |1 + \operatorname{cosec} x| + 2 \int \frac{\cot x}{1 + \operatorname{cosec} x} dx \\ &= -\ln |1 + \operatorname{cosec} x| + 2 \int \frac{\cot x}{1 + \operatorname{cosec} x} dx \quad \leftarrow \text{NOTING "TOP" OF "BOTTOM" IS "1" OR "1-x"} \\ &= -\ln |1 + \operatorname{cosec} x| + 2 \int \frac{\cot x}{1 + \operatorname{cosec} x} dx \\ \text{NOW THE NUMERATOR OF THE FRACTION "BOTTOM" DIFFERENTIATES TO TOP} \\ &= -\ln |1 + \operatorname{cosec} x| + 2 \ln |1 + \operatorname{cosec} x| + C \\ &= \ln |(1 + \operatorname{cosec} x)^2| - \ln |1 + \operatorname{cosec} x| + C \\ &= \ln (1 + \operatorname{cosec} x)^2 - \ln |1 + \operatorname{cosec} x| + C \\ &= \ln (1 + \operatorname{cosec} x)^2 - \ln \left| \frac{1 + \operatorname{cosec} x}{\operatorname{cosec} x} \right| + C \\ &= \ln (1 + \operatorname{cosec} x)^2 + \ln \left| \frac{\operatorname{cosec} x}{1 + \operatorname{cosec} x} \right| + C \\ &= \ln \left| (1 + \operatorname{cosec} x)^2 \times \frac{\operatorname{cosec} x}{1 + \operatorname{cosec} x} \right| + C \\ &= \ln \left| (1 + \operatorname{cosec} x) \operatorname{cosec} x \right| + C \quad \text{I.E. } f(x) = \operatorname{cosec} x \end{aligned}$$

Question 283 (****)

$$I = \int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi} \frac{6}{\sin x + \sin 2x} dx$$

Use appropriate integration techniques to show that

$$I = A \ln N + B \ln M,$$

where A , B , N and M are integers to be found.

$$\boxed{I = 8 \ln 2 - 3 \ln 3}$$

The image shows a handwritten solution for the integral $I = \int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi} \frac{6}{\sin x + \sin 2x} dx$. The solution is divided into two main parts: substitution and partial fractions.

By Substitution:

Let $u = \cos 2x$. Then $\frac{du}{dx} = -2 \sin 2x$, so $dx = -\frac{du}{2 \sin 2x}$.

When $x = \frac{1}{3}\pi$, $u = \cos \frac{2}{3}\pi = -\frac{1}{2}$.
 When $x = \frac{1}{2}\pi$, $u = \cos \pi = -1$.

The integral becomes:

$$I = \int_{-\frac{1}{2}}^{-1} \frac{6}{\sin 2x + 2 \sin 2x \cos 2x} \left(-\frac{du}{2 \sin 2x}\right) = \int_{-\frac{1}{2}}^{-1} \frac{-3}{\sin 2x (1 + \cos 2x)} du$$

Since $\sin 2x = \sqrt{1 - u^2}$ and $\cos 2x = u$, we have:

$$I = \int_{-\frac{1}{2}}^{-1} \frac{-3}{\sqrt{1 - u^2} (1 + u)} du$$

By Partial Fractions:

We decompose the integrand into partial fractions:

$$\frac{-3}{(1-u)(1+u)} = \frac{A}{1-u} + \frac{B}{1+u}$$

Solving for A and B :

$$-3 = A(1+u) + B(1-u)$$

Let $u = 1$: $-3 = A(2) \Rightarrow A = -\frac{3}{2}$
 Let $u = -1$: $-3 = B(2) \Rightarrow B = -\frac{3}{2}$

Thus, the integral becomes:

$$I = \int_{-\frac{1}{2}}^{-1} \left(\frac{-\frac{3}{2}}{1-u} + \frac{-\frac{3}{2}}{1+u} \right) du$$

Evaluating the integral:

$$I = \left[-\frac{3}{2} \ln|1-u| - \frac{3}{2} \ln|1+u| \right]_{-\frac{1}{2}}^{-1}$$

At $u = -1$: $-\frac{3}{2} \ln|1-(-1)| - \frac{3}{2} \ln|1+(-1)| = -\frac{3}{2} \ln 2 - \frac{3}{2} \ln 0$ (undefined)
 At $u = -\frac{1}{2}$: $-\frac{3}{2} \ln|1-(-\frac{1}{2})| - \frac{3}{2} \ln|1+(-\frac{1}{2})| = -\frac{3}{2} \ln \frac{3}{2} - \frac{3}{2} \ln \frac{1}{2}$

Therefore, the final result is:

$$I = 8 \ln 2 - 3 \ln 3$$

Question 284 (****)

Use the substitution $x = \tan\left(\frac{1}{2}\theta\right)$, to find a simplified expression for

$$\int x \arccos\left[\frac{1-x^2}{1+x^2}\right] dx.$$

$$\boxed{}, -x + (1+x^2) \arctan x + \text{constant}$$

CONSIDER THE SUBSTITUTION GIVEN

$$x = \tan\left(\frac{1}{2}\theta\right)$$

$$dx = \frac{1}{2} \sec^2\left(\frac{1}{2}\theta\right) d\theta$$

OR $\left[\frac{dx}{d\theta} = \frac{1}{2} (1 + \tan^2\left(\frac{1}{2}\theta\right)) \right]$

OR $\left[\frac{dx}{d\theta} = \frac{1}{2} (1 + x^2) \right]$

(ONE SIMPLER SEE UNITARY PAPER II BOTTOM FOR THE QUESTION)

$$\frac{1-x^2}{1+x^2} = \frac{1 - \tan^2\left(\frac{1}{2}\theta\right)}{1 + \tan^2\left(\frac{1}{2}\theta\right)}$$

$$= \frac{1 - \tan^2\left(\frac{1}{2}\theta\right)}{\sec^2\left(\frac{1}{2}\theta\right)}$$

$$= \frac{1}{\sec^2\left(\frac{1}{2}\theta\right)} - \frac{\tan^2\left(\frac{1}{2}\theta\right)}{\sec^2\left(\frac{1}{2}\theta\right)}$$

$$= \cos^2\left(\frac{1}{2}\theta\right) - \sin^2\left(\frac{1}{2}\theta\right)$$

$$= \cos(\theta)$$

TRANSFORMING THE INTEGRAL WE HAVE

$$\int x \arccos\left(\frac{1-x^2}{1+x^2}\right) dx = \int \tan\left(\frac{1}{2}\theta\right) \arccos(\cos(\theta)) \left[\frac{1}{2} \sec^2\left(\frac{1}{2}\theta\right) d\theta\right]$$

$$= \int \frac{1}{2} \tan\left(\frac{1}{2}\theta\right) \arccos(\cos(\theta)) \sec^2\left(\frac{1}{2}\theta\right) d\theta$$

INTEGRATION BY PARTS

$\frac{1}{2}\theta$	$\frac{1}{2}$
$\tan\left(\frac{1}{2}\theta\right)$	$\tan\left(\frac{1}{2}\theta\right) \sec^2\left(\frac{1}{2}\theta\right)$

$$= \frac{1}{2}\theta \tan\left(\frac{1}{2}\theta\right) - \int \frac{1}{2} \tan^2\left(\frac{1}{2}\theta\right) d\theta$$

$$= \frac{1}{2}\theta \tan\left(\frac{1}{2}\theta\right) - \frac{1}{2} \int \sec^2\theta - 1 d\theta$$

$$= \frac{1}{2}\theta \tan\left(\frac{1}{2}\theta\right) - \frac{1}{2} \left[2 \tan\left(\frac{1}{2}\theta\right) - \theta \right] + C$$

$$= \frac{1}{2}\theta \tan\left(\frac{1}{2}\theta\right) - \tan\left(\frac{1}{2}\theta\right) + \frac{1}{2}\theta + C$$

$$= \frac{1}{2}\theta (1 + \tan^2\left(\frac{1}{2}\theta\right)) - \tan\left(\frac{1}{2}\theta\right) + C$$

$x = \tan\left(\frac{1}{2}\theta\right)$

$\arctan x = \frac{1}{2}\theta$

$$= [\arctan x] [1+x^2] - x + C$$

$$= -x + (1+x^2) \arctan x + C$$

Question 285 (****)

By using a suitable trigonometric substitution, or otherwise, find

$$\int \frac{(3x^2 + 5x)\sqrt{x}}{(x+1)^2} dx.$$

$$\boxed{}, \frac{2x^{\frac{5}{2}}}{x+1} + C$$

SOLVE BY THE SUBSTITUTION $\sqrt{x} = \tan \theta$

$\int \frac{(3x^2 + 5x)\sqrt{x}}{(x+1)^2} dx$

$= \int \frac{(3\tan^4 \theta + 5\tan^3 \theta) \tan \theta}{(\tan^2 \theta + 1)^2} (2\tan \theta \sec^2 \theta d\theta)$

$= \int \frac{2\tan^6 \theta \sec^2 \theta (3\tan \theta + 5\tan^2 \theta)}{\sec^4 \theta} d\theta$

$= \int \frac{6\tan^7 \theta \sec^2 \theta + 10\tan^8 \theta \sec^2 \theta}{\sec^4 \theta} d\theta$

$= \int \frac{6\tan^7 \theta}{\sec^2 \theta} + \frac{10\tan^8 \theta}{\sec^2 \theta} d\theta = \int 6\tan^7 \theta \cos^2 \theta + 10\tan^8 \theta \cos^2 \theta d\theta$

$= \int \frac{6\sin^7 \theta}{\cos^2 \theta} \times \cos^2 \theta + \frac{10\sin^8 \theta}{\cos^2 \theta} \times \cos^2 \theta d\theta$

$= \int \frac{6\sin^7 \theta}{\cos^2 \theta} + \frac{10\sin^8 \theta}{\cos^2 \theta} d\theta$

$= \int \frac{6(1-\cos^2 \theta)^3 + 10(1-\cos^2 \theta)^4}{\cos^2 \theta} d\theta$

$= \int \frac{6(1-3\cos^2 \theta + 3\cos^4 \theta - \cos^6 \theta) + 10(1-4\cos^2 \theta + 6\cos^4 \theta - 4\cos^6 \theta + \cos^8 \theta)}{\cos^2 \theta} d\theta$

$= \int \frac{6 - 18\cos^2 \theta + 18\cos^4 \theta - 6\cos^6 \theta + 10 - 40\cos^2 \theta + 60\cos^4 \theta - 40\cos^6 \theta + 10\cos^8 \theta}{\cos^2 \theta} d\theta$

$= \int (6\sec^2 \theta - 8\sec^2 \theta + 4\sec^2 \theta - 2 d\theta)$

BRINGING THE INTEGRAL AS FOLLOWS

$= \int (6\sec^2 \theta (1 + \tan^2 \theta) - 8\sec^2 \theta + 4(\frac{1}{2} + \frac{1}{2}\sec^2 \theta) - 2) d\theta$

$= \int (6\sec^2 \theta + 6\sec^2 \theta \tan^2 \theta - 8\sec^2 \theta + 2 + \sec^2 \theta - 2) d\theta$

$= \int 6\sec^2 \theta \tan^2 \theta - 2\sec^2 \theta + 2\cos^2 \theta d\theta$

$= 2\tan^3 \theta - 2\tan \theta + \sin 2\theta + C$

$= 2\tan^3 \theta - 2\tan \theta + 2\sin \theta \cos \theta + C$

$= 2\tan^3 \theta - 2\tan \theta + \frac{2\sin \theta \cos \theta}{\cos^2 \theta} + C$

$= 2\tan^3 \theta - 2\tan \theta + \frac{2\tan \theta}{1 + \tan^2 \theta} + C$

$= 2x^{\frac{3}{2}} - 2x^{\frac{1}{2}} + \frac{2x^{\frac{1}{2}}}{1+x} + C$

$= \frac{2x^{\frac{3}{2}}}{1+x} + C$

Question 286 (****)

The function f is defined as

$$f(x) \equiv 2^{\ln x}, \quad x \in [1, \infty).$$

Show, with details workings, that

$$\int_1^e f(x) \, dx = \frac{2e-1}{1+\ln 2}.$$

, proof

TRANSFORM INTO EXPONENTIALS AS FOLLOWS

$$\int_1^e 2^{\ln x} \, dx = \int_1^e e^{\ln(2^{\ln x})} \, dx = \int_1^e e^{(\ln x)(\ln 2)} \, dx$$

MANIPULATE FURTHER

$$= \int_1^e (e^{\ln x})^{\ln 2} \, dx = \int_1^e x^{\ln 2} \, dx$$

INTEGRATE AND EVALUATE

$$= \left[\frac{1}{1+\ln 2} x^{1+\ln 2} \right]_1^e = \frac{1}{1+\ln 2} \left[e^{1+\ln 2} - 1^{1+\ln 2} \right]$$

$$= \frac{1}{1+\ln 2} \left[e \times e^{\ln 2} - 1 \right] = \frac{1}{1+\ln 2} [2e - 1]$$

$$\therefore \int_1^e 2^{\ln x} \, dx = \frac{2e-1}{1+\ln 2}$$

Question 287 (****)

A function F is defined by the integral

$$F(x) \equiv \int_1^x \frac{e^t}{t} dt, \quad x \geq 1.$$

Find a simplified expression, in terms of F , for

$$\int_1^x \frac{e^t}{t+a} dt,$$

where a is a positive constant.

$$\boxed{}, \quad \int_1^x \frac{e^t}{t+a} dt = e^{-a} [F(x+a) - F(a+1)]$$

Handwritten solution showing the derivation of the result:

$$F(x) = \int_1^x \frac{e^t}{t} dt$$

... By substitution

$$\int_1^x \frac{e^t}{t+a} dt = \int_{a+1}^{x+a} \frac{e^{u-a}}{u} du$$

Let $u = t+a$
 $du = dt$
 $t=1 \rightarrow u=a+1$
 $t=x \rightarrow u=x+a$

$$= e^{-a} \int_{a+1}^{x+a} \frac{e^u}{u} du = e^{-a} [F(x+a) - F(a+1)]$$

Question 288 (****)

It is given that

$$u^2 = \frac{1-x^2}{(1-x)^2}, \quad x \neq \pm 1.$$

a) Show clearly that ...

i. ... $x = \frac{u^2 - 1}{u^2 + 1}.$

ii. ... $1 - x^2 = \frac{4u^2}{(u^2 + 1)^2}$

iii. ... $\frac{dx}{du} = \frac{4u}{(u^2 + 1)^2}.$

b) Hence show further that

$$\int \frac{3}{(4x+5)\sqrt{1-x^2}-3(1-x^2)} dx = \frac{2\sqrt{1-x}}{\sqrt{1-x}-3\sqrt{1+x}} + \text{constant}.$$

□, proof

a) i) TRANSFORM AS RATIONAL

$$u^2 = \frac{1-x^2}{(1-x)^2} = \frac{(1-x)(1+x)}{(1-x)^2} = \frac{1+x}{1-x}$$

$$\Rightarrow u^2(1-x) = 1+x$$

$$\Rightarrow u^2 - xu^2 = 1+x$$

$$\Rightarrow u^2 - 1 = xu^2 + x$$

$$\Rightarrow u^2 - 1 = x(u^2 + 1)$$

$$\Rightarrow x = \frac{u^2 - 1}{u^2 + 1} \quad \text{As required}$$

ii) TRANSFORM AS RATIONAL

$$1-x^2 = 1 - \left(\frac{u^2-1}{u^2+1}\right)^2 = 1 - \frac{u^4 - 2u^2 + 1}{u^4 + 2u^2 + 1}$$

$$= \frac{u^4 + 2u^2 + 1 - (u^4 - 2u^2 + 1)}{u^4 + 2u^2 + 1}$$

$$= \frac{u^4 + 2u^2 + 1 - u^4 + 2u^2 - 1}{u^4 + 2u^2 + 1}$$

$$= \frac{4u^2}{(u^2 + 1)^2}$$

$$= \frac{4u^2}{(u^2 + 1)^2} \quad \text{As required}$$

iii) DIFFERENTIATE (2) WITH RESPECT TO 'u'

$$x = \frac{u^2 - 1}{u^2 + 1} = \frac{(u^2 + 1) - 2}{u^2 + 1} = 1 - \frac{2}{u^2 + 1}$$

$$\frac{dx}{du} = 0 + 2(u^2 + 1)^{-2} \left(\frac{du}{du}\right)$$

$$\frac{dx}{du} = \frac{4u}{(u^2 + 1)^2} \quad \text{As required}$$

b) CONVERT THE EQUATION FROM PART (a)

$$\int \frac{3}{(4x+5)\sqrt{1-x^2}-3(1-x^2)} dx$$

$$= \int \frac{3}{\left(4\left(\frac{u^2-1}{u^2+1}\right) + 5\right) \left(\frac{4u^2}{(u^2+1)^2}\right) - 3\left(\frac{4u^2}{(u^2+1)^2}\right)} \times \frac{4u}{(u^2+1)^2} du$$

$$= \int \frac{4\left(\frac{4u^2-4}{u^2+1} + 5\right) \frac{4u^2}{(u^2+1)^2} - 3\left(\frac{4u^2}{(u^2+1)^2}\right)}{du} du$$

$$= \int \frac{\frac{16u^2-16}{u^2+1} + 20u^2 - \frac{12u^2}{u^2+1}}{(u^2+1)^3} du$$

$$= \int \frac{4u^2 + 20u^2}{(u^2+1)^3} du = \int \frac{24u^2}{(u^2+1)^3} du = \int \frac{6}{(u^2+1)^2} du$$

$$= \int \frac{6}{u^4 + 2u^2 + 1} du = \int \frac{6}{(u^2+1)^2} du = \int \frac{6}{(u^2+1)^2} du$$

$$= -2(3u+1)^{-1} + C = \frac{-2}{3u+1} + C = \frac{2}{1-3u} + C$$

$$= \frac{2}{1-3\left(\frac{u^2-1}{u^2+1}\right)} + C = \frac{2(1-x)}{1-x-3(1-x)} + C$$

$$= \frac{2(1-x)}{(1-x)-3(1-x)(1+x)} + C = \frac{2\sqrt{1-x}}{\sqrt{1-x}-3\sqrt{1+x}} + C$$

