INTEGRATION STRUCTURED EXAM QUESTIONS PART I

Question 1 (**)

Evaluate each of the following integrals, giving the answers in exact form.

a) $\displaystyle\int_0^2 \frac{1}{\sqrt{4x+1}}\,dx$.

b) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos 3x\,dx$.

$\boxed{1}$, $\boxed{-\frac{1}{3}}$

(a) $\displaystyle\int_0^2 \frac{1}{\sqrt{4x+1}}\,dx = \int_0^2 (4x+1)^{-\frac{1}{2}}\,dx = \left[\frac{1}{2}(4x+1)^{\frac{1}{2}}\right]_0^2 = \left[\frac{1}{2}\sqrt{4x+1}\right]_0^2$

$= \frac{1}{2}\times 3 - \frac{1}{2} = \frac{3}{2} - \frac{1}{2} = 1$

(b) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos 3x\,dx = \left[\frac{1}{3}\sin 3x\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \frac{1}{3}\sin\pi - \frac{1}{3}\sin\frac{\pi}{2} = -\frac{1}{3}$

Question 2 (**)

By using the substitution $u = 4 + 3x^2$, or otherwise, find

$$\int \frac{2x}{\left(4+3x^2\right)^2}\,dx .$$

$\boxed{-\frac{1}{3\left(4+3x^2\right)} + C}$

$\displaystyle\int \frac{2x}{(4+3x^2)^2}\,dx = \int \frac{2x}{u^2}\,\frac{du}{6x}$

$\displaystyle = \int \frac{1}{3}u^{-2}\,du = -\frac{1}{3}u^{-1} + C$

$\displaystyle = -\frac{1}{3}(4+3x^2)^{-1} + C = -\frac{1}{3(4+3x^2)} + C$

$u = 4+3x^2$

$\frac{du}{dx} = 6x$

$du = 6x\,dx$

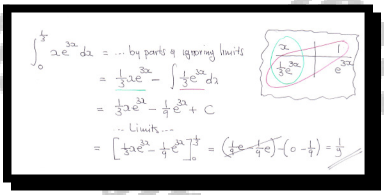
$dx = \frac{du}{6x}$

Question 3 **(**)**

Show clearly that

$$\int_0^{\frac{1}{3}} x\mathrm{e}^{3x} \, dx = \frac{1}{9}.$$

C4 , proof



Question 4 **(**)**

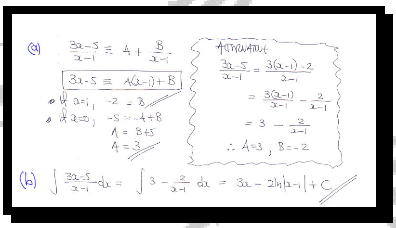
$$\frac{3x-5}{x-1} \equiv A + \frac{B}{x-1}.$$

a) Determine the value of each of the constants A and B.

b) Hence find

$$\int \frac{3x-5}{x-1} \, dx.$$

$\boxed{A=3}$, $\boxed{B=-2}$, $\boxed{3x-2\ln|x-1|+C}$



Question 5 **(**)**

Evaluate each of the following integrals, giving the answers in exact form.

a) $\displaystyle\int_0^4 e^{\frac{1}{2}x}\,dx$.

b) $\displaystyle\int_0^{\frac{\pi}{4}} \cos\left(3x+\frac{\pi}{4}\right)\,dx$.

C4, $\boxed{2\left(e^2-1\right)}$, $\boxed{-\frac{\sqrt{2}}{6}}$

(a) $\displaystyle\int_0^4 e^{\frac{1}{2}x}\,dx = \left[2e^{\frac{1}{2}x}\right]_0^4 = 2e^2-2 = 2(e^2-1)$

(b) $\displaystyle\int_0^{\frac{\pi}{4}} \cos\left(3x+\frac{\pi}{4}\right)dx = \left[\frac{1}{3}\sin\left(3x+\frac{\pi}{4}\right)\right]_0^{\frac{\pi}{4}} = \frac{1}{3}\sin\pi - \frac{1}{3}\sin\frac{\pi}{4}$

$= -\frac{1}{3}\times\frac{\sqrt{2}}{2} = -\frac{\sqrt{2}}{6}$

Question 6 (**)

$$\frac{5x+13}{(2x+1)(x+4)} \equiv \frac{A}{2x+1} + \frac{B}{x+4}.$$

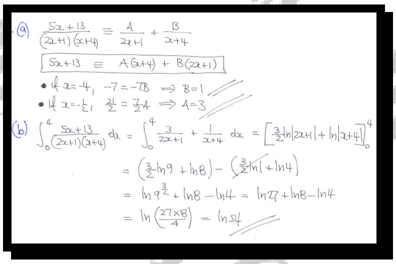
a) Determine the value of each of the constants A and B.

b) Evaluate

$$\int_0^4 \frac{5x+13}{(2x+1)(x+4)}\, dx,$$

giving the answer as a single simplified natural logarithm.

$A=3$, $B=1$, $\ln 54$

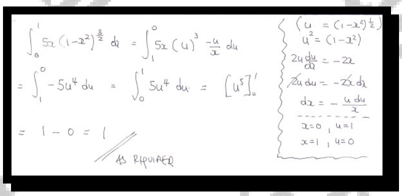


Question 7 (**)

By using the substitution $u^2 = 1 - x^2$, or otherwise, show that

$$\int_0^1 5x\left(1-x^2\right)^{\frac{3}{2}}\, dx = 1.$$

proof



Question 8 **(**)**

Use integration by parts to find the value of

$$\int_0^{\frac{\pi}{4}} 4x\cos 4x \ dx .$$

$\boxed{}$, $\boxed{-\frac{1}{2}}$

$$\int_0^{\frac{\pi}{4}} 4x\cos 4x \, dx = \dots \text{by parts \& ignoring limits}$$
$$= x\sin 4x - \int \sin 4x \, dx$$
$$= x\sin 4x + \frac{1}{4}\cos 4x + C$$

$4x$	4
$\frac{1}{4}\sin 4x$	$\cos 4x$

... Limits ...

$$= \left[x\sin 4x + \frac{1}{4}\cos 4x\right]_0^{\frac{\pi}{4}}$$
$$= \left(\frac{\pi}{4}\sin\pi + \frac{1}{4}\cos\pi\right) - \left(0 + \frac{1}{4}\cos 0\right)$$
$$= -\frac{1}{4} - \left(\frac{1}{4}\right) = -\frac{1}{2}$$

Question 9 **(**)**

By using the substitution $u = 1 - x^2$, or otherwise, find

$$\int \frac{12x}{\left(1-x^2\right)^{\frac{3}{2}}} \, dx .$$

$\boxed{}$, $\boxed{\frac{12}{\sqrt{1-x^2}} + C}$

$$u = 1 - x^2, \quad \frac{du}{dx} = -2x, \quad dx = \frac{du}{-2x}$$

$$\int \frac{12x}{(1-x^2)^{\frac{3}{2}}}\,dx = \int \frac{12x}{u^{3/2}} \frac{du}{-2x}$$
$$= \int \frac{-6}{u^{\frac{3}{2}}}\,du = \int -6u^{-\frac{3}{2}}\,du$$
$$= 12u^{-\frac{1}{2}} + C = \frac{12}{\sqrt{1-x^2}} + C$$

Question 10 **(**)**

Evaluate each of the following integrals, giving the answers in exact form.

a) $\displaystyle\int_0^3 \frac{4}{2x+3}\, dx$.

b) $\displaystyle\int_0^{\frac{\pi}{6}} \sin\left(4x+\frac{\pi}{6}\right) dx$.

$\boxed{\ln 9}$, $\boxed{\dfrac{\sqrt{3}}{4}}$

(a) $\displaystyle\int_0^3 \frac{4}{2x+3}\,dx = \Big[2\ln|2x+3|\Big]_0^3 = 2\ln 9 - 2\ln 3$
$= 2\ln 9 - \ln 9 = \ln 9$ (or $2\ln 3$)

(b) $\displaystyle\int_0^{\frac{\pi}{6}} \sin\left(4x+\frac{\pi}{6}\right) dx = \left[-\frac{1}{4}\cos\left(4x+\frac{\pi}{6}\right)\right]_0^{\frac{\pi}{6}} = \frac{1}{4}\left[\cos\left(4x+\frac{\pi}{6}\right)\right]_{\frac{\pi}{6}}^{0}$
$= \frac{1}{4}\left[\cos\frac{\pi}{6} - \cos\frac{5\pi}{6}\right] = \frac{1}{4}\left[\frac{\sqrt{3}}{2} - \left(-\frac{\sqrt{3}}{2}\right)\right]$
$= \frac{1}{4}\times\sqrt{3} = \frac{\sqrt{3}}{4}$

Question 11 **(**)**

Use a trigonometric identity to integrate

$$\int \frac{1}{1+\cos 2x}\, dx.$$

$\boxed{\frac{1}{2}\tan x + C}$

$\displaystyle\int \frac{1}{1+\cos 2x}\,dx = \int \frac{1}{1+(2\cos^2 x - 1)}\,dx$
$\displaystyle= \int \frac{1}{2\cos^2 x}\,dx$
$\displaystyle= \int \frac{1}{2}\sec^2 x\,dx$
$= \frac{1}{2}\tan x + C$

Question 12 **(**)**

$$\frac{30}{(x+3)(9-2x)} \equiv \frac{A}{x+3} + \frac{B}{9-2x}.$$

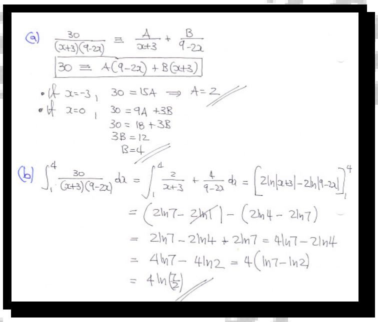
a) Determine the value of each of the constants A and B.

b) Evaluate

$$\int_1^4 \frac{30}{(x+3)(9-2x)}\, dx,$$

giving the answer as a single simplified natural logarithm.

$\boxed{A=2}$, $\boxed{B=4}$, $\boxed{4\ln\left(\frac{7}{2}\right) = \ln\left(\frac{2401}{16}\right)}$



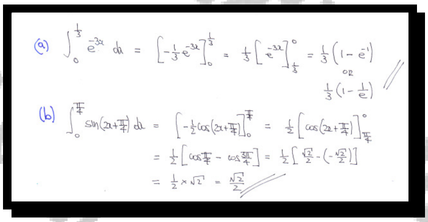
Question 13 (**)

Evaluate each of the following integrals, giving the answers in exact form.

a) $\displaystyle\int_0^{\frac{1}{3}} e^{-3x}\ dx$.

b) $\displaystyle\int_0^{\frac{\pi}{4}} \sin\left(2x+\frac{\pi}{4}\right)\ dx$.

$\boxed{\frac{1}{3}\left(1-e^{-1}\right)}$, $\boxed{\frac{\sqrt{2}}{2}}$



(a) $\int_0^{\frac{1}{3}} e^{-3x}\,dx = \left[-\frac{1}{3}e^{-3x}\right]_0^{\frac{1}{3}} = \frac{1}{3}\left[e^{-3x}\right]_{\frac{1}{3}}^{0} = \frac{1}{3}\left(1-e^{-1}\right)$ OR $\frac{1}{3}\left(1-\frac{1}{e}\right)$

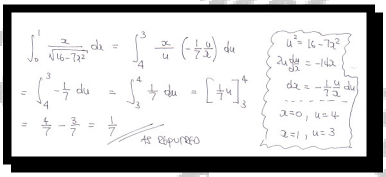
(b) $\int_0^{\frac{\pi}{4}} \sin\left(2x+\frac{\pi}{4}\right)dx = \left[-\frac{1}{2}\cos\left(2x+\frac{\pi}{4}\right)\right]_0^{\frac{\pi}{4}} = \frac{1}{2}\left[\cos\left(2x+\frac{\pi}{4}\right)\right]_{\frac{\pi}{4}}^{0}$
$= \frac{1}{2}\left[\cos\frac{\pi}{4} - \cos\frac{3\pi}{4}\right] = \frac{1}{2}\left[\frac{\sqrt{2}}{2} - \left(-\frac{\sqrt{2}}{2}\right)\right]$
$= \frac{1}{2}\times\sqrt{2} = \frac{\sqrt{2}}{2}$

Question 14 (**+)

By using the substitution $u^2 = 16-7x^2$, or otherwise, show that

$$\int_0^1 \frac{x}{\sqrt{16-7x^2}}\ dx = \frac{1}{7}.$$

$\boxed{}$, $\boxed{\text{proof}}$



$\int_0^1 \frac{x}{\sqrt{16-7x^2}}\,dx = \int_4^3 \frac{x}{u}\left(-\frac{u}{7x}\right)du$
$= \int_4^3 -\frac{1}{7}\,du = \int_3^4 \frac{1}{7}\,du = \left[\frac{1}{7}u\right]_3^4$
$= \frac{4}{7} - \frac{3}{7} = \frac{1}{7}$ AS REQUIRED

$u^2 = 16-7x^2$
$2u\frac{du}{dx} = -14x$
$dx = -\frac{u}{7x}\,du$
$x=0,\ u=4$
$x=1,\ u=3$

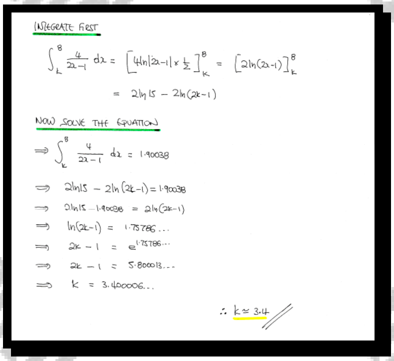
Question 15 **(**+)**

Determine the value of the positive constant k given further that

$$\int_{k}^{8} \frac{4}{2x-1}\,dx = 1.90038.$$

Give the value of k to an appropriate degree of accuracy.

$k \approx 3.4$


INTEGRATE FIRST

$\int_k^8 \frac{4}{2x-1}\,dx = \left[4\ln|2x-1| \times \frac{1}{2}\right]_k^8 = \left[2\ln(2x-1)\right]_k^8$

$= 2\ln 15 - 2\ln(2k-1)$

NOW SOLVE THE EQUATION

$\Rightarrow \int_k^8 \frac{4}{2x-1}\,dx = 1.90038$

$\Rightarrow 2\ln 15 - 2\ln(2k-1) = 1.90038$

$\Rightarrow 2\ln 15 - 1.90038 = 2\ln(2k-1)$

$\Rightarrow \ln(2k-1) = 1.75786...$

$\Rightarrow 2k-1 = e^{1.75786...}$

$\Rightarrow 2k-1 = 5.80003...$

$\Rightarrow k = 3.400006...$

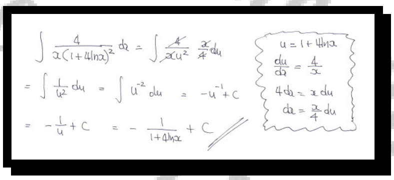
$\therefore k \approx 3.4$

Question 16 **(**+)**

By using the substitution $u = 1 + 4\ln x$, or otherwise, find

$$\int \frac{4}{x(1+4\ln x)^2}\,dx.$$

$-\frac{1}{1+4\ln x} + C$


$\int \frac{4}{x(1+4\ln x)^2}\,dx = \int \frac{4}{xu^2}\,\frac{x}{4}\,du$

$u = 1 + 4\ln x$, $\frac{du}{dx} = \frac{4}{x}$, $4\,dx = x\,du$, $dx = \frac{x}{4}\,du$

$= \int \frac{1}{u^2}\,du = \int u^{-2}\,du = -u^{-1} + C$

$= -\frac{1}{u} + C = -\frac{1}{1+4\ln x} + C$

Question 17 (**+)

$$\frac{8x}{4x-3} \equiv A + \frac{B}{4x-3}.$$

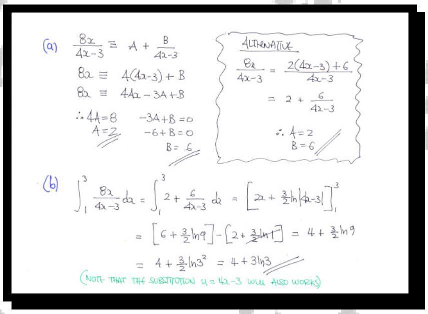
a) Determine the value of each of the constants A and B.

b) Hence, or otherwise, evaluate

$$\int_1^3 \frac{8x}{4x-3} \, dx,$$

giving the answer in terms of natural logarithms.

$\boxed{A=2}$, $\boxed{B=6}$, $\boxed{4+3\ln 3}$

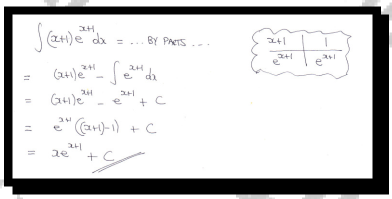


Question 18 (**+)

Use an appropriate integration method to find

$$\int (x+1)e^{x+1} \, dx.$$

$\boxed{}$, $\boxed{xe^{x+1}+C}$



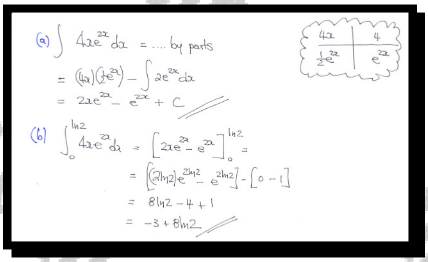
Question 19 **(**+)**

$$f(x) = 4x\mathrm{e}^{2x}.$$

a) Use integration by parts to find $\int f(x)\,dx$.

b) Find an exact vale for $\int_{0}^{\ln 2} f(x)\,dx$.

$\boxed{}$, $\boxed{2x\mathrm{e}^{2x} - \mathrm{e}^{2x} + C}$, $\boxed{-3 + 8\ln 2}$



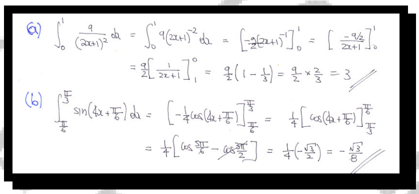
Question 20 **(**+)**

Evaluate each of the following integrals, giving the answers in exact form.

a) $\int_{0}^{1} \frac{9}{(2x+1)^2}\,dx$.

b) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin\left(4x + \frac{\pi}{6}\right)\,dx$.

$\boxed{3}$, $\boxed{-\frac{\sqrt{3}}{8}}$



Question 21 **(**+)**

By using the substitution $u = \ln x$, or otherwise, find an exact value for

$$\int_{e}^{3} \frac{1}{x \ln x} \, dx .$$

$\boxed{\ln(\ln 3)}$

$$\int_{e}^{3} \frac{1}{x \ln x} \, dx = \int_{1}^{\ln 3} \frac{1}{xu} \, x \, du = \int_{1}^{\ln 3} \frac{1}{u} \, du$$

$$\left[\ln|u| \right]_{1}^{\ln 3} = \ln(\ln 3) - \ln 1 = \ln(\ln 3)$$

$u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$

$dx = x \, du$

$x = e, \ u = 1$

$x = 3, \ u = \ln 3$

Question 22 **(**+)**

$$f(x) \equiv \frac{x-5}{x^2+5x+4}.$$

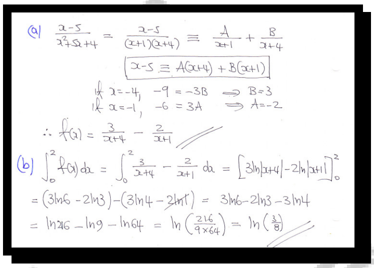
a) Express $f(x)$ in partial fractions.

b) Find the value of

$$\int_0^2 f(x)\ dx\ ,$$

giving the answer as a single simplified logarithm.

$\boxed{}$, $\boxed{f(x) \equiv \frac{3}{x+4} - \frac{2}{x+1}}$, $\boxed{\int_0^2 f(x)\ dx = \ln\left(\frac{3}{8}\right)}$

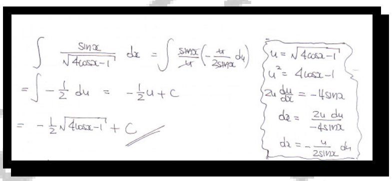


Question 23 **(**+)**

By using the substitution $u^2 = 4\cos x - 1$, or otherwise, find

$$\int \frac{\sin x}{\sqrt{4\cos x - 1}}\ dx\ .$$

$\boxed{-\frac{1}{2}\sqrt{4\cos x - 1} + C}$



Question 24 (**+)

Use the substitution $u = \sqrt{2x-7}$ to find

$$\int_4^8 \frac{6x}{\sqrt{2x-7}} \, dx .$$

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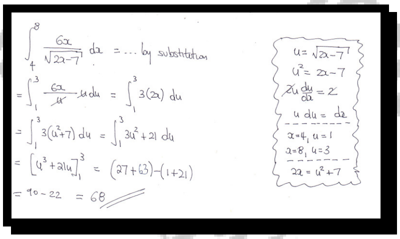


Question 25 (**+)

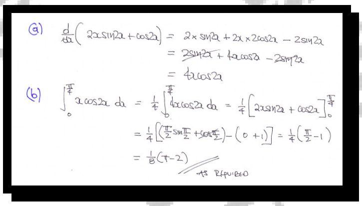
$$y = 2x\sin 2x + \cos 2x , \ x \in \mathbb{R}$$

a) Find an expression for $\frac{dy}{dx}$.

b) Hence show that

$$\int_0^{\frac{\pi}{4}} x\cos 2x \, dx = \frac{1}{8}(\pi - 2).$$

$$\frac{dy}{dx} = 4x\cos 2x$$



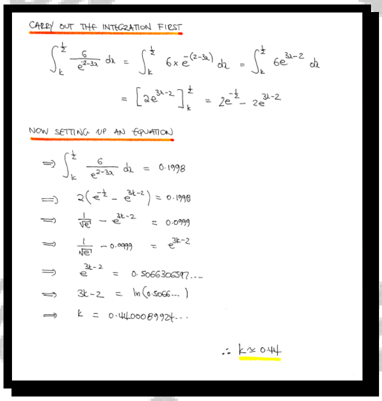
Question 26 **(**+)**

Determine the value of the positive constant k given further that

$$\int_{k}^{\frac{1}{2}} \frac{6}{e^{2-3x}}\,dx = 0.1998.$$

Give the value of k to an appropriate degree of accuracy.

$\boxed{}$, $\boxed{k \approx 0.44}$



CARRY OUT THE INTEGRATION FIRST

$$\int_{k}^{\frac{1}{2}} \frac{6}{e^{2-3x}}\,dx = \int_{k}^{\frac{1}{2}} 6 \times e^{-(2-3x)}\,dx = \int_{k}^{\frac{1}{2}} 6e^{3x-2}\,dx$$

$$= \left[2e^{3x-2}\right]_{k}^{\frac{1}{2}} = 2e^{-\frac{1}{2}} - 2e^{3k-2}$$

NOW SETTING UP AN EQUATION

$$\Rightarrow \int_{k}^{\frac{1}{2}} \frac{6}{e^{2-3x}}\,dx = 0.1998$$

$$\Rightarrow 2\left(e^{-\frac{1}{2}} - e^{3k-2}\right) = 0.1998$$

$$\Rightarrow \frac{1}{\sqrt{e}} - e^{3k-2} = 0.0999$$

$$\Rightarrow \frac{1}{\sqrt{e}} - 0.0999 = e^{3k-2}$$

$$\Rightarrow e^{3k-2} = 0.5066306597\ldots$$

$$\Rightarrow 3k-2 = \ln(0.5066\ldots)$$

$$\Rightarrow k = 0.4400089924\ldots$$

$$\therefore k \simeq 0.44$$

Question 27 (***)

$$y = \frac{3x}{2 + x - x^2}$$

a) Calculate the three missing values of y in the following table.

x	0	0.25	0.5	0.75	1
y	0				1.5

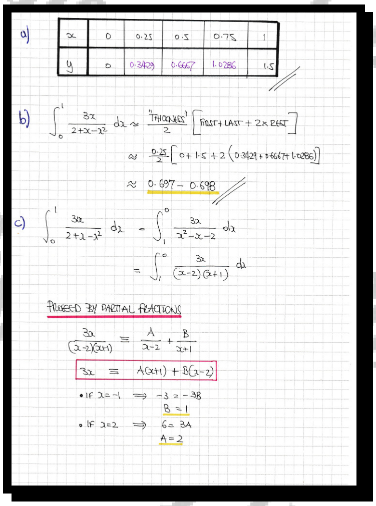
b) Use the trapezium rule with all the values from the completed table of part **(a)** to find an estimate for

$$\int_0^1 \frac{3x}{2 + x - x^2}\, dx .$$

c) Use a suitable method to find the exact value of

$$\int_0^1 \frac{3x}{2 + x - x^2}\, dx .$$

$\boxed{}$, $\boxed{0.3429}$, $\boxed{0.6667}$, $\boxed{1.0286}$, $\boxed{\ln 2}$



a)

x	0	0.25	0.5	0.75	1
y	0	0.3429	0.6667	1.0286	1.5

b) $\int_0^1 \frac{3x}{2+x-x^2}\,dx \approx \frac{\text{"THICKNESS"}}{2}\left[\text{FIRST} + \text{LAST} + 2 \times \text{REST}\right]$

$\approx \frac{0.25}{2}\left[0 + 1.5 + 2(0.3429 + 0.6667 + 1.0286)\right]$

$\approx 0.697 - 0.698$

c) $\int_0^1 \frac{3x}{2+x-x^2}\,dx = \int_1^0 \frac{3x}{x^2-x-2}\,dx$

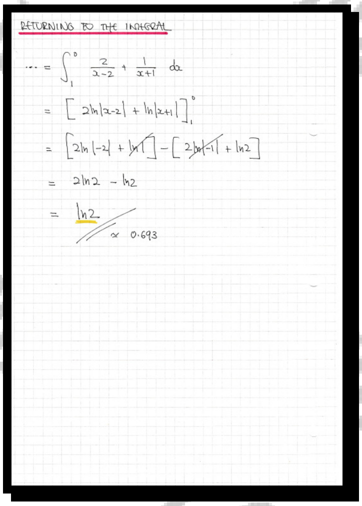
$= \int_1^0 \frac{3x}{(x-2)(x+1)}\,dx$

PROCEED BY PARTIAL FRACTIONS

$\frac{3x}{(x-2)(x+1)} \equiv \frac{A}{x-2} + \frac{B}{x+1}$

$3x \equiv A(x+1) + B(x-2)$

- IF $x=-1 \Rightarrow -3 = -3B$, $B = 1$
- IF $x=2 \Rightarrow 6 = 3A$, $A = 2$



RETURNING TO THE INTEGRAL

$\dots = \int_1^0 \frac{2}{x-2} + \frac{1}{x+1}\,dx$

$= \left[2\ln|x-2| + \ln|x+1|\right]_1^0$

$= \left[2\ln|-2| + \ln 1\right] - \left[2\ln|-1| + \ln 2\right]$

$= 2\ln 2 - \ln 2$

$= \ln 2$

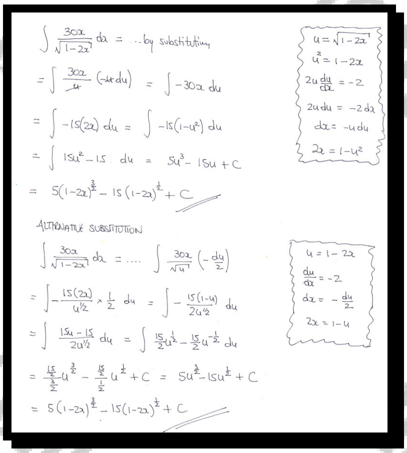
≈ 0.693

Question 28 **(***)**

Use a suitable substitution to find

$$\int \frac{30x}{\sqrt{1-2x}}\, dx .$$

$$\boxed{5(1-2x)^{\frac{3}{2}} - 15(1-2x)^{\frac{1}{2}} + C}$$



$\int \frac{30x}{\sqrt{1-2x}}\,dx = \dots$ by substitution

$= \int \frac{30x}{u}(-u\,du) = \int -30x\,du$

$= \int -15(2x)\,du = \int -15(1-u^2)\,du$

$= \int 15u^2 - 15\,du = 5u^3 - 15u + C$

$= 5(1-2x)^{\frac{3}{2}} - 15(1-2x)^{\frac{1}{2}} + C$

$u = \sqrt{1-2x}$

$u^2 = 1-2x$

$2u\frac{du}{dx} = -2$

$2u\,du = -2\,dx$

$dx = -u\,du$

$2x = 1-u^2$

ALTERNATIVE SUBSTITUTION

$\int \frac{30x}{\sqrt{1-2x}}\,dx = \dots \int \frac{30x}{\sqrt{u}}\left(-\frac{du}{2}\right)$

$= \int -\frac{15(2x)}{u^{1/2}} \times \frac{1}{2}\,du = \int -\frac{15(1-u)}{2u^{1/2}}\,du$

$= \int \frac{15u-15}{2u^{1/2}}\,du = \int \frac{15}{2}u^{\frac{1}{2}} - \frac{15}{2}u^{-\frac{1}{2}}\,du$

$= \frac{\frac{15}{2}}{\frac{3}{2}}u^{\frac{3}{2}} - \frac{\frac{15}{2}}{\frac{1}{2}}u^{\frac{1}{2}} + C = 5u^{\frac{3}{2}} - 15u^{\frac{1}{2}} + C$

$= 5(1-2x)^{\frac{3}{2}} - 15(1-2x)^{\frac{1}{2}} + C$

$u = 1-2x$

$\frac{du}{dx} = -2$

$dx = -\frac{du}{2}$

$2x = 1-u$

Question 29 **(***)**

$$\frac{x^2+3}{x-1} \equiv Ax+B+\frac{C}{x-1}.$$

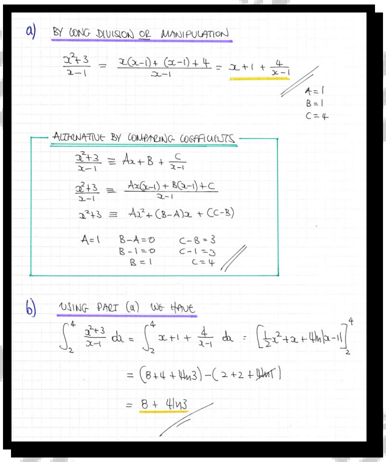
a) Determine the value of each of the constants A, B and C.

b) Hence, or otherwise, evaluate

$$\int_2^4 \frac{x^2+3}{x-1}\,dx,$$

giving the answer in terms of natural logarithms.

[illegible], $\boxed{A=1}$, $\boxed{B=1}$, $\boxed{C=4}$, $\boxed{8+4\ln 3}$



a) BY LONG DIVISION OR MANIPULATION

$$\frac{x^2+3}{x-1} = \frac{x(x-1)+(x-1)+4}{x-1} = x+1+\frac{4}{x-1}$$

$A=1$
$B=1$
$C=4$

ALTERNATIVE BY COMPARING COEFFICIENTS

$$\frac{x^2+3}{x-1} \equiv Ax+B+\frac{C}{x-1}$$

$$\frac{x^2+3}{x-1} \equiv \frac{Ax(x-1)+B(x-1)+C}{x-1}$$

$$x^2+3 \equiv Ax^2+(B-A)x+(C-B)$$

$A=1$ $\quad$ $B-A=0$, $B-1=0$, $B=1$ $\quad$ $C-B=3$, $C-1=3$, $C=4$

b) USING PART (a) WE HAVE

$$\int_2^4 \frac{x^2+3}{x-1}\,dx = \int_2^4 x+1+\frac{4}{x-1}\,dx = \left[\frac{1}{2}x^2+x+4\ln|x-1|\right]_2^4$$

$$= (8+4+4\ln 3)-(2+2+4\ln 1)$$

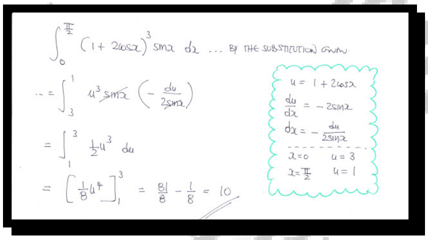
$$= 8+4\ln 3$$

Question 30 (***)

Use the substitution $u = 1 + 2\cos x$ to find

$$\int_0^{\frac{\pi}{2}} (1 + 2\cos x)^3 \sin x \; dx .$$

$\boxed{10}$

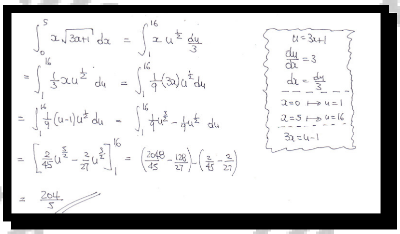


Question 31 (***)

By using the substitution $u = 3x + 1$, or otherwise, find

$$\int_0^5 x\sqrt{3x+1} \; dx .$$

$\boxed{\frac{204}{5} = 40.8}$

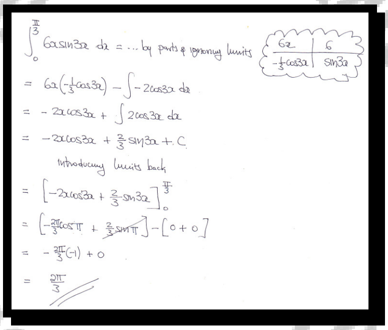


Question 32 (***)

Use an appropriate integration method to find an exact value for

$$\int_0^{\frac{\pi}{3}} 6x \sin 3x \ dx .$$

$\boxed{}$, $\boxed{\frac{2\pi}{3}}$

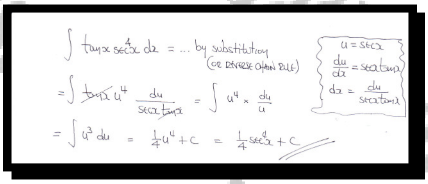


Question 33 (***)

By using the substitution $u = \sec x$, or otherwise, find

$$\int \tan x \sec^4 x \ dx .$$

$\boxed{\frac{1}{4}\sec^4 x + C}$



Question 34 (***)

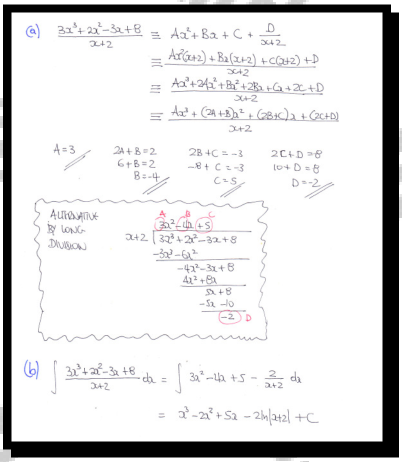
$$\frac{3x^3+2x^2-3x+8}{x+2} \equiv Ax^2+Bx+C+\frac{D}{x+2}.$$

a) Find the value of each of the constants A, B, C and D.

b) Hence find

$$\int \frac{3x^3+2x^2-3x+8}{x+2}\,dx.$$

$\boxed{A=3}$, $\boxed{B=-4}$, $\boxed{C=5}$, $\boxed{D=-2}$, $\boxed{x^3-2x^2+5x-2\ln|x+2|+C}$



Question 35 (***)

$$f(x) \equiv \frac{5}{3x^2 - 5x}.$$

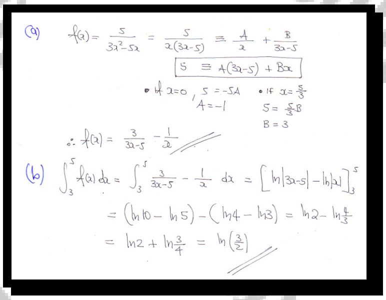
a) Express $f(x)$ in partial fractions.

b) Find the value of

$$\int_3^5 f(x)\ dx\ ,$$

giving the answer as a single simplified logarithm.

$\boxed{}$, $\boxed{f(x) \equiv \frac{3}{3x-5} - \frac{1}{x}}$, $\boxed{\ln\left(\frac{3}{2}\right)}$


(a) $f(x) = \frac{5}{3x^2-5x} = \frac{5}{x(3x-5)} \equiv \frac{A}{x} + \frac{B}{3x-5}$

$5 \equiv A(3x-5) + Bx$

• If $x=0$, $5 = -5A$, $A = -1$ • If $x = \frac{5}{3}$, $5 = \frac{5}{3}B$, $B = 3$

$\therefore f(x) = \frac{3}{3x-5} - \frac{1}{x}$

(b) $\int_3^5 f(x)\,dx = \int_3^5 \frac{3}{3x-5} - \frac{1}{x}\,dx = \left[\ln|3x-5| - \ln|x|\right]_3^5$

$= (\ln 10 - \ln 5) - (\ln 4 - \ln 3) = \ln 2 - \ln\frac{4}{3}$

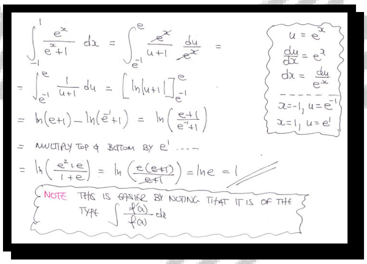
$= \ln 2 + \ln\frac{3}{4} = \ln\left(\frac{3}{2}\right)$

Question 36 **(***)**

By using the substitution $u = e^x$, or otherwise, show clearly that

$$\int_{-1}^{1} \frac{e^x}{e^x+1}\, dx = 1 .$$

proof

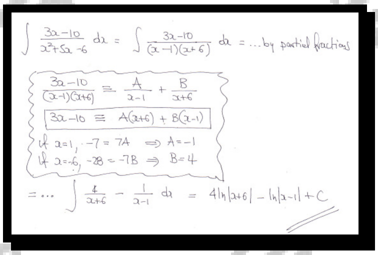


Question 37 **(***)**

Find an expression for the integral

$$\int \frac{3x-10}{x^2+5x-6}\, dx .$$

$4\ln|x+6| - \ln|x-1| + C$

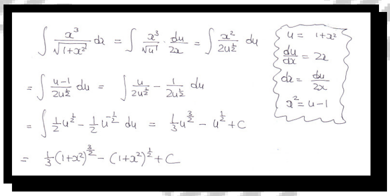


Question 38 (***)

By using the substitution $u = 1 + x^2$, or otherwise, find

$$\int \frac{x^3}{\sqrt{1+x^2}}\, dx .$$

$$\boxed{\frac{1}{3}\left(1+x^2\right)^{\frac{3}{2}} - \left(1+x^2\right)^{\frac{1}{2}} + C}$$

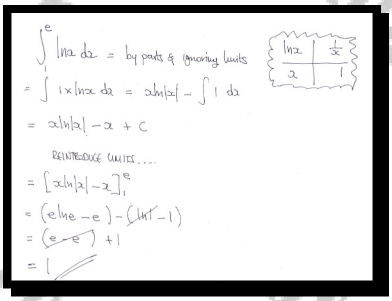


Question 39 (***)

Use integration by parts to find the value of

$$\int_1^{e} \ln x \ dx .$$

$\boxed{}$, $\boxed{1}$

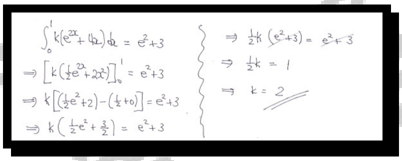


Question 40 (***)

Find the value of the constant k given that

$$\int_0^1 k\left(e^{2x}+4x\right)\,dx = e^2+3.$$

$\boxed{k=2}$



$\int_0^1 k(e^{2x}+4x)\,dx = e^2+3$
$\Rightarrow \left[k\left(\tfrac{1}{2}e^{2x}+2x^2\right)\right]_0^1 = e^2+3$
$\Rightarrow k\left[\left(\tfrac{1}{2}e^2+2\right)-\left(\tfrac{1}{2}+0\right)\right] = e^2+3$
$\Rightarrow k\left(\tfrac{1}{2}e^2+\tfrac{3}{2}\right) = e^2+3$
$\Rightarrow \tfrac{1}{2}k\left(e^2+3\right) = e^2+3$
$\Rightarrow \tfrac{1}{2}k = 1$
$\Rightarrow k = 2$

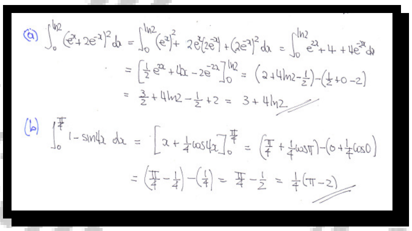
Question 41 (***)

Evaluate each of the following integrals, giving the answers in exact form.

a) $\displaystyle\int_0^{\ln 2} \left(e^{x}+2e^{-x}\right)^2\,dx.$

b) $\displaystyle\int_0^{\frac{\pi}{4}} 1-\sin 4x\,dx.$

$\boxed{3+4\ln 2}$, $\boxed{\frac{1}{4}(\pi-2)}$



(a) $\int_0^{\ln 2}(e^x+2e^{-x})^2\,dx = \int_0^{\ln 2}(e^x)^2+2e^x(2e^{-x})+(2e^{-x})^2\,dx = \int_0^{\ln 2} e^{2x}+4+4e^{-2x}\,dx$
$= \left[\tfrac{1}{2}e^{2x}+4x-2e^{-2x}\right]_0^{\ln 2} = \left(2+4\ln 2-\tfrac{1}{2}\right)-\left(\tfrac{1}{2}+0-2\right)$
$= \tfrac{3}{2}+4\ln 2-\tfrac{1}{2}+2 = 3+4\ln 2$

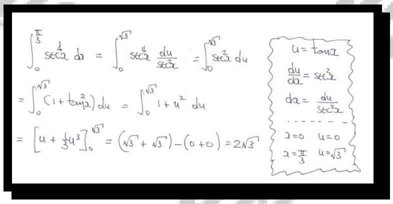
(b) $\int_0^{\frac{\pi}{4}} 1-\sin 4x\,dx = \left[x+\tfrac{1}{4}\cos 4x\right]_0^{\frac{\pi}{4}} = \left(\tfrac{\pi}{4}+\tfrac{1}{4}\cos\pi\right)-\left(0+\tfrac{1}{4}\cos 0\right)$
$= \left(\tfrac{\pi}{4}-\tfrac{1}{4}\right)-\left(\tfrac{1}{4}\right) = \tfrac{\pi}{4}-\tfrac{1}{2} = \tfrac{1}{4}(\pi-2)$

Question 42 (***)

By using the substitution $u = \tan x$ and the trigonometric identity $1 + \tan^2 x = \sec^2 x$, show clearly that

$$\int_0^{\frac{\pi}{3}} \sec^4 x \ dx = 2\sqrt{3}.$$

$\boxed{}$, $\boxed{\text{proof}}$



Question 43 (***)

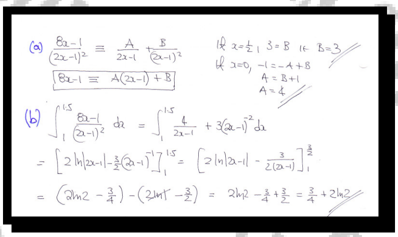
$$\frac{8x-1}{(2x-1)^2} \equiv \frac{A}{2x-1} + \frac{B}{(2x-1)^2}.$$

a) Determine the value of each of the constants A and B.

b) Hence find the exact value of

$$\int_1^{1.5} \frac{8x-1}{(2x-1)^2} \ dx.$$

$\boxed{A=4}$, $\boxed{B=3}$, $\boxed{\frac{3}{4} + 2\ln 2}$

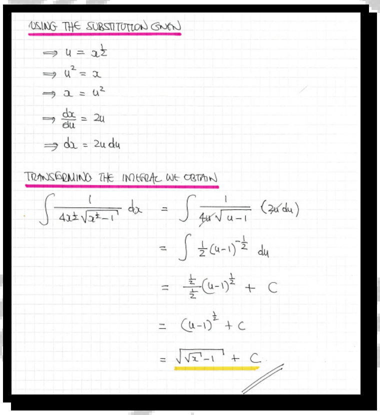


Question 44 (***)

By using the substitution $u = x^{\frac{1}{2}}$, or otherwise, find

$$\int \frac{1}{4x^{\frac{1}{2}}\sqrt{x^{\frac{1}{2}}-1}}\,dx\,.$$

$$\boxed{}\,,\ \boxed{\left(x^{\frac{1}{2}}-1\right)^{\frac{1}{2}}+C}$$



USING THE SUBSTITUTION GIVEN

$\Rightarrow u = x^{\frac{1}{2}}$

$\Rightarrow u^2 = x$

$\Rightarrow x = u^2$

$\Rightarrow \frac{dx}{du} = 2u$

$\Rightarrow dx = 2u\,du$

TRANSFORMING THE INTEGRAL WE OBTAIN

$$\int \frac{1}{4x^{\frac{1}{2}}\sqrt{x^{\frac{1}{2}}-1}}\,dx = \int \frac{1}{4u\sqrt{u-1}}\,(2u\,du)$$

$$= \int \tfrac{1}{2}(u-1)^{-\frac{1}{2}}\,du$$

$$= \frac{\frac{1}{2}}{\frac{1}{2}}(u-1)^{\frac{1}{2}} + C$$

$$= (u-1)^{\frac{1}{2}} + C$$

$$= \sqrt{\sqrt{x}-1} + C$$

Question 44 (***)

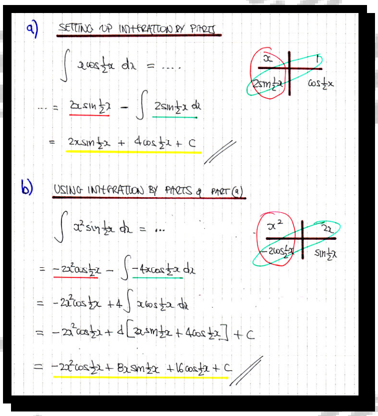
a) Use integration by parts to find

$$\int x\cos\left(\tfrac{1}{2}x\right)\,dx\,.$$

b) Hence determine

$$\int x^2\sin\left(\tfrac{1}{2}x\right)\,dx\,.$$

$$\boxed{2x\sin\left(\tfrac{1}{2}x\right)+4\cos\left(\tfrac{1}{2}x\right)+C}\,,$$
$$\boxed{-2x^2\cos\left(\tfrac{1}{2}x\right)+8x\sin\left(\tfrac{1}{2}x\right)+16\cos\left(\tfrac{1}{2}x\right)+C}$$

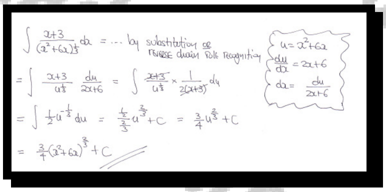


Question 45 (***)

By using the substitution $u = x^2 + 6x$, or otherwise find

$$\int \frac{x+3}{\left(x^2+6x\right)^{\frac{1}{3}}}\,dx\,.$$

$$\boxed{\frac{1}{2}\left(x^2+6x\right)^{\frac{2}{3}}+C}$$



$\int \frac{x+3}{(x^2+6x)^{\frac{1}{3}}}\,dx = \ldots$ by substitution or reverse chain rule recognition

$u = x^2+6x$, $\frac{du}{dx} = 2x+6$, $dx = \frac{du}{2x+6}$

$= \int \frac{x+3}{u^{\frac{1}{3}}}\,\frac{du}{2x+6} = \int \frac{x+3}{u^{\frac{1}{3}}} \times \frac{1}{2(x+3)}\,du$

$= \int \frac{1}{2}u^{-\frac{1}{3}}\,du = \frac{\frac{1}{2}}{\frac{2}{3}}u^{\frac{2}{3}} + C = \frac{3}{4}u^{\frac{2}{3}} + C$

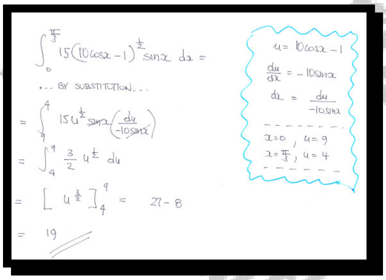
$= \frac{3}{4}(x^2+6x)^{\frac{2}{3}} + C$

Question 46 (***)

Use the substitution $u = 10\cos x - 1$ to find

$$\int_0^{\frac{\pi}{3}} 15\left(10\cos x - 1\right)^{\frac{1}{2}} \sin x \;\, dx\,.$$

$$\boxed{19}$$



$\int_0^{\frac{\pi}{3}} 15(10\cos x - 1)^{\frac{1}{2}} \sin x\,dx =$

... BY SUBSTITUTION ...

$u = 10\cos x - 1$, $\frac{du}{dx} = -10\sin x$, $dx = \frac{du}{-10\sin x}$

$x = 0,\ u = 9$; $x = \frac{\pi}{3},\ u = 4$

$= \int_9^4 15u^{\frac{1}{2}} \sin x \left(\frac{du}{-10\sin x}\right)$

$= \int_4^9 \frac{3}{2}u^{\frac{1}{2}}\,du$

$= \left[u^{\frac{3}{2}}\right]_4^9 = 27 - 8$

$= 19$

Question 47 (***)

$$\frac{2x^2-x+6}{x^2(3-2x)} \equiv \frac{A}{x}+\frac{B}{x^2}+\frac{C}{3-2x}.$$

a) Determine the value of each of the constants A, B and C.

b) Evaluate

$$\int_2^3 \frac{2x^2-x+6}{x^2(3-2x)}\,dx,$$

giving the answer in the form $p-\ln q$, where p and q are constants.

$\boxed{A=1}$, $\boxed{B=2}$, $\boxed{C=4}$, $\boxed{\frac{1}{3}-\ln 6}$

(a) $\frac{2x^2-x+6}{x^2(3-2x)} \equiv \frac{A}{x}+\frac{B}{x^2}+\frac{C}{3-2x}$

$\boxed{2x^2-x+6 \equiv Ax(3-2x)+B(3-2x)+Cx^2}$

- If $x=\frac{3}{2}$, $\frac{9}{2}-\frac{3}{2}+6=\frac{9}{4}C$, $9=\frac{9}{4}C$, $C=4$
- If $x=0$, $6=3B$, $B=2$
- If $x=1$, $2-1+6=A+B+C$, $7=A+2+4$, $A=1$

(b) $\int_2^3 \frac{2x^2-x+6}{x^2(3-2x)}\,dx = \int_2^3 \frac{1}{x}+\frac{2}{x^2}+\frac{4}{3-2x}\,dx$

$= \left[\ln|x|-\frac{2}{x}-2\ln|3-2x|\right]_2^3 = \left(\ln 3-\frac{2}{3}-2\ln 3\right)-\left(\ln 2-1-2\ln 1\right)$

$= \ln 3-2\ln 3-\ln 2-\frac{2}{3}+1 = \frac{1}{3}-\ln 3-\ln 2$

$= \frac{1}{3}-(\ln 3+\ln 2) = \frac{1}{3}-\ln 6$

Question 48 (***+)

x	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$
y	0	0.1309	0.4534	0.7854		0.6545

The table above shows tabulated values for the equation

$$y = x\sin 2x \,,\ 0 \le x \le \frac{5\pi}{12}.$$

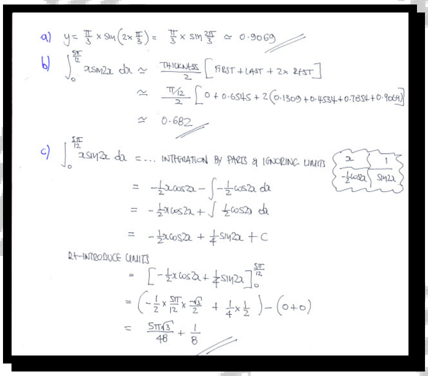
a) Complete the missing value in the table.

b) Use the trapezium rule with all the values from the table to find an approximate value for

$$\int_0^{\frac{5\pi}{12}} x\sin 2x \ dx .$$

c) Use integration by parts to find an exact value for $\int_0^{\frac{5\pi}{12}} x\sin 2x \ dx$.

C.A , 0.9069 , 0.682 , $\frac{5\pi\sqrt{3}}{48}+\frac{1}{8}$


a) $y = \frac{\pi}{3} \times \sin\left(2\times\frac{\pi}{3}\right) = \frac{\pi}{3}\times\sin\frac{2\pi}{3} \simeq 0.9069$

b) $\int_0^{\frac{5\pi}{12}} x\sin 2x \, dx \simeq \frac{\text{THICKNESS}}{2}\left[\text{FIRST} + \text{LAST} + 2\times\text{REST}\right]$

$\simeq \frac{\pi/12}{2}\left[0 + 0.6545 + 2(0.1309 + 0.4534 + 0.7854 + 0.9069)\right]$

$\simeq 0.682$

c) $\int_0^{\frac{5\pi}{12}} x\sin 2x \, dx = \dots$ INTEGRATION BY PARTS & IGNORING LIMITS

x	1
$-\frac{1}{2}\cos 2x$	$\sin 2x$

$= -\frac{1}{2}x\cos 2x - \int -\frac{1}{2}\cos 2x \, dx$

$= -\frac{1}{2}x\cos 2x + \int \frac{1}{2}\cos 2x \, dx$

$= -\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + C$

RE-INTRODUCE LIMITS

$= \left[-\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x\right]_0^{\frac{5\pi}{12}}$

$= \left(-\frac{1}{2}\times\frac{5\pi}{12}\times\frac{-\sqrt{3}}{2} + \frac{1}{4}\times\frac{1}{2}\right) - (0+0)$

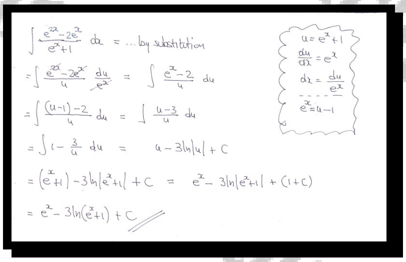
$= \frac{5\pi\sqrt{3}}{48} + \frac{1}{8}$

Question 49 **(***+)**

By using the substitution $u = e^x + 1$, or otherwise, find

$$\int \frac{e^{2x} - 2e^x}{e^x + 1} \, dx .$$

$\boxed{}$, $\boxed{e^x - 3\ln\left(e^x + 1\right) + C}$



$\int \frac{e^{2x}-2e^{x}}{e^{x}+1}\,dx = \dots$ by substitution

$= \int \frac{e^{2x}-2e^{x}}{u}\,\frac{du}{e^{x}} = \int \frac{e^{x}-2}{u}\,du$

$= \int \frac{(u-1)-2}{u}\,du = \int \frac{u-3}{u}\,du$

$= \int 1-\frac{3}{u}\,du = u - 3\ln|u| + C$

$= (e^{x}+1) - 3\ln|e^{x}+1| + C = e^{x} - 3\ln|e^{x}+1| + (1+C)$

$= e^{x} - 3\ln(e^{x}+1) + C$

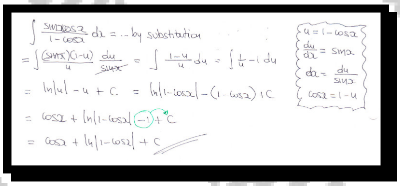
$u = e^{x}+1$, $\frac{du}{dx} = e^{x}$, $dx = \frac{du}{e^{x}}$, $e^{x} = u-1$

Question 50 **(***+)**

By using the substitution $u = 1 - \cos x$, or otherwise, find

$$\int \frac{\sin x \cos x}{1 - \cos x} \, dx .$$

$\boxed{\cos x + \ln\left(1 - \cos x\right) + C}$



$\int \frac{\sin x\cos x}{1-\cos x}\,dx = \dots$ by substitution

$= \int \frac{(\sin x)(1-u)}{u}\,\frac{du}{\sin x} = \int \frac{1-u}{u}\,du = \int \frac{1}{u}-1\,du$

$= \ln|u| - u + C = \ln|1-\cos x| - (1-\cos x) + C$

$= \cos x + \ln|1-\cos x| - 1 + C$

$= \cos x + \ln|1-\cos x| + C$

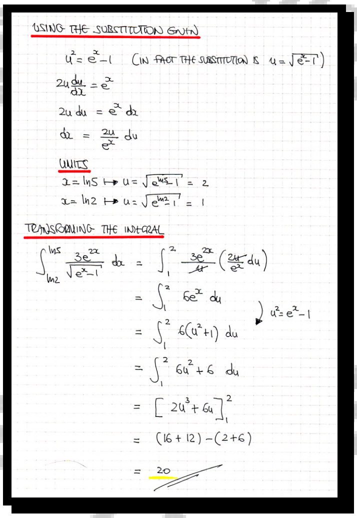
$u = 1-\cos x$, $\frac{du}{dx} = \sin x$, $dx = \frac{du}{\sin x}$, $\cos x = 1-u$

Question 51 (***+)

By using the substitution $u^2 = \mathrm{e}^x - 1$, or otherwise, find

$$\int_{\ln 2}^{\ln 5} \frac{3\mathrm{e}^{2x}}{\sqrt{\mathrm{e}^x - 1}}\, dx .$$

V.P. , 20



USING THE SUBSTITUTION GIVEN

$u^2 = e^x - 1$ (IN FACT THE SUBSTITUTION IS $u = \sqrt{e^x - 1}$)

$2u\frac{du}{dx} = e^x$

$2u\,du = e^x\,dx$

$dx = \frac{2u}{e^x}\,du$

LIMITS

$x = \ln 5 \mapsto u = \sqrt{e^{\ln 5} - 1} = 2$

$x = \ln 2 \mapsto u = \sqrt{e^{\ln 2} - 1} = 1$

TRANSFORMING THE INTEGRAL

$$\int_{\ln 2}^{\ln 5} \frac{3e^{2x}}{\sqrt{e^x - 1}}\,dx = \int_1^2 \frac{3e^{2x}}{u}\left(\frac{2u}{e^x}\,du\right) = \int_1^2 6e^x\,du = \int_1^2 6(u^2+1)\,du = \int_1^2 6u^2 + 6\,du = \left[2u^3 + 6u\right]_1^2 = (16+12) - (2+6) = 20$$

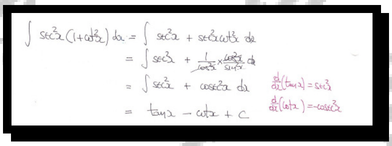
($u^2 = e^x - 1$)

Question 52 (***+)

Use trigonometric identities to find

$$\int \sec^2 x\left(1 + \cot^2 x\right)\, dx .$$

$\tan x - \cot x + C$



$$\int \sec^2 x(1+\cot^2 x)\,dx = \int \sec^2 x + \sec^2 x\cot^2 x\,dx = \int \sec^2 x + \frac{1}{\cos^2 x} \times \frac{\cos^2 x}{\sin^2 x}\,dx = \int \sec^2 x + \operatorname{cosec}^2 x\,dx = \tan x - \cot x + C$$

$\frac{d}{dx}(\tan x) = \sec^2 x$

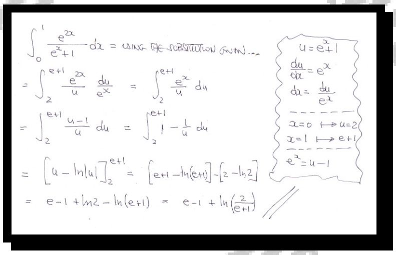
$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

Question 53 **(***+)**

By using the substitution $u = e^x + 1$, or otherwise, find the exact value of

$$\int_0^1 \frac{e^{2x}}{e^x + 1} \, dx .$$

$$\boxed{e - 1 + \ln\left(\frac{2}{1+e}\right)}$$

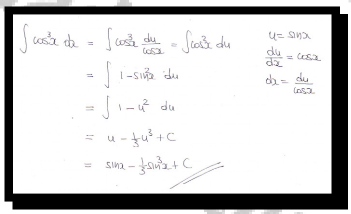


Question 54 **(***+)**

By using the substitution $u = \sin x$, or otherwise, find

$$\int \cos^3 x \, dx .$$

$\boxed{}$, $\boxed{\sin x - \frac{1}{3}\sin^3 x + C}$



Question 55 (***+)

$$I = \int (x-1)(4-x)^{\frac{1}{2}}\ dx,\ x \in \mathbb{R},\ x \le 4.$$

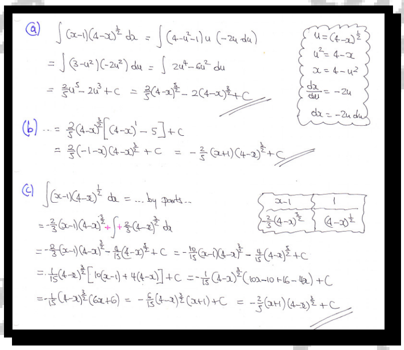
a) Use the substitution $u = (4-x)^{\frac{1}{2}}$ to find an expression for I.

b) Show that the answer of part **(a)** can be written as

$$I = -\frac{2}{5}(x+1)(4-x)^{\frac{3}{2}} + C.$$

c) Use integration by parts to verify the answer of part **(b)**.

C4T , $I = \frac{2}{5}(4-x)^{\frac{5}{2}} - 2(4-x)^{\frac{3}{2}} + C$


(a) $\int (x-1)(4-x)^{\frac12}\,dx = \int (4-u^2-1)u(-2u\,du)$

$= \int (3-u^2)(-2u^2)\,du = \int 2u^4-6u^2\,du$

$= \frac25 u^5 - 2u^3 + C = \frac25(4-x)^{\frac52} - 2(4-x)^{\frac32}+C$

$u=(4-x)^{\frac12}$, $u^2=4-x$, $x=4-u^2$, $\frac{dx}{du}=-2u$, $dx=-2u\,du$

(b) $\dots = \frac25(4-x)^{\frac32}\left[(4-x)^1-5\right]+C$

$=\frac25(-1-x)(4-x)^{\frac32}+C = -\frac25(x+1)(4-x)^{\frac32}+C$

(c) $\int(x-1)(4-x)^{\frac12}\,dx = \dots$ by parts...