Question 289 (****)

By using the substitution $x = 2 \tan^2 \theta$, or otherwise, find

$$\int \frac{2-x}{\sqrt{x}(x+2)^2} dx.$$

$$\boxed{}, \frac{2\sqrt{x}}{x+2} + C$$

$$\int \frac{2-x}{\sqrt{x}(x+2)^2} dx$$
 By substitution

$$x = 2 \tan^2 \theta$$

$$\frac{dx}{d\theta} = 4 \tan \theta \sec^2 \theta$$

$$dx = 4 \tan \theta \sec^2 \theta d\theta$$

$$\frac{x}{2} = \tan^2 \theta$$

$$\tan \theta = \left(\frac{x}{2}\right)^{\frac{1}{2}}$$

$$= \int \frac{2 - 2 \tan^2 \theta}{\sqrt{2 \tan^2 \theta} (2 \tan^2 \theta + 2)^2} (4 \tan \theta \sec^2 \theta d\theta)$$

$$= \int \frac{8(1 - \tan^2 \theta) \tan \theta \sec^2 \theta}{\sqrt{2} \tan \theta \times 4 (\tan^2 \theta + 1)^2} d\theta = \frac{2}{\sqrt{2}} \int \frac{(1 - \tan^2 \theta) \sec^2 \theta}{(\sec^2 \theta)^2} d\theta$$

$$= \sqrt{2} \int \frac{(1 - \tan^2 \theta) \sec^2 \theta}{\sec^4 \theta} d\theta = \sqrt{2} \int \frac{1 - \tan^2 \theta}{\sec^2 \theta} d\theta$$

$$= \sqrt{2} \int \frac{1}{\sec^2 \theta} - \frac{\tan^2 \theta}{\sec^2 \theta} d\theta = \sqrt{2} \int \cos^2 \theta - \frac{\sin^2 \theta}{\cos^2 \theta} \cos^2 \theta d\theta$$

$$= \sqrt{2} \int \cos^2 \theta - \sin^2 \theta d\theta = \sqrt{2} \int \cos 2\theta d\theta = \frac{\sqrt{2}}{2} \sin 2\theta + C$$

$$= \frac{\sqrt{2}}{2} (2 \sin \theta \cos \theta) + C = \sqrt{2} \sin \theta \cos \theta + C = \sqrt{2} \cos \theta \sin \theta + C$$

$$= \sqrt{2} \cos \theta \tan \theta + C = \sqrt{2} \frac{\tan \theta}{\sec \theta} + C = \sqrt{2} \frac{\frac{x}{2}}{\sqrt{1 + \frac{x}{2}}} + C$$

$$= \sqrt{2} \frac{\frac{\sqrt{x}}{2}}{1 + \frac{x}{2}} + C = \frac{\sqrt{2}}{1 + \frac{x}{2}} + C = \frac{2\sqrt{x}}{2+x} + C$$

Question 290 **(****)**

It is given that

$$\sqrt{5-4x-x^2} = (1-x)u, \quad x \neq 1, \quad x \neq -5.$$

a) Show clearly that ...

i. ... $x = \frac{u^2 - 5}{u^2 + 1}$.

ii. ... $dx = \frac{12u}{(u^2 + 1)^2} du$.

b) Hence show further that

$$\int \frac{x}{(5-4x-x^2)^{\frac{3}{2}}} dx = \int \frac{u^2-5}{18u^2} du.$$

c) Find a simplified expression for

$$\int \frac{x}{(5-4x-x^2)^{\frac{3}{2}}} dx .$$

$$\frac{1}{9}, \frac{5-2x}{9\sqrt{5-4x-x^2}} + C$$

[illegible]

By using an appropriate trigonometric substitution, or otherwise, find an exact value for the following integral.

$$\int_7^9 \sqrt{\frac{x-7}{11-x}} \, dx.$$

 \square , $\pi - 2$

QUESTION

Find the value of the integral $\int_7^9 \frac{\sqrt{x-7}}{11-x} dx$

SOLUTION

Let $x = a \cos^2 \theta + b \sin^2 \theta$

Here $a = 7$ and $b = 11$

Let $x = 7 \cos^2 \theta + 11 \sin^2 \theta$

$\Rightarrow x = 7 + 4 \sin^2 \theta$

$\Rightarrow dx = 8 \sin \theta \cos \theta d\theta$

When $x = 7$, $\sin \theta = 0$, $\theta = 0$

When $x = 11$, $\sin^2 \theta = 1$, $\theta = \frac{\pi}{2}$

$\therefore \int_7^9 \frac{\sqrt{x-7}}{11-x} dx = \int_0^{\frac{\pi}{2}} \frac{\sqrt{7+4 \sin^2 \theta - 7}}{11 - (7+4 \sin^2 \theta)} \cdot 8 \sin \theta \cos \theta d\theta$

$= \int_0^{\frac{\pi}{2}} \frac{2 \sin \theta}{4 \cos^2 \theta} \cdot 8 \sin \theta \cos \theta d\theta$

$= \int_0^{\frac{\pi}{2}} \frac{16 \sin^2 \theta}{4 \cos \theta} d\theta$

$= 4 \int_0^{\frac{\pi}{2}} \frac{\sin^2 \theta}{\cos \theta} d\theta$

$= 4 \int_0^{\frac{\pi}{2}} \frac{1 - \cos^2 \theta}{\cos \theta} d\theta$

$= 4 \int_0^{\frac{\pi}{2}} \left(\frac{1}{\cos \theta} - \cos \theta \right) d\theta$

$= 4 \left[\ln |\sec \theta + \tan \theta| - \sin \theta \right]_0^{\frac{\pi}{2}}$

$= 4 \left[\ln |\sec \frac{\pi}{2} + \tan \frac{\pi}{2}| - \sin \frac{\pi}{2} \right] - 4 \left[\ln |\sec 0 + \tan 0| - \sin 0 \right]$

$= 4 \left[\ln |\infty| - 1 \right] - 4 \left[\ln |1| - 0 \right]$

$= 4 \left[\ln \infty - 1 \right] - 4 \left[0 - 0 \right]$

$= 4 \ln \infty - 4$

ANSWERS

1. $\int_0^{\frac{\pi}{2}} \frac{1 + 4 \sin^2 \theta - 1}{11 - (7 + 4 \sin^2 \theta)} \cdot 8 \sin \theta \cos \theta d\theta$

$= \int_0^{\frac{\pi}{2}} \frac{4 \sin^2 \theta}{4 \cos^2 \theta} \cdot 8 \sin \theta \cos \theta d\theta$

$= \int_0^{\frac{\pi}{2}} \frac{32 \sin^2 \theta}{4 \cos \theta} d\theta$

$= \int_0^{\frac{\pi}{2}} \frac{8 \sin^2 \theta}{\cos \theta} d\theta$

$= \int_0^{\frac{\pi}{2}} \frac{8(1 - \cos^2 \theta)}{\cos \theta} d\theta$

$= 8 \int_0^{\frac{\pi}{2}} \left(\frac{1}{\cos \theta} - \cos \theta \right) d\theta$

$= 8 \left[\ln |\sec \theta + \tan \theta| - \sin \theta \right]_0^{\frac{\pi}{2}}$

$= 8 \left[\ln |\sec \frac{\pi}{2} + \tan \frac{\pi}{2}| - \sin \frac{\pi}{2} \right] - 8 \left[\ln |\sec 0 + \tan 0| - \sin 0 \right]$

$= 8 \left[\ln \infty - 1 \right] - 8 \left[0 - 0 \right]$

$= 8 \ln \infty - 8$

Question 292 (****)

$$I = \int_0^{\infty} \frac{1}{\left(x + \sqrt{x^2 + 1}\right)^2} dx.$$

- a) Use the substitution $u = x + \sqrt{x^2 + 1}$ to find the value of I .
- b) Verify the answer to part (a) by a trigonometric substitution.

, $\frac{3}{8}$

a) SPLIT WITH THE SUBSTITUTION GIVEN

$$u = x + \sqrt{x^2 + 1}$$

$$u - x = \sqrt{x^2 + 1}$$

$$u^2 - 2ux + x^2 = x^2 + 1$$

$$u^2 - 1 = 2ux$$

$$x = \frac{u^2 - 1}{2u}$$

$$\frac{dx}{du} = \frac{1}{2} + \frac{1}{2u^2}$$

$$dx = \frac{1}{2} \left(1 + \frac{1}{u^2}\right) du$$

$$dx = \frac{1}{2} \left(\frac{u^2 + 1}{u^2}\right) du$$

Have to find limits

$$\int_0^{\infty} \frac{1}{(\sqrt{x^2 + 1} + x)^2} dx = \int_1^{\infty} \frac{1}{u^2} \times \frac{1}{2} \left(\frac{u^2 + 1}{u^2}\right) du$$

$$= \int_1^{\infty} \frac{1}{2} \frac{u^2 + 1}{u^4} du = \frac{1}{2} \int_1^{\infty} \frac{1}{u^2} + \frac{1}{u^4} du$$

$$= \frac{1}{2} \left[-\frac{1}{2u^2} - \frac{1}{4u^3} \right]_1^{\infty} = \frac{1}{2} \left[\frac{1}{2u^2} + \frac{1}{4u^3} \right]_1^{\infty}$$

$$= \frac{1}{2} \left[\left(\frac{1}{2} + \frac{1}{4}\right) - 0 \right] = \frac{1}{2} \times \frac{3}{4} = \frac{3}{8}$$

b) BY A TRIGONOMETRIC SUBSTITUTION

$$\int_0^{\infty} \frac{1}{(\sqrt{x^2 + 1} + x)^2} dx = \dots$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{(1 + \tan^2 \theta + \tan \theta)^2} (\sec^2 \theta) d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sec^2 \theta}{(\sec \theta + \tan \theta)^2} d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sec \theta}{\left(\frac{1}{\cos \theta} + \frac{\sin \theta}{\cos \theta}\right)^2} d\theta = \int_0^{\frac{\pi}{2}} \frac{\sec \theta}{(1 + \sin \theta)^2} d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos \theta}{(1 + \sin \theta)^2} d\theta = \int_0^{\frac{\pi}{2}} \cos \theta (1 + \sin \theta)^{-2} d\theta$$

BY RECOGNITION

$$= \left[-\frac{1}{2} (1 + \sin \theta)^{-2} \right]_0^{\frac{\pi}{2}} = \left[-\frac{1}{2(1 + \sin \theta)^2} \right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2} - \frac{1}{8} = \frac{3}{8}$$

Question 293 (****)

Find an exact value for the following integral.

$$\int_0^{\pi} x \sin^3 x \, dx.$$

$$\boxed{\frac{5\pi}{3}}, \quad \boxed{\frac{2\pi}{3}}$$

• Let $I = \int_0^{\pi} x \sin^3 x \, dx$
 • Let $u = \pi - x$
 $du = -dx$
 $u=0 \rightarrow x=\pi$
 $u=\pi \rightarrow x=0$
 • Hence we have
 $\Rightarrow I = \int_{\pi}^0 (\pi - x) (\sin(\pi - x))^3 (-dx)$
 $\Rightarrow I = \int_{\pi}^0 (\pi - x) [\sin \pi \cos x - \sin \pi \sin x] dx$ $\sin \pi = 0$
 $\cos \pi = -1$
 $\Rightarrow I = \int_{\pi}^0 (\pi - x) \sin^3 x \, dx$
 $\Rightarrow I = \pi \int_0^{\pi} \sin^3 x \, dx - \int_0^{\pi} x \sin^3 x \, dx$
 $\Rightarrow I = \pi \int_0^{\pi} \sin x \sin^2 x \, dx - I$
 $\Rightarrow 2I = \pi \int_0^{\pi} \sin x (1 - \cos^2 x) \, dx$
 $\Rightarrow 2I = \pi \int_0^{\pi} \sin x - \sin x \cos^2 x \, dx$
 $\Rightarrow 2I = \pi \left[-\cos x + \frac{1}{3} \cos^3 x \right]_0^{\pi}$
 $\Rightarrow I = \frac{\pi}{2} \left[(1 - \frac{1}{3}) - (-1 + \frac{1}{3}) \right]$
 $\Rightarrow \int_0^{\pi} x \sin^3 x \, dx = \frac{\pi}{2} \times \frac{4}{3} = \frac{2\pi}{3}$

Determine, as an exact simplified fraction, the value of the following integral.

$$\int_{\frac{3}{2}}^{\frac{5}{2}} (4x^2 - 16x + 15)^4 dx.$$

$$\boxed{}, \frac{128}{315}$$

[illegible]

Question 295 (****)

Find an exact value for

$$\int_0^{\pi} \frac{x \sin x}{\sqrt{4 - \cos^2 x}} dx.$$

You may assume without proof that

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \arcsin\left(\frac{x}{a}\right) + \text{constant}.$$

$$\boxed{}, \quad \boxed{\frac{\pi^2}{6}}$$

Let $I = \int_0^{\pi} \frac{x \sin x}{\sqrt{4 - \cos^2 x}} dx$.
 Use the substitution $x = \pi - y \Leftrightarrow y = \pi - x$
 $dx = -dy$
 Limits: $x=0 \rightarrow y=\pi$
 $x=\pi \rightarrow y=0$
 $\sin x = \sin(\pi - y) = \sin \pi \cos y - \cos \pi \sin y = \sin y$
 $\cos x = [\cos(\pi - y)] = [\cos \pi \cos y - \sin \pi \sin y] = -\cos y$
 Hence let also $dx = -dy$
 $I = \int_{\pi}^0 \frac{(\pi - y) \sin y}{\sqrt{4 - \cos^2 y}} (-dy) = \int_0^{\pi} \frac{(\pi - y) \sin y}{\sqrt{4 - \cos^2 y}} dy$
 $I = \int_0^{\pi} \frac{\pi \sin y}{\sqrt{4 - \cos^2 y}} dy - \int_0^{\pi} \frac{y \sin y}{\sqrt{4 - \cos^2 y}} dy$
 $I = \pi \int_0^{\pi} \frac{\sin y}{\sqrt{4 - \cos^2 y}} dy - I$
 $2I = \pi \int_0^{\pi} \frac{\sin y}{\sqrt{4 - \cos^2 y}} dy$
 Substitution again
 $v = \cos y$
 $\frac{dv}{dy} = -\sin y$ when $y=0 \rightarrow v=1$
 $y=\pi \rightarrow v=-1$
 $dy = -\frac{dv}{\sin y}$
 $\Rightarrow 2I = \pi \int_1^{-1} \frac{-1}{\sqrt{4 - v^2}} dv = \pi \int_{-1}^1 \frac{1}{\sqrt{4 - v^2}} dv$
 $\Rightarrow 2I = \pi \left[\arcsin \frac{v}{2} \right]_{-1}^1$
 $\Rightarrow 2I = \pi \left[\arcsin \frac{1}{2} - \arcsin \left(-\frac{1}{2}\right) \right]$
 $\Rightarrow 2I = \pi \times \frac{\pi}{6}$
 $\Rightarrow \int_0^{\pi} \frac{x \sin x}{\sqrt{4 - \cos^2 x}} dx = \frac{\pi^2}{6}$

Question 296 (****)

A family of functions, known as the Chebyshev polynomials of the first kind $T_n(x)$, is defined as

$$T_n(x) = \cos(n \arccos x), \quad -1 \leq x \leq 1, \quad n \in \mathbb{N}.$$

Evaluate the following integral

$$\int_{-1}^1 \frac{T_n(x) T_m(x)}{\sqrt{1-x^2}} dx.$$

,

• START BY THE DEFINITION OF $T_n(x)$

$$\int_{-1}^1 \frac{T_n(x) T_m(x)}{\sqrt{1-x^2}} dx = \int_{-1}^1 \frac{\cos(n \arccos x) \cos(m \arccos x)}{\sqrt{1-x^2}} dx$$

• PROCEED BY A SUBSTITUTION

$$= \int_0^\pi \frac{\cos n\theta \cos m\theta}{\sqrt{1-\cos^2\theta}} (-\sin\theta d\theta)$$

$$= \int_0^\pi \frac{\cos n\theta \cos m\theta}{\sin\theta} \sin\theta d\theta$$

$$= \int_0^\pi \cos n\theta \cos m\theta d\theta$$

$$\begin{aligned} x &= \cos\theta \implies \theta = \arccos x \\ dx &= -\sin\theta d\theta \\ x = -1 &\implies \theta = \pi \\ x = 1 &\implies \theta = 0 \end{aligned}$$

• USING THE COMPOUND ANGLE IDENTITIES

$$\begin{aligned} \cos(n\theta + m\theta) &= \cos n\theta \cos m\theta - \sin n\theta \sin m\theta \\ \cos(n\theta - m\theta) &= \cos n\theta \cos m\theta + \sin n\theta \sin m\theta \end{aligned} \quad \text{Hence}$$

$$2\cos n\theta \cos m\theta = \cos(n\theta + m\theta) + \cos(n\theta - m\theta)$$

• RETURNING TO THE INTEGRAL

$$= \int_0^\pi \frac{1}{2} [\cos(n+m)\theta] + \cos(n-m)\theta d\theta$$

$$= \left[\frac{1}{2(n+m)} \sin(n+m)\theta + \frac{1}{2(n-m)} \sin(n-m)\theta \right]_0^\pi = 0 //$$