$x-1$	1
$-\frac23(4-x)^{\frac32}$	$(4-x)^{\frac12}$

$= -\frac23(x-1)(4-x)^{\frac32} + \int +\frac23(4-x)^{\frac32}\,dx$

$= -\frac23(x-1)(4-x)^{\frac32} - \frac4{15}(4-x)^{\frac52}+C = -\frac{10}{15}(x-1)(4-x)^{\frac32} - \frac4{15}(4-x)^{\frac52}+C$

$= -\frac1{15}(4-x)^{\frac32}\left[10(x-1)+4(4-x)\right]+C = -\frac1{15}(4-x)^{\frac32}(10x-10+16-4x)+C$

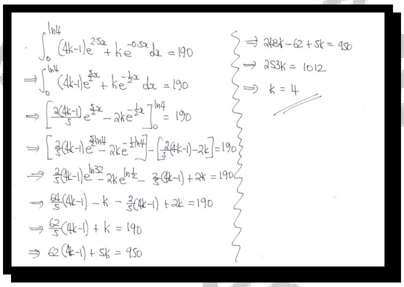
$= -\frac1{15}(4-x)^{\frac32}(6x+6) = -\frac6{15}(4-x)^{\frac32}(x+1)+C = -\frac25(x+1)(4-x)^{\frac32}+C$

Question 56 (*+)**

Find the value of the constant k given that

$$\int_0^{\ln 4} (4k-1)\mathrm{e}^{2.5x} + k\,\mathrm{e}^{-0.5x} \ dx = 190 .$$

$\boxed{k=4}$



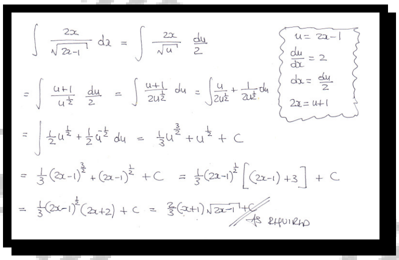
$$\int_0^{\ln 4} (4k-1)e^{2.5x} + ke^{-0.5x}\,dx = 190$$
$$\Rightarrow \int_0^{\ln 4} (4k-1)e^{\frac{5}{2}x} + ke^{-\frac{1}{2}x}\,dx = 190$$
$$\Rightarrow \left[\frac{2(4k-1)}{5}e^{\frac{5}{2}x} - 2ke^{-\frac{1}{2}x}\right]_0^{\ln 4} = 190$$
$$\Rightarrow \left[\frac{2}{5}(4k-1)e^{\frac{5}{2}\ln 4} - 2ke^{-\frac{1}{2}\ln 4}\right] - \left[\frac{2}{5}(4k-1) - 2k\right] = 190$$
$$\Rightarrow \frac{2}{5}(4k-1)e^{\ln 32} - 2ke^{\ln\frac{1}{2}} - \frac{2}{5}(4k-1) + 2k = 190$$
$$\Rightarrow \frac{64}{5}(4k-1) - k - \frac{2}{5}(4k-1) + 2k = 190$$
$$\Rightarrow \frac{62}{5}(4k-1) + k = 190$$
$$\Rightarrow 62(4k-1) + 5k = 950$$
$$\Rightarrow 248k - 62 + 5k = 950$$
$$\Rightarrow 253k = 1012$$
$$\Rightarrow k = 4$$

Question 57 (*+)**

By using the substitution $u = 2x-1$, or otherwise, show that

$$\int \frac{2x}{\sqrt{2x-1}}\, dx = \frac{2}{3}(x+1)\sqrt{2x-1} + C .$$

$\boxed{\text{proof}}$



$u = 2x-1$, $\frac{du}{dx} = 2$, $dx = \frac{du}{2}$, $2x = u+1$

$$\int \frac{2x}{\sqrt{2x-1}}\,dx = \int \frac{2x}{\sqrt{u}}\,\frac{du}{2}$$
$$= \int \frac{u+1}{u^{\frac{1}{2}}}\,\frac{du}{2} = \int \frac{u+1}{2u^{\frac{1}{2}}}\,du = \int \frac{u}{2u^{\frac{1}{2}}} + \frac{1}{2u^{\frac{1}{2}}}\,du$$
$$= \int \frac{1}{2}u^{\frac{1}{2}} + \frac{1}{2}u^{-\frac{1}{2}}\,du = \frac{1}{3}u^{\frac{3}{2}} + u^{\frac{1}{2}} + C$$
$$= \frac{1}{3}(2x-1)^{\frac{3}{2}} + (2x-1)^{\frac{1}{2}} + C = \frac{1}{3}(2x-1)^{\frac{1}{2}}\left[(2x-1)+3\right] + C$$
$$= \frac{1}{3}(2x-1)^{\frac{1}{2}}(2x+2) + C = \frac{2}{3}(x+1)\sqrt{2x-1} + C$$

AS REQUIRED

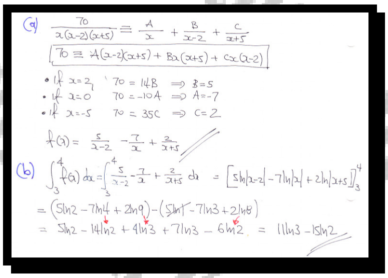
Question 58 (***+)

$$f(x) \equiv \frac{70}{x(x-2)(x+5)}.$$

a) Express $f(x)$ in partial fractions.

b) Show that $\int_3^4 f(x)\, dx$ can be written in the form $p\ln 3 + q\ln 2$, where p and q are integers to be found.

$$\boxed{f(x) \equiv \frac{2}{x+5} + \frac{5}{x-2} - \frac{7}{x}}, \quad \boxed{\int_3^4 f(x)\, dx = 11\ln 3 - 15\ln 2}$$

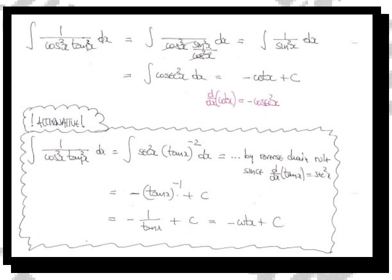


Question 59 (***+)

Use trigonometric identities to find

$$\int \frac{1}{\cos^2 x \tan^2 x}\, dx.$$

$$\boxed{-\cot x + C}$$



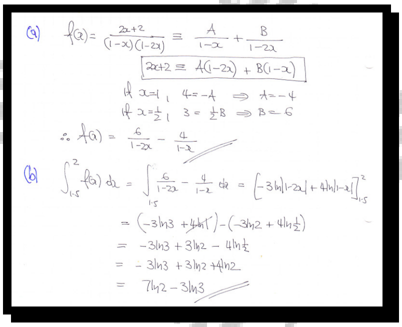
Question 60 (***+)

$$f(x) \equiv \frac{2x+2}{(1-x)(1-2x)}.$$

a) Express $f(x)$ in partial fractions.

b) Show that $\int_{1.5}^{2} f(x)\ dx$ can be written in the form $p\ln 2 + q\ln 3$, where p and q are integers to be found.

$$\boxed{f(x) \equiv \frac{6}{1-2x} - \frac{4}{1-x}}, \quad \boxed{\int_{1.5}^{2} f(x)\ dx = 7\ln 2 - 3\ln 3}$$

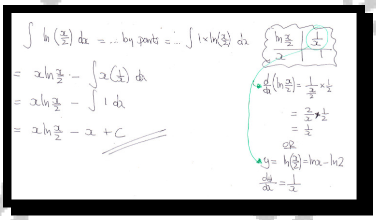


Question 61 (***+)

Use a suitable method to find

$$\int \ln\left(\tfrac{x}{2}\right) dx.$$

$$\boxed{x\ln\left(\tfrac{x}{2}\right) - x + C}$$



Question 62 (***+)

$$f(x) \equiv \frac{32-17x}{(x+1)(3x-4)^2}.$$

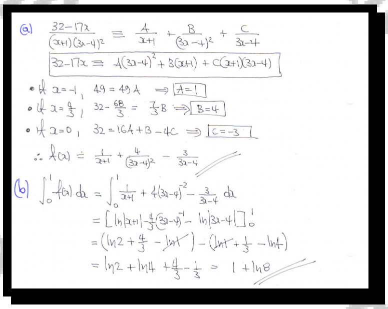
a) Express $f(x)$ in partial fractions.

b) Show that

$$\int_0^1 f(x)\ dx,$$

can be evaluated in the form $p + \ln q$, where p and q are integers to be found.

$$\boxed{f(x) \equiv \frac{4}{(3x-4)^2} - \frac{3}{(3x-4)} + \frac{1}{(x+1)}}, \quad \boxed{\int_0^1 f(x)\ dx = 1 + \ln 8}$$



(a) $\frac{32-17x}{(x+1)(3x-4)^2} \equiv \frac{A}{x+1} + \frac{B}{(3x-4)^2} + \frac{C}{3x-4}$

$32-17x \equiv A(3x-4)^2 + B(x+1) + C(x+1)(3x-4)$

- If $x=-1$, $49 = 49A \Rightarrow A=1$
- If $x=\frac{4}{3}$; $32 - \frac{68}{3} = \frac{7}{3}B \Rightarrow B=4$
- If $x=0$, $32 = 16A + B - 4C \Rightarrow C=-3$

$\therefore f(x) = \frac{1}{x+1} + \frac{4}{(3x-4)^2} - \frac{3}{3x-4}$

(b) $\int_0^1 f(x)\,dx = \int_0^1 \frac{1}{x+1} + 4(3x-4)^{-2} - \frac{3}{3x-4}\,dx$

$= \left[\ln|x+1| - \frac{4}{3}(3x-4)^{-1} - \ln|3x-4|\right]_0^1$

$= \left(\ln 2 + \frac{4}{3} - \ln 1\right) - \left(\ln 1 + \frac{1}{3} - \ln 4\right)$

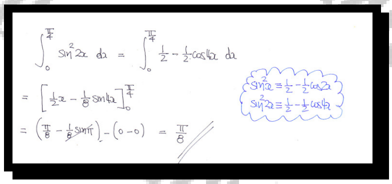
$= \ln 2 + \ln 4 + \frac{4}{3} - \frac{1}{3} = 1 + \ln 8$

Question 63 (*+)**

Use a trigonometric identity to find the exact value of

$$\int_0^{\frac{\pi}{4}} \sin^2 2x \ dx .$$

$\boxed{\dfrac{\pi}{8}}$

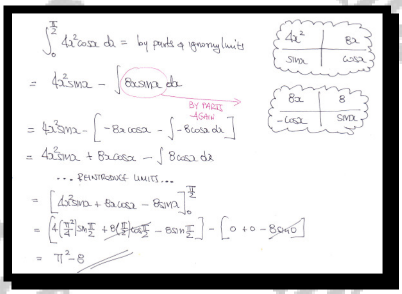


Question 64 (*+)**

Use integration by parts **twice** to find an exact value for

$$\int_0^{\frac{\pi}{2}} 4x^2 \cos x \ dx .$$

$\boxed{}$, $\boxed{\pi^2 - 8}$



Question 65 (***+)

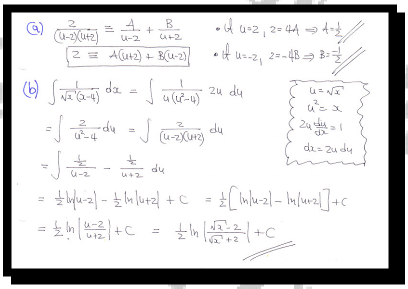
$$\frac{2}{(u-2)(u+2)} \equiv \frac{A}{u-2} + \frac{B}{u+2}.$$

a) Find the value of A and B in the above identity.

b) By using the substitution $u = \sqrt{x}$, or otherwise, find

$$\int \frac{1}{\sqrt{x}(x-4)}\, dx.$$

$\boxed{A = \frac{1}{2}}$, $\boxed{B = -\frac{1}{2}}$, $\boxed{\frac{1}{2}\ln\left|\frac{\sqrt{x}-2}{\sqrt{x}-2}\right| + C}$

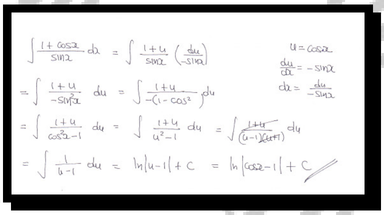


Question 66 (***+)

By using the substitution $u = \cos x$, or otherwise, find

$$\int \frac{1+\cos x}{\sin x}\, dx.$$

$\boxed{\ln|\cos x - 1| + C}$

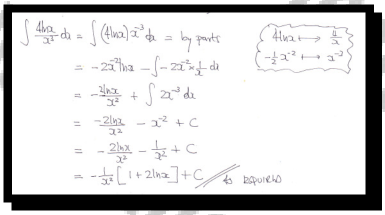


Question 67 **(***+)**

Use integration by parts to show that

$$\int \frac{4\ln x}{x^3}\,dx = -\frac{1+2\ln x}{x^2} + C\,.$$

$\boxed{\sin x - \frac{1}{3}\sin^3 x + C}$

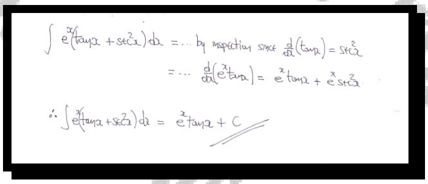


Question 68 **(***+)**

By considering the differentiation of a product of two appropriate functions, find

$$\int e^x\left(\tan x + \sec^2 x\right)dx\,.$$

$\boxed{e^x \tan x + C}$



Question 69 **(***+)**

$$\frac{2x^2-3}{(x-1)^2} \equiv A + \frac{B}{x-1} + \frac{C}{(x-1)^2}.$$

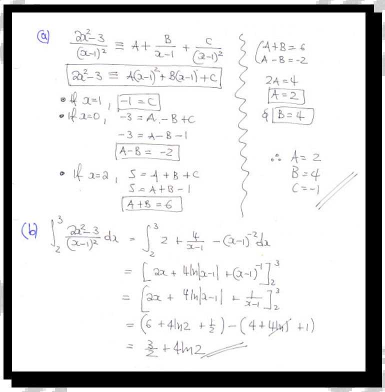
a) Determine the value of each of the constants A, B and C.

b) Evaluate

$$\int_2^3 \frac{2x^2-3}{(x-1)^2}\, dx,$$

giving the answer in the form $p + \ln q$, where p and q are constants.

$\boxed{A = 2}$, $\boxed{B = 4}$, $\boxed{C = -1}$, $\boxed{\frac{3}{2} + \ln 16}$

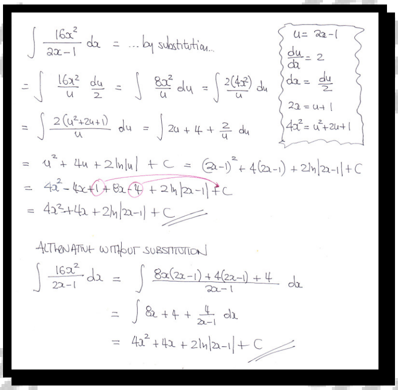


Question 70 (*+)**

By using the substitution $u = 2x - 1$, or otherwise, find

$$\int \frac{16x^2}{2x-1}\,dx.$$

$$\boxed{(2x-1)^2 + 4(2x-1) + 2\ln|2x-1| + C \;=\; 4x^2 + 4x + 2\ln|2x-1| + C}$$

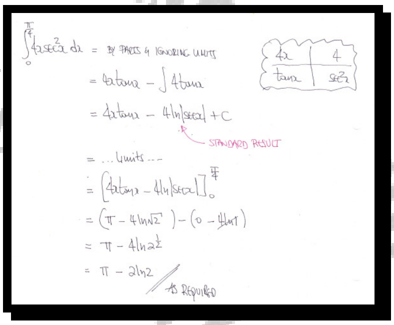


Question 71 (*+)**

Use integration by parts to show that

$$\int_0^{\frac{\pi}{4}} 4x\sec^2 x \;\, dx = \pi - 2\ln 2.$$

proof



Question 72 **(***+)**

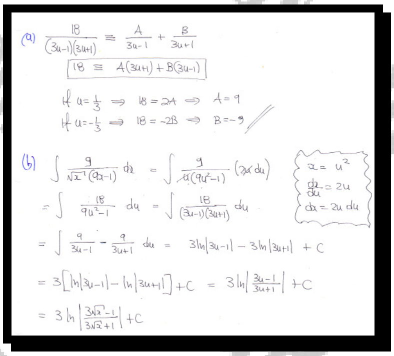
$$\frac{18}{(3u-1)(3u+1)} \equiv \frac{A}{3u-1} + \frac{B}{3u+1}.$$

a) Find the value of A and B in the above identity.

b) By using the substitution $x = u^2$, or otherwise, find

$$\int \frac{9}{\sqrt{x}(9x-1)}\, dx.$$

$\boxed{A=9}$, $\boxed{B=-9}$, $\boxed{3\ln\left|\frac{3\sqrt{x}-1}{3\sqrt{x}+1}\right|+C}$


(a) $\frac{18}{(3u-1)(3u+1)} \equiv \frac{A}{3u-1} + \frac{B}{3u+1}$

$\boxed{18 \equiv A(3u+1) + B(3u-1)}$

If $u = \frac{1}{3} \Rightarrow 18 = 2A \Rightarrow A = 9$

If $u = -\frac{1}{3} \Rightarrow 18 = -2B \Rightarrow B = -9$

(b) $\int \frac{9}{\sqrt{x}(9x-1)}\, dx = \int \frac{9}{u(9u^2-1)}(2u\, du)$

$x = u^2$, $\frac{dx}{du} = 2u$, $dx = 2u\, du$

$= \int \frac{18}{9u^2-1}\, du = \int \frac{18}{(3u-1)(3u+1)}\, du$

$= \int \frac{9}{3u-1} - \frac{9}{3u+1}\, du = 3\ln|3u-1| - 3\ln|3u+1| + C$

$= 3\left[\ln|3u-1| - \ln|3u+1|\right] + C = 3\ln\left|\frac{3u-1}{3u+1}\right| + C$

$= 3\ln\left|\frac{3\sqrt{x}-1}{3\sqrt{x}+1}\right| + C$

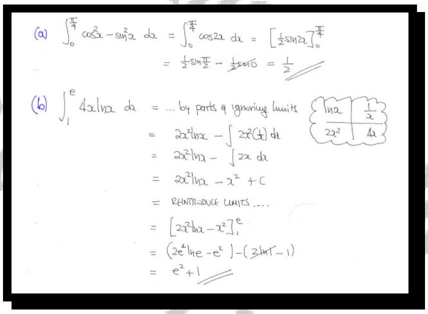
Question 73 **(***+)**

Use an appropriate integration method to find an exact value for each of the following integrals

a) $\displaystyle\int_0^{\frac{\pi}{4}} \cos^2 x - \sin^2 x \ \, dx$.

b) $\displaystyle\int_1^{e} 4x\ln x \ \, dx$.

$\boxed{\frac{1}{2}}$, $\boxed{e^2+1}$


(a) $\int_0^{\frac{\pi}{4}} \cos^2 x - \sin^2 x \, dx = \int_0^{\frac{\pi}{4}} \cos 2x \, dx = \left[\frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{4}}$

$= \frac{1}{2}\sin\frac{\pi}{2} - \frac{1}{2}\sin 0 = \frac{1}{2}$

(b) $\int_1^{e} 4x\ln x \, dx = \dots$ by parts & ignoring limits

$\ln x$	$\frac{1}{x}$
$2x^2$	$4x$

$= 2x^2\ln x - \int 2x^2\left(\frac{1}{x}\right)dx$

$= 2x^2\ln x - \int 2x \, dx$

$= 2x^2\ln x - x^2 + C$

= REINTRODUCE LIMITS ....

$= \left[2x^2\ln x - x^2\right]_1^{e}$

$= (2e^2\ln e - e^2) - (2\ln 1 - 1)$

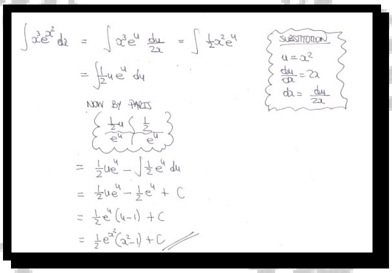
$= e^2 + 1$

Question 74 **(***+)**

Use the substitution $u = x^2$, followed by integration by parts to find

$$\int x^3 e^{x^2} \, dx .$$

$\boxed{}$, $\boxed{\frac{1}{2}e^{x^2}\left(x^2-1\right)+C}$


$\int x^3 e^{x^2} dx = \int x^3 e^{u} \frac{du}{2x} = \int \frac{1}{2}x^2 e^{u}$

$= \int \frac{1}{2} u e^{u} \, du$

SUBSTITUTION: $u = x^2$, $\frac{du}{dx} = 2x$, $dx = \frac{du}{2x}$

NOW BY PARTS

$\frac{1}{2}u$	$\frac{1}{2}$
e^u	e^u

$= \frac{1}{2}ue^{u} - \int \frac{1}{2}e^{u} \, du$

$= \frac{1}{2}ue^{u} - \frac{1}{2}e^{u} + C$

$= \frac{1}{2}e^{u}(u-1) + C$

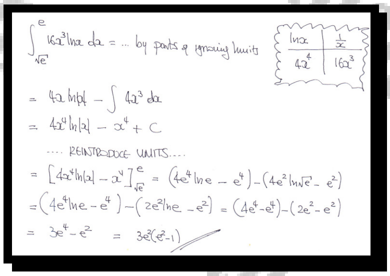
$= \frac{1}{2}e^{x^2}(x^2-1) + C$

Question 75 (***+)

Use integration by parts to find the exact value of

$$\int_{\sqrt{e}}^{e} 16x^3 \ln x \ dx .$$

$$\boxed{3e^2\left(e^2-1\right)}$$



$\int_{\sqrt{e}}^{e} 16x^3 \ln x \, dx = \ldots$ by parts & ignoring limits

$\ln x$	$\frac{1}{x}$
$4x^4$	$16x^3$

$= 4x^4 \ln|x| - \int 4x^3 \, dx$

$= 4x^4 \ln|x| - x^4 + C$

.... REINTRODUCE LIMITS ....

$= \left[4x^4 \ln|x| - x^4\right]_{\sqrt{e}}^{e} = \left(4e^4 \ln e - e^4\right) - \left(4e^2 \ln\sqrt{e} - e^2\right)$

$= \left(4e^4 \ln e - e^4\right) - \left(2e^2 \ln e - e^2\right) = \left(4e^4 - e^4\right) - \left(2e^2 - e^2\right)$

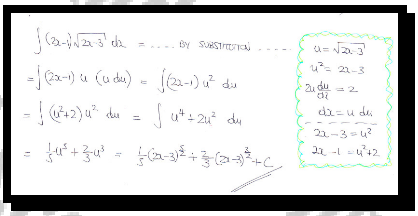
$= 3e^4 - e^2 = 3e^2(e^2-1)$

Question 76 (***+)

By using the substitution $u = \sqrt{2x-3}$, or otherwise, find an expression for

$$\int (2x-1)\sqrt{2x-3} \ dx .$$

$$\boxed{\frac{1}{5}(2x-3)^{\frac{5}{2}} + \frac{2}{3}(2x-3)^{\frac{3}{2}} + C}$$



$\int (2x-1)\sqrt{2x-3} \, dx = \ldots$ BY SUBSTITUTION $\ldots$

$= \int (2x-1)\, u \,(u \, du) = \int (2x-1) u^2 \, du$

$= \int (u^2+2) u^2 \, du = \int u^4 + 2u^2 \, du$

$= \frac{1}{5}u^5 + \frac{2}{3}u^3 = \frac{1}{5}(2x-3)^{\frac{5}{2}} + \frac{2}{3}(2x-3)^{\frac{3}{2}} + C$

$u = \sqrt{2x-3}$
$u^2 = 2x-3$
$2u\frac{du}{dx} = 2$
$dx = u \, du$
$2x-3 = u^2$
$2x-1 = u^2+2$

Question 77 (***+)

$$\frac{3x^2-2x+1}{2x(x-1)^2} \equiv \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}.$$

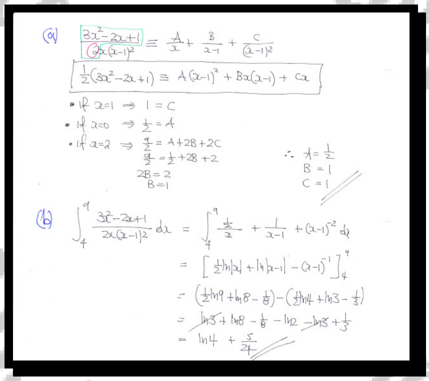
a) Determine the value of each of the constants A, B and C.

b) Evaluate

$$\int_4^9 \frac{3x^2-2x+1}{2x(x-1)^2}\, dx,$$

giving the answer in the form $p + q\ln 4$, where p and q are constants.

$\boxed{A=\frac{1}{2}}$, $\boxed{B=1}$, $\boxed{C=1}$, $\boxed{\frac{5}{24}+\ln 4}$



Question 78 (***+)

It is given that

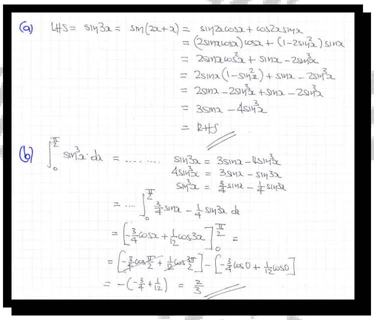
$$\sin 3x \equiv 3\sin x - 4\sin^3 x\,.$$

a) Prove the above trigonometric identity, by writing $\sin 3x$ as $\sin(2x + x)$.

b) Hence, or otherwise, find the exact value of

$$\int_0^{\frac{\pi}{2}} \sin^3 x \;\; dx\,.$$

$\boxed{\frac{2}{3}}$



(a) $\text{LHS} = \sin 3x = \sin(2x+x) = \sin 2x \cos x + \cos 2x \sin x$

$= (2\sin x\cos x)\cos x + (1-2\sin^2 x)\sin x$

$= 2\sin x\cos^2 x + \sin x - 2\sin^3 x$

$= 2\sin x(1-\sin^2 x) + \sin x - 2\sin^3 x$

$= 2\sin x - 2\sin^3 x + \sin x - 2\sin^3 x$

$= 3\sin x - 4\sin^3 x$

$= \text{RHS}$

(b) $\int_0^{\frac{\pi}{2}} \sin^3 x \, dx = \ldots\ldots$

$\sin 3x = 3\sin x - 4\sin^3 x$

$4\sin^3 x = 3\sin x - \sin 3x$

$\sin^3 x = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$

$= \ldots \int_0^{\frac{\pi}{2}} \frac{3}{4}\sin x - \frac{1}{4}\sin 3x \, dx$

$= \left[-\frac{3}{4}\cos x + \frac{1}{12}\cos 3x\right]_0^{\frac{\pi}{2}} =$

$= \left[-\frac{3}{4}\cos\frac{\pi}{2} + \frac{1}{12}\cos\frac{3\pi}{2}\right] - \left[-\frac{3}{4}\cos 0 + \frac{1}{12}\cos 0\right]$

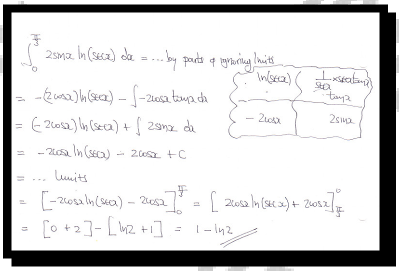
$= -\left(-\frac{3}{4} + \frac{1}{12}\right) = \frac{2}{3}$

Question 79 (*+)**

Use integration by parts to find an exact value for

$$\int_{1}^{\frac{\pi}{3}} 2\sin x \ln(\sec x) \ dx .$$

$\boxed{1-\ln 2}$



$\int_0^{\frac{\pi}{3}} 2\sin x \ln(\sec x)\,dx = \dots$ by parts & ignoring limits

$\ln(\sec x)$	$\frac{1}{\sec x} \times \sec x \tan x$ $\tan x$
$-2\cos x$	$2\sin x$

$= -(2\cos x)\ln(\sec x) - \int -2\cos x \tan x\,dx$

$= (-2\cos x)\ln(\sec x) + \int 2\sin x\,dx$

$= -2\cos x \ln(\sec x) - 2\cos x + C$

$= \dots$ limits

$= \left[-2\cos x \ln(\sec x) - 2\cos x\right]_0^{\frac{\pi}{3}} = \left[2\cos x \ln(\sec x) + 2\cos x\right]_{\frac{\pi}{3}}^{0}$

$= [0+2] - [\ln 2 + 1] = 1 - \ln 2$

Question 80 (***+)

x	0	$\frac{\pi}{18}$	$\frac{\pi}{9}$	$\frac{\pi}{6}$
y	0	0.1632		0.2500

The table above shows tabulated values for the equation

$$y = \sin x \cos 2x, \ 0 \le x \le \frac{\pi}{6}.$$

a) Complete the missing value in the table. **(1)**

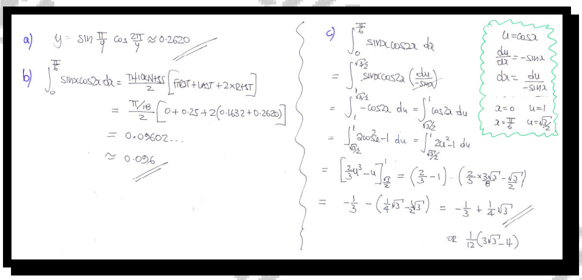
b) Use the trapezium rule with all the values from the table to find an approximate value for

$$\int_0^{\frac{\pi}{6}} \sin x \cos 2x \ dx. \quad \textbf{(3)}$$

c) By using the substitution $u = \cos x$, or otherwise, find an exact value for **(6)**

$$\int_0^{\frac{\pi}{6}} \sin x \cos 2x \ dx.$$

$\boxed{}$, $\boxed{0.2620}$, $\boxed{0.096}$, $\boxed{\frac{1}{12}\left(3\sqrt{3}-4\right)}$



Question 81 (***+)

$$\frac{12x^2+x+3}{(6x+1)(2x^2+1)} \equiv \frac{A}{6x+1}+\frac{Bx+C}{2x^2+1}.$$

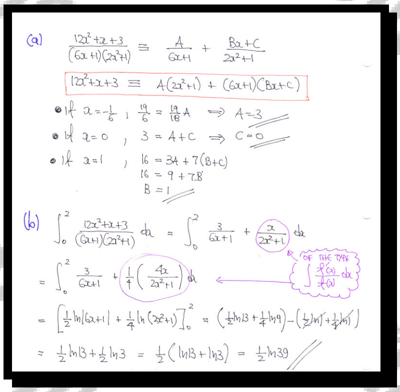
a) Determine the value of each of the constants A, B and C.

b) Evaluate

$$\int_0^2 \frac{12x^2+x+3}{(6x+1)(2x^2+1)}\,dx,$$

giving the answer in the form $p\ln q$, where p and q are constants.

$\boxed{A=3}$, $\boxed{B=1}$, $\boxed{C=0}$, $\boxed{\frac{1}{2}\ln 39}$



Question 82 **(***+)**

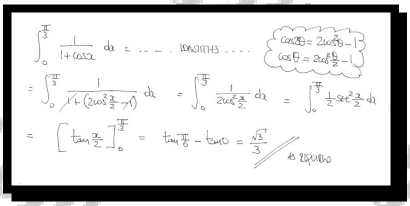
By using the trigonometric identity

$$\cos 2\theta \equiv 2\cos^2\theta - 1$$

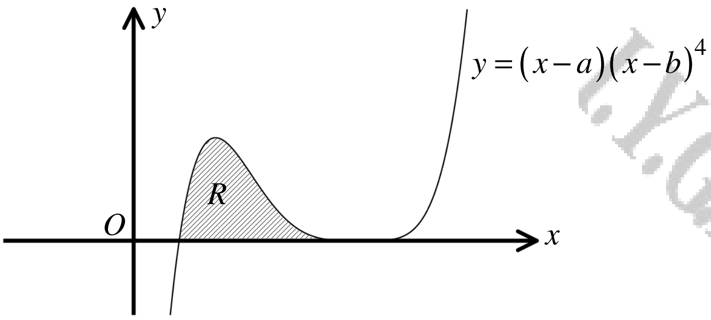
and the fact that $\frac{d}{dx}(\tan x) = \sec^2 x$, show clearly that

$$\int_0^{\frac{\pi}{3}} \frac{1}{1+\cos x}\ dx = \frac{\sqrt{3}}{3}.$$

proof



Question 83 **(***+)**



The figure above shows the graph of the curve with equation

$$y=(x-a)(x-b)^4,$$

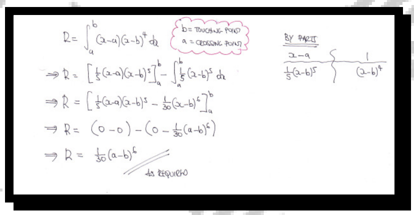
where a and b are positive constants.

The shaded region R is bounded by the curve and the x axis.

By using integration by parts, or otherwise, show that the area of the shaded region is

$$\frac{1}{30}(a-b)^6.$$

proof

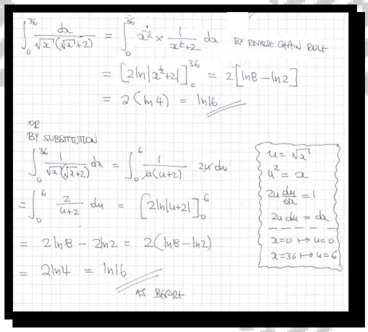


Question 84 (*+)**

By using the substitution $u = \sqrt{x}$, or otherwise, show that

$$\int_0^{36} \frac{1}{\sqrt{x}\left(\sqrt{x}+2\right)} \, dx = \ln 16 .$$

proof

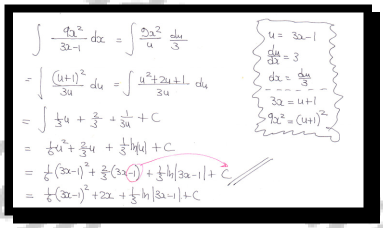


Question 85 (*+)**

By using the substitution $u = 3x-1$, or otherwise, find

$$\int \frac{9x^2}{3x-1} \, dx .$$

$$\frac{1}{6}(3x-1)^2 + \frac{2}{3}(3x-1) + \frac{1}{3}\ln|3x-1| + C$$

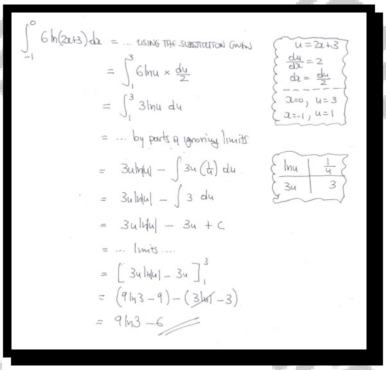


Question 86 (*+)**

By using the substitution $u = 2x + 3$, or otherwise, show clearly that

$$\int_{-1}^{0} 6\ln(2x+3)\ dx = 9\ln 3 - 6.$$

proof



$\int_{-1}^{0} 6\ln(2x+3)\,dx = \dots$ USING THE SUBSTITUTION GIVEN

$= \int_1^3 6\ln u \times \frac{du}{2}$

$= \int_1^3 3\ln u\ du$

$= \dots$ by parts & ignoring limits

$= 3u\ln|u| - \int 3u\left(\frac{1}{u}\right) du$

$= 3u\ln|u| - \int 3\ du$

$= 3u\ln|u| - 3u + C$

$= \dots$ limits $\dots$

$= \left[3u\ln|u| - 3u\right]_1^3$

$= (9\ln 3 - 9) - (3\ln 1 - 3)$

$= 9\ln 3 - 6$

$u = 2x+3$, $\frac{du}{dx} = 2$, $dx = \frac{du}{2}$; $x=0,\ u=3$; $x=-1,\ u=1$

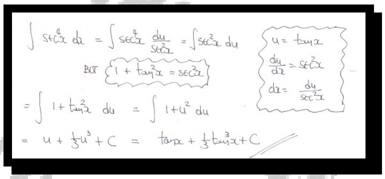
$\ln u$	$\frac{1}{u}$
$3u$	3

Question 87 (*+)**

By using the substitution $u = \tan x$, or otherwise, find

$$\int \sec^4 x\ dx.$$

$\tan x + \frac{1}{3}\tan^3 x + C$



$\int \sec^4 x\,dx = \int \sec^4 x \frac{du}{\sec^2 x} = \int \sec^2 x\,du$

BUT $1 + \tan^2 x = \sec^2 x$

$= \int 1 + \tan^2 x\ du = \int 1 + u^2\ du$

$= u + \frac{1}{3}u^3 + C = \tan x + \frac{1}{3}\tan^3 x + C$

$u = \tan x$, $\frac{du}{dx} = \sec^2 x$, $dx = \frac{du}{\sec^2 x}$

Question 88 (***+)

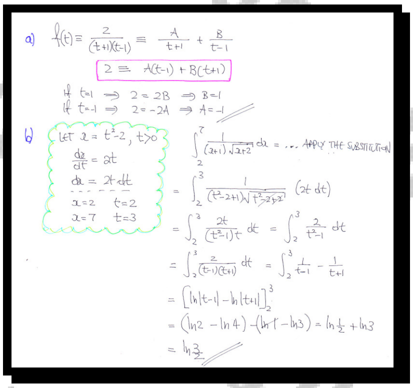
$$f(t) \equiv \frac{2}{(t-1)(t+1)} \equiv \frac{A}{(t-1)} + \frac{B}{(t+1)}.$$

a) Find the value of each of the constants A and B in the above identity.

b) Use the substitution $x = t^2 - 2$, $t > 0$ to show that

$$\int_4^9 \frac{1}{(x+1)\sqrt{x+2}}\, dx = \ln\frac{3}{2}.$$

$\boxed{A=2}$, $\boxed{B=2}$



Question 89 (***+)

x	0	$\frac{2\pi}{5}$	$\frac{4\pi}{5}$	$\frac{6\pi}{5}$	$\frac{8\pi}{5}$	2π
y	0	0.2031	0.8602			0

The table above shows some tabulated values for the equation

$$y = \sin^3\left(\tfrac{1}{2}x\right), \ 0 \le x \le 2\pi .$$

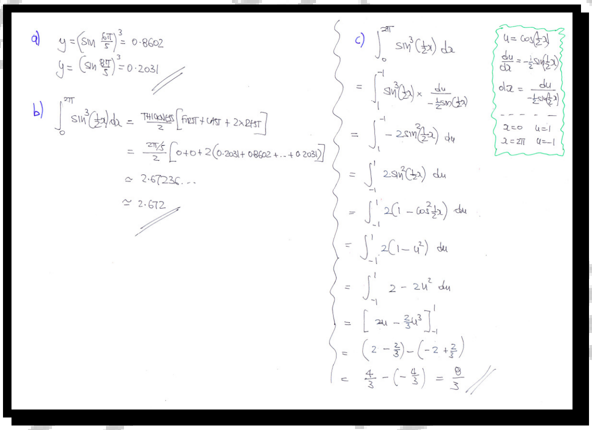
a) Complete the missing values in the table.

b) Use the trapezium rule with all the values from the table to find an approximate value for

$$\int_0^{2\pi} \sin^3\left(\tfrac{1}{2}x\right) \, dx .$$

c) By using the substitution $u = \cos\left(\tfrac{1}{2}x\right)$, or otherwise, find the value of the integral of part (b).

[illegible], 0.8602, 0.2031, 2.672, $\frac{8}{3}$



a) $y = \left(\sin \frac{6\pi}{5}\right)^3 = 0.8602$

$y = \left(\sin \frac{8\pi}{5}\right)^3 = 0.2031$

b) $\int_0^{2\pi} \sin^3\left(\frac{1}{2}x\right) dx = \frac{\text{THICKNESS}}{2}\left[\text{FIRST} + \text{LAST} + 2 \times \text{REST}\right]$

$= \frac{2\pi/5}{2}\left[0 + 0 + 2(0.2031 + 0.8602 + \dots + 0.2031)\right]$

$\simeq 2.67236\dots$

$\simeq 2.672$

c) $\int_0^{2\pi} \sin^3\left(\frac{1}{2}x\right) dx$

$= \int_1^{-1} \sin^3\left(\frac{1}{2}x\right) \times \frac{du}{-\frac{1}{2}\sin\left(\frac{1}{2}x\right)}$

$= \int_1^{-1} -2\sin^2\left(\frac{1}{2}x\right) du$

$= \int_{-1}^{1} 2\sin^2\left(\frac{1}{2}x\right) du$

$= \int_{-1}^{1} 2\left(1 - \cos^2\frac{1}{2}x\right) du$

$= \int_{-1}^{1} 2\left(1 - u^2\right) du$

$= \int_{-1}^{1} 2 - 2u^2 \, du$

$= \left[2u - \frac{2}{3}u^3\right]_{-1}^{1}$

$= \left(2 - \frac{2}{3}\right) - \left(-2 + \frac{2}{3}\right)$

$= \frac{4}{3} - \left(-\frac{4}{3}\right) = \frac{8}{3}$

$u = \cos\left(\frac{1}{2}x\right)$

$\frac{du}{dx} = -\frac{1}{2}\sin\left(\frac{1}{2}x\right)$

$dx = \frac{du}{-\frac{1}{2}\sin\left(\frac{1}{2}x\right)}$

$x = 0 \quad u = 1$

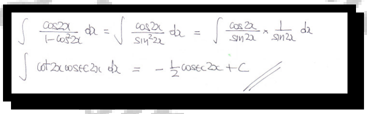
$x = 2\pi \quad u = -1$

Question 90 (***+)

Use trigonometric identities to integrate

$$\int \frac{\cos 2x}{1-\cos^2 2x}\,dx\,.$$

$\boxed{}$, $\boxed{-\frac{1}{2}\operatorname{cosec} 2x + C}$



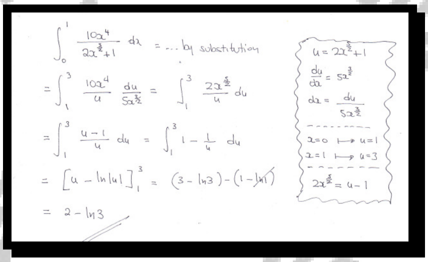
$$\int \frac{\cos 2x}{1-\cos^2 2x}\,dx = \int \frac{\cos 2x}{\sin^2 2x}\,dx = \int \frac{\cos 2x}{\sin 2x}\times\frac{1}{\sin 2x}\,dx$$
$$\int \cot 2x \operatorname{cosec} 2x\,dx = -\frac{1}{2}\operatorname{cosec} 2x + C$$

Question 91 (****)

By using the substitution $u = 2x^{\frac{5}{2}} + 1$, or otherwise, find an exact simplified value for

$$\int_0^1 \frac{10x^4}{2x^{\frac{5}{2}}+1}\,dx\,.$$

$\boxed{}$, $\boxed{2-\ln 3}$



$$\int_0^1 \frac{10x^4}{2x^{\frac{5}{2}}+1}\,dx = \dots \text{by substitution}$$
$$= \int_1^3 \frac{10x^4}{u}\,\frac{du}{5x^{\frac{3}{2}}} = \int_1^3 \frac{2x^{\frac{5}{2}}}{u}\,du$$
$$= \int_1^3 \frac{u-1}{u}\,du = \int_1^3 1-\frac{1}{u}\,du$$
$$= \left[u - \ln|u|\right]_1^3 = (3-\ln 3)-(1-\ln 1)$$
$$= 2-\ln 3$$

$u = 2x^{\frac{5}{2}}+1$

$\frac{du}{dx} = 5x^{\frac{3}{2}}$

$dx = \frac{du}{5x^{\frac{3}{2}}}$

$x=0 \mapsto u=1$

$x=1 \mapsto u=3$

$2x^{\frac{5}{2}} = u-1$

Question 92 (****)

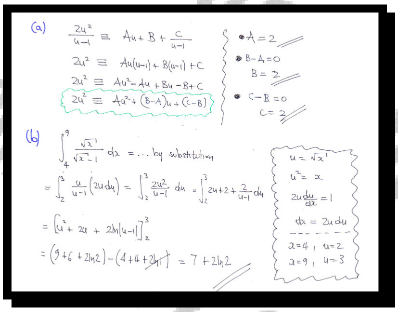
$$\frac{2u^2}{u-1} \equiv Au + B + \frac{C}{u-1}.$$

a) Find the value of each of the constants A, B and C in the above identity.

b) Use the substitution $u = \sqrt{x}$ to show

$$\int_4^9 \frac{\sqrt{x}}{\sqrt{x}-1}\, dx = 7 + 2\ln 2.$$

$\boxed{A=2}$, $\boxed{B=2}$, $\boxed{C=2}$



Question 93 (****)

$$\int_1^9 \frac{1}{2x\left(1+\sqrt{x}\right)}\,dx\,.$$

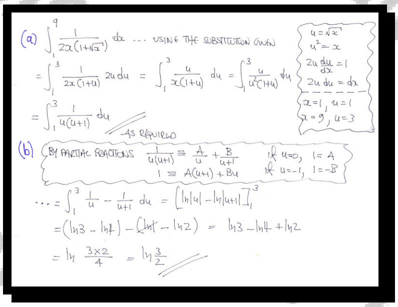
a) Show that the substitution $u=\sqrt{x}$ transforms the above integral to

$$\int_{x_1}^{x_2} \frac{1}{u(u+1)}\,du\,,$$

where x_1 and x_2 are constants to be found.

b) Hence find an exact value for the original integral.

$\boxed{\ln\left(\frac{3}{2}\right)}$



(a) $\int_1^9 \frac{1}{2x(1+\sqrt{x})}\,dx$ ... USING THE SUBSTITUTION GIVEN

$u=\sqrt{x}$, $u^2=x$, $2u\frac{du}{dx}=1$, $2u\,du=dx$

$x=1,\ u=1$; $x=9,\ u=3$

$=\int_1^3 \frac{1}{2x(1+u)}\,2u\,du = \int_1^3 \frac{u}{x(1+u)}\,du = \int_1^3 \frac{u}{u^2(1+u)}\,du$

$=\int_1^3 \frac{1}{u(u+1)}\,du$ AS REQUIRED

(b) BY PARTIAL FRACTIONS $\frac{1}{u(u+1)} \equiv \frac{A}{u}+\frac{B}{u+1}$

$1 \equiv A(u+1)+Bu$

if $u=0$, $1=A$; if $u=-1$, $1=-B$

$\dots=\int_1^3 \frac{1}{u}-\frac{1}{u+1}\,du = \Big[\ln|u|-\ln|u+1|\Big]_1^3$

$=(\ln 3-\ln 4)-(\ln 1-\ln 2) = \ln 3-\ln 4+\ln 2$

$=\ln\frac{3\times 2}{4} = \ln\frac{3}{2}$

Question 94 (****)

$$f(x) \equiv \frac{x^3}{x^2-4}, \quad x \neq \pm 2 .$$

a) Use a suitable substitution to show that

$$\int_{\sqrt{6}}^{\sqrt{8}} f(x) \, dx = 1 + \ln 4 .$$

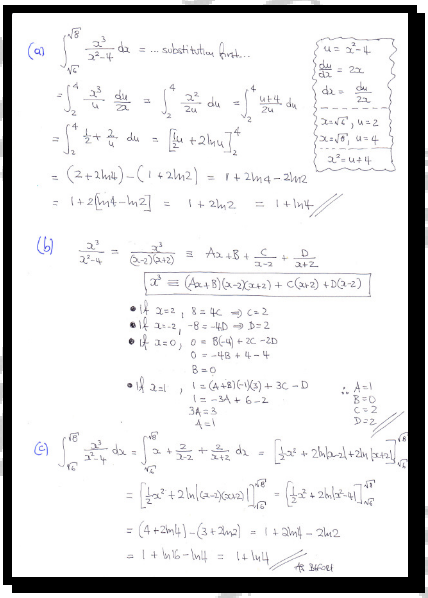
b) Express $f(x)$ in the form

$$Ax + B + \frac{C}{x-2} + \frac{D}{x+2},$$

where A, B, C and D are constants to be found.

c) Use the result part (b) to verify the result of part (a).

$\boxed{A=1}$, $\boxed{B=0}$, $\boxed{C=2}$, $\boxed{D=2}$


(a) $\int_{\sqrt{6}}^{\sqrt{8}} \frac{x^3}{x^2-4}\,dx = \ldots$ substitution first...

$u = x^2-4$, $\frac{du}{dx} = 2x$, $dx = \frac{du}{2x}$; $x=\sqrt{6},\ u=2$; $x=\sqrt{8},\ u=4$; $x^2 = u+4$

$= \int_2^4 \frac{x^3}{u}\frac{du}{2x} = \int_2^4 \frac{x^2}{2u}\,du = \int_2^4 \frac{u+4}{2u}\,du$

$= \int_2^4 \frac{1}{2} + \frac{2}{u}\,du = \left[\frac{1}{2}u + 2\ln u\right]_2^4$

$= (2+2\ln 4) - (1+2\ln 2) = 1 + 2\ln 4 - 2\ln 2$

$= 1 + 2[\ln 4 - \ln 2] = 1 + 2\ln 2 = 1 + \ln 4$

(b) $\frac{x^3}{x^2-4} = \frac{x^3}{(x-2)(x+2)} \equiv Ax + B + \frac{C}{x-2} + \frac{D}{x+2}$

$\boxed{x^3 \equiv (Ax+B)(x-2)(x+2) + C(x+2) + D(x-2)}$

- if $x=2$, $8 = 4C \Rightarrow C = 2$
- if $x=-2$, $-8 = -4D \Rightarrow D = 2$
- if $x=0$, $0 = B(-4) + 2C - 2D$; $0 = -4B + 4 - 4$; $B = 0$
- if $x=1$, $1 = (A+B)(-1)(3) + 3C - D$; $1 = -3A + 6 - 2$; $3A = 3$; $A = 1$

$\therefore A=1,\ B=0,\ C=2,\ D=2$

(c) $\int_{\sqrt{6}}^{\sqrt{8}} \frac{x^3}{x^2-4}\,dx = \int_{\sqrt{6}}^{\sqrt{8}} x + \frac{2}{x-2} + \frac{2}{x+2}\,dx = \left[\frac{1}{2}x^2 + 2\ln|x-2| + 2\ln|x+2|\right]_{\sqrt{6}}^{\sqrt{8}}$

$= \left[\frac{1}{2}x^2 + 2\ln|(x-2)(x+2)|\right]_{\sqrt{6}}^{\sqrt{8}} = \left[\frac{1}{2}x^2 + 2\ln|x^2-4|\right]_{\sqrt{6}}^{\sqrt{8}}$

$= (4+2\ln 4) - (3+2\ln 2) = 1 + 2\ln 4 - 2\ln 2$

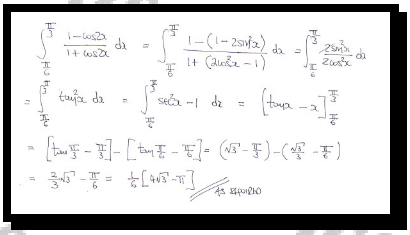
$= 1 + \ln 16 - \ln 4 = 1 + \ln 4$ AS BEFORE

Question 95 **(****)**

By using the cosine double angle identities and the fact that $\frac{d}{dx}(\tan x) = \sec^2 x$, show clearly that

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1-\cos 2x}{1+\cos 2x} \, dx = \frac{1}{6}\left(4\sqrt{3}-\pi\right).$$

proof



$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1-\cos 2x}{1+\cos 2x}\,dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1-(1-2\sin^2 x)}{1+(2\cos^2 x-1)}\,dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{2\sin^2 x}{2\cos^2 x}\,dx$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \tan^2 x\,dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^2 x - 1\,dx = \left[\tan x - x\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= \left[\tan\frac{\pi}{3} - \frac{\pi}{3}\right] - \left[\tan\frac{\pi}{6} - \frac{\pi}{6}\right] = \left(\sqrt{3} - \frac{\pi}{3}\right) - \left(\frac{\sqrt{3}}{3} - \frac{\pi}{6}\right)$$

$$= \frac{2}{3}\sqrt{3} - \frac{\pi}{6} = \frac{1}{6}\left[4\sqrt{3} - \pi\right]$$

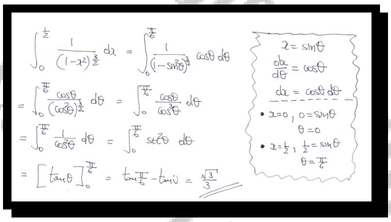
AS REQUIRED

Question 96 **(****)**

Use the substitution $x = \sin\theta$ to find the exact value of

$$\int_{0}^{\frac{1}{2}} \frac{1}{\left(1-x^2\right)^{\frac{3}{2}}} \, dx .$$

$\frac{\sqrt{3}}{3}$



$$\int_0^{\frac{1}{2}} \frac{1}{(1-x^2)^{\frac{3}{2}}}\,dx = \int_0^{\frac{\pi}{6}} \frac{1}{(1-\sin^2\theta)^{\frac{3}{2}}}\cos\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{6}} \frac{\cos\theta}{(\cos^2\theta)^{\frac{3}{2}}}\,d\theta = \int_0^{\frac{\pi}{6}} \frac{\cos\theta}{\cos^3\theta}\,d\theta$$

$$= \int_0^{\frac{\pi}{6}} \frac{1}{\cos^2\theta}\,d\theta = \int_0^{\frac{\pi}{6}} \sec^2\theta\,d\theta$$

$$= \left[\tan\theta\right]_0^{\frac{\pi}{6}} = \tan\frac{\pi}{6} - \tan 0 = \frac{\sqrt{3}}{3}$$

$x = \sin\theta$

$\frac{dx}{d\theta} = \cos\theta$

$dx = \cos\theta\,d\theta$

- $x = 0$, $0 = \sin\theta$, $\theta = 0$
- $x = \frac{1}{2}$, $\frac{1}{2} = \sin\theta$, $\theta = \frac{\pi}{6}$

Question 97 **(****)**

$$f(x) \equiv \frac{1}{x\left(x^2+1\right)}, \quad x \neq 0.$$

a) Use the substitution $x = \tan\theta$ to find

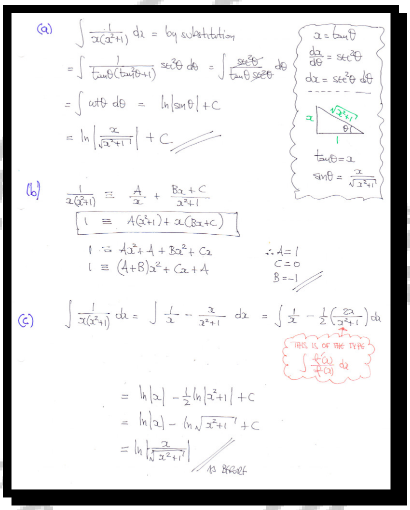
$$\int f(x)\, dx.$$

b) Find the value of each of the constants A, B and C, so that

$$f(x) \equiv \frac{A}{x} + \frac{Bx+C}{x^2+1}.$$

c) Use the result of part (b) to independently verify the answer of part (a).

$$\boxed{\ln\left(\frac{x}{\sqrt{x^2+1}}\right)}, \ \boxed{A=1}, \ \boxed{B=-1}, \ \boxed{C=0}$$



(a) $\int \frac{1}{x(x^2+1)}\,dx$ = by substitution

$= \int \frac{1}{\tan\theta(\tan^2\theta+1)} \sec^2\theta \, d\theta = \int \frac{\sec^2\theta}{\tan\theta\sec^2\theta}\,d\theta$

$= \int \cot\theta \, d\theta = \ln|\sin\theta| + C$

$= \ln\left|\frac{x}{\sqrt{x^2+1}}\right| + C$

$x = \tan\theta$

$\frac{dx}{d\theta} = \sec^2\theta$

$dx = \sec^2\theta \, d\theta$

$\tan\theta = x$

$\sin\theta = \frac{x}{\sqrt{x^2+1}}$

(b) $\frac{1}{x(x^2+1)} \equiv \frac{A}{x} + \frac{Bx+C}{x^2+1}$

$1 \equiv A(x^2+1) + x(Bx+C)$

$1 \equiv Ax^2 + A + Bx^2 + Cx$

$1 \equiv (A+B)x^2 + Cx + A$

$\therefore A = 1$, $C = 0$, $B = -1$

(c) $\int \frac{1}{x(x^2+1)}\,dx = \int \frac{1}{x} - \frac{x}{x^2+1}\,dx = \int \frac{1}{x} - \frac{1}{2}\left(\frac{2x}{x^2+1}\right)dx$

THIS IS OF THE TYPE $\int \frac{f'(x)}{f(x)}\,dx$

$= \ln|x| - \frac{1}{2}\ln|x^2+1| + C$

$= \ln|x| - \ln\sqrt{x^2+1} + C$

$= \ln\left|\frac{x}{\sqrt{x^2+1}}\right|$

AS BEFORE

Question 98 (****)

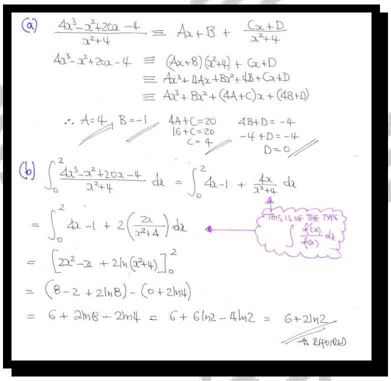
$$\frac{4x^3 - x^2 + 20x - 4}{x^2 + 4} \equiv Ax + B + \frac{Cx + D}{x^2 + 4}.$$

a) Determine the value of each of the constants A, B, C and D.

b) Hence show that

$$\int_0^2 \frac{4x^3 - x^2 + 20x - 4}{x^2 + 4}\, dx = 6 + 2\ln 2,$$

$\boxed{A = 4}$, $\boxed{B = -1}$, $\boxed{C = 4}$, $\boxed{D = 0}$


(a) $\frac{4x^3-x^2+20x-4}{x^2+4} \equiv Ax+B+\frac{Cx+D}{x^2+4}$

$4x^3-x^2+20x-4 \equiv (Ax+B)(x^2+4)+Cx+D \equiv Ax^3+4Ax+Bx^2+4B+Cx+D \equiv Ax^3+Bx^2+(4A+C)x+(4B+D)$

$\therefore A=4$, $B=-1$; $4A+C=20$, $16+C=20$, $C=4$; $4B+D=-4$, $-4+D=-4$, $D=0$

(b) $\int_0^2 \frac{4x^3-x^2+20x-4}{x^2+4}\,dx = \int_0^2 4x-1+\frac{4x}{x^2+4}\,dx$

$= \int_0^2 4x-1+2\left(\frac{2x}{x^2+4}\right)dx$ ← THIS IS OF THE TYPE $\int \frac{f'(x)}{f(x)}\,dx$

$= \left[2x^2-x+2\ln(x^2+4)\right]_0^2$

$= (8-2+2\ln 8)-(0+2\ln 4)$

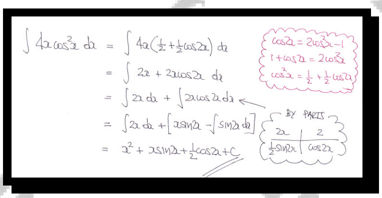
$= 6+2\ln 8-2\ln 4 = 6+6\ln 2-4\ln 2 = 6+2\ln 2$ // AS REQUIRED

Question 99 (****)

Use a cosine double identity and integration by parts to find

$$\int 4x\cos^2 x\, dx.$$

$\boxed{x^2 + x\sin 2x + \frac{1}{2}\cos 2x + C}$


$\int 4x\cos^2 x\,dx = \int 4x\left(\frac{1}{2}+\frac{1}{2}\cos 2x\right)dx$

$= \int 2x+2x\cos 2x\,dx$

$= \int 2x\,dx+\int 2x\cos 2x\,dx$

$= \int 2x\,dx+\left[x\sin 2x-\int \sin 2x\,dx\right]$

$= x^2+x\sin 2x+\frac{1}{2}\cos 2x+C$

$\cos 2x = 2\cos^2 x-1$; $1+\cos 2x = 2\cos^2 x$; $\cos^2 x = \frac{1}{2}+\frac{1}{2}\cos 2x$

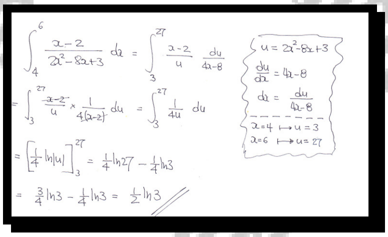
BY PARTS: $2x$ | 2; $\frac{1}{2}\sin 2x$ | $\cos 2x$

Question 100 (****)

By using the substitution $u = 2x^2 - 8x + 3$, or otherwise, find the exact value

$$\int_4^6 \frac{x-2}{2x^2-8x+3}\,dx .$$

$$\boxed{\frac{1}{2}\ln 3}$$



Question 101 (****)

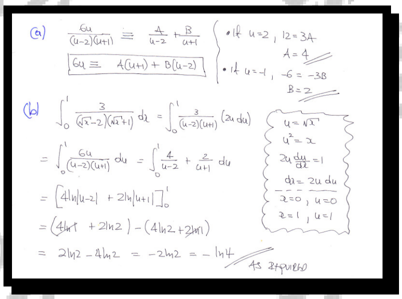
$$\frac{6u}{(u-2)(u+1)} \equiv \frac{A}{u-2} + \frac{B}{u+1} .$$

a) Find the value of each of the constants A and B in the above identity.

b) By using the substitution $u = \sqrt{x}$, or otherwise, show that

$$\int_0^1 \frac{3}{\left(\sqrt{x}-2\right)\left(\sqrt{x}+1\right)}\,dx = -\ln 4 .$$

$$\boxed{A=4},\ \boxed{B=2}$$



Question 102 (****)

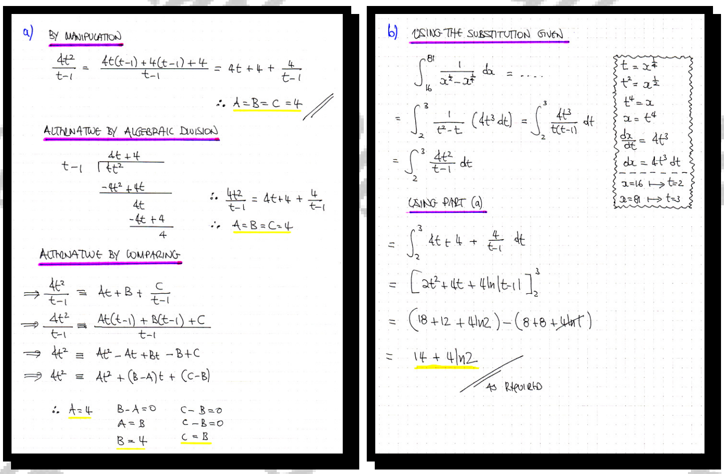
$$\frac{4t^2}{t-1} \equiv At + B + \frac{C}{t-1}.$$

a) Determine the value of each of the constants A, B and C.

b) Use the substitution $t = x^{\frac{1}{4}}$ to show

$$\int_{16}^{81} \frac{1}{x^{\frac{1}{2}} - x^{\frac{1}{4}}} \, dx = 14 + 4\ln 2.$$

$\boxed{}$, $\boxed{A=4}$, $\boxed{B=4}$, $\boxed{C=4}$



Question 103 (****)

It is given that

$$\int_{k}^{2k} \frac{3x-5}{x(x-1)} \, dx = \ln 72,$$

determine the value of k, $0 < k < 1$.

$\boxed{}$, $\boxed{k = \frac{1}{4}}$

START BY PARTIAL FRACTIONS

$$\frac{3x-5}{x(x-1)} \equiv \frac{A}{x} + \frac{B}{x-1}$$

$$\boxed{3x-5 \equiv A(x-1) + Bx}$$

IF $x=1 \Rightarrow -2 = B$
$\Rightarrow B = -2$

IF $x=0 \Rightarrow -5 = -A$
$\Rightarrow A = 5$

HENCE THE INTEGRAL BECOMES

$$\Rightarrow \int_{k}^{2k} \frac{3x-5}{x(x-1)} \, dx = \ln 72$$

$$\Rightarrow \int_{k}^{2k} \frac{5}{x} - \frac{2}{x-1} \, dx = \ln 72$$

$$\Rightarrow \Big[5\ln|x| - 2\ln|x-1| \Big]_{k}^{2k} = \ln 72$$

$$\Rightarrow \Big[5\ln|2k| - 2\ln|2k-1| \Big] - \Big[5\ln|k| - 2\ln|k-1| \Big] = \ln 72$$

$$\Rightarrow 5\ln|2k| - 2\ln|2k-1| - 5\ln|k| + 2\ln|k-1| = \ln 72$$

$$\Rightarrow 5\ln\left|\frac{2k}{k}\right| + 2\ln\left|\frac{k-1}{2k-1}\right| = \ln 72$$

$$\Rightarrow 5\ln 2 + 2\ln\left|\frac{k-1}{2k-1}\right| = \ln 72$$

$$\Rightarrow 2\ln\left|\frac{k-1}{2k-1}\right| = \ln 72 - 5\ln 2$$

$$\Rightarrow 2\ln\left|\frac{k-1}{2k-1}\right| = \ln 72 - \ln 32$$

$$\Rightarrow 2\ln\left|\frac{k-1}{2k-1}\right| = \ln\frac{9}{4}$$

$$\Rightarrow 2\ln\left|\frac{k-1}{2k-1}\right| = \ln\left(\frac{3}{2}\right)^2$$

$$\Rightarrow 2\ln\left|\frac{k-1}{2k-1}\right| = 2\ln\frac{3}{2}$$

$$\Rightarrow \frac{k-1}{2k-1} = \frac{3}{2}$$

$$\Rightarrow 6k-3 = 2k-2$$

$$\Rightarrow 4k = 1$$

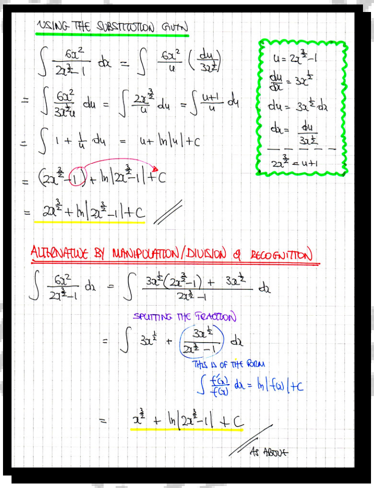
$$\Rightarrow k = \frac{1}{4}$$

Question 104 (****)

By using the substitution $u = 2x^{\frac{3}{2}} - 1$, or otherwise, find an expression for the integral

$$\int \frac{6x^2}{2x^{\frac{3}{2}} - 1}\, dx .$$

$\boxed{}$, $\boxed{2x^{\frac{1}{2}} + \ln\left|2x^{\frac{3}{2}} - 1\right| + C}$

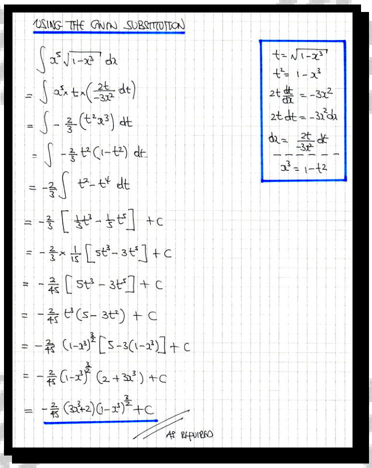


Question 105 (****)

Use the substitution $t=\sqrt{1-x^{3}}$ to show that

$$\int x^{5}\sqrt{1-x^{3}}\,dx=-\frac{2}{45}\left(3x^{3}+2\right)\left(1-x^{3}\right)^{\frac{3}{2}}+C.$$

proof


USING THE GIVEN SUBSTITUTION

$$\int x^{5}\sqrt{1-x^{3}}\,dx$$
$$=\int x^{5}\times t\times\left(\frac{2t}{-3x^{2}}\,dt\right)$$
$$=\int -\frac{2}{3}\left(t^{2}x^{3}\right)dt$$
$$=\int -\frac{2}{3}t^{2}\left(1-t^{2}\right)dt$$
$$=-\frac{2}{3}\int t^{2}-t^{4}\,dt$$
$$=-\frac{2}{3}\left[\frac{1}{3}t^{3}-\frac{1}{5}t^{5}\right]+C$$
$$=-\frac{2}{3}\times\frac{1}{15}\left[5t^{3}-3t^{5}\right]+C$$
$$=-\frac{2}{45}\left[5t^{3}-3t^{5}\right]+C$$
$$=-\frac{2}{45}t^{3}\left(5-3t^{2}\right)+C$$
$$=-\frac{2}{45}\left(1-x^{3}\right)^{\frac{3}{2}}\left[5-3\left(1-x^{3}\right)\right]+C$$
$$=-\frac{2}{45}\left(1-x^{3}\right)^{\frac{3}{2}}\left(2+3x^{3}\right)+C$$
$$=-\frac{2}{45}\left(3x^{3}+2\right)\left(1-x^{3}\right)^{\frac{3}{2}}+C$$

AS REQUIRED

$t=\sqrt{1-x^{3}}$

$t^{2}=1-x^{3}$

$2t\frac{dt}{dx}=-3x^{2}$

$2t\,dt=-3x^{2}\,dx$

$dx=\frac{2t}{-3x^{2}}\,dt$

$x^{3}=1-t^{2}$

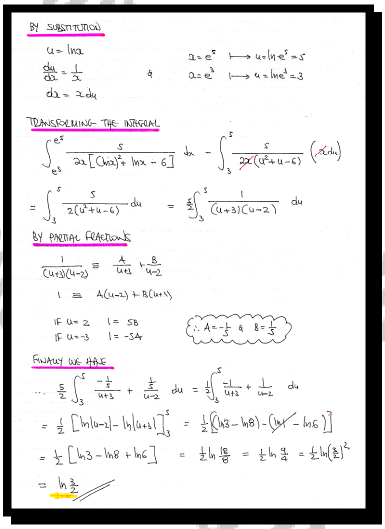
Question 106 (****)

Use an appropriate substitution, followed by partial fractions, to show that

$$\int_{e^3}^{e^5} \frac{5}{2x\left[(\ln x)^2 + \ln x - 6\right]}\, dx = \ln\left(\frac{3}{2}\right).$$

You may assume that the integral converges.