As $\sin k\theta = 0, \quad k \in \mathbb{Z} \quad \nmid \quad n+m, n-m \in \mathbb{Z}$

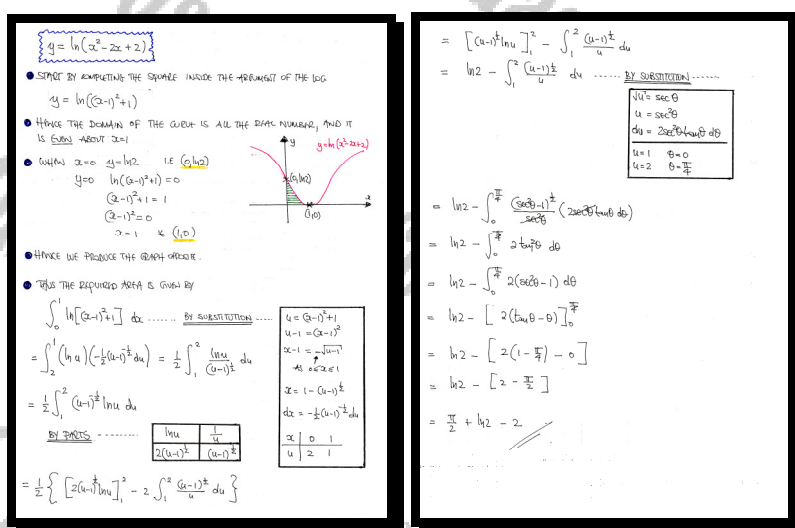
Question 297 (****)

The function $y = f(x)$ is defined in the largest possible real domain by

$$f(x) \equiv \ln[x^2 - 2x + 2].$$

Sketch the graph of $f(x)$ and determine an exact simplified value for the area of the finite region bounded by the graph of $f(x)$ and the coordinate axes.

$$\boxed{}, \quad \boxed{\frac{1}{2}\pi - 2 + \ln 2}$$



Question 298 (****)

$$I = \int_1^3 (3-x)^7 (x-1)^7 dx.$$

Show that

$$I = \frac{(7!)^2 \times 2^{15}}{15!}.$$

 , proof

• Let $I(7,7) = \int_1^3 (3-x)^7 (x-1)^7 dx$

• Proceed by integration by parts

$\frac{(3-x)^7}{\frac{1}{8}(x-1)^8}$	$\frac{-7(3-x)^6}{(x-1)^7}$
--------------------------------------	-----------------------------

$\Rightarrow I(7,7) = \left[\frac{(3-x)^7}{\frac{1}{8}(x-1)^8} \right]_1^3 + \frac{7}{8} \int_1^3 (x-1)^8 (3-x)^6 dx$

$\Rightarrow I(7,7) = \frac{7}{8} I(6,8) = \frac{7}{8} \int_1^3 (3-x)^6 (x-1)^8 dx$

• BY PARTS AGAIN

$\frac{(3-x)^6}{\frac{1}{8}(x-1)^9}$	$\frac{-6(3-x)^5}{(x-1)^8}$
--------------------------------------	-----------------------------

$\Rightarrow I(7,7) = \frac{7}{8} \left\{ \left[\frac{(3-x)^6}{\frac{1}{8}(x-1)^9} \right]_1^3 + \frac{6}{8} \int_1^3 (3-x)^5 (x-1)^8 dx \right\}$

$\Rightarrow I(7,7) = \frac{7}{8} \times \frac{6}{8} \times I(5,9)$

• RECOGNISING THE PATTERN BY PARTS

$\Rightarrow I(7,7) = \frac{1}{8} \times \frac{6}{8} \times \frac{5}{8} \times \frac{4}{8} \times \frac{3}{8} \times \frac{2}{8} \times \frac{1}{8} \times I(0,14)$

$\Rightarrow I(7,7) = \frac{7!}{14 \times 13 \times 12 \times \dots \times 1 \times 10 \times 9 \times 8} I(0,14)$

$\Rightarrow I(7,7) = \frac{7! \times 7! \times 2^{15}}{14 \times 13 \times 12 \times \dots \times 1 \times 10 \times 9 \times 8} I(0,14)$

$\Rightarrow I(7,7) = \frac{7! \times 7!}{14!} I(0,14)$

$\Rightarrow I(7,7) = \frac{(7!)^2}{14!} \int_1^3 (3-x)^0 (x-1)^0 dx$

$\Rightarrow I(7,7) = \frac{(7!)^2}{14!} \int_1^3 (x-1)^0 dx$

$\Rightarrow I(7,7) = \frac{(7!)^2}{14!} \left[\frac{1}{1} (x-1)^1 \right]_1^3$

$\Rightarrow I(7,7) = \frac{(7!)^2}{14!} \times \frac{1}{1} \times 2^{15}$

$\Rightarrow I(7,7) = \frac{(7!)^2}{14!} \times 2^{15}$

$\therefore \int_1^3 (3-x)^7 (x-1)^7 dx = \frac{(7!)^2 \times 2^{15}}{15!}$

Question 299 (*****)

Use integration by parts to find a simplified expression for

$$\int \left(1+x - \frac{1}{x}\right) e^{x+\frac{1}{x}} dx.$$

$$\boxed{}, \boxed{x e^{x+\frac{1}{x}} + C}$$

• LOOKING AT THE INTEGRAND WE SUSPECT THAT A TERM CONTAINING $e^{x+\frac{1}{x}}$ COULD BE INTEGRATED
 • SEEK AN EXACT DIFFERENTIAL
 $\frac{d}{dx} \left[e^{x+\frac{1}{x}} \right] = \left(e^{x+\frac{1}{x}} \right) \times \left(1 - \frac{1}{x^2} \right)$
 • REWRITE THE INTEGRAND AS FOLLOWS

$$\int \left(1+x - \frac{1}{x} \right) e^{x+\frac{1}{x}} dx = \int \left[\left(1+x - \frac{1}{x} \right) e^{x+\frac{1}{x}} \right] dx$$

$$= \int e^{x+\frac{1}{x}} dx + \int x e^{x+\frac{1}{x}} dx - \int \frac{1}{x} e^{x+\frac{1}{x}} dx$$

$$= \int e^{x+\frac{1}{x}} dx + \int x \left[\left(1 - \frac{1}{x^2} \right) e^{x+\frac{1}{x}} \right] dx$$

x	1
$e^{x+\frac{1}{x}}$	$\left(1 - \frac{1}{x^2} \right) e^{x+\frac{1}{x}}$

$$= \int e^{x+\frac{1}{x}} dx + \left[x e^{x+\frac{1}{x}} - \int e^{x+\frac{1}{x}} dx \right]$$

$$= x e^{x+\frac{1}{x}} + C$$

Question 300 (*****)

Use trigonometric identities to find a simplified expression for

$$\int \frac{\sin^8 x - \cos^8 x}{1 - \frac{1}{2} \sin^2 2x} dx.$$

$$\boxed{}, \boxed{-\frac{1}{2} \sin 2x + C}$$

$$\int \frac{\sin^8 x - \cos^8 x}{1 - \frac{1}{2} \sin^2 2x} dx$$
 • RECOGNISE THE DIFFERENCE OF SQUARES IN THE NUMERATOR & THE DOUBLE ANGLE IN THE DENOMINATOR

$$= \int \frac{(\sin^4 x - \cos^4 x)(\sin^4 x + \cos^4 x)}{1 - \frac{1}{2} (2 \sin x \cos x)^2} dx = \int \frac{(\sin^4 x - \cos^4 x)(\sin^4 x + \cos^4 x)}{1 - 2 \sin^2 x \cos^2 x} dx$$
 • NEXT CREATE A PERFECT SQUARE IN THE DENOMINATOR AS FOLLOWS

$$= \int \frac{(\sin^4 x - \cos^4 x)(\sin^4 x + \cos^4 x)}{1^2 - 2 \sin^2 x \cos^2 x} dx = \int \frac{(\sin^4 x - \cos^4 x)(\sin^4 x + \cos^4 x)}{(1 - \sin^2 x \cos^2 x)^2} dx$$
 • EXPAND THE DIFFERENCE OF SQUARES IN THE NUMERATOR & THE BRACKET IN THE DENOMINATOR

$$= \int \frac{(\sin^4 x - \cos^4 x)(\sin^4 x + \cos^4 x)}{\sin^4 x + \cos^4 x + 2 \sin^2 x \cos^2 x - 2 \sin^2 x \cos^2 x} dx$$

$$= \int \frac{\sin^4 x - \cos^4 x}{\sin^4 x + \cos^4 x} dx = \int \cos 2x dx$$

$$= -\frac{1}{2} \sin 2x + C$$

Question 301 (****)

By using an appropriate substitution followed by trigonometric identities, show that

$$\int_0^{\pi} \frac{x \tan x}{\tan x + \sec x} dx = \frac{1}{2} \pi (\pi - 2).$$

S.P., proof

• SET BY A SUBSTITUTION

$$\begin{aligned} 2 &= \pi - \theta & \tan(\pi - \theta) &= \frac{\tan \pi - \tan \theta}{1 + \tan \pi \tan \theta} = -\tan \theta \\ dx &= -d\theta \\ 2 &= \pi \rightarrow \theta = 0 & \sec(\pi - \theta) &= \frac{1}{\cos(\pi - \theta)} = \frac{1}{-\cos \theta} = -\sec \theta \\ 2 &= 0 \rightarrow \theta = \pi & & \end{aligned}$$

• THIS THE INTEGRAL CAN BE TRANSFORMED TO

$$\begin{aligned} \int_0^{\pi} \frac{x \tan x}{\tan x + \sec x} dx &= \int_{\pi}^0 \frac{(\pi - \theta)(-\tan \theta)}{-\sec \theta - \tan \theta} (-d\theta) \\ &= \int_0^{\pi} \frac{\theta \tan \theta - \pi \tan \theta}{-\sec \theta - \tan \theta} d\theta \\ &= \int_0^{\pi} \frac{\pi \tan \theta - \theta \tan \theta}{\sec \theta + \tan \theta} d\theta \\ &= \int_0^{\pi} \frac{\pi \tan \theta}{\sec \theta + \tan \theta} d\theta - \int_0^{\pi} \frac{\theta \tan \theta}{\sec \theta + \tan \theta} d\theta \end{aligned}$$

• COLLECTING THE RESULT SO FINE USE HAVE

$$\Rightarrow I = \int_0^{\pi} \frac{\tan \theta}{\sec \theta + \tan \theta} d\theta - I \quad \text{using } I = \int_0^{\pi} \frac{x \tan x}{\tan x + \sec x} dx$$

$$\Rightarrow 2I = \int_0^{\pi} \frac{\tan \theta}{\sec \theta + \tan \theta} d\theta$$

$$\Rightarrow 2I = \int_0^{\pi} \frac{\tan \theta (\sec \theta - \tan \theta)}{(\sec \theta + \tan \theta)(\sec \theta - \tan \theta)} d\theta$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \frac{\sec \theta \tan \theta - \tan^2 \theta}{\sec^2 \theta - \tan^2 \theta} d\theta$$

$$1 + \tan^2 \theta \equiv \sec^2 \theta$$

$$\sec \theta - \tan \theta \equiv 1$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \sec \theta \tan \theta - \tan^2 \theta d\theta$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \sec \theta \tan \theta - (\sec^2 \theta - 1) d\theta$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \sec \theta \tan \theta - \sec^2 \theta + 1 d\theta$$

$$\Rightarrow 2I = \pi \left[\sec \theta - \tan \theta + \theta \right]_0^{\pi}$$

$$\Rightarrow 2I = \pi \left[(-1 - 0 + \pi) - (1 - 0 + 0) \right]$$

$$\Rightarrow 2I = \pi \left[\pi - 2 \right]$$

$$\Rightarrow I = \frac{1}{2} \pi (\pi - 2)$$

$$\therefore \int_0^{\pi} \frac{x \tan x}{\tan x + \sec x} dx = \frac{1}{2} \pi (\pi - 2)$$

Question 302 (****)

Show that if n is an integer, then

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \cos(2n\theta) \cos \theta \, d\theta = \frac{2(-1)^n}{1-(2n)^2}$$

 , proof

$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(2n\theta) \cos \theta \, d\theta = \frac{2(-1)^n}{1-(2n)^2}, \quad n \in \mathbb{Z}$

- START BY OBSERVING THAT THE INTEGRAND IS EVEN

$$I_n = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(2n\theta) \cos \theta \, d\theta = \int_0^{\frac{\pi}{2}} 2 \cos 2n\theta \cos \theta \, d\theta$$
- USE THE COSINE COMPOUND ANGLE IDENTITIES

$$\begin{aligned} \cos(2n\theta + \theta) &= \cos 2n\theta \cos \theta - \sin 2n\theta \sin \theta \\ \cos(2n\theta - \theta) &= \cos 2n\theta \cos \theta + \sin 2n\theta \sin \theta \\ \hline \cos(2n\theta + \theta) + \cos(2n\theta - \theta) &= 2 \cos 2n\theta \cos \theta \\ \cos(2n\theta) \cos \theta &= \frac{1}{2} [\cos(2n+1)\theta + \cos(2n-1)\theta] \end{aligned}$$

$$\Rightarrow I_n = \int_0^{\frac{\pi}{2}} [\cos(2n+1)\theta + \cos(2n-1)\theta] \, d\theta$$

$$\Rightarrow I_n = \left[\frac{1}{2n+1} \sin(2n+1)\theta + \frac{1}{2n-1} \sin(2n-1)\theta \right]_0^{\frac{\pi}{2}}$$

$$\Rightarrow I_n = \frac{1}{2n+1} \sin\left[\frac{(2n+1)\pi}{2}\right] + \frac{1}{2n-1} \sin\left[\frac{(2n-1)\pi}{2}\right]$$

$$\Rightarrow I_n = \frac{1}{2n+1} (-1)^n + \frac{1}{2n-1} (-1)^{n+1}$$

$$\Rightarrow I_n = (-1)^n \left[\frac{1}{2n+1} - \frac{1}{2n-1} \right]$$

$$\Rightarrow I_n = (-1)^n \left[\frac{(2n-1) - (2n+1)}{(2n+1)(2n-1)} \right]$$

$$\Rightarrow I_n = (-1)^n \left[\frac{-2}{4n^2 - 1} \right]$$

$$\Rightarrow I_n = (-1)^n \times \frac{2}{1-4n^2}$$

$$\Rightarrow I_n = \frac{2(-1)^n}{1-4n^2} \quad \text{✓ required}$$

Question 303 (****)

$$\int_0^{\frac{1}{2}\pi} \frac{1}{1 + \tan^n x} dx, \quad n \in \mathbb{Q}.$$

Find the value of the above integral, for all values of n

$$\boxed{\frac{\pi}{4}}$$

Handwritten solution for the integral problem:

$$\int_0^{\frac{\pi}{2}} \frac{1}{1 + \tan^n x} dx = \int_0^{\frac{\pi}{2}} \frac{1}{1 + \frac{\sin^n x}{\cos^n x}} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos^n x}{\cos^n x + \sin^n x} dx$$

Also by substitution $x = \frac{\pi}{2} - x$

$$dx = -dx$$

$$x=0 \rightarrow x = \frac{\pi}{2}$$

$$x=\frac{\pi}{2} \rightarrow x=0$$

$$\text{Also } \sin(\frac{\pi}{2} - x) = \cos(x)$$

$$\cos(\frac{\pi}{2} - x) = \sin(x)$$

$$= \int_{\frac{\pi}{2}}^0 \frac{\sin^n x}{\sin^n x + \cos^n x} (-dx) = \int_0^{\frac{\pi}{2}} \frac{\sin^n x}{\sin^n x + \cos^n x} dx$$

This

$$I = \int_0^{\frac{\pi}{2}} \frac{1}{1 + \tan^n x} dx = \int_0^{\frac{\pi}{2}} \frac{\cos^n x}{\sin^n x + \cos^n x} dx$$

$$I = \dots \text{substitution} = \int_0^{\frac{\pi}{2}} \frac{\sin^n x}{\sin^n x + \cos^n x} dx$$

$$2I = \int_0^{\frac{\pi}{2}} \frac{\cos^n x}{\sin^n x + \cos^n x} dx + \int_0^{\frac{\pi}{2}} \frac{\sin^n x}{\sin^n x + \cos^n x} dx$$

$$2I = \int_0^{\frac{\pi}{2}} 1 dx$$

$$2I = \frac{\pi}{2}$$

$$I = \frac{\pi}{4}$$

Question 304 (****)

By suitably rewriting the numerator of the integrand, find a simplified expression for the following integral.

$$\int \frac{12 \sin x - 5 \cos x}{2 \sin x - 3 \cos x} dx.$$

$$\boxed{3x + 2 \ln |2 \sin x - 3 \cos x| + C}$$

$\int \frac{12 \sin x - 5 \cos x}{2 \sin x - 3 \cos x} dx = \dots$ **WHENGETT the follows**
 • $\frac{d}{dx}(2 \sin x - 3 \cos x) = 2 \cos x + 3 \sin x$
 • THIS $12 \sin x - 5 \cos x \equiv A(2 \sin x - 3 \cos x) + B(2 \cos x + 3 \sin x)$
 So it's "split" so it's easier to find the coefficients
 $\Rightarrow 12 \sin x - 5 \cos x \equiv (2A + 3B) \sin x + (2B - 3A) \cos x$
 $\Rightarrow \begin{cases} 2A + 3B = 12 \\ -3A + 2B = -5 \end{cases} \Rightarrow \begin{cases} 6A + 9B = 36 \\ -6A + 4B = -10 \end{cases} \Rightarrow \begin{cases} 13B = 26 \\ B = 2 \end{cases}$
 $\Rightarrow A = 3$
 $\therefore \int \frac{12 \sin x - 5 \cos x}{2 \sin x - 3 \cos x} dx = \int \frac{3(2 \sin x - 3 \cos x) + 2(2 \cos x + 3 \sin x)}{2 \sin x - 3 \cos x} dx$
 $= \int \left(3 + \frac{2(2 \cos x + 3 \sin x)}{2 \sin x - 3 \cos x} \right) dx$
 $= 3x + 2 \ln |2 \sin x - 3 \cos x| + C$