[ ], [proof]


BY SUBSTITUTION

$u = \ln x$, $\frac{du}{dx} = \frac{1}{x}$, $dx = x\,du$ & $x = e^5 \longmapsto u = \ln e^5 = 5$, $x = e^3 \longmapsto u = \ln e^3 = 3$

TRANSFORMING THE INTEGRAL

$\int_{e^3}^{e^5} \frac{5}{2x[(\ln x)^2 + \ln x - 6]}\,dx = \int_3^5 \frac{5}{2x(u^2+u-6)}(x\,du)$

$= \int_3^5 \frac{5}{2(u^2+u-6)}\,du = \frac{5}{2}\int_3^5 \frac{1}{(u+3)(u-2)}\,du$

BY PARTIAL FRACTIONS

$\frac{1}{(u+3)(u-2)} \equiv \frac{A}{u+3} + \frac{B}{u-2}$

$1 \equiv A(u-2) + B(u+3)$

IF $u = 2$, $1 = 5B$; IF $u = -3$, $1 = -5A$; $\therefore A = -\frac{1}{5}$ & $B = \frac{1}{5}$

FINALLY WE HAVE

$\dots \frac{5}{2}\int_3^5 \frac{-\frac{1}{5}}{u+3} + \frac{\frac{1}{5}}{u-2}\,du = \frac{1}{2}\int_3^5 \frac{-1}{u+3} + \frac{1}{u-2}\,du$

$= \frac{1}{2}\left[\ln|u-2| - \ln|u+3|\right]_3^5 = \frac{1}{2}\left[(\ln 3 - \ln 8) - (\ln 1 - \ln 6)\right]$

$= \frac{1}{2}\left[\ln 3 - \ln 8 + \ln 6\right] = \frac{1}{2}\ln\frac{18}{8} = \frac{1}{2}\ln\frac{9}{4} = \frac{1}{2}\ln\left(\frac{3}{2}\right)^2$

$= \ln\frac{3}{2}$

Question 107 (****)

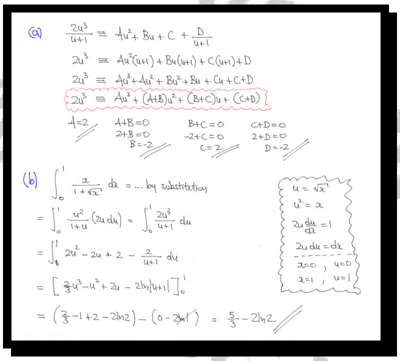
$$\frac{2u^3}{u+1} \equiv Au^2 + Bu + C + \frac{D}{u+1}.$$

a) Find the value of each of the constants A, B and C in the above identity.

b) Use the substitution $u = \sqrt{x}$ to show

$$\int_0^1 \frac{x}{1+\sqrt{x}}\, dx = \frac{5}{3} - 2\ln 2.$$

$\boxed{A=2}$, $\boxed{B=-2}$, $\boxed{C=2}$, $\boxed{D=-2}$


(a) $\frac{2u^3}{u+1} \equiv Au^2 + Bu + C + \frac{D}{u+1}$

$2u^3 \equiv Au^2(u+1) + Bu(u+1) + C(u+1) + D$

$2u^3 \equiv Au^3 + Au^2 + Bu^2 + Bu + Cu + C + D$

$2u^3 \equiv Au^3 + (A+B)u^2 + (B+C)u + (C+D)$

$A=2$; $A+B=0$, $2+B=0$, $B=-2$; $B+C=0$, $-2+C=0$, $C=2$; $C+D=0$, $2+D=0$, $D=-2$

(b) $\int_0^1 \frac{x}{1+\sqrt{x}}\, dx = \dots$ by substitution

$= \int_0^1 \frac{u^2}{1+u}(2u\, du) = \int_0^1 \frac{2u^3}{u+1}\, du$

$= \int_0^1 2u^2 - 2u + 2 - \frac{2}{u+1}\, du$

$= \left[\frac{2}{3}u^3 - u^2 + 2u - 2\ln|u+1|\right]_0^1$

$= \left(\frac{2}{3} - 1 + 2 - 2\ln 2\right) - (0 - 2\ln 1) = \frac{5}{3} - 2\ln 2$

$u = \sqrt{x}$, $u^2 = x$, $2u\frac{du}{dx} = 1$, $2u\, du = dx$; $x=0,\ u=0$; $x=1,\ u=1$

Question 109 (****)

$$\int \frac{\cos x}{1-\cos x}\,dx.$$

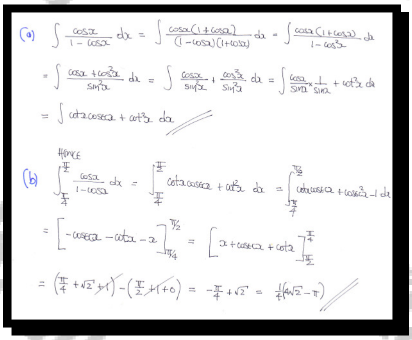
a) Show by multiplying the numerator and denominator of the integrand by $(1+\cos x)$, that the above integral can eventually be written as

$$\int \cot x \operatorname{cosec} x + \cot^2 x \, dx.$$

b) Hence show further that

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos x}{1-\cos x}\,dx = \frac{1}{4}\left(4\sqrt{2}-\pi\right).$$

[ ], [proof]


(a) $\int \frac{\cos x}{1-\cos x}\,dx = \int \frac{\cos x(1+\cos x)}{(1-\cos x)(1+\cos x)}\,dx = \int \frac{\cos x(1+\cos x)}{1-\cos^2 x}\,dx$

$= \int \frac{\cos x+\cos^2 x}{\sin^2 x}\,dx = \int \frac{\cos x}{\sin^2 x}+\frac{\cos^2 x}{\sin^2 x}\,dx = \int \frac{\cos x}{\sin x}\times\frac{1}{\sin x}+\cot^2 x\,dx$

$= \int \cot x \operatorname{cosec} x + \cot^2 x\,dx$

HENCE

(b) $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos x}{1-\cos x}\,dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot x\operatorname{cosec} x+\cot^2 x\,dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot x\operatorname{cosec} x+\operatorname{cosec}^2 x-1\,dx$

$= \left[-\operatorname{cosec} x-\cot x-x\right]_{\pi/4}^{\pi/2} = \left[x+\operatorname{cosec} x+\cot x\right]_{\frac{\pi}{2}}^{\frac{\pi}{4}}$

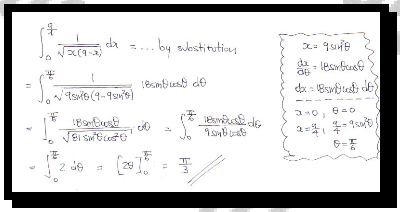
$= \left(\frac{\pi}{4}+\sqrt{2}+1\right)-\left(\frac{\pi}{2}+1+0\right) = -\frac{\pi}{4}+\sqrt{2} = \frac{1}{4}\left(4\sqrt{2}-\pi\right)$

Question 109 (****)

By using the substitution $x = 9\sin^2\theta$, or otherwise, find the exact value of

$$\int_0^{\frac{9}{4}} \frac{1}{\sqrt{x(9-x)}}\, dx .$$

$\boxed{\frac{\pi}{3}}$



Question 110 (****)

It is given that

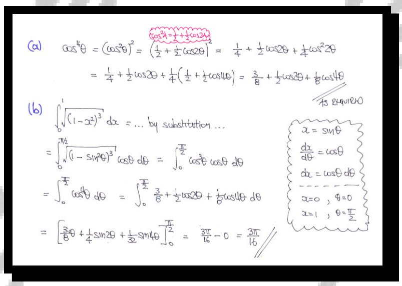
$$\cos^4\theta \equiv \frac{3}{8} + \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta .$$

a) Prove the validity of the above trigonometric identity.

b) Use the substitution $u = \sin\theta$ to show

$$\int_0^1 \sqrt{\left(1-x^2\right)^3}\, dx = \frac{3\pi}{16} .$$

$\boxed{\text{proof}}$



Question 111 (****)

$$y = \frac{x^2 + 2x - 2}{x^2 - 2x + 2}.$$

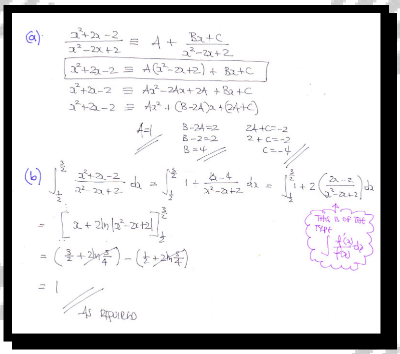
a) Find the value of each of the constants A, B and C, so that

$$y \equiv A + \frac{Bx + C}{x^2 - 2x + 2}.$$

b) Hence, or otherwise, show that

$$\int_{\frac{1}{2}}^{\frac{3}{2}} y \ dx = 1.$$

$\boxed{A = 1}$, $\boxed{B = 4}$, $\boxed{C = -4}$



Question 112 (****)

$$f(x) \equiv \frac{2}{x+\sqrt{2x-1}}, \quad x \geq \frac{1}{2}.$$

a) Use the substitution $u=\sqrt{2x-1}$ transforms to show

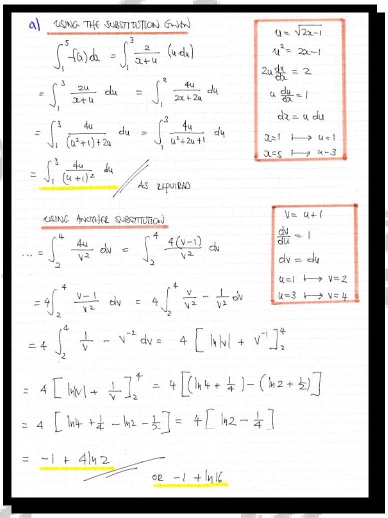
$$\int_1^5 f(x)\ dx \equiv \int_{u_1}^{u_2} \frac{4u}{(u+1)^2}\ du,$$

where u_1 and u_2 are constants to be found.

b) By using another suitable substitution, or otherwise, show that

$$\int_1^5 f(x)\ dx = -1+\ln 16.$$

[ ], [proof]


a) USING THE SUBSTITUTION GIVEN

$u=\sqrt{2x-1}$, $u^2=2x-1$, $2u\frac{du}{dx}=2$, $u\frac{du}{dx}=1$, $dx=u\,du$, $x=1 \mapsto u=1$, $x=5 \mapsto u=3$

$$\int_1^5 f(x)\,dx = \int_1^3 \frac{2}{x+u}(u\,du)$$

$$= \int_1^3 \frac{2u}{x+u}\,du = \int_1^3 \frac{4u}{2x+2u}\,du$$

$$= \int_1^3 \frac{4u}{(u^2+1)+2u}\,du = \int_1^3 \frac{4u}{u^2+2u+1}\,du$$

$$= \int_1^3 \frac{4u}{(u+1)^2}\,du$$ // AS REQUIRED

USING ANOTHER SUBSTITUTION

$v=u+1$, $\frac{dv}{du}=1$, $dv=du$, $u=1 \mapsto v=2$, $u=3 \mapsto v=4$

$$\dots = \int_2^4 \frac{4u}{v^2}\,dv = \int_2^4 \frac{4(v-1)}{v^2}\,dv$$

$$= 4\int_2^4 \frac{v-1}{v^2}\,dv = 4\int_2^4 \frac{v}{v^2}-\frac{1}{v^2}\,dv$$

$$= 4\int_2^4 \frac{1}{v}-v^{-2}\,dv = 4\left[\ln|v|+v^{-1}\right]_2^4$$

$$= 4\left[\ln|v|+\frac{1}{v}\right]_2^4 = 4\left[\left(\ln 4+\frac{1}{4}\right)-\left(\ln 2+\frac{1}{2}\right)\right]$$

$$= 4\left[\ln 4+\frac{1}{4}-\ln 2-\frac{1}{2}\right] = 4\left[\ln 2-\frac{1}{4}\right]$$

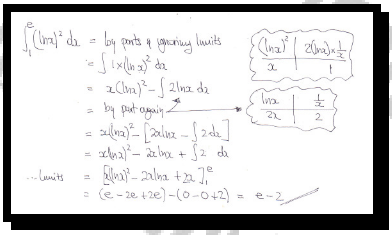
$$= -1+4\ln 2$$ // OR $-1+\ln 16$

Question 113 **(****)**

Use integration by parts find an exact value in terms of e, for the integral

$$\int_1^{e} (\ln x)^2 \ dx .$$

$\boxed{e-2}$

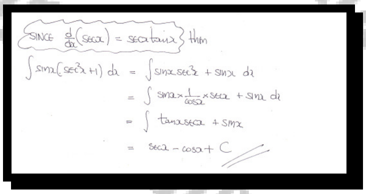


Question 114 **(****)**

Use the fact that $\frac{d}{dx}(\sec x) = \sec x \tan x$, to find

$$\int \sin x\left(1+\sec^2 x\right) dx .$$

$\boxed{\sec x - \cos x + C}$



Question 115 (****)

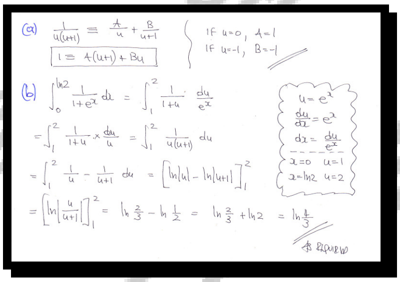
$$\frac{1}{u(u+1)} \equiv \frac{A}{u} + \frac{B}{u+1}.$$

a) Find the value of A and B in the above identity.

b) By using the substitution $u = e^x$, or otherwise, show that

$$\int_0^{\ln 2} \frac{1}{1+e^x}\, dx = \ln\left(\frac{4}{3}\right).$$

$\boxed{A=1}$, $\boxed{B=-1}$



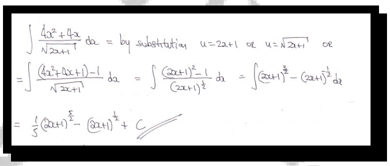
Question 116 (****)

By completing the square in the expression $4x^2 + 4x$, or otherwise, show that

$$\int \frac{4x^2+4x}{\sqrt{2x+1}}\, dx = A(2x+1)^{\frac{5}{2}} + B(2x+1)^{\frac{1}{2}} + C,$$

where A and B are constants to be found and C is the arbitrary constant of the integration.

$\boxed{A=\frac{1}{5}}$, $\boxed{B=-1}$



Question 117 (****)

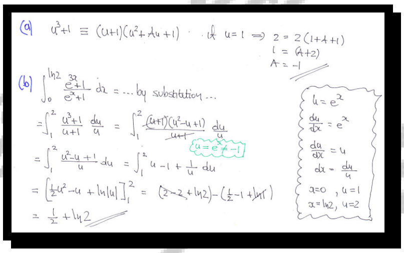
$$u^3+1 \equiv (u+1)\left(u^2+Au+1\right).$$

a) Determine the value of A in the above identity.

b) Use the substitution $u=e^x$ to show

$$\int_0^{\ln 2} \frac{e^{3x}+1}{e^x+1}\,dx = \frac{1}{2}+\ln 2.$$

$\boxed{A=1}$, $\boxed{B=\frac{1}{2}}$, $\boxed{C=-\frac{1}{2}}$



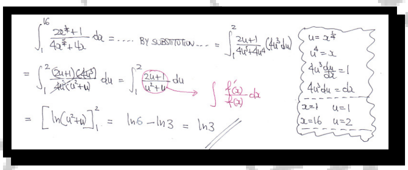
Question 118 (****)

Use the substitution $u=x^{\frac{1}{4}}$ to find

$$\int_1^{16} \frac{2x^{\frac{1}{4}}+1}{4x^{\frac{5}{4}}+4x}\,dx,$$

giving the final answer as an exact simplified natural logarithm.

$\boxed{\ln 3}$



Question 119 (****)

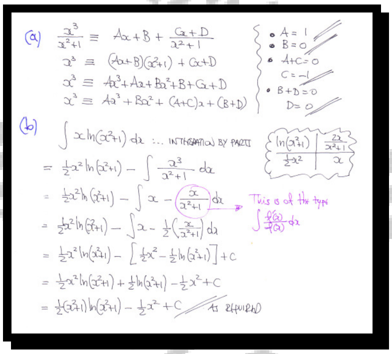
$$\frac{x^3}{x^2+1} \equiv Ax + B + \frac{Cx+D}{x^2+1}.$$

a) Determine the value of each of the constants A, B, C and D.

b) Use integration by parts to show that

$$\int x\ln\left(x^2+1\right)\,dx = \tfrac{1}{2}\left(x^2+1\right)\ln\left(x^2+1\right) - \tfrac{1}{2}x^2 + C.$$

$\boxed{A=1}$, $\boxed{B=0}$, $\boxed{C=-1}$, $\boxed{D=0}$

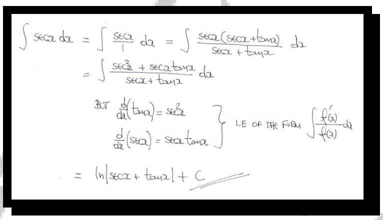


Question 120 (****)

By writing $\sec x$ as the fraction $\dfrac{\sec x}{1}$ and multiplying the numerator and the denominator by $(\sec x + \tan x)$, find

$$\int \sec x\,dx.$$

$\boxed{\ln\left|\sec x + \tan x\right| + C}$



Question 121 (****)

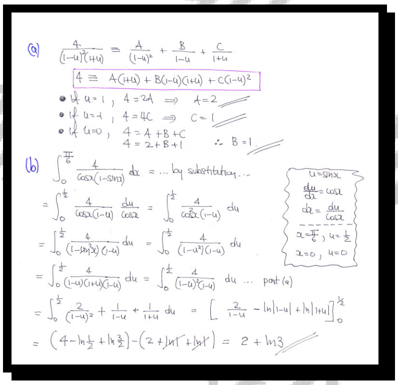
$$\frac{4}{(1-u)^2(1+u)} \equiv \frac{A}{(1-u)^2} + \frac{B}{1-u} + \frac{C}{1+u}.$$

a) Find the value of A, B and C in the above identity.

b) Hence by using a suitable substitution find the exact value of

$$\int_0^{\frac{\pi}{6}} \frac{4}{\cos x(1-\sin x)}\, dx.$$

$\boxed{}$, $\boxed{A=2}$, $\boxed{B=1}$, $\boxed{C=1}$, $\boxed{2+\ln 3}$

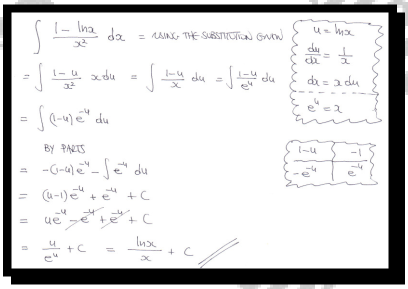


Question 122 **(****)**

Use the substitution $u = \ln x$, followed by integration by parts to find

$$\int \frac{1-\ln x}{x^2}\, dx .$$

$$\boxed{\frac{\ln x}{x} + C}$$



Question 123 **(****)**

$$y = \arctan x, \quad x \in \mathbb{R} .$$

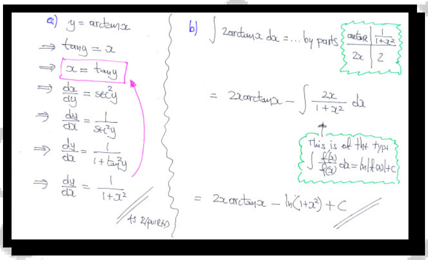
a) By writing the above equation as $x = \tan y$, show clearly that

$$\frac{dy}{dx} = \frac{1}{1+x^2} .$$

b) Use integration by parts and the result of part (a) to find

$$\int 2\arctan x \, dx .$$

$$\boxed{2x\arctan x - \ln\left(1+x^2\right) + C}$$



Question 124 (****)

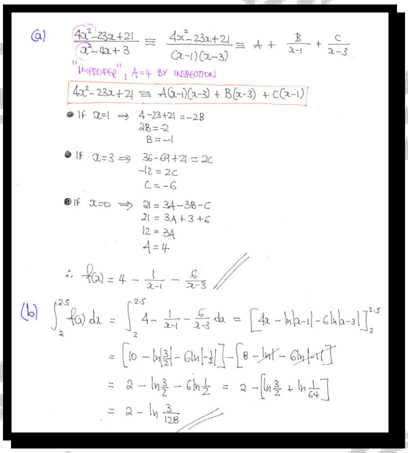
$$f(x) \equiv \frac{4x^2 - 23x + 21}{x^2 - 4x + 3}, \; x \neq 1, \; x \neq 3.$$

a) Express $f(x)$ in partial fractions.

b) Hence find an exact value for

$$\int_2^{2.5} f(x) \, dx .$$

$$\boxed{f(x) \equiv 4 - \frac{1}{x-1} - \frac{6}{x-3}}, \; \boxed{2 - \ln\left(\frac{3}{128}\right)}$$


(a) $\frac{4x^2-23x+21}{x^2-4x+3} \equiv \frac{4x^2-23x+21}{(x-1)(x-3)} \equiv A + \frac{B}{x-1} + \frac{C}{x-3}$

"IMPROPER", A=4 BY INSPECTION

$4x^2-23x+21 \equiv A(x-1)(x-3) + B(x-3) + C(x-1)$

- IF $x=1 \Rightarrow 4-23+21 = -2B$, $2B=-2$, $B=-1$
- IF $x=3 \Rightarrow 36-69+21 = 2C$, $-12 = 2C$, $C=-6$
- IF $x=0 \Rightarrow 21 = 3A-3B-C$, $21 = 3A+3+6$, $12 = 3A$, $A=4$

$\therefore f(x) = 4 - \frac{1}{x-1} - \frac{6}{x-3}$

(b) $\int_2^{2.5} f(x)\,dx = \int_2^{2.5} 4 - \frac{1}{x-1} - \frac{6}{x-3}\,dx = \left[4x - \ln|x-1| - 6\ln|x-3|\right]_2^{2.5}$

$= \left[10 - \ln\left|\frac{3}{2}\right| - 6\ln\left|-\frac{1}{2}\right|\right] - \left[8 - \ln 1 - 6\ln|-1|\right]$

$= 2 - \ln\frac{3}{2} - 6\ln\frac{1}{2} = 2 - \left[\ln\frac{3}{2} + \ln\frac{1}{64}\right]$

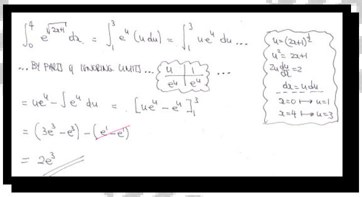
$= 2 - \ln\frac{3}{128}$

Question 125 (****)

By using the substitution $u = (2x+1)^{\frac{1}{2}}$, or otherwise, show that

$$\int_0^4 e^{\sqrt{2x+1}} \, dx = 2e^3 .$$

proof


$\int_0^4 e^{\sqrt{2x+1}} dx = \int_1^3 e^u (u\,du) = \int_1^3 ue^u \, du \ldots$

... BY PARTS & IGNORING LIMITS ...

$= ue^u - \int e^u \, du = [ue^u - e^u]_1^3$

$= (3e^3 - e^3) - (e - e)$

$= 2e^3$

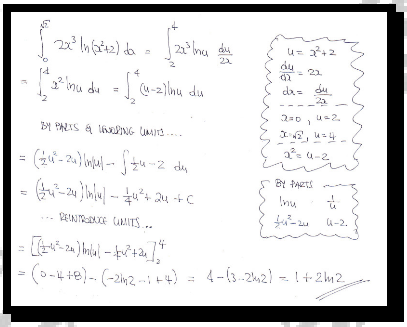
$u = (2x+1)^{\frac{1}{2}}$, $u^2 = 2x+1$, $2u\frac{du}{dx} = 2$, $dx = u\,du$, $x=0 \mapsto u=1$, $x=4 \mapsto u=3$

Question 126 (****)

Use the substitution $u = x^2 + 2$, followed by integration by parts to show that

$$\int 2x^3 \ln\left(x^2+2\right) dx = 1 + 2\ln 2 .$$

proof


$\int_0^{\sqrt{2}} 2x^3 \ln(x^2+2)\,dx = \int_2^4 2x^3 \ln u \frac{du}{2x}$

$= \int_2^4 x^2 \ln u \, du = \int_2^4 (u-2)\ln u \, du$

BY PARTS & IGNORING LIMITS ...

$= \left(\frac{1}{2}u^2 - 2u\right)\ln|u| - \int \frac{1}{2}u - 2 \, du$

$= \left(\frac{1}{2}u^2 - 2u\right)\ln|u| - \frac{1}{4}u^2 + 2u + C$

... REINTRODUCE LIMITS ...

$= \left[\left(\frac{1}{2}u^2 - 2u\right)\ln|u| - \frac{1}{4}u^2 + 2u\right]_2^4$

$= (0 - 4 + 8) - (-2\ln 2 - 1 + 4) = 4 - (3 - 2\ln 2) = 1 + 2\ln 2$

$u = x^2 + 2$, $\frac{du}{dx} = 2x$, $dx = \frac{du}{2x}$, $x = 0, u = 2$, $x = \sqrt{2}, u = 4$, $x^2 = u - 2$

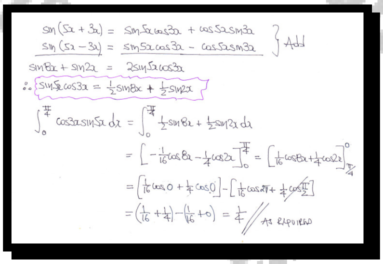
BY PARTS: $\ln u$, $\frac{1}{u}$; $\frac{1}{2}u^2 - 2u$, $u - 2$

Question 127 (****)

By considering the trigonometric expansions of $\sin(5x+3x)$ and $\sin(5x-3x)$, show clearly that

$$\int_0^{\frac{\pi}{4}} \cos 3x \sin 5x \ dx = \frac{1}{4}.$$

proof

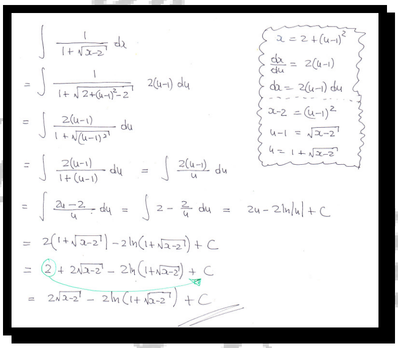


Question 128 (****)

By using the substitution $x = 2+(u-1)^2$, or otherwise, find

$$\int \frac{1}{1+\sqrt{x-2}}\, dx.$$

$$2\sqrt{x-2} + 2\ln\left(1+\sqrt{x-2}\right) + C$$



Question 129 (****)

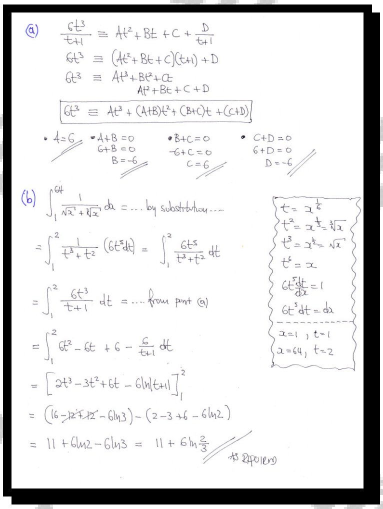
$$\frac{6t^3}{t+1} \equiv At^2 + Bt + C + \frac{D}{t+1}.$$

a) Determine the value of each of the constants A, B, C and D.

b) Use the substitution $t = x^{\frac{1}{6}}$ to show

$$\int_1^{64} \frac{1}{\sqrt{x} + \sqrt[3]{x}} \, dx = 11 + 6\ln\left(\frac{2}{3}\right).$$

$\boxed{A = 6}$, $\boxed{B = -6}$, $\boxed{C = 6}$, $\boxed{D = -6}$


(a) $\frac{6t^3}{t+1} \equiv At^2 + Bt + C + \frac{D}{t+1}$

$6t^3 \equiv (At^2 + Bt + C)(t+1) + D$

$6t^3 \equiv At^3 + Bt^2 + Ct + At^2 + Bt + C + D$

$6t^3 \equiv At^3 + (A+B)t^2 + (B+C)t + (C+D)$

• $A = 6$ • $A + B = 0$, $6 + B = 0$, $B = -6$ • $B + C = 0$, $-6 + C = 0$, $C = 6$ • $C + D = 0$, $6 + D = 0$, $D = -6$

(b) $\int_1^{64} \frac{1}{\sqrt{x} + \sqrt[3]{x}} \, dx = \ldots$ by substitution ...

$t = x^{\frac{1}{6}}$, $t^2 = x^{\frac{1}{3}} = \sqrt[3]{x}$, $t^3 = x^{\frac{1}{2}} = \sqrt{x}$, $t^6 = x$, $6t^5 \frac{dt}{dx} = 1$, $6t^5 dt = dx$

$x = 1$, $t = 1$; $x = 64$, $t = 2$

$= \int_1^2 \frac{1}{t^3 + t^2} (6t^5 dt) = \int_1^2 \frac{6t^5}{t^3 + t^2} dt$

$= \int_1^2 \frac{6t^3}{t+1} dt = \ldots$ from part (a)

$= \int_1^2 6t^2 - 6t + 6 - \frac{6}{t+1} dt$

$= \left[2t^3 - 3t^2 + 6t - 6\ln|t+1|\right]_1^2$

$= (16 - 12 + 12 - 6\ln 3) - (2 - 3 + 6 - 6\ln 2)$

$= 11 + 6\ln 2 - 6\ln 3 = 11 + 6\ln\frac{2}{3}$ as required

Question 130 (****)

$$f(x) = 8x\ln(2x+1), \ x > -\frac{1}{2}.$$

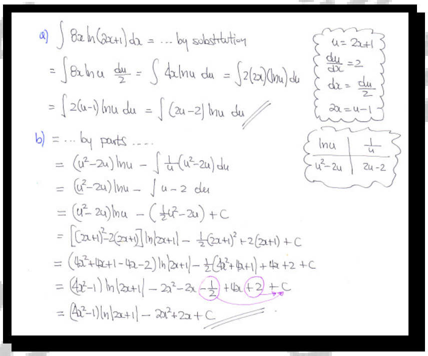
a) Use the substitution $u = 2x+1$ to show that

$$\int f(x)\,dx = \int (2u-2)\ln u \, du.$$

b) Use integration by parts to show that

$$\int f(x)\,dx = \left(4x^2-1\right)\ln(2x+1) - 2x^2 + 2x + C.$$

proof



a) $\int 8x\ln(2x+1)\,dx = \dots$ by substitution

$= \int 8x\ln u \frac{du}{2} = \int 4x\ln u\,du = \int 2(2x)(\ln u)\,du$

$= \int 2(u-1)\ln u\,du = \int (2u-2)\ln u\,du$

$u = 2x+1$, $\frac{du}{dx} = 2$, $dx = \frac{du}{2}$, $2x = u-1$

b) $= \dots$ by parts $\dots$

$\ln u$	$\frac{1}{u}$
u^2-2u	$2u-2$

$= (u^2-2u)\ln u - \int \frac{1}{u}(u^2-2u)\,du$

$= (u^2-2u)\ln u - \int u-2\,du$

$= (u^2-2u)\ln u - \left(\frac{1}{2}u^2-2u\right) + C$

$= \left[(2x+1)^2-2(2x+1)\right]\ln|2x+1| - \frac{1}{2}(2x+1)^2 + 2(2x+1) + C$

$= (4x^2+4x+1-4x-2)\ln|2x+1| - \frac{1}{2}(4x^2+4x+1) + 4x + 2 + C$

$= (4x^2-1)\ln|2x+1| - 2x^2 - 2x - \frac{1}{2} + 4x + 2 + C$

$= (4x^2-1)\ln|2x+1| - 2x^2 + 2x + C$

Question 131 (****)

$$y = \frac{(1+\sin x)^2}{\cos^2 x}.$$

a) Calculate the two missing values of the following table.

x	$\frac{\pi}{6}$		$\frac{\pi}{4}$	$\frac{7\pi}{24}$	$\frac{\pi}{3}$
y	3		5.8284	8.6784	13.9282

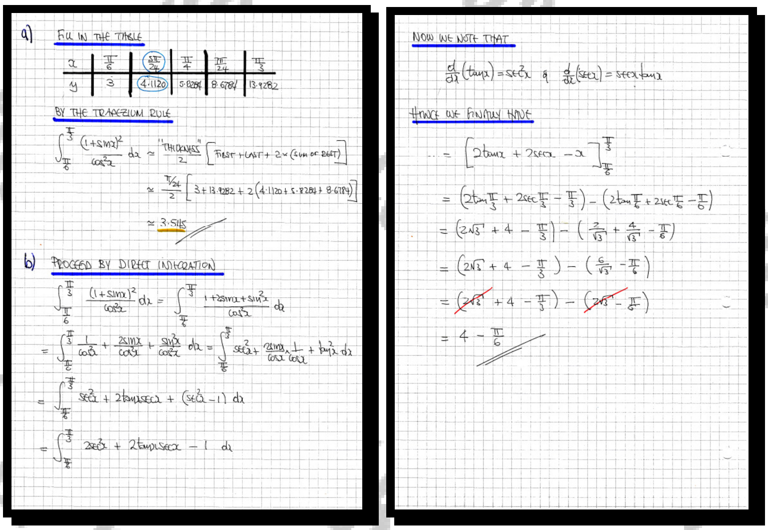
b) Use the trapezium rule with all the values from the completed table of part **(a)** to find an estimate for

$$\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{(1+\sin x)^2}{\cos^2 x}\, dx.$$

c) Use trigonometric identities to find the exact value of

$$\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{(1+\sin x)^2}{\cos^2 x}\, dx.$$

$\boxed{}$, $\boxed{x = \frac{5\pi}{24}}$, $\boxed{4.112}$, $\boxed{4 - \frac{1}{6}\pi}$


a) FILL IN THE TABLE

x	$\frac{\pi}{6}$	$\frac{5\pi}{24}$	$\frac{\pi}{4}$	$\frac{7\pi}{24}$	$\frac{\pi}{3}$
y	3	4.1120	5.8284	8.6784	13.9282

BY THE TRAPEZIUM RULE

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(1+\sin x)^2}{\cos^2 x}\,dx \simeq \frac{\text{"THICKNESS"}}{2}\left[\text{FIRST} + \text{LAST} + 2\times(\text{SUM OF REST})\right]$$

$$\simeq \frac{\pi/24}{2}\left[3 + 13.9282 + 2(4.1120 + 5.8284 + 8.6784)\right]$$

$$\simeq 3.545$$

b) PROCEED BY DIRECT INTEGRATION

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(1+\sin x)^2}{\cos^2 x}\,dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1+2\sin x+\sin^2 x}{\cos^2 x}\,dx$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{\cos^2 x} + \frac{2\sin x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x}\,dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^2 x + \frac{2\sin x}{\cos x}\times\frac{1}{\cos x} + \tan^2 x\,dx$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^2 x + 2\tan x\sec x + (\sec^2 x - 1)\,dx$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 2\sec^2 x + 2\tan x\sec x - 1\,dx$$

NOW WE NOTE THAT

$$\frac{d}{dx}(\tan x) = \sec^2 x \quad \& \quad \frac{d}{dx}(\sec x) = \sec x\tan x$$

HENCE WE FINALLY HAVE

$$\dots = \left[2\tan x + 2\sec x - x\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= \left(2\tan\frac{\pi}{3} + 2\sec\frac{\pi}{3} - \frac{\pi}{3}\right) - \left(2\tan\frac{\pi}{6} + 2\sec\frac{\pi}{6} - \frac{\pi}{6}\right)$$

$$= \left(2\sqrt{3} + 4 - \frac{\pi}{3}\right) - \left(\frac{2}{\sqrt{3}} + \frac{4}{\sqrt{3}} - \frac{\pi}{6}\right)$$

$$= \left(2\sqrt{3} + 4 - \frac{\pi}{3}\right) - \left(\frac{6}{\sqrt{3}} - \frac{\pi}{6}\right)$$

$$= \left(2\sqrt{3} + 4 - \frac{\pi}{3}\right) - \left(2\sqrt{3} - \frac{\pi}{6}\right)$$

$$= 4 - \frac{\pi}{6}$$

Question 132 (****)

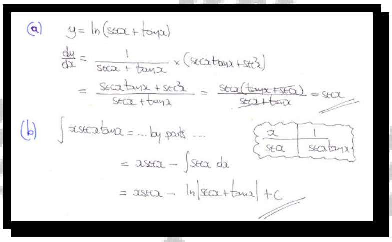
$$y = \ln(\sec x + \tan x).$$

a) Express $\frac{dy}{dx}$ as a single trigonometric function.

b) Hence find

$$\int x \sec x \tan x \, dx .$$

$$\boxed{x\sec x - \ln\left|\sec x + \tan x\right| + C}$$

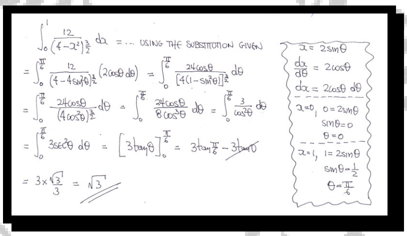


Question 133 (****)

Use the substitution $x = 2\sin\theta$ to find the exact value of

$$\int_0^1 \frac{12}{\left(4-x^2\right)^{\frac{3}{2}}} \, dx .$$

$$\boxed{\sqrt{3}}$$



Question 134 (****)

$$\frac{u^2}{u^2-9} \equiv A + \frac{B}{u-3} + \frac{C}{u+3}.$$

a) Find the value of A, B and C in the above identity.

b) By using the substitution $u = \sqrt{x^2+9}$, or otherwise, find

$$\int \frac{\sqrt{x^2+9}}{x}\, dx.$$

$\boxed{A=1}$, $\boxed{B=\frac{3}{2}}$, $\boxed{C=-\frac{3}{2}}$, $\boxed{\sqrt{x^2+9}+\frac{3}{2}\ln\left|\frac{\sqrt{x^2+9}-3}{\sqrt{x^2+9}+3}\right|+C}$

(a) $\frac{u^2}{u^2-9} = \frac{u^2}{(u-3)(u+3)} \equiv A + \frac{B}{u-3} + \frac{C}{u+3}$

$\boxed{u^2 \equiv A(u-3)(u+3) + B(u+3) + C(u-3)}$

If $u=3$, $9 = 6B \Rightarrow B = \frac{3}{2}$

If $u=-3$, $9 = -6C \Rightarrow C = -\frac{3}{2}$

If $u=0$, $0 = -9A + 3B - 3C$

$9A = \frac{9}{2} - \left(-\frac{9}{2}\right)$

$9A = 9$

$A = 1$

(b) $u = \sqrt{x^2+9}$, $u^2 = x^2+9$, $2u\frac{du}{dx} = 2x$, $dx = \frac{u}{x}\,du$, $x^2 = u^2-9$

$\int \frac{\sqrt{x^2+9}}{x}\,dx = \int \frac{u}{x}\frac{u}{x}\,du$

$= \int \frac{u^2}{x^2}\,du = \int \frac{u^2}{u^2-9}\,du$

$= \int 1 + \frac{3/2}{u-3} - \frac{3/2}{u+3}\,du$

$= u + \frac{3}{2}\ln|u-3| - \frac{3}{2}\ln|u+3| + C$

$= u + \frac{3}{2}\ln\left|\frac{u-3}{u+3}\right| + C$

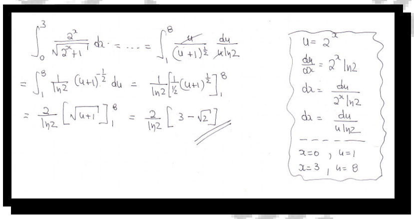
$= \sqrt{x^2+9} + \frac{3}{2}\ln\left|\frac{\sqrt{x^2+9}-3}{\sqrt{x^2+9}+3}\right| + C$

Question 135 (****)

By using the substitution $u = 2^x$, or otherwise, find an exact value for

$$\int_0^3 \frac{2^x}{\sqrt{2^x+1}}\,dx.$$

$$\boxed{\frac{2\left(3-\sqrt{2}\right)}{\ln 2}}$$


$$\int_0^3 \frac{2^x}{\sqrt{2^x+1}}\,dx = \dots = \int_1^8 \frac{u}{(u+1)^{\frac{1}{2}}}\,\frac{du}{u\ln 2}$$
$$= \int_1^8 \frac{1}{\ln 2}(u+1)^{-\frac{1}{2}}\,du = \frac{1}{\ln 2}\left[\frac{1}{\frac{1}{2}}(u+1)^{\frac{1}{2}}\right]_1^8$$
$$= \frac{2}{\ln 2}\left[\sqrt{u+1}\right]_1^8 = \frac{2}{\ln 2}\left[3-\sqrt{2}\right]$$

$u = 2^x$, $\frac{du}{dx} = 2^x \ln 2$, $dx = \frac{du}{2^x \ln 2}$, $dx = \frac{du}{u \ln 2}$; $x=0,\ u=1$; $x=3,\ u=8$

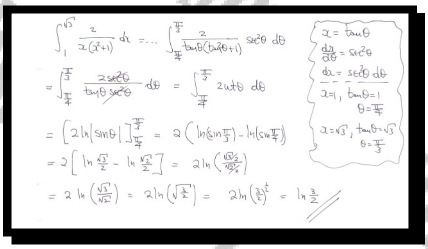
Question 136 (****)

Use the substitution $x = \tan\theta$ to show that

$$\int_1^{\sqrt{3}} \frac{2}{x\left(x^2+1\right)}\,dx = \ln\left(\frac{m}{n}\right),$$

where m and n are integers.

$\boxed{\text{proof}}$


$$\int_1^{\sqrt{3}} \frac{2}{x(x^2+1)}\,dx = \dots \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{2}{\tan\theta(\tan^2\theta+1)}\sec^2\theta\,d\theta$$
$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{2\sec^2\theta}{\tan\theta\sec^2\theta}\,d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} 2\cot\theta\,d\theta$$
$$= \left[2\ln|\sin\theta|\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}} = 2\left(\ln\left(\sin\tfrac{\pi}{3}\right) - \ln\left(\sin\tfrac{\pi}{4}\right)\right)$$
$$= 2\left[\ln\frac{\sqrt{3}}{2} - \ln\frac{\sqrt{2}}{2}\right] = 2\ln\left(\frac{\sqrt{3}/2}{\sqrt{2}/2}\right)$$
$$= 2\ln\left(\frac{\sqrt{3}}{\sqrt{2}}\right) = 2\ln\left(\sqrt{\frac{3}{2}}\right) = 2\ln\left(\frac{3}{2}\right)^{\frac{1}{2}} = \ln\frac{3}{2}$$

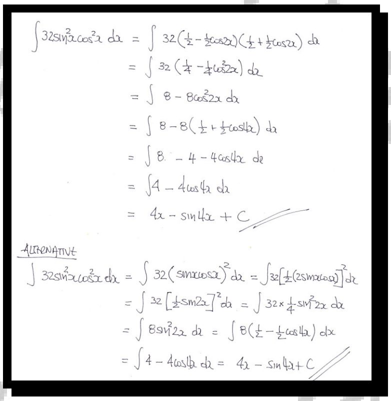
$x = \tan\theta$, $\frac{dx}{d\theta} = \sec^2\theta$, $dx = \sec^2\theta\,d\theta$; $x=1,\ \tan\theta = 1,\ \theta = \frac{\pi}{4}$; $x=\sqrt{3},\ \tan\theta=\sqrt{3},\ \theta=\frac{\pi}{3}$

Question 137 (****)

Use trigonometric identities to find

$$\int 32\sin^2 x\cos^2 x \, dx .$$

$$\boxed{4x-\sin 4x+C}$$

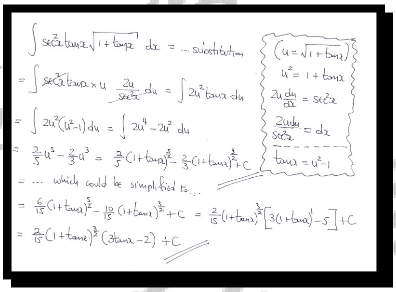


Question 138 (****)

By using the substitution $u^2 = 1+\tan x$, or otherwise, find

$$\int \sec^2 x\tan x\sqrt{1+\tan x} \, dx .$$

$$\boxed{\frac{2}{15}(3\tan x-2)(1+\tan x)^{\frac{3}{2}}}$$



Question 139 (****)

$$\frac{2u^2}{(u-1)(u+1)} \equiv A + \frac{B}{u+1} + \frac{C}{u-1}.$$

a) Find the value of A, B and C in the above identity.

b) By using the substitution $u^2 = x+1$, or otherwise, find an exact value for

$$\int_3^8 \frac{\sqrt{x+1}}{x}\, dx.$$

The table below shows some tabulated values for the equation $y = \dfrac{\sqrt{x+1}}{x}$, $3 \le x \le 8$.

x	3	4	5	6	7	8
y	0.6667	0.5590	0.4899		0.4041	0.3750

c) Complete the missing value in the table.

[continues overleaf]

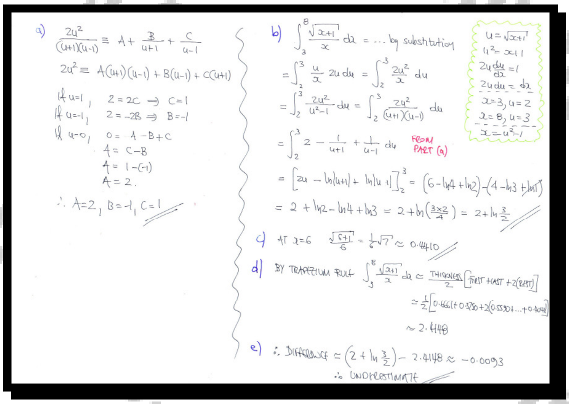
[continued from overleaf]

d) Use the trapezium rule with all the values from the table to find an approximate value for

$$\int_{3}^{8} \frac{\sqrt{x+1}}{x}\, dx .$$

e) Calculate the difference between the exact value, found in part (b), and the trapezium rule estimate, found in part (d), and hence state whether the trapezium rule produces an overestimate or an underestimate.

$\boxed{}$, $\boxed{A=2}$, $\boxed{B=-1}$, $\boxed{C=-1}$, $\boxed{2+\ln\left(\frac{3}{2}\right)}$, $\boxed{0.4410}$, $\boxed{2.4148}$, $\boxed{0.0093}$



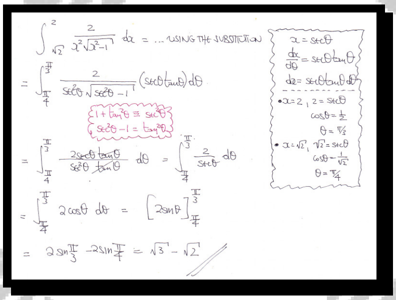
Question 140 (****)

Use the substitution $x = \sec\theta$ to show that

$$\int_{\sqrt{2}}^{2} \frac{2}{x^2\sqrt{x^2-1}}\,dx = \sqrt{m} - \sqrt{n}\,,$$

where m and n are integers.

$$\boxed{\sqrt{3}-\sqrt{2}}$$

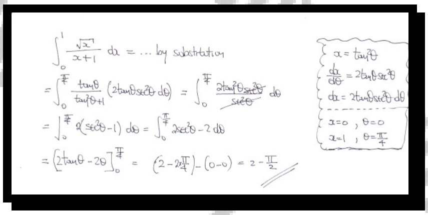


Question 141 (****)

Use the substitution $x = \tan^2\theta$ to find an exact value for

$$\int_{0}^{1} \frac{\sqrt{x}}{x+1}\,dx\,.$$

$$\boxed{2-\frac{\pi}{2}}$$



Question 142 (****)

$$y = \frac{e^{2x}}{e^x + 1}, \quad x \in \mathbb{R}$$

a) Calculate the missing values of x and y in the following table.

x	$\ln 2$	x_2	x_3	x_4	$\ln 8$
y	1.333	y_2	y_3	y_4	7.111

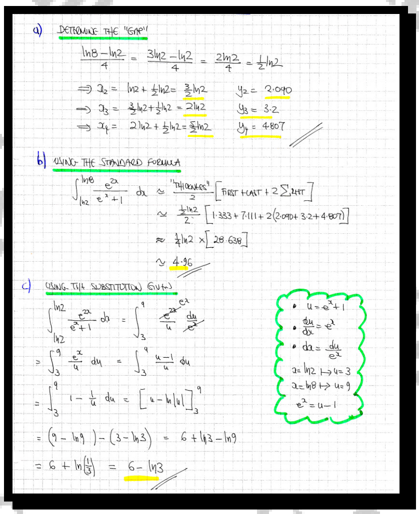
b) Use the trapezium rule with all the values from the completed table of part **(a)** to find an estimate for

$$\int_{\ln 2}^{\ln 8} \frac{e^{2x}}{e^x + 1} \, dx.$$

c) Use the substitution $u = e^x + 1$ to find an exact simplified value for

$$\int_{\ln 2}^{\ln 8} \frac{e^{2x}}{e^x + 1} \, dx.$$

$\boxed{}$, $\boxed{\frac{3}{2}\ln 2, \ 2\ln 2, \ \frac{5}{2}\ln 2, \ 2.090, \ 3.2, \ 4.807}$, $\boxed{\approx 4.96}$ $\boxed{6 - \ln 3}$



Question 143 (****)

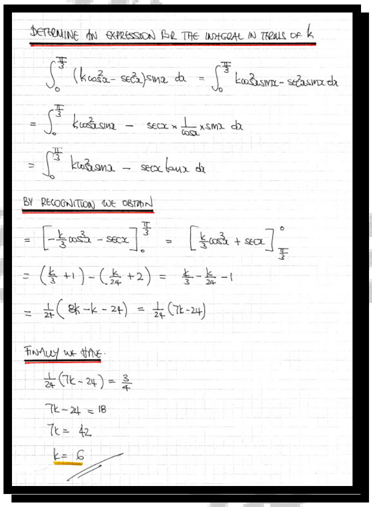
It is given that the value of

$$\int_0^{\frac{1}{3}\pi} \left(k\cos^2 x - \sec^2 x\right)\sin x \ dx\,,$$

is 2, where k is a non zero constant.

Determine the value of k.

$\boxed{}$, $\boxed{k=6}$



DETERMINE AN EXPRESSION FOR THE INTEGRAL IN TERMS OF k

$$\int_0^{\frac{\pi}{3}} (k\cos^2 x - \sec^2 x)\sin x \, dx = \int_0^{\frac{\pi}{3}} k\cos^2 x\sin x - \sec^2 x\sin x \, dx$$

$$= \int_0^{\frac{\pi}{3}} k\cos^2 x\sin x - \sec x \times \frac{1}{\cos x} \times \sin x \, dx$$

$$= \int_0^{\frac{\pi}{3}} k\cos^2 x\sin x - \sec x\tan x \, dx$$

BY RECOGNITION WE OBTAIN

$$= \left[-\frac{k}{3}\cos^3 x - \sec x\right]_0^{\frac{\pi}{3}} = \left[\frac{k}{3}\cos^3 x + \sec x\right]_{\frac{\pi}{3}}^{0}$$

$$= \left(\frac{k}{3}+1\right) - \left(\frac{k}{24}+2\right) = \frac{k}{3} - \frac{k}{24} - 1$$

$$= \frac{1}{24}(8k - k - 24) = \frac{1}{24}(7k-24)$$

FINALLY WE HAVE

$$\frac{1}{24}(7k-24) = \frac{3}{4}$$

$$7k - 24 = 18$$

$$7k = 42$$

$$k = 6$$

Question 144 (****)

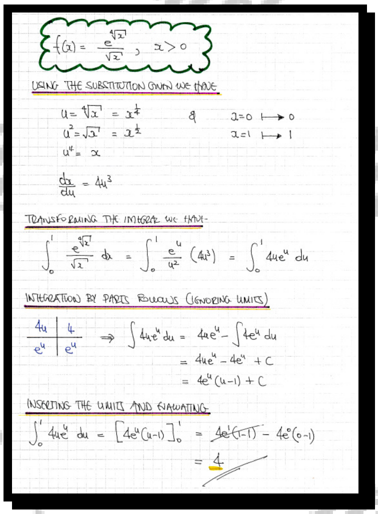
$$f(x)=\frac{e^{\sqrt[4]{x}}}{\sqrt{x}}, \quad x\in\mathbb{R},\ x>0.$$

Find the value of

$$\int_0^1 f(x)\ dx,$$

given further that the integral exists.

$\boxed{}$, $\boxed{4}$



Question 145 (****)

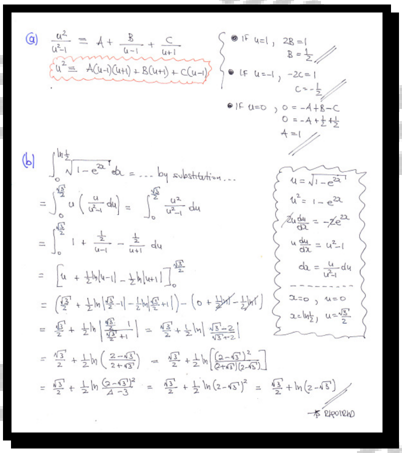
$$\frac{u^2}{u^2-1} \equiv A + \frac{B}{u-1} + \frac{C}{u+1}.$$

a) Find the value of A, B and C in the above identity.

b) Use the substitution $u = \sqrt{1-e^{2x}}$ to show

$$\int_0^{\ln\frac{1}{2}} \sqrt{1-e^{2x}} \; dx = \frac{\sqrt{3}}{2} + \ln\left(2-\sqrt{3}\right).$$

$\boxed{A=1}$, $\boxed{B=\frac{1}{2}}$, $\boxed{C=-\frac{1}{2}}$

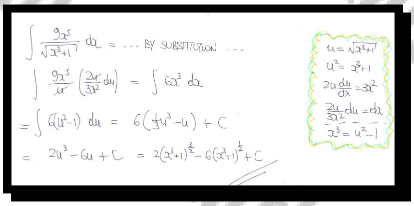


Question 146 **(****)**

By using the substitution $u = \sqrt{x^3+1}$, or otherwise, find an expression for

$$\int \frac{9x^5}{\sqrt{x^3+1}}\,dx .$$

$$\boxed{2\left(x^3+1\right)^{\frac{3}{2}} - 6\left(x^3+1\right)^{\frac{1}{2}} + C}$$



$\int \frac{9x^5}{\sqrt{x^3+1}}\,dx = \dots$ BY SUBSTITUTION $\dots$

$\int \frac{9x^5}{u}\left(\frac{2u}{3x^2}\,du\right) = \int 6x^3\,dx$

$= \int 6(u^2-1)\,du = 6\left(\frac{1}{3}u^3 - u\right) + C$

$= 2u^3 - 6u + C = 2(x^3+1)^{\frac{3}{2}} - 6(x^3+1)^{\frac{1}{2}} + C$

$u = \sqrt{x^3+1}$

$u^2 = x^3+1$

$2u\frac{du}{dx} = 3x^2$

$\frac{2u}{3x^2}\,du = dx$

$x^3 = u^2-1$

Question 147 (****)

It is given that

$$\sin(A+B) \equiv \sin A \cos B + \cos A \sin B.$$

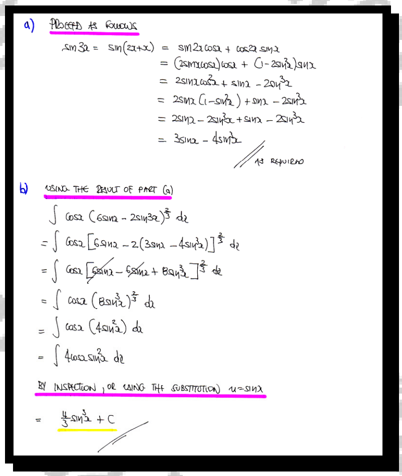
a) Use the above trigonometric identity to show that

$$\sin 3x \equiv 3\sin x - 4\sin^3 x.$$

b) Hence find

$$\int \cos x (6\sin x - 2\sin 3x)^{\frac{2}{3}} \, dx.$$

$\boxed{}$, $\boxed{\frac{4}{3}\sin^3 x + C}$



a) PROCEED AS FOLLOWS

$\sin 3x = \sin(2x+x) = \sin 2x \cos x + \cos 2x \sin x$
$= (2\sin x \cos x)\cos x + (1-2\sin^2 x)\sin x$
$= 2\sin x \cos^2 x + \sin x - 2\sin^3 x$
$= 2\sin x(1-\sin^2 x) + \sin x - 2\sin^3 x$
$= 2\sin x - 2\sin^3 x + \sin x - 2\sin^3 x$
$= 3\sin x - 4\sin^3 x$

AS REQUIRED

b) USING THE RESULT OF PART (a)

$\int \cos x (6\sin x - 2\sin 3x)^{\frac{2}{3}} \, dx$
$= \int \cos x [6\sin x - 2(3\sin x - 4\sin^3 x)]^{\frac{2}{3}} \, dx$
$= \int \cos x [6\sin x - 6\sin x + 8\sin^3 x]^{\frac{2}{3}} \, dx$
$= \int \cos x (8\sin^3 x)^{\frac{2}{3}} \, dx$
$= \int \cos x (4\sin^2 x) \, dx$
$= \int 4\cos x \sin^2 x \, dx$

BY INSPECTION, OR USING THE SUBSTITUTION $u = \sin x$

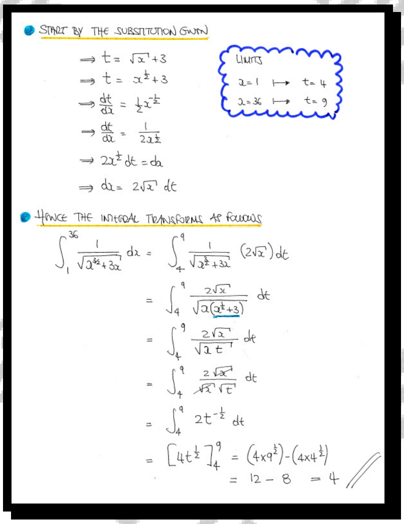
$= \frac{4}{3}\sin^3 x + C$

Question 148 (****)

Use the substitution $t = 3 + \sqrt{x}$ to find the value of the following integral

$$\int_1^{36} \frac{1}{\sqrt{x^{\frac{3}{2}} + 3x}} \, dx \, .$$

4


START BY THE SUBSTITUTION GIVEN

$\Rightarrow t = \sqrt{x} + 3$

$\Rightarrow t = x^{\frac{1}{2}} + 3$

$\Rightarrow \frac{dt}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$

$\Rightarrow \frac{dt}{dx} = \frac{1}{2x^{\frac{1}{2}}}$

$\Rightarrow 2x^{\frac{1}{2}}\,dt = dx$

$\Rightarrow dx = 2\sqrt{x}\,dt$

LIMITS: $x = 1 \mapsto t = 4$, $x = 36 \mapsto t = 9$

HENCE THE INTEGRAL TRANSFORMS AS FOLLOWS

$$\int_1^{36} \frac{1}{\sqrt{x^{\frac{3}{2}} + 3x}}\,dx = \int_4^9 \frac{1}{\sqrt{x^{\frac{3}{2}} + 3x}}(2\sqrt{x})\,dt = \int_4^9 \frac{2\sqrt{x}}{\sqrt{x(x^{\frac{1}{2}}+3)}}\,dt = \int_4^9 \frac{2\sqrt{x}}{\sqrt{x\,t}}\,dt = \int_4^9 \frac{2\sqrt{x}}{\sqrt{x}\sqrt{t}}\,dt = \int_4^9 2t^{-\frac{1}{2}}\,dt$$

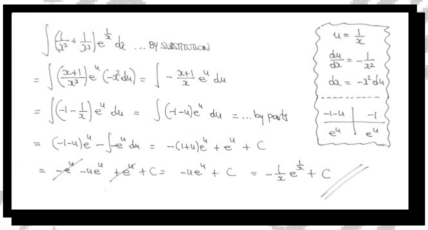
$$= \left[4t^{\frac{1}{2}}\right]_4^9 = (4\times 9^{\frac{1}{2}}) - (4\times 4^{\frac{1}{2}}) = 12 - 8 = 4$$

Question 149 (****)

By using the substitution $u = \frac{1}{x}$, or otherwise, show that

$$\int_0^4 \left(\frac{1}{x^2} + \frac{1}{x^3}\right) e^{\frac{1}{x}} \, dx = -\frac{1}{x} e^{\frac{1}{x}} + C \, .$$

proof


$\int \left(\frac{1}{x^2}+\frac{1}{x^3}\right)e^{\frac{1}{x}}\,dx$ ... BY SUBSTITUTION

$= \int \left(\frac{x+1}{x^3}\right)e^u(-x^2\,du) = \int -\frac{x+1}{x}e^u\,du$

$= \int \left(-1-\frac{1}{x}\right)e^u\,du = \int (-1-u)e^u\,du = \ldots$ by parts

$= (-1-u)e^u - \int -e^u\,du = -(1+u)e^u + e^u + C$

$= -e^u - ue^u + e^u + C = -ue^u + C = -\frac{1}{x}e^{\frac{1}{x}} + C$

$u = \frac{1}{x}$, $\frac{du}{dx} = -\frac{1}{x^2}$, $dx = -x^2\,du$

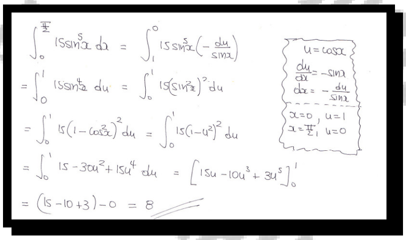
$-1-u$	-1
e^u	e^u

Question 150 **(****)**

By using the substitution $u = \cos x$, or otherwise, show clearly that

$$\int_0^{\frac{\pi}{2}} 15\cos^5 x \; dx = 8 .$$

proof



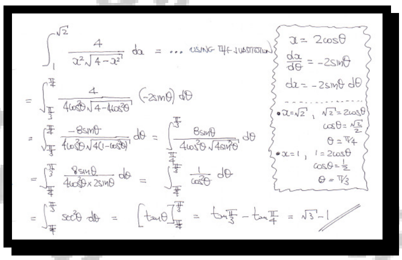
Question 151 **(****)**

Use the substitution $x = 2\cos\theta$ to show that

$$\int_1^{\sqrt{2}} \frac{4}{x^2\sqrt{4-x^2}} \; dx = \sqrt{m} - n ,$$

where m and n are integers.

proof



Question 152 (****)

$$y = \frac{4x+3}{3x+4}, \quad x \neq -\frac{4}{3}.$$

a) Calculate the five missing values of x and y in the following table.

$\boldsymbol{x}$	0				32
$\boldsymbol{y}$	$\frac{3}{4}$	$\frac{35}{29}$	$\frac{67}{52}$		

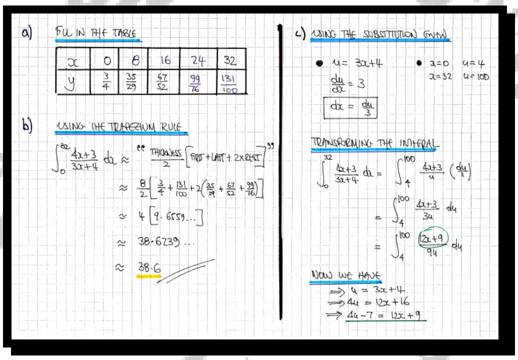
b) Use the trapezium rule with all the values from the completed table of part (**a**) to find an estimate for

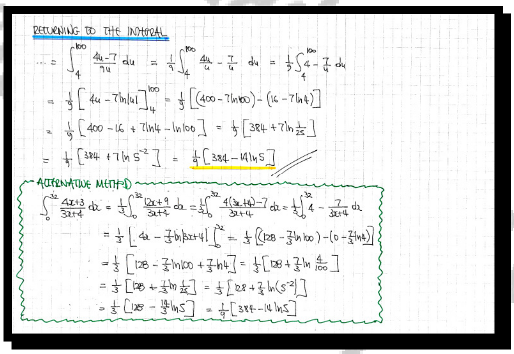
$$\int_0^{32} \frac{4x+3}{3x+4}\, dx.$$

c) Use the substitution $u = 3x+4$ to find the exact value of

$$\int_0^{32} \frac{4x+3}{3x+4}\, dx.$$

$\boxed{}$, $\boxed{8, 16, 24 \quad \frac{99}{76}, \frac{131}{100}}$, $\boxed{38.6}$, $\boxed{\frac{128}{3} - \frac{14}{9}\ln 5 \;=\; \frac{1}{9}\left[384 - 14\ln 5\right]}$





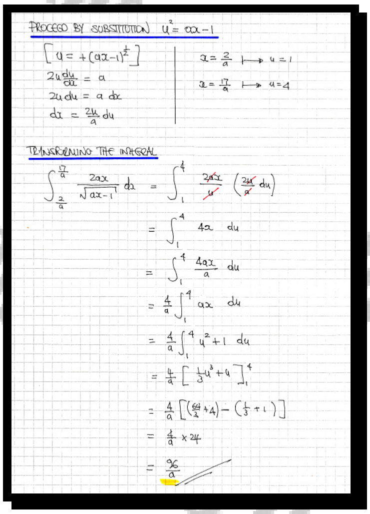
Question 153 **(****)**

Determine, in terms of a, the value of the following integral.

$$\int_{\frac{2}{a}}^{\frac{17}{a}} \frac{2ax}{\sqrt{ax-1}}\, dx\ ,\ a \neq 0\, .$$

You may find the substitution $u^2 = ax - 1$ useful in this question.

$\boxed{}$, $\boxed{\dfrac{96}{a}}$


PROCEED BY SUBSTITUTION $u^2 = ax - 1$

$\left[u = +(ax-1)^{\frac{1}{2}}\right]$

$2u\frac{du}{dx} = a$

$2u\,du = a\,dx$

$dx = \frac{2u}{a}\,du$

$x = \frac{2}{a} \longmapsto u = 1$

$x = \frac{17}{a} \longmapsto u = 4$

TRANSFORMING THE INTEGRAL

$$\int_{\frac{2}{a}}^{\frac{17}{a}} \frac{2ax}{\sqrt{ax-1}}\,dx = \int_1^4 \frac{2ax}{u}\left(\frac{2u}{a}\,du\right)$$

$$= \int_1^4 4x\,du$$

$$= \int_1^4 \frac{4ax}{a}\,du$$

$$= \frac{4}{a}\int_1^4 ax\,du$$

$$= \frac{4}{a}\int_1^4 u^2+1\,du$$

$$= \frac{4}{a}\left[\frac{1}{3}u^3+u\right]_1^4$$

$$= \frac{4}{a}\left[\left(\frac{64}{3}+4\right)-\left(\frac{1}{3}+1\right)\right]$$

$$= \frac{4}{a}\times 24$$

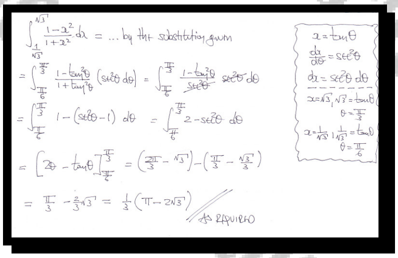
$$= \frac{96}{a}$$

Question 154 (****)

Use the substitution $x = \tan\theta$ to show that

$$\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1-x^2}{1+x^2}\ dx = \frac{1}{3}\left(\pi - 2\sqrt{3}\right).$$

proof

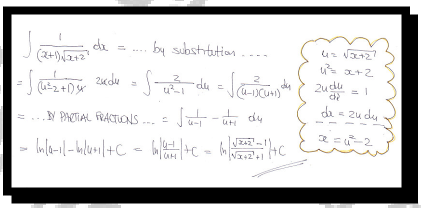


Question 155 (****)

By using the substitution $u = \sqrt{x+2}$, or otherwise, find an expression for

$$\int \frac{1}{(x+1)\sqrt{x+2}}\ dx.$$

$$\ln\left|\frac{\sqrt{x+2}-1}{\sqrt{x+2}+1}\right| + C$$



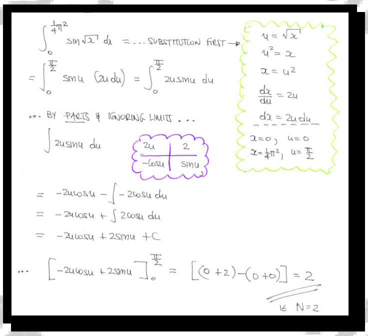
Question 156 (****)

Use appropriate integration techniques to show that

$$\int_0^{\frac{1}{4}\pi^2} \sin\sqrt{x}\ dx = N ,$$

where N is a positive integer.

$\boxed{}$, $\boxed{N=2}$

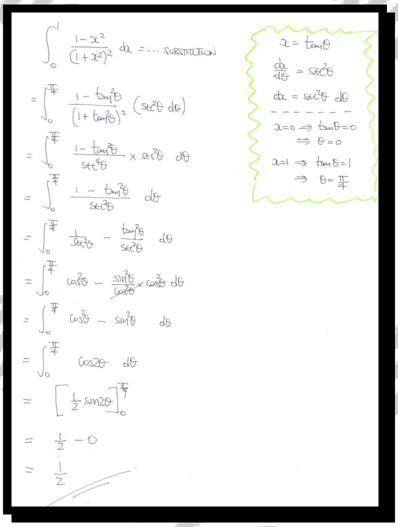


Question 157 (****)

By using the substitution $x = \tan\theta$, or otherwise, find the value of

$$\int_0^1 \frac{1-x^2}{\left(1-x^2\right)^2}\, dx .$$

$\boxed{\frac{1}{2}}$



$$\int_0^1 \frac{1-x^2}{(1+x^2)^2}\,dx = \dots \text{SUBSTITUTION}$$

$x = \tan\theta$

$\frac{dx}{d\theta} = \sec^2\theta$

$dx = \sec^2\theta \, d\theta$

$x=0 \Rightarrow \tan\theta = 0 \Rightarrow \theta = 0$

$x=1 \Rightarrow \tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$

$$= \int_0^{\frac{\pi}{4}} \frac{1-\tan^2\theta}{(1+\tan^2\theta)^2}\left(\sec^2\theta\, d\theta\right)$$

$$= \int_0^{\frac{\pi}{4}} \frac{1-\tan^2\theta}{\sec^4\theta} \times \sec^2\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{1-\tan^2\theta}{\sec^2\theta}\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{\sec^2\theta} - \frac{\tan^2\theta}{\sec^2\theta}\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \cos^2\theta - \frac{\sin^2\theta}{\cos^2\theta} \times \cos^2\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \cos^2\theta - \sin^2\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \cos 2\theta \, d\theta$$

$$= \left[\tfrac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{4}}$$

$$= \tfrac{1}{2} - 0$$

$$= \tfrac{1}{2}$$

Question 158 (****)

$$y = \left(1+\cot^2 x\right)\sec^2 x\,, \quad 0 < x < \tfrac{1}{2}\pi\,.$$

a) Calculate the three missing values of x in the following table.

x	$\frac{1}{6}\pi$				$\frac{1}{3}\pi$
y	$\frac{16}{3}$	$32-16\sqrt{3}$	4	$32-16\sqrt{3}$	$\frac{16}{3}$

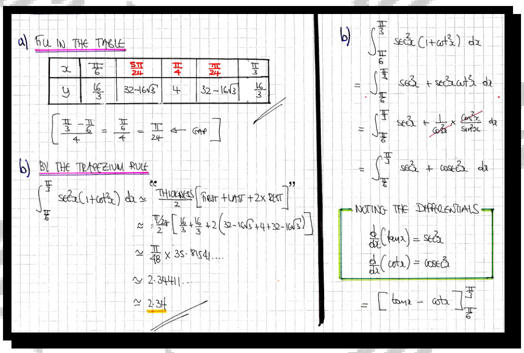
b) Use the trapezium rule with all the values from the completed table of part **(a)** to find an estimate for

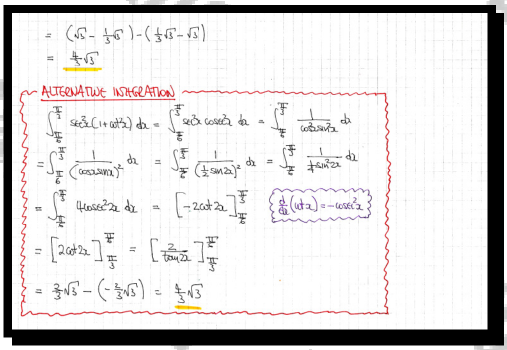
$$\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \left(1+\cot^2 x\right)\sec^2 x \; dx\,.$$

c) Use an appropriate integration method to find an exact simplified value for

$$\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \left(1+\cot^2 x\right)\sec^2 x \; dx\,.$$

$\boxed{\frac{5\pi}{24}, \frac{\pi}{4}, \frac{7\pi}{24}}$, $\boxed{2.34}$, $\boxed{\frac{4}{3}\sqrt{3}}$





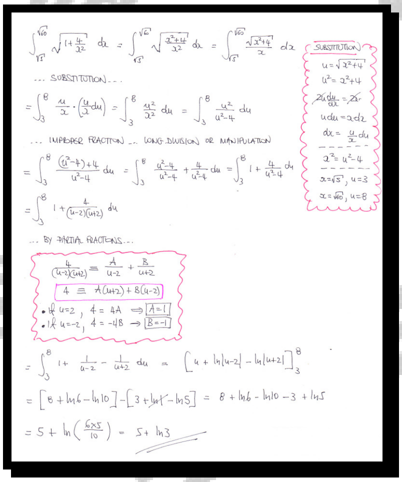
Question 159 (****+)

Use appropriate integration techniques to evaluate

$$\int_{\sqrt{5}}^{\sqrt{60}} \sqrt{1+\frac{4}{x^2}}\, dx .$$

Give the answer in the form $a+b\ln 3$, where a and b are positive integers.

$\boxed{5+\ln 3}$

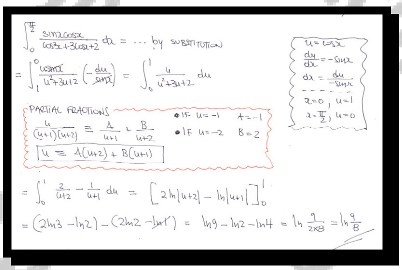


Question 160 (****+)

Use a suitable substitution to show that

$$\int_0^{\frac{\pi}{2}} \frac{\sin x \cos x}{\cos^2 x + 3\cos x + 2}\, dx = \ln\left(\frac{9}{8}\right).$$

$\boxed{\text{proof}}$

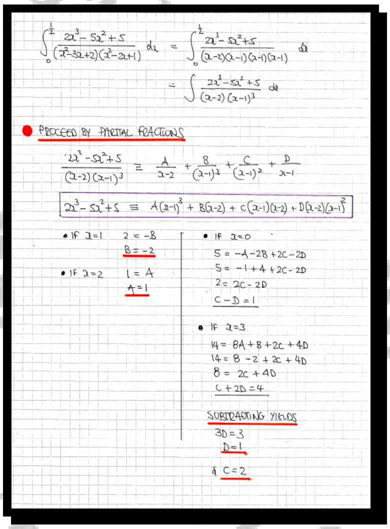


Question 161 (****)

Use partial fractions to determine, in exact simplified form, the value of the following integral.

$$\int_0^{\frac{1}{2}} \frac{2x^3-5x^2+5}{\left(x^2-3x+2\right)\left(x^2-2x+1\right)}\ dx .$$

$\boxed{}$, $\boxed{5+\ln\left(\frac{3}{8}\right)}$



$$\int_0^{\frac{1}{2}} \frac{2x^3-5x^2+5}{(x^2-3x+2)(x^2-2x+1)}\,dx = \int_0^{\frac{1}{2}} \frac{2x^3-5x^2+5}{(x-2)(x-1)(x-1)(x-1)}\,dx = \int \frac{2x^3-5x^2+5}{(x-2)(x-1)^3}\,dx$$

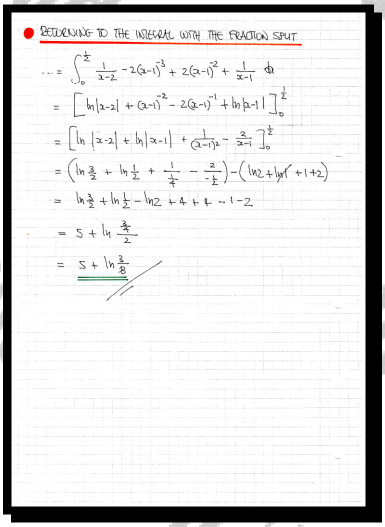
PROCEED BY PARTIAL FRACTIONS

$$\frac{2x^3-5x^2+5}{(x-2)(x-1)^3} \equiv \frac{A}{x-2} + \frac{B}{(x-1)^3} + \frac{C}{(x-1)^2} + \frac{D}{x-1}$$

$$2x^3-5x^2+5 \equiv A(x-1)^3 + B(x-2) + C(x-1)(x-2) + D(x-2)(x-1)^2$$

- IF $x=1$: $2=-B$, $B=-2$
- IF $x=2$: $1=A$, $A=1$
- IF $x=0$: $5=-A-2B+2C-2D$, $5=-1+4+2C-2D$, $2=2C-2D$, $C-D=1$
- IF $x=3$: $14=8A+B+2C+4D$, $14=8-2+2C+4D$, $8=2C+4D$, $C+2D=4$

SUBTRACTING YIELDS: $3D=3$, $D=1$ & $C=2$



RETURNING TO THE INTEGRAL WITH THE FRACTION SPLIT

$$\ldots = \int_0^{\frac{1}{2}} \frac{1}{x-2} - 2(x-1)^{-3} + 2(x-1)^{-2} + \frac{1}{x-1}\,dx$$

$$= \Big[\ln|x-2| + (x-1)^{-2} - 2(x-1)^{-1} + \ln|x-1|\Big]_0^{\frac{1}{2}}$$

$$= \Big[\ln|x-2| + \ln|x-1| + \frac{1}{(x-1)^2} - \frac{2}{x-1}\Big]_0^{\frac{1}{2}}$$

$$= \left(\ln\frac{3}{2} + \ln\frac{1}{2} + \frac{1}{\frac{1}{4}} - \frac{2}{-\frac{1}{2}}\right) - \left(\ln 2 + \ln 1 + 1 + 2\right)$$

$$= \ln\frac{3}{2} + \ln\frac{1}{2} - \ln 2 + 4 + 4 - 1 - 2$$

$$= 5 + \ln\frac{\frac{3}{4}}{2}$$

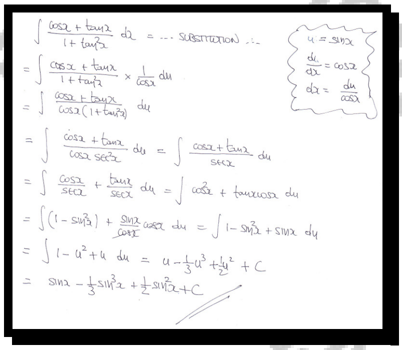
$$= 5 + \ln\frac{3}{8}$$

Question 162 (****)

Use the substitution $u = \sin x$ to find an expression for

$$\int \frac{\cos x + \tan x}{1 + \tan^2 x}\, dx .$$

$$\boxed{\sin x - \tfrac{1}{3}\sin^3 x + \tfrac{1}{2}\sin^2 x + C}$$

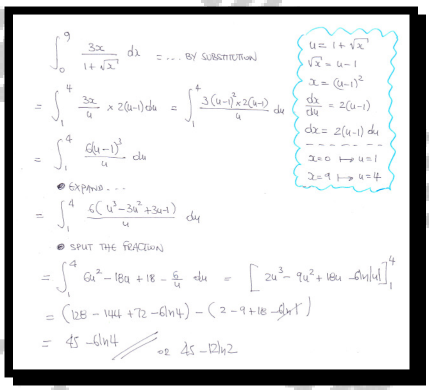


Question 163 (****+)

Use the substitution $u = 1 + \sqrt{x}$ to evaluate

$$\int_0^9 \frac{3x}{1+\sqrt{x}}\, dx .$$

$$\boxed{45 - 12\ln 2}$$



Question 164 (****+)

$$y = \frac{x^2}{2x+1}, \quad x \neq -\frac{1}{2}$$

a) Calculate the two missing values of y in the following table.

x	0	0.1	0.2	0.3	0.4	0.5
y	0	$\frac{1}{120}$	$\frac{1}{35}$			$\frac{1}{8}$

b) Use the trapezium rule with all the values from the completed table of part **(a)** to find an estimate, correct to 4 significant figures, for the following integral.

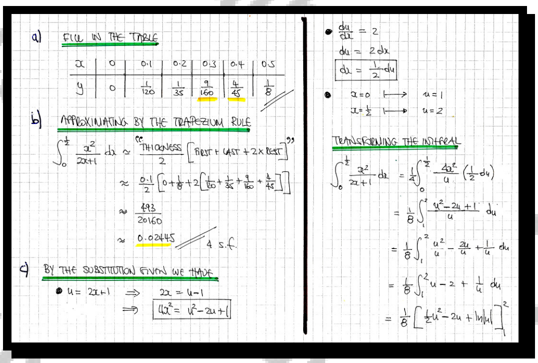
$$\int_0^{\frac{1}{2}} \frac{x^2}{2x+1}\, dx .$$

d) Use the substitution $u = 2x+1$ to find an exact simplified value for

$$\int_0^{\frac{1}{2}} \frac{x^2}{2x+1}\, dx .$$

e) Hence deduce, by referring to parts **(b) and (c)**, the approximate value of $\ln 2$ correct to 2 significant figures.

$\boxed{}$, $\boxed{\frac{9}{160}, \frac{4}{45}}$, $\boxed{0.02445}$, $\boxed{\frac{1}{16}[-1+2\ln 2]}$, $\boxed{\ln 2 \approx 0.70}$


a) FILL IN THE TABLE

x	0	0.1	0.2	0.3	0.4	0.5
y	0	$\frac{1}{120}$	$\frac{1}{35}$	$\frac{9}{160}$	$\frac{4}{45}$	$\frac{1}{8}$

b) APPROXIMATING BY THE TRAPEZIUM RULE

$\int_0^{\frac{1}{2}} \frac{x^2}{2x+1}\,dx \approx$ "$\frac{\text{THICKNESS}}{2}$ [FIRST + LAST + 2×REST]"

$\approx \frac{0.1}{2}\left[0+\frac{1}{8}+2\left[\frac{1}{120}+\frac{1}{35}+\frac{9}{160}+\frac{4}{45}\right]\right]$

$\approx \frac{493}{20160}$

≈ 0.02445 4 s.f.

c) BY THE SUBSTITUTION GIVEN WE HAVE

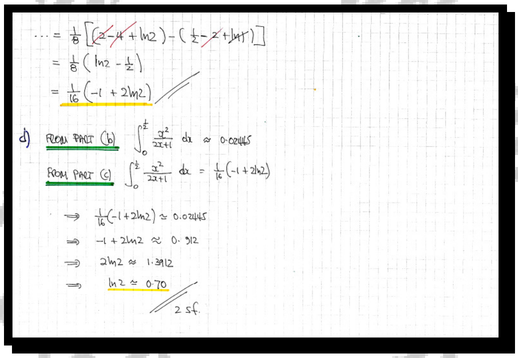
• $u = 2x+1 \Rightarrow 2x = u-1 \Rightarrow 4x^2 = u^2-2u+1$

• $\frac{du}{dx} = 2$, $du = 2\,dx$, $dx = \frac{1}{2}du$

• $x=0 \mapsto u=1$, $x=\frac{1}{2} \mapsto u=2$

TRANSFORMING THE INTEGRAL

$\int_0^{\frac{1}{2}} \frac{x^2}{2x+1}\,dx = \int_1^2 \frac{\frac{4x^2}{4}}{u}\left(\frac{1}{2}du\right) = \frac{1}{8}\int_1^2 \frac{u^2-2u+1}{u}\,du = \frac{1}{8}\int_1^2 \frac{u^2}{u}-\frac{2u}{u}+\frac{1}{u}\,du = \frac{1}{8}\int_1^2 u-2+\frac{1}{u}\,du = \frac{1}{8}\left[\frac{1}{2}u^2-2u+\ln|u|\right]_1^2$


$\dots = \frac{1}{8}\left[(2-4+\ln 2)-(\frac{1}{2}-2+\ln 1)\right] = \frac{1}{8}(\ln 2-\frac{1}{2}) = \frac{1}{16}(-1+2\ln 2)$

d) FROM PART (b) $\int_0^{\frac{1}{2}} \frac{x^2}{2x+1}\,dx \approx 0.02445$

FROM PART (c) $\int_0^{\frac{1}{2}} \frac{x^2}{2x+1}\,dx = \frac{1}{16}(-1+2\ln 2)$

$\Rightarrow \frac{1}{16}(-1+2\ln 2) \approx 0.02445$

$\Rightarrow -1+2\ln 2 \approx 0.312$

$\Rightarrow 2\ln 2 \approx 1.3912$

$\Rightarrow \ln 2 \approx 0.70$ 2 s.f.

Question 165 (****+)

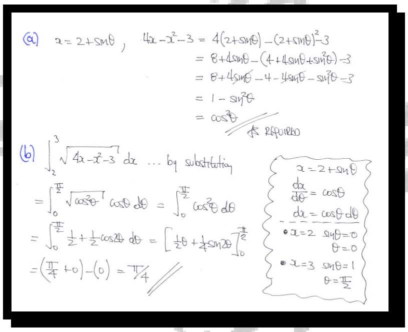
$$f(x) = -x^2 + 4x - 3, \; 1 \le x \le 3.$$

a) Show clearly that $f(2+\sin\theta) = \cos^2\theta$.

b) Hence find the exact value of

$$\int_2^3 \sqrt{f(x)} \, dx.$$

$\boxed{\dfrac{\pi}{4}}$



Question 166 (****+)

$$f(x) = \ln\left(\frac{1+x}{1-x}\right), \ |x| < 1.$$

a) Show that $f(x)$ is an odd function.

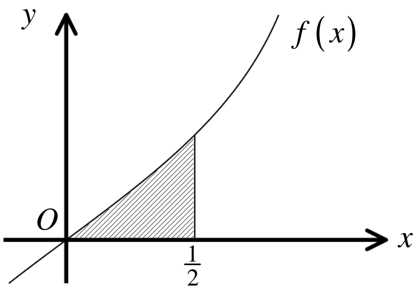
b) Find an expression for $f'(x)$ as a single simplified fraction, showing further that $f'(x)$ is an even function.

c) Determine an expression for $f^{-1}(x)$.

[continues overleaf]

[continued from overleaf]

The figure below shows part of the graph of $f(x)$.

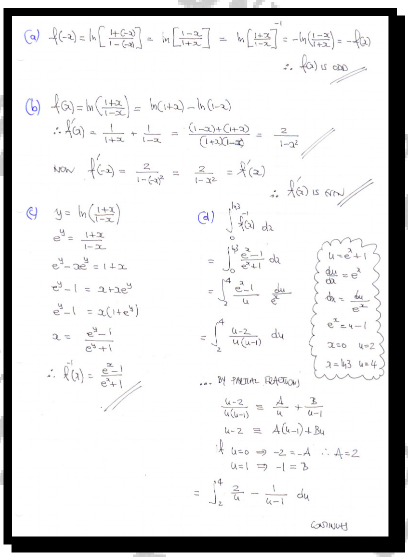


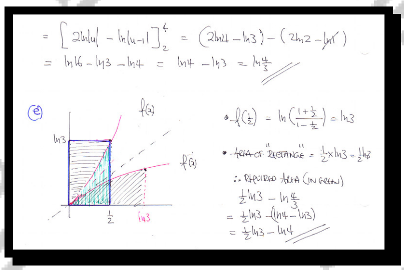
d) Use the substitution $u = e^x + 1$ to find the exact value of

$$\int_0^{\ln 3} f^{-1}(x)\ dx .$$

e) Hence find an exact value for the area of the shaded region, bounded by $f(x)$, the coordinate axes and the line $x = \frac{1}{2}$.

$\boxed{f'(x) = \frac{2}{1-x^2}}$, $\boxed{f'(x) = \frac{e^x - 1}{e^x + 1}}$, $\boxed{\ln\left(\frac{4}{3}\right)}$, $\boxed{\text{area} = \frac{1}{2}\ln 3 - 2\ln 2 \approx 0.262}$





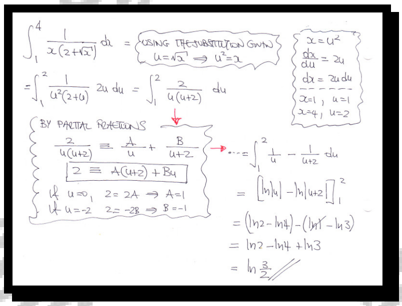
Question 167 **(****+)**

By using the substitution $u = \sqrt{x}$ find

$$\int_1^4 \frac{1}{x\left(2+\sqrt{x}\right)} \; dx \, ,$$

giving the answer as an exact single natural logarithm.

$\ln\left(\frac{3}{2}\right)$

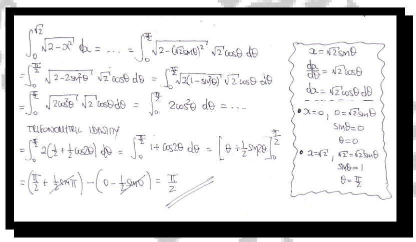


Question 168 **(****+)**

Use the substitution $x = \sqrt{2}\sin\theta$ to show that

$$\int_0^{\sqrt{2}} \sqrt{2-x^2} \;\; dx = \frac{\pi}{2}.$$

proof



Question 169 (****+)

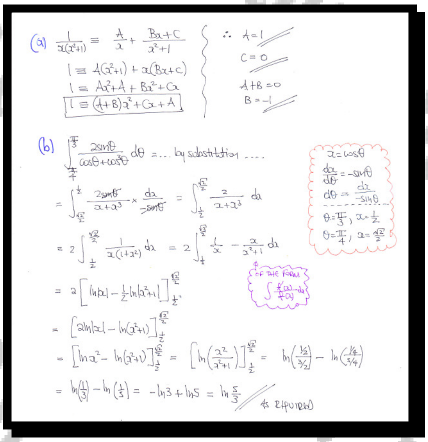
$$\frac{1}{x\left(x^2+1\right)} \equiv \frac{A}{x} + \frac{Bx+C}{x^2+1}.$$

a) Find the value of each of the constants A, B and C.

b) Use the substitution $x = \cos\theta$ to show

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{2\sin\theta}{\cos\theta + \cos^3\theta}\, d\theta = \ln\left(\frac{5}{3}\right).$$

$\boxed{A=1}$, $\boxed{B=-1}$, $\boxed{C=0}$

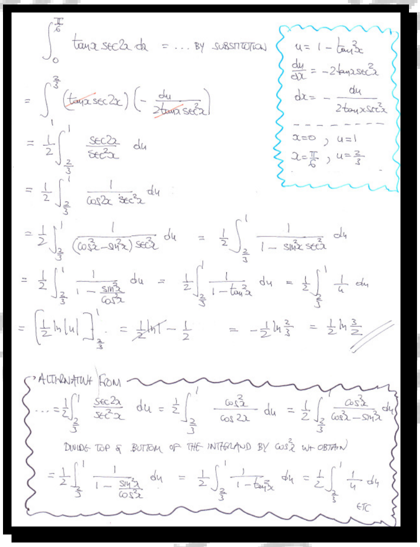


Question 170 (****+)

By using the substitution $u = 1 - \tan^2 x$, or otherwise, find the exact value of

$$\int_0^{\frac{\pi}{6}} \tan x \sec 2x \; dx .$$

$$\boxed{\frac{1}{2}\ln\left(\frac{3}{2}\right)}$$



$\int_0^{\frac{\pi}{6}} \tan x \sec 2x \, dx = \dots$ BY SUBSTITUTION

$u = 1 - \tan^2 x$

$\frac{du}{dx} = -2\tan x \sec^2 x$

$dx = -\frac{du}{2\tan x \sec^2 x}$

$x = 0, \; u = 1$

$x = \frac{\pi}{6}, \; u = \frac{2}{3}$

$= \int_1^{\frac{2}{3}} (\tan x \sec 2x)\left(-\frac{du}{2\tan x \sec^2 x}\right)$

$= \frac{1}{2}\int_{\frac{2}{3}}^{1} \frac{\sec 2x}{\sec^2 x} \, du$

$= \frac{1}{2}\int_{\frac{2}{3}}^{1} \frac{1}{\cos 2x \sec^2 x} \, du$

$= \frac{1}{2}\int_{\frac{2}{3}}^{1} \frac{1}{(\cos^2 x - \sin^2 x)\sec^2 x} \, du = \frac{1}{2}\int_{\frac{2}{3}}^{1} \frac{1}{1 - \sin^2 x \sec^2 x} \, du$

$= \frac{1}{2}\int_{\frac{2}{3}}^{1} \frac{1}{1 - \frac{\sin^2 x}{\cos^2 x}} \, du = \frac{1}{2}\int_{\frac{2}{3}}^{1} \frac{1}{1 - \tan^2 x} \, du = \frac{1}{2}\int_{\frac{2}{3}}^{1} \frac{1}{u} \, du$

$= \left[\frac{1}{2}\ln|u|\right]_{\frac{2}{3}}^{1} = \frac{1}{2}\ln 1 - \frac{1}{2} \dots = -\frac{1}{2}\ln\frac{2}{3} = \frac{1}{2}\ln\frac{3}{2}$

ALTERNATIVE FROM

$\dots = \frac{1}{2}\int_{\frac{2}{3}}^{1} \frac{\sec 2x}{\sec^2 x} \, du = \frac{1}{2}\int_{\frac{2}{3}}^{1} \frac{\cos^2 x}{\cos 2x} \, du = \frac{1}{2}\int_{\frac{2}{3}}^{1} \frac{\cos^2 x}{\cos^2 x - \sin^2 x} \, du$

DIVIDE TOP & BOTTOM OF THE INTEGRAND BY $\cos^2 x$ WE OBTAIN

$= \frac{1}{2}\int_{\frac{2}{3}}^{1} \frac{1}{1 - \frac{\sin^2 x}{\cos^2 x}} \, du = \frac{1}{2}\int_{\frac{2}{3}}^{1} \frac{1}{1 - \tan^2 x} \, du = \frac{1}{2}\int_{\frac{2}{3}}^{1} \frac{1}{u} \, du$

ETC

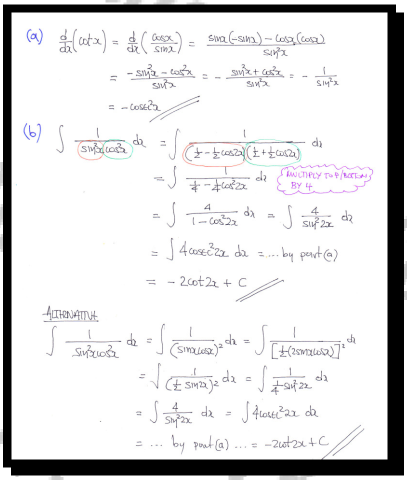
Question 171 (****+)

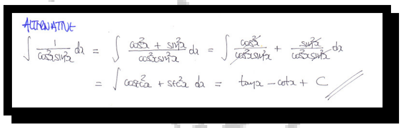
a) Show clearly that $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$.

b) Use trigonometric identities to find

$$\int \frac{1}{\sin^2 x \cos^2 x} \, dx .$$

$\boxed{-2\cot 2x + C}$



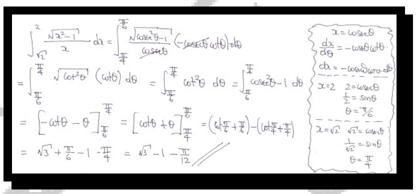


Question 172 (****+)

Use the substitution $x = \text{cosec}\,\theta$ to find the exact value of

$$\int_{\sqrt{2}}^{2} \frac{\sqrt{x^2-1}}{x} \, dx .$$

$\boxed{\sqrt{3} - 1 - \frac{\pi}{12}}$



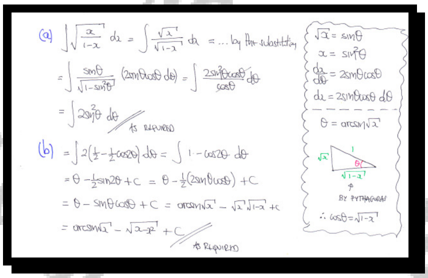
Question 173 **(****+)**

a) Write down an expression for $\frac{d}{dx}\left(e^{\cos x}\right)$.

b) By using integration by parts, or otherwise, show that

$$\int e^{\cos x}\cos x \sin x \, dx = e^{x}\left(1-\cos x\right)+\text{constant} .$$

$$\boxed{\frac{d}{dx}\left(e^{\cos x}\right)=-e^{\cos x}\sin x}$$

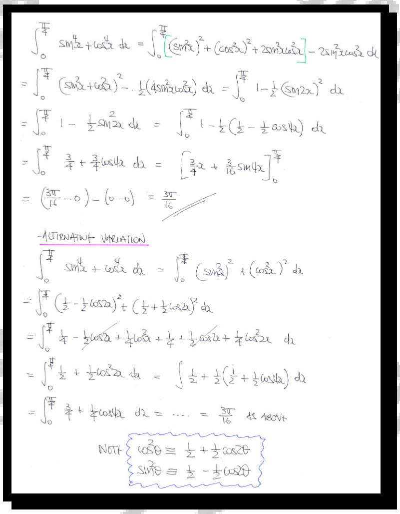


Question 174 (****+)

By using trigonometric identities, show that

$$\int_0^{\frac{\pi}{4}} \sin^4 x + \cos^4 x \ \ dx = \frac{3\pi}{16}.$$

proof



$$\int_0^{\frac{\pi}{4}} \sin^4 x + \cos^4 x \, dx = \int_0^{\frac{\pi}{4}} \left[(\sin^2 x)^2 + (\cos^2 x)^2 + 2\sin^2 x\cos^2 x\right] - 2\sin^2 x\cos^2 x \, dx$$

$$= \int_0^{\frac{\pi}{4}} (\sin^2 x + \cos^2 x)^2 - \tfrac{1}{2}(4\sin^2 x\cos^2 x)\, dx = \int_0^{\frac{\pi}{4}} 1 - \tfrac{1}{2}(\sin 2x)^2 \, dx$$

$$= \int_0^{\frac{\pi}{4}} 1 - \tfrac{1}{2}\sin^2 2x \, dx = \int_0^{\frac{\pi}{4}} 1 - \tfrac{1}{2}\left(\tfrac{1}{2} - \tfrac{1}{2}\cos 4x\right) dx$$

$$= \int_0^{\frac{\pi}{4}} \tfrac{3}{4} + \tfrac{3}{4}\cos 4x \, dx = \left[\tfrac{3}{4}x + \tfrac{3}{16}\sin 4x\right]_0^{\frac{\pi}{4}}$$

$$= \left(\tfrac{3\pi}{16} - 0\right) - (0 - 0) = \tfrac{3\pi}{16}$$

ALTERNATIVE VARIATION

$$\int_0^{\frac{\pi}{4}} \sin^4 x + \cos^4 x \, dx = \int_0^{\frac{\pi}{4}} (\sin^2 x)^2 + (\cos^2 x)^2 \, dx$$

$$= \int_0^{\frac{\pi}{4}} \left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2x\right)^2 + \left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2x\right)^2 dx$$

$$= \int_0^{\frac{\pi}{4}} \tfrac{1}{4} - \tfrac{1}{2}\cos 2x + \tfrac{1}{4}\cos^2 2x + \tfrac{1}{4} + \tfrac{1}{2}\cos 2x + \tfrac{1}{4}\cos^2 2x \, dx$$

$$= \int_0^{\frac{\pi}{4}} \tfrac{1}{2} + \tfrac{1}{2}\cos^2 2x \, dx = \int \tfrac{1}{2} + \tfrac{1}{2}\left(\tfrac{1}{2} + \tfrac{1}{2}\cos 4x\right) dx$$

$$= \int_0^{\frac{\pi}{4}} \tfrac{3}{4} + \tfrac{1}{4}\cos 4x \, dx = \dots = \tfrac{3\pi}{16} \text{ AS ABOVE}$$

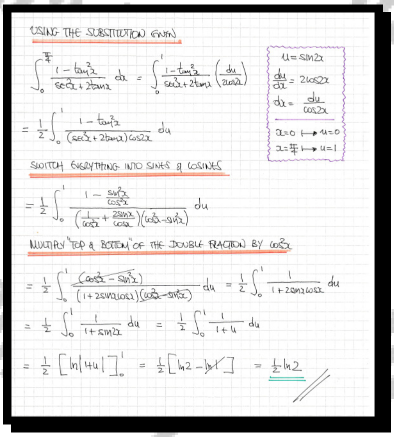
NOTE
$\cos^2\theta \equiv \tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta$
$\sin^2\theta \equiv \tfrac{1}{2} - \tfrac{1}{2}\cos 2\theta$

Question 175 (****+)

By using the substitution $u = \sin 2x$, or otherwise, find an exact simplified value for the following trigonometric integral.

$$\int_0^{\frac{1}{4}\pi} \frac{1-\tan^2 x}{\sec^2 x + 2\tan x}\, dx .$$

$\boxed{}$, $\boxed{\frac{1}{2}\ln 2}$

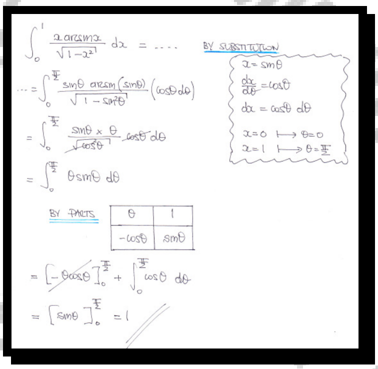


Question 176 (****+)

By using a suitable substitution, or otherwise, find the value of

$$\int_0^{1} \frac{x \arcsin x}{\sqrt{1-x^2}}\, dx .$$

$\boxed{1}$



Question 177 (****+)

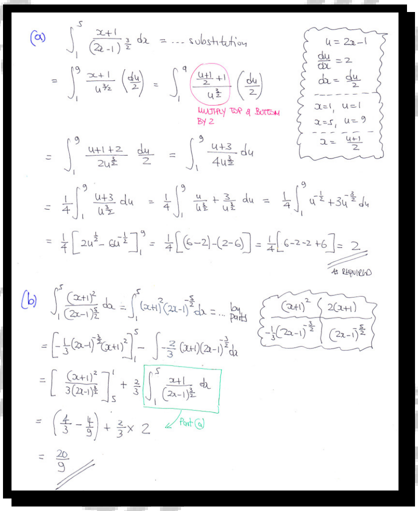
a) Use the substitution $u = 2x - 1$ to show that

$$\int_{1}^{5} \frac{x+1}{(2x-1)^{\frac{3}{2}}} \, dx = 2 .$$

b) By using integration by parts and the result of part **(a)**, find the value of

$$\int_{1}^{5} \frac{(x+1)^2}{(2x-1)^{\frac{5}{2}}} \, dx .$$

$\boxed{\frac{20}{9}}$

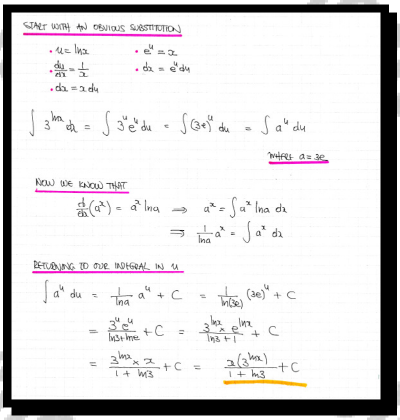


Question 178 (****+)

Use the substitution $u = \ln x$ to show that

$$\int 3^{\ln x}\, dx = \frac{x\left(3^{\ln x}\right)}{1+\ln 3} + \text{constant}.$$

V, N-W, proof


START WITH AN OBVIOUS SUBSTITUTION

- $u = \ln x$
- $\frac{du}{dx} = \frac{1}{x}$
- $dx = x\,du$
- $e^u = x$
- $dx = e^u\,du$

$$\int 3^{\ln x}\,dx = \int 3^u e^u\,du = \int (3e)^u\,du = \int a^u\,du$$

WHERE $a = 3e$

NOW WE KNOW THAT

$$\frac{d}{dx}(a^x) = a^x \ln a \implies a^x = \int a^x \ln a\,dx$$

$$\implies \frac{1}{\ln a}a^x = \int a^x\,dx$$

RETURNING TO OUR INTEGRAL IN u

$$\int a^u\,du = \frac{1}{\ln a}a^u + C = \frac{1}{\ln(3e)}(3e)^u + C$$

$$= \frac{3^u e^u}{\ln 3 + \ln e} + C = \frac{3^{\ln x} \times e^{\ln x}}{\ln 3 + 1} + C$$

$$= \frac{3^{\ln x} \times x}{1 + \ln 3} + C = \frac{x\left(3^{\ln x}\right)}{1 + \ln 3} + C$$

Question 179 (****+)

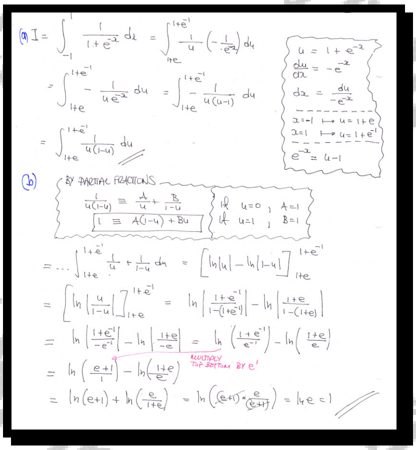
$$J = \int_{-1}^{1} \frac{1}{1+e^{-x}} \, dx \, .$$

a) Show that the substitution $u = 1 + e^{-x}$ transforms J into

$$\int_{1+e}^{1+e^{-1}} \frac{1}{u(1-u)} \, du \, .$$

b) By expressing $\dfrac{1}{u(1-u)}$ into partial fractions show clearly that $J = 1$.

proof



(a) $I = \int_{-1}^{1} \frac{1}{1+e^{-x}}\,dx = \int_{1+e}^{1+e^{-1}} \frac{1}{u}\left(-\frac{1}{e^{-x}}\right) du = \int_{1+e}^{1+e^{-1}} -\frac{1}{ue^{-x}}\,du = \int_{1+e}^{1+e^{-1}} -\frac{1}{u(u-1)}\,du = \int_{1+e}^{1+e^{-1}} \frac{1}{u(1-u)}\,du$

$u = 1+e^{-x}$, $\frac{du}{dx} = -e^{-x}$, $dx = \frac{du}{-e^{-x}}$, $x=-1 \mapsto u = 1+e$, $x = 1 \mapsto u = 1+e^{-1}$, $e^{-x} = u-1$

(b) BY PARTIAL FRACTIONS

$\frac{1}{u(1-u)} \equiv \frac{A}{u} + \frac{B}{1-u}$, $1 \equiv A(1-u) + Bu$; if $u=0$, $A=1$; if $u=1$, $B=1$

$= \dots \int_{1+e}^{1+e^{-1}} \frac{1}{u} + \frac{1}{1-u}\,du = \Big[\ln|u| - \ln|1-u|\Big]_{1+e}^{1+e^{-1}}$

$= \left[\ln\left|\frac{u}{1-u}\right|\right]_{1+e}^{1+e^{-1}} = \ln\left|\frac{1+e^{-1}}{1-(1+e^{-1})}\right| - \ln\left|\frac{1+e}{1-(1+e)}\right|$

$= \ln\left|\frac{1+e^{-1}}{-e^{-1}}\right| - \ln\left|\frac{1+e}{-e}\right| = \ln\left(\frac{1+e^{-1}}{e^{-1}}\right) - \ln\left(\frac{1+e}{e}\right)$

$= \ln\left(\frac{e+1}{1}\right) - \ln\left(\frac{1+e}{e}\right)$ MULTIPLY THE BOTTOM BY e^{1}

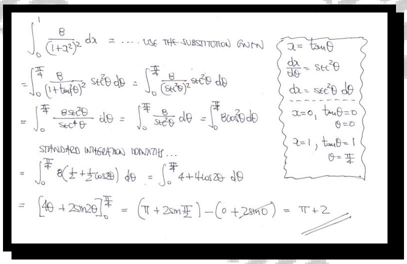
$= \ln(e+1) + \ln\left(\frac{e}{1+e}\right) = \ln\left((e+1)\times\frac{e}{(e+1)}\right) = \ln e = 1$

Question 180 (****+)

Use the substitution $x = \tan\theta$ to find the exact value of

$$\int_0^1 \frac{8}{\left(1+x^2\right)^2}\ dx.$$

, $\boxed{\pi + 2}$



$\int_0^1 \frac{8}{(1+x^2)^2}\,dx = \dots$ USE THE SUBSTITUTION GIVEN

$= \int_0^{\frac{\pi}{4}} \frac{8}{(1+\tan^2\theta)^2}\sec^2\theta\,d\theta = \int_0^{\frac{\pi}{4}} \frac{8}{(\sec^2\theta)^2}\sec^2\theta\,d\theta$

$= \int_0^{\frac{\pi}{4}} \frac{8\sec^2\theta}{\sec^4\theta}\,d\theta = \int_0^{\frac{\pi}{4}} \frac{8}{\sec^2\theta}\,d\theta = \int_0^{\frac{\pi}{4}} 8\cos^2\theta\,d\theta$

STANDARD INTEGRATION IDENTITIES …

$= \int_0^{\frac{\pi}{4}} 8\left(\frac{1}{2}+\frac{1}{2}\cos 2\theta\right)d\theta = \int_0^{\frac{\pi}{4}} 4+4\cos 2\theta\,d\theta$

$= \left[4\theta + 2\sin 2\theta\right]_0^{\frac{\pi}{4}} = \left(\pi + 2\sin\frac{\pi}{2}\right) - \left(0 + 2\sin 0\right) = \pi + 2$

$x = \tan\theta$, $\frac{dx}{d\theta} = \sec^2\theta$, $dx = \sec^2\theta\,d\theta$

$x=0$, $\tan\theta = 0$, $\theta = 0$

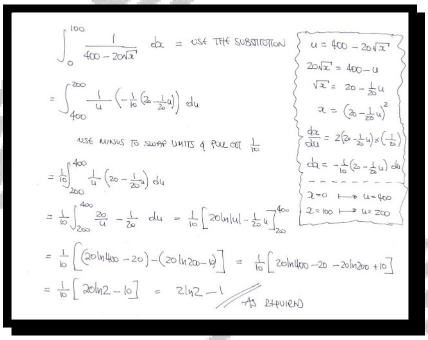
$x=1$, $\tan\theta = 1$, $\theta = \frac{\pi}{4}$

Question 181 (****+)

Use the substitution $u = 400 - 20\sqrt{x}$ to show that

$$\int_0^{100} \frac{1}{400-20\sqrt{x}}\ dx = -1 + 2\ln 2.$$

proof



$\int_0^{100} \frac{1}{400-20\sqrt{x}}\,dx =$ USE THE SUBSTITUTION

$= \int_{400}^{200} \frac{1}{u}\left(-\frac{1}{10}\left(20-\frac{1}{20}u\right)\right)du$

USE MINUS TO SWAP LIMITS & PULL OUT $\frac{1}{10}$

$= \frac{1}{10}\int_{200}^{400} \frac{1}{u}\left(20-\frac{1}{20}u\right)du$

$= \frac{1}{10}\int_{200}^{400} \frac{20}{u} - \frac{1}{20}\,du = \frac{1}{10}\left[20\ln|u| - \frac{1}{20}u\right]_{200}^{400}$

$= \frac{1}{10}\left[(20\ln 400 - 20) - (20\ln 200 - 10)\right] = \frac{1}{10}\left[20\ln 400 - 20 - 20\ln 200 + 10\right]$

$= \frac{1}{10}\left[20\ln 2 - 10\right] = 2\ln 2 - 1$ AS REQUIRED

$u = 400 - 20\sqrt{x}$, $20\sqrt{x} = 400 - u$, $\sqrt{x} = 20 - \frac{1}{20}u$, $x = \left(20 - \frac{1}{20}u\right)^2$

$\frac{dx}{du} = 2\left(20 - \frac{1}{20}u\right)\times\left(-\frac{1}{20}\right)$, $dx = -\frac{1}{10}\left(20-\frac{1}{20}u\right)du$

$x = 0 \mapsto u = 400$

$x = 100 \mapsto u = 200$

Question 182 (****+)

It is given that

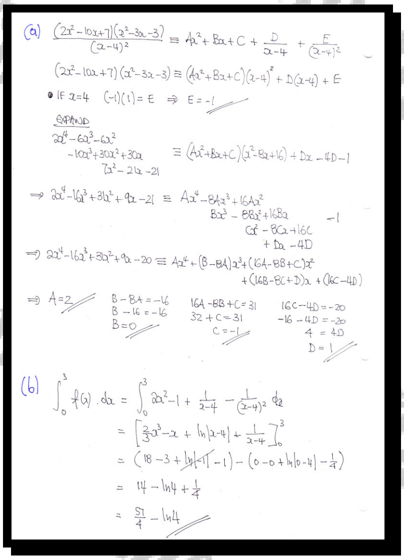
$$\frac{\left(2x^2-10x+7\right)\left(x^2-3x-3\right)}{(x-4)^2} \equiv Ax^2+Bx+C+\frac{D}{x-4}+\frac{E}{(x-4)^2}.$$

a) Find the value of A, B, C, D and E in the above identity.

b) Hence find the exact value of

$$\int_0^3 f(x)\ dx.$$

$\boxed{A=2}$, $\boxed{B=0}$, $\boxed{C=-1}$, $\boxed{D=1}$, $\boxed{E=-1}$, $\boxed{\frac{57}{4}-\ln 4}$


(a) $\frac{(2x^2-10x+7)(x^2-3x-3)}{(x-4)^2} \equiv Ax^2+Bx+C+\frac{D}{x-4}+\frac{E}{(x-4)^2}$

$(2x^2-10x+7)(x^2-3x-3) \equiv (Ax^2+Bx+C)(x-4)^2+D(x-4)+E$

• IF $x=4$ $(-1)(1)=E \Rightarrow E=-1$

EXPAND

$2x^4-6x^3-6x^2$
$-10x^3+30x^2+30x$
$7x^2-21x-21$ $\equiv (Ax^2+Bx+C)(x^2-8x+16)+Dx-4D-1$

$\Rightarrow 2x^4-16x^3+31x^2+9x-21 \equiv Ax^4-8Ax^3+16Ax^2 + Bx^3-8Bx^2+16Bx + Cx^2-8Cx+16C + Dx-4D-1$

$\Rightarrow 2x^4-16x^3+31x^2+9x-20 \equiv Ax^4+(B-8A)x^3+(16A-8B+C)x^2+(16B-8C+D)x+(16C-4D)$

$\Rightarrow A=2$; $B-8A=-16$, $B-16=-16$, $B=0$; $16A-8B+C=31$, $32+C=31$, $C=-1$; $16C-4D=-20$, $-16-4D=-20$, $4=4D$, $D=1$

(b) $\int_0^3 f(x)\,dx = \int_0^3 2x^2-1+\frac{1}{x-4}-\frac{1}{(x-4)^2}\,dx$

$= \left[\frac{2}{3}x^3-x+\ln|x-4|+\frac{1}{x-4}\right]_0^3$

$= \left(18-3+\ln|-1|-1\right)-\left(0-0+\ln|0-4|-\frac{1}{4}\right)$

$= 14-\ln 4+\frac{1}{4}$

$= \frac{57}{4}-\ln 4$

Question 183 (****+)

$$x = \frac{1}{2}(-1 + 4\tan\theta), \quad -\frac{1}{2}\pi < \theta < \frac{1}{2}\pi \ .$$

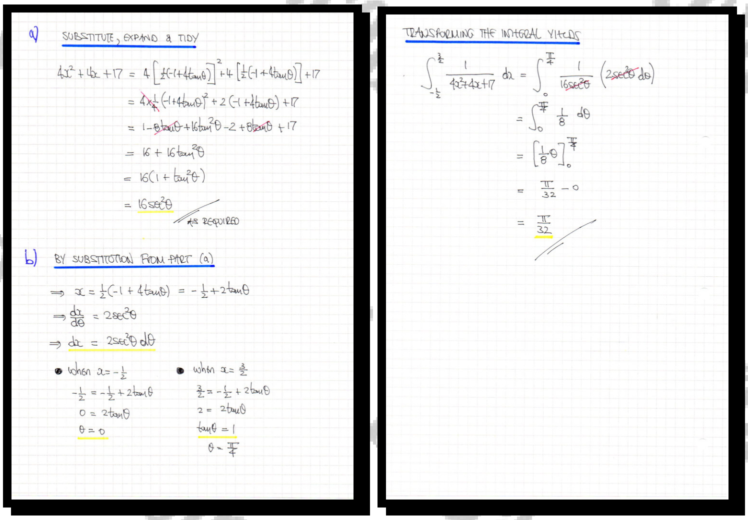
a) Use trigonometric identities to show that

$$4x^2 + 4x + 17 = 16\sec^2\theta \ .$$

b) Hence find the exact value of

$$\int_{-\frac{1}{2}}^{\frac{3}{2}} \frac{1}{4x^2 + 4x + 17} \, dx \ .$$

SYN-I , $\boxed{\frac{\pi}{32}}$



a) SUBSTITUTE, EXPAND & TIDY

$4x^2+4x+17 = 4\left[\frac{1}{2}(-1+4\tan\theta)\right]^2 + 4\left[\frac{1}{2}(-1+4\tan\theta)\right] + 17$

$= 4 \times \frac{1}{4}(-1+4\tan\theta)^2 + 2(-1+4\tan\theta) + 17$

$= 1 - 8\tan\theta + 16\tan^2\theta - 2 + 8\tan\theta + 17$

$= 16 + 16\tan^2\theta$

$= 16(1+\tan^2\theta)$

$= 16\sec^2\theta$ — AS REQUIRED

b) BY SUBSTITUTION FROM PART (a)

$\Rightarrow x = \frac{1}{2}(-1+4\tan\theta) = -\frac{1}{2} + 2\tan\theta$

$\Rightarrow \frac{dx}{d\theta} = 2\sec^2\theta$

$\Rightarrow dx = 2\sec^2\theta \, d\theta$

- when $x = -\frac{1}{2}$: $-\frac{1}{2} = -\frac{1}{2} + 2\tan\theta$, $0 = 2\tan\theta$, $\theta = 0$
- when $x = \frac{3}{2}$: $\frac{3}{2} = -\frac{1}{2} + 2\tan\theta$, $2 = 2\tan\theta$, $\tan\theta = 1$, $\theta = \frac{\pi}{4}$

TRANSFORMING THE INTEGRAL YIELDS

$\int_{-\frac{1}{2}}^{\frac{3}{2}} \frac{1}{4x^2+4x+17}\,dx = \int_0^{\frac{\pi}{4}} \frac{1}{16\sec^2\theta}(2\sec^2\theta\,d\theta)$

$= \int_0^{\frac{\pi}{4}} \frac{1}{8}\,d\theta$

$= \left[\frac{1}{8}\theta\right]_0^{\frac{\pi}{4}}$

$= \frac{\pi}{32} - 0$

$= \frac{\pi}{32}$

Question 184 (****+)

It is given that

$$\sin(A+B) \equiv \sin A\cos B + \cos A\sin B.$$

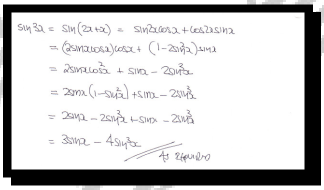
Use the above trigonometric identity to show that

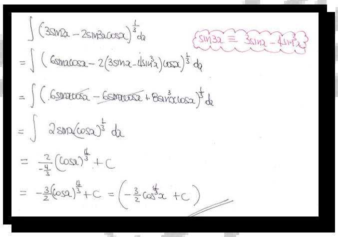
$$\sin 3x \equiv 3\sin x - 4\sin^3 x,$$

and hence find

$$\int \sqrt[3]{3\sin 2x - 2\sin 3x\cos x}\, dx.$$

$$\boxed{-\frac{3}{2}\sin^{\frac{4}{3}} x + C}$$



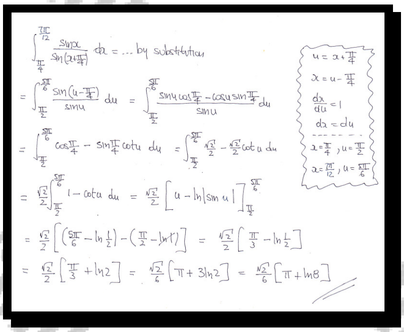


Question 185 (****+)

Use the substitution $u = x + \frac{\pi}{4}$ to show that

$$\int_{\frac{\pi}{4}}^{\frac{7\pi}{12}} \frac{\sin x}{\sin\left(x + \frac{\pi}{4}\right)} \, dx = \frac{\sqrt{2}}{6}\left(\pi + \ln 8\right).$$

proof



$$\int_{\frac{\pi}{4}}^{\frac{7\pi}{12}} \frac{\sin x}{\sin(x+\frac{\pi}{4})} \, dx = \dots \text{ by substitution}$$

$u = x + \frac{\pi}{4}$, $x = u - \frac{\pi}{4}$, $\frac{dx}{du} = 1$, $dx = du$

$x = \frac{\pi}{4}, u = \frac{\pi}{2}$; $x = \frac{7\pi}{12}, u = \frac{5\pi}{6}$

$$= \int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} \frac{\sin(u - \frac{\pi}{4})}{\sin u} \, du = \int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} \frac{\sin u \cos\frac{\pi}{4} - \cos u \sin\frac{\pi}{4}}{\sin u} \, du$$

$$= \int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} \cos\frac{\pi}{4} - \sin\frac{\pi}{4}\cot u \, du = \int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\cot u \, du$$

$$= \frac{\sqrt{2}}{2}\int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} 1 - \cot u \, du = \frac{\sqrt{2}}{2}\Big[u - \ln|\sin u|\Big]_{\frac{\pi}{2}}^{\frac{5\pi}{6}}$$

$$= \frac{\sqrt{2}}{2}\left[\left(\frac{5\pi}{6} - \ln\frac{1}{2}\right) - \left(\frac{\pi}{2} - \ln 1\right)\right] = \frac{\sqrt{2}}{2}\left[\frac{\pi}{3} - \ln\frac{1}{2}\right]$$

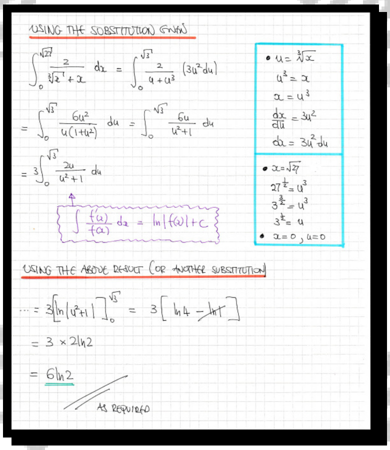
$$= \frac{\sqrt{2}}{2}\left[\frac{\pi}{3} + \ln 2\right] = \frac{\sqrt{2}}{6}\left[\pi + 3\ln 2\right] = \frac{\sqrt{2}}{6}\left[\pi + \ln 8\right]$$

Question 186 (****+)

By using the substitution $u=\sqrt[3]{x}$, or otherwise, show that

$$\int_0^{\sqrt{27}} \frac{2}{x+\sqrt[3]{x}}\,dx = 6\ln 2\,.$$

proof



Question 187 **(****+)**

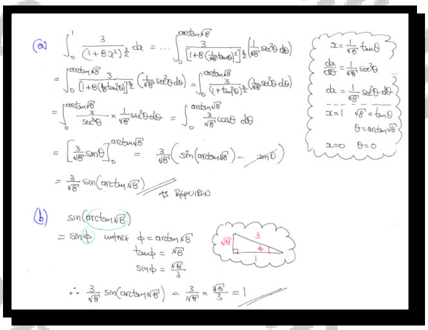
$$I = \int_0^1 \frac{3}{\left(1+8x^2\right)^{\frac{3}{2}}}\, dx$$

a) Use the substitution $x = \frac{1}{\sqrt{8}}\tan\theta$ to show that

$$I = \frac{3}{\sqrt{8}}\sin\left(\arctan\sqrt{8}\right).$$

b) Show, presenting detailed calculations, that $I = 1$.

proof



Question 188 (****+)

$$I = \int_{1}^{2} \frac{1}{x^2 - x\sqrt{x^2-1}}\, dx .$$

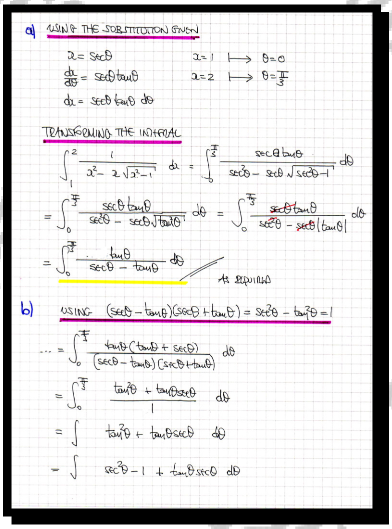
a) Show that the substitution $x = \sec\theta$ transforms I to

$$I = \int_{0}^{\frac{1}{3}\pi} \frac{\tan\theta}{\sec\theta - \tan\theta}\, d\theta .$$

b) Hence use trigonometric identities to show that

$$I = 1 + \sqrt{3} - \tfrac{1}{3}\pi .$$

[ ], [proof]


a) USING THE SUBSTITUTION GIVEN

$x = \sec\theta$, $\frac{dx}{d\theta} = \sec\theta\tan\theta$, $dx = \sec\theta\tan\theta\, d\theta$

$x = 1 \mapsto \theta = 0$, $x = 2 \mapsto \theta = \frac{\pi}{3}$

TRANSFORMING THE INTEGRAL

$\int_1^2 \frac{1}{x^2 - x\sqrt{x^2-1}}\,dx = \int_0^{\frac{\pi}{3}} \frac{\sec\theta\tan\theta}{\sec^2\theta - \sec\theta\sqrt{\sec^2\theta-1}}\,d\theta$

$= \int_0^{\frac{\pi}{3}} \frac{\sec\theta\tan\theta}{\sec^2\theta - \sec\theta\sqrt{\tan^2\theta}}\,d\theta = \int_0^{\frac{\pi}{3}} \frac{\sec\theta\tan\theta}{\sec^2\theta - \sec\theta|\tan\theta|}\,d\theta$

$= \int_0^{\frac{\pi}{3}} \frac{\tan\theta}{\sec\theta - \tan\theta}\,d\theta$ AS REQUIRED

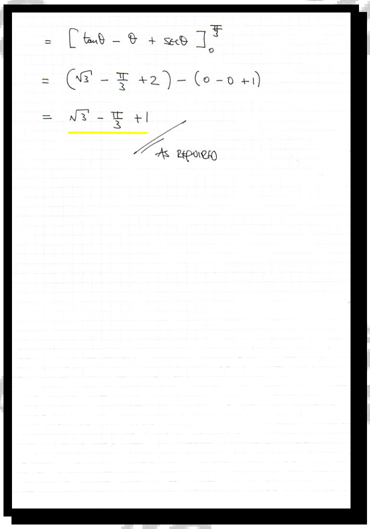
b) USING $(\sec\theta - \tan\theta)(\sec\theta + \tan\theta) = \sec^2\theta - \tan^2\theta = 1$

$\dots = \int_0^{\frac{\pi}{3}} \frac{\tan\theta(\tan\theta + \sec\theta)}{(\sec\theta - \tan\theta)(\sec\theta + \tan\theta)}\,d\theta$

$= \int_0^{\frac{\pi}{3}} \frac{\tan^2\theta + \tan\theta\sec\theta}{1}\,d\theta$

$= \int \tan^2\theta + \tan\theta\sec\theta\,d\theta$

$= \int \sec^2\theta - 1 + \tan\theta\sec\theta\,d\theta$


$= \left[\tan\theta - \theta + \sec\theta\right]_0^{\frac{\pi}{3}}$

$= \left(\sqrt{3} - \frac{\pi}{3} + 2\right) - (0 - 0 + 1)$

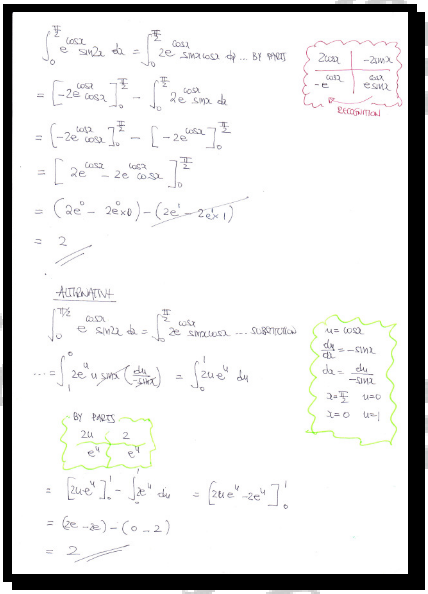
$= \sqrt{3} - \frac{\pi}{3} + 1$ AS REQUIRED

Question 189 (****+)

Use integration by parts to find the value of

$$\int_0^{\frac{\pi}{2}} e^{\cos x} \sin 2x \ dx \, .$$

$\boxed{2}$


$\int_0^{\frac{\pi}{2}} e^{\cos x} \sin 2x \, dx = \int_0^{\frac{\pi}{2}} 2e^{\cos x} \sin x \cos x \, dx$ ... BY PARTS

$= \left[-2e^{\cos x}\cos x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 2e^{\cos x}\sin x \, dx$

$= \left[-2e^{\cos x}\cos x\right]_0^{\frac{\pi}{2}} - \left[-2e^{\cos x}\right]_0^{\frac{\pi}{2}}$

$= \left[2e^{\cos x} - 2e^{\cos x}\cos x\right]_0^{\frac{\pi}{2}}$

$= (2e^0 - 2e^0 \times 0) - (2e^1 - 2e^1 \times 1)$

$= 2$

$2\cos x$ | $-2\sin x$
$-e^{\cos x}$ | $e^{\cos x}\sin x$
RECOGNITION

ALTERNATIVE

$\int_0^{\pi/2} e^{\cos x} \sin 2x \, dx = \int_0^{\frac{\pi}{2}} 2e^{\cos x} \sin x \cos x$ ... SUBSTITUTION

$\dots = \int_1^0 2e^u u \sin x \left(\frac{du}{-\sin x}\right) = \int_0^1 2ue^u \, du$

$u = \cos x$
$\frac{du}{dx} = -\sin x$
$dx = \frac{du}{-\sin x}$
$x = \frac{\pi}{2} \quad u = 0$
$x = 0 \quad u = 1$

BY PARTS
$2u$ | 2
e^u | e^u

$= \left[2ue^u\right]_0^1 - \int_0^1 2e^u \, du = \left[2ue^u - 2e^u\right]_0^1$

$= (2e - 2e) - (0 - 2)$

$= 2$

Question 190 (****+)

$$\sin 2x \equiv \frac{2\tan x}{1+\tan^2 x}.$$

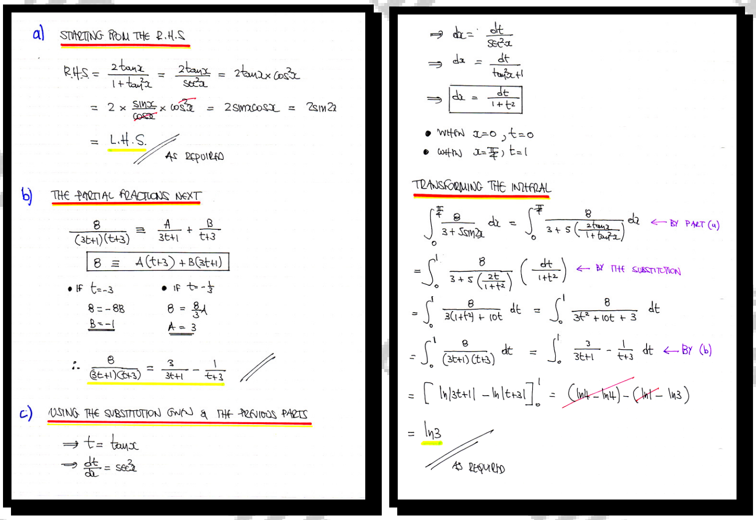
a) Prove the validity of the above trigonometric identity.

b) Express $\dfrac{8}{(3t+1)(t+3)}$ into partial fractions.

c) Hence use the substitution $t = \tan x$ to show that

$$\int_0^{\frac{\pi}{4}} \frac{8}{3+5\sin 2x}\,dx = \ln 3.$$

SYN, $\boxed{\dfrac{8}{(3t+1)(t+3)} = \dfrac{3}{3t+1} - \dfrac{1}{t+3}}$



a) STARTING FROM THE R.H.S

$\text{R.H.S} = \frac{2\tan x}{1+\tan^2 x} = \frac{2\tan x}{\sec^2 x} = 2\tan x \times \cos^2 x$

$= 2 \times \frac{\sin x}{\cos x} \times \cos^2 x = 2\sin x\cos x = \sin 2x$

$= \text{L.H.S.}$ AS REQUIRED

b) THE PARTIAL FRACTIONS NEXT

$\frac{8}{(3t+1)(t+3)} \equiv \frac{A}{3t+1} + \frac{B}{t+3}$

$8 \equiv A(t+3) + B(3t+1)$

• IF $t=-3$: $8=-8B$, $B=-1$ • IF $t=-\frac{1}{3}$: $8=\frac{8}{3}A$, $A=3$

$\therefore \frac{8}{(3t+1)(t+3)} = \frac{3}{3t+1} - \frac{1}{t+3}$

c) USING THE SUBSTITUTION GIVEN & THE PREVIOUS PARTS

$\Rightarrow t = \tan x$

$\Rightarrow \frac{dt}{dx} = \sec^2 x$

$\Rightarrow dx = \frac{dt}{\sec^2 x}$

$\Rightarrow dx = \frac{dt}{\tan^2 x+1}$

$\Rightarrow dx = \frac{dt}{1+t^2}$

• WHEN $x=0$, $t=0$
• WHEN $x=\frac{\pi}{4}$, $t=1$

TRANSFORMING THE INTEGRAL

$\int_0^{\frac{\pi}{4}} \frac{8}{3+5\sin 2x}\,dx = \int_0^{\frac{\pi}{4}} \frac{8}{3+5\left(\frac{2\tan x}{1+\tan^2 x}\right)}\,dx$ ← BY PART (a)

$= \int_0^1 \frac{8}{3+5\left(\frac{2t}{1+t^2}\right)}\left(\frac{dt}{1+t^2}\right)$ ← BY THE SUBSTITUTION

$= \int_0^1 \frac{8}{3(1+t^2)+10t}\,dt = \int_0^1 \frac{8}{3t^2+10t+3}\,dt$

$= \int_0^1 \frac{8}{(3t+1)(t+3)}\,dt = \int_0^1 \frac{3}{3t+1} - \frac{1}{t+3}\,dt$ ← BY (b)

$= \left[\ln|3t+1| - \ln|t+3|\right]_0^1 = (\ln 4 - \ln 4) - (\ln 1 - \ln 3)$

$= \ln 3$ AS REQUIRED

Question 191 (****+)

It is given that for some constants A and B

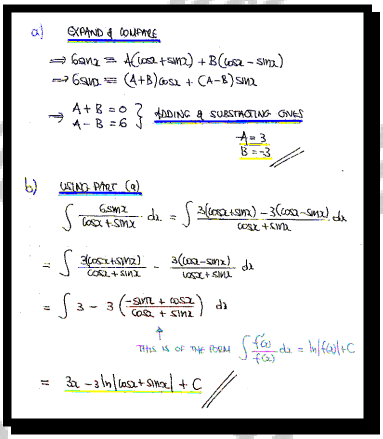
$$6\sin x \equiv A(\cos x+\sin x)+B(\cos x-\sin x).$$

a) Find the value of A and the value of B.

b) Hence find

$$\int \frac{6\sin x}{\cos x+\sin x}\,dx .$$

[illegible], $\boxed{A=3}$, $\boxed{B=-3}$, $\boxed{3x-3\ln|\cos x+\sin x|+C}$

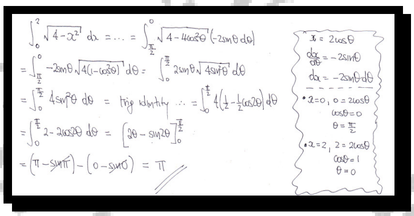


Question 192 (****+)

Use the substitution $x=2\sin\theta$ to show that

$$\int_0^2 \sqrt{4-x^2}\;dx=\pi .$$

proof

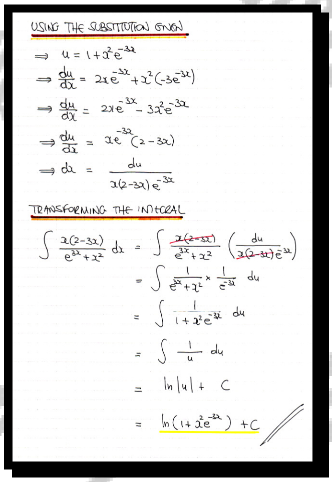


Question 193 (****+)

Use the substitution $u = 1 + x^2 e^{-3x}$ to find an expression for

$$\int \frac{x(2-3x)}{e^{3x}+x^2}\,dx .$$

$\boxed{}$, $\boxed{\ln\left(1+x^2e^{-3x}\right)+C}$



USING THE SUBSTITUTION GIVEN

$\Rightarrow u = 1 + x^2 e^{-3x}$

$\Rightarrow \frac{du}{dx} = 2xe^{-3x} + x^2(-3e^{-3x})$

$\Rightarrow \frac{du}{dx} = 2xe^{-3x} - 3x^2e^{-3x}$

$\Rightarrow \frac{du}{dx} = xe^{-3x}(2-3x)$

$\Rightarrow dx = \frac{du}{x(2-3x)e^{-3x}}$

TRANSFORMING THE INTEGRAL

$\int \frac{x(2-3x)}{e^{3x}+x^2}\,dx = \int \frac{x(2-3x)}{e^{3x}+x^2}\left(\frac{du}{x(2-3x)e^{-3x}}\right)$

$= \int \frac{1}{e^{3x}+x^2} \times \frac{1}{e^{-3x}}\,du$

$= \int \frac{1}{1+x^2e^{-3x}}\,du$

$= \int \frac{1}{u}\,du$

$= \ln|u| + C$

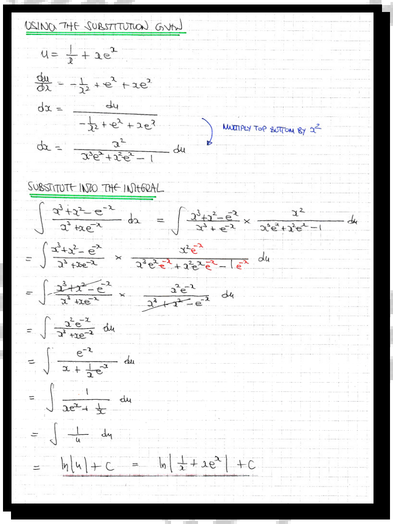
$= \ln(1+x^2e^{-3x}) + C$

Question 194 (****+)

Use the substitution $u = \frac{1}{x} + x\mathrm{e}^{x}$ to find an expression for

$$\int \frac{x^3 + x^2 - \mathrm{e}^{-x}}{x^3 + x\mathrm{e}^{-x}}\, dx .$$

$\boxed{}$, $\boxed{\ln\left|\frac{1}{x} + x\mathrm{e}^{x}\right| + C}$



Question 195 (****+)

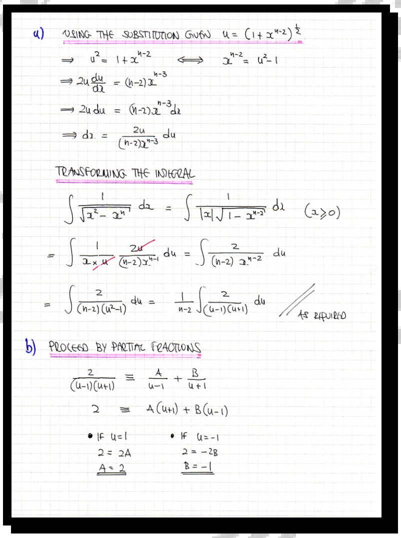
$$\int \frac{1}{\sqrt{x^2 + x^n}}\, dx\,,\ n \neq 2\,,\ x \geq 0\,.$$

a) Show that the substitution $u^2 = 1 + x^{n-2}$ transforms above integral into

$$\frac{1}{n-2}\int \frac{2}{(u-1)(u+1)}\, du\,.$$

b) Use partial fractions to find, in terms of x and n, an integrated expression for the original integral.