Question 305 (****)

Find an exact value for the following integral

$$\int_0^{\frac{1}{2}\pi} \frac{1}{1 + (\tan x)^{\sqrt{2}e}} dx.$$

$$\square, \boxed{\frac{1}{4}\pi}$$

The SPANISH $\sqrt{2}e$ could have been REDUNDANT!! (Two ways), let $x = \frac{\pi}{4} - y$

$$\int_0^{\frac{\pi}{2}} \frac{1}{1 + (\tan x)^{\sqrt{2}e}} dx = \int_0^{\frac{\pi}{2}} \frac{1}{1 + \tan^{\sqrt{2}e} x} dx = \int_0^{\frac{\pi}{2}} \frac{1}{1 + \frac{\sin^{\sqrt{2}e} x}{\cos^{\sqrt{2}e} x}} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos^{\sqrt{2}e} x}{\cos^{\sqrt{2}e} x + \sin^{\sqrt{2}e} x} dx$$

Now use A SUBSTITUTION

$$\begin{aligned} x &= \frac{\pi}{4} - y \\ dx &= -dy \\ x=0 \rightarrow y &= \frac{\pi}{4} \\ x=\frac{\pi}{2} \rightarrow y &= 0 \end{aligned} \quad \dots = \int_{\frac{\pi}{4}}^0 \frac{\cos^{\sqrt{2}e}(\frac{\pi}{4}-y)}{\cos^{\sqrt{2}e}(\frac{\pi}{4}-y) + \sin^{\sqrt{2}e}(\frac{\pi}{4}-y)} (-dy)$$

$$= \int_0^{\frac{\pi}{4}} \frac{\sin^{\sqrt{2}e} y}{\sin^{\sqrt{2}e} y + \cos^{\sqrt{2}e} y} dy$$

This is the VERY SIMILAR

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos^{\sqrt{2}e} x}{\cos^{\sqrt{2}e} x + \sin^{\sqrt{2}e} x} dx = \int_0^{\frac{\pi}{2}} \frac{\sin^{\sqrt{2}e} y}{\sin^{\sqrt{2}e} y + \cos^{\sqrt{2}e} y} dy$$

$$2I = \int_0^{\frac{\pi}{2}} \frac{\cos^{\sqrt{2}e} x + \sin^{\sqrt{2}e} x}{\cos^{\sqrt{2}e} x + \sin^{\sqrt{2}e} x} dx$$

$$2I = \int_0^{\frac{\pi}{2}} 1 dx$$

$$2I = \frac{\pi}{2}$$

$$I = \frac{\pi}{4}$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{1}{1 + (\tan x)^{\sqrt{2}e}} dx = \frac{\pi}{4}$$

Question 306 (*****)

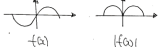
$$I = \int_{-\infty}^{\infty} \left| x^3 (2^{-x^2}) \right| dx$$

It is given that $I \approx 2$.

Use this fact to estimate the value of $\ln 2$ correct to 1 significant figure.

$$\boxed{}, \ln 2 \approx 0.7$$

• As $x^3 \cdot 2^{-x^2}$ is odd, the modulus in the integrand will be even

E.g. 

• Hence we have

$$\int_{-\infty}^{\infty} |x^3 (2^{-x^2})| dx = 2 \int_0^{\infty} x^3 (2^{-x^2}) dx = \int_0^{\infty} 2x^3 (2^{-x^2}) dx$$

• By substitution first

$$\begin{aligned} \dots &= \int_0^{\infty} 2x^3 (2^{-x^2}) \left(\frac{du}{dx} \right) dx \\ &= \int_0^{\infty} x^2 (2^{-u}) du \\ &= \int_0^{\infty} -u (2^{-u}) du \\ &= \int_0^{\infty} u (2^{-u}) du \end{aligned}$$

$$\begin{aligned} u &= x^2 \\ \frac{du}{dx} &= 2x \\ dx &= \frac{du}{2x} \\ x=0 &\rightarrow u=0 \\ x=\infty &\rightarrow u=\infty \end{aligned}$$

• By parts next

$$\begin{aligned} \dots &= \left[\frac{u(2^{-u})}{\ln 2} \right]_0^{\infty} - \int_0^{\infty} \frac{2^{-u}}{\ln 2} du \\ &= 0 - 0 - \frac{1}{\ln 2} \int_0^{\infty} 2^{-u} du \end{aligned}$$

$$\begin{array}{c|c} u & 1 \\ \hline \frac{2^{-u}}{\ln 2} & 2^{-u} \end{array}$$

$$\begin{aligned} &= -\frac{1}{\ln 2} \left[\frac{2^{-u}}{\ln 2} \right]_0^{\infty} \\ &= -\frac{1}{\ln 2} \left[0 - \frac{1}{\ln 2} \right] \\ &= \frac{1}{(\ln 2)^2} \end{aligned}$$

• Finally we have

$$\begin{aligned} \frac{1}{(\ln 2)^2} &\approx 2 \quad (1 \text{ sf}) \\ (\ln 2)^2 &\approx \frac{1}{2} \approx 0.49 \\ \ln 2 &\approx 0.7 \quad (1 \text{ sf}) \end{aligned}$$

Question 307 (****)

The definite integral I is defined in terms of the constant k , where $k \neq 0$, $k \neq \pm 1$.

$$I = \int_0^{\frac{1}{2}\pi} \frac{1}{1+k^2 \tan^2 x} dx.$$

Use appropriate integration techniques to show that

$$I = \frac{\pi}{2(k+1)}.$$

, proof

• START BY A SUBSTITUTION

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+k^2 \tan^2 x} dx \dots \int_0^{\infty} \frac{du}{1+u^2}$$

$$= \int_0^{\infty} \frac{1}{1+u^2} \times \frac{1}{k(1+\tan^2 x)} du$$

$$= \int_0^{\infty} \frac{1}{1+u^2} \times \frac{1}{k(1+\frac{u^2}{k^2})} du$$

$$= \int_0^{\infty} \frac{1}{1+u^2} \times \frac{k^2}{k(k^2+u^2)} du = \int_0^{\infty} \frac{k}{(k^2+u^2)(k^2+u^2)} du$$

• NOW BY PARTIAL FRACTIONS

$$\frac{k}{(k^2+u^2)(k^2+u^2)} = \frac{A}{k^2+u^2} + \frac{B}{k^2+u^2}$$

$$\Rightarrow k = (A+u^2)(k^2+u^2) + (B+u^2)(k^2+u^2)$$

$$\Rightarrow k = \{A^2 + B^2 + A^2u^2 + B^2u^2 + 2ABu^2 + A^2u^4 + B^2u^4\}$$

$$\Rightarrow k = (A^2+B^2)u^4 + (2AB)u^2 + (A^2+B^2)$$

- $A^2+B^2=0 \Rightarrow A^2=B^2=0 \Rightarrow A=B=0$
- $2AB=1 \Rightarrow A=B=\frac{1}{2}$

• RETURNING TO THE INTEGRAL WE NOW HAVE

$$\int_0^{\infty} \frac{k}{(k^2+u^2)(k^2+u^2)} du = \int_0^{\infty} \frac{k}{u^2+1} - \frac{k}{u^2+k^2} du$$

$$= \frac{k}{k^2-1} \int_0^{\infty} \frac{1}{u^2+1} - \frac{1}{u^2+k^2} du = \frac{k}{k^2-1} \left[\arctan u - \frac{1}{k} \arctan \frac{u}{k} \right]_0^{\infty}$$

$$= \frac{k}{k^2-1} \left[\left(\frac{\pi}{2} - \frac{\pi}{2k} \right) - 0 \right] = \frac{k}{k^2-1} \times \frac{\pi}{2} \times \left(1 - \frac{1}{k} \right)$$

$$= \frac{k}{(k-1)(k+1)} \times \frac{\pi}{2} \times \frac{k-1}{k} = \frac{\pi}{2(k+1)}$$

Question 308 (****)

By suitably rewriting the numerator of the integrand, find a simplified expression for the following integral.

$$\int \frac{3 \cos x + 2 \sin x}{2 \cos x + 3 \sin x} dx.$$

$$\boxed{\frac{12}{13}x + \frac{5}{13} \ln |2 \cos x + 3 \sin x| + C}$$

$$\int \frac{3 \cos x + 2 \sin x}{2 \cos x + 3 \sin x} dx = ?$$

- MANIPULATE AS: $\frac{d}{dx} [2 \cos x + 3 \sin x] = -2 \sin x + 3 \cos x$
- DETERMINE THE NUMERATOR AS: $3 \cos x + 2 \sin x \equiv A(2 \cos x + 3 \sin x) + B(3 \cos x - 2 \sin x)$
so it can be divided by the denominator
- Hence: $\begin{cases} 2A + 3B = 3 \\ 3A - 2B = 2 \end{cases} \times 2 \rightarrow \begin{cases} 4A + 6B = 6 \\ 3A - 2B = 2 \end{cases} \rightarrow \begin{cases} 4A + 6B = 6 \\ 14A - 6B = 6 \end{cases} \rightarrow \begin{cases} 4A + 6B = 6 \\ 14A = 12 \end{cases}$

$$\begin{aligned} 4 \times 2 \left(\frac{12}{14} \right) + 3B &= 3 \\ 24 + 3B &= 21 \\ 3B &= -3 \\ B &= -1 \end{aligned}$$
- RETURNING TO THE INTEGRAL:

$$\begin{aligned} &= \int \frac{12}{13} \left(\frac{2 \cos x + 3 \sin x}{2 \cos x + 3 \sin x} \right) + \frac{5}{13} \left(\frac{3 \cos x - 2 \sin x}{2 \cos x + 3 \sin x} \right) dx \\ &= \int \frac{12}{13} + \frac{5}{13} \left(\frac{3 \cos x - 2 \sin x}{2 \cos x + 3 \sin x} \right) dx \\ &= \frac{12}{13}x + \frac{5}{13} \ln |2 \cos x + 3 \sin x| + C \end{aligned}$$

Question 309 (****)

Find the value of the following definite integral.

$$\int_{80^{-\frac{1}{4}}}^{15^{-\frac{1}{4}}} \frac{1}{x^2 (1+x^4)^{\frac{3}{4}}} dx$$

 ,

As the domain of integration is positive we may differentiate out of the radical.

$$\begin{aligned} \int_{80^{-\frac{1}{4}}}^{15^{-\frac{1}{4}}} \frac{1}{x^2 (1+x^4)^{\frac{3}{4}}} dx &= \int_{80^{-\frac{1}{4}}}^{15^{-\frac{1}{4}}} \frac{1}{x^2 (x^4)^{\frac{3}{4}} (1+x^4)^{\frac{3}{4}}} dx \\ &= \int_{80^{-\frac{1}{4}}}^{15^{-\frac{1}{4}}} \frac{1}{x^2 x^3 (1+x^4)^{\frac{3}{4}}} dx = \int_{80^{-\frac{1}{4}}}^{15^{-\frac{1}{4}}} \frac{1}{x^5 (1+x^4)^{\frac{3}{4}}} dx \\ &= \int_{80^{-\frac{1}{4}}}^{15^{-\frac{1}{4}}} x^{-5} (1+x^4)^{-\frac{3}{4}} dx \end{aligned}$$

By RESUBSTITUTION OR U-SUBSTITUTION $u = x^4$

$$\frac{d}{dx} [(1+x^4)^{-\frac{3}{4}}] = -\frac{3}{4} (1+x^4)^{-\frac{7}{4}} \times (4x^3) = -3x^3 (1+x^4)^{-\frac{7}{4}}$$

INTRODUCING THE MULTIPLYING EQUATION

$$\begin{aligned} &= - \left[(1+x^4)^{-\frac{3}{4}} \right]_{80^{-\frac{1}{4}}}^{15^{-\frac{1}{4}}} = - \left[(1+(15)^4)^{-\frac{3}{4}} - (1+(80)^4)^{-\frac{3}{4}} \right] \\ &= - (16^4 - 81^4) = - (2 - 3) = 1 \end{aligned}$$

Question 310 (****)

Find in exact simplified form the value of

$$\int_0^1 \frac{\sqrt{1-x}}{1-\sqrt{x}} dx.$$

You may assume that the integral converges.

$$\boxed{v}, \boxed{\square}, \boxed{\frac{1}{2}(\pi+4)}$$

ENDING THE LIMITS OF ASSUME CONVERGENCE, START BY DIFFERENTIATING

THE INTEGRAND

$$\int_0^1 \frac{\sqrt{1-x}}{1-\sqrt{x}} dx = \int_0^1 \frac{\sqrt{1-x} \cdot (1+\sqrt{x})}{(1-\sqrt{x})(1+\sqrt{x})} dx = \int_0^1 \frac{\sqrt{1-x}(1+\sqrt{x})}{1-x} dx$$

$$= \int_0^1 \frac{1+\sqrt{1-x}}{1-\sqrt{x}} dx$$

NOW USE A SUBSTITUTION ON THE DENOMINATOR

$$= \int_0^1 \frac{1+\sqrt{1-x}}{1-u^2} (-2u du)$$

$$= \int_0^1 -2 + 2\sqrt{1-u^2} du$$

$$= \int_0^1 2 + 2\sqrt{1-u^2} du$$

$$= [2u]_0^1 + 2 \int_0^1 \sqrt{1-u^2} du$$

$$= 2 + 2 \int_0^1 \sqrt{1-u^2} du$$

BY A TRIGONOMETRIC SUBSTITUTION

$$= 2 + 2 \int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2 \theta} (\cos \theta d\theta)$$

$$= 2 + 2 \int_0^{\frac{\pi}{2}} \cos \theta \cos \theta d\theta$$

$$= 2 + 2 \int_0^{\frac{\pi}{2}} \cos^2 \theta d\theta$$

$$= 2 + 2 \int_0^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2} d\theta$$

BOXED RESULTS

$$u = \sqrt{1-x}$$

$$u^2 = 1-x$$

$$2u = 1-u^2$$

$$dx = -2u du$$

$$x=0 \rightarrow u=1$$

$$x=1 \rightarrow u=0$$

$$\theta = \arcsin u$$

$$\sin \theta = u$$

$$d\theta = \frac{1}{\sqrt{1-u^2}} du$$

$$\cos \theta = \sqrt{1-u^2}$$

$$u=1 \rightarrow \theta = \frac{\pi}{2}$$

$$u=0 \rightarrow \theta = 0$$

$$= 2 + \int_0^{\frac{\pi}{2}} (1 + \cos 2\theta) d\theta$$

$$= 2 + \left[\theta + \frac{1}{2} \sin 2\theta \right]_0^{\frac{\pi}{2}}$$

$$= 2 + \left[\left(\frac{\pi}{2} + \frac{1}{2} \sin \pi \right) - (0+0) \right]$$

$$= 2 + \frac{\pi}{2}$$

$$= \frac{1}{2}(\pi+4)$$

Question 311 (****)

Find an exact simplified value for

$$\int_{\sqrt{e}}^e \ln(\ln x) + \frac{1}{(\ln x)^2} dx.$$

$$\boxed{}, \boxed{e^{\frac{1}{2}}(2 + \ln 2) - e}$$

$\int_{\sqrt{e}}^e \ln(\ln x) + \frac{1}{(\ln x)^2} dx = \int_{\sqrt{e}}^e \ln(\ln x) dx + \int_{\sqrt{e}}^e \frac{1}{(\ln x)^2} dx$

• ATTACHMENT INTEGRATION BY PARTS ON THE FIRST INTEGRAL

$\ln(\ln x)$	$\frac{1}{\ln x} \times \frac{1}{x}$
u	v

$$\int_{\sqrt{e}}^e \ln(\ln x) dx = \left[x \ln(\ln x) \right]_{\sqrt{e}}^e - \int_{\sqrt{e}}^e \frac{1}{\ln x} dx$$

$$= 0 - \sqrt{e} \ln\left(\frac{1}{2}\right) - \int_{\sqrt{e}}^e \frac{1}{\ln x} dx$$

$$= e^{\frac{1}{2}} \ln 2 - \int_{\sqrt{e}}^e \frac{1}{\ln x} dx$$

2nd part - might be able to cancel this integral

• PROCEED WITH THE NEXT INTEGRAL ALSO BY PARTS

$\frac{(\ln x)^{-2}}{x}$	$-2(\ln x)^{-3} \times \frac{1}{x}$
u	v

$$\int_{\sqrt{e}}^e \frac{1}{(\ln x)^2} dx = \left[\frac{x}{(\ln x)^2} \right]_{\sqrt{e}}^e + 2 \int_{\sqrt{e}}^e \frac{1}{\ln x} dx$$

THIS PART IS IF WE REVERSE THE PARTS OF THE FIRST, WE CAN ATTACH THE INTEGRAL BY PARTS BY ATTACHING WITH A POSITIVE SIGN

• SO DO THE PARTS IN REVERSE AS FOLLOWS

$\frac{(\ln x)^{-1}}{x}$	$-(\ln x)^{-2} \times \frac{1}{x}$
u	v

NOT AN OX CHANGE

$$\Rightarrow \int_{\sqrt{e}}^e \frac{1}{(\ln x)^2} dx = \left[\frac{x}{(\ln x)^2} \right]_{\sqrt{e}}^e + \int_{\sqrt{e}}^e \frac{1}{\ln x} dx$$

$$\Rightarrow \int_{\sqrt{e}}^e \frac{1}{(\ln x)^2} dx = \int_{\sqrt{e}}^e \frac{1}{\ln x} dx - \left[\frac{x}{\ln x} \right]_{\sqrt{e}}^e$$

$$\Rightarrow \int_{\sqrt{e}}^e \frac{1}{(\ln x)^2} dx = \int_{\sqrt{e}}^e \frac{1}{\ln x} dx - e + 2e^{\frac{1}{2}}$$

• COLLECTING THE RESULTS SO FAR

$$\int_{\sqrt{e}}^e \ln(\ln x) dx = e^{\frac{1}{2}} \ln 2 - \int_{\sqrt{e}}^e \frac{1}{\ln x} dx$$

$$\int_{\sqrt{e}}^e \frac{1}{(\ln x)^2} dx = \int_{\sqrt{e}}^e \frac{1}{\ln x} dx - e + 2e^{\frac{1}{2}}$$

ADDING THE RESULTS

$$\int_{\sqrt{e}}^e \ln(\ln x) + \frac{1}{(\ln x)^2} dx = e^{\frac{1}{2}} \ln 2 - e + 2e^{\frac{1}{2}}$$

$$= e^{\frac{1}{2}}(2 + \ln 2) - e$$

Question 312 (****)

By using appropriate substitutions, or otherwise, show that

$$\int_0^{\frac{1}{2}} \frac{\ln(1+2x)}{1+4x^2} dx = \frac{\pi \ln 2}{16}.$$

□, proof

THE ANSWER SUGGESTS A TANGENT SUBSTITUTION

$$1+4x^2 = 1+(2x)^2 = 1+\tan^2 \theta = \sec^2 \theta$$

- $2x = \tan \theta$ [0 - and $\pi/2$ is the substitution]
- $2dx = \sec^2 \theta$
- $x=0 \mapsto \theta=0$
- $x=\frac{1}{2} \mapsto \theta=\frac{\pi}{4}$

TRANSFORMING THE INTEGRAL

$$\int_0^{\frac{1}{2}} \frac{\ln(1+2x)}{1+4x^2} dx = \int_0^{\frac{\pi}{4}} \frac{\ln(1+\tan \theta)}{\sec^2 \theta} (\sec^2 \theta d\theta)$$

$$I = \frac{1}{2} \int_0^{\frac{\pi}{4}} \ln(1+\tan \theta) d\theta$$

ANOTHER SUBSTITUTION

- $\theta = \frac{\pi}{4} - \phi \iff \phi = \frac{\pi}{4} - \theta$
- $d\theta = -d\phi \iff d\phi = -d\theta$
- $\theta=0 \mapsto \phi=\frac{\pi}{4}$
- $\theta=\frac{\pi}{4} \mapsto \phi=0$

THIS WE NOW HAVE

$$I = \frac{1}{2} \int_{\frac{\pi}{4}}^0 \ln(1+\tan(\frac{\pi}{4}-\phi)) (-d\phi) = \frac{1}{2} \int_0^{\frac{\pi}{4}} \ln\left[1 + \frac{\tan \frac{\pi}{4} - \tan \phi}{1 + \tan \frac{\pi}{4} \tan \phi}\right] d\phi$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{4}} \ln\left[1 + \frac{1 - \tan \phi}{1 + \tan \phi}\right] d\phi = \frac{1}{2} \int_0^{\frac{\pi}{4}} \ln\left[\frac{1 + \tan \phi + 1 - \tan \phi}{1 + \tan \phi}\right] d\phi$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{4}} \ln\left[\frac{2}{1 + \tan \phi}\right] d\phi$$

SPLIT THE LOG, OBSERVING THE DEFINITION OF \int

$$\Rightarrow I = \frac{1}{2} \int_0^{\frac{\pi}{4}} \ln(1 + \tan \phi) d\phi$$

$$\Rightarrow I = \frac{1}{2} \int_0^{\frac{\pi}{4}} \ln 2 - \ln(1 + \tan \phi) d\phi$$

$$\Rightarrow I = \frac{1}{2} \int_0^{\frac{\pi}{4}} \ln 2 d\phi - \frac{1}{2} \int_0^{\frac{\pi}{4}} \ln(1 + \tan \phi) d\phi$$

$$\Rightarrow I = \frac{1}{2} \ln 2 \times \frac{\pi}{4} - I$$

$$\Rightarrow 2I = \frac{\pi \ln 2}{8}$$

$$\Rightarrow I = \frac{\pi \ln 2}{16}$$

$\therefore \int_0^{\frac{1}{2}} \frac{\ln(1+2x)}{1+4x^2} dx = \frac{\pi \ln 2}{16}$ A BEAUTY

Question 313 (****)

Use appropriate integration techniques to show that

$$\int_0^1 4x \arctan x \, dx = \pi - 2.$$

\$

, proof

$\int_0^1 4x \arctan x = \pi - 2$

• By substitution

$x = \tan \theta$
$\theta = \arctan x$
$\frac{dx}{d\theta} = \sec^2 \theta \Rightarrow dx = \sec^2 \theta \, d\theta$
$x=0 \mapsto \theta=0$
$x=1 \mapsto \theta=\frac{\pi}{4}$

$$= \int_0^{\frac{\pi}{4}} 4 \tan \theta \arctan(\tan \theta) \times \sec^2 \theta \, d\theta$$

$$= \int_0^{\frac{\pi}{4}} 4\theta \tan \theta \sec^2 \theta \, d\theta$$

• Integration by parts next

θ	1
$2 \tan \theta$	$4 \tan \theta \sec^2 \theta$

$$= \left[2\theta \tan \theta \right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} 2 \tan^2 \theta \, d\theta$$

$$= \left(\frac{\pi}{2} - 0 \right) - \int_0^{\frac{\pi}{4}} 2(\sec^2 \theta - 1) \, d\theta$$

$$= \frac{\pi}{2} - \left[2 \sec \theta - 2\theta \right]_0^{\frac{\pi}{4}} = \frac{\pi}{2} - \left[(2 - \frac{\pi}{2}) - 0 \right]$$

$$= \pi - 2$$

Question 314 (****)

$$I = \int_0^{\frac{1}{2}} \frac{\ln(1+2x)}{1+4x^2} dx.$$

a) Use an appropriate trigonometric substitution to show that

$$I = \int_0^{\frac{\pi}{4}} \frac{1}{2} \ln \sqrt{2} + \frac{1}{2} \ln \left[\frac{\cos\left(\theta - \frac{1}{4}\pi\right)}{\cos \theta} \right] d\theta.$$

b) Show further that

$$I = \frac{\pi \ln 2}{16} + \int_{-\frac{\pi}{8}}^{\frac{\pi}{8}} \frac{1}{2} \ln \left[\frac{\cos\left(\varphi - \frac{1}{8}\pi\right)}{\cos\left(\varphi + \frac{1}{8}\pi\right)} \right] d\varphi.$$

c) Deduce that

$$I = \frac{\pi \ln 2}{16}.$$

V, , proof

a) LOOKING AT THE DENOMINATOR

$$\int_0^{\frac{1}{2}} \frac{\ln(1+2x)}{1+4x^2} dx = \int_0^{\frac{\pi}{4}} \frac{\ln(1+\tan \theta)}{1+\tan^2 \theta} (\sec^2 \theta) d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{\ln(1+\tan \theta)}{\sec^2 \theta} (\sec^2 \theta) d\theta = \int_0^{\frac{\pi}{4}} \ln(1+\tan \theta) d\theta$$

$$= \int_0^{\frac{\pi}{4}} \ln\left(1 + \frac{\sin \theta}{\cos \theta}\right) d\theta = \int_0^{\frac{\pi}{4}} \ln\left(\frac{\cos \theta + \sin \theta}{\cos \theta}\right) d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{2} \ln(\cos \theta + \sin \theta) - \frac{1}{2} \ln(\cos \theta) d\theta$$

MANIPULATE TO A SIMPLER FORM BY INTEGRATING

$$= \int_0^{\frac{\pi}{4}} \frac{1}{2} \ln\left[\sqrt{2}\left(\frac{1}{\sqrt{2}}\cos \theta + \frac{1}{\sqrt{2}}\sin \theta\right)\right] - \frac{1}{2} \ln(\cos \theta) d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{2} \ln\left[\sqrt{2}\left(\cos\left(\theta - \frac{\pi}{4}\right)\right)\right] - \frac{1}{2} \ln(\cos \theta) d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{2} \ln \sqrt{2} + \frac{1}{2} \ln\left[\cos\left(\theta - \frac{\pi}{4}\right)\right] - \frac{1}{2} \ln(\cos \theta) d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{2} \ln \sqrt{2} + \frac{1}{2} \ln\left[\frac{\cos(\theta - \frac{\pi}{4})}{\cos \theta}\right] d\theta$$

b) SPOT THE NEEDED & USE A TRIGONOMETRIC SUBSTITUTION

$$= \int_0^{\frac{\pi}{4}} \frac{1}{2} \ln \sqrt{2} d\theta + \frac{1}{2} \int_0^{\frac{\pi}{4}} \ln\left[\frac{\cos(\theta - \frac{\pi}{4})}{\cos \theta}\right] d\theta$$

$$= \frac{1}{2} \ln \sqrt{2} \cdot \frac{\pi}{4} + \frac{1}{2} \int_0^{\frac{\pi}{4}} \ln\left[\frac{\cos(\theta - \frac{\pi}{4})}{\cos \theta}\right] d\theta$$

ANOTHER SUBSTITUTION

- $\phi = \theta - \frac{\pi}{4} \Leftrightarrow \theta = \phi + \frac{\pi}{4}$
- $d\phi = d\theta$
- $\theta = 0 \rightarrow \phi = -\frac{\pi}{4}$
- $\theta = \frac{\pi}{4} \rightarrow \phi = 0$

LOOKING AT THE FINAL ANSWER GIVEN WE SUSPECT WE HAVE AN ODD INTEGRAND IN THE SUBSTITUTION VARIABLE - LET $f(\phi)$ BE THE INTEGRAND

$$f(\phi) = \ln\left[\frac{\cos(\phi + \frac{\pi}{4})}{\cos(\phi + \frac{\pi}{4} + \frac{\pi}{4})}\right] = \ln\left[\frac{\cos(\phi + \frac{\pi}{4})}{\cos(\phi + \frac{\pi}{2})}\right] = \dots \cos \text{ IS ODD}$$

$$= \ln\left[\frac{\cos(\phi + \frac{\pi}{4})}{-\sin(\phi + \frac{\pi}{4})}\right] = -\ln\left[\frac{\cos(\phi + \frac{\pi}{4})}{\sin(\phi + \frac{\pi}{4})}\right] = -f(\phi)$$

$$\therefore \int_{-\frac{\pi}{4}}^0 \ln\left[\frac{\cos(\phi + \frac{\pi}{4})}{\sin(\phi + \frac{\pi}{4})}\right] d\phi = 0$$

$$\therefore I = \int_0^{\frac{\pi}{4}} \frac{1}{2} \ln \sqrt{2} d\theta = \frac{1}{2} \ln \sqrt{2} \cdot \frac{\pi}{4}$$

As Expected

Question 315 (****)

$$I = \int \frac{2x+1}{\sqrt{x-1}} dx.$$

a) Show that

$$I = \frac{4}{3}(x-1)^{\frac{3}{2}} + 6(x-1)^{\frac{1}{2}} + \text{constant}.$$

You may not use any substitution or integration by parts.

b) Determine the value of a , given that

$$\int_2^a \frac{2x+1}{\sqrt{x-1}} dx = 102.$$

$$\boxed{}, \boxed{a=17}$$

1) GIVEN THAT WE MAY NOT USE SUBSTITUTION OR INTEGRATION BY PARTS, WE CAN MANIPULATE AS FOLLOWS:

$$\frac{2x+1}{\sqrt{x-1}} = \frac{2x-1}{(x-1)^{\frac{1}{2}}} + \frac{2(x-1)+3}{(x-1)^{\frac{1}{2}}} = \frac{2(x-1)}{(x-1)^{\frac{1}{2}}} + \frac{3}{(x-1)^{\frac{1}{2}}}$$

$$= 2(x-1)^{\frac{1}{2}} + 3(x-1)^{-\frac{1}{2}}$$

$$\therefore \int \frac{2x+1}{\sqrt{x-1}} dx = \int 2(x-1)^{\frac{1}{2}} + 3(x-1)^{-\frac{1}{2}} dx = \frac{2}{\frac{3}{2}}(x-1)^{\frac{3}{2}} + \frac{3}{-\frac{1}{2}}(x-1)^{\frac{1}{2}} + C$$

$$= \frac{4}{3}(x-1)^{\frac{3}{2}} + 6(x-1)^{\frac{1}{2}} + C$$

2) NOW PROCEED WITH THE LIMITS:

$$\int_2^a \frac{2x+1}{\sqrt{x-1}} dx = \left[\frac{4}{3}(x-1)^{\frac{3}{2}} + 6(x-1)^{\frac{1}{2}} \right]_2^a = 102$$

$$\frac{4}{3}(a-1)^{\frac{3}{2}} + 6(a-1)^{\frac{1}{2}} - \left[\frac{4}{3}(1)^{\frac{3}{2}} + 6(1)^{\frac{1}{2}} \right] = 102$$

$$\frac{4}{3}(a-1)^{\frac{3}{2}} + 6(a-1)^{\frac{1}{2}} - 4 - 6 = 102$$

$$\frac{4}{3}(a-1)^{\frac{3}{2}} + 6(a-1)^{\frac{1}{2}} - 10 = 102$$

$$\frac{4}{3}(a-1)^{\frac{3}{2}} + 6(a-1)^{\frac{1}{2}} = 112$$

$$2(a-1)^{\frac{3}{2}} + 9(a-1)^{\frac{1}{2}} = 112$$

$$2(a-1)^{\frac{3}{2}} + 9(a-1)^{\frac{1}{2}} - 112 = 0$$

Now MULTIPLY $A = (a-1)^{\frac{1}{2}}$ FOR SIMPLICITY, $a > 3 \Rightarrow A > 1$

$$\Rightarrow 2A^3 + 9A - 112 = 0$$

TRY $A=1$: $2(1)^3 + 9(1) - 112 \neq 0$
 $A=2$: $2(2)^3 + 9(2) - 112 \neq 0$
 $A=4$: $2(4)^3 + 9(4) - 112 = 0$
 $(A-4)$ IS A FACTOR

BY LONG DIVISION OR FURTHER MANIPULATION:

$$2A^3 + 9A - 112 = 0$$

$$2A^3(A-4) + 8A(A-4) + 41(A-4) = 0$$

$$(A-4)(2A^2 + 8A + 41) = 0$$

ONLY SOLUTION IS $A=4$, i.e. $(a-1)^{\frac{1}{2}} = 4$
 $a-1 = 16$
 $a = 17$

Question 316 (****)

$$J = \int_0^1 \frac{(x^2 + 1)e^x}{(x+1)^2} dx.$$

Show that $J = 1$, proof

$J = \int_0^1 \frac{x^2+1}{(x+1)^2} e^x dx \dots$ FIRSTLY PARTIAL FRACTIONS ON THE INTEGRAND IGNORING e^x

$$\frac{x^2+1}{(x+1)^2} \equiv \frac{A}{x+1} + \frac{C}{(x+1)^2}$$

$$x^2+1 \equiv A(x+1) + B(x+1) + C$$

$$x^2+1 \equiv Ax^2 + 2Ax + A + Bx + B + C$$

$$x^2+1 \equiv Ax^2 + (2A+B)x + (A+B+C)$$

$\bullet \underline{A=1}$	$\bullet 2A+B=0$ $2+B=0$ $\underline{B=-2}$	$\bullet A+B+C=1$ $1-2+C=1$ $\underline{C=2}$
---------------------------	---	---

$\Rightarrow J = \int_0^1 e^x \left(\frac{1}{x+1} - \frac{2}{(x+1)^2} \right) dx$
 $\Rightarrow J = \int_0^1 e^x dx - \int_0^1 \frac{2e^x}{(x+1)^2} dx + \int_0^1 \frac{2e^x}{(x+1)^2} dx$
 $\Rightarrow J = [e^x]_0^1 - \left\{ \left[\frac{2e^x}{x+1} \right]_0^1 + \int_0^1 \frac{2e^x}{(x+1)^2} dx \right\} + \int_0^1 \frac{2e^x}{(x+1)^2} dx$
 $\Rightarrow J = (e-1) - (e-2)$
 $\Rightarrow J = 1$

Question 317 (*****)

By using an appropriate substitution or substitutions, show that

$$\int_{\sqrt{\ln 2}}^{\sqrt{\ln 3}} \frac{4x \sin(x^2)}{\sin(x^2) + \sin(\ln 6 - x^2)} dx = \ln\left(\frac{3}{2}\right).$$

QED

,

proof

Handwritten solution for Question 317:

Start by A substitution

$$y = x^2$$

$$\frac{dy}{dx} = 2x$$

$$dx = \frac{dy}{2x}$$

$$x = \sqrt{\ln 2} \rightarrow y = \ln 2$$

$$x = \sqrt{\ln 3} \rightarrow y = \ln 3$$

...

$$= \int_{\ln 2}^{\ln 3} \frac{4x \sin(x^2)}{\sin(x^2) + \sin(\ln 6 - x^2)} \left(\frac{dy}{2x}\right)$$

$$= \int_{\ln 2}^{\ln 3} \frac{2 \sin y}{\sin y + \sin(\ln 6 - y)} dy$$

$$= \int_{\ln 2}^{\ln 3} \frac{2 \sin(\ln 6 - y)}{\sin y + \sin(\ln 6 - y)} (-dy)$$

Another substitution

$$v = \ln 6 - u$$

$$u = \ln 6 - v$$

$$du = -dv$$

$$u = \ln 2 \rightarrow v = \ln 6 - \ln 2 = \ln 3$$

$$u = \ln 3 \rightarrow v = \ln 6 - \ln 3 = \ln 2$$

Combining the results

$$\Rightarrow I = \int_{\ln 2}^{\ln 3} \frac{2 \sin u}{\sin u + \sin(\ln 6 - u)} du = \int_{\ln 2}^{\ln 3} \frac{2 \sin(\ln 6 - u)}{\sin u + \sin(\ln 6 - u)} du$$

$$\Rightarrow 2I = \int_{\ln 2}^{\ln 3} \frac{2 \sin u - 2 \sin(\ln 6 - u)}{\sin u + \sin(\ln 6 - u)} du$$

$$\Rightarrow 2I = \int_{\ln 2}^{\ln 3} 1 du = [u]_{\ln 2}^{\ln 3} = \ln 3 - \ln 2$$

$$\Rightarrow \int_{\ln 2}^{\ln 3} \frac{4x \sin(x^2)}{\sin(x^2) + \sin(\ln 6 - x^2)} dx = \ln \frac{3}{2}$$

Question 318 (****)

$$f(x) = \begin{cases} x - [x] & x \in \mathbb{R}, [x] = 2k + 1, k \in \mathbb{Z} \\ -x + [x] + 1 & x \in \mathbb{R}, [x] = 2k, k \in \mathbb{Z} \end{cases}$$

where $[x]$ is defined as the greatest integer less or equal to x .