$\boxed{}$, $\boxed{\frac{1}{n-2}\ln\left|\frac{\sqrt{1+x^{n-2}}-1}{\sqrt{1+x^{n-2}}+1}\right| + C}$



a) USING THE SUBSTITUTION GIVEN $u = (1 + x^{n-2})^{\frac{1}{2}}$

$\Rightarrow u^2 = 1 + x^{n-2} \iff x^{n-2} = u^2 - 1$

$\Rightarrow 2u\frac{du}{dx} = (n-2)x^{n-3}$

$\Rightarrow 2u\,du = (n-2)x^{n-3}\,dx$

$\Rightarrow dx = \frac{2u}{(n-2)x^{n-3}}\,du$

TRANSFORMING THE INTEGRAL

$\int \frac{1}{\sqrt{x^2 - x^n}}\,dx = \int \frac{1}{|x|\sqrt{1 - x^{n-2}}}\,dx \quad (x \geq 0)$

$= \int \frac{1}{x \times u}\,\frac{2u}{(n-2)x^{n-1}}\,du = \int \frac{2}{(n-2)\,x^{n-2}}\,du$

$= \int \frac{2}{(n-2)(u^2-1)}\,du = \frac{1}{n-2}\int \frac{2}{(u-1)(u+1)}\,du$ // AS REQUIRED

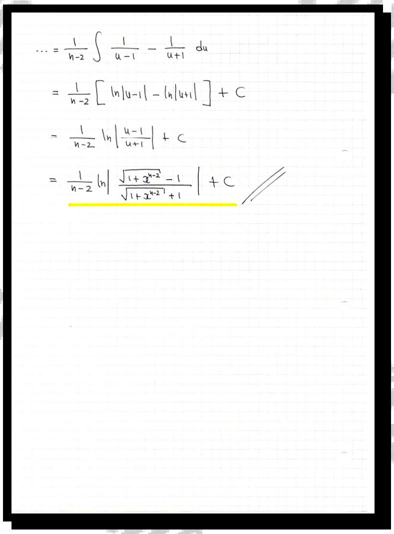
b) PROCEED BY PARTIAL FRACTIONS

$\frac{2}{(u-1)(u+1)} \equiv \frac{A}{u-1} + \frac{B}{u+1}$

$2 \equiv A(u+1) + B(u-1)$

• IF $u=1$: $2 = 2A$, $A = 2$

• IF $u=-1$: $2 = -2B$, $B = -1$



$\dots = \frac{1}{n-2}\int \frac{1}{u-1} - \frac{1}{u+1}\,du$

$= \frac{1}{n-2}\left[\ln|u-1| - \ln|u+1|\right] + C$

$= \frac{1}{n-2}\ln\left|\frac{u-1}{u+1}\right| + C$

$= \frac{1}{n-2}\ln\left|\frac{\sqrt{1+x^{n-2}}-1}{\sqrt{1+x^{n-2}}+1}\right| + C$ //

Question 196 (****+)

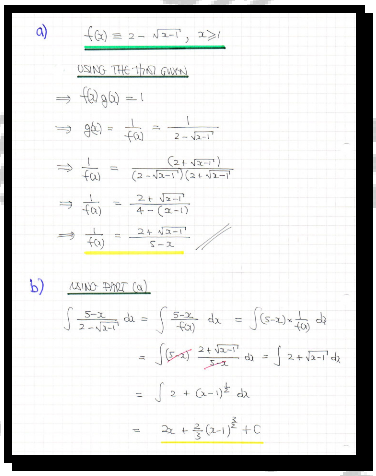
$$f(x) \equiv 2 - \sqrt{x-1}, \quad x \geq 1.$$

a) Find a simplified expression for $g(x)$ so that $f(x)g(x) = 1$.

b) Hence, or otherwise, find

$$\int \frac{5-x}{2-\sqrt{x-1}} \, dx .$$

MP , $g(x) \equiv \dfrac{2+\sqrt{x-1}}{5-x}$, $2x + \frac{1}{2}(x-1)^{\frac{3}{2}} + C$



a) $f(x) \equiv 2 - \sqrt{x-1}, \; x \geq 1$

USING THE THING GIVEN

$\Rightarrow f(x)g(x) = 1$

$\Rightarrow g(x) = \dfrac{1}{f(x)} = \dfrac{1}{2-\sqrt{x-1}}$

$\Rightarrow \dfrac{1}{f(x)} = \dfrac{(2+\sqrt{x-1})}{(2-\sqrt{x-1})(2+\sqrt{x-1})}$

$\Rightarrow \dfrac{1}{f(x)} = \dfrac{2+\sqrt{x-1}}{4-(x-1)}$

$\Rightarrow \dfrac{1}{f(x)} = \dfrac{2+\sqrt{x-1}}{5-x}$

b) USING PART (a)

$\displaystyle\int \frac{5-x}{2-\sqrt{x-1}} \, dx = \int \frac{5-x}{f(x)} \, dx = \int (5-x) \times \frac{1}{f(x)} \, dx$

$\displaystyle = \int (5-x) \frac{2+\sqrt{x-1}}{5-x} \, dx = \int 2 + \sqrt{x-1} \, dx$

$\displaystyle = \int 2 + (x-1)^{\frac{1}{2}} \, dx$

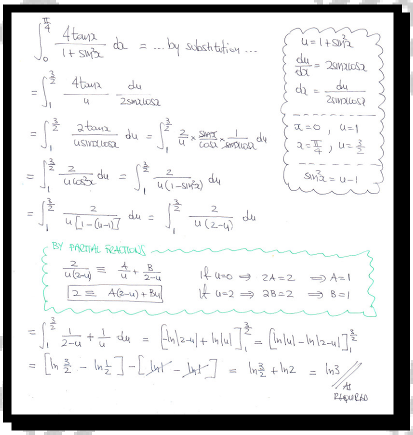
$= 2x + \frac{2}{3}(x-1)^{\frac{3}{2}} + C$

Question 197 (****+)

By using the substitution $u = 1 + \sin^2 x$, or otherwise, show clearly that

$$\int_0^{\frac{\pi}{4}} \frac{4\tan x}{1+\sin^2 x}\, dx = \ln 3 .$$

proof



$$\int_0^{\frac{\pi}{4}} \frac{4\tan x}{1+\sin^2 x}\, dx = \dots \text{by substitution} \dots$$

$u = 1+\sin^2 x$

$\frac{du}{dx} = 2\sin x\cos x$

$dx = \frac{du}{2\sin x\cos x}$

$x = 0,\ u = 1$

$x = \frac{\pi}{4},\ u = \frac{3}{2}$

$\sin^2 x = u - 1$

$$= \int_1^{\frac{3}{2}} \frac{4\tan x}{u} \frac{du}{2\sin x\cos x}$$

$$= \int_1^{\frac{3}{2}} \frac{2\tan x}{u\sin x\cos x}\, du = \int_1^{\frac{3}{2}} \frac{2}{u} \times \frac{\sin x}{\cos x} \times \frac{1}{\sin x\cos x}\, du$$

$$= \int_1^{\frac{3}{2}} \frac{2}{u\cos^2 x}\, du = \int_1^{\frac{3}{2}} \frac{2}{u(1-\sin^2 x)}\, du$$

$$= \int_1^{\frac{3}{2}} \frac{2}{u[1-(u-1)]}\, du = \int_1^{\frac{3}{2}} \frac{2}{u(2-u)}\, du$$

BY PARTIAL FRACTIONS

$\frac{2}{u(2-u)} \equiv \frac{A}{u} + \frac{B}{2-u}$

$2 \equiv A(2-u) + Bu$

If $u=0 \Rightarrow 2A = 2 \Rightarrow A = 1$

If $u=2 \Rightarrow 2B = 2 \Rightarrow B = 1$

$$= \int_1^{\frac{3}{2}} \frac{1}{2-u} + \frac{1}{u}\, du = \Big[-\ln|2-u| + \ln|u|\Big]_1^{\frac{3}{2}} = \Big[\ln|u| - \ln|2-u|\Big]_1^{\frac{3}{2}}$$

$$= \left[\ln\frac{3}{2} - \ln\frac{1}{2}\right] - \left[\ln 1 - \ln 1\right] = \ln\frac{3}{2} + \ln 2 = \ln 3$$

AS REQUIRED

Question 198 **(****+)**

$$\sec x \equiv \frac{\cos x}{1-\sin^2 x}.$$

a) Prove the validity of the above trigonometric identity.

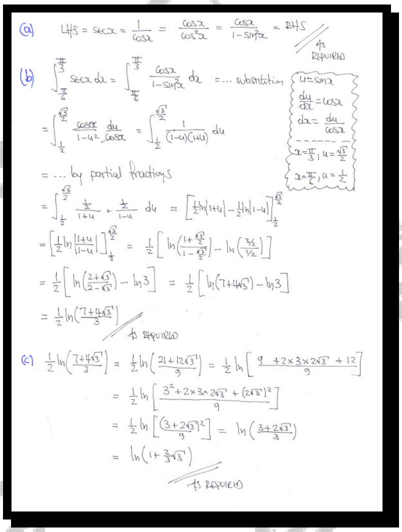
b) Use the substitution $u = \sin x$ to show that

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec x \ dx = \frac{1}{2}\ln\left(\frac{7+4\sqrt{3}}{3}\right).$$

c) Show clearly that

$$\frac{1}{2}\ln\left(\frac{7+4\sqrt{3}}{3}\right) = \ln\left(1+\frac{2}{3}\sqrt{3}\right)$$

proof



(a) $\text{LHS} = \sec x = \frac{1}{\cos x} = \frac{\cos x}{\cos^2 x} = \frac{\cos x}{1-\sin^2 x} = \text{RHS}$ AS REQUIRED

(b) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec x \, dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos x}{1-\sin^2 x} \, dx = \dots$ substitution

$u = \sin x$, $\frac{du}{dx} = \cos x$, $dx = \frac{du}{\cos x}$; $x = \frac{\pi}{3}, u = \frac{\sqrt{3}}{2}$; $x = \frac{\pi}{6}, u = \frac{1}{2}$

$= \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{\cos x}{1-u^2} \frac{du}{\cos x} = \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{1}{(1-u)(1+u)} \, du$

$= \dots$ by partial fractions

$= \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{\frac{1}{2}}{1+u} + \frac{\frac{1}{2}}{1-u} \, du = \left[\frac{1}{2}\ln|1+u| - \frac{1}{2}\ln|1-u|\right]_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}$

$= \left[\frac{1}{2}\ln\left|\frac{1+u}{1-u}\right|\right]_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} = \frac{1}{2}\left[\ln\left(\frac{1+\frac{\sqrt{3}}{2}}{1-\frac{\sqrt{3}}{2}}\right) - \ln\left(\frac{3/2}{1/2}\right)\right]$

$= \frac{1}{2}\left[\ln\left(\frac{2+\sqrt{3}}{2-\sqrt{3}}\right) - \ln 3\right] = \frac{1}{2}\left[\ln(7+4\sqrt{3}) - \ln 3\right]$

$= \frac{1}{2}\ln\left(\frac{7+4\sqrt{3}}{3}\right)$ AS REQUIRED

(c) $\frac{1}{2}\ln\left(\frac{7+4\sqrt{3}}{3}\right) = \frac{1}{2}\ln\left(\frac{21+12\sqrt{3}}{9}\right) = \frac{1}{2}\ln\left[\frac{9+2\times3\times2\sqrt{3}+12}{9}\right]$

$= \frac{1}{2}\ln\left[\frac{3^2+2\times3\times2\sqrt{3}+(2\sqrt{3})^2}{9}\right]$

$= \frac{1}{2}\ln\left[\frac{(3+2\sqrt{3})^2}{9}\right] = \ln\left(\frac{3+2\sqrt{3}}{3}\right)$

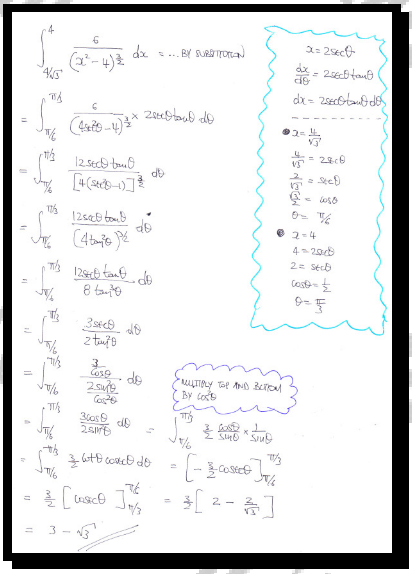
$= \ln\left(1+\frac{2}{3}\sqrt{3}\right)$ AS REQUIRED

Question 199 (****+)

Use the substitution $x = 2\sec\theta$, to find the exact value of

$$\int_{\frac{4}{\sqrt{3}}}^{4} \frac{6}{\left(x^2-4\right)^{\frac{3}{2}}}\,dx\,.$$

$\boxed{3-\sqrt{3}}$

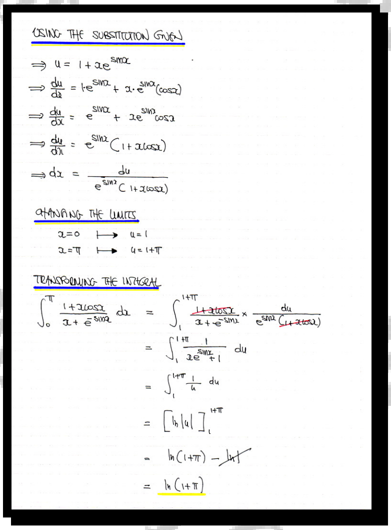


Question 200 (****+)

Use the substitution $u = 1 + x\mathrm{e}^{\sin x}$ to find an exact simplified value for the following definite integral.

$$\int_0^{\pi} \frac{1 + x\cos x}{x + \mathrm{e}^{-\sin x}} \, dx \, .$$

, $\boxed{\ln(1+\pi)}$



USING THE SUBSTITUTION GIVEN

$\Rightarrow u = 1 + xe^{\sin x}$

$\Rightarrow \frac{du}{dx} = 1e^{\sin x} + x \cdot e^{\sin x}(\cos x)$

$\Rightarrow \frac{du}{dx} = e^{\sin x} + xe^{\sin x}\cos x$

$\Rightarrow \frac{du}{dx} = e^{\sin x}(1 + x\cos x)$

$\Rightarrow dx = \frac{du}{e^{\sin x}(1 + x\cos x)}$

CHANGING THE LIMITS

$x = 0 \longmapsto u = 1$

$x = \pi \longmapsto u = 1+\pi$

TRANSFORMING THE INTEGRAL

$\int_0^{\pi} \frac{1 + x\cos x}{x + e^{-\sin x}} \, dx = \int_1^{1+\pi} \frac{1 + x\cos x}{x + e^{-\sin x}} \times \frac{du}{e^{\sin x}(1 + x\cos x)}$

$= \int_1^{1+\pi} \frac{1}{xe^{\sin x} + 1} \, du$

$= \int_1^{1+\pi} \frac{1}{u} \, du$

$= \left[\ln|u|\right]_1^{1+\pi}$

$= \ln(1+\pi) - \ln 1$

$= \ln(1+\pi)$

Question 201 (****+)

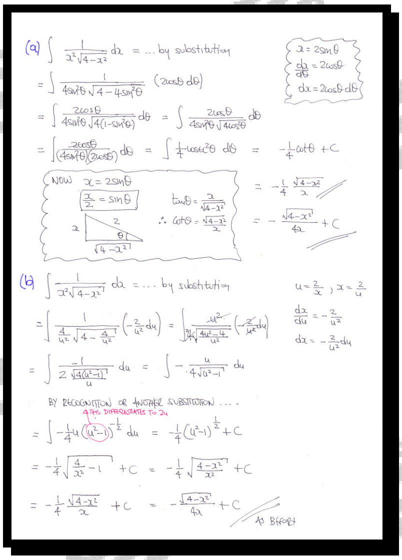
$$I = \int \frac{1}{x^2\sqrt{4-x^2}}\,dx\,.$$

a) Use the substitution $x = 2\sin\theta$ to show clearly that

$$I = -\frac{\sqrt{4-x^2}}{4x} + C\,.$$

b) Verify the answer to part **(a)** by using the substitution $u = \dfrac{2}{x}$.

proof

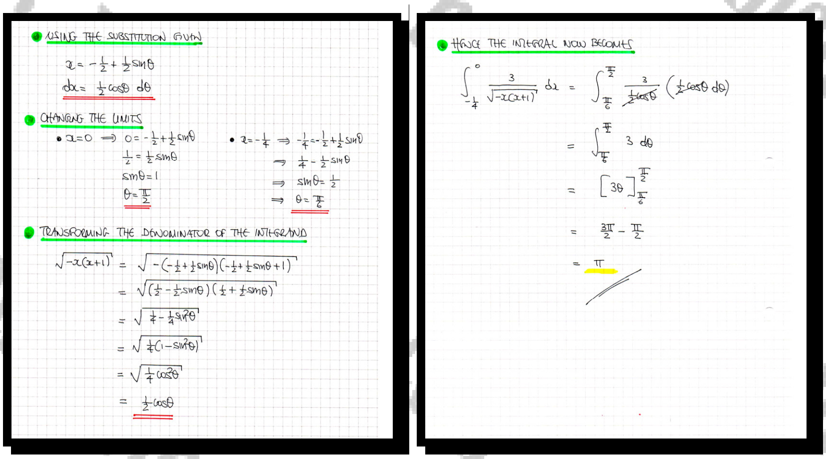


Question 202 (****+)

By using the substitution $x = -\frac{1}{2} + \frac{1}{2}\sin\theta$, or otherwise, find the exact value of

$$\int_{-\frac{1}{4}}^{0} \frac{3}{\sqrt{-x(x+1)}}\, dx .$$

$\boxed{}$, $\boxed{\pi}$

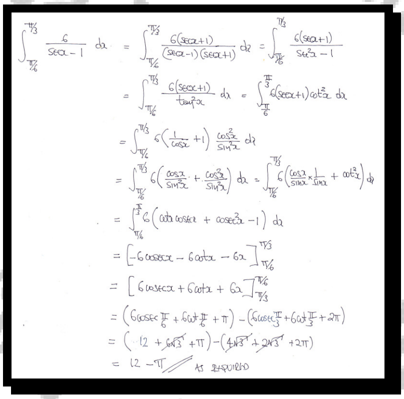


Question 203 (****+)

By using multiplying the numerator and denominator of the integrand by $(\sec x + 1)$, and manipulating it further by various trigonometric identities, show clearly that

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{6}{\sec x - 1}\, dx = 12 - \pi .$$

proof



$$\int_{\pi/6}^{\pi/3} \frac{6}{\sec x - 1}\,dx = \int_{\pi/6}^{\pi/3} \frac{6(\sec x+1)}{(\sec x-1)(\sec x+1)}\,dx = \int_{\pi/6}^{\pi/3} \frac{6(\sec x+1)}{\sec^2 x - 1}$$
$$= \int_{\pi/6}^{\pi/3} \frac{6(\sec x+1)}{\tan^2 x}\,dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 6(\sec x+1)\cot^2 x\,dx$$
$$= \int_{\pi/6}^{\pi/3} 6\left(\frac{1}{\cos x}+1\right)\frac{\cos^2 x}{\sin^2 x}\,dx$$
$$= \int_{\pi/6}^{\pi/3} 6\left(\frac{\cos x}{\sin^2 x}+\frac{\cos^2 x}{\sin^2 x}\right)dx = \int_{\pi/6}^{\pi/3} 6\left(\frac{\cos x}{\sin x}\times\frac{1}{\sin x}+\cot^2 x\right)dx$$
$$= \int_{\pi/6}^{\frac{\pi}{3}} 6\left(\cot x\,\mathrm{cosec}\,x + \mathrm{cosec}^2 x - 1\right)dx$$
$$= \Big[-6\,\mathrm{cosec}\,x - 6\cot x - 6x\Big]_{\pi/6}^{\pi/3}$$
$$= \Big[6\,\mathrm{cosec}\,x + 6\cot x + 6x\Big]_{\pi/3}^{\pi/6}$$
$$= \left(6\,\mathrm{cosec}\tfrac{\pi}{6} + 6\cot\tfrac{\pi}{6} + \pi\right) - \left(6\,\mathrm{cosec}\tfrac{\pi}{3} + 6\cot\tfrac{\pi}{3} + 2\pi\right)$$
$$= \left(12 + 6\sqrt{3} + \pi\right) - \left(4\sqrt{3} + 2\sqrt{3} + 2\pi\right)$$
$$= 12 - \pi$$

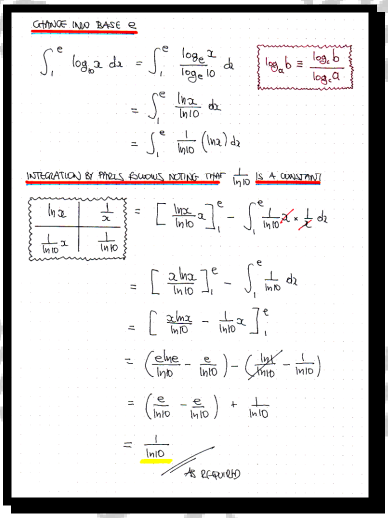
AS REQUIRED

Question 204 **(****+)**

By changing the base of the logarithmic integrand into base e and further using integration by parts, show that

$$\int_{1}^{e} \log_{10} x \ dx = \frac{1}{\ln 10}.$$

, proof



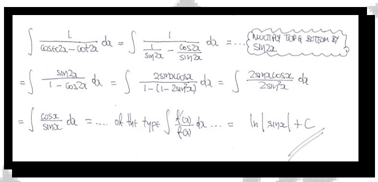
Question 205 **(****+)**

Use trigonometric identities to find

$$\int \frac{1}{\operatorname{cosec} 2x - \cot 2x} \ dx,$$

giving the answer in the form $\ln|f(x)|$

$\ln|\sin x| + C$

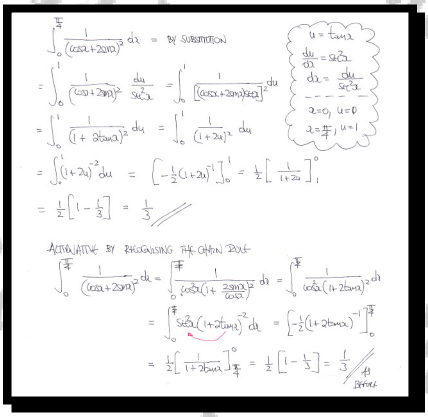


Question 206 (****+)

By using the substitution $u = \tan x$, or otherwise, find the exact value of

$$\int_0^{\frac{\pi}{4}} \frac{1}{(\cos x + 2\sin x)^2}\, dx .$$

$\boxed{\frac{1}{3}}$



$$\int_0^{\frac{\pi}{4}} \frac{1}{(\cos x+2\sin x)^2}\,dx = \text{BY SUBSTITUTION}$$

$u = \tan x$, $\frac{du}{dx} = \sec^2 x$, $dx = \frac{du}{\sec^2 x}$

$x = 0,\ u = 0$; $x = \frac{\pi}{4},\ u = 1$

$$= \int_0^1 \frac{1}{(\cos x+2\sin x)^2}\,\frac{du}{\sec^2 x} = \int_0^1 \frac{1}{[(\cos x+2\sin x)\sec x]^2}\,du$$

$$= \int_0^1 \frac{1}{(1+2\tan x)^2}\,du = \int_0^1 \frac{1}{(1+2u)^2}\,du$$

$$= \int_0^1 (1+2u)^{-2}\,du = \left[-\tfrac{1}{2}(1+2u)^{-1}\right]_0^1 = \tfrac{1}{2}\left[\frac{1}{1+2u}\right]_1^0$$

$$= \tfrac{1}{2}\left[1-\tfrac{1}{3}\right] = \tfrac{1}{3}$$

ALTERNATIVE BY RECOGNISING THE CHAIN RULE

$$\int_0^{\frac{\pi}{4}} \frac{1}{(\cos x+2\sin x)^2}\,dx = \int_0^{\frac{\pi}{4}} \frac{1}{\cos^2 x\left(1+\frac{2\sin x}{\cos x}\right)^2}\,dx = \int_0^{\frac{\pi}{4}} \frac{1}{\cos^2 x(1+2\tan x)^2}\,dx$$

$$= \int_0^{\frac{\pi}{4}} \sec^2 x(1+2\tan x)^{-2}\,dx = \left[-\tfrac{1}{2}(1+2\tan x)^{-1}\right]_0^{\frac{\pi}{4}}$$

$$= \tfrac{1}{2}\left[\frac{1}{1+2\tan x}\right]_{\frac{\pi}{4}}^0 = \tfrac{1}{2}\left[1-\tfrac{1}{3}\right] = \tfrac{1}{3}$$

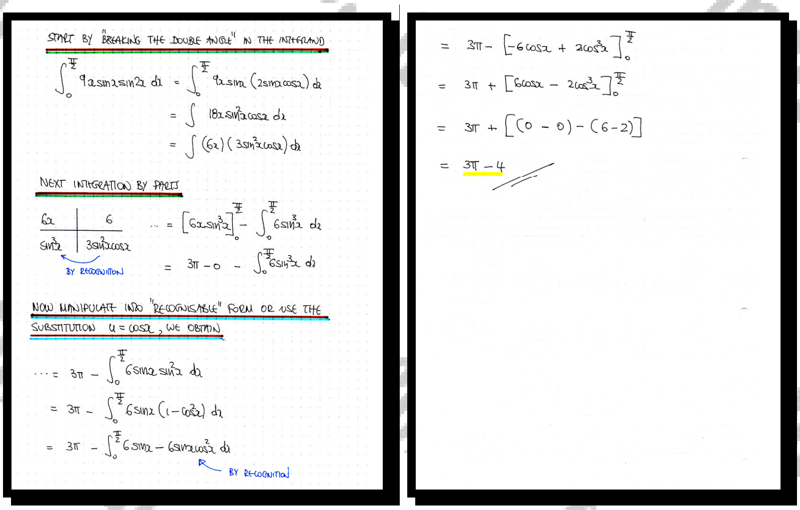
AS BEFORE

Question 207 (****+)

Use trigonometric identities and integration by parts to find an exact value for

$$\int_0^{\frac{\pi}{2}} 9x\sin x\sin 2x \; dx .$$

$\boxed{3\pi - 4}$



START BY "BREAKING THE DOUBLE ANGLE" IN THE INTEGRAND

$\int_0^{\frac{\pi}{2}} 9x\sin x\sin 2x\,dx = \int_0^{\frac{\pi}{2}} 9x\sin x(2\sin x\cos x)\,dx$

$= \int 18x\sin^2 x\cos x\,dx$

$= \int (6x)(3\sin^2 x\cos x)\,dx$

NEXT INTEGRATION BY PARTS

$6x$	6
$\sin^3 x$	$3\sin^2 x\cos x$

BY RECOGNITION

$\dots = \left[6x\sin^3 x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 6\sin^3 x\,dx$

$= 3\pi - 0 - \int_0^{\frac{\pi}{2}} 6\sin^3 x\,dx$

NOW MANIPULATE INTO "RECOGNISABLE" FORM OR USE THE SUBSTITUTION $u = \cos x$, WE OBTAIN

$\dots = 3\pi - \int_0^{\frac{\pi}{2}} 6\sin x\sin^2 x\,dx$

$= 3\pi - \int_0^{\frac{\pi}{2}} 6\sin x(1-\cos^2 x)\,dx$

$= 3\pi - \int_0^{\frac{\pi}{2}} 6\sin x - 6\sin x\cos^2 x\,dx$

BY RECOGNITION

$= 3\pi - \left[-6\cos x + 2\cos^3 x\right]_0^{\frac{\pi}{2}}$

$= 3\pi + \left[6\cos x - 2\cos^3 x\right]_0^{\frac{\pi}{2}}$

$= 3\pi + \left[(0-0)-(6-2)\right]$

$= 3\pi - 4$

Question 208 (****+)

$$I \equiv \int \frac{1}{1+\sin 2x}\,dx\,.$$

a) Integrate I by multiplying the numerator and denominator of the integrand by $(1-\sin 2x)$.

b) Hence evaluate

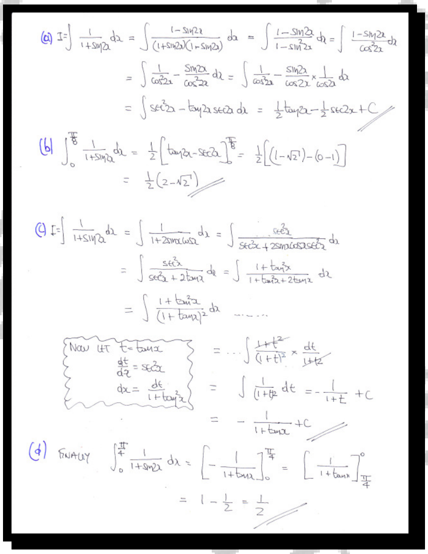
$$\int_0^{\frac{\pi}{8}} \frac{1}{1+\sin 2x}\,dx\,.$$

c) Use the substitution $t = \tan x$ to integrate I .

d) Hence evaluate

$$\int_0^{\frac{\pi}{4}} \frac{1}{1+\sin 2x}\,dx\,.$$

$\boxed{\frac{1}{2}\tan 2x - \frac{1}{2}\sec 2x + C}$, $\boxed{\frac{1}{2}\left(2-\sqrt{2}\right)}$, $\boxed{-\frac{1}{1+\tan x}+C}$, $\boxed{\frac{1}{2}}$



(a) $I = \int \frac{1}{1+\sin 2x}\,dx = \int \frac{1-\sin 2x}{(1+\sin 2x)(1-\sin 2x)}\,dx = \int \frac{1-\sin 2x}{1-\sin^2 2x}\,dx = \int \frac{1-\sin 2x}{\cos^2 2x}\,dx$

$= \int \frac{1}{\cos^2 2x} - \frac{\sin 2x}{\cos^2 2x}\,dx = \int \frac{1}{\cos^2 2x} - \frac{\sin 2x}{\cos 2x} \times \frac{1}{\cos 2x}\,dx$

$= \int \sec^2 2x - \tan 2x \sec 2x\,dx = \frac{1}{2}\tan 2x - \frac{1}{2}\sec 2x + C$

(b) $\int_0^{\frac{\pi}{8}} \frac{1}{1+\sin 2x}\,dx = \frac{1}{2}\left[\tan 2x - \sec 2x\right]_0^{\frac{\pi}{8}} = \frac{1}{2}\left[(1-\sqrt{2}) - (0-1)\right]$

$= \frac{1}{2}(2-\sqrt{2})$

(c) $I = \int \frac{1}{1+\sin 2x}\,dx = \int \frac{1}{1+2\sin x\cos x}\,dx = \int \frac{\sec^2 x}{\sec^2 x + 2\sin x\cos x\sec^2 x}\,dx$

$= \int \frac{\sec^2 x}{\sec^2 x + 2\tan x}\,dx = \int \frac{1+\tan^2 x}{1+\tan^2 x + 2\tan x}\,dx$

$= \int \frac{1+\tan^2 x}{(1+\tan x)^2}\,dx \ \ldots\ldots$

Now let $t = \tan x$, $\frac{dt}{dx} = \sec^2 x$, $dx = \frac{dt}{1+\tan^2 x}$

$= \ldots \int \frac{1+t^2}{(1+t)^2} \times \frac{dt}{1+t^2}$

$= \int \frac{1}{(1+t)^2}\,dt = -\frac{1}{1+t} + C$

$= -\frac{1}{1+\tan x} + C$

(d) Finally $\int_0^{\frac{\pi}{4}} \frac{1}{1+\sin 2x}\,dx = \left[-\frac{1}{1+\tan x}\right]_0^{\frac{\pi}{4}} = \left[\frac{1}{1+\tan x}\right]_{\frac{\pi}{4}}^{0}$

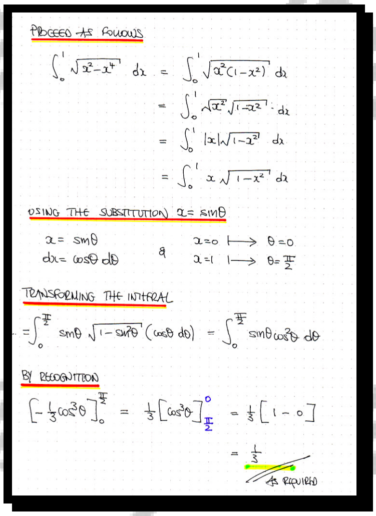
$= 1 - \frac{1}{2} = \frac{1}{2}$

Question 209 (****+)

Show clearly that

$$\int_0^1 \sqrt{x^2 - x^4}\, dx = \frac{1}{3}.$$

proof


PROCEED AS FOLLOWS

$$\int_0^1 \sqrt{x^2-x^4}\,dx = \int_0^1 \sqrt{x^2(1-x^2)}\,dx = \int_0^1 \sqrt{x^2}\sqrt{1-x^2}\,dx = \int_0^1 |x|\sqrt{1-x^2}\,dx = \int_0^1 x\sqrt{1-x^2}\,dx$$

USING THE SUBSTITUTION $x = \sin\theta$

$x = \sin\theta$, $dx = \cos\theta\,d\theta$ & $x = 0 \mapsto \theta = 0$, $x = 1 \mapsto \theta = \frac{\pi}{2}$

TRANSFORMING THE INTEGRAL

$$= \int_0^{\frac{\pi}{2}} \sin\theta\sqrt{1-\sin^2\theta}\,(\cos\theta\,d\theta) = \int_0^{\frac{\pi}{2}} \sin\theta\cos^2\theta\,d\theta$$

BY RECOGNITION

$$\left[-\frac{1}{3}\cos^3\theta\right]_0^{\frac{\pi}{2}} = \frac{1}{3}\left[\cos^3\theta\right]_{\frac{\pi}{2}}^{0} = \frac{1}{3}[1-0] = \frac{1}{3}$$

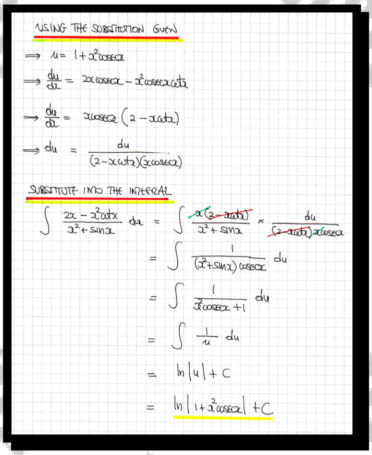
AS REQUIRED

Question 210 (****+)

Use the substitution $u = 1 + x^2 \operatorname{cosec} x$ to find an expression for

$$\int \frac{2x - x^2 \cot x}{x^2 + \sin x}\,dx .$$

$\boxed{}$, $\boxed{\ln\left|1 + x^2 \operatorname{cosec} x\right| + C}$



USING THE SUBSTITUTION GIVEN

$\Rightarrow u = 1 + x^2 \operatorname{cosec} x$

$\Rightarrow \frac{du}{dx} = 2x\operatorname{cosec} x - x^2 \operatorname{cosec} x \cot x$

$\Rightarrow \frac{du}{dx} = x\operatorname{cosec} x\,(2 - x\cot x)$

$\Rightarrow du = \frac{du}{(2 - x\cot x)(x\operatorname{cosec} x)}$

SUBSTITUTE INTO THE INTEGRAL

$$\int \frac{2x - x^2\cot x}{x^2 + \sin x}\,dx = \int \frac{x(2 - x\cot x)}{x^2 + \sin x} \times \frac{du}{(2 - x\cot x)x\operatorname{cosec} x}$$

$$= \int \frac{1}{(x^2 + \sin x)\operatorname{cosec} x}\,du$$

$$= \int \frac{1}{x^2\operatorname{cosec} x + 1}\,du$$

$$= \int \frac{1}{u}\,du$$

$$= \ln|u| + C$$

$$= \ln\left|1 + x^2\operatorname{cosec} x\right| + C$$

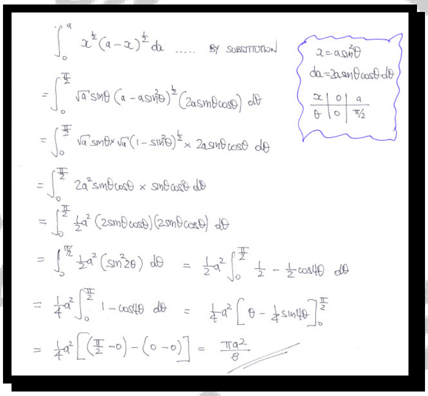
Question 211 (****+)

Use a suitable trigonometric substitution to find an exact simplified value for

$$\int_0^a x^{\frac{1}{2}}\sqrt{a-x}\ dx,$$

where a is a positive constant.

$$\boxed{\frac{1}{8}\pi a^2}$$



$$\int_0^a x^{\frac{1}{2}}(a-x)^{\frac{1}{2}}\,dx \ \ldots\ldots \text{ BY SUBSTITUTION}$$

$x = a\sin^2\theta$, $dx = 2a\sin\theta\cos\theta\,d\theta$; $x: 0 \to a$, $\theta: 0 \to \frac{\pi}{2}$

$$= \int_0^{\frac{\pi}{2}} \sqrt{a}\sin\theta\,(a - a\sin^2\theta)^{\frac{1}{2}}(2a\sin\theta\cos\theta)\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{a}\sin\theta \times \sqrt{a}(1-\sin^2\theta)^{\frac{1}{2}} \times 2a\sin\theta\cos\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} 2a^2\sin\theta\cos\theta \times \sin\theta\cos\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2}a^2(2\sin\theta\cos\theta)(2\sin\theta\cos\theta)\,d\theta$$

$$= \int_0^{\pi/2} \frac{1}{2}a^2(\sin^2 2\theta)\,d\theta = \frac{1}{2}a^2\int_0^{\frac{\pi}{2}} \frac{1}{2} - \frac{1}{2}\cos 4\theta\,d\theta$$

$$= \frac{1}{4}a^2\int_0^{\frac{\pi}{2}} 1 - \cos 4\theta\,d\theta = \frac{1}{4}a^2\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{4}a^2\left[\left(\frac{\pi}{2}-0\right)-(0-0)\right] = \frac{\pi a^2}{8}$$

Question 212 (****+)

$$f(u) \equiv \frac{1}{u^2+5u+6}.$$

a) Express $f(u)$ into partial fractions.

$$I = \int_{\arcsin\frac{3}{5}}^{\arccos\frac{3}{5}} \frac{1}{(\sin x + 2\cos x)(\sin x + 3\cos x)}\, dx.$$

b) Express I in the form

$$I = \int_{\arcsin\frac{3}{5}}^{\arccos\frac{3}{5}} \frac{\sec^2 x}{g(\tan x)}\, dx,$$

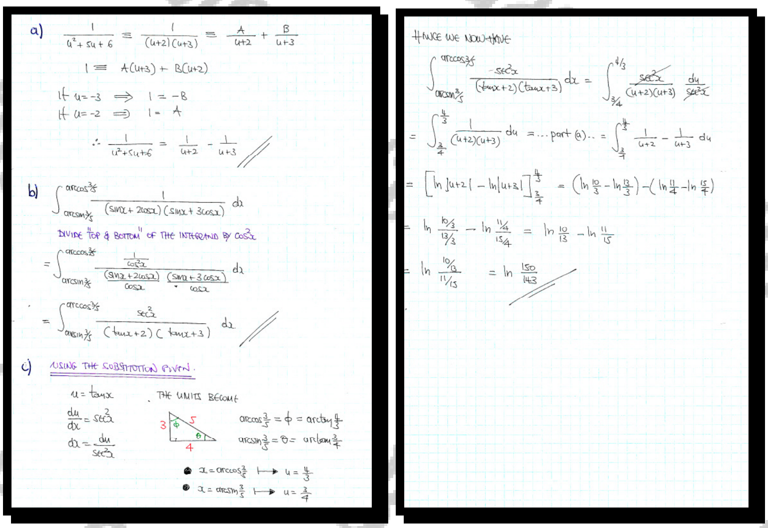
where g is a function to be found.

c) Hence show that

$$I = \ln\left(\frac{a}{b}\right),$$

where a and b are positive integers to be found.

MP2, $f(u) \equiv \frac{1}{u+2} - \frac{1}{u+3}$, $g(\tan x) \equiv (2+\tan x)(3+\tan x)$, $I = \ln\left(\frac{150}{143}\right)$


a) $\frac{1}{u^2+5u+6} = \frac{1}{(u+2)(u+3)} = \frac{A}{u+2} + \frac{B}{u+3}$

$1 \equiv A(u+3) + B(u+2)$

If $u=-3 \Rightarrow 1 = -B$

If $u=-2 \Rightarrow 1 = A$

$\therefore \frac{1}{u^2+5u+6} = \frac{1}{u+2} - \frac{1}{u+3}$

b) $\int_{\arcsin\frac{3}{5}}^{\arccos\frac{3}{5}} \frac{1}{(\sin x+2\cos x)(\sin x+3\cos x)}\,dx$

DIVIDE "TOP & BOTTOM" OF THE INTEGRAND BY $\cos^2 x$

$= \int_{\arcsin\frac{3}{5}}^{\arccos\frac{3}{5}} \frac{\frac{1}{\cos^2 x}}{\frac{(\sin x+2\cos x)}{\cos x}\cdot\frac{(\sin x+3\cos x)}{\cos x}}\,dx$

$= \int_{\arcsin\frac{3}{5}}^{\arccos\frac{3}{5}} \frac{\sec^2 x}{(\tan x+2)(\tan x+3)}\,dx$

c) USING THE SUBSTITUTION GIVEN

$u = \tan x$, $\frac{du}{dx} = \sec^2 x$, $dx = \frac{du}{\sec^2 x}$

THE LIMITS BECOME

$\arccos\frac{3}{5} = \phi = \arctan\frac{4}{3}$, $\arcsin\frac{3}{5} = \theta = \arctan\frac{3}{4}$

$x = \arccos\frac{3}{5} \mapsto u = \frac{4}{3}$

$x = \arcsin\frac{3}{5} \mapsto u = \frac{3}{4}$

HENCE WE NOW HAVE

$\int_{\arcsin\frac{3}{5}}^{\arccos\frac{3}{5}} \frac{\sec^2 x}{(\tan x+2)(\tan x+3)}\,dx = \int_{3/4}^{4/3} \frac{\sec^2 x}{(u+2)(u+3)}\,\frac{du}{\sec^2 x}$

$= \int_{3/4}^{4/3} \frac{1}{(u+2)(u+3)}\,du = \ldots$ part (a) $\ldots = \int_{3/4}^{4/3} \frac{1}{u+2} - \frac{1}{u+3}\,du$

$= \Big[\ln|u+2| - \ln|u+3|\Big]_{3/4}^{4/3} = \left(\ln\frac{10}{3} - \ln\frac{13}{3}\right) - \left(\ln\frac{11}{4} - \ln\frac{15}{4}\right)$

$= \ln\frac{10/3}{13/3} - \ln\frac{11/4}{15/4} = \ln\frac{10}{13} - \ln\frac{11}{15}$

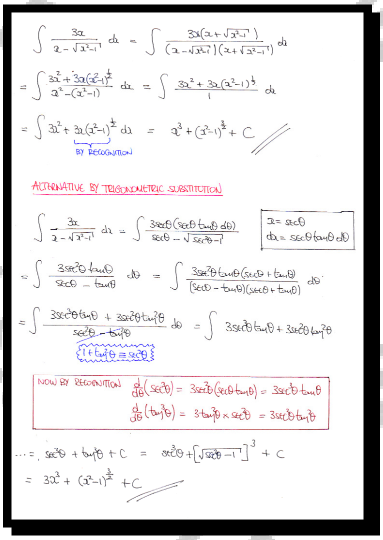
$= \ln\frac{10/13}{11/15} = \ln\frac{150}{143}$

Question 213 (****+)

Find in exact simplified form an expression for

$$\int \frac{3x}{x-\sqrt{x^2-1}}\,dx .$$

$\boxed{}$, $\boxed{x^3+\left(x^2-1\right)^{\frac{3}{2}}+C}$



$$\int \frac{3x}{x-\sqrt{x^2-1}}\,dx = \int \frac{3x\left(x+\sqrt{x^2-1}\right)}{\left(x-\sqrt{x^2-1}\right)\left(x+\sqrt{x^2-1}\right)}\,dx$$

$$= \int \frac{3x^2+3x(x^2-1)^{\frac{1}{2}}}{x^2-(x^2-1)}\,dx = \int \frac{3x^2+3x(x^2-1)^{\frac{1}{2}}}{1}\,dx$$

$$= \int 3x^2+3x(x^2-1)^{\frac{1}{2}}\,dx = x^3+(x^2-1)^{\frac{3}{2}}+C$$

BY RECOGNITION

ALTERNATIVE BY TRIGONOMETRIC SUBSTITUTION

$$\int \frac{3x}{x-\sqrt{x^2-1}}\,dx = \int \frac{3\sec\theta(\sec\theta\tan\theta\,d\theta)}{\sec\theta-\sqrt{\sec^2\theta-1}}$$

$x=\sec\theta$
$dx=\sec\theta\tan\theta\,d\theta$

$$= \int \frac{3\sec^2\theta\tan\theta}{\sec\theta-\tan\theta}\,d\theta = \int \frac{3\sec^2\theta\tan\theta(\sec\theta+\tan\theta)}{(\sec\theta-\tan\theta)(\sec\theta+\tan\theta)}\,d\theta$$

$$= \int \frac{3\sec^3\theta\tan\theta+3\sec^2\theta\tan^2\theta}{\sec^2\theta-\tan^2\theta}\,d\theta = \int 3\sec^3\theta\tan\theta+3\sec^2\theta\tan^2\theta$$

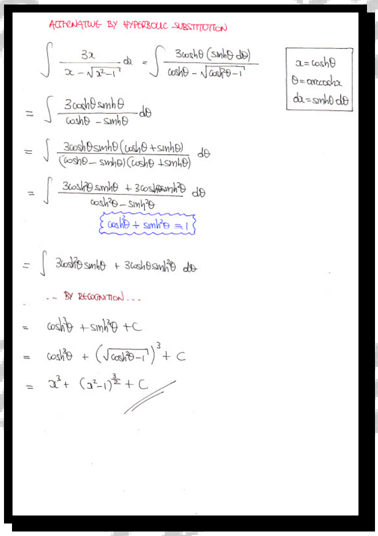
$\{1+\tan^2\theta\equiv\sec^2\theta\}$

NOW BY RECOGNITION $\frac{d}{d\theta}(\sec^3\theta)=3\sec^2\theta(\sec\theta\tan\theta)=3\sec^3\theta\tan\theta$

$\frac{d}{d\theta}(\tan^3\theta)=3\tan^2\theta\times\sec^2\theta=3\sec^2\theta\tan^2\theta$

$$\dots = \sec^3\theta+\tan^3\theta+C = \sec^3\theta+\left[\sqrt{\sec^2\theta-1}\right]^3+C$$

$$= 3x^3+(x^2-1)^{\frac{3}{2}}+C$$



ALTERNATIVE BY HYPERBOLIC SUBSTITUTION

$$\int \frac{3x}{x-\sqrt{x^2-1}}\,dx = \int \frac{3\cosh\theta(\sinh\theta\,d\theta)}{\cosh\theta-\sqrt{\cosh^2\theta-1}}$$

$x=\cosh\theta$
$\theta=\text{arcosh}\,x$
$dx=\sinh\theta\,d\theta$

$$= \int \frac{3\cosh\theta\sinh\theta}{\cosh\theta-\sinh\theta}\,d\theta$$

$$= \int \frac{3\cosh\theta\sinh\theta(\cosh\theta+\sinh\theta)}{(\cosh\theta-\sinh\theta)(\cosh\theta+\sinh\theta)}\,d\theta$$

$$= \int \frac{3\cosh^2\theta\sinh\theta+3\cosh\theta\sinh^2\theta}{\cosh^2\theta-\sinh^2\theta}\,d\theta$$

$\{\cosh^2\theta+\sinh^2\theta\equiv1\}$

$$= \int 3\cosh^2\theta\sinh\theta+3\cosh\theta\sinh^2\theta\,d\theta$$

... BY RECOGNITION ...

$$= \cosh^3\theta+\sinh^3\theta+C$$

$$= \cosh^3\theta+\left(\sqrt{\cosh^2\theta-1}\right)^3+C$$

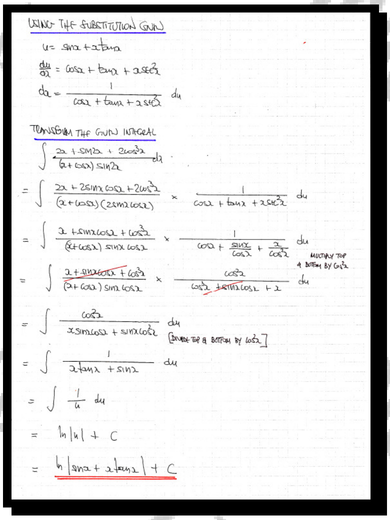
$$= x^3+(x^2-1)^{\frac{3}{2}}+C$$

Question 214 (****+)

Use the substitution $u=\sin x + x\tan x$ to find an expression for

$$\int \frac{2x+\sin 2x+2\cos^3 x}{(x+\cos x)\sin 2x}\,dx\,.$$

$\boxed{}$, $\boxed{\ln\left|\sin x + x\tan x\right| + C}$



USING THE SUBSTITUTION GIVEN

$u=\sin x + x\tan x$

$\frac{du}{dx}=\cos x+\tan x+x\sec^2 x$

$dx=\frac{1}{\cos x+\tan x+x\sec^2 x}\,du$

TRANSFORM THE GIVEN INTEGRAL

$\int \frac{2x+\sin 2x+2\cos^3 x}{(x+\cos x)\sin 2x}\,dx$

$=\int \frac{2x+2\sin x\cos x+2\cos^3 x}{(x+\cos x)(2\sin x\cos x)} \times \frac{1}{\cos x+\tan x+x\sec^2 x}\,du$

$=\int \frac{x+\sin x\cos x+\cos^3 x}{(x+\cos x)\sin x\cos x} \times \frac{1}{\cos x+\frac{\sin x}{\cos x}+\frac{x}{\cos^2 x}}\,du$ MULTIPLY TOP & BOTTOM BY $\cos^2 x$

$=\int \frac{x+\sin x\cos x+\cos^3 x}{(x+\cos x)\sin x\cos x} \times \frac{\cos^2 x}{\cos^3 x+\sin x\cos x+x}\,du$

$=\int \frac{\cos^2 x}{x\sin x\cos x+\sin x\cos^2 x}\,du$ [DIVIDE TOP & BOTTOM BY $\cos^2 x$]

$=\int \frac{1}{x\tan x+\sin x}\,du$

$=\int \frac{1}{u}\,du$

$=\ln|u|+C$

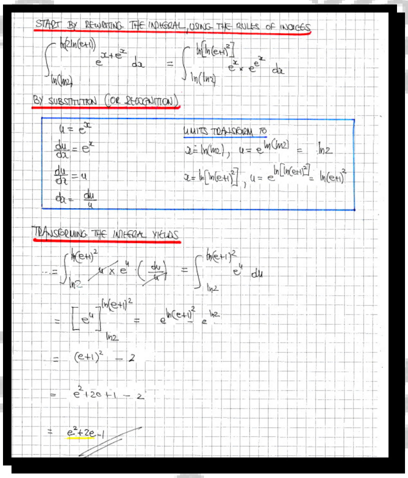
$=\ln\left|\sin x+x\tan x\right|+C$

Question 215 (****+)

Find, in exact simplified form, the value of

$$\int_{\ln(\ln 2)}^{\ln\left[2\ln(\mathrm{e}+1)\right]} \mathrm{e}^{x+\mathrm{e}^{x}} \, dx \, .$$

$\boxed{}$, $\boxed{\mathrm{e}^2+2\mathrm{e}-1}$

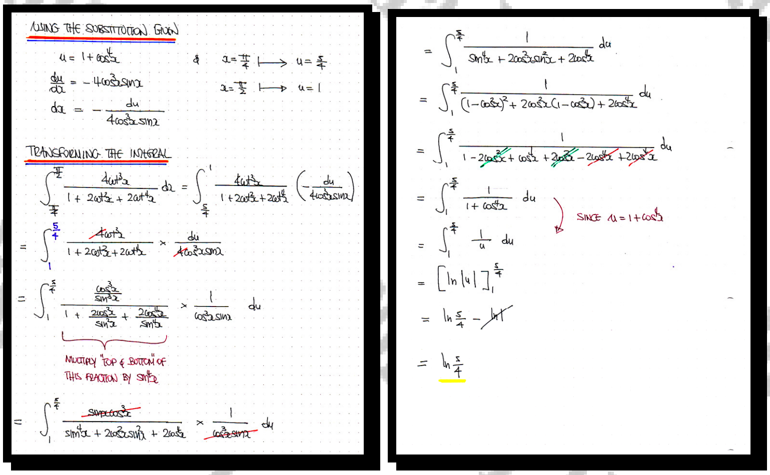


Question 216 (*****)

By using the substitution $u = 1 + \cos^4 x$, or otherwise, find the exact value of

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{4\cot^3 x}{1 + 2\cot^2 x + 2\cot^4 x} \, dx .$$

$\boxed{}$, $\boxed{\ln\left(\frac{5}{4}\right)}$



Question 217 (*****)

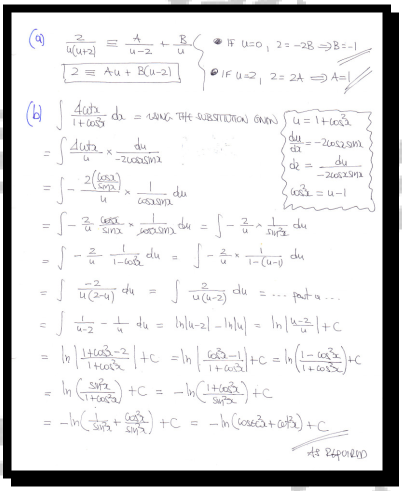
$$\frac{2}{u(u-2)} \equiv \frac{A}{u-2} + \frac{B}{u}.$$

a) Find the value of each of the constants A and B.

b) By using the substitution $u = 1+\cos^2 x$, or otherwise, show clearly that

$$\int \frac{4\cot x}{1+\cos^2 x}\, dx = -\ln\left(\operatorname{cosec}^2 x + \cot^2 x\right) + C.$$

$\boxed{A=1}$, $\boxed{B=1}$


(a) $\frac{2}{u(u+2)} \equiv \frac{A}{u-2} + \frac{B}{u}$

$\boxed{2 \equiv Au + B(u-2)}$

• IF $u=0$, $2=-2B \Rightarrow B=-1$

• IF $u=2$, $2=2A \Rightarrow A=1$

(b) $\int \frac{4\cot x}{1+\cos^2 x}\,dx$ = USING THE SUBSTITUTION GIVEN

$u = 1+\cos^2 x$

$\frac{du}{dx} = -2\cos x\sin x$

$dx = \frac{du}{-2\cos x\sin x}$

$\cos^2 x = u-1$

$= \int \frac{4\cot x}{u} \times \frac{du}{-2\cos x\sin x}$

$= \int -\frac{2\left(\frac{\cos x}{\sin x}\right)}{u} \times \frac{1}{\cos x\sin x}\,du$

$= \int -\frac{2}{u}\frac{\cos x}{\sin x} \times \frac{1}{\cos x\sin x}\,du = \int -\frac{2}{u} \times \frac{1}{\sin^2 x}\,du$

$= \int -\frac{2}{u}\frac{1}{1-\cos^2 x}\,du = \int -\frac{2}{u} \times \frac{1}{1-(u-1)}\,du$

$= \int \frac{-2}{u(2-u)}\,du = \int \frac{2}{u(u-2)}\,du = \dots$ part a $\dots$

$= \int \frac{1}{u-2} - \frac{1}{u}\,du = \ln|u-2| - \ln|u| = \ln\left|\frac{u-2}{u}\right| + C$

$= \ln\left|\frac{1+\cos^2 x-2}{1+\cos^2 x}\right| + C = \ln\left|\frac{\cos^2 x-1}{1+\cos^2 x}\right| + C = \ln\left(\frac{1-\cos^2 x}{1+\cos^2 x}\right) + C$

$= \ln\left(\frac{\sin^2 x}{1+\cos^2 x}\right) + C = -\ln\left(\frac{1+\cos^2 x}{\sin^2 x}\right) + C$

$= -\ln\left(\frac{1}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x}\right) + C = -\ln\left(\operatorname{cosec}^2 x + \cot^2 x\right) + C$

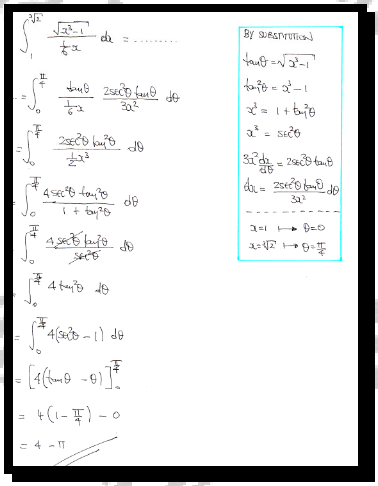
AS REQUIRED

Question 218 (*****)

By using the substitution $\tan\theta = \sqrt{x^3 - 1}$, or otherwise, find an exact value for the following integral.

$$\int_{1}^{\sqrt[3]{2}} \frac{\sqrt{x^3-1}}{\frac{1}{6}x} \, dx .$$

$\boxed{}$, $\boxed{4-\pi}$

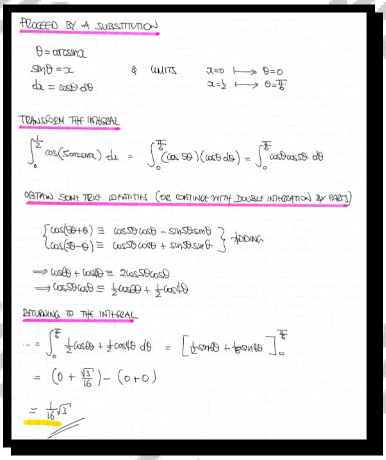


Question 219 (*****)

Use appropriate integration methods to find an exact simplified value for

$$\int_0^{\frac{1}{2}} \cos(5\arcsin x) \, dx .$$

V, □, $\boxed{\dfrac{\sqrt{3}}{16}}$



PROCEED BY A SUBSTITUTION

$\theta = \arcsin x$

$\sin\theta = x$ & LIMITS $x=0 \longmapsto \theta = 0$

$dx = \cos\theta \, d\theta$ $\qquad x=\frac{1}{2} \longmapsto \theta = \frac{\pi}{6}$

TRANSFORM THE INTEGRAL

$$\int_0^{\frac{1}{2}} \cos(5\arcsin x)\,dx = \int_0^{\frac{\pi}{6}} (\cos 5\theta)(\cos\theta\,d\theta) = \int_0^{\frac{\pi}{6}} \cos\theta\cos 5\theta \, d\theta$$

OBTAIN SOME TRIG IDENTITIES (OR CONTINUE WITH DOUBLE INTEGRATION BY PARTS)

$$\left.\begin{array}{l}\cos(5\theta+\theta) \equiv \cos 5\theta\cos\theta - \sin 5\theta\sin\theta \\ \cos(5\theta-\theta) \equiv \cos 5\theta\cos\theta + \sin 5\theta\sin\theta\end{array}\right\} \text{ADDING}$$

$\Longrightarrow \cos 6\theta + \cos 4\theta \equiv 2\cos 5\theta\cos\theta$

$\Longrightarrow \cos 5\theta\cos\theta \equiv \frac{1}{2}\cos 6\theta + \frac{1}{2}\cos 4\theta$

RETURNING TO THE INTEGRAL

$$\dots = \int_0^{\frac{\pi}{6}} \tfrac{1}{2}\cos 6\theta + \tfrac{1}{2}\cos 4\theta \, d\theta = \left[\tfrac{1}{12}\sin 6\theta + \tfrac{1}{8}\sin 4\theta\right]_0^{\frac{\pi}{6}}$$

$$= \left(0 + \frac{\sqrt{3}}{16}\right) - (0+0)$$

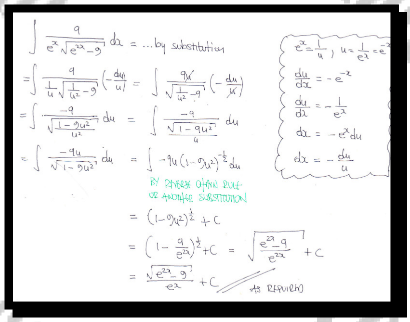
$$= \frac{1}{16}\sqrt{3}$$

Question 220 (*****)

By using the substitution $e^x = \frac{1}{u}$, or otherwise, show clearly that

$$\int \frac{9}{e^x\sqrt{e^{2x}-9}}\,dx = \frac{\sqrt{e^{2x}-9}}{e^x} + C.$$

proof

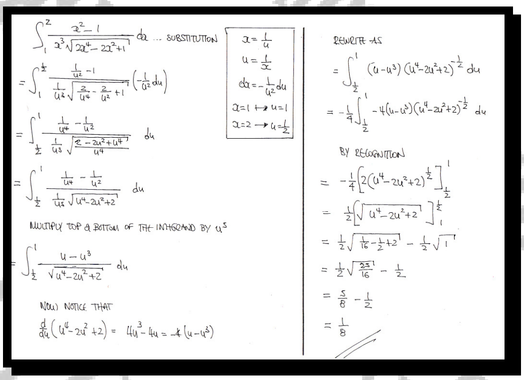


Question 221 (*****)

By using a reciprocal substitution, or otherwise, find the value of the following integral.

$$\int_1^2 \frac{x^2-1}{x^3\sqrt{2x^4-2x^2+1}}\,dx.$$

[ ], $\frac{1}{8}$

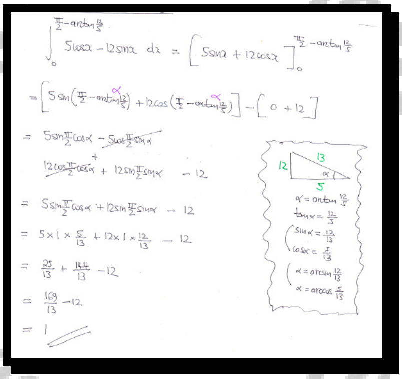


Question 222 **(*****) non calculator**

Show clearly that

$$\int_0^{\frac{\pi}{2}-\arctan\frac{12}{5}} 5\cos x - 12\sin x \ dx = 1.$$

proof

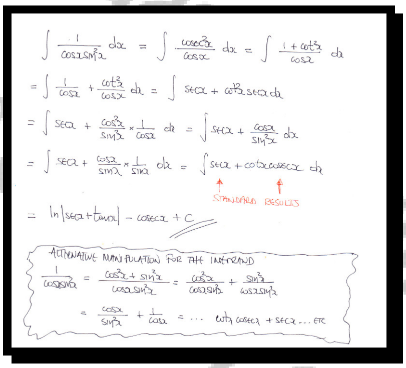


Question 223 **(*****)**

Use trigonometric identities to find

$$\int \frac{1}{\cos x \sin^2 x} \, dx.$$

$\ln|\sec x + \tan x| - \text{cosec}\, x + C$



Question 224 (*****)

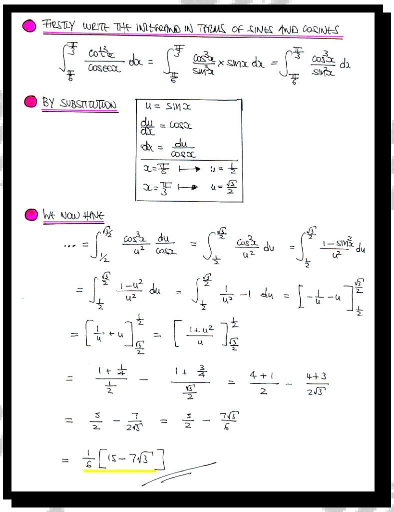
$$I = \int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{\cot^3 x}{\operatorname{cosec} x}\ dx .$$

Use appropriate integration techniques to show that

$$I = \frac{1}{6}\left[a + b\sqrt{3}\right],$$

where a and b are integers to be found.

$I = \frac{1}{6}\left[15 - 7\sqrt{3}\right]$

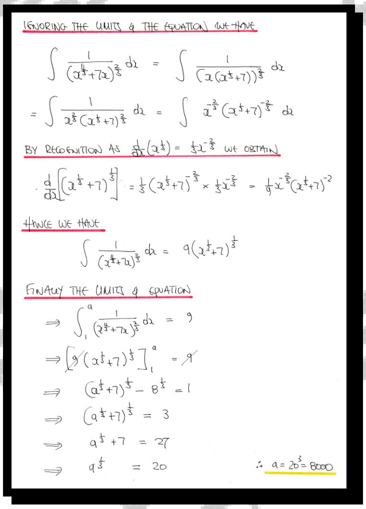


Question 225 (*****)

$$I = \int_{1}^{a} \frac{1}{\left(x^{\frac{4}{3}} + 7x\right)^{\frac{2}{3}}} \, dx .$$

Given that $I = 9$, determine the value a.