Find the value of

$$\frac{\pi^2}{8} \int_{-8}^8 f(x) \cos(\pi x) dx.$$

,

$f(x) = \begin{cases} x - [x] & \text{if } [x] \text{ is odd} \\ -x + [x] + 1 & \text{if } [x] \text{ is even} \end{cases}$

• FIRSTLY WE PRODUCE A QUICK SKETCH FOR THE GRAPH OF $f(x)$, BY TRYING SOME VALUES FOR x .

• WE NOTICE FROM THE GRAPH THAT: $f(x)$ IS EVEN.
 $f(x)$ IS PERIODIC (PERIOD 2).

THIS $\frac{\pi^2}{8} \int_{-8}^8 f(x) \cos(\pi x) dx = \frac{\pi^2}{4} \int_0^8 f(x) \cos(\pi x) dx$

• NOW COS(2x) HAS PERIOD $\frac{2\pi}{2} = \pi$, SO THE TWO FUNCTIONS IN THE INTEGRAL SHARE PERIODS.

$$\dots = 4 \times \frac{\pi^2}{4} \int_0^2 f(x) \cos(\pi x) dx = \pi^2 \int_0^2 f(x) \cos(\pi x) dx$$

$$= \pi^2 \left(\int_0^1 (1-x) \cos(\pi x) dx + \int_1^2 (x-1) \cos(\pi x) dx \right)$$

• PROCESSED BY PARTS OF USE & "u/v" SUBSTITUTION IN THE SECOND INTEGRAL.

$\begin{aligned} \text{Let } u &= 1-x \\ du &= -dx \\ x=1 &\rightarrow 0 \\ x=2 &\rightarrow 1 \end{aligned}$	$\begin{aligned} \text{Also } \cos(\pi x) &= \cos(\pi(1-u)) \\ &= \cos(\pi u + \pi) \\ &= \cos(\pi u) \cos(\pi) - \sin(\pi u) \sin(\pi) \\ &= -\cos(\pi u) \end{aligned}$
---	---

$$\dots = \pi^2 \int_0^1 (1-x) \cos(\pi x) dx + \pi^2 \int_1^2 (x-1) \cos(\pi x) dx$$

$$= \pi^2 \int_0^1 (1-x) \cos(\pi x) dx$$

• FINALLY (INTEGRATION BY PARTS)

$\begin{aligned} u &= 1-x \\ du &= -dx \\ x=0 &\rightarrow 1 \\ x=1 &\rightarrow 0 \end{aligned}$	$\begin{aligned} v &= \cos(\pi x) \\ dv &= -\pi \sin(\pi x) dx \end{aligned}$
---	---

$$= \pi^2 \left[\frac{1-x}{\pi} \sin(\pi x) \right]_0^1 + \frac{\pi^2}{\pi} \int_0^1 \sin(\pi x) dx$$

$$= 2\pi \int_0^1 \sin(\pi x) dx$$

$$= 2\pi \left[-\frac{1}{\pi} \cos(\pi x) \right]_0^1$$

$$= 2 \left[\cos(\pi x) \right]_0^1$$

$$= 2 [1 - \cos(\pi)]$$

$$= 4$$

Question 319 (****)

By using symmetry arguments, find the exact value of the following integral

$$\int_0^{\pi} e^{\cos x} [\sin(\cos x) + \cos(\cos x)] \sin x \, dx.$$

$$\boxed{}, \boxed{e(\cos 1 + \sin 1) - 1}$$

$$\int_0^{\pi} e^{\cos x} [\sin(\cos x) + \cos(\cos x)] \sin x \, dx$$

$$= \int_0^{\pi} e^{\cos x} \sin(\cos x) \sin x \, dx + \int_0^{\pi} e^{\cos x} \cos(\cos x) \sin x \, dx$$

$$= 2 \int_0^{\frac{\pi}{2}} e^{\cos x} \sin(\cos x) \sin x \, dx$$

By substitution $u = \cos x$
 $\frac{du}{dx} = -\sin x$
 $dx = -\frac{du}{\sin x}$

$$= 2 \int_1^0 e^u \sin u \left(-\frac{du}{\sin u}\right) = 2 \int_0^1 e^u \sin u \, du$$

BY PARTS (OR COMPLEX NUMBERS METHOD)

$$\int e^u \sin u \, du = e^u \cos u + \int e^u \sin u \, du$$

$$\int e^u \cos u \, du = e^u \sin u - \int e^u \sin u \, du$$

$$2 \int e^u \sin u \, du = e^u \cos u + e^u \sin u + C$$

$$\int e^u \sin u \, du = \frac{1}{2} e^u (\cos u + \sin u) + C$$

RETURNING TO THE DEFINITE INTEGRAL

$$= 2 \int_0^1 e^u \sin u \, du = [e^u (\cos u + \sin u)]_0^1$$

$$= e^1 (\cos 1 + \sin 1) - e^0 (\cos 0 + \sin 0)$$

$$= e \cos 1 + e \sin 1 - 1$$

Question 320 (****)

$$I = \int_0^1 \left[\prod_{r=1}^{10} (x+r) \right] \left[\sum_{r=1}^{10} \left(\frac{1}{x+r} \right) \right] dx.$$

Show by a detailed method that

$$I = a \times b!,$$

where a and b are positive integers to be found.

$$\boxed{}, \quad a = b = 10$$

The image shows two pages of handwritten work. The left page uses differentiation to find the integral. It starts with the integral $I = \int_0^1 \left[\prod_{r=1}^{10} (x+r) \right] \left[\sum_{r=1}^{10} \frac{1}{x+r} \right] dx$. It then defines $u = \prod_{r=1}^{10} (x+r)$ and finds $\frac{du}{dx} = u \sum_{r=1}^{10} \frac{1}{x+r}$. This leads to $\frac{du}{u} = \left(\sum_{r=1}^{10} \frac{1}{x+r} \right) dx$, which integrates to $\ln u = \ln(x+1) + \ln(x+2) + \dots + \ln(x+10)$. The integral then becomes $\int_0^1 u \frac{du}{u} = \int_0^1 du = u \Big|_0^1 = 10!$. The right page uses integration by parts, setting $u = \prod_{r=1}^{10} (x+r)$ and $v = \sum_{r=1}^{10} \frac{1}{x+r}$. It finds $\frac{dv}{dx} = -\sum_{r=1}^{10} \frac{1}{(x+r)^2}$. The integration by parts formula is applied, and the boundary terms are evaluated at $x=0$ and $x=1$, leading to the same result of $10!$.

Question 321 (*****)

$$I = \int \sqrt{\tan x} \, dx.$$

- a) Use a suitable substitution to show that

$$I = \int \frac{1 + \frac{1}{u^2}}{\left(u - \frac{1}{u}\right)^2 + 2} dx + I = \int \frac{1 - \frac{1}{u^2}}{\left(u + \frac{1}{u}\right)^2 - 2} dx.$$

- b) By using a further substitution in each of the integrals of part (a) find a simplified expression for I , in terms of x .

You may assume without proof that

$$\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \arctan \left[\frac{x}{a} \right] + \text{constant}.$$

$$\boxed{}, \quad \frac{1}{\sqrt{2}} \arctan \left[\frac{\tan x - 1}{\sqrt{2} \tan x} \right] + \frac{1}{2\sqrt{2}} \ln \left[\frac{\tan x - \sqrt{2} \tan x + 1}{\tan x + \sqrt{2} \tan x + 1} \right] + C$$

a) $\int \sqrt{\tan x} \, dx \dots$ BY SUBSTITUTION

$u = \sqrt{\tan x}$
 $u^2 = \tan x$
 $2u \, du = \sec^2 x \, dx$
 $2u \, du = (1 + \tan^2 x) \, dx$
 $2u \, du = (1 + u^4) \, dx$
 $dx = \frac{2u}{1 + u^4} \, du$

$I = \int \frac{1 + \frac{1}{u^2}}{\left(u - \frac{1}{u}\right)^2 + 2} dx + I = \int \frac{1 - \frac{1}{u^2}}{\left(u + \frac{1}{u}\right)^2 - 2} dx$

b) NOW APPLYING SUBSTITUTION TO EACH INTEGRAL

$V = u - \frac{1}{u}$
 $dV = \left(1 + \frac{1}{u^2}\right) du$

$W = u + \frac{1}{u}$
 $dW = \left(1 - \frac{1}{u^2}\right) du$

$I = \int \frac{1 + \frac{1}{u^2}}{V^2 + 2} \frac{dV}{1 + \frac{1}{u^2}} + \int \frac{1 - \frac{1}{u^2}}{W^2 - 2} \frac{dW}{1 - \frac{1}{u^2}}$

$= \int \frac{1}{V^2 + 2} dV + \int \frac{1}{W^2 - 2} dW$

$= \frac{1}{\sqrt{2}} \arctan \frac{V}{\sqrt{2}} + \int \frac{1}{(W - \sqrt{2})(W + \sqrt{2})} dW$

$= \frac{1}{\sqrt{2}} \arctan \left(\frac{u^2 - 1}{\sqrt{2} u} \right) + \int \frac{\frac{1}{2\sqrt{2}}}{W - \sqrt{2}} - \frac{\frac{1}{2\sqrt{2}}}{W + \sqrt{2}} dW$

$= \frac{1}{\sqrt{2}} \arctan \left(\frac{u^2 - 1}{\sqrt{2} u} \right) + \frac{1}{2\sqrt{2}} \int \frac{1}{W - \sqrt{2}} - \frac{1}{W + \sqrt{2}} dW$

$= \frac{1}{\sqrt{2}} \arctan \left(\frac{u^2 - 1}{\sqrt{2} u} \right) + \frac{1}{2\sqrt{2}} \ln \left[\frac{W - \sqrt{2}}{W + \sqrt{2}} \right] + C$

FINALLY REVERSING THE FIRST SUBSTITUTION WE OBTAIN

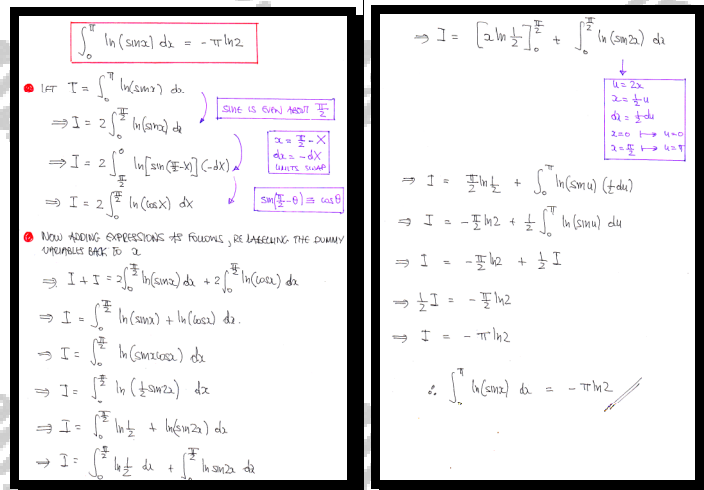
$= \frac{1}{\sqrt{2}} \arctan \left[\frac{\tan x - 1}{\sqrt{2} \tan x} \right] + \frac{1}{2\sqrt{2}} \ln \left[\frac{\tan x - \sqrt{2} \tan x + 1}{\tan x + \sqrt{2} \tan x + 1} \right] + C$

Question 322 (****)

By using an appropriate substitution or substitutions, show that

$$\int_0^{\pi} \ln(\sin x) \, dx = -\pi \ln 2.$$

 , proof



Handwritten solution for Question 322:

Method 1 (Left side):

$$\int_0^{\pi} \ln(\sin x) \, dx = -\pi \ln 2$$

Let $I = \int_0^{\pi} \ln(\sin x) \, dx$.

Since sine is symmetric about $\frac{\pi}{2}$, we can write:

$$I = 2 \int_0^{\frac{\pi}{2}} \ln(\sin x) \, dx$$

Using the identity $\sin(\frac{\pi}{2} - x) = \cos x$, we have:

$$I = 2 \int_0^{\frac{\pi}{2}} \ln(\cos x) \, dx$$

Now adding the two expressions for I , we get:

$$2I = 2 \int_0^{\frac{\pi}{2}} \ln(\sin x \cos x) \, dx$$

$$I = \int_0^{\frac{\pi}{2}} \ln(\sin 2x) \, dx$$

Using the substitution $u = 2x$, $du = 2dx$, we have:

$$I = \frac{1}{2} \int_0^{\pi} \ln(\sin u) \, du$$

Since $I = \int_0^{\pi} \ln(\sin x) \, dx$, we have:

$$I = -\pi \ln 2$$

Method 2 (Right side):

$$I = \left[2 \ln \frac{1}{2} \right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \ln(\sin 2x) \, dx$$

Using the substitution $u = 2x$, $du = 2dx$, we have:

$$I = -\pi \ln 2 + \frac{1}{2} \int_0^{\pi} \ln(\sin u) \, du$$

Since $I = \int_0^{\pi} \ln(\sin x) \, dx$, we have:

$$I = -\pi \ln 2$$

Question 323 (****)

It is given that

$$I = \int_{\frac{1}{2}\pi}^{\pi} \frac{3 + \cos x}{13 + 3\cos x + 2\sin x} dx \quad \text{and} \quad J = \int_{\frac{1}{2}\pi}^{\pi} \frac{2 + \sin x}{13 + 3\cos x + 2\sin x} dx.$$

By considering two linear combinations in I and J , show that

$$I = \frac{1}{26} \left[3\pi - \ln \left(\frac{81}{16} \right) \right],$$

and find a similar expression for J .

$$\boxed{}, \quad I = \frac{1}{13} \left[\pi + \ln \left(\frac{27}{8} \right) \right]$$

Handwritten solution for Question 323:

Given:

$$I = \int_{\frac{1}{2}\pi}^{\pi} \frac{3 + \cos x}{13 + 3\cos x + 2\sin x} dx$$

$$J = \int_{\frac{1}{2}\pi}^{\pi} \frac{2 + \sin x}{13 + 3\cos x + 2\sin x} dx$$

• CREATE A NEW INTEGRAL AS FOLLOWS

$$3I + 2J = \int_{\frac{1}{2}\pi}^{\pi} \frac{9 + 3\cos x}{13 + 3\cos x + 2\sin x} dx + \int_{\frac{1}{2}\pi}^{\pi} \frac{4 + 2\sin x}{13 + 3\cos x + 2\sin x} dx$$

$$= \int_{\frac{1}{2}\pi}^{\pi} \frac{13 + 3\cos x + 2\sin x}{13 + 3\cos x + 2\sin x} dx$$

$$= \int_{\frac{1}{2}\pi}^{\pi} 1 dx$$

$$= \frac{\pi}{2}$$

• CREATE ANOTHER NEW INTEGRAL

$$2I - 3J = \int_{\frac{1}{2}\pi}^{\pi} \frac{2 + 2\cos x}{13 + 3\cos x + 2\sin x} dx - \int_{\frac{1}{2}\pi}^{\pi} \frac{6 + 3\sin x}{13 + 3\cos x + 2\sin x} dx$$

$$= \int_{\frac{1}{2}\pi}^{\pi} \frac{2\cos x - 3\sin x}{13 + 3\cos x + 2\sin x} dx$$

$$= \left[\ln [13 + 3\cos x + 2\sin x] \right]_{\frac{1}{2}\pi}^{\pi}$$

$$= \ln 10 - \ln 15$$

$$= \ln \frac{2}{3}$$

• THENCE WE FIND

$$\begin{array}{rcl} 3I + 2J & = & \frac{\pi}{2} \quad \times 3 \\ 2I - 3J & = & \ln \frac{2}{3} \quad \times 2 \\ \hline 9I + 4J & = & \frac{3\pi}{2} \\ 4I - 6J & = & 2\ln \frac{2}{3} \end{array}$$

$$13I = \frac{3\pi}{2} + 2\ln \frac{2}{3}$$

$$I = \frac{1}{13} \left[\frac{3\pi}{2} + 2\ln \frac{2}{3} \right]$$

$$I = \frac{1}{13} \left[\frac{3\pi}{2} - 4\ln \frac{3}{2} \right]$$

$$I = \frac{1}{13} \left[\frac{3\pi}{2} - \ln \frac{81}{16} \right]$$

• FIND SIMILAR

$$\begin{array}{rcl} 3I + 2J & = & \frac{\pi}{2} \quad \times 2 \\ 2I - 3J & = & \ln \frac{2}{3} \quad \times 3 \\ \hline 6I + 4J & = & \pi \\ 4I - 9J & = & 3\ln \frac{2}{3} \end{array}$$

$$13J = \pi - 3\ln \frac{2}{3}$$

$$J = \frac{1}{13} \left[\pi - 3\ln \frac{2}{3} \right]$$

$$J = \frac{1}{13} \left[\pi + \ln \frac{27}{8} \right]$$

Question 324 (****)