[ ] , $a = 8000$


IGNORING THE LIMITS & THE EQUATION WE HAVE

$$\int \frac{1}{(x^{\frac{4}{3}}+7x)^{\frac{2}{3}}}\,dx = \int \frac{1}{(x(x^{\frac{1}{3}}+7))^{\frac{2}{3}}}\,dx$$

$$= \int \frac{1}{x^{\frac{2}{3}}(x^{\frac{1}{3}}+7)^{\frac{2}{3}}}\,dx = \int x^{-\frac{2}{3}}(x^{\frac{1}{3}}+7)^{-\frac{2}{3}}\,dx$$

BY RECOGNITION AS $\frac{d}{dx}(x^{\frac{1}{3}}) = \frac{1}{3}x^{-\frac{2}{3}}$ WE OBTAIN

$$\frac{d}{dx}\left[(x^{\frac{1}{3}}+7)^{\frac{1}{3}}\right] = \frac{1}{3}(x^{\frac{1}{3}}+7)^{-\frac{2}{3}} \times \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{9}x^{-\frac{2}{3}}(x^{\frac{1}{3}}+7)^{-2}$$

HENCE WE HAVE

$$\int \frac{1}{(x^{\frac{4}{3}}+7x)^{\frac{2}{3}}}\,dx = 9(x^{\frac{1}{3}}+7)^{\frac{1}{3}}$$

FINALLY THE LIMITS & EQUATION

$$\Rightarrow \int_1^a \frac{1}{(x^{\frac{4}{3}}+7x)^{\frac{2}{3}}}\,dx = 9$$

$$\Rightarrow \left[9(x^{\frac{1}{3}}+7)^{\frac{1}{3}}\right]_1^a = 9$$

$$\Rightarrow (a^{\frac{1}{3}}+7)^{\frac{1}{3}} - 8^{\frac{1}{3}} = 1$$

$$\Rightarrow (a^{\frac{1}{3}}+7)^{\frac{1}{3}} = 3$$

$$\Rightarrow a^{\frac{1}{3}}+7 = 27$$

$$\Rightarrow a^{\frac{1}{3}} = 20$$

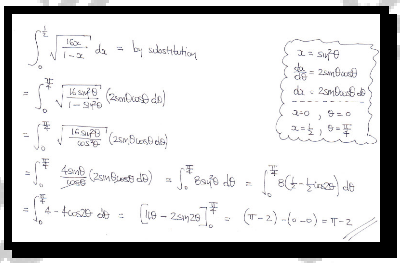
$\therefore a = 20^3 = 8000$

Question 226 (*****)

By using a suitable trigonometric substitution, show clearly that

$$\int_{0}^{\frac{1}{2}} \sqrt{\frac{16x}{1-x}} \, dx = \pi - 2 .$$

proof


$$\int_0^{\frac{1}{2}} \sqrt{\frac{16x}{1-x}}\,dx = \text{by substitution}$$

$x = \sin^2\theta$, $\frac{dx}{d\theta} = 2\sin\theta\cos\theta$, $dx = 2\sin\theta\cos\theta\,d\theta$; $x=0,\ \theta=0$; $x=\frac{1}{2},\ \theta=\frac{\pi}{4}$

$$= \int_0^{\frac{\pi}{4}} \sqrt{\frac{16\sin^2\theta}{1-\sin^2\theta}}\,(2\sin\theta\cos\theta\,d\theta)$$

$$= \int_0^{\frac{\pi}{4}} \sqrt{\frac{16\sin^2\theta}{\cos^2\theta}}\,(2\sin\theta\cos\theta\,d\theta)$$

$$= \int_0^{\frac{\pi}{4}} \frac{4\sin\theta}{\cos\theta}(2\sin\theta\cos\theta\,d\theta) = \int_0^{\frac{\pi}{4}} 8\sin^2\theta\,d\theta = \int_0^{\frac{\pi}{4}} 8\left(\frac{1}{2}-\frac{1}{2}\cos2\theta\right)d\theta$$

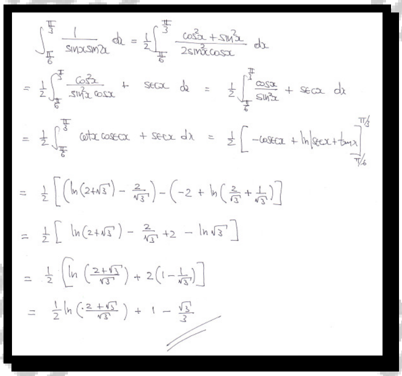
$$= \int_0^{\frac{\pi}{4}} 4 - 4\cos2\theta\,d\theta = \left[4\theta - 2\sin2\theta\right]_0^{\frac{\pi}{4}} = (\pi-2)-(0-0) = \pi-2$$

Question 227 (*****)

Find an exact simplified value for

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{\sin x \sin 2x} \, dx .$$

$$\boxed{\frac{1}{2}\ln\left[\frac{2+\sqrt{3}}{\sqrt{3}}\right]+1-\frac{\sqrt{3}}{3}}$$



$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{\sin x \sin 2x}\,dx = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos^2 x+\sin^2 x}{2\sin^2 x\cos x}\,dx$$

$$= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos^2 x}{\sin^2 x\cos x} + \sec x\,dx = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos x}{\sin^2 x} + \sec x\,dx$$

$$= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cot x\csc x + \sec x\,dx = \frac{1}{2}\Big[-\csc x + \ln|\sec x+\tan x|\Big]_{\pi/6}^{\pi/3}$$

$$= \frac{1}{2}\left[\left(\ln(2+\sqrt{3}) - \frac{2}{\sqrt{3}}\right) - \left(-2 + \ln\left(\frac{2}{\sqrt{3}}+\frac{1}{\sqrt{3}}\right)\right)\right]$$

$$= \frac{1}{2}\left[\ln(2+\sqrt{3}) - \frac{2}{\sqrt{3}} + 2 - \ln\sqrt{3}\right]$$

$$= \frac{1}{2}\left[\ln\left(\frac{2+\sqrt{3}}{\sqrt{3}}\right) + 2\left(1-\frac{1}{\sqrt{3}}\right)\right]$$

$$= \frac{1}{2}\ln\left(\frac{2+\sqrt{3}}{\sqrt{3}}\right) + 1 - \frac{\sqrt{3}}{3}$$

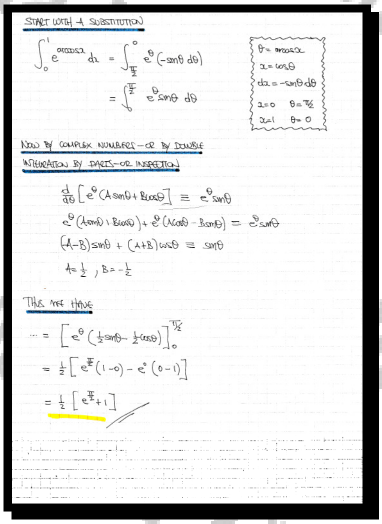
Question 228 (*****)

Evaluate the following definite integral.

$$\int_0^1 e^{\arccos x} \; dx .$$

Give the answer in exact simplified form.

$\boxed{}$, $\boxed{\text{proof}}$



START WITH A SUBSTITUTION

$$\int_0^1 e^{\arccos x}\,dx = \int_{\frac{\pi}{2}}^{0} e^{\theta}(-\sin\theta\,d\theta) = \int_0^{\frac{\pi}{2}} e^{\theta}\sin\theta\,d\theta$$

$\theta = \arccos x$, $x = \cos\theta$, $dx = -\sin\theta\,d\theta$, $x=0 \;\; \theta = \frac{\pi}{2}$, $x=1 \;\; \theta = 0$

NOW BY COMPLEX NUMBERS – OR BY DOUBLE INTEGRATION BY PARTS – OR INSPECTION

$$\frac{d}{d\theta}\left[e^{\theta}(A\sin\theta + B\cos\theta)\right] \equiv e^{\theta}\sin\theta$$

$$e^{\theta}(A\sin\theta + B\cos\theta) + e^{\theta}(A\cos\theta - B\sin\theta) \equiv e^{\theta}\sin\theta$$

$$(A-B)\sin\theta + (A+B)\cos\theta \equiv \sin\theta$$

$$A = \tfrac{1}{2}, \; B = -\tfrac{1}{2}$$

THUS WE HAVE

$$\dots = \left[e^{\theta}\left(\tfrac{1}{2}\sin\theta - \tfrac{1}{2}\cos\theta\right)\right]_0^{\frac{\pi}{2}}$$

$$= \tfrac{1}{2}\left[e^{\frac{\pi}{2}}(1-0) - e^{0}(0-1)\right]$$

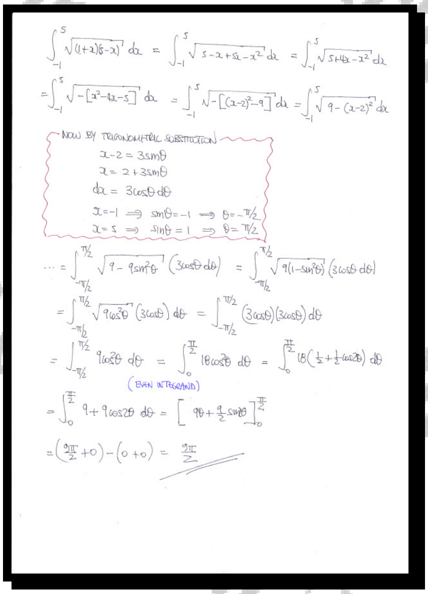
$$= \tfrac{1}{2}\left[e^{\frac{\pi}{2}} + 1\right]$$

Question 229 (*****)

Use a suitable trigonometric substitution to find the exact value of

$$\int_{-1}^{5} \sqrt{(1+x)(5-x)} \, dx .$$

$$\boxed{\frac{9\pi}{2}}$$



$$\int_{-1}^{5} \sqrt{(1+x)(5-x)}\,dx = \int_{-1}^{5} \sqrt{5-x+5x-x^2}\,dx = \int_{-1}^{5} \sqrt{5+4x-x^2}\,dx$$

$$= \int_{-1}^{5} \sqrt{-[x^2-4x-5]}\,dx = \int_{-1}^{5} \sqrt{-[(x-2)^2-9]}\,dx = \int_{-1}^{5} \sqrt{9-(x-2)^2}\,dx$$

NOW BY TRIGONOMETRIC SUBSTITUTION

$x-2 = 3\sin\theta$

$x = 2+3\sin\theta$

$dx = 3\cos\theta\,d\theta$

$x=-1 \Rightarrow \sin\theta=-1 \Rightarrow \theta=-\pi/2$

$x=5 \Rightarrow \sin\theta=1 \Rightarrow \theta=\pi/2$

$$\dots = \int_{-\pi/2}^{\pi/2} \sqrt{9-9\sin^2\theta}\,(3\cos\theta\,d\theta) = \int_{-\pi/2}^{\pi/2} \sqrt{9(1-\sin^2\theta)}\,(3\cos\theta\,d\theta)$$

$$= \int_{-\pi/2}^{\pi/2} \sqrt{9\cos^2\theta}\,(3\cos\theta)\,d\theta = \int_{-\pi/2}^{\pi/2} (3\cos\theta)(3\cos\theta)\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2} 9\cos^2\theta\,d\theta = \int_{0}^{\frac{\pi}{2}} 18\cos^2\theta\,d\theta = \int_{0}^{\frac{\pi}{2}} 18\left(\tfrac{1}{2}+\tfrac{1}{2}\cos 2\theta\right)d\theta$$

(EVEN INTEGRAND)

$$= \int_{0}^{\frac{\pi}{2}} 9+9\cos 2\theta\,d\theta = \left[9\theta+\tfrac{9}{2}\sin 2\theta\right]_{0}^{\frac{\pi}{2}}$$

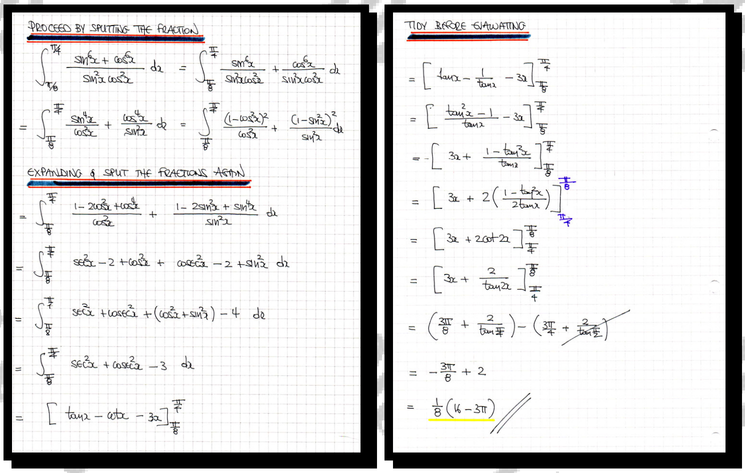
$$= \left(\tfrac{9\pi}{2}+0\right)-(0+0) = \tfrac{9\pi}{2}$$

Question 230 (*****)

By using trigonometric identities, show that

$$\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cos^2 x} \, dx = \frac{1}{8}(16 - 3\pi).$$

, proof

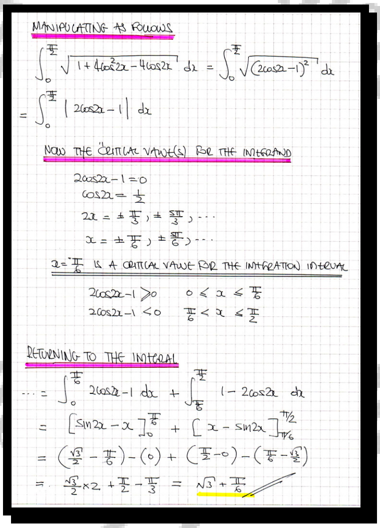


Question 231 (*****)

Find, in exact simplified form, the value of the following integral.

$$\int_0^{\frac{1}{2}\pi} \sqrt{1+4\cos^2 2x - 4\cos 2x} \; dx .$$

$\boxed{}$, $\boxed{\frac{1}{6}\pi + \sqrt{3}}$

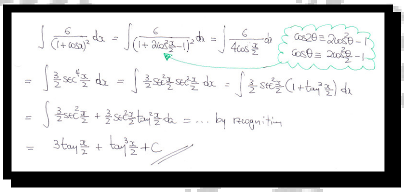


Question 232 (*****)

By using a suitable cosine double angle trigonometric identity find

$$\int \frac{6}{(1+\cos x)^2} \, dx .$$

$$\boxed{3\tan\frac{x}{2} + \tan^3\frac{x}{2} + C}$$

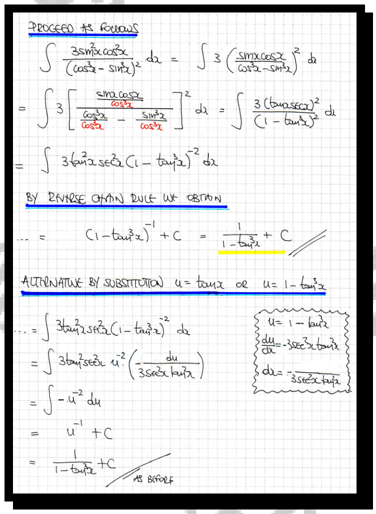


Question 233 (*****)

By expressing the integrand in the form $\sec^2 x\, f(\tan x)$, or otherwise, find a simplified expression for the following integral.

$$\int \frac{3\sin^2 x \cos^2 x}{\left(\cos^3 x - \sin^3 x\right)^2}\, dx .$$

$\boxed{}$, $\boxed{\frac{1}{1-\tan^3 x}+C}$



Question 234 (*****)

$$\sec x \equiv \frac{1+\tan^2\left(\frac{x}{2}\right)}{1-\tan^2\left(\frac{x}{2}\right)}.$$

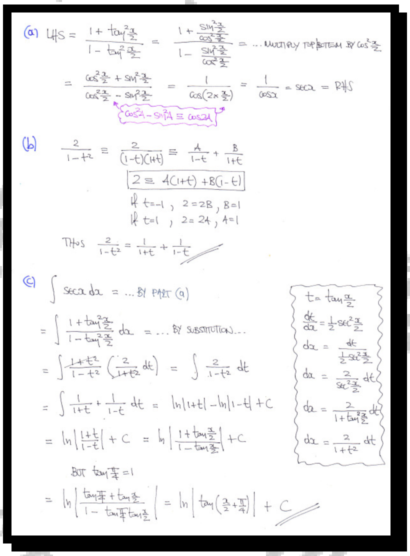
a) Prove the validity of the above trigonometric identity.

b) Express $\dfrac{2}{1-t^2}$ into partial fractions.

c) Hence use the substitution $t=\tan\left(\frac{x}{2}\right)$ to show that

$$\int \sec x \, dx = \ln\left|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right| + C.$$

$$\boxed{\frac{2}{1-t^2}=\frac{1}{1+t}+\frac{1}{1-t}}$$


(a) LHS $= \dfrac{1+\tan^2\frac{x}{2}}{1-\tan^2\frac{x}{2}} = \dfrac{1+\frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}}{1-\frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}} = \ldots$ MULTIPLY TOP/BOTTOM BY $\cos^2\frac{x}{2}$

$= \dfrac{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}} = \dfrac{1}{\cos(2\times\frac{x}{2})} = \dfrac{1}{\cos x} = \sec x = $ RHS

$\cos^2 A - \sin^2 A \equiv \cos 2A$

(b) $\dfrac{2}{1-t^2} \equiv \dfrac{2}{(1-t)(1+t)} \equiv \dfrac{A}{1-t}+\dfrac{B}{1+t}$

$2 \equiv A(1+t)+B(1-t)$

If $t=-1$, $2=2B$, $B=1$

If $t=1$, $2=2A$, $A=1$

THUS $\dfrac{2}{1-t^2}=\dfrac{1}{1+t}+\dfrac{1}{1-t}$

(c) $\int \sec x\,dx = \ldots$ BY PART (a)

$= \int \dfrac{1+\tan^2\frac{x}{2}}{1-\tan^2\frac{x}{2}}\,dx = \ldots$ BY SUBSTITUTION...

$= \int \dfrac{1+t^2}{1-t^2}\left(\dfrac{2}{1+t^2}\,dt\right) = \int \dfrac{2}{1-t^2}\,dt$

$= \int \dfrac{1}{1+t}+\dfrac{1}{1-t}\,dt = \ln|1+t| - \ln|1-t| + C$

$= \ln\left|\dfrac{1+t}{1-t}\right| + C = \ln\left|\dfrac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right| + C$

BUT $\tan\frac{\pi}{4}=1$

$= \ln\left|\dfrac{\tan\frac{\pi}{4}+\tan\frac{x}{2}}{1-\tan\frac{\pi}{4}\tan\frac{x}{2}}\right| = \ln\left|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right| + C$

$t=\tan\frac{x}{2}$

$\frac{dt}{dx}=\frac{1}{2}\sec^2\frac{x}{2}$

$dx = \dfrac{dt}{\frac{1}{2}\sec^2\frac{x}{2}}$

$dx = \dfrac{2}{\sec^2\frac{x}{2}}\,dt$

$dx = \dfrac{2}{1+\tan^2\frac{x}{2}}\,dt$

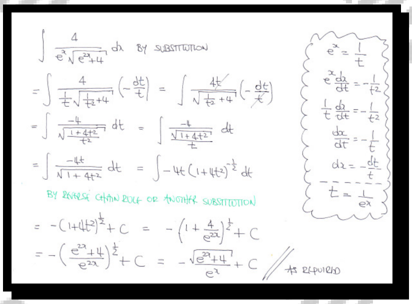
$dx = \dfrac{2}{1+t^2}\,dt$

Question 235 (*****)

By using the substitution $e^x = \dfrac{1}{t}$, or otherwise, show clearly that

$$\int \frac{4}{e^x \sqrt{e^{2x}+4}}\, dx = -\frac{\sqrt{e^{2x}+4}}{e^x} + C\,.$$

proof



$$\int \frac{4}{e^x\sqrt{e^{2x}+4}}\,dx \quad \text{BY SUBSTITUTION}$$

$$= \int \frac{4}{\frac{1}{t}\sqrt{\frac{1}{t^2}+4}}\left(-\frac{dt}{t}\right) = \int \frac{4t}{\sqrt{\frac{1}{t^2}+4}}\left(-\frac{dt}{t}\right)$$

$$= \int \frac{-4}{\sqrt{\frac{1+4t^2}{t^2}}}\,dt = \int \frac{-4}{\frac{\sqrt{1+4t^2}}{t}}\,dt$$

$$= \int \frac{-4t}{\sqrt{1+4t^2}}\,dt = \int -4t(1+4t^2)^{-\frac{1}{2}}\,dt$$

BY REVERSE CHAIN RULE OR ANOTHER SUBSTITUTION

$$= -(1+4t^2)^{\frac{1}{2}} + C = -\left(1+\frac{4}{e^{2x}}\right)^{\frac{1}{2}} + C$$

$$= -\left(\frac{e^{2x}+4}{e^{2x}}\right)^{\frac{1}{2}} + C = -\frac{\sqrt{e^{2x}+4}}{e^x} + C \quad \text{// AS REQUIRED}$$

$e^x = \frac{1}{t}$

$e^x\frac{dx}{dt} = -\frac{1}{t^2}$

$\frac{1}{t}\frac{dx}{dt} = -\frac{1}{t^2}$

$\frac{dx}{dt} = -\frac{1}{t}$

$dx = -\frac{dt}{t}$

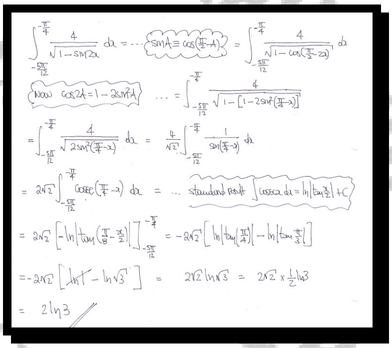
$t = \frac{1}{e^x}$

Question 236 (*****)

Use the fact that $\sin A \equiv \cos\left(\dfrac{\pi}{2} - A\right)$ and other trigonometric identities to show that

$$\int_{-\frac{5\pi}{12}}^{-\frac{\pi}{4}} \frac{4}{\sqrt{1-\sin 2x}}\, dx = 2\ln 3\,.$$

proof



$$\int_{-\frac{5\pi}{12}}^{-\frac{\pi}{4}} \frac{4}{\sqrt{1-\sin 2x}}\,dx = \dots \left\{\sin A \equiv \cos\left(\tfrac{\pi}{2}-A\right)\right\} = \int_{-\frac{5\pi}{12}}^{-\frac{\pi}{4}} \frac{4}{\sqrt{1-\cos\left(\frac{\pi}{2}-2x\right)}}\,dx$$

$$\left\{\text{NOW } \cos 2A = 1-2\sin^2 A\right\} \quad \dots = \int_{-\frac{5\pi}{12}}^{-\frac{\pi}{4}} \frac{4}{\sqrt{1-\left[1-2\sin^2\left(\frac{\pi}{4}-x\right)\right]}}$$

$$= \int_{-\frac{5\pi}{12}}^{-\frac{\pi}{4}} \frac{4}{\sqrt{2\sin^2\left(\frac{\pi}{4}-x\right)}}\,dx = \frac{4}{\sqrt{2}}\int_{-\frac{5\pi}{12}}^{-\frac{\pi}{4}} \frac{1}{\sin\left(\frac{\pi}{4}-x\right)}\,dx$$

$$= 2\sqrt{2}\int_{-\frac{5\pi}{12}}^{-\frac{\pi}{4}} \operatorname{cosec}\left(\tfrac{\pi}{4}-x\right)\,dx = \dots \text{ standard result } \int \operatorname{cosec} x\,dx = \ln\left|\tan\tfrac{x}{2}\right| + C$$

$$= 2\sqrt{2}\left[-\ln\left|\tan\left(\tfrac{\pi}{8}-\tfrac{x}{2}\right)\right|\right]_{-\frac{5\pi}{12}}^{-\frac{\pi}{4}} = -2\sqrt{2}\left[\ln\left|\tan\left(\tfrac{\pi}{4}\right)\right| - \ln\left|\tan\tfrac{\pi}{3}\right|\right]$$

$$= -2\sqrt{2}\left[\ln 1 - \ln\sqrt{3}\right] = 2\sqrt{2}\ln\sqrt{3} = 2\sqrt{2}\times\tfrac{1}{2}\ln 3$$

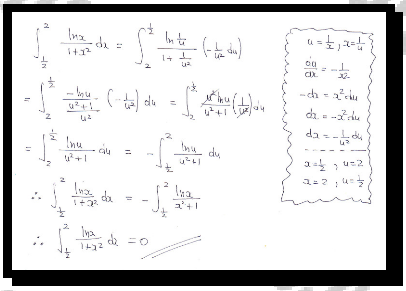
$$= 2\ln 3 \quad //$$

Question 237 (*****)

Use the substitution $x = \frac{1}{u}$ to find the value of

$$\int_{\frac{1}{2}}^{2} \frac{\ln x}{1+x^2} \, dx .$$

$\boxed{0}$

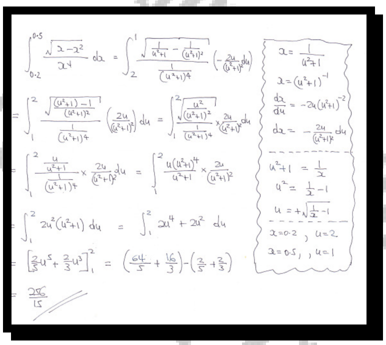


Question 238 (*****)

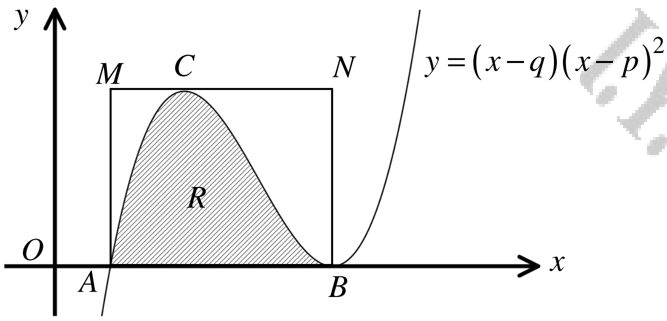
Use the substitution $x = \frac{1}{u^2+1}$ to show that

$$\int_{0.2}^{0.5} \frac{\sqrt{x-x^2}}{x^4} \, dx = \frac{256}{15} .$$

$\boxed{\text{proof}}$



Question 239 (*****)



The figure above shows the graph of the curve with equation

$$y=(x-q)(x-p)^2,$$

where p and q are positive constants.

The curve meets the x axis at the points A and B. The region R, shown shaded in the figure, is bounded by the curve and the x axis.

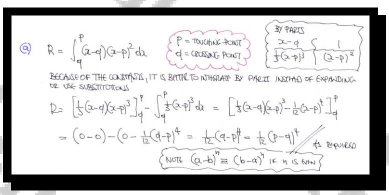
a) Show that the area of the shaded region is

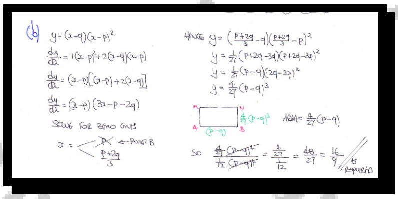
$$\frac{1}{12}(p-q)^4.$$

The point C is the local maximum of the curve. The rectangle $AMCNB$ is such so that MCN is parallel to the x axis and both AM and BN are parallel to the y axis.

b) Show that the area of the rectangle $AMCNB$ is $\frac{16}{9}$ times as large as the area of R, regardless of the values of p and q.

proof



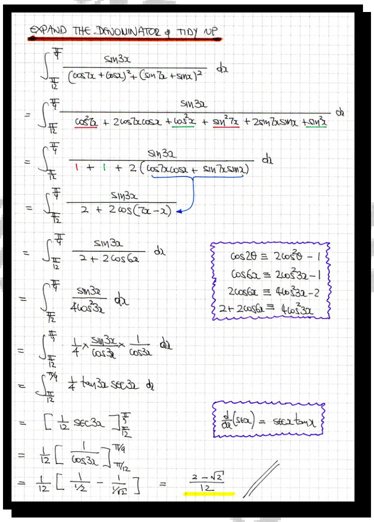


Question 240 (*****)

$$f(x) = \frac{\sin 3x}{(\cos 7x + \cos x)^2 + (\sin 7x + \sin x)^2}, \quad x \in \mathbb{R}.$$

Use trigonometric identities to find the exact value of $\displaystyle\int_{\frac{1}{12}\pi}^{\frac{1}{9}\pi} f(x)\, dx$.

$\boxed{}$, $\boxed{\dfrac{2-\sqrt{2}}{12}}$



EXPAND THE DENOMINATOR & TIDY UP

$\displaystyle\int_{\frac{\pi}{12}}^{\frac{\pi}{9}} \frac{\sin 3x}{(\cos 7x + \cos x)^2 + (\sin 7x + \sin x)^2}\, dx$

$= \displaystyle\int_{\frac{\pi}{12}}^{\frac{\pi}{9}} \frac{\sin 3x}{\cos^2 7x + 2\cos 7x \cos x + \cos^2 x + \sin^2 7x + 2\sin 7x \sin x + \sin^2 x}\, dx$

$= \displaystyle\int_{\frac{\pi}{12}}^{\frac{\pi}{9}} \frac{\sin 3x}{1 + 1 + 2(\cos 7x \cos x + \sin 7x \sin x)}\, dx$

$= \displaystyle\int_{\frac{\pi}{12}}^{\frac{\pi}{9}} \frac{\sin 3x}{2 + 2\cos(7x - x)}$

$= \displaystyle\int_{\frac{\pi}{12}}^{\frac{\pi}{9}} \frac{\sin 3x}{2 + 2\cos 6x}\, dx$

$\cos 2\theta \equiv 2\cos^2\theta - 1$
$\cos 6x \equiv 2\cos^2 3x - 1$
$2\cos 6x \equiv 4\cos^2 3x - 2$
$2 + 2\cos 6x \equiv 4\cos^2 3x$

$= \displaystyle\int_{\frac{\pi}{12}}^{\frac{\pi}{9}} \frac{\sin 3x}{4\cos^2 3x}\, dx$

$= \displaystyle\int_{\frac{\pi}{12}}^{\frac{\pi}{9}} \frac{1}{4} \times \frac{\sin 3x}{\cos 3x} \times \frac{1}{\cos 3x}\, dx$

$= \displaystyle\int_{\frac{\pi}{12}}^{\pi/9} \frac{1}{4} \tan 3x \sec 3x\, dx$

$= \left[\frac{1}{12}\sec 3x\right]_{\frac{\pi}{12}}^{\frac{\pi}{9}}$

$\frac{d}{dx}(\sec x) = \sec x \tan x$

$= \frac{1}{12}\left[\frac{1}{\cos 3x}\right]_{\pi/12}^{\pi/9}$

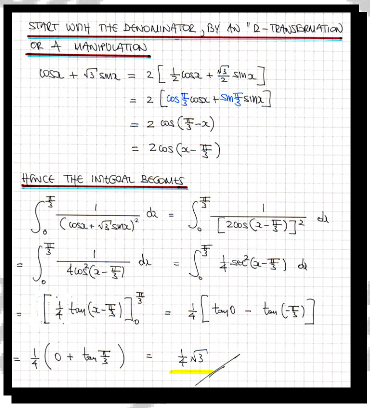
$= \frac{1}{12}\left[\frac{1}{1/2} - \frac{1}{1/\sqrt{2}}\right] = \frac{2-\sqrt{2}}{12}$

Question 241 (*****)

Use a suitable trigonometric manipulation to find an exact simplified answer for the following integral.

$$\int_{0}^{\frac{\pi}{3}} \frac{1}{\left(\cos x+\sqrt{3}\sin x\right)^{2}}\ dx .$$

$\boxed{}$, $\boxed{\frac{1}{4}\sqrt{3}}$


START WITH THE DENOMINATOR, BY AN "R-TRANSFORMATION" OR A MANIPULATION

$\cos x + \sqrt{3}\sin x = 2\left[\frac{1}{2}\cos x + \frac{\sqrt{3}}{2}\sin x\right]$
$= 2\left[\cos\frac{\pi}{3}\cos x + \sin\frac{\pi}{3}\sin x\right]$
$= 2\cos\left(\frac{\pi}{3} - x\right)$
$= 2\cos\left(x - \frac{\pi}{3}\right)$

HENCE THE INTEGRAL BECOMES

$\int_0^{\frac{\pi}{3}} \frac{1}{(\cos x + \sqrt{3}\sin x)^2}\, dx = \int_0^{\frac{\pi}{3}} \frac{1}{\left[2\cos\left(x-\frac{\pi}{3}\right)\right]^2}\, dx$

$= \int_0^{\frac{\pi}{3}} \frac{1}{4\cos^2\left(x-\frac{\pi}{3}\right)}\, dx = \int_0^{\frac{\pi}{3}} \frac{1}{4}\sec^2\left(x-\frac{\pi}{3}\right)\, dx$

$= \left[\frac{1}{4}\tan\left(x-\frac{\pi}{3}\right)\right]_0^{\frac{\pi}{3}} = \frac{1}{4}\left[\tan 0 - \tan\left(-\frac{\pi}{3}\right)\right]$

$= \frac{1}{4}\left(0 + \tan\frac{\pi}{3}\right) = \frac{1}{4}\sqrt{3}$

Question 242 (*****)

$$f(x) = 3\sin x - \cos x + 3, \ x \in \mathbb{R}.$$

$$g(x) = \sin x + \cos x, \ x \in \mathbb{R}.$$

a) Express $f(x)$ in the form

$$A \times g(x) + B \times g'(x) + 3,$$

where A and B are constants.

b) Express $g(x)$ in the form

$$R\cos(x - \varphi),$$

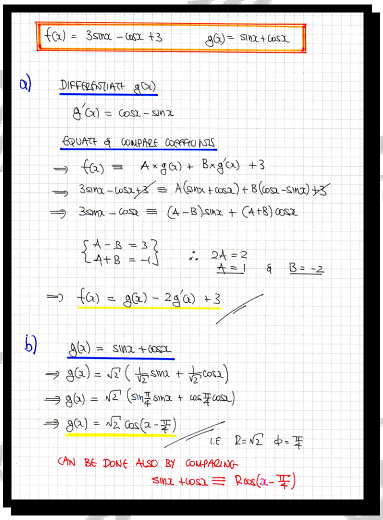
where R and φ are positive constants.

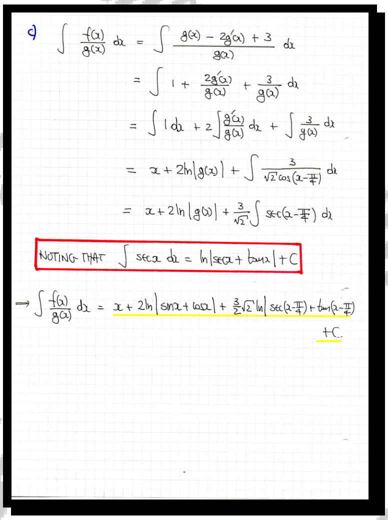
c) Hence find a simplified expression for

$$\int \frac{f(x)}{g(x)} \, dx.$$

$\boxed{}$, $\boxed{A = 1}$, $\boxed{B = -2}$, $\boxed{R = \sqrt{2}}$, $\boxed{\varphi = \frac{1}{4}\pi}$,

$$\boxed{x - 2\ln|\sin x + \cos x| + \frac{3}{2}\sqrt{2}\ln\left|\sec\left(x - \frac{1}{4}\pi\right) + \tan\left(x - \frac{1}{4}\pi\right)\right| + C}$$



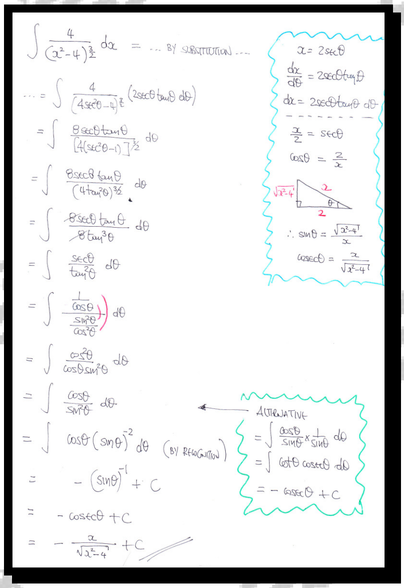


Question 243 (*****)

Use the substitution $x = 2\sec\theta$, to find a simplified expression for

$$\int \frac{6}{\left(x^2-4\right)^{\frac{3}{2}}}\,dx\,.$$

$$\boxed{-\frac{x}{\sqrt{x^2-4}}+C}$$

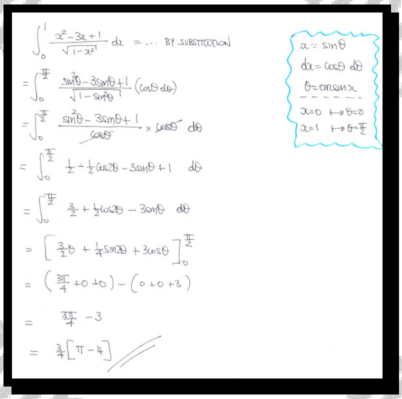


Question 244 (*****)

Find an exact simplified value for

$$\int_0^1 \frac{x^2-3x+1}{\sqrt{1-x^2}}\,dx\,.$$

$$\boxed{\frac{3}{4}[\pi-4]}$$

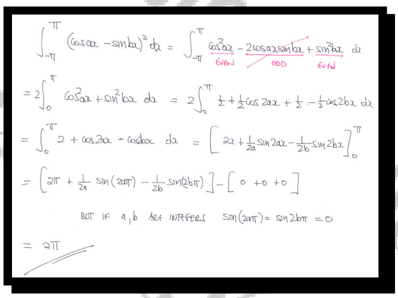


Question 245 (*****)

Given that a and b are integers, evaluate

$$\int_{-\pi}^{\pi} (\cos ax - \sin bx)^2 \, dx\,.$$

$$\boxed{2\pi}$$



Question 246 (*****)

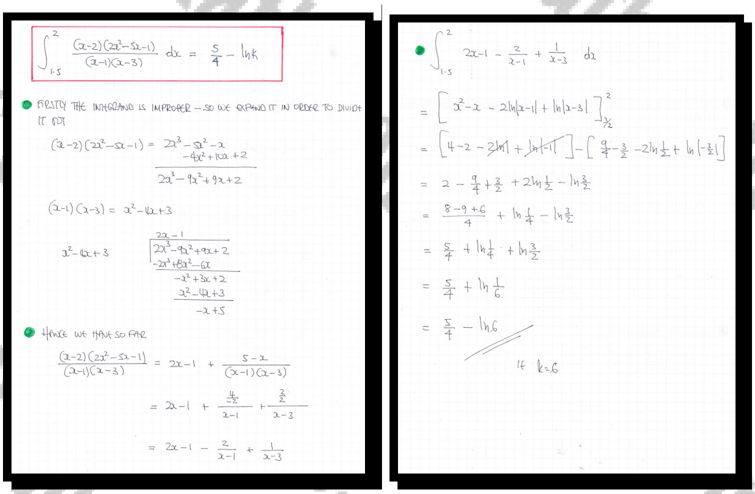
$$I=\int_{1.5}^{2}\frac{(x-2)\left(2x^2-5x-1\right)}{(x-1)(x-3)}\,dx\,.$$

Use appropriate integrations techniques to show that

$$I=\frac{5}{4}-\ln k\,,$$

where k is a positive integer.

$\boxed{}$, $\boxed{k=6}$


$\int_{1.5}^{2}\frac{(x-2)(2x^2-5x-1)}{(x-1)(x-3)}\,dx=\frac{5}{4}-\ln k$

FIRSTLY THE INTEGRAND IS IMPROPER – SO WE EXPAND IT IN ORDER TO DIVIDE IT OUT

$(x-2)(2x^2-5x-1)=2x^3-5x^2-x-4x^2+10x+2=2x^3-9x^2+9x+2$

$(x-1)(x-3)=x^2-4x+3$

$2x^3-9x^2+9x+2$ divided by x^2-4x+3 gives quotient $2x-1$, remainder $-x+5$

HENCE WE HAVE SO FAR

$\frac{(x-2)(2x^2-5x-1)}{(x-1)(x-3)}=2x-1+\frac{5-x}{(x-1)(x-3)}=2x-1+\frac{\frac{4}{-2}}{x-1}+\frac{\frac{2}{2}}{x-3}=2x-1-\frac{2}{x-1}+\frac{1}{x-3}$

$\int_{1.5}^{2}2x-1-\frac{2}{x-1}+\frac{1}{x-3}\,dx$

$=\left[x^2-x-2\ln|x-1|+\ln|x-3|\right]_{3/2}^{2}$

$=\left[4-2-2\ln 1+\ln|-1|\right]-\left[\frac{9}{4}-\frac{3}{2}-2\ln\frac{1}{2}+\ln\left|-\frac{3}{2}\right|\right]$

$=2-\frac{9}{4}+\frac{3}{2}+2\ln\frac{1}{2}-\ln\frac{3}{2}$

$=\frac{8-9+6}{4}+\ln\frac{1}{4}-\ln\frac{3}{2}$

$=\frac{5}{4}+\ln\frac{1}{4}+\ln\frac{2}{3}$

$=\frac{5}{4}+\ln\frac{1}{6}$

$=\frac{5}{4}-\ln 6$

$\therefore k=6$

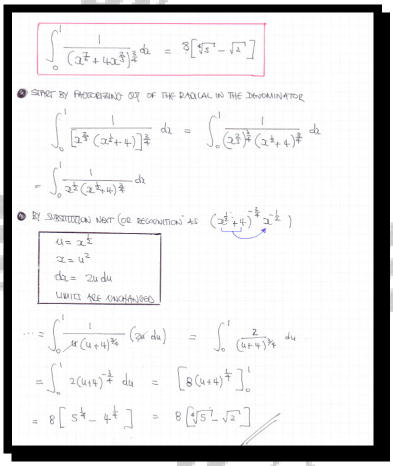
Question 247 (*****)

$$I = \int_0^1 \left(x^{\frac{7}{6}} + 4x^{\frac{2}{3}} \right)^{-\frac{3}{4}} \, dx .$$

Use appropriate integration techniques to show that

$$I = 8\left[\sqrt[4]{5} - \sqrt{2} \right].$$

[ ], [proof]



$$\int_0^1 \frac{1}{\left(x^{\frac{7}{6}} + 4x^{\frac{2}{3}}\right)^{\frac{3}{4}}} \, dx = 8\left[\sqrt[4]{5} - \sqrt{2}\right]$$

- START BY FACTORIZING OUT OF THE RADICAL IN THE DENOMINATOR

$$\int_0^1 \frac{1}{\left[x^{\frac{2}{3}}\left(x^{\frac{1}{2}}+4\right)\right]^{\frac{3}{4}}} \, dx = \int_0^1 \frac{1}{\left(x^{\frac{2}{3}}\right)^{\frac{3}{4}}\left(x^{\frac{1}{2}}+4\right)^{\frac{3}{4}}} \, dx$$

$$= \int_0^1 \frac{1}{x^{\frac{1}{2}}\left(x^{\frac{1}{2}}+4\right)^{\frac{3}{4}}} \, dx$$

- BY SUBSTITUTION NEXT (OR RECOGNITION AS $\left(x^{\frac{1}{2}}+4\right)^{-\frac{3}{4}} x^{-\frac{1}{2}}$)

$u = x^{\frac{1}{2}}$
$x = u^2$
$dx = 2u \, du$
LIMITS ARE UNCHANGED

$$\dots = \int_0^1 \frac{1}{u(u+4)^{3/4}} (2u \, du) = \int_0^1 \frac{2}{(u+4)^{3/4}} \, du$$

$$= \int_0^1 2(u+4)^{-\frac{3}{4}} \, du = \left[8(u+4)^{\frac{1}{4}} \right]_0^1$$

$$= 8\left[5^{\frac{1}{4}} - 4^{\frac{1}{4}} \right] = 8\left[\sqrt[4]{5} - \sqrt{2}\right]$$

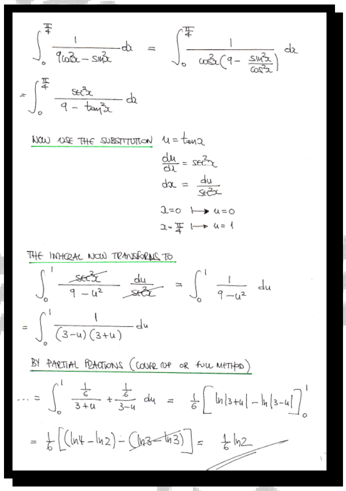
Question 248 (*****)

$$I = \int_0^{\frac{1}{4}\pi} \frac{1}{9\cos^2 x - \sin^2 x} \, dx \, .$$

By using a tangent substitution, or otherwise, show that

$$I = \frac{1}{6}\ln 2 \, .$$

[ ], [proof]



$$\int_0^{\frac{\pi}{4}} \frac{1}{9\cos^2 x - \sin^2 x}\,dx = \int_0^{\frac{\pi}{4}} \frac{1}{\cos^2 x\left(9 - \frac{\sin^2 x}{\cos^2 x}\right)}\,dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{9 - \tan^2 x}\,dx$$

NOW USE THE SUBSTITUTION $u = \tan x$

$\frac{du}{dx} = \sec^2 x$

$dx = \frac{du}{\sec^2 x}$

$x = 0 \mapsto u = 0$

$x = \frac{\pi}{4} \mapsto u = 1$

THE INTEGRAL NOW TRANSFORMS TO

$$\int_0^1 \frac{\sec^2 x}{9 - u^2}\,\frac{du}{\sec^2 x} = \int_0^1 \frac{1}{9 - u^2}\,du$$

$$= \int_0^1 \frac{1}{(3-u)(3+u)}\,du$$

BY PARTIAL FRACTIONS (COVER UP OR FULL METHOD)

$$\ldots = \int_0^1 \frac{\frac{1}{6}}{3+u} + \frac{\frac{1}{6}}{3-u}\,du = \frac{1}{6}\Big[\ln|3+u| - \ln|3-u|\Big]_0^1$$

$$= \frac{1}{6}\Big[(\ln 4 - \ln 2) - (\ln 3 - \ln 3)\Big] = \frac{1}{6}\ln 2$$

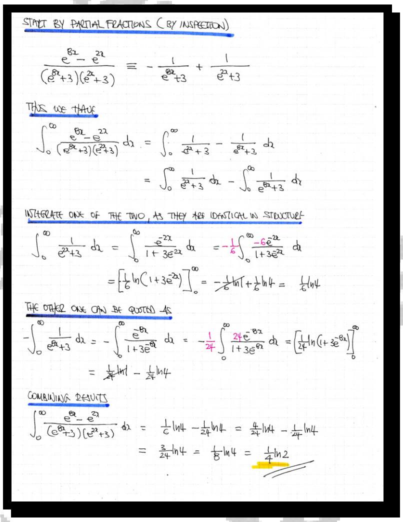
Question 249 (*****)

Find an exact simplified value for the following definite integral.

$$\int_0^{\infty} \frac{e^{8x}-e^{2x}}{\left(e^{8x}+3\right)\left(e^{2x}+3\right)}\, dx .$$

You may assume without proof that the integral converges.

V, $\square$, $\boxed{\frac{1}{4}\ln 2}$


START BY PARTIAL FRACTIONS (BY INSPECTION)

$$\frac{e^{8x}-e^{2x}}{(e^{8x}+3)(e^{2x}+3)} \equiv -\frac{1}{e^{8x}+3}+\frac{1}{e^{2x}+3}$$

THUS WE HAVE

$$\int_0^{\infty}\frac{e^{8x}-e^{2x}}{(e^{8x}+3)(e^{2x}+3)}\,dx = \int_0^{\infty}\frac{1}{e^{2x}+3}-\frac{1}{e^{8x}+3}\,dx = \int_0^{\infty}\frac{1}{e^{2x}+3}\,dx-\int_0^{\infty}\frac{1}{e^{8x}+3}\,dx$$

INTEGRATE ONE OF THE TWO, AS THEY ARE IDENTICAL IN STRUCTURE

$$\int_0^{\infty}\frac{1}{e^{2x}+3}\,dx = \int_0^{\infty}\frac{e^{-2x}}{1+3e^{-2x}}\,dx = -\frac{1}{6}\int_0^{\infty}\frac{-6e^{-2x}}{1+3e^{-2x}}\,dx$$

$$= \left[-\frac{1}{6}\ln\left(1+3e^{-2x}\right)\right]_0^{\infty} = -\frac{1}{6}\ln 1+\frac{1}{6}\ln 4 = \frac{1}{6}\ln 4$$

THE OTHER ONE CAN BE QUOTED AS

$$-\int_0^{\infty}\frac{1}{e^{8x}+3}\,dx = -\int_0^{\infty}\frac{e^{-8x}}{1+3e^{-8x}}\,dx = -\frac{1}{24}\int_0^{\infty}\frac{24e^{-8x}}{1+3e^{-8x}}\,dx = \left[\frac{1}{24}\ln\left(1+3e^{-8x}\right)\right]_0^{\infty}$$

$$= \frac{1}{24}\ln 1-\frac{1}{24}\ln 4$$

COMBINING RESULTS

$$\int_0^{\infty}\frac{e^{8x}-e^{2x}}{(e^{8x}+3)(e^{2x}+3)}\,dx = \frac{1}{6}\ln 4-\frac{1}{24}\ln 4 = \frac{4}{24}\ln 4-\frac{1}{24}\ln 4 = \frac{3}{24}\ln 4 = \frac{1}{8}\ln 4 = \frac{1}{4}\ln 2$$

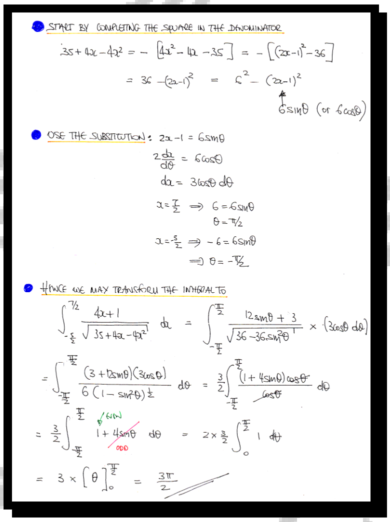
Question 250 (*****)

$$I = \int_{-\frac{5}{2}}^{\frac{7}{2}} \frac{4x+1}{\sqrt{35+4x-4x^2}} \, dx .$$

By writing $35+4x-4x^2$ in completed the square form, followed by a suitable trigonometric substitution, show that

$$I = \frac{3}{2}\pi .$$

[ ], [proof]



START BY COMPLETING THE SQUARE IN THE DENOMINATOR

$$35+4x-4x^2 = -\left[4x^2-4x-35\right] = -\left[(2x-1)^2-36\right]$$

$$= 36-(2x-1)^2 = 6^2-(2x-1)^2$$

$6\sin\theta$ (or $6\cos\theta$)

USE THE SUBSTITUTION: $2x-1 = 6\sin\theta$

$$2\frac{dx}{d\theta} = 6\cos\theta$$

$$dx = 3\cos\theta \, d\theta$$

$$x=\frac{7}{2} \Rightarrow 6 = 6\sin\theta, \quad \theta = \pi/2$$

$$x=-\frac{5}{2} \Rightarrow -6 = 6\sin\theta \Rightarrow \theta = -\pi/2$$

HENCE WE MAY TRANSFORM THE INTEGRAL TO

$$\int_{-\frac{5}{2}}^{7/2} \frac{4x+1}{\sqrt{35+4x-4x^2}} \, dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{12\sin\theta+3}{\sqrt{36-36\sin^2\theta}} \times (3\cos\theta \, d\theta)$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(3+12\sin\theta)(3\cos\theta)}{6(1-\sin^2\theta)^{\frac{1}{2}}} \, d\theta = \frac{3}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(1+4\sin\theta)\cos\theta}{\cos\theta} \, d\theta$$

$$= \frac{3}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1+4\sin\theta \, d\theta = 2\times\frac{3}{2}\int_{0}^{\frac{\pi}{2}} 1 \, d\theta$$

EVEN, ODD

$$= 3\times\left[\theta\right]_0^{\frac{\pi}{2}} = \frac{3\pi}{2}$$

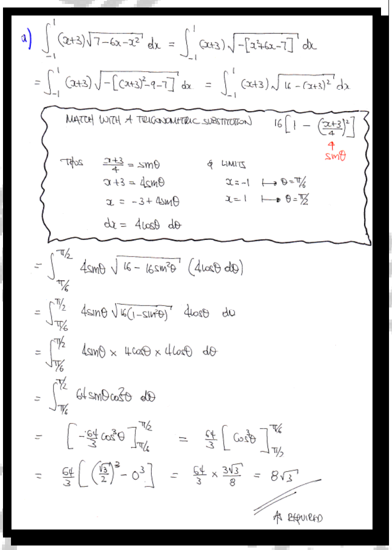
Question 251 (*****)

$$I=\int_{-1}^{1}(x+3)\sqrt{7-6x-x^2}\ dx$$

a) Use a suitable trigonometric substitution to show that $I=8\sqrt{3}$.

b) Verify the answer of part **(a)** by an alternative method.

[ ], proof



a) $\int_{-1}^{1}(x+3)\sqrt{7-6x-x^2}\,dx=\int_{-1}^{1}(x+3)\sqrt{-[x^2+6x-7]}\,dx$

$=\int_{-1}^{1}(x+3)\sqrt{-[(x+3)^2-9-7]}\,dx=\int_{-1}^{1}(x+3)\sqrt{16-(x+3)^2}\,dx$

MATCH WITH A TRIGONOMETRIC SUBSTITUTION $16\left[1-\left(\frac{x+3}{4}\right)^2\right]$ ↑ $\sin\theta$

THUS $\frac{x+3}{4}=\sin\theta$ & LIMITS

$x+3=4\sin\theta$ $x=-1 \mapsto \theta=\frac{\pi}{6}$

$x=-3+4\sin\theta$ $x=1 \mapsto \theta=\frac{\pi}{2}$

$dx=4\cos\theta\,d\theta$

$=\int_{\pi/6}^{\pi/2}4\sin\theta\sqrt{16-16\sin^2\theta}\,(4\cos\theta\,d\theta)$

$=\int_{\pi/6}^{\pi/2}4\sin\theta\sqrt{16(1-\sin^2\theta)}\,4\cos\theta\,d\theta$

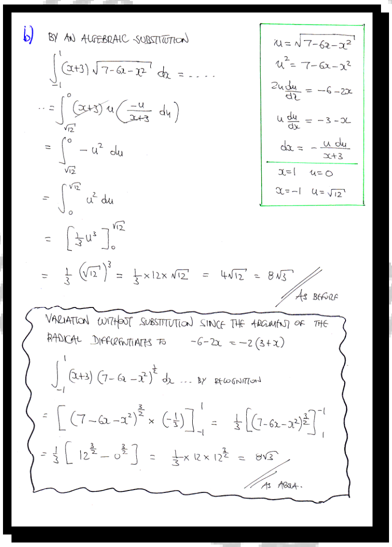
$=\int_{\pi/6}^{\pi/2}4\sin\theta\times4\cos\theta\times4\cos\theta\,d\theta$

$=\int_{\pi/6}^{\pi/2}64\sin\theta\cos^2\theta\,d\theta$

$=\left[-\frac{64}{3}\cos^3\theta\right]_{\pi/6}^{\pi/2}=\frac{64}{3}\left[\cos^3\theta\right]_{\pi/2}^{\pi/6}$

$=\frac{64}{3}\left[\left(\frac{\sqrt{3}}{2}\right)^3-0^3\right]=\frac{64}{3}\times\frac{3\sqrt{3}}{8}=8\sqrt{3}$

AS REQUIRED



b) BY AN ALGEBRAIC SUBSTITUTION

$\int_{-1}^{1}(x+3)\sqrt{7-6x-x^2}\,dx=\dots$

$\dots=\int_{\sqrt{12}}^{0}(x+3)u\left(\frac{-u}{x+3}\,du\right)$

$=\int_{\sqrt{12}}^{0}-u^2\,du$

$=\int_{0}^{\sqrt{12}}u^2\,du$

$=\left[\frac{1}{3}u^3\right]_0^{\sqrt{12}}$

$=\frac{1}{3}(\sqrt{12})^3=\frac{1}{3}\times12\times\sqrt{12}=4\sqrt{12}=8\sqrt{3}$

AS BEFORE

$u=\sqrt{7-6x-x^2}$

$u^2=7-6x-x^2$

$2u\frac{du}{dx}=-6-2x$

$u\frac{du}{dx}=-3-x$

$dx=-\frac{u\,du}{x+3}$

$x=1 \quad u=0$

$x=-1 \quad u=\sqrt{12}$

VARIATION WITHOUT SUBSTITUTION SINCE THE ARGUMENT OF THE RADICAL DIFFERENTIATES TO $-6-2x=-2(3+x)$

$\int_{-1}^{1}(x+3)(7-6x-x^2)^{\frac{1}{2}}\,dx\dots$ BY RECOGNITION

$=\left[(7-6x-x^2)^{\frac{3}{2}}\times\left(-\frac{1}{3}\right)\right]_{-1}^{1}=\frac{1}{3}\left[(7-6x-x^2)^{\frac{3}{2}}\right]_{1}^{-1}$

$=\frac{1}{3}\left[12^{\frac{3}{2}}-0^{\frac{3}{2}}\right]=\frac{1}{3}\times12\times12^{\frac{1}{2}}=8\sqrt{3}$

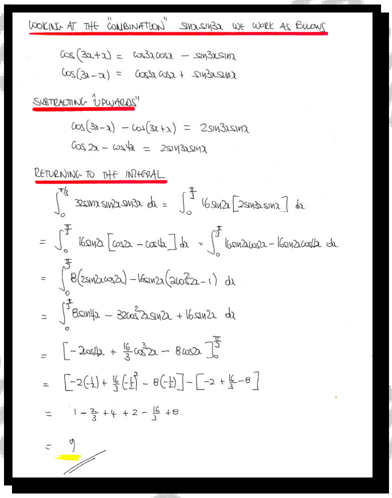
AS ABOVE.

Question 252 (*****)

Use trigonometric identities to find the value of

$$\int_0^{\frac{\pi}{3}} 32\sin x \sin 2x \sin 3x \ dx .$$

MP2 , 9



LOOKING AT THE "COMBINATION" sinx sin3x WE WORK AS FOLLOWS

$\cos(3x+x) = \cos 3x \cos x - \sin 3x \sin x$

$\cos(3x-x) = \cos 3x \cos x + \sin 3x \sin x$

SUBTRACTING "UPWARDS"

$\cos(3x-x) - \cos(3x+x) = 2\sin 3x \sin x$

$\cos 2x - \cos 4x = 2\sin 3x \sin x$

RETURNING TO THE INTEGRAL

$\int_0^{\pi/3} 32\sin x \sin 2x \sin 3x \, dx = \int_0^{\frac{\pi}{3}} 16\sin 2x \left[2\sin 3x \sin x\right] dx$

$= \int_0^{\frac{\pi}{3}} 16\sin 2x \left[\cos 2x - \cos 4x\right] dx = \int_0^{\frac{\pi}{3}} 16\sin 2x\cos 2x - 16\sin 2x \cos 4x \, dx$

$= \int_0^{\frac{\pi}{3}} 8(2\sin 2x\cos 2x) - 16\sin 2x(2\cos^2 2x - 1) \, dx$

$= \int_0^{\frac{\pi}{3}} 8\sin 4x - 32\cos^2 2x \sin 2x + 16\sin 2x \, dx$

$= \left[-2\cos 4x + \frac{16}{3}\cos^3 2x - 8\cos 2x\right]_0^{\frac{\pi}{3}}$

$= \left[-2(-\tfrac{1}{2}) + \tfrac{16}{3}(-\tfrac{1}{2})^3 - 8(-\tfrac{1}{2})\right] - \left[-2 + \tfrac{16}{3} - 8\right]$

$= 1 - \frac{2}{3} + 4 + 2 - \frac{16}{3} + 8$

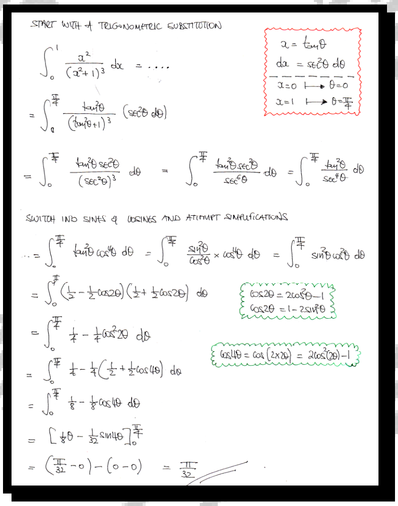
$= 9$

Question 253 (*****)

Use suitable integration techniques to show that

$$\int_0^1 \frac{x^2}{\left(x^2+1\right)^3}\, dx = \frac{\pi}{32}.$$

proof



START WITH A TRIGONOMETRIC SUBSTITUTION

$\int_0^1 \frac{x^2}{(x^2+1)^3}\,dx = \dots$

$x = \tan\theta$, $dx = \sec^2\theta\, d\theta$; $x=0 \mapsto \theta=0$, $x=1 \mapsto \theta=\frac{\pi}{4}$

$= \int_0^{\frac{\pi}{4}} \frac{\tan^2\theta}{(\tan^2\theta+1)^3}(\sec^2\theta\, d\theta)$

$= \int_0^{\frac{\pi}{4}} \frac{\tan^2\theta\sec^2\theta}{(\sec^2\theta)^3}\,d\theta = \int_0^{\frac{\pi}{4}} \frac{\tan^2\theta\sec^2\theta}{\sec^6\theta}\,d\theta = \int_0^{\frac{\pi}{4}} \frac{\tan^2\theta}{\sec^4\theta}\,d\theta$

SWITCH INTO SINES & COSINES AND ATTEMPT SIMPLIFICATIONS

$\dots = \int_0^{\frac{\pi}{4}} \tan^2\theta\cos^4\theta\,d\theta = \int_0^{\frac{\pi}{4}} \frac{\sin^2\theta}{\cos^2\theta}\times\cos^4\theta\,d\theta = \int_0^{\frac{\pi}{4}} \sin^2\theta\cos^2\theta\,d\theta$

$= \int_0^{\frac{\pi}{4}} \left(\frac{1}{2}-\frac{1}{2}\cos2\theta\right)\left(\frac{1}{2}+\frac{1}{2}\cos2\theta\right)d\theta$

$\cos2\theta = 2\cos^2\theta-1$, $\cos2\theta = 1-2\sin^2\theta$

$= \int_0^{\frac{\pi}{4}} \frac{1}{4}-\frac{1}{4}\cos^2 2\theta\,d\theta$

$\cos4\theta = \cos(2\times2\theta) = 2\cos^2(2\theta)-1$

$= \int_0^{\frac{\pi}{4}} \frac{1}{4}-\frac{1}{4}\left(\frac{1}{2}+\frac{1}{2}\cos4\theta\right)d\theta$

$= \int_0^{\frac{\pi}{4}} \frac{1}{8}-\frac{1}{8}\cos4\theta\,d\theta$

$= \left[\frac{1}{8}\theta - \frac{1}{32}\sin4\theta\right]_0^{\frac{\pi}{4}}$

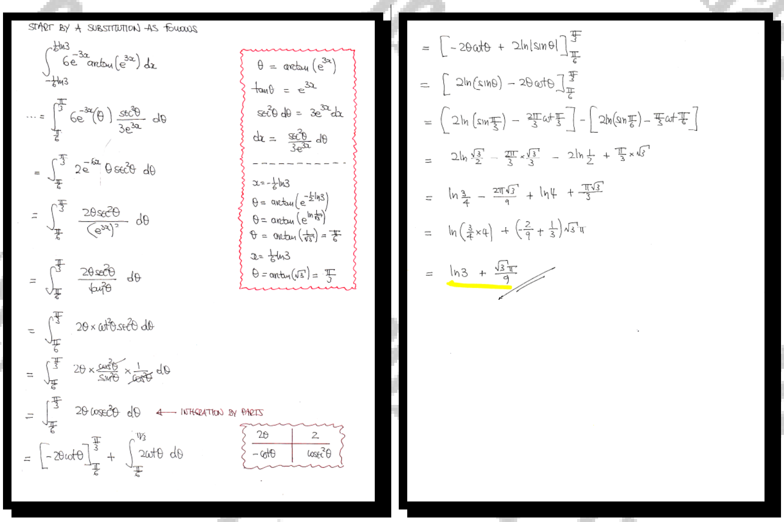
$= \left(\frac{\pi}{32}-0\right)-(0-0) = \frac{\pi}{32}$

Question 254 (*****)

Use suitable integration techniques to show that

$$\int_{-\frac{1}{6}\ln 3}^{\frac{1}{6}\ln 3} 6e^{-3x}\arctan\left(e^{3x}\right)\,dx = \ln 3+\frac{\pi\sqrt{3}}{9}.$$

proof



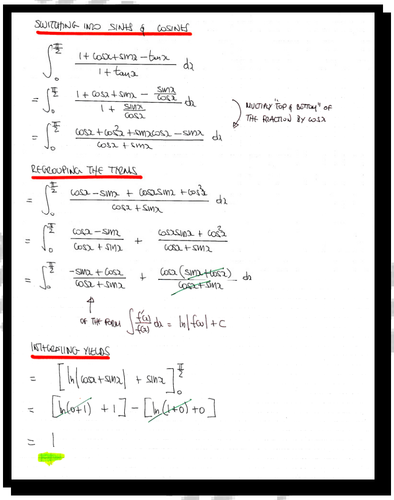
Question 255 (*****)

Use suitable integration techniques to show that

$$\int_{0}^{\frac{1}{2}\pi} \frac{1+\cos x+\sin x-\tan x}{1+\tan x}\, dx = 1.$$

You may assume that the above integral converges.

[ ], [proof]

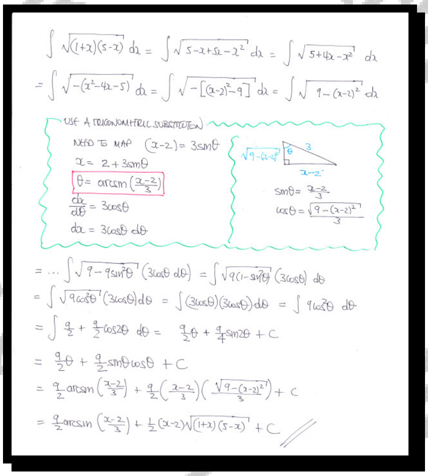


Question 256 (*****)

Use a suitable trigonometric substitution to find a simplified expression for

$$\int \sqrt{(1+x)(5-x)}\, dx .$$

$$\boxed{\frac{9}{2}\arcsin\left(\frac{x-2}{3}\right)+\frac{1}{2}(x-2)\sqrt{(1+x)(5-x)}+C}$$



$$\int \sqrt{(1+x)(5-x)}\,dx = \int \sqrt{5-x+5x-x^2}\,dx = \int \sqrt{5+4x-x^2}\,dx$$

$$= \int \sqrt{-(x^2-4x-5)}\,dx = \int \sqrt{-[(x-2)^2-9]}\,dx = \int \sqrt{9-(x-2)^2}\,dx$$

USE A TRIGONOMETRIC SUBSTITUTION

NEED TO MAP $(x-2) = 3\sin\theta$

$x = 2+3\sin\theta$

$\theta = \arcsin\left(\frac{x-2}{3}\right)$

$\frac{dx}{d\theta} = 3\cos\theta$

$dx = 3\cos\theta\, d\theta$

$\sin\theta = \frac{x-2}{3}$

$\cos\theta = \frac{\sqrt{9-(x-2)^2}}{3}$

$$= \dots \int \sqrt{9-9\sin^2\theta}\,(3\cos\theta\, d\theta) = \int \sqrt{9(1-\sin^2\theta)}\,(3\cos\theta)\, d\theta$$

$$= \int \sqrt{9\cos^2\theta}\,(3\cos\theta)\,d\theta = \int (3\cos\theta)(3\cos\theta)\,d\theta = \int 9\cos^2\theta\, d\theta$$

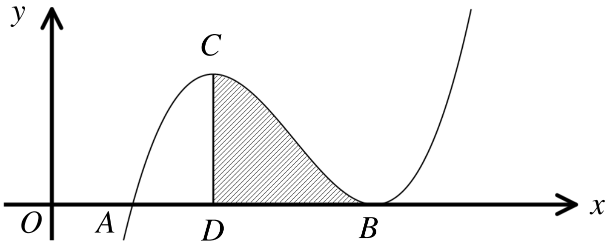
$$= \int \frac{9}{2}+\frac{9}{2}\cos 2\theta\, d\theta = \frac{9}{2}\theta + \frac{9}{4}\sin 2\theta + C$$

$$= \frac{9}{2}\theta + \frac{9}{2}\sin\theta\cos\theta + C$$

$$= \frac{9}{2}\arcsin\left(\frac{x-2}{3}\right) + \frac{9}{2}\left(\frac{x-2}{3}\right)\left(\frac{\sqrt{9-(x-2)^2}}{3}\right) + C$$

$$= \frac{9}{2}\arcsin\left(\frac{x-2}{3}\right) + \frac{1}{2}(x-2)\sqrt{(1+x)(5-x)} + C$$

Question 257 (*****)



The figure above shows a cubic curve that crosses the x axis at $A(a,0)$ and touches the x axis at $B(b,0)$, where a and b are positive constants. The point C is a local maximum of the curve.

a) Find the x coordinate of C, in terms of a and b.

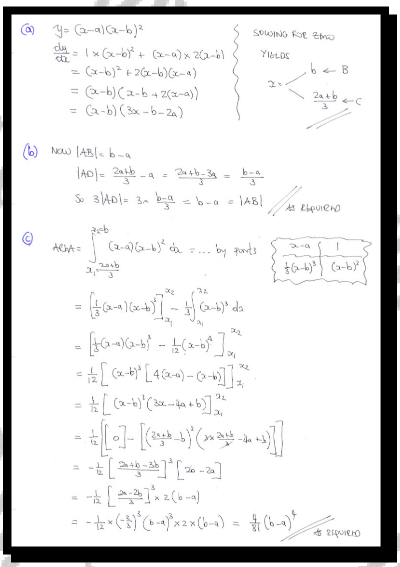
The point D lies on the x axis so that CD is parallel to the y axis.

b) Show that $|AB| = 3|AD|$.

The region R is bounded by the curve, the line segment CD and the x axis.

c) Use integration by parts to show that the area of R is $\frac{4}{81}(b-a)^4$.

$$\boxed{x = \tfrac{1}{3}(2a+b)}$$



(a) $y = (x-a)(x-b)^2$

$\frac{dy}{dx} = 1 \times (x-b)^2 + (x-a) \times 2(x-b)$

$= (x-b)^2 + 2(x-b)(x-a)$

$= (x-b)(x-b+2(x-a))$

$= (x-b)(3x-b-2a)$

SOLVING FOR ZERO YIELDS $x = b \leftarrow B$, $x = \frac{2a+b}{3} \leftarrow C$

(b) NOW $|AB| = b-a$

$|AD| = \frac{2a+b}{3} - a = \frac{2a+b-3a}{3} = \frac{b-a}{3}$

So $3|AD| = 3 \times \frac{b-a}{3} = b-a = |AB|$ // AS REQUIRED

(c) AREA $= \int_{x_1=\frac{2a+b}{3}}^{x_2=b} (x-a)(x-b)^2 \, dx = \dots$ by parts

$x-a$	1
$\frac{1}{3}(x-b)^3$	$(x-b)^2$

$= \left[\frac{1}{3}(x-a)(x-b)^3\right]_{x_1}^{x_2} - \frac{1}{3}\int_{x_1}^{x_2} (x-b)^3 \, dx$

$= \left[\frac{1}{3}(x-a)(x-b)^3 - \frac{1}{12}(x-b)^4\right]_{x_1}^{x_2}$

$= \frac{1}{12}\left[(x-b)^3\left[4(x-a)-(x-b)\right]\right]_{x_1}^{x_2}$

$= \frac{1}{12}\left[(x-b)^3(3x-4a+b)\right]_{x_1}^{x_2}$

$= \frac{1}{12}\left[[0] - \left[\left(\frac{2a+b}{3}-b\right)^3\left(3 \times \frac{2a+b}{3} - 4a + b\right)\right]\right]$

$= -\frac{1}{12}\left[\frac{2a+b-3b}{3}\right]^3\left[2b-2a\right]$

$= -\frac{1}{12}\left[\frac{2a-2b}{3}\right]^3 \times 2(b-a)$

$= -\frac{1}{12} \times \left(-\frac{2}{3}\right)^3 (b-a)^3 \times 2 \times (b-a) = \frac{4}{81}(b-a)^4$ // AS REQUIRED

Question 258 (*****)

By considering the derivatives of $e^x \sin x$ and $e^x \cos x$, find

$$\int e^x (2\cos x - 3\sin x)\, dx .$$

$$\boxed{\frac{1}{2}e^x(5\cos x - \sin x) + C}$$

$\frac{d}{dx}(e^x \sin x) = e^x \sin x + e^x \cos x$
$\frac{d}{dx}(e^x \cos x) = e^x \cos x - e^x \sin x$ } add & subtract giving

$\frac{d}{dx}(e^x \sin x + e^x \cos x) = 2e^x \cos x$
$\frac{d}{dx}(e^x \sin x - e^x \cos x) = 2e^x \sin x$ } $\Rightarrow$ $\frac{d}{dx}\left(\frac{1}{2}e^x(\sin x + \cos x)\right) = e^x \cos x$, $\frac{d}{dx}\left(\frac{1}{2}e^x(\sin x - \cos x)\right) = e^x \sin x$

Hence $2e^x\cos x - 3e^x \sin x = 2\frac{d}{dx}\left(\frac{1}{2}e^x(\sin x+\cos x)\right) - 3\frac{d}{dx}\left(\frac{1}{2}e^x(\sin x - \cos x)\right)$

$2e^x\cos x - 3e^x \sin x = \frac{d}{dx}\left[e^x(\sin x+\cos x) - \frac{3}{2}e^x(\sin x - \cos x)\right]$

$2e^x\cos x - 3e^x \sin x = \frac{d}{dx}\left[\frac{1}{2}e^x\left[2\sin x + 2\cos x - 3\sin x + 3\cos x\right]\right]$

$2e^x\cos x - 3e^x \sin x = \frac{d}{dx}\left[\frac{1}{2}e^x(5\cos x - \sin x)\right]$

$\therefore \int e^x(2\cos x - 3\sin x)\,dx = \frac{1}{2}e^x(5\cos x - \sin x) + C$

Question 259 (*****)

Use integration by parts and suitable trigonometric identities to find

$$\int \sec^3 x\, dx .$$

$\boxed{}$, $\boxed{\frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C}$

- $\int \sec^3 x\,dx = \int \sec^2 x \sec x\,dx \ldots$

BY PARTS

$\sec x$	$\sec x \tan x$
$\tan x$	$\sec^2 x$

$\ldots = \sec x \tan x - \int \sec x \tan^2 x\,dx$

$= \sec x \tan x - \int \sec x(\sec^2 x - 1)\,dx$

$= \sec x \tan x - \int \sec^3 x - \sec x\,dx$

$= \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx$

$= \sec x \tan x - \int \sec^3 x\,dx + \ln|\sec x + \tan x|$

- COLLECTING THE RESULTS

$\int \sec^3 x\,dx = \sec x \tan x - \int \sec^3 x\,dx + \ln|\sec x + \tan x|$

$2\int \sec^3 x\,dx = \sec x \tan x + \ln|\sec x + \tan x| + C$

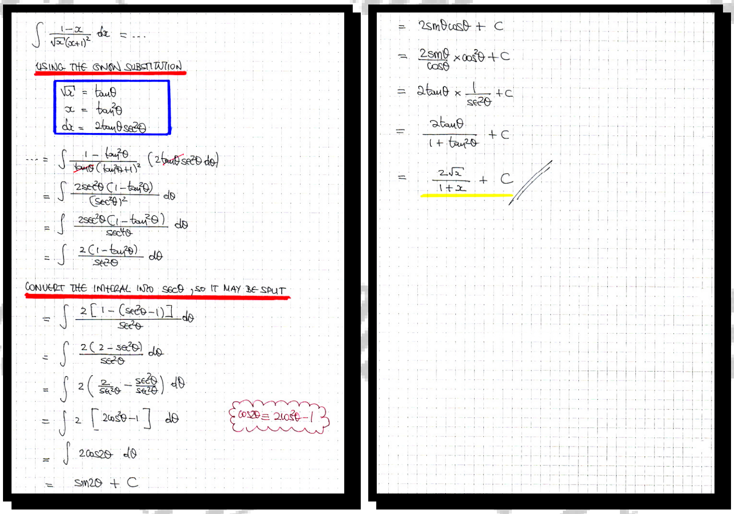
$\int \sec^3 x\,dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C$

Question 260 (*****)

By using the substitution $\sqrt{x} = \tan\theta$, or otherwise, find a simplified expression for the following integral.

$$\int \frac{1-x}{\sqrt{x}\,(x+1)^2}\, dx .$$

$\boxed{}$, $\boxed{\dfrac{2\sqrt{x}}{x+1}+C}$



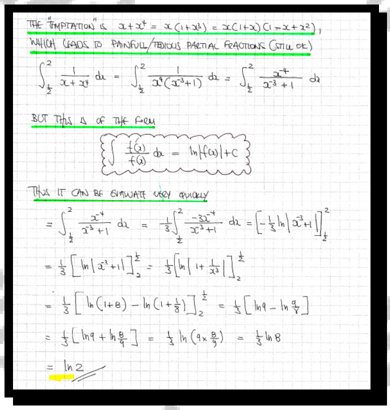
Question 261 (*****)

Find the value of the following definite integral.

$$\int_{\frac{1}{2}}^{2} \frac{1}{x+x^4}\, dx \ .$$

Give the answer in the form $\ln k$, where k is a positive integer.

$\boxed{}$, $\boxed{k=2}$



THE "TEMPTATION" IS $x+x^4 = x(1+x^3) = x(1+x)(1-x+x^2)$, WHICH LEADS TO PAINFULL/TEDIOUS PARTIAL FRACTIONS (STILL OK)

$$\int_{\frac{1}{2}}^{2} \frac{1}{x+x^4}\,dx = \int_{\frac{1}{2}}^{2} \frac{1}{x^4(x^{-3}+1)}\,dx = \int_{\frac{1}{2}}^{2} \frac{x^{-4}}{x^{-3}+1}\,dx$$

BUT THIS IS OF THE FORM

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)|+C$$

THUS IT CAN BE EVALUATE VERY QUICKLY

$$= \int_{\frac{1}{2}}^{2} \frac{x^{-4}}{x^{-3}+1}\,dx = -\frac{1}{3}\int_{\frac{1}{2}}^{2} \frac{-3x^{-4}}{x^{-3}+1}\,dx = \left[-\frac{1}{3}\ln|x^{-3}+1|\right]_{\frac{1}{2}}^{2}$$

$$= \frac{1}{3}\left[\ln|x^{-3}+1|\right]_{2}^{\frac{1}{2}} = \frac{1}{3}\left[\ln\left|1+\frac{1}{x^3}\right|\right]_{2}^{\frac{1}{2}}$$

$$= \frac{1}{3}\left[\ln(1+8) - \ln\left(1+\frac{1}{8}\right)\right]_{2}^{\frac{1}{2}} = \frac{1}{3}\left[\ln 9 - \ln\frac{9}{8}\right]$$

$$= \frac{1}{3}\left[\ln 9 + \ln\frac{8}{9}\right] = \frac{1}{3}\ln\left(9\times\frac{8}{9}\right) = \frac{1}{3}\ln 8$$

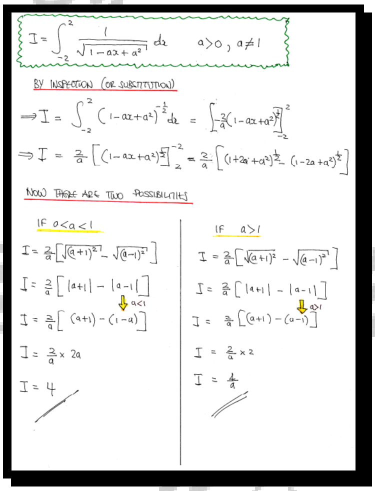
$$= \ln 2$$

Question 262 (*****)

$$I = \int_{-2}^{2} \frac{1}{\sqrt{1-ax+a^2}}\, dx\,, \ a>0\,, \ a \neq 0\,.$$

Find the two possible values of I, giving the answer in terms of a where appropriate.

$$\boxed{\quad}\,, \quad \boxed{\begin{array}{ll} I = 4 & 0<a<1 \\ I = \dfrac{4}{a} & a>1 \end{array}}$$


$I = \int_{-2}^{2} \frac{1}{\sqrt{1-ax+a^2}}\,dx \qquad a>0,\ a \neq 1$

BY INSPECTION (OR SUBSTITUTION)

$\Rightarrow I = \int_{-2}^{2} (1-ax+a^2)^{-\frac{1}{2}}\,dx = \left[-\frac{2}{a}(1-ax+a^2)^{\frac{1}{2}}\right]_{-2}^{2}$

$\Rightarrow I = \frac{2}{a}\left[(1-ax+a^2)^{\frac{1}{2}}\right]_{2}^{-2} = \frac{2}{a}\left[(1+2a+a^2)^{\frac{1}{2}} - (1-2a+a^2)^{\frac{1}{2}}\right]$

NOW THERE ARE TWO POSSIBILITIES

IF $0<a<1$

$I = \frac{2}{a}\left[\sqrt{(a+1)^2} - \sqrt{(a-1)^2}\right]$

$I = \frac{2}{a}\left[|a+1| - |a-1|\right]$ ($a<1$)

$I = \frac{2}{a}\left[(a+1) - (1-a)\right]$

$I = \frac{2}{a} \times 2a$

$I = 4$

IF $a>1$

$I = \frac{2}{a}\left[\sqrt{(a+1)^2} - \sqrt{(a-1)^2}\right]$

$I = \frac{2}{a}\left[|a+1| - |a-1|\right]$ ($a>1$)

$I = \frac{2}{a}\left[(a+1) - (a-1)\right]$

$I = \frac{2}{a} \times 2$

$I = \frac{4}{a}$

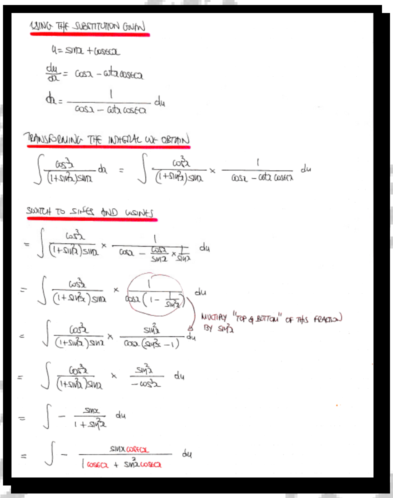
Question 263 (*****)

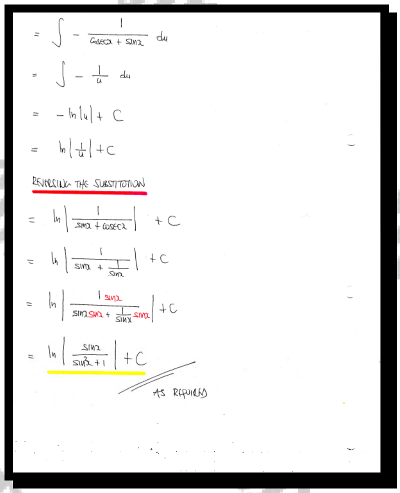
$$I = \int \frac{\cos^3 x}{\left(1+\sin^2 x\right)\sin x}\,dx .$$

By using the substitution $u = \sin x + \operatorname{cosec} x$, or otherwise, show that

$$I = \ln\left|\frac{\sin x}{1+\sin^2 x}\right| + \text{constant}$$

[ ], [proof]





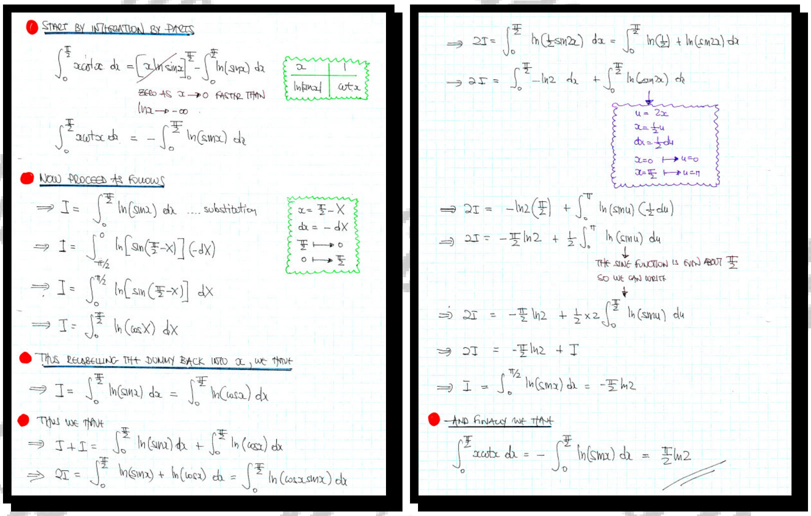
Question 264 (*****)

$$I = \int_{0}^{\frac{1}{2}\pi} x \cot x \ dx .$$

Use appropriate integration techniques to show that

$$I = \frac{1}{2}\pi \ln 2 .$$

[ ] , [proof]



- START BY INTEGRATION BY PARTS

$\int_0^{\frac{\pi}{2}} x\cot x \, dx = \left[x\ln(\sin x)\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \ln(\sin x) \, dx$

ZERO AS $x \to 0$ FASTER THAN $\ln x \to -\infty$

x	1
$\ln(\sin x)$	$\cot x$

$\int_0^{\frac{\pi}{2}} x\cot x \, dx = -\int_0^{\frac{\pi}{2}} \ln(\sin x) \, dx$

- NOW PROCEED AS FOLLOWS

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \ln(\sin x) \, dx$ ..... substitution

$\Rightarrow I = \int_{\pi/2}^{0} \ln\left[\sin\left(\frac{\pi}{2} - X\right)\right](-dX)$

$\Rightarrow I = \int_0^{\pi/2} \ln\left[\sin\left(\frac{\pi}{2} - X\right)\right] dX$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \ln(\cos X) \, dX$

$x = \frac{\pi}{2} - X$, $dx = -dX$, $\frac{\pi}{2} \mapsto 0$, $0 \mapsto \frac{\pi}{2}$

- THUS RELABELLING THE DUMMY BACK INTO x, WE HAVE

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \ln(\sin x) \, dx = \int_0^{\frac{\pi}{2}} \ln(\cos x) \, dx$

- THUS WE HAVE

$\Rightarrow I + I = \int_0^{\frac{\pi}{2}} \ln(\sin x) \, dx + \int_0^{\frac{\pi}{2}} \ln(\cos x) \, dx$

$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \ln(\sin x) + \ln(\cos x) \, dx = \int_0^{\frac{\pi}{2}} \ln(\cos x \sin x) \, dx$

$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \ln\left(\frac{1}{2}\sin 2x\right) dx = \int_0^{\frac{\pi}{2}} \ln\left(\frac{1}{2}\right) + \ln(\sin 2x) \, dx$

$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} -\ln 2 \, dx + \int_0^{\frac{\pi}{2}} \ln(\sin 2x) \, dx$

$u = 2x$, $x = \frac{1}{2}u$, $dx = \frac{1}{2}du$, $x = 0 \mapsto u = 0$, $x = \frac{\pi}{2} \mapsto u = \pi$

$\Rightarrow 2I = -\ln 2\left(\frac{\pi}{2}\right) + \int_0^{\pi} \ln(\sin u)\left(\frac{1}{2}du\right)$

$\Rightarrow 2I = -\frac{\pi}{2}\ln 2 + \frac{1}{2}\int_0^{\pi} \ln(\sin u) \, du$

THE SINE FUNCTION IS EVEN ABOUT $\frac{\pi}{2}$ SO WE CAN WRITE

$\Rightarrow 2I = -\frac{\pi}{2}\ln 2 + \frac{1}{2} \times 2\int_0^{\frac{\pi}{2}} \ln(\sin u) \, du$

$\Rightarrow 2I = -\frac{\pi}{2}\ln 2 + I$

$\Rightarrow I = \int_0^{\pi/2} \ln(\sin x) \, dx = -\frac{\pi}{2}\ln 2$

- AND FINALLY WE HAVE

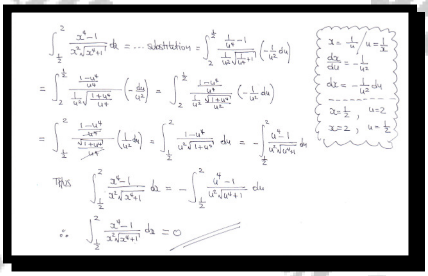
$\int_0^{\frac{\pi}{2}} x\cot x \, dx = -\int_0^{\frac{\pi}{2}} \ln(\sin x) \, dx = \frac{\pi}{2}\ln 2$

Question 265 (*****)

Use the substitution $x = \frac{1}{u}$ to find the value of

$$\int_{\frac{1}{2}}^{2} \frac{x^4 - 1}{x^2\sqrt{x^4+1}}\, dx .$$

$\boxed{0}$



$$\int_{\frac{1}{2}}^{2} \frac{x^4-1}{x^2\sqrt{x^4+1}}\,dx = \ldots \text{substitution} = \int_{2}^{\frac{1}{2}} \frac{\frac{1}{u^4}-1}{\frac{1}{u^2}\sqrt{\frac{1}{u^4}+1}}\left(-\frac{1}{u^2}du\right)$$

$$= \int_{2}^{\frac{1}{2}} \frac{\frac{1-u^4}{u^4}}{\frac{1}{u^2}\sqrt{\frac{1+u^4}{u^4}}}\left(-\frac{du}{u^2}\right) = \int_{2}^{\frac{1}{2}} \frac{\frac{1-u^4}{u^4}}{\frac{1}{u^2}\frac{\sqrt{1+u^4}}{u^2}}\left(-\frac{1}{u^2}du\right)$$

$$= \int_{\frac{1}{2}}^{2} \frac{\frac{1-u^4}{u^4}}{\frac{\sqrt{1+u^4}}{u^4}}\left(\frac{1}{u^2}du\right) = \int_{\frac{1}{2}}^{2} \frac{1-u^4}{u^2\sqrt{1+u^4}}\,du = -\int_{\frac{1}{2}}^{2} \frac{u^4-1}{u^2\sqrt{u^4+1}}\,du$$

$x = \frac{1}{u}$ / $u = \frac{1}{x}$

$\frac{dx}{du} = -\frac{1}{u^2}$

$dx = -\frac{1}{u^2}du$

$x = \frac{1}{2}$, $u = 2$

$x = 2$, $u = \frac{1}{2}$

THUS $\int_{\frac{1}{2}}^{2} \frac{x^4-1}{x^2\sqrt{x^4+1}}\,dx = -\int_{\frac{1}{2}}^{2} \frac{u^4-1}{u^2\sqrt{u^4+1}}\,du$

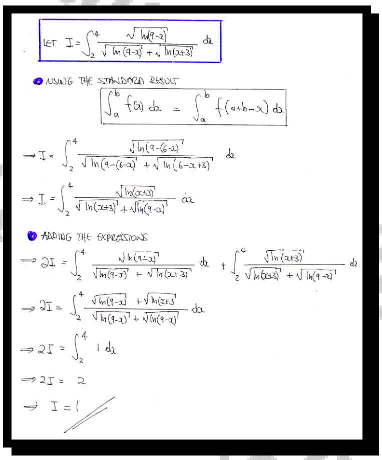
$\therefore \int_{\frac{1}{2}}^{2} \frac{x^4-1}{x^2\sqrt{x^4+1}}\,dx = 0$

Question 266 (*****)

Use a suitable substitution to find the value of

$$\int_{2}^{4} \frac{\sqrt{\ln(9-x)}}{\sqrt{\ln(9-x)} + \sqrt{\ln(3+x)}}\, dx .$$

$\boxed{}$, $\boxed{1}$



LET $I = \int_{2}^{4} \frac{\sqrt{\ln(9-x)}}{\sqrt{\ln(9-x)} + \sqrt{\ln(x+3)}}\,dx$

- USING THE STANDARD RESULT

$$\int_{a}^{b} f(x)\,dx = \int_{a}^{b} f(a+b-x)\,dx$$

$\Rightarrow I = \int_{2}^{4} \frac{\sqrt{\ln(9-(6-x))}}{\sqrt{\ln(9-(6-x))} + \sqrt{\ln(6-x+3)}}\,dx$

$\Rightarrow I = \int_{2}^{4} \frac{\sqrt{\ln(x+3)}}{\sqrt{\ln(x+3)} + \sqrt{\ln(9-x)}}\,dx$

- ADDING THE EXPRESSIONS

$\Rightarrow 2I = \int_{2}^{4} \frac{\sqrt{\ln(9-x)}}{\sqrt{\ln(9-x)} + \sqrt{\ln(x+3)}}\,dx + \int_{2}^{4} \frac{\sqrt{\ln(x+3)}}{\sqrt{\ln(x+3)} + \sqrt{\ln(9-x)}}\,dx$

$\Rightarrow 2I = \int_{2}^{4} \frac{\sqrt{\ln(9-x)} + \sqrt{\ln(x+3)}}{\sqrt{\ln(9-x)} + \sqrt{\ln(9-x)}}\,dx$

$\Rightarrow 2I = \int_{2}^{4} 1\,dx$

$\Rightarrow 2I = 2$

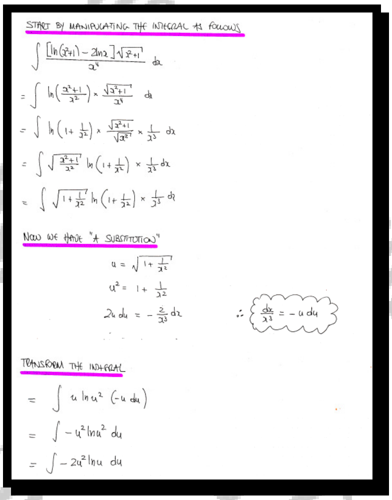
$\Rightarrow I = 1$

Question 267 (*****)

Use an appropriate substitution followed by integration by parts to find a simplified expression for

$$\int \frac{\left[\ln\left(x^2+1\right)-2\ln x\right]\sqrt{x^2+1}}{x^4}\,dx .$$

$$\boxed{}\,, \quad \boxed{\frac{2}{9x^3}\left(x^2+1\right)^{\frac{3}{2}}\left[1-3\ln\left(\frac{x^2+1}{x^2}\right)\right]+C}$$


START BY MANIPULATING THE INTEGRAL AS FOLLOWS

$$\int \frac{\left[\ln(x^2+1)-2\ln x\right]\sqrt{x^2+1}}{x^4}\,dx$$

$$= \int \ln\left(\frac{x^2+1}{x^2}\right) \times \frac{\sqrt{x^2+1}}{x^4}\,dx$$

$$= \int \ln\left(1+\frac{1}{x^2}\right) \times \frac{\sqrt{x^2+1}}{\sqrt{x^2}} \times \frac{1}{x^3}\,dx$$

$$= \int \sqrt{\frac{x^2+1}{x^2}}\ln\left(1+\frac{1}{x^2}\right) \times \frac{1}{x^3}\,dx$$

$$= \int \sqrt{1+\frac{1}{x^2}}\ln\left(1+\frac{1}{x^2}\right) \times \frac{1}{x^3}\,dx$$

NOW WE HAVE "A SUBSTITUTION"

$$u = \sqrt{1+\frac{1}{x^2}}$$

$$u^2 = 1+\frac{1}{x^2}$$

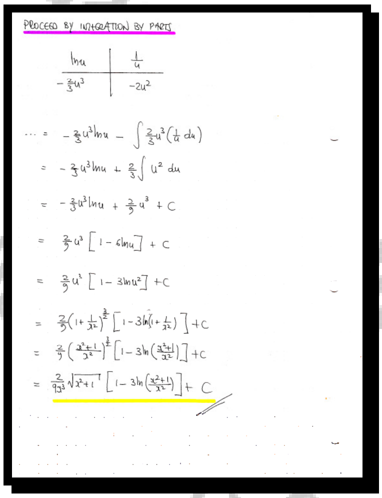
$$2u\,du = -\frac{2}{x^3}\,dx \qquad \therefore \frac{dx}{x^3} = -u\,du$$

TRANSFORM THE INTEGRAL

$$= \int u\ln u^2\,(-u\,du)$$

$$= \int -u^2\ln u^2\,du$$

$$= \int -2u^2\ln u\,du$$


PROCEED BY INTEGRATION BY PARTS

$\ln u$	$\frac{1}{u}$
$-\frac{2}{3}u^3$	$-2u^2$

$$\dots = -\frac{2}{3}u^3\ln u - \int \frac{2}{3}u^3\left(\frac{1}{u}\,du\right)$$

$$= -\frac{2}{3}u^3\ln u + \frac{2}{3}\int u^2\,du$$

$$= -\frac{2}{3}u^3\ln u + \frac{2}{9}u^3 + C$$

$$= \frac{2}{9}u^3\left[1-6\ln u\right] + C$$

$$= \frac{2}{9}u^3\left[1-3\ln u^2\right] + C$$

$$= \frac{2}{9}\left(1+\frac{1}{x^2}\right)^{\frac{3}{2}}\left[1-3\ln\left(1+\frac{1}{x^2}\right)\right]+C$$

$$= \frac{2}{9}\left(\frac{x^2+1}{x^2}\right)^{\frac{3}{2}}\left[1-3\ln\left(\frac{x^2+1}{x^2}\right)\right]+C$$

$$= \frac{2}{9x^3}\sqrt{x^2+1}\left[1-3\ln\left(\frac{x^2+1}{x^2}\right)\right]+C$$

Question 268 (*****)

Use appropriate integration techniques to show that

$$\int_0^{\frac{1}{2}\pi} \frac{\sin^2 x}{\sin x + \cos x}\, dx = \frac{1}{\sqrt{2}} \ln\left(1+\sqrt{2}\right).$$

S.Y., proof

- LET $I = \int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sin x + \cos x}$

BY SUBSTITUTION

$x = \frac{\pi}{2} - y$, $y = \frac{\pi}{2} - x$ $\Rightarrow dx = -dy$ & $x = \frac{\pi}{2} \mapsto y = 0$, $x = 0 \mapsto y = \frac{\pi}{2}$

$$\Rightarrow I = \int_{\frac{\pi}{2}}^{0} \frac{\sin^2\left(\frac{\pi}{2}-y\right)}{\sin\left(\frac{\pi}{2}-y\right) + \cos\left(\frac{\pi}{2}-y\right)}(-dy)$$

$\sin\left(\frac{\pi}{2}-A\right) = \cos A$, $\cos\left(\frac{\pi}{2}-A\right) \equiv \sin A$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\cos^2 y}{\cos y + \sin y}\, dy$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sin x + \cos x}\, dx = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\sin x + \cos x}\, dx$$

- ADDING WE OBTAIN

$$\Rightarrow I + I = \int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sin x + \cos x} + \frac{\cos^2 x}{\sin x + \cos x}\, dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \frac{1}{\sin x + \cos x}\, dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{2}\left[\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right]}\, dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{2}\left[\cos x \cos\frac{\pi}{4} + \sin x \sin\frac{\pi}{4}\right]}\, dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{2}\cos\left(x-\frac{\pi}{4}\right)}\, dx$$

$$\Rightarrow 2I = \frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{2}} \sec\left(x-\frac{\pi}{4}\right) dx$$

- STANDARD RESULT

$$\int \sec\theta \, d\theta = \ln|\sec\theta + \tan\theta| + C$$

$$\Rightarrow I = \frac{1}{2\sqrt{2}}\left[\ln\left|\sec\left(x-\frac{\pi}{4}\right) + \tan\left(x-\frac{\pi}{4}\right)\right|\right]_0^{\frac{\pi}{2}}$$

$$\Rightarrow I = \frac{1}{2\sqrt{2}}\left[\ln\left[\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right] - \ln\left[\sec\left(-\frac{\pi}{4}\right) + \tan\left(-\frac{\pi}{4}\right)\right]\right]$$

$$\Rightarrow I = \frac{1}{2\sqrt{2}}\left[\ln(\sqrt{2}+1) - \ln(\sqrt{2}-1)\right]$$

$$\Rightarrow I = \frac{1}{2\sqrt{2}}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$$

$$\Rightarrow I = \frac{1}{2\sqrt{2}}\ln\left[\frac{(\sqrt{2}+1)(\sqrt{2}+1)}{(\sqrt{2}-1)(\sqrt{2}+1)}\right]$$

$$\Rightarrow I = \frac{1}{2\sqrt{2}}\ln\left[\frac{(\sqrt{2}+1)^2}{2-1}\right]$$

$$\Rightarrow I = \frac{1}{2\sqrt{2}}\ln(\sqrt{2}+1)^2$$

$$\Rightarrow I = \frac{1}{\sqrt{2}}\ln(\sqrt{2}+1)$$

AS REQUIRED

ALTERNATIVE FROM THE POINT WHERE THE INTEGRAL IS OBTAINED

$$\frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{1}{\cos x + \sin x}\, dx$$

BY LITTLE t IDENTITIES

- LET $t = \tan\frac{x}{2}$

$\frac{dt}{dx} = \sec^2\frac{x}{2} \times \frac{1}{2}$

$\frac{dt}{dx} = \frac{1}{2}\left(1 + \tan^2\frac{x}{2}\right)$

$\frac{dt}{dx} = \frac{1}{2}(1+t^2)$

$dx = \frac{dt}{\frac{1}{2}(1+t^2)}$

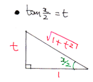
$dx = \frac{2}{1+t^2}\, dt$

- LIMITS

$x = 0 \mapsto t = 0$

$x = \frac{\pi}{2} \mapsto t = 1$

- $\tan\frac{x}{2} = t$



- $\sin x = 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)$

$= 2\left(\frac{t}{\sqrt{1+t^2}}\right)\left(\frac{1}{\sqrt{1+t^2}}\right)$

$= \frac{2t}{1+t^2}$

- $\cos x = \cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right)$

$= \left(\frac{1}{\sqrt{1+t^2}}\right)^2 - \left(\frac{t}{\sqrt{1+t^2}}\right)^2$

$= \frac{1-t^2}{1+t^2}$

$$= \frac{1}{2}\int_0^1 \frac{1}{\frac{1-t^2}{1+t^2} + \frac{2t}{1+t^2}}\left(\frac{2}{1+t^2}\, dt\right) = \frac{1}{2}\int_0^1 \frac{2}{1-t^2+2t}\, dt$$

$$= \int_0^1 \frac{1}{-(t^2-2t-1)}\, dt = \int_1^0 \frac{1}{(t-1)^2-2}\, dt$$

$$= \int_1^0 \frac{1}{(t-1-\sqrt{2})(t-1+\sqrt{2})}\, dt$$

NOW BY PARTIAL FRACTIONS (BY COVER UP)

$$= \int_1^0 \frac{\frac{1}{1+\sqrt{2}-1+\sqrt{2}}}{(t-1-\sqrt{2})} + \frac{\frac{1}{1-\sqrt{2}-1-\sqrt{2}}}{t-1+\sqrt{2}}\, dt$$

$$= \int_1^0 \frac{\frac{1}{2\sqrt{2}}}{t-1-\sqrt{2}} - \frac{\frac{1}{2\sqrt{2}}}{t-1+\sqrt{2}}\, dt$$

$$= \frac{1}{2\sqrt{2}}\int_1^0 \frac{1}{t-1-\sqrt{2}} - \frac{1}{t-1+\sqrt{2}}\, dt$$

$$= \frac{1}{2\sqrt{2}}\left[\ln\left|\frac{t-1-\sqrt{2}}{t-1+\sqrt{2}}\right|\right]_1^0$$

$$= \frac{1}{2\sqrt{2}}\left[\ln\left|\frac{-1-\sqrt{2}}{-1+\sqrt{2}}\right| - \ln\left|\frac{-\sqrt{2}}{\sqrt{2}}\right|\right]$$

$$= \frac{1}{2\sqrt{2}}\ln\left[\frac{-1-\sqrt{2}}{-1+\sqrt{2}} \times -1\right]$$

$$= \frac{1}{2\sqrt{2}}\ln\left[\frac{\sqrt{2}+1}{\sqrt{2}-1}\right]$$

$$= \frac{1}{2\sqrt{2}}\ln\left[\frac{(\sqrt{2}+1)(\sqrt{2}+1)}{(\sqrt{2}-1)(\sqrt{2}+1)}\right]$$

$$= \frac{1}{2\sqrt{2}}\ln\frac{(\sqrt{2}+1)^2}{2-1}$$

$$= \frac{1}{2\sqrt{2}}\ln(\sqrt{2}+1)^2$$

$$= \frac{1}{\sqrt{2}}\ln(\sqrt{2}+1)$$

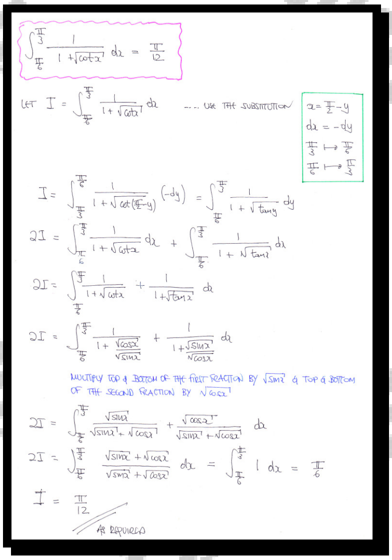
AS BEFORE

Question 269 (*****)

Use appropriate integration techniques to show that

$$\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{1}{1+\sqrt{\cot x}}\,dx = \frac{\pi}{12}.$$

proof


$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{1+\sqrt{\cot x}}\,dx = \frac{\pi}{12}$$

LET $I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{1+\sqrt{\cot x}}\,dx$ .... USE THE SUBSTITUTION

$x = \frac{\pi}{2} - y$

$dx = -dy$

$\frac{\pi}{3} \mapsto \frac{\pi}{6}$

$\frac{\pi}{6} \mapsto \frac{\pi}{3}$

$$I = \int_{\frac{\pi}{3}}^{\frac{\pi}{6}} \frac{1}{1+\sqrt{\cot(\frac{\pi}{2}-y)}}(-dy) = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{1+\sqrt{\tan y}}\,dy$$

$$2I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{1+\sqrt{\cot x}}\,dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{1+\sqrt{\tan x}}\,dx$$

$$2I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{1+\sqrt{\cot x}} + \frac{1}{1+\sqrt{\tan x}}\,dx$$

$$2I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{1+\frac{\sqrt{\cos x}}{\sqrt{\sin x}}} + \frac{1}{1+\frac{\sqrt{\sin x}}{\sqrt{\cos x}}}\,dx$$

MULTIPLY TOP & BOTTOM OF THE FIRST FRACTION BY $\sqrt{\sin x}$ & TOP & BOTTOM OF THE SECOND FRACTION BY $\sqrt{\cos x}$

$$2I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}} + \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$$

$$2I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 1\,dx = \frac{\pi}{6}$$

$$I = \frac{\pi}{12}$$

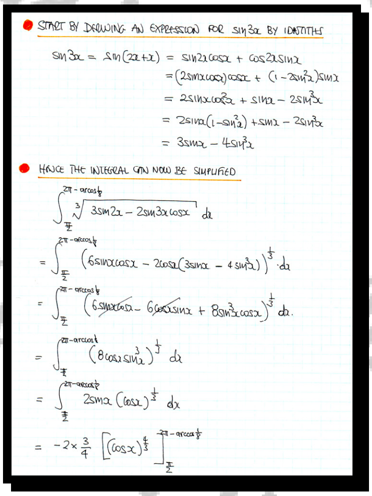
AS REQUIRED

Question 270 (*****)

Use appropriate integration techniques to find an exact answer for the following definite integral.

$$\int_{\frac{1}{2}\pi}^{2\pi-\arccos\frac{1}{8}} \sqrt[3]{3\sin 2x - 2\sin 3x \cos x} \;\; dx$$

$\boxed{}$, $\boxed{-\frac{3}{32}}$

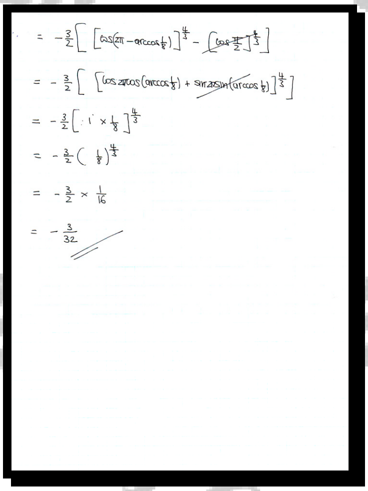


START BY DERIVING AN EXPRESSION FOR $\sin 3x$ BY IDENTITIES

$$\sin 3x = \sin(2x+x) = \sin 2x\cos x + \cos 2x \sin x$$
$$= (2\sin x\cos x)\cos x + (1-2\sin^2 x)\sin x$$
$$= 2\sin x\cos^2 x + \sin x - 2\sin^3 x$$
$$= 2\sin x(1-\sin^2 x) + \sin x - 2\sin^3 x$$
$$= 3\sin x - 4\sin^3 x$$

HENCE THE INTEGRAL CAN NOW BE SIMPLIFIED

$$\int_{\frac{\pi}{2}}^{2\pi-\arccos\frac{1}{8}} \sqrt[3]{3\sin 2x - 2\sin 3x\cos x}\, dx$$
$$= \int_{\frac{\pi}{2}}^{2\pi-\arccos\frac{1}{8}} \left(6\sin x\cos x - 2\cos x(3\sin x - 4\sin^3 x)\right)^{\frac{1}{3}} dx$$
$$= \int_{\frac{\pi}{2}}^{2\pi-\arccos\frac{1}{8}} \left(6\sin x\cos x - 6\cos x\sin x + 8\sin^3 x\cos x\right)^{\frac{1}{3}} dx$$
$$= \int_{\frac{\pi}{2}}^{2\pi-\arccos\frac{1}{8}} \left(8\cos x\sin^3 x\right)^{\frac{1}{3}} dx$$
$$= \int_{\frac{\pi}{2}}^{2\pi-\arccos\frac{1}{8}} 2\sin x\,(\cos x)^{\frac{1}{3}}\, dx$$
$$= -2\times\frac{3}{4}\left[(\cos x)^{\frac{4}{3}}\right]_{\frac{\pi}{2}}^{2\pi-\arccos\frac{1}{8}}$$



$$= -\frac{3}{2}\left[\left[\cos\left(2\pi-\arccos\tfrac{1}{8}\right)\right]^{\frac{4}{3}} - \left[\cos\tfrac{\pi}{2}\right]^{\frac{4}{3}}\right]$$
$$= -\frac{3}{2}\left[\left[\cos 2\pi\cos\left(\arccos\tfrac{1}{8}\right) + \sin 2\pi\sin\left(\arccos\tfrac{1}{8}\right)\right]^{\frac{4}{3}}\right]$$
$$= -\frac{3}{2}\left[1\times\frac{1}{8}\right]^{\frac{4}{3}}$$
$$= -\frac{3}{2}\left(\frac{1}{8}\right)^{\frac{4}{3}}$$
$$= -\frac{3}{2}\times\frac{1}{16}$$
$$= -\frac{3}{32}$$

Question 271 (*****)

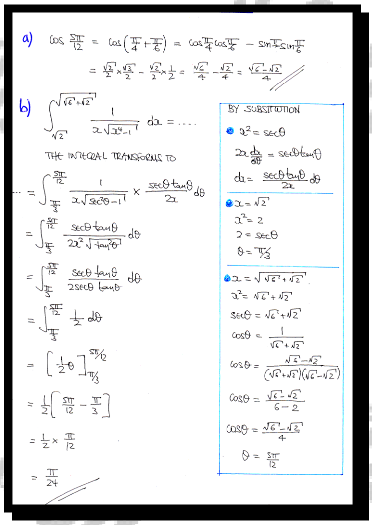
a) Use the compound angle identity $\cos(A+B)$ to show that

$$\cos\left(\frac{5\pi}{12}\right)=\frac{\sqrt{6}-\sqrt{2}}{4}$$

b) Use a suitable trigonometric substitution to find the exact value of

$$\int_{\sqrt{2}}^{\sqrt{\sqrt{6}+\sqrt{2}}} \frac{2}{x\sqrt{x^4-1}}\,dx\,.$$

[illegible] , $\boxed{\frac{\pi}{24}}$



a) $\cos\frac{5\pi}{12} = \cos\left(\frac{\pi}{4}+\frac{\pi}{6}\right) = \cos\frac{\pi}{4}\cos\frac{\pi}{6} - \sin\frac{\pi}{4}\sin\frac{\pi}{6}$

$= \frac{\sqrt{2}}{2}\times\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\times\frac{1}{2} = \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \frac{\sqrt{6}-\sqrt{2}}{4}$

b) $\int_{\sqrt{2}}^{\sqrt{\sqrt{6}+\sqrt{2}}} \frac{1}{x\sqrt{x^4-1}}\,dx = \ldots$

THE INTEGRAL TRANSFORMS TO

$\ldots = \int_{\frac{\pi}{3}}^{\frac{5\pi}{12}} \frac{1}{x\sqrt{\sec^2\theta-1}} \times \frac{\sec\theta\tan\theta}{2x}\,d\theta$

$= \int_{\frac{\pi}{3}}^{\frac{5\pi}{12}} \frac{\sec\theta\tan\theta}{2x^2\sqrt{\tan^2\theta}}\,d\theta$

$= \int_{\frac{\pi}{3}}^{\frac{5\pi}{12}} \frac{\sec\theta\tan\theta}{2\sec\theta\tan\theta}\,d\theta$

$= \int_{\frac{\pi}{3}}^{\frac{5\pi}{12}} \frac{1}{2}\,d\theta$

$= \left[\frac{1}{2}\theta\right]_{\pi/3}^{5\pi/12}$

$= \frac{1}{2}\left[\frac{5\pi}{12} - \frac{\pi}{3}\right]$

$= \frac{1}{2}\times\frac{\pi}{12}$

$= \frac{\pi}{24}$

BY SUBSTITUTION

- $x^2 = \sec\theta$
  $2x\frac{dx}{d\theta} = \sec\theta\tan\theta$
  $dx = \frac{\sec\theta\tan\theta}{2x}\,d\theta$
- $x = \sqrt{2}$
  $x^2 = 2$
  $2 = \sec\theta$
  $\theta = \pi/3$
- $x = \sqrt{\sqrt{6}+\sqrt{2}}$
  $x^2 = \sqrt{6}+\sqrt{2}$
  $\sec\theta = \sqrt{6}+\sqrt{2}$
  $\cos\theta = \frac{1}{\sqrt{6}+\sqrt{2}}$
  $\cos\theta = \frac{\sqrt{6}-\sqrt{2}}{(\sqrt{6}+\sqrt{2})(\sqrt{6}-\sqrt{2})}$
  $\cos\theta = \frac{\sqrt{6}-\sqrt{2}}{6-2}$
  $\cos\theta = \frac{\sqrt{6}-\sqrt{2}}{4}$
  $\theta = \frac{5\pi}{12}$

Question 272 (*****)

It is given that the functions of x, $u(x)$ and $v(x)$ satisfy

$$\int u(x)v(x)\,dx = \left[\int u(x)\,dx\right]\times\left[\int v(x)\,dx\right], \text{ for } x\in\mathbb{R},\ x\neq 0,\ x\neq 1.$$

a) Show clearly that

$$\frac{\int u(x)\,dx}{u(x)} + \frac{\int v(x)\,dx}{v(x)} = 1.$$

b) Given further that

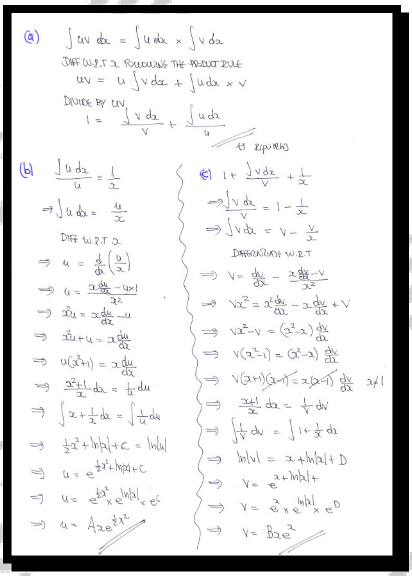
$$\frac{\int u(x)\,dx}{u(x)} = \frac{1}{x},$$

show that

$$u(x) = Axe^{\frac{1}{2}x^2}, \text{ where } A \text{ is an arbitrary constant.}$$

c) Determine a similar expression for $v(x)$.

$$\boxed{v(x) = Bxe^{x}}$$


(a) $\int uv\,dx = \int u\,dx \times \int v\,dx$

DIFF W.R.T x FOLLOWING THE PRODUCT RULE

$uv = u\int v\,dx + \int u\,dx \times v$

DIVIDE BY uv

$1 = \frac{\int v\,dx}{v} + \frac{\int u\,dx}{u}$

AS REQUIRED

(b) $\frac{\int u\,dx}{u} = \frac{1}{x}$

$\Rightarrow \int u\,dx = \frac{u}{x}$

DIFF W.R.T x

$\Rightarrow u = \frac{d}{dx}\left(\frac{u}{x}\right)$

$\Rightarrow u = \frac{x\frac{du}{dx} - u\times 1}{x^2}$

$\Rightarrow x^2u = x\frac{du}{dx} - u$

$\Rightarrow x^2u + u = x\frac{du}{dx}$

$\Rightarrow u(x^2+1) = x\frac{du}{dx}$

$\Rightarrow \frac{x^2+1}{x}\,dx = \frac{1}{u}\,du$

$\Rightarrow \int x + \frac{1}{x}\,dx = \int \frac{1}{u}\,du$

$\Rightarrow \frac{1}{2}x^2 + \ln|x| + C = \ln|u|$

$\Rightarrow u = e^{\frac{1}{2}x^2 + \ln|x| + C}$

$\Rightarrow u = e^{\frac{1}{2}x^2} \times e^{\ln|x|} \times e^{C}$

$\Rightarrow u = Axe^{\frac{1}{2}x^2}$

(c) $1 + \frac{\int v\,dx}{v} + \frac{1}{x}$

$\Rightarrow \frac{\int v\,dx}{v} = 1 - \frac{1}{x}$

$\Rightarrow \int v\,dx = v - \frac{v}{x}$

DIFFERENTIATE W.R.T

$\Rightarrow v = \frac{dv}{dx} - \frac{x\frac{dv}{dx} - v}{x^2}$

$\Rightarrow vx^2 = x^2\frac{dv}{dx} - x\frac{dv}{dx} + v$

$\Rightarrow vx^2 - v = (x^2 - x)\frac{dv}{dx}$

$\Rightarrow v(x^2-1) = (x^2-x)\frac{dv}{dx}$

$\Rightarrow v(x+1)(x-1) = x(x-1)\frac{dv}{dx}$ $x\neq 1$

$\Rightarrow \frac{x+1}{x}\,dx = \frac{1}{v}\,dv$

$\Rightarrow \int \frac{1}{v}\,dv = \int 1 + \frac{1}{x}\,dx$

$\Rightarrow \ln|v| = x + \ln|x| + D$

$\Rightarrow v = e^{x + \ln|x| +}$

$\Rightarrow v = e^{x} \times e^{\ln|x|} \times e^{D}$

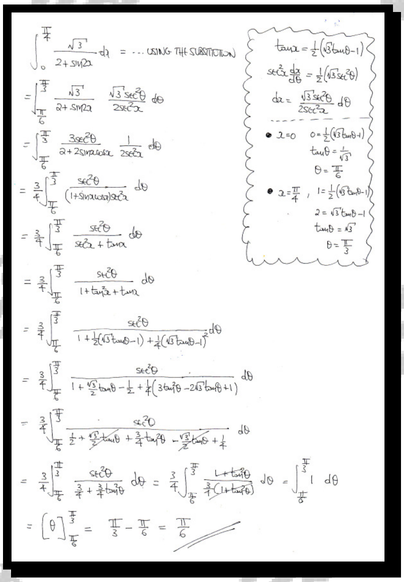
$\Rightarrow v = Bxe^{x}$

Question 273 **(*****)**

Use the substitution $\tan x = \frac{1}{2}\left(-1+\sqrt{3}\tan\theta\right)$ to find the exact value of

$$\int_0^{\frac{\pi}{4}} \frac{\sqrt{3}}{2+\sin 2x}\, dx .$$

$\boxed{}$, $\boxed{\frac{\pi}{6}}$



$$\int_0^{\frac{\pi}{4}} \frac{\sqrt{3}}{2+\sin 2x}\,dx = \dots \text{ USING THE SUBSTITUTION}$$

$\tan x = \frac{1}{2}\left(\sqrt{3}\tan\theta - 1\right)$

$\sec^2 x \frac{dx}{d\theta} = \frac{1}{2}\left(\sqrt{3}\sec^2\theta\right)$

$dx = \frac{\sqrt{3}\sec^2\theta}{2\sec^2 x}\,d\theta$

- $x=0$: $0 = \frac{1}{2}\left(\sqrt{3}\tan\theta - 1\right)$, $\tan\theta = \frac{1}{\sqrt{3}}$, $\theta = \frac{\pi}{6}$
- $x=\frac{\pi}{4}$: $1 = \frac{1}{2}\left(\sqrt{3}\tan\theta - 1\right)$, $2 = \sqrt{3}\tan\theta - 1$, $\tan\theta = \sqrt{3}$, $\theta = \frac{\pi}{3}$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{3}}{2+\sin 2x}\,\frac{\sqrt{3}\sec^2\theta}{2\sec^2 x}\,d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{3\sec^2\theta}{2+2\sin x\cos x}\,\frac{1}{2\sec^2 x}\,d\theta$$

$$= \frac{3}{4}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sec^2\theta}{(1+\sin x\cos x)\sec^2 x}\,d\theta$$

$$= \frac{3}{4}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sec^2\theta}{\sec^2 x+\tan x}\,d\theta$$

$$= \frac{3}{4}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sec^2\theta}{1+\tan^2 x+\tan x}\,d\theta$$

$$= \frac{3}{4}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sec^2\theta}{1+\frac{1}{2}(\sqrt{3}\tan\theta-1)+\frac{1}{4}(\sqrt{3}\tan\theta-1)^2}\,d\theta$$

$$= \frac{3}{4}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sec^2\theta}{1+\frac{\sqrt{3}}{2}\tan\theta-\frac{1}{2}+\frac{1}{4}(3\tan^2\theta-2\sqrt{3}\tan\theta+1)}\,d\theta$$

$$= \frac{3}{4}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sec^2\theta}{\frac{1}{2}+\frac{\sqrt{3}}{2}\tan\theta+\frac{3}{4}\tan^2\theta-\frac{\sqrt{3}}{2}\tan\theta+\frac{1}{4}}\,d\theta$$

$$= \frac{3}{4}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sec^2\theta}{\frac{3}{4}+\frac{3}{4}\tan^2\theta}\,d\theta = \frac{3}{4}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1+\tan^2\theta}{\frac{3}{4}(1+\tan^2\theta)}\,d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 1\,d\theta$$

$$= \left[\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \frac{\pi}{3}-\frac{\pi}{6} = \frac{\pi}{6}$$

Question 274 (*****)

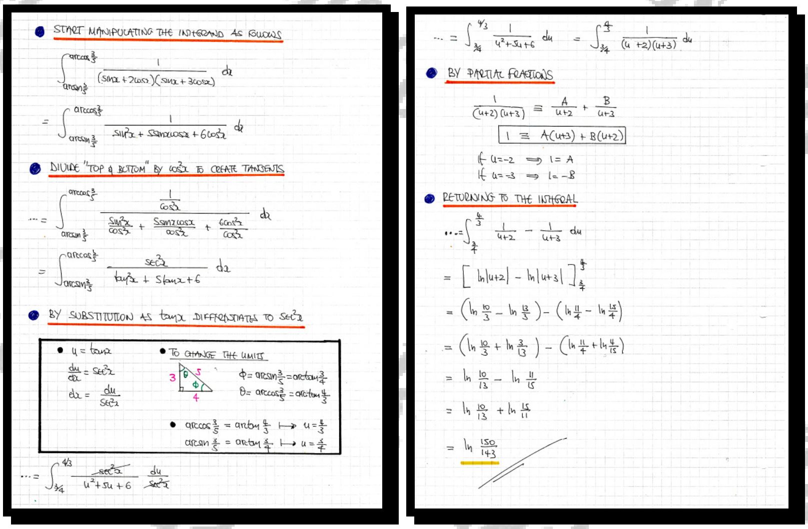
$$I = \int_{\arcsin\frac{3}{5}}^{\arccos\frac{3}{5}} \frac{1}{(\sin x + 2\cos x)(\sin x + 3\cos x)}\, dx\,.$$

Use appropriate integration techniques to show that

$$I = \ln\left(\frac{a}{b}\right),$$

where a and b are positive integers to be found.

$$\boxed{I = \ln\left(\frac{150}{143}\right)}$$



Question 275 (*****)

Use appropriate integration techniques to show that

$$\int_0^1 \frac{1}{x+\sqrt{1-x^2}}\,dx = \frac{\pi}{4}.$$

proof

USING A TRIGONOMETRIC SUBSTITUTION

$x = \sin\theta$, $dx = \cos\theta\,d\theta$, $x=0 \mapsto \theta=0$, $x=1 \mapsto \theta=\frac{\pi}{2}$

TRANSFORMING THE INTEGRAL

$$\int_0^1 \frac{1}{x+\sqrt{1-x^2}}\,dx = \int_0^{\frac{\pi}{2}} \frac{1}{\sin\theta+\sqrt{1-\sin^2\theta}}\,(\cos\theta\,d\theta) = \int_0^{\frac{\pi}{2}} \frac{\cos\theta}{\sin\theta+\cos\theta}\,d\theta$$

PROCEED AS FOLLOWS

LET $I = \int_0^{\frac{\pi}{2}} \frac{\cos\theta}{\sin\theta+\cos\theta}\,d\theta$

USING $\int_\alpha^\beta f(x)\,dx \equiv \int_\alpha^\beta f(\alpha+\beta-x)\,dx$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\cos(\frac{\pi}{2}-\theta)}{\sin(\frac{\pi}{2}-\theta)+\cos(\frac{\pi}{2}-\theta)}\,d\theta$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sin\theta}{\cos\theta+\sin\theta}\,d\theta$$

THUS WE NOW HAVE

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \frac{\cos\theta}{\sin\theta+\cos\theta}\,d\theta + \int_0^{\frac{\pi}{2}} \frac{\sin\theta}{\cos\theta+\sin\theta}\,d\theta$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \frac{\cos\theta+\sin\theta}{\cos\theta+\sin\theta}\,d\theta$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 1\,d\theta$$

$$\Rightarrow 2I = \frac{\pi}{2}$$

$$\Rightarrow I = \frac{\pi}{4}$$

$$\therefore \int_0^1 \frac{1}{x+\sqrt{1-x^2}}\,dx = \frac{\pi}{4}$$

ALTERNATIVE AFTER THE TRIGONOMETRIC SUBSTITUTION

$$\int_0^1 \frac{1}{x+\sqrt{1-x^2}}\,dx = \cdots = \int_0^{\frac{\pi}{2}} \frac{\cos\theta}{\sin\theta+\cos\theta}\,d\theta$$

$$= \frac{1}{2}\int \frac{2\cos\theta}{\sin\theta+\cos\theta}\,d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\cos\theta-\sin\theta+\sin\theta+\cos\theta}{\sin\theta+\cos\theta}\,d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\cos\theta-\sin\theta}{\sin\theta+\cos\theta} + \frac{\sin\theta+\cos\theta}{\sin\theta+\cos\theta}\,d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\cos\theta-\sin\theta}{\sin\theta+\cos\theta} + 1\,d\theta$$

THIS IS OF THE FORM $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)|+C$

$$= \frac{1}{2}\Big[\ln|\sin\theta+\cos\theta| + \theta\Big]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\Big[(\ln 1 + \frac{\pi}{2}) - (\ln 1 + 0)\Big]$$

$$= \frac{\pi}{4}$$

AS BEFORE

Question 276 (*****)

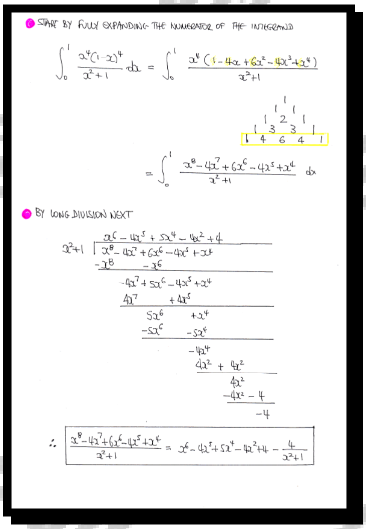
Use polynomial division to find the exact value of

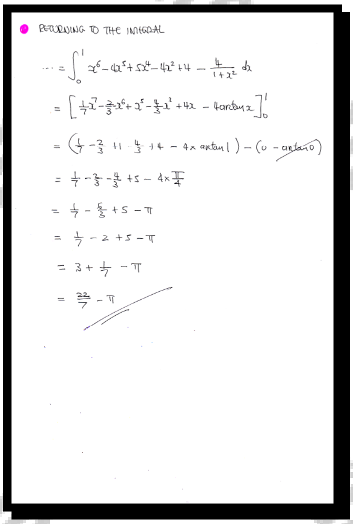
$$\int_0^1 \frac{x^4(1-x)^4}{x^2+1}\,dx\,.$$

You may assume that

$$\int \frac{1}{1+x^2}\,dx = \arctan x + \text{constant}\,.$$

$\boxed{}$, $\boxed{\dfrac{22}{7}-\pi}$





Question 277 (*****)

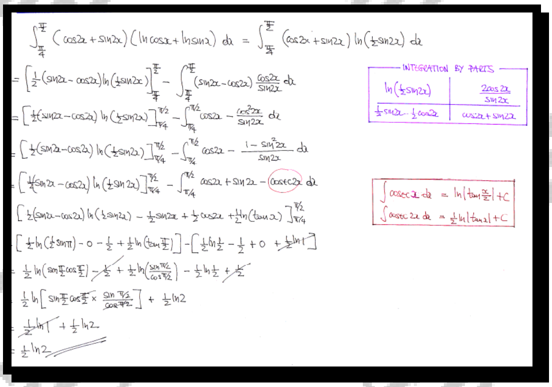
Use integration by parts to find a simplified exact value for

$$\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} (\cos 2x + \sin 2x)(\ln \cos x + \ln \sin x) \ dx .$$

You may assume that

$$\int \operatorname{cosec} x \ dx = \ln\left|\tan\left(\frac{1}{2}x\right)\right| + \text{constant} .$$

$\boxed{}$, $\boxed{\frac{1}{2}\ln 2}$



$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\cos 2x + \sin 2x)(\ln\cos x + \ln\sin x)\,dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\cos 2x + \sin 2x)\ln\left(\tfrac{1}{2}\sin 2x\right)dx$$

$$= \left[\tfrac{1}{2}(\sin 2x - \cos 2x)\ln\left(\tfrac{1}{2}\sin 2x\right)\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\sin 2x - \cos 2x)\frac{\cos 2x}{\sin 2x}\,dx$$

$$= \left[\tfrac{1}{2}(\sin 2x - \cos 2x)\ln\left(\tfrac{1}{2}\sin 2x\right)\right]_{\pi/4}^{\pi/2} - \int_{\pi/4}^{\pi/2} \cos 2x - \frac{\cos^2 2x}{\sin 2x}\,dx$$

$$= \left[\tfrac{1}{2}(\sin 2x - \cos 2x)\ln\left(\tfrac{1}{2}\sin 2x\right)\right]_{\pi/4}^{\pi/2} - \int_{\pi/4}^{\pi/2} \cos 2x - \frac{1 - \sin^2 2x}{\sin 2x}\,dx$$

$$= \left[\tfrac{1}{2}(\sin 2x - \cos 2x)\ln\left(\tfrac{1}{2}\sin 2x\right)\right]_{\pi/4}^{\pi/2} - \int_{\pi/4}^{\pi/2} \cos 2x + \sin 2x - \operatorname{cosec} 2x\,dx$$

$$= \left[\tfrac{1}{2}(\sin 2x - \cos 2x)\ln\left(\tfrac{1}{2}\sin 2x\right) - \tfrac{1}{2}\sin 2x + \tfrac{1}{2}\cos 2x + \tfrac{1}{2}\ln(\tan x)\right]_{\pi/4}^{\pi/2}$$

$$= \left[\tfrac{1}{2}\ln\left(\tfrac{1}{2}\sin\pi\right) - 0 - \tfrac{1}{2} + \tfrac{1}{2}\ln\left(\tan\tfrac{\pi}{2}\right)\right] - \left[\tfrac{1}{2}\ln\tfrac{1}{2} - \tfrac{1}{2} + 0 + \tfrac{1}{2}\ln 1\right]$$

$$= \tfrac{1}{2}\ln\left(\sin\tfrac{\pi}{2}\cos\tfrac{\pi}{2}\right) - \tfrac{1}{2} + \tfrac{1}{2}\ln\left(\frac{\sin\pi/2}{\cos\pi/2}\right) - \tfrac{1}{2}\ln\tfrac{1}{2} + \tfrac{1}{2}$$

$$= \tfrac{1}{2}\ln\left[\sin\tfrac{\pi}{2}\cos\tfrac{\pi}{2} \times \frac{\sin\pi/2}{\cos\pi/2}\right] + \tfrac{1}{2}\ln 2$$

$$= \tfrac{1}{2}\ln 1 + \tfrac{1}{2}\ln 2$$

$$= \tfrac{1}{2}\ln 2$$

INTEGRATION BY PARTS

$\ln\left(\tfrac{1}{2}\sin 2x\right)$	$\frac{2\cos 2x}{\sin 2x}$
$\tfrac{1}{2}\sin 2x - \tfrac{1}{2}\cos 2x$	$\cos 2x + \sin 2x$

$\int \operatorname{cosec} x\,dx = \ln\left|\tan\frac{x}{2}\right| + C$

$\int \operatorname{cosec} 2x\,dx = \frac{1}{2}\ln|\tan x| + C$

Question 278 (*****)

It is given that

$$x^2 + x + 2 = (u - x)^2 .$$

a) Show clearly that …

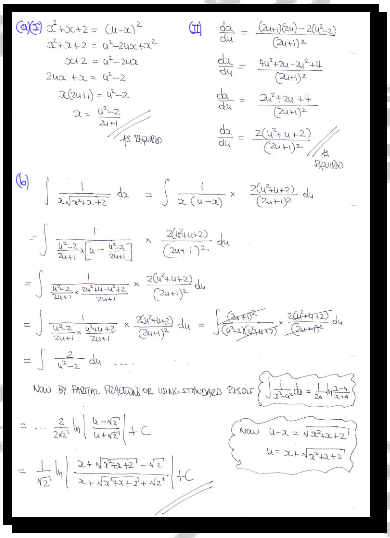
i. … $x = \dfrac{u^2 - 2}{2u + 1}$.