By using an appropriate substitution or substitutions, show that

$$\int_0^1 \frac{\ln(x+1)}{1+x^2} dx = \frac{\pi \ln 2}{8}.$$

 , proof

$\int_0^1 \frac{\ln(x+1)}{1+x^2} dx = \frac{\pi \ln 2}{8}$

Let $I = \int_0^1 \frac{\ln(x+1)}{1+x^2} dx$

BY SUBSTITUTION

$x = \tan \theta$
$dx = \sec^2 \theta d\theta$
$x=0 \mapsto \theta=0$
$x=1 \mapsto \theta=\frac{\pi}{4}$

$\Rightarrow I = \int_0^{\frac{\pi}{4}} \frac{\ln(1+\tan \theta)}{1+\tan^2 \theta} (\sec^2 \theta d\theta)$

$\Rightarrow I = \int_0^{\frac{\pi}{4}} \frac{\ln(1+\tan \theta)}{\sec^2 \theta} (\sec^2 \theta d\theta)$

$\Rightarrow I = \int_0^{\frac{\pi}{4}} \ln(1+\tan \theta) d\theta$

ANOTHER SUBSTITUTION

$\theta = \frac{\pi}{4} - \phi$ OR $\phi = \frac{\pi}{4} - \theta$
$d\theta = -d\phi$
$\theta=0 \mapsto \phi = \frac{\pi}{4}$
$\theta=\frac{\pi}{4} \mapsto \phi=0$

$\Rightarrow I = \int_{\frac{\pi}{4}}^0 \ln[1+\tan(\frac{\pi}{4}-\phi)] (-d\phi)$

$\Rightarrow I = \int_0^{\frac{\pi}{4}} \ln[1+\tan(\frac{\pi}{4}-\phi)] d\phi$

NOW MANIPULATING THE ARGUMENT OF THE LOG BY THE TANGENT COMPOUND FORMULA

$\tan(\frac{\pi}{4}-\phi) = \frac{\tan \frac{\pi}{4} - \tan \phi}{1 + \tan \frac{\pi}{4} \tan \phi} = \frac{1 - \tan \phi}{1 + \tan \phi}$

TRYING THE ARGUMENT FURTHER

$1 + \tan(\frac{\pi}{4}-\phi) = 1 + \frac{1 - \tan \phi}{1 + \tan \phi} = \frac{1 + \tan \phi + 1 - \tan \phi}{1 + \tan \phi} = \frac{2}{1 + \tan \phi}$

$\therefore I = \int_0^{\frac{\pi}{4}} \frac{\ln(1+x)}{1+x^2} dx = \int_0^{\frac{\pi}{4}} \ln(1+\tan \theta) d\theta = \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan \theta}\right) d\theta$

THUS USING THE LAST TWO FORMS IN A COMMON DOUBLE INTEGRAL

$\Rightarrow 2I = \int_0^{\frac{\pi}{4}} \ln(1+\tan \theta) d\theta + \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan \theta}\right) d\theta$

$\Rightarrow 2I = \int_0^{\frac{\pi}{4}} \ln\left[(1+\tan \theta) \cdot \frac{2}{1+\tan \theta}\right] d\theta$

$\Rightarrow 2I = \int_0^{\frac{\pi}{4}} \ln(2) d\theta = \int_0^{\frac{\pi}{4}} \ln 2 d\theta$

$\Rightarrow 2 \int_0^1 \frac{\ln(1+x)}{1+x^2} dx = \frac{\pi}{4} \ln 2$

$\Rightarrow \int_0^1 \frac{\ln(1+x)}{1+x^2} dx = \frac{\pi \ln 2}{8}$

Question 325 (****)

It is given that

$$I = \int_0^{\frac{1}{3}} \frac{32x^2}{(x^2-1)(x+1)^3} dx.$$

Show that $I = \frac{7}{6} - 2\ln 2$.

□, proof

$$\int_0^{\frac{1}{3}} \frac{32x^2}{(x^2-1)(x+1)^3} dx = \int_0^{\frac{1}{3}} \frac{32x^2}{(x-1)(x+1)^4} dx$$

CONTINUE WITH A SIMPLE SUBSTITUTION

$$\begin{aligned} u &= x+1 \Leftrightarrow x = u-1 \\ du &= dx \\ x=0 &\rightarrow u=1 \\ x=\frac{1}{3} &\rightarrow u=\frac{4}{3} \end{aligned}$$

$$\int_1^{\frac{4}{3}} \frac{32(u-1)^2}{(u-2)u^4} du = \int_1^{\frac{4}{3}} \frac{32}{u^4} \cdot \frac{u^2-2u+1}{u-2} du$$

MANIPULATE BY DIVIDING "BACKWARDS" AS SHOWN BELOW

$$\begin{aligned} -2+u & \quad \frac{-\frac{1}{2} + \frac{3}{2}u - \frac{1}{2}u^2 - \frac{1}{2}u^3}{-1 + \frac{3}{2}u} \\ & \quad \frac{-\frac{1}{2}u + u^2}{+\frac{3}{2}u - \frac{1}{2}u^2} \\ & \quad \frac{-\frac{1}{2}u^2 + \frac{1}{2}u^3}{-\frac{1}{2}u^2 + \frac{1}{2}u^3} \\ & \quad \frac{-\frac{1}{2}u^3 + \frac{1}{2}u^4}{-\frac{1}{2}u^3} \end{aligned}$$

(We skip the 'left' the 32 with u^2-2u+1)

RETURNING TO THE INTEGRAL WE OBTAIN

$$\begin{aligned} \dots &= \int_1^{\frac{4}{3}} \frac{32}{u^4} \left[-\frac{1}{2} + \frac{3}{2}u - \frac{1}{2}u^2 - \frac{1}{2}u^3 + \frac{1}{2}u^4 \right] du \\ &= \int_1^{\frac{4}{3}} \left[-\frac{16}{u^4} + \frac{48}{u^3} - \frac{16}{u^2} - \frac{16}{u} + \frac{16}{u-2} \right] du \\ &= \left[\frac{16}{303} - \frac{12}{u^2} + \frac{48}{u} - 2\ln|u| + 2\ln|u-2| \right]_1^{\frac{4}{3}} \\ &= \left[\frac{16}{3 \times 27} - \frac{12}{\frac{16}{9}} + \frac{48}{\frac{4}{3}} - 2\ln\frac{4}{3} + 2\ln\left|\frac{4}{3}-2\right| \right] - \dots \\ &\dots = \left[\frac{16}{3} - 12 + 4 - 2\ln\frac{4}{3} + 2\ln\left|\frac{4}{3}-2\right| \right] \\ &= \frac{16}{3} - \frac{12 \times 9}{16} + \frac{4 \times 3}{4} - 2\ln\frac{4}{3} + 2\ln\frac{4}{3} - \frac{16}{3} + 12 - 4 \\ &= \frac{16 \times 3}{24} - \frac{3 \times 9}{4} + 3 - 2\ln\frac{4}{3} - 2\ln\frac{4}{3} - \frac{16}{3} + 8 \\ &= \frac{9}{4} - \frac{27}{4} - \frac{16}{3} + 11 - 2\ln\left(\frac{4}{3} \times \frac{3}{4}\right) \\ &= 11 - \frac{16}{3} - \frac{16}{3} - 2\ln 2 \\ &= 11 - \frac{32}{3} - \frac{16}{3} - 2\ln 2 \\ &= \frac{36 - 32 - 16}{3} - 2\ln 2 = \frac{7}{6} - 2\ln 2 \end{aligned}$$

Question 326 (****)

$$I = \int_0^{\frac{1}{2}\pi} 4 \sin x \sqrt{\cos 2x} \, dx.$$

By using an appropriate substitution or substitutions, show that

$$I = 2 - \sqrt{2} \ln(1 + \sqrt{2}).$$

☐ , ☐ proof

The image shows two handwritten solutions for the integral $I = \int_0^{\frac{1}{2}\pi} 4 \sin x \sqrt{\cos 2x} \, dx$.

Left Solution (Trigonometric Substitution):

- Start with TRIGONOMETRIC IDENTITIES: $I = \int_0^{\frac{1}{2}\pi} 4 \sin x \sqrt{\cos 2x} \, dx = 2 - \sqrt{2} \ln(1 + \sqrt{2})$
- STANDARD TRIGONOMETRIC IDENTITIES: $\Rightarrow I = \int_0^{\frac{1}{2}\pi} 4 \sin x \sqrt{2 \cos^2 x - 1} \, dx$
- SUBSTITUTION ALERT: $u = \cos x$, $\frac{du}{dx} = -\sin x$, $dx = \frac{du}{-\sin x}$
- ADJUST THE LIMITS: $x=0 \rightarrow u=1$, $x=\frac{1}{2}\pi \rightarrow u=0$
- NEXT CONTINUE WITH A TRIGONOMETRIC (OR HYPERBOLIC) SUBSTITUTION: $\Rightarrow I = \int_1^0 4 \sqrt{2u^2 - 1} \frac{du}{-1} = \int_0^1 4 \sqrt{2u^2 - 1} \, du$
- Let $u = \frac{1}{\sqrt{2}} \sec \theta$, $du = \frac{1}{\sqrt{2}} \sec \theta \tan \theta \, d\theta$
- When $u=1$, $\theta=0$; when $u=0$, $\theta=\frac{\pi}{4}$
- At THREE IS NO CANCEL IDENTITY OR SUBSTITUTION PROCESS BY ANY
- $\Rightarrow I = \int_0^{\frac{\pi}{4}} \left[\frac{1}{\sqrt{2}} \tan \theta \right] \left[\frac{1}{\sqrt{2}} \sec \theta \right] \left[\frac{1}{\sqrt{2}} \sec \theta \tan \theta \, d\theta \right] = \int_0^{\frac{\pi}{4}} \frac{1}{2} \tan^2 \theta \sec^2 \theta \, d\theta$

Right Solution (Hyperbolic Substitution):

- $\Rightarrow I = \left[\frac{1}{10} x \ln x - 0 \right] - \frac{1}{10} \int_0^{\frac{1}{2}\pi} \sec \theta \sec \theta \, d\theta$
- $\Rightarrow I = 4 - \frac{1}{\sqrt{2}} \int_0^{\frac{\pi}{4}} \sec \theta (1 + \tan^2 \theta) \, d\theta$
- $\Rightarrow I = 4 - \frac{1}{\sqrt{2}} \left[\int_0^{\frac{\pi}{4}} \sec \theta \, d\theta + \int_0^{\frac{\pi}{4}} \sec \theta \tan^2 \theta \, d\theta \right]$
- $\Rightarrow I = 4 - \frac{1}{\sqrt{2}} \left[\ln |\sec \theta + \tan \theta| \right]_0^{\frac{\pi}{4}} - \frac{1}{\sqrt{2}} \int_0^{\frac{\pi}{4}} \sec \theta \tan^2 \theta \, d\theta$
- $\Rightarrow I = 4 - \frac{1}{\sqrt{2}} \left[\ln(\sqrt{2} + 1) - 0 \right] - \frac{1}{\sqrt{2}} \int_0^{\frac{\pi}{4}} \sec \theta \tan^2 \theta \, d\theta$
- THIS: $\Rightarrow I = 2 - \frac{1}{\sqrt{2}} \ln(\sqrt{2} + 1)$
- $\Rightarrow I = 2 - \sqrt{2} \ln(1 + \sqrt{2})$

Question 327 (****)

Find as an exact fraction the value of I ,

$$I = \frac{\int_0^1 (1-x^{20})^{50} dx}{\int_0^1 (1-x^{20})^{51} dx}.$$

$$\boxed{}, \frac{1021}{1020}$$

• Let $I_{50} = \int_0^1 (1-x^{20})^{50} dx$ & $I_{51} = \int_0^1 (1-x^{20})^{51} dx$
 $I_{50} - \int_0^1 (1-x^{20})^{50} dx = \int_0^1 (1-x^{20})^{50} (1-x^{20})^{50} dx$
 $= \int_0^1 (1-x^{20})^{50} - x^{20} (1-x^{20})^{50} dx = \left(\int_0^1 (1-x^{20})^{50} dx - \int_0^1 x^{20} (1-x^{20})^{50} dx \right)$
 $= I_{50} - \int_0^1 x^{20} (1-x^{20})^{50} dx$
 • This is integrable by RECOGNITION (use the one you did in integration by parts)

$$= I_{50} - \left\{ \left[-\frac{1}{1020} (1-x^{20})^{51} \right]_0^1 + \frac{1}{1020} \int_0^1 (1-x^{20})^{51} dx \right\}$$

 $= I_{50} - \frac{1}{1020} I_{51}$
 • Finally DERIVATIVE THE DERIVATIVE
 $I_{51} = I_{50} - \frac{1}{1020} I_{51}$
 $1020 I_{51} = 1020 I_{50} - I_{51}$
 $1021 I_{51} = 1020 I_{50}$
 $\frac{I_{50}}{I_{51}} = \frac{1021}{1020}$

Question 328 (**)**

The integral I is defined as

$$I = \int_0^{\pi} \frac{\sin^2 x}{1 + \cos^2 x} dx.$$

a) Show by a detailed method that

$$I + \pi = \int_0^{\frac{1}{2}\pi} \frac{4}{1 + \cos^2 x} dx.$$

b) Hence, find the value of I in exact simplified form.

c) Verify the answer obtained in part (b) by an alternative method by first writing the integrand of I as a function of $\cot^2 x$.

$$\boxed{}, \quad I = \pi(\sqrt{2} - 1)$$

a) PROCESSED AT ROULETTE

$$\begin{aligned} &\Rightarrow \int_0^{\pi} \frac{\sin^2 x}{1 + \cos^2 x} dx + \int_0^{\pi} \frac{\cos^2 x}{1 + \cos^2 x} dx = \int_0^{\pi} \frac{\sin^2 x + \cos^2 x}{1 + \cos^2 x} dx \\ &\Rightarrow \int_0^{\pi} \frac{\sin^2 x}{1 + \cos^2 x} dx + \int_0^{\pi} \frac{(1 + \cos^2 x) - 1}{1 + \cos^2 x} dx = \int_0^{\pi} \frac{1}{1 + \cos^2 x} dx \\ &\Rightarrow \int_0^{\pi} \frac{\sin^2 x}{1 + \cos^2 x} dx + \int_0^{\pi} \frac{1}{1 + \cos^2 x} dx = \int_0^{\pi} \frac{1}{1 + \cos^2 x} dx \\ &\Rightarrow \int_0^{\pi} \frac{\sin^2 x}{1 + \cos^2 x} dx = \int_0^{\pi} \frac{1}{1 + \cos^2 x} dx - \int_0^{\pi} \frac{1}{1 + \cos^2 x} dx \\ &\Rightarrow \int_0^{\pi} \frac{\sin^2 x}{1 + \cos^2 x} dx = 2 \int_0^{\frac{\pi}{2}} \frac{1}{1 + \cos^2 x} dx \quad \text{SINCE DISCONTINUITY ABOUT } \frac{\pi}{2} \\ &\Rightarrow \int_0^{\pi} \frac{\sin^2 x}{1 + \cos^2 x} dx + \pi = \int_0^{\frac{\pi}{2}} \frac{4}{1 + \cos^2 x} dx \quad \text{AS REQUIRED} \\ &\Rightarrow I + \pi = \int_0^{\frac{\pi}{2}} \frac{4}{1 + \cos^2 x} dx \end{aligned}$$

CONTINUE BY NOTICING "TOP A BOTTOM" OF THE INTEGRAND BY $\sec^2 x$ (HINT YOU NOTICED THE DISCONTINUITY ABOUT $\frac{\pi}{2}$)

$$\begin{aligned} &\Rightarrow I + \pi = \int_0^{\frac{\pi}{2}} \frac{4 \sec^2 x}{\sec^2 x + 1} dx \\ &\Rightarrow I + \pi = \int_0^{\frac{\pi}{2}} \frac{4 \sec^2 x}{(1 + \tan^2 x) + 1} dx \end{aligned}$$

$$\begin{aligned} &\Rightarrow I + \pi = \int_0^{\frac{\pi}{2}} \frac{4 \sec^2 x}{2 + \tan^2 x} dx \\ &\text{NEXT CREATE AN ARCTAN DERIVATIVE (BY INSPECTION) OR USE A SUBSTITUTION} \\ &\Rightarrow I + \pi = \int_0^{\frac{\pi}{2}} \frac{4 \sec^2 x}{(\tan^2 x) + (\sqrt{2})^2} dx \\ &\Rightarrow I + \pi = \left[\frac{4}{\sqrt{2}} \arctan\left(\frac{\tan x}{\sqrt{2}}\right) \right]_0^{\frac{\pi}{2}} \\ &\Rightarrow I + \pi = \frac{4\sqrt{2}}{\sqrt{2}} \left[\arctan\left(\frac{\tan(\frac{\pi}{2})}{\sqrt{2}}\right) - \arctan\left(\frac{\tan(0)}{\sqrt{2}}\right) \right] \\ &\Rightarrow I + \pi = 2\sqrt{2} \left[\frac{\pi}{2} - 0 \right] \\ &\Rightarrow I + \pi = \pi\sqrt{2} \\ &\Rightarrow I = \pi\sqrt{2} - \pi \\ &\Rightarrow I = \pi(\sqrt{2} - 1) \quad \text{AS REQUIRED} \end{aligned}$$

d) WE NEED A TANGENT-TAN SUBSTITUTION - SIMPLY BY CRAFTING: GET BY DIVIDING TOP A BOTTOM BY $\sin^2 x$

$$\int_0^{\pi} \frac{\sin^2 x}{1 + \cos^2 x} dx = \int_0^{\pi} \frac{1}{\frac{1}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x}} dx$$

(NOTE THAT THERE IS NO DISCONTINUITY AT $\frac{\pi}{2}$)

$$\begin{aligned} &= \int_0^{\pi} \frac{1}{(1 + \cot^2 x) + \cot^2 x} dx = \int_0^{\pi} \frac{1}{1 + 2\cot^2 x} dx \\ &\text{BY SUBSTITUTION NEXT} \\ &\bullet u = \cot x \\ &\Rightarrow du = -\csc^2 x dx \\ &\Rightarrow dx = \frac{du}{-\csc^2 x} \\ &\Rightarrow dx = -\frac{du}{1 + u^2} \\ &\Rightarrow dx = -\frac{du}{1 + u^2} \end{aligned}$$

TRANSFORMING THE INTEGRAL

$$\dots = \int_{\infty}^{-\infty} \frac{1}{1 + 2u^2} \left(-\frac{du}{1 + u^2}\right) = \int_{-\infty}^{\infty} \frac{1}{(1 + 2u^2)(1 + u^2)} du$$

PARTIAL FRACTIONS (FULL METHOD OR INSPECTION)

$$\dots = \int_{-\infty}^{\infty} \frac{2}{(1 + 2u^2)(1 + u^2)} du = \int_{-\infty}^{\infty} \frac{1}{u^2 + \frac{1}{2}} - \frac{1}{u^2 + 1} du$$

$$= \int_{-\infty}^{\infty} \frac{1}{u^2 + \frac{1}{2}} - \frac{1}{u^2 + 1} du$$

$$\begin{aligned} &= \int_{-\infty}^{\infty} \frac{1}{u^2 + \left(\frac{1}{\sqrt{2}}\right)^2} - \frac{1}{u^2 + 1} du \\ &\text{EVEN INTEGRAND IN A SYMMETRIC DOMAIN} \\ &= 2 \int_0^{\infty} \frac{1}{u^2 + \left(\frac{1}{\sqrt{2}}\right)^2} - \frac{1}{u^2 + 1} du \\ &= 2 \left[\frac{1}{\frac{1}{\sqrt{2}}} \arctan\left(\frac{u}{\frac{1}{\sqrt{2}}}\right) - \arctan u \right]_0^{\infty} \\ &= 2 \left[\sqrt{2} \arctan(\sqrt{2}u) - \arctan u \right]_0^{\infty} \\ &= 2 \left[\left(\sqrt{2} \times \frac{\pi}{2} - \frac{\pi}{2} \right) - (0 - 0) \right] \\ &= \pi\sqrt{2} - \pi \\ &= \pi(\sqrt{2} - 1) \quad \text{AS REQUIRED} \end{aligned}$$

ALTERNATIVE METHOD

$$\begin{aligned} \int_0^{\pi} \frac{\sin^2 x}{1 + \cos^2 x} dx &= \int_0^{\pi} \frac{\sin^2 x}{2 - \sin^2 x} dx = \text{GIVEN NEXT } \frac{\pi}{2} \dots \\ &= \int_0^{\frac{\pi}{2}} \frac{2 \sin^2 x}{2 - \sin^2 x} dx = \int_0^{\frac{\pi}{2}} \frac{2 \sin^2 x}{2 - \sin^2 x} dx \quad \text{THIS IS BACK AND WORKS WELL} \end{aligned}$$

Question 329 (*****)

By using an appropriate substitution or substitutions, followed by partial fractions show that

$$\int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{9 + 16 \sin 2x} dx = \frac{\ln 3}{20}.$$

 , proof

• $\int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{9 + 16 \sin 2x} dx = \text{BY SUBSTITUTION} \dots$

<p>LET $u = \sin x - \cos x$</p> <p>$\frac{du}{dx} = \cos x + \sin x$</p> <p>$\frac{du}{\cos x + \sin x} = dx$</p> <p>$x=0 \rightarrow u=1$</p> <p>$x=\frac{\pi}{4} \rightarrow u=0$</p>	<p>ALSO $u^2 = (\sin x - \cos x)^2$</p> <p>$\Rightarrow u^2 = \sin^2 x + \cos^2 x - 2 \sin x \cos x$</p> <p>$\Rightarrow u^2 = 1 - \sin 2x$</p> <p>$\Rightarrow 16 \sin 2x = 1 - u^2$</p> <p>$\Rightarrow 9 + 16 \sin 2x = 9 - 16u^2$</p>
---	--

$\dots \int_1^0 \frac{\sin x + \cos x}{9 - 16u^2} \left(\frac{du}{\cos x + \sin x} \right) = \int_1^0 \frac{1}{(5-4u)(5+4u)} du$

• BY PARTIAL FRACTIONS (COVER UP METHOD)

$\dots = \int_1^0 \left(\frac{\frac{1}{10}}{5+4u} + \frac{\frac{1}{10}}{5-4u} \right) du = \frac{1}{10} \int_1^0 \left(\frac{1}{5+4u} + \frac{1}{5-4u} \right) du$

$= \frac{1}{10} \left[\frac{1}{4} \ln|5+4u| - \frac{1}{4} \ln|5-4u| \right]_1^0$

$= \frac{1}{40} \left[\ln|5+4u| - \ln|5-4u| \right]_1^0$

$= \frac{1}{40} \left[(\ln 5 - \ln 1) - (\ln 1 - \ln 9) \right] = \frac{1}{40} \ln 9 = \frac{1}{20} \ln 3$ ✓ REPOSED

ALTERNATIVE METHOD

• USING $\int_a^b f(x) dx \equiv \int_a^b f(a+b-x) dx$

$\int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{9 + 16 \sin 2x} dx = \int_0^{\frac{\pi}{4}} \frac{\sin(\frac{\pi}{4}-x) + \cos(\frac{\pi}{4}-x)}{9 + 16 \sin(\frac{\pi}{4}-x)} dx$

$= \int_0^{\frac{\pi}{4}} \frac{\sin x \cos x - \cos x \sin x + \sin x \cos x + \cos x \sin x}{9 + 16 \sin 2x} dx$

$\dots = \int_0^{\frac{\pi}{4}} \frac{2 \sin x \cos x}{9 + 16 \sin 2x} dx = \int_0^{\frac{\pi}{4}} \frac{\sqrt{2} \sin 2x}{9 + 16 \sin 2x} dx$

$\dots = \int_0^{\frac{\pi}{4}} \frac{\sqrt{2} \sin 2x}{9 + 16(1 - \sin^2 2x)} dx = \int_0^{\frac{\pi}{4}} \frac{\sqrt{2} \sin 2x}{25 - 32 \sin^2 2x} dx$

• BY SUBSTITUTION

$u = \sin 2x$

$\frac{du}{dx} = 2 \cos 2x$

$\frac{du}{2 \cos 2x} = dx$

$x=0 \rightarrow u=0$

$x=\frac{\pi}{4} \rightarrow u=1$

$\dots = \int_0^1 \frac{\sqrt{2} u}{25 - 32u^2} \frac{du}{2 \cos 2x}$

$\dots = \frac{\sqrt{2}}{2} \int_0^1 \frac{u}{(5+\sqrt{8}u)(5-\sqrt{8}u)} du$

• PARTIAL FRACTIONS (BY COVER UP)

$\dots = \sqrt{2} \int_0^1 \left(\frac{\frac{1}{10}}{5+\sqrt{8}u} + \frac{\frac{1}{10}}{5-\sqrt{8}u} \right) du = \frac{\sqrt{2}}{10} \int_0^1 \left(\frac{1}{5+\sqrt{8}u} + \frac{1}{5-\sqrt{8}u} \right) du$

$= \frac{\sqrt{2}}{10} \left[\frac{1}{\sqrt{8}} \ln|5+\sqrt{8}u| - \frac{1}{\sqrt{8}} \ln|5-\sqrt{8}u| \right]_0^1$

$= \frac{1}{40} \left[\ln|5+\sqrt{8}| - \ln|5-\sqrt{8}| \right] = \frac{1}{40} \ln 9 = \frac{1}{20} \ln 3$ ✓

Question 330 (****)

$$I = \int \frac{\sec^2 x}{\sqrt{\sec x + \tan x}} dx.$$

Without using a verification approach, show that

$$I = (\sec x + \tan x)^{\frac{1}{2}} - \frac{1}{3}(\sec x + \tan x)^{\frac{3}{2}} + \text{constant}.$$

You may consider the substitution $u = \sec x + \tan x$ useful at some stages in the manipulation of the integrand.

☐ , ☐ proof

LOOKING AT THE SUBSTITUTION SUGGESTED $\frac{d}{dx}(\sec x + \tan x) = \sec x + \sec^2 x$

$$\int \frac{\sec^2 x}{\sqrt{\sec x + \tan x}} dx = \frac{1}{2} \int \frac{\sec^2 x + \sec x \tan x - \sec x \tan x + \sec^2 x}{\sqrt{\sec x + \tan x}} dx$$

$$= \frac{1}{2} \int \frac{\sec^2 x + \sec x \tan x}{\sqrt{\sec x + \tan x}} dx + \frac{1}{2} \int \frac{\sec^2 x - \sec x \tan x}{\sqrt{\sec x + \tan x}} dx$$

ADVANTAGE

MANIPULATE THE SECOND INTEGRAL AS FOLLOWS & NOTE THAT $1 + \tan^2 x = \sec^2 x$

$$= \frac{1}{2} \int \frac{\sec^2 x + \sec x \tan x}{\sqrt{\sec x + \tan x}} dx + \frac{1}{2} \int \frac{\sec^2 x - \tan x \sec x}{(\sec x + \tan x)^2 (\sec x + \tan x)} dx$$

$$= \frac{1}{2} \int \frac{\sec^2 x + \sec x \tan x}{\sqrt{\sec x + \tan x}} dx + \frac{1}{2} \int \frac{\sec^2 x (1 - \tan^2 x)}{(\sec x + \tan x)^3} dx$$

$$= \frac{1}{2} \int \frac{\sec^2 x + \sec x \tan x}{\sqrt{\sec x + \tan x}} dx + \frac{1}{2} \int \frac{\sec^2 x (\sec x - \tan x)}{(\sec x + \tan x)^3} dx$$

$$= \frac{1}{2} \int \frac{\sec^2 x + \sec x \tan x}{\sqrt{\sec x + \tan x}} dx + \frac{1}{2} \int \frac{\sec^3 x - \sec^2 x \tan x}{(\sec x + \tan x)^3} dx$$

$$= \frac{1}{2} \int \frac{\sec^2 x + \sec x \tan x}{\sqrt{\sec x + \tan x}} dx + \frac{1}{2} \int \frac{\sec^2 x (1 - \tan^2 x)}{(\sec x + \tan x)^3} dx$$

THE NEW TWO INTEGRALS

FINALLY BY SUBSTITUTION OR RECOGNITION

$$= \frac{1}{2} \int (\sec^2 x + \sec x \tan x) \sqrt{\sec x + \tan x} dx + \frac{1}{2} \int (\sec^2 x + \sec x \tan x) (\sec x + \tan x)^{-\frac{3}{2}} dx$$

$$= \frac{1}{2} \cdot \frac{1}{\frac{3}{2}} (\sec x + \tan x)^{\frac{3}{2}} + \frac{1}{2} \cdot \frac{1}{-\frac{1}{2}} (\sec x + \tan x)^{-\frac{1}{2}} + C$$

$$= \frac{1}{3} (\sec x + \tan x)^{\frac{3}{2}} - \frac{1}{2} (\sec x + \tan x)^{\frac{1}{2}} + C$$

As Required

Question 331 (****)

Use an appropriate integration method to determine an antiderivative for the following indefinite integral.

$$\int \frac{x^2(x^4+1)}{\sqrt[4]{x^4+1}} dx$$

$$\boxed{\sqrt{}}, \boxed{}, \boxed{\frac{1}{6}(x^8+2x^4)^{\frac{1}{2}}+C}$$

MANIPULATE AS FOLLOWS

$$\begin{aligned} \int \frac{x^2(x^4+1)}{\sqrt[4]{x^4+1}} dx &= \int x^2(x^4+1)^{\frac{1}{4}} dx = \int x^2(x^4+1)^{\frac{1}{4}} dx \\ &= \int (x^2+2)(x^4+1)^{\frac{1}{4}} dx = \int (x^2+2) \left[\frac{1}{5}(x^4+1)^{\frac{5}{4}} \right] dx \\ &= \int (x^2+2)(x^4+1)^{\frac{1}{4}} dx \end{aligned}$$


NOTE THAT $\frac{d}{dx}(x^4+1) = 4x^3 = 4(x^2+2)x$

$$\begin{aligned} &= \frac{1}{5} \int (x^2+2)(x^4+1)^{\frac{1}{4}} dx \\ &= \frac{1}{5} \times \frac{1}{4} (x^4+1)^{\frac{5}{4}} + C \\ &= \frac{1}{20} (x^4+1)^{\frac{5}{4}} + C \end{aligned}$$

Question 332 (****)

Use partial fractions followed by integration by parts to show that

$$\int_0^{\infty} \left[\frac{x^2 + 3x + 3}{(x+1)^3} \right] e^{-x} \sin x \, dx = \frac{1}{2}.$$

 , proof

• SPLIT BY PARTIAL FRACTIONS FIRST

$$\frac{x^2 + 3x + 3}{(x+1)^3} \equiv \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$$

$$x^2 + 3x + 3 \equiv A(x+1)^2 + B(x+1) + C$$

$$x^2 + 3x + 3 \equiv Ax^2 + 2Ax + A + Bx + B + C$$

$$x^2 + 3x + 3 \equiv Ax^2 + (2A+B)x + (A+B+C)$$

4thrice $A = B = C = 1$

• NEXT WE FIND THE INTEGRAL OF $e^{-x} \sin x$, BY PARTS TWICE

$$\int e^{-x} \sin x \, dx = -e^{-x} \cos x - \int e^{-x} \cos x \, dx$$

$\frac{e^{-x}}{-\cos x}$	$-\frac{e^{-x}}{\sin x}$
--------------------------	--------------------------

$$\int e^{-x} \cos x \, dx = -e^{-x} \cos x - \left[e^{-x} \sin x + \int e^{-x} \sin x \, dx \right]$$

$$\int e^{-x} \sin x \, dx = -e^{-x} \cos x - e^{-x} \sin x - \int e^{-x} \sin x \, dx$$

$$2 \int e^{-x} \sin x \, dx = -e^{-x} (\cos x + \sin x) + C$$

$$\int e^{-x} \sin x \, dx = -\frac{1}{2} e^{-x} (\cos x + \sin x) + C$$

• NEXT PUT THE INTEGRAL INTO 3 & 4 (ONLY) ON INTEGRATION BY PARTS IN THE FIRST & THIRD, BUT NOT IN THE SECOND (LEFT FOR ONEHAND)

$$\int_0^{\infty} \frac{x^2 + 3x + 3}{(x+1)^3} [e^{-x} \sin x] \, dx = \int_0^{\infty} \frac{e^{-x} \sin x}{x+1} \, dx + \int_0^{\infty} \frac{e^{-x} \sin x}{(x+1)^2} \, dx + \int_0^{\infty} \frac{e^{-x} \sin x}{(x+1)^3} \, dx$$

$\frac{1}{x+1}$	$-\frac{1}{(x+1)^2}$	$\frac{e^{-x} \sin x}{x+1}$	$-\frac{e^{-x} \sin x}{(x+1)^2}$
$-\frac{1}{2} e^{-x} (\cos x + \sin x)$	$\frac{e^{-x} \sin x}{x+1}$	$-\frac{e^{-x} \sin x}{(x+1)^2}$	$-\frac{e^{-x} (\cos x - \sin x)}{(x+1)^3}$

• $\int_0^{\infty} \frac{x^2 + 3x + 3}{(x+1)^3} [e^{-x} \sin x] \, dx = \left[-\frac{1}{2} \frac{e^{-x}}{x+1} (\cos x + \sin x) \right]_0^{\infty} - \int_0^{\infty} \frac{e^{-x} (\cos x - \sin x)}{(x+1)^3} \, dx$

$$+ \int_0^{\infty} \frac{e^{-x} \sin x}{(x+1)^2} \, dx$$

$$\left[-\frac{1}{2} \frac{e^{-x}}{x+1} \right]_0^{\infty} + \int_0^{\infty} \frac{e^{-x} \sin x}{(x+1)^2} \, dx + \int_0^{\infty} \frac{e^{-x} \sin x}{(x+1)^3} \, dx$$

$$= [0 - (-\frac{1}{2})] + \int_0^{\infty} \frac{e^{-x} \sin x}{(x+1)^2} \, dx - \frac{1}{2} \frac{e^{-x} \cos x}{(x+1)^2} \, dx$$

$$+ \int_0^{\infty} \frac{e^{-x} \sin x}{(x+1)^3} \, dx$$

$$[0 - (0)] + \int_0^{\infty} \frac{e^{-x} \sin x}{(x+1)^2} \, dx - \frac{1}{2} \frac{e^{-x} \cos x}{(x+1)^2} \, dx$$

$$= \frac{1}{2}$$