ii. … $\dfrac{dx}{du} = \dfrac{2\left(u^2 + u + 2\right)}{(2u+1)^2}$.

b) Find a simplified expression for

$$\int \frac{1}{x\sqrt{x^2 + x + 2}}\, dx .$$

$\square$, $\boxed{\dfrac{1}{\sqrt{2}} \ln \left| \dfrac{x + \sqrt{x^2 + x + 2} - \sqrt{2}}{x + \sqrt{x^2 + x + 2} + \sqrt{2}} \right| + C}$



(a)(I) $x^2+x+2 = (u-x)^2$

$x^2+x+2 = u^2-2ux+x^2$

$x+2 = u^2-2ux$

$2ux + x = u^2-2$

$x(2u+1) = u^2-2$

$x = \dfrac{u^2-2}{2u+1}$ AS REQUIRED

(II) $\dfrac{dx}{du} = \dfrac{(2u+1)(2u)-2(u^2-2)}{(2u+1)^2}$

$\dfrac{dx}{du} = \dfrac{4u^2+2u-2u^2+4}{(2u+1)^2}$

$\dfrac{dx}{du} = \dfrac{2u^2+2u+4}{(2u+1)^2}$

$\dfrac{dx}{du} = \dfrac{2(u^2+u+2)}{(2u+1)^2}$ AS REQUIRED

(b) $\displaystyle\int \frac{1}{x\sqrt{x^2+x+2}}\,dx = \int \frac{1}{x(u-x)} \times \frac{2(u^2+u+2)}{(2u+1)^2}\,du$

$= \displaystyle\int \frac{1}{\frac{u^2-2}{2u+1}\left[u - \frac{u^2-2}{2u+1}\right]} \times \frac{2(u^2+u+2)}{(2u+1)^2}\,du$

$= \displaystyle\int \frac{1}{\frac{u^2-2}{2u+1} \times \frac{2u^2+u-u^2+2}{2u+1}} \times \frac{2(u^2+u+2)}{(2u+1)^2}\,du$

$= \displaystyle\int \frac{1}{\frac{u^2-2}{2u+1} \times \frac{u^2+u+2}{2u+1}} \times \frac{2(u^2+u+2)}{(2u+1)^2}\,du = \int \frac{(2u+1)^2}{(u^2-2)(u^2+u+2)} \times \frac{2(u^2+u+2)}{(2u+1)^2}\,du$

$= \displaystyle\int \frac{2}{u^2-2}\,du \ \dots$

NOW BY PARTIAL FRACTIONS OR USING STANDARD RESULT $\left\{ \displaystyle\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\ln\frac{x-a}{x+a} \right\}$

$= \dots \dfrac{2}{2\sqrt{2}} \ln \left| \dfrac{u-\sqrt{2}}{u+\sqrt{2}} \right| + C$

NOW $u - x = \sqrt{x^2+x+2}$

$u = x + \sqrt{x^2+x+2}$

$= \dfrac{1}{\sqrt{2}} \ln \left| \dfrac{x+\sqrt{x^2+x+2}-\sqrt{2}}{x+\sqrt{x^2+x+2}+\sqrt{2}} \right| + C$

Question 279 (*****)

It is given that a and b are distinct real constants and λ is a real parameter.

a) Starting by the relationship between two functions of x, $f(x)$ and $g(x)$

$$\left[\lambda f(x)+g(x)\right]^2 \geq 0,$$

show clearly that

$$\lambda^2\int_a^b \left[f(x)\right]^2\ dx+2\lambda\int_a^b f(x)g(x)\ dx+\int_a^b \left[g(x)\right]^2\ dx \geq 0.$$

b) Deduce the Cauchy Schwarz inequality for integrals

$$\left[\int_a^b f(x)g(x)\ dx\right]^2 \leq \left[\int_a^b \left[f(x)\right]^2\ dx\right]\left[\int_a^b \left[g(x)\right]^2\ dx\right].$$

[continues overleaf]

[continued from overleaf]

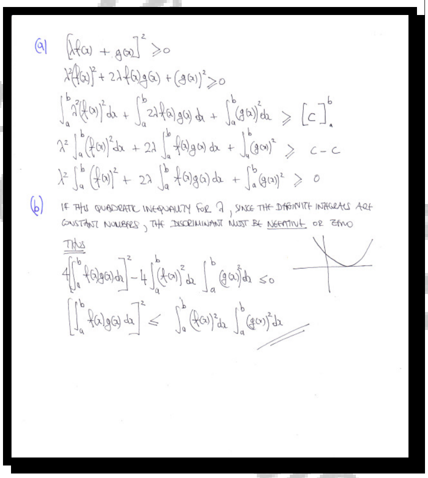
c) By letting $f(x)=\sqrt{\sin x}$ and $g(x)=1$, show that

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin x}\ \ dx \le \sqrt{\frac{\pi}{2}}\ .$$

d) By letting $f(x)=\sqrt{\sqrt{\sin x}}$ and $g(x)=\cos x$, show that

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin x}\ \ dx \ge \frac{64}{25\pi}\ .$$

proof



(a) $[\lambda f(x)+g(x)]^2 \geqslant 0$

$\lambda^2(f(x))^2+2\lambda f(x)g(x)+(g(x))^2 \geqslant 0$

$\int_a^b \lambda^2(f(x))^2\,dx+\int_a^b 2\lambda f(x)g(x)\,dx+\int_a^b (g(x))^2\,dx \geqslant [C]_a^b$

$\lambda^2\int_a^b (f(x))^2\,dx+2\lambda\int_a^b f(x)g(x)\,dx+\int_a^b (g(x))^2 \geqslant C-C$

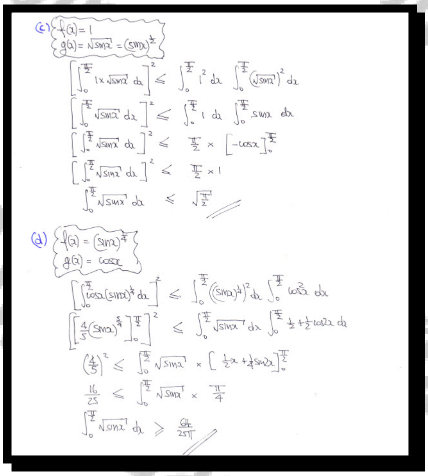
$\lambda^2\int_a^b (f(x))^2+2\lambda\int_a^b f(x)g(x)\,dx+\int_a^b (g(x))^2 \geqslant 0$

(b) IF THIS QUADRATIC INEQUALITY FOR λ, SINCE THE DEFINITE INTEGRALS ARE CONSTANT NUMBERS, THE DISCRIMINANT MUST BE NEGATIVE OR ZERO

THUS

$4\left[\int_a^b f(x)g(x)\,dx\right]^2-4\int_a^b (f(x))^2\,dx\int_a^b (g(x))^2\,dx \leqslant 0$

$\left[\int_a^b f(x)g(x)\,dx\right]^2 \leqslant \int_a^b (f(x))^2\,dx\int_a^b (g(x))^2\,dx$



(c) $f(x)=1$, $g(x)=\sqrt{\sin x}=(\sin x)^{\frac{1}{2}}$

$\left[\int_0^{\frac{\pi}{2}} 1\times\sqrt{\sin x}\,dx\right]^2 \leqslant \int_0^{\frac{\pi}{2}} 1^2\,dx\int_0^{\frac{\pi}{2}} (\sqrt{\sin x})^2\,dx$

$\left[\int_0^{\frac{\pi}{2}} \sqrt{\sin x}\,dx\right]^2 \leqslant \int_0^{\frac{\pi}{2}} 1\,dx\int_0^{\frac{\pi}{2}} \sin x\,dx$

$\left[\int_0^{\frac{\pi}{2}} \sqrt{\sin x}\,dx\right]^2 \leqslant \frac{\pi}{2}\times\left[-\cos x\right]_0^{\frac{\pi}{2}}$

$\left[\int_0^{\frac{\pi}{2}} \sqrt{\sin x}\,dx\right]^2 \leqslant \frac{\pi}{2}\times 1$

$\int_0^{\frac{\pi}{2}} \sqrt{\sin x}\,dx \leqslant \sqrt{\frac{\pi}{2}}$

(d) $f(x)=(\sin x)^{\frac{1}{4}}$, $g(x)=\cos x$

$\left[\int_0^{\frac{\pi}{2}} \cos x(\sin x)^{\frac{1}{4}}\,dx\right]^2 \leqslant \int_0^{\frac{\pi}{2}} \left((\sin x)^{\frac{1}{4}}\right)^2 dx\int_0^{\frac{\pi}{2}} \cos^2 x\,dx$

$\left[\left[\frac{4}{5}(\sin x)^{\frac{5}{4}}\right]_0^{\frac{\pi}{2}}\right]^2 \leqslant \int_0^{\frac{\pi}{2}} \sqrt{\sin x}\,dx\int_0^{\frac{\pi}{2}} \frac{1}{2}+\frac{1}{2}\cos 2x\,dx$

$\left(\frac{4}{5}\right)^2 \leqslant \int_0^{\frac{\pi}{2}} \sqrt{\sin x}\times\left[\frac{1}{2}x+\frac{1}{4}\sin 2x\right]_0^{\frac{\pi}{2}}$

$\frac{16}{25} \leqslant \int_0^{\frac{\pi}{2}} \sqrt{\sin x}\times\frac{\pi}{4}$

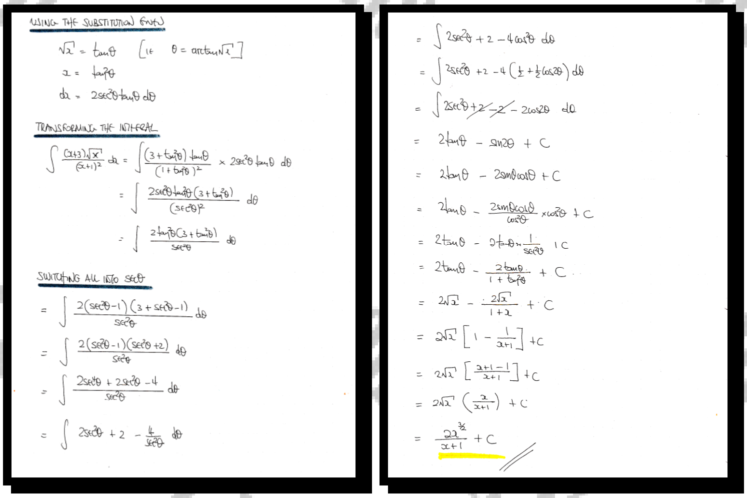
$\int_0^{\frac{\pi}{2}} \sqrt{\sin x}\,dx \geqslant \frac{64}{25\pi}$

Question 280 (*****)

By using the substitution $\sqrt{x} = \tan\theta$, or otherwise, find

$$\int \frac{(x+3)\sqrt{x}}{(x+1)^2}\,dx.$$

$\boxed{}$, $\boxed{\dfrac{2x^{\frac{3}{2}}}{x+1}+C}$



USING THE SUBSTITUTION GIVEN

$\sqrt{x} = \tan\theta \quad [\text{i.e. } \theta = \arctan\sqrt{x}]$

$x = \tan^2\theta$

$dx = 2\sec^2\theta\tan\theta\,d\theta$

TRANSFORMING THE INTEGRAL

$\int \frac{(x+3)\sqrt{x}}{(x+1)^2}\,dx = \int \frac{(3+\tan^2\theta)\tan\theta}{(1+\tan^2\theta)^2} \times 2\sec^2\theta\tan\theta\,d\theta$

$= \int \frac{2\sec^2\theta\tan^2\theta(3+\tan^2\theta)}{(\sec^2\theta)^2}\,d\theta$

$= \int \frac{2\tan^2\theta(3+\tan^2\theta)}{\sec^2\theta}\,d\theta$

SWITCHING ALL INTO $\sec\theta$

$= \int \frac{2(\sec^2\theta-1)(3+\sec^2\theta-1)}{\sec^2\theta}\,d\theta$

$= \int \frac{2(\sec^2\theta-1)(\sec^2\theta+2)}{\sec^2\theta}\,d\theta$

$= \int \frac{2\sec^4\theta+2\sec^2\theta-4}{\sec^2\theta}\,d\theta$

$= \int 2\sec^2\theta + 2 - \frac{4}{\sec^2\theta}\,d\theta$

$= \int 2\sec^2\theta + 2 - 4\cos^2\theta\,d\theta$

$= \int 2\sec^2\theta + 2 - 4\left(\frac{1}{2}+\frac{1}{2}\cos 2\theta\right)d\theta$

$= \int 2\sec^2\theta + 2 - 2 - 2\cos 2\theta\,d\theta$

$= 2\tan\theta - \sin 2\theta + C$

$= 2\tan\theta - 2\sin\theta\cos\theta + C$

$= 2\tan\theta - \frac{2\sin\theta\cos\theta}{\cos^2\theta} \times \cos^2\theta + C$

$= 2\tan\theta - 2\tan\theta \times \frac{1}{\sec^2\theta} + C$

$= 2\tan\theta - \frac{2\tan\theta}{1+\tan^2\theta} + C$

$= 2\sqrt{x} - \frac{2\sqrt{x}}{1+x} + C$

$= 2\sqrt{x}\left[1 - \frac{1}{x+1}\right] + C$

$= 2\sqrt{x}\left[\frac{x+1-1}{x+1}\right] + C$

$= 2\sqrt{x}\left(\frac{x}{x+1}\right) + C$

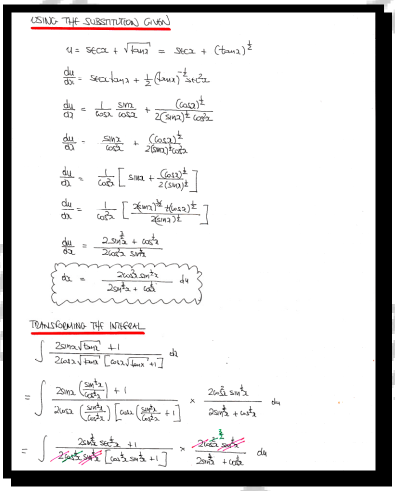
$= \frac{2x^{\frac{3}{2}}}{x+1} + C$

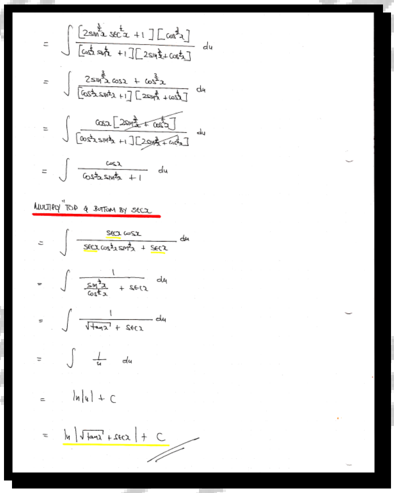
Question 281 (*****)

By using the substitution $u = \sec x + \sqrt{\tan x}$, or otherwise, find

$$\int \frac{1 + 2\sin x\sqrt{\tan x}}{2\left[1 + \cos x\sqrt{\tan x}\right]\cos x\sqrt{\tan x}}\, dx .$$

$\boxed{}$, $\boxed{\ln\left|\sec x + \sqrt{\tan x}\right| + C}$





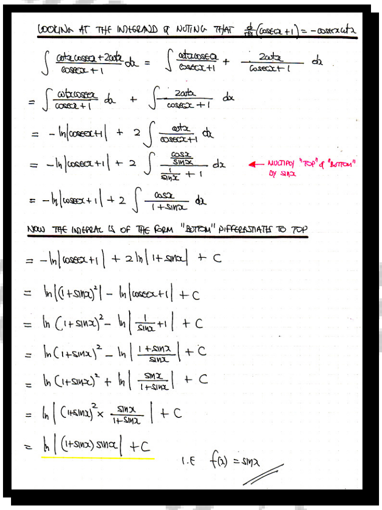
Question 282 (*****)

It is given that

$$\int \frac{\cot x \operatorname{cosec} x + 2\cot x}{1+\operatorname{cosec} x}\, dx \equiv \ln\left|\left[1+f(x)\right]f(x)\right| + \text{ constant}.$$

Using integration techniques, determine an expression for $f(x)$.

$\boxed{}$, $\boxed{f(x) = \sin x}$



LOOKING AT THE INTEGRAND & NOTING THAT $\frac{d}{dx}(\operatorname{cosec}x+1) = -\operatorname{cosec}x\cot x$

$$\int \frac{\cot x\operatorname{cosec}x+2\cot x}{\operatorname{cosec}x+1}\,dx = \int \frac{\cot x\operatorname{cosec}x}{\operatorname{cosec}x+1} + \frac{2\cot x}{\operatorname{cosec}x+1}\,dx$$

$$= \int \frac{\cot x\operatorname{cosec}x}{\operatorname{cosec}x+1}\,dx + \int \frac{2\cot x}{\operatorname{cosec}x+1}\,dx$$

$$= -\ln|\operatorname{cosec}x+1| + 2\int \frac{\cot x}{\operatorname{cosec}x+1}\,dx$$

$$= -\ln|\operatorname{cosec}x+1| + 2\int \frac{\frac{\cos x}{\sin x}}{\frac{1}{\sin x}+1}\,dx$$ ← MULTIPLY "TOP" & "BOTTOM" BY $\sin x$

$$= -\ln|\operatorname{cosec}x+1| + 2\int \frac{\cos x}{1+\sin x}\,dx$$

NOW THE INTEGRAL IS OF THE FORM "BOTTOM" DIFFERENTIATES TO TOP

$$= -\ln|\operatorname{cosec}x+1| + 2\ln|1+\sin x| + C$$

$$= \ln|(1+\sin x)^2| - \ln|\operatorname{cosec}x+1| + C$$

$$= \ln(1+\sin x)^2 - \ln\left|\frac{1}{\sin x}+1\right| + C$$

$$= \ln(1+\sin x)^2 - \ln\left|\frac{1+\sin x}{\sin x}\right| + C$$

$$= \ln(1+\sin x)^2 + \ln\left|\frac{\sin x}{1+\sin x}\right| + C$$

$$= \ln\left|(1+\sin x)^2 \times \frac{\sin x}{1+\sin x}\right| + C$$

$$= \ln|(1+\sin x)\sin x| + C$$

I.E. $f(x) = \sin x$

Question 283 (*****)

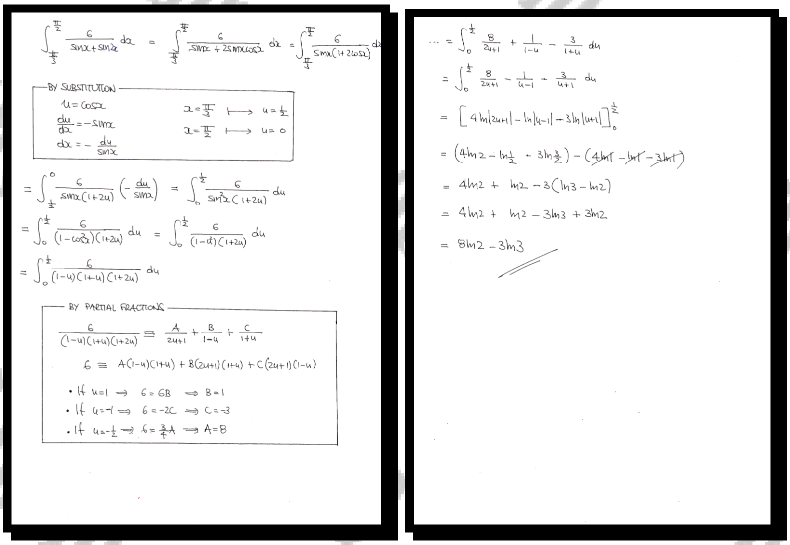
$$I = \int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi} \frac{6}{\sin x + \sin 2x} \, dx$$

Use appropriate integration techniques to show that

$$I = A \ln N + B \ln M \ ,$$

where A, B, N and M are integers to be found.

SP-X , $I = 8\ln 2 - 3\ln 3$



$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{6}{\sin x+\sin 2x}\,dx = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{6}{\sin x + 2\sin x\cos x}\,dx = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{6}{\sin x(1+2\cos x)}\,dx$$

BY SUBSTITUTION

$u = \cos x$, $\frac{du}{dx} = -\sin x$, $dx = -\frac{du}{\sin x}$

$x = \frac{\pi}{3} \mapsto u = \frac{1}{2}$

$x = \frac{\pi}{2} \mapsto u = 0$

$$= \int_{\frac{1}{2}}^{0} \frac{6}{\sin x(1+2u)}\left(-\frac{du}{\sin x}\right) = \int_{0}^{\frac{1}{2}} \frac{6}{\sin^2 x(1+2u)}\,du$$

$$= \int_{0}^{\frac{1}{2}} \frac{6}{(1-\cos^2 x)(1+2u)}\,du = \int_{0}^{\frac{1}{2}} \frac{6}{(1-u^2)(1+2u)}\,du$$

$$= \int_{0}^{\frac{1}{2}} \frac{6}{(1-u)(1+u)(1+2u)}\,du$$

BY PARTIAL FRACTIONS

$$\frac{6}{(1-u)(1+u)(1+2u)} \equiv \frac{A}{2u+1} + \frac{B}{1-u} + \frac{C}{1+u}$$

$$6 \equiv A(1-u)(1+u) + B(2u+1)(1+u) + C(2u+1)(1-u)$$

- If $u=1 \Rightarrow 6 = 6B \Rightarrow B = 1$
- If $u=-1 \Rightarrow 6 = -2C \Rightarrow C = -3$
- If $u=-\frac{1}{2} \Rightarrow 6 = \frac{3}{4}A \Rightarrow A = 8$

$$\dots = \int_{0}^{\frac{1}{2}} \frac{8}{2u+1} + \frac{1}{1-u} - \frac{3}{1+u}\,du$$

$$= \int_{0}^{\frac{1}{2}} \frac{8}{2u+1} - \frac{1}{u-1} - \frac{3}{u+1}\,du$$

$$= \Big[4\ln|2u+1| - \ln|u-1| - 3\ln|u+1|\Big]_0^{\frac{1}{2}}$$

$$= \left(4\ln 2 - \ln\tfrac{1}{2} - 3\ln\tfrac{3}{2}\right) - (4\ln 1 - \ln 1 - 3\ln 1)$$

$$= 4\ln 2 + \ln 2 - 3(\ln 3 - \ln 2)$$

$$= 4\ln 2 + \ln 2 - 3\ln 3 + 3\ln 2$$

$$= 8\ln 2 - 3\ln 3$$

Question 284 (*****)

Use the substitution $x = \tan\left(\frac{1}{2}\theta\right)$, to find a simplified expression for

$$\int x \arccos\left[\frac{1-x^2}{1+x^2}\right] dx .$$

$\boxed{\quad}$, $\boxed{-x + \left(1+x^2\right)\arctan x + \text{constant}}$

USING THE SUBSTITUTION GIVEN

$x = \tan\left(\frac{1}{2}\theta\right)$

$dx = \frac{1}{2}\sec^2\left(\frac{1}{2}\theta\right)\, d\theta$

OR $\left[dx = \frac{1}{2}\left[1 + \tan^2\left(\frac{1}{2}\theta\right)\right] d\theta \right]$

OR $\left[dx = \frac{1}{2}\left(1 + x^2\right) d\theta \right]$

(WE SHALL SEE WHICH FORM IS BETTER FOR THIS QUESTION)

$$\frac{1-x^2}{1+x^2} = \frac{1-\tan^2\left(\frac{1}{2}\theta\right)}{1+\tan^2\left(\frac{1}{2}\theta\right)} = \frac{1-\tan^2\left(\frac{1}{2}\theta\right)}{\sec^2\left(\frac{1}{2}\theta\right)} = \frac{1}{\sec^2\left(\frac{1}{2}\theta\right)} - \frac{\tan^2\left(\frac{1}{2}\theta\right)}{\sec^2\left(\frac{1}{2}\theta\right)}$$

$$= \cos^2\left(\tfrac{1}{2}\theta\right) - \frac{\sin^2\left(\frac{1}{2}\theta\right)}{\cos^2\left(\frac{1}{2}\theta\right)}\cos^2\left(\tfrac{1}{2}\theta\right) = \cos^2\left(\tfrac{1}{2}\theta\right) - \sin^2\left(\tfrac{1}{2}\theta\right) = \cos\theta$$

TRANSFORMING THE INTEGRAL WE HAVE

$$\int x \arccos\left(\frac{1-x^2}{1+x^2}\right) dx = \int \tan\left(\tfrac{1}{2}\theta\right) \arccos\left(\cos\theta\right)\left[\tfrac{1}{2}\sec^2\left(\tfrac{1}{2}\theta\right) d\theta\right]$$

$$= \int \tfrac{1}{2}\theta \tan\left(\tfrac{1}{2}\theta\right)\sec^2\left(\tfrac{1}{2}\theta\right)\, d\theta$$

INTEGRATION BY PARTS

$\frac{1}{2}\theta$	$\frac{1}{2}$
$\tan^2\left(\frac{1}{2}\theta\right)$	$\tan\left(\frac{1}{2}\theta\right)\sec^2\left(\frac{1}{2}\theta\right)$

$$\ldots = \tfrac{1}{2}\theta\tan^2\left(\tfrac{1}{2}\theta\right) - \int \tfrac{1}{2}\tan^2\left(\tfrac{1}{2}\theta\right) d\theta$$

$$= \tfrac{1}{2}\theta\tan^2\left(\tfrac{1}{2}\theta\right) - \tfrac{1}{2}\int \sec^2\tfrac{\theta}{2} - 1 \ d\theta$$

$$= \tfrac{1}{2}\theta\tan^2\left(\tfrac{1}{2}\theta\right) - \tfrac{1}{2}\left[2\tan\left(\tfrac{1}{2}\theta\right) - \theta\right] + C$$

$$= \tfrac{1}{2}\theta\tan^2\left(\tfrac{1}{2}\theta\right) - \tan\left(\tfrac{1}{2}\theta\right) + \tfrac{1}{2}\theta + C$$

$$= \tfrac{1}{2}\theta\left(1 + \tan^2\left(\tfrac{1}{2}\theta\right)\right) - \tan\left(\tfrac{1}{2}\theta\right) + C$$

$x = \tan\frac{1}{2}\theta$

$\arctan x = \frac{1}{2}\theta$

$$= \left[\arctan x\right]\left[1+x^2\right] - x + C$$

$$= -x + \left(1+x^2\right)\arctan x + C$$

Question 285 (*****)

By using a suitable trigonometric substitution, or otherwise, find

$$\int \frac{\left(3x^2+5x\right)\sqrt{x}}{(x+1)^2}\,dx .$$

$\boxed{}$, $\boxed{\dfrac{2x^{\frac{5}{2}}}{x+1}+C}$

START BY THE SUBSTITUTION $\sqrt{x}=\tan\theta$

$\sqrt{x}=\tan\theta$

$x=\tan^2\theta$

$dx=2\tan\theta\sec^2\theta\,d\theta$

$$\int \frac{(3x^2+5x)\sqrt{x}}{(x+1)^2}\,dx$$

$$=\int \frac{(3\tan^4\theta+5\tan^2\theta)\tan\theta}{(\tan^2\theta+1)^2}\left(2\tan\theta\sec^2\theta\,d\theta\right)$$

$$=\int \frac{2\tan^2\theta\sec^2\theta\,(3\tan^4\theta+5\tan^2\theta)}{\sec^4\theta}\,d\theta$$

$$=\int \frac{6\tan^6\theta\sec^2\theta+10\tan^4\theta\sec^2\theta}{\sec^4\theta}\,d\theta$$

$$=\int \frac{6\tan^6\theta}{\sec^2\theta}+\frac{10\tan^4\theta}{\sec^2\theta}\,d\theta=\int 6\tan^6\theta\cos^2\theta+10\tan^4\theta\cos^2\theta\,d\theta$$

$$=\int \frac{6\sin^6\theta}{\cos^6\theta}\times\cos^2\theta+\frac{10\sin^4\theta}{\cos^4\theta}\times\cos^2\theta\,d\theta$$

$$=\int \frac{6\sin^6\theta}{\cos^4\theta}+\frac{10\sin^4\theta}{\cos^2\theta}\,d\theta$$

$$=\int \frac{6(1-\cos^2\theta)^3}{\cos^4\theta}+\frac{10(1-\cos^2\theta)^2}{\cos^2\theta}\,d\theta$$

$$=\int \frac{6(1-3\cos^2\theta+3\cos^4\theta-\cos^6\theta)}{\cos^4\theta}+\frac{10(1-2\cos^2\theta+\cos^4\theta)}{\cos^2\theta}\,d\theta$$

$$=\int \frac{6}{\cos^4\theta}-\frac{18}{\cos^2\theta}+18-6\cos^2\theta+\frac{10}{\cos^2\theta}-20+10\cos^2\theta\,d\theta$$

$$=\int 6\sec^4\theta-8\sec^2\theta+4\cos^2\theta-2\,d\theta$$

REWRITING THE INTEGRAND AS FOLLOWS

$$=\int 6\sec^2\theta(1+\tan^2\theta)-8\sec^2\theta+4\left(\tfrac{1}{2}+\tfrac{1}{2}\cos2\theta\right)-2\,d\theta$$

$$=\int 6\sec^2\theta+6\sec^2\theta\tan^2\theta-8\sec^2\theta+2+2\cos2\theta-2\,d\theta$$

$$=\int 6\sec^2\theta\tan^2\theta-2\sec^2\theta+2\cos2\theta\,d\theta$$

$$=2\tan^3\theta-2\tan\theta+\sin2\theta+C$$

$$=2\tan^3\theta-2\tan\theta+2\sin\theta\cos\theta+C$$

$$=2\tan^3\theta-2\tan\theta+\frac{2\sin\theta\cos^2\theta}{\cos\theta}+C$$

$$=2\tan^3\theta-2\tan\theta+2\tan\theta\cos^2\theta+C$$

$$=2\tan^3\theta-2\tan\theta+\frac{2\tan\theta}{\sec^2\theta}+C$$

$$=2\tan^3\theta-2\tan\theta+\frac{2\tan\theta}{1+\tan^2\theta}+C$$

$$=2x^{\frac{3}{2}}-2x^{\frac{1}{2}}+\frac{2x^{\frac{1}{2}}}{1+x}+C$$

$$=\frac{2x^{\frac{3}{2}}+2x^{\frac{5}{2}}-2x^{\frac{1}{2}}-2x^{\frac{3}{2}}+2x^{\frac{1}{2}}}{1+x}+C$$

$$=\frac{2x^{\frac{5}{2}}}{1+x}+C$$

Question 286 (*****)

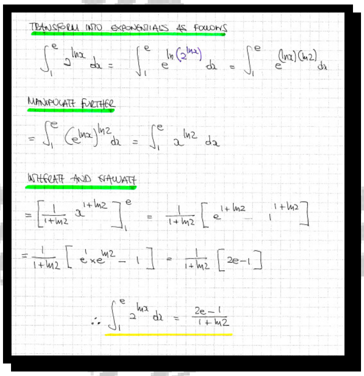
The function f is defined as

$$f(x) \equiv 2^{\ln x}, \quad x \in [1, \infty).$$

Show, with details workings, that

$$\int_{1}^{e} f(x)\, dx = \frac{2e-1}{1+\ln 2}.$$

[ ], [proof]



Question 287 (*****)

A function F is defined by the integral

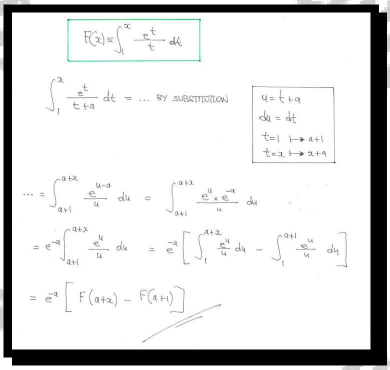
$$F(x) \equiv \int_1^x \frac{e^t}{t}\,dt\,,\ \ x \geq 1.$$

Find a simplified expression, in terms of F, for

$$\int_1^x \frac{e^t}{t+a}\,dt\,,$$

where a is a positive constant.

$$\boxed{\int_1^x \frac{e^t}{t+a}\,dt = e^{-a}\left[F(x+a) - F(a+1)\right]}$$



Question 288 (*****)

It is given that

$$u^2 = \frac{1-x^2}{(1-x)^2}, \quad x \neq \pm 1.$$

a) Show clearly that …

i. … $x = \dfrac{u^2-1}{u^2+1}$.

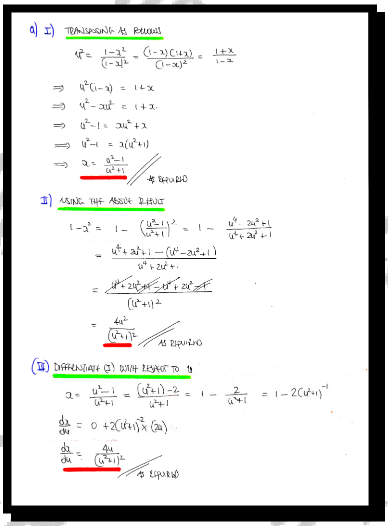
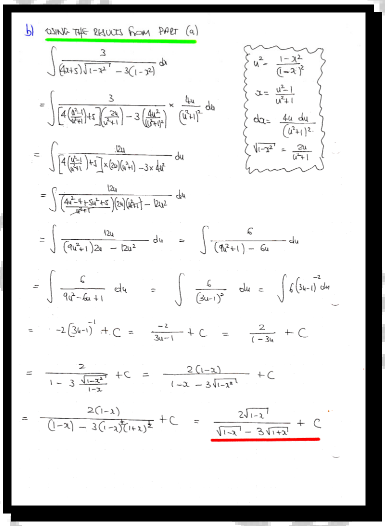
ii. … $1-x^2 = \dfrac{4u^2}{\left(u^2+1\right)^2}$

iii. … $\dfrac{dx}{du} = \dfrac{4u}{\left(u^2+1\right)^2}$.

b) Hence show further that

$$\int \frac{3}{(4x+5)\sqrt{1-x^2} - 3\left(1-x^2\right)}\, dx = \frac{2\sqrt{1-x}}{\sqrt{1-x} - 3\sqrt{1+x}} + \text{constant}.$$

[ ] , [proof]

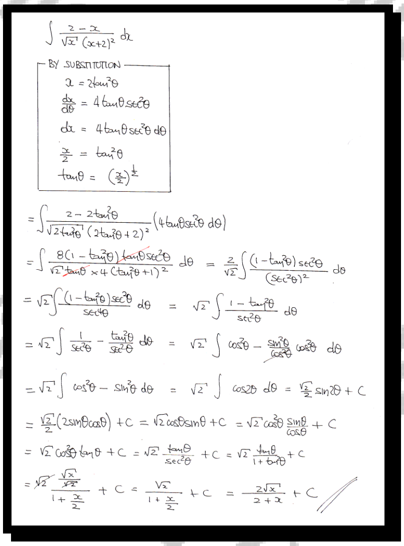



Question 289 (*****)

By using the substitution $x = 2\tan^2\theta$, or otherwise, find

$$\int \frac{2-x}{\sqrt{x}\,(x+2)^2}\,dx\,.$$

$\boxed{}$, $\boxed{\dfrac{2\sqrt{x}}{x+2} + C}$



$$\int \frac{2-x}{\sqrt{x}\,(x+2)^2}\,dx$$

BY SUBSTITUTION

$$x = 2\tan^2\theta$$
$$\frac{dx}{d\theta} = 4\tan\theta\sec^2\theta$$
$$dx = 4\tan\theta\sec^2\theta\,d\theta$$
$$\frac{x}{2} = \tan^2\theta$$
$$\tan\theta = \left(\frac{x}{2}\right)^{\frac{1}{2}}$$

$$= \int \frac{2-2\tan^2\theta}{\sqrt{2\tan^2\theta}\,(2\tan^2\theta+2)^2}\left(4\tan\theta\sec^2\theta\,d\theta\right)$$

$$= \int \frac{8(1-\tan^2\theta)\tan\theta\sec^2\theta}{\sqrt{2}\tan\theta \times 4(\tan^2\theta+1)^2}\,d\theta = \frac{2}{\sqrt{2}}\int \frac{(1-\tan^2\theta)\sec^2\theta}{(\sec^2\theta)^2}\,d\theta$$

$$= \sqrt{2}\int \frac{(1-\tan^2\theta)\sec^2\theta}{\sec^4\theta}\,d\theta = \sqrt{2}\int \frac{1-\tan^2\theta}{\sec^2\theta}\,d\theta$$

$$= \sqrt{2}\int \frac{1}{\sec^2\theta} - \frac{\tan^2\theta}{\sec^2\theta}\,d\theta = \sqrt{2}\int \cos^2\theta - \frac{\sin^2\theta}{\cos^2\theta}\cos^2\theta\,d\theta$$

$$= \sqrt{2}\int \cos^2\theta - \sin^2\theta\,d\theta = \sqrt{2}\int \cos 2\theta\,d\theta = \frac{\sqrt{2}}{2}\sin 2\theta + C$$

$$= \frac{\sqrt{2}}{2}(2\sin\theta\cos\theta) + C = \sqrt{2}\cos\theta\sin\theta + C = \sqrt{2}\cos^2\theta\,\frac{\sin\theta}{\cos\theta} + C$$

$$= \sqrt{2}\cos^2\theta\tan\theta + C = \sqrt{2}\,\frac{\tan\theta}{\sec^2\theta} + C = \sqrt{2}\,\frac{\tan\theta}{1+\tan^2\theta} + C$$

$$= \sqrt{2}\,\frac{\frac{\sqrt{x}}{\sqrt{2}}}{1+\frac{x}{2}} + C = \frac{\sqrt{x}}{1+\frac{x}{2}} + C = \frac{2\sqrt{x}}{2+x} + C$$

Question 290 (*****)

It is given that

$$\sqrt{5-4x-x^2} = (1-x)u \,,\ x \neq 1,\ x \neq -5 .$$

a) Show clearly that …

i. … $x = \dfrac{u^2-5}{u^2+1}$.

ii. … $dx = \dfrac{12u}{\left(u^2+1\right)^2}\,du$.

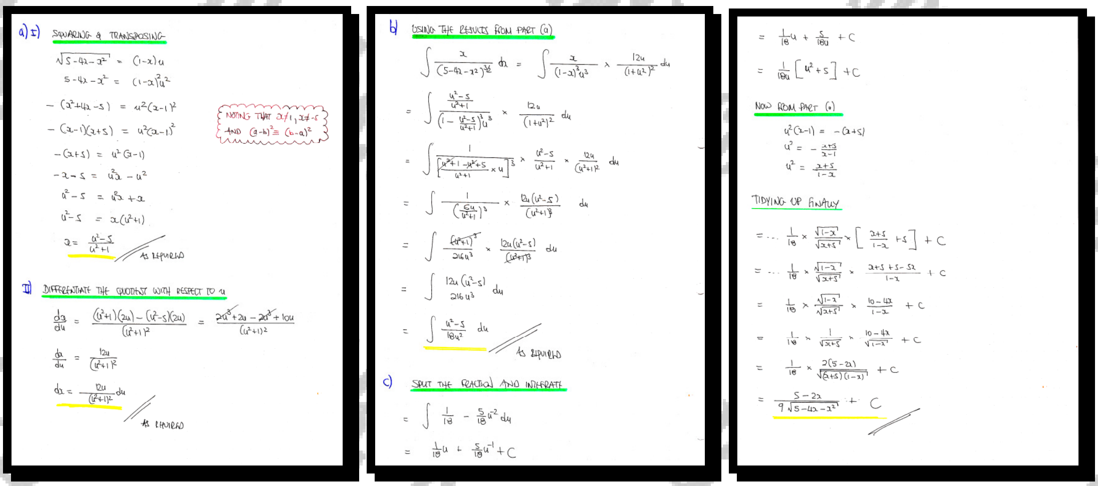
b) Hence show further that

$$\int \frac{x}{\left(5-4x-x^2\right)^{\frac{3}{2}}}\,dx = \int \frac{u^2-5}{18u^2}\,du .$$

c) Find a simplified expression for

$$\int \frac{x}{\left(5-4x-x^2\right)^{\frac{3}{2}}}\,dx .$$

$$\boxed{\frac{5-2x}{9\sqrt{5-4x-x^2}}+C}$$

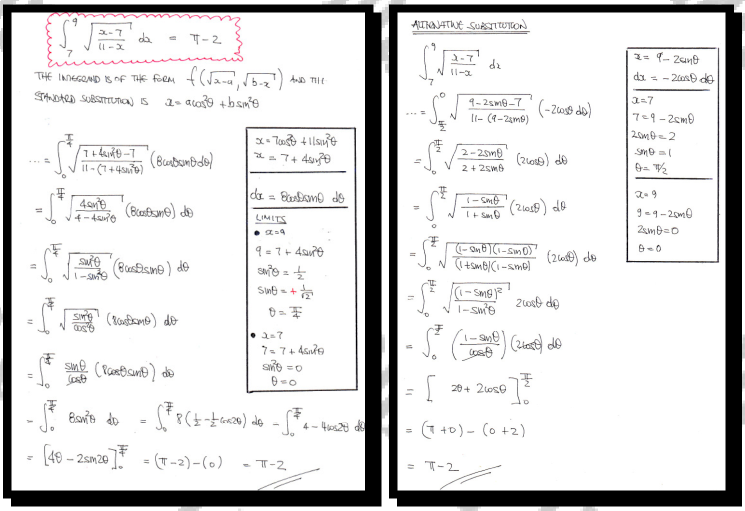


Question 291 (*****)

By using an appropriate trigonometric substitution, or otherwise, find an exact value for the following integral.

$$\int_{7}^{9} \sqrt{\frac{x-7}{11-x}} \, dx \, .$$

$\boxed{\quad}$, $\boxed{\pi - 2}$



$\int_7^9 \sqrt{\frac{x-7}{11-x}}\,dx = \pi - 2$

THE INTEGRAND IS OF THE FORM $f\left(\sqrt{x-a}, \sqrt{b-x}\right)$ AND THE STANDARD SUBSTITUTION IS $x = a\cos^2\theta + b\sin^2\theta$

$x = 7\cos^2\theta + 11\sin^2\theta$, $x = 7 + 4\sin^2\theta$, $dx = 8\cos\theta\sin\theta\,d\theta$

LIMITS: $x=9$: $9 = 7 + 4\sin^2\theta$, $\sin^2\theta = \frac{1}{2}$, $\sin\theta = +\frac{1}{\sqrt{2}}$, $\theta = \frac{\pi}{4}$; $x=7$: $7 = 7 + 4\sin^2\theta$, $\sin^2\theta = 0$, $\theta = 0$

$\dots = \int_0^{\frac{\pi}{4}} \sqrt{\frac{7+4\sin^2\theta - 7}{11-(7+4\sin^2\theta)}}\,(8\cos\theta\sin\theta\,d\theta)$

$= \int_0^{\frac{\pi}{4}} \sqrt{\frac{4\sin^2\theta}{4-4\sin^2\theta}}\,(8\cos\theta\sin\theta)\,d\theta$

$= \int_0^{\frac{\pi}{4}} \sqrt{\frac{\sin^2\theta}{1-\sin^2\theta}}\,(8\cos\theta\sin\theta)\,d\theta$

$= \int_0^{\frac{\pi}{4}} \sqrt{\frac{\sin^2\theta}{\cos^2\theta}}\,(8\cos\theta\sin\theta)\,d\theta$

$= \int_0^{\frac{\pi}{4}} \frac{\sin\theta}{\cos\theta}\,(8\cos\theta\sin\theta)\,d\theta$

$= \int_0^{\frac{\pi}{4}} 8\sin^2\theta\,d\theta = \int_0^{\frac{\pi}{4}} 8\left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right)d\theta = \int_0^{\frac{\pi}{4}} 4 - 4\cos 2\theta\,d\theta$

$= \left[4\theta - 2\sin 2\theta\right]_0^{\frac{\pi}{4}} = (\pi - 2) - (0) = \pi - 2$

ALTERNATIVE SUBSTITUTION

$x = 9 - 2\sin\theta$, $dx = -2\cos\theta\,d\theta$; $x = 7$: $7 = 9 - 2\sin\theta$, $2\sin\theta = 2$, $\sin\theta = 1$, $\theta = \frac{\pi}{2}$; $x = 9$: $9 = 9 - 2\sin\theta$, $2\sin\theta = 0$, $\theta = 0$

$\int_7^9 \sqrt{\frac{x-7}{11-x}}\,dx$

$\dots = \int_{\frac{\pi}{2}}^{0} \sqrt{\frac{9-2\sin\theta-7}{11-(9-2\sin\theta)}}\,(-2\cos\theta\,d\theta)$

$= \int_0^{\frac{\pi}{2}} \sqrt{\frac{2-2\sin\theta}{2+2\sin\theta}}\,(2\cos\theta)\,d\theta$

$= \int_0^{\frac{\pi}{2}} \sqrt{\frac{1-\sin\theta}{1+\sin\theta}}\,(2\cos\theta)\,d\theta$

$= \int_0^{\frac{\pi}{2}} \sqrt{\frac{(1-\sin\theta)(1-\sin\theta)}{(1+\sin\theta)(1-\sin\theta)}}\,(2\cos\theta)\,d\theta$

$= \int_0^{\frac{\pi}{2}} \sqrt{\frac{(1-\sin\theta)^2}{1-\sin^2\theta}}\,2\cos\theta\,d\theta$

$= \int_0^{\frac{\pi}{2}} \left(\frac{1-\sin\theta}{\cos\theta}\right)(2\cos\theta)\,d\theta$

$= \left[2\theta + 2\cos\theta\right]_0^{\frac{\pi}{2}}$

$= (\pi + 0) - (0 + 2)$

$= \pi - 2$

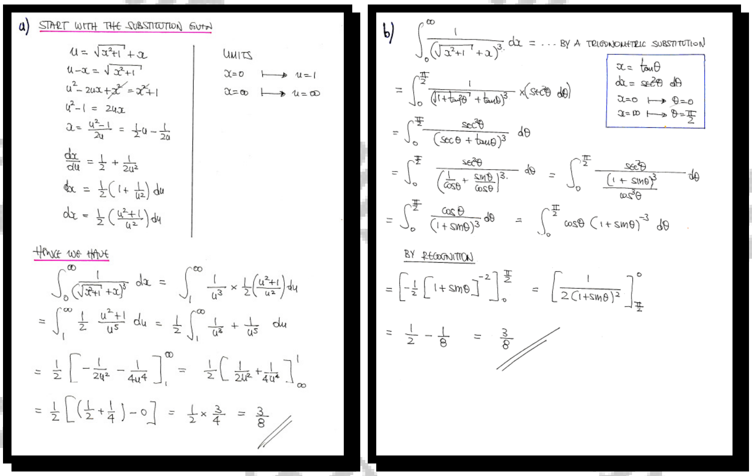
Question 292 (*****)

$$I = \int_0^{\infty} \frac{1}{\left(x+\sqrt{x^2+1}\right)^2}\, dx\,.$$

a) Use the substitution $u = x+\sqrt{x^2+1}$ to find the value of I.

b) Verify the answer to part **(a)** by a trigonometric substitution.

$\boxed{}$, $\boxed{\frac{3}{8}}$



a) START WITH THE SUBSTITUTION GIVEN

$u = \sqrt{x^2+1} + x$

$u - x = \sqrt{x^2+1}$

$u^2 - 2ux + x^2 = x^2 + 1$

$u^2 - 1 = 2ux$

$x = \frac{u^2-1}{2u} = \frac{1}{2}u - \frac{1}{2u}$

$\frac{dx}{du} = \frac{1}{2} + \frac{1}{2u^2}$

$dx = \frac{1}{2}\left(1+\frac{1}{u^2}\right) du$

$dx = \frac{1}{2}\left(\frac{u^2+1}{u^2}\right) du$

LIMITS

$x=0 \mapsto u=1$

$x=\infty \mapsto u=\infty$

HENCE WE HAVE

$\int_0^{\infty} \frac{1}{(\sqrt{x^2+1}+x)^3}\, dx = \int_1^{\infty} \frac{1}{u^3} \times \frac{1}{2}\left(\frac{u^2+1}{u^2}\right) du$

$= \int_1^{\infty} \frac{1}{2}\,\frac{u^2+1}{u^5}\, du = \frac{1}{2}\int_1^{\infty} \frac{1}{u^3} + \frac{1}{u^5}\, du$

$= \frac{1}{2}\left[-\frac{1}{2u^2} - \frac{1}{4u^4}\right]_1^{\infty} = \frac{1}{2}\left[\frac{1}{2u^2} + \frac{1}{4u^4}\right]_{\infty}^{1}$

$= \frac{1}{2}\left[\left(\frac{1}{2}+\frac{1}{4}\right) - 0\right] = \frac{1}{2} \times \frac{3}{4} = \frac{3}{8}$

b) $\int_0^{\infty} \frac{1}{(\sqrt{x^2+1}+x)^3}\, dx = \ldots$ BY A TRIGONOMETRIC SUBSTITUTION

$x = \tan\theta$

$dx = \sec^2\theta\, d\theta$

$x=0 \mapsto \theta=0$

$x=\infty \mapsto \theta=\frac{\pi}{2}$

$= \int_0^{\frac{\pi}{2}} \frac{1}{(\sqrt{1+\tan^2\theta}+\tan\theta)^3} \times (\sec^2\theta\, d\theta)$

$= \int_0^{\frac{\pi}{2}} \frac{\sec^2\theta}{(\sec\theta+\tan\theta)^3}\, d\theta$

$= \int_0^{\frac{\pi}{2}} \frac{\sec^2\theta}{\left(\frac{1}{\cos\theta}+\frac{\sin\theta}{\cos\theta}\right)^3}\, d\theta = \int_0^{\frac{\pi}{2}} \frac{\sec^2\theta}{\frac{(1+\sin\theta)^3}{\cos^3\theta}}\, d\theta$

$= \int_0^{\frac{\pi}{2}} \frac{\cos\theta}{(1+\sin\theta)^3}\, d\theta = \int_0^{\frac{\pi}{2}} \cos\theta\,(1+\sin\theta)^{-3}\, d\theta$

BY RECOGNITION

$= \left[-\frac{1}{2}\left[1+\sin\theta\right]^{-2}\right]_0^{\frac{\pi}{2}} = \left[\frac{1}{2(1+\sin\theta)^2}\right]_{\frac{\pi}{2}}^{0}$

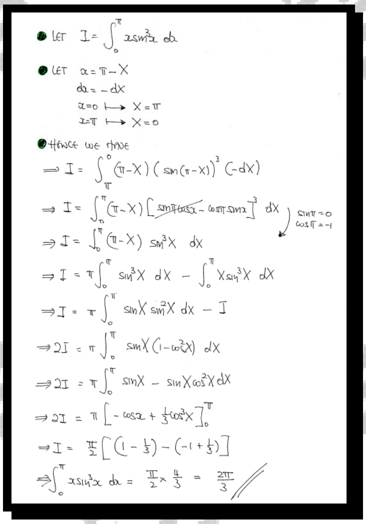
$= \frac{1}{2} - \frac{1}{8} = \frac{3}{8}$

Question 293 (*****)

Find an exact value for the following integral.

$$\int_0^{\pi} x\sin^3 x \; dx .$$

SP , $\boxed{\dfrac{2\pi}{3}}$



- LET $I = \int_0^{\pi} x\sin^3 x \, dx$
- LET $x = \pi - X$
  $dx = -dX$
  $x = 0 \mapsto X = \pi$
  $x = \pi \mapsto X = 0$
- HENCE WE HAVE

$\Rightarrow I = \int_{\pi}^{0} (\pi - X)(\sin(\pi - X))^3 (-dX)$

$\Rightarrow I = \int_0^{\pi} (\pi - X)\left[\sin\pi\cos x - \cos\pi\sin x\right]^3 dX$ ($\sin\pi = 0$, $\cos\pi = -1$)

$\Rightarrow I = \int_0^{\pi} (\pi - X)\sin^3 X \, dX$

$\Rightarrow I = \pi\int_0^{\pi} \sin^3 X \, dX - \int_0^{\pi} X\sin^3 X \, dX$

$\Rightarrow I = \pi\int_0^{\pi} \sin X \sin^2 X \, dX - I$

$\Rightarrow 2I = \pi\int_0^{\pi} \sin X(1 - \cos^2 X) \, dX$

$\Rightarrow 2I = \pi\int_0^{\pi} \sin X - \sin X\cos^2 X \, dX$

$\Rightarrow 2I = \pi\left[-\cos x + \frac{1}{3}\cos^3 X\right]_0^{\pi}$

$\Rightarrow I = \frac{\pi}{2}\left[\left(1 - \frac{1}{3}\right) - \left(-1 + \frac{1}{3}\right)\right]$

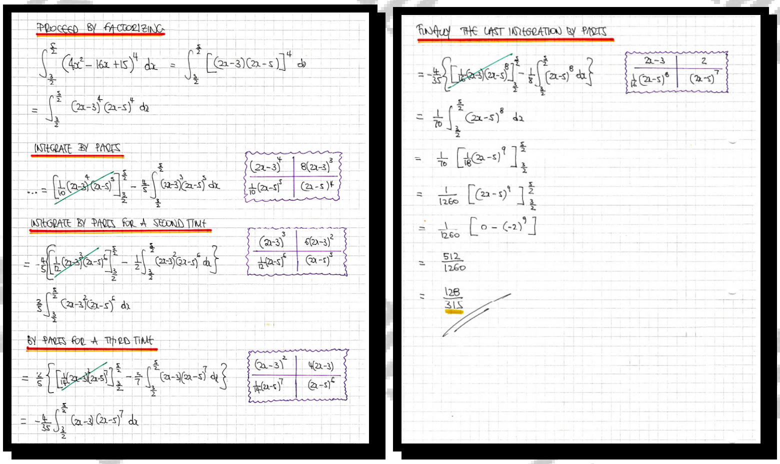
$\Rightarrow \int_0^{\pi} x\sin^3 x \, dx = \frac{\pi}{2} \times \frac{4}{3} = \frac{2\pi}{3}$

Question 294 (*****)

Determine, as an exact simplified fraction, the value of the following integral.

$$\int_{\frac{3}{2}}^{\frac{5}{2}} \left(4x^2 - 16x + 15\right)^4 \, dx .$$

$\boxed{}$, $\boxed{\frac{128}{315}}$



Question 295 (*****)

Find an exact value for

$$\int_0^{\pi} \frac{x \sin x}{\sqrt{4-\cos^2 x}} \, dx .$$

You my assume without proof that

$$\int \frac{1}{\sqrt{a^2-x^2}} \, dx = \arcsin\left(\frac{x}{a}\right) + \text{constant} .$$

$\boxed{}$, $\boxed{\dfrac{\pi^2}{6}}$

- LET $I = \int_0^{\pi} \frac{x \sin x}{\sqrt{4-\cos^2 x}} \, dx$
- USE THE SUBSTITUTION

$x = \pi - y \iff y = \pi - x$

$dx = -dy$

LIMITS $x = 0 \mapsto y = \pi$, $x = \pi \mapsto y = 0$

$\sin x = \sin(\pi - y) = \sin\pi\cos y - \cos\pi\sin y = \sin y$

$\cos^2 x = [\cos(\pi - y)]^2 = [\cos\pi\cos y - \sin\pi\sin y]^2 = \cos^2 y$

- HENCE WE NOW OBTAIN

$$I = \int_{\pi}^{0} \frac{(\pi - y)\sin y}{\sqrt{4-\cos^2 y}}(-dy) = \int_0^{\pi} \frac{(\pi - y)\sin y}{\sqrt{4-\cos^2 y}} \, dy$$

$$I = \int_0^{\pi} \frac{\pi \sin y}{\sqrt{4-\cos^2 y}} \, dy - \int_0^{\pi} \frac{y \sin y}{\sqrt{4-\cos^2 y}} \, dy$$

$$I = \pi \int_0^{\pi} \sin y \left(4 - \cos^2 y\right)^{-\frac{1}{2}} dy - I$$

$$2I = \pi \int_0^{\pi} \sin y \left(4 - \cos^2 y\right)^{-\frac{1}{2}} dy$$

- SUBSTITUTION AGAIN

$v = \cos y$

$\frac{dv}{dy} = -\sin y$

$dy = -\frac{dv}{\sin y}$

WITH LIMITS $y = 0 \mapsto v = 1$, $y = \pi \mapsto v = -1$

$$\Rightarrow 2I = \pi \int_1^{-1} \frac{\sin y}{\sqrt{4 - v^2}} \left(-\frac{dv}{\sin y}\right)$$

$$\Rightarrow I = \frac{\pi}{2} \int_{-1}^{1} \frac{1}{\sqrt{4 - v^2}} \, dv$$

EVEN INTEGRAND IN A SYMMETRICAL DOMAIN

$$\Rightarrow I = \frac{\pi}{2} \times 2 \int_0^{1} \frac{1}{\sqrt{4 - v^2}} \, dv$$

$$\Rightarrow I = \pi \int_0^{1} \frac{1}{\sqrt{2^2 - v^2}} \, dv$$

$$\Rightarrow I = \pi \left[\arcsin \frac{v}{2}\right]_0^1$$

$$\Rightarrow I = \pi \left[\arcsin\frac{1}{2} - \arcsin 0\right]$$

$$\Rightarrow I = \pi \times \frac{\pi}{6}$$

$$\Rightarrow \int_0^{\pi} \frac{x \sin x}{\sqrt{4-\cos^2 x}} \, dx = \frac{\pi^2}{6}$$

Question 296 (*****)

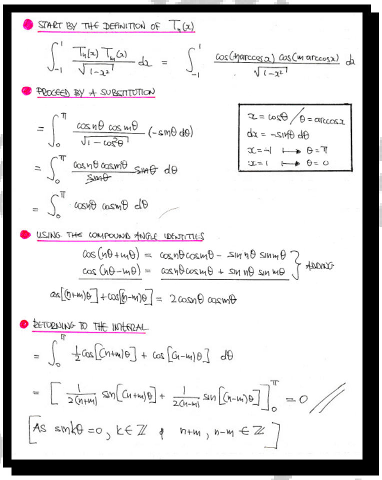
A family of functions, known as the Chebyshev polynomials of the first kind $T_n(x)$, is defined as

$$T_n(x) = \cos(n \arccos x), \quad -1 \le x \le 1, \quad n \in \mathbb{N}.$$

Evaluate the following integral

$$\int_{-1}^{1} \frac{T_n(x)\,T_m(x)}{\sqrt{1-x^2}}\,dx.$$

$\boxed{}$, $\boxed{0}$



START BY THE DEFINITION OF $T_n(x)$

$$\int_{-1}^{1} \frac{T_n(x)\,T_m(x)}{\sqrt{1-x^2}}\,dx = \int_{-1}^{1} \frac{\cos(n\arccos x)\cos(m\arccos x)}{\sqrt{1-x^2}}\,dx$$

PROCEED BY A SUBSTITUTION

$x = \cos\theta$ / $\theta = \arccos x$

$dx = -\sin\theta\,d\theta$

$x = -1 \mapsto \theta = \pi$

$x = 1 \mapsto \theta = 0$

$$= \int_0^{\pi} \frac{\cos n\theta \cos m\theta}{\sqrt{1-\cos^2\theta}}(-\sin\theta\,d\theta)$$

$$= \int_0^{\pi} \frac{\cos n\theta \cos m\theta}{\sin\theta}\sin\theta\,d\theta$$

$$= \int_0^{\pi} \cos n\theta \cos m\theta\,d\theta$$

USING THE COMPOUND ANGLE IDENTITIES

$$\cos(n\theta + m\theta) = \cos n\theta\cos m\theta - \sin n\theta \sin m\theta$$

$$\cos(n\theta - m\theta) = \cos n\theta\cos m\theta + \sin n\theta \sin m\theta$$

ADDING

$$\cos[(n+m)\theta] + \cos[(n-m)\theta] = 2\cos n\theta \cos m\theta$$

RETURNING TO THE INTEGRAL

$$= \int_0^{\pi} \tfrac{1}{2}\cos[(n+m)\theta] + \cos[(n-m)\theta]\,d\theta$$

$$= \left[\frac{1}{2(n+m)}\sin[(n+m)\theta] + \frac{1}{2(n-m)}\sin[(n-m)\theta]\right]_0^{\pi} = 0$$

[As $\sin k\theta = 0$, $k \in \mathbb{Z}$ & $n+m$, $n-m \in \mathbb{Z}$]

Question 297 (*****)

The function $y = f(x)$ is defined in the largest possible real domain by

$$f(x) \equiv \ln\left[x^2 - 2x + 2\right].$$

Sketch the graph of $f(x)$ and determine an exact simplified value for the area of the finite region bounded by the graph of $f(x)$ and the coordinate axes.

$\boxed{}$, $\boxed{\frac{1}{2}\pi - 2 + \ln 2}$

$y = \ln(x^2 - 2x + 2)$

- START BY COMPLETING THE SQUARE INSIDE THE ARGUMENT OF THE LOG

$y = \ln((x-1)^2 + 1)$

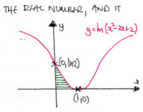
- HENCE THE DOMAIN OF THE CURVE IS ALL THE REAL NUMBER, AND IT IS EVEN ABOUT $x=1$
- WHEN $x=0$ $y=\ln 2$ I.E $(0,\ln 2)$

$y=0$ $\ln((x-1)^2+1)=0$

$(x-1)^2+1=1$

$(x-1)^2=0$

$x=1$ IE $(1,0)$



- HENCE WE PRODUCE THE GRAPH OPPOSITE.
- THUS THE REQUIRED AREA IS GIVEN BY

$\int_0^1 \ln[(x-1)^2+1]\,dx$ .... BY SUBSTITUTION .....

$u=(x-1)^2+1$
$u-1=(x-1)^2$
$x-1=-\sqrt{u-1}$ ↑ AS $0 \le x \le 1$
$x = 1-(u-1)^{\frac{1}{2}}$
$dx = -\frac{1}{2}(u-1)^{-\frac{1}{2}}\,du$

x	0	1
u	2	1

$= \int_2^1 (\ln u)\left(-\frac{1}{2}(u-1)^{-\frac{1}{2}}\,du\right) = \frac{1}{2}\int_1^2 \frac{\ln u}{(u-1)^{\frac{1}{2}}}\,du$

$= \frac{1}{2}\int_1^2 (u-1)^{-\frac{1}{2}}\ln u\,du$

BY PARTS ....

$\ln u$	$\frac{1}{u}$
$2(u-1)^{\frac{1}{2}}$	$(u-1)^{-\frac{1}{2}}$

$= \frac{1}{2}\left\{\left[2(u-1)^{\frac{1}{2}}\ln u\right]_1^2 - 2\int_1^2 \frac{(u-1)^{\frac{1}{2}}}{u}\,du\right\}$

$= \left[(u-1)^{\frac{1}{2}}\ln u\right]_1^2 - \int_1^2 \frac{(u-1)^{\frac{1}{2}}}{u}\,du$

$= \ln 2 - \int_1^2 \frac{(u-1)^{\frac{1}{2}}}{u}\,du$ ...... BY SUBSTITUTION ......

$\sqrt{u} = \sec\theta$
$u = \sec^2\theta$
$du = 2\sec^2\theta\tan\theta\,d\theta$
$u=1$ $\theta=0$
$u=2$ $\theta=\frac{\pi}{4}$

$= \ln 2 - \int_0^{\frac{\pi}{4}} \frac{(\sec^2\theta-1)^{\frac{1}{2}}}{\sec^2\theta}(2\sec^2\theta\tan\theta\,d\theta)$

$= \ln 2 - \int_0^{\frac{\pi}{4}} 2\tan^2\theta\,d\theta$

$= \ln 2 - \int_0^{\frac{\pi}{4}} 2(\sec^2\theta - 1)\,d\theta$

$= \ln 2 - \left[2(\tan\theta - \theta)\right]_0^{\frac{\pi}{4}}$

$= \ln 2 - \left[2\left(1-\frac{\pi}{4}\right) - 0\right]$

$= \ln 2 - \left[2 - \frac{\pi}{2}\right]$

$= \frac{\pi}{2} + \ln 2 - 2$

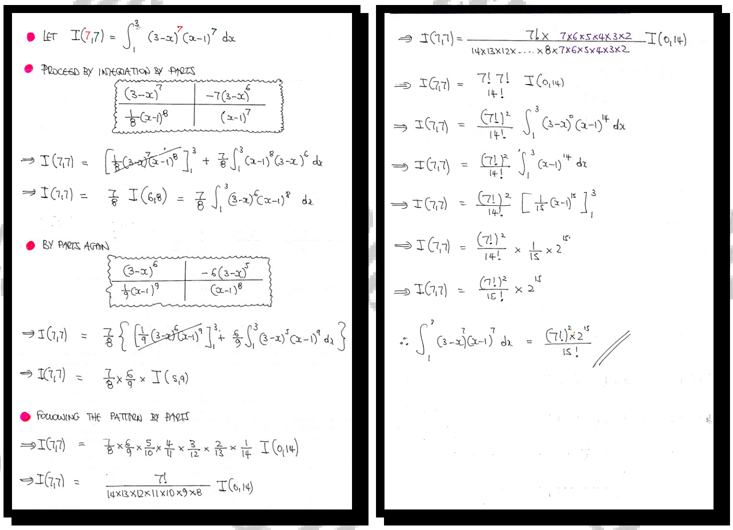
Question 298 (*****)

$$I = \int_{1}^{3} (3-x)^{7}(x-1)^{7}\ dx .$$

Show that

$$I = \frac{(7!)^{2} \times 2^{15}}{15!} .$$

[ ] , [proof]

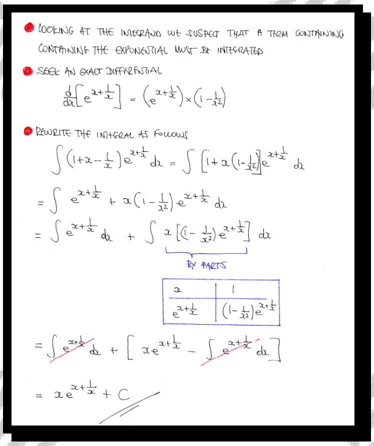


Question 299 (*****)

Use integration by parts to find a simplified expression for

$$\int \left(1+x-\frac{1}{x}\right)e^{x+\frac{1}{x}}\,dx .$$

$\boxed{}$, $\boxed{xe^{x+\frac{1}{x}}+C}$



LOOKING AT THE INTEGRAND WE SUSPECT THAT A TERM CONTAINING CONTAINING THE EXPONENTIAL MUST BE INTEGRATED

SEEK AN EXACT DIFFERENTIAL

$$\frac{d}{dx}\left[e^{x+\frac{1}{x}}\right] = \left(e^{x+\frac{1}{x}}\right)\times\left(1-\frac{1}{x^2}\right)$$

REWRITE THE INTEGRAL AS FOLLOWS

$$\int\left(1+x-\frac{1}{x}\right)e^{x+\frac{1}{x}}\,dx = \int\left[1+x\left(1-\frac{1}{x^2}\right)\right]e^{x+\frac{1}{x}}\,dx$$

$$= \int e^{x+\frac{1}{x}} + x\left(1-\frac{1}{x^2}\right)e^{x+\frac{1}{x}}\,dx$$

$$= \int e^{x+\frac{1}{x}}\,dx + \int x\left[\left(1-\frac{1}{x^2}\right)e^{x+\frac{1}{x}}\right]dx$$

BY PARTS

x	1
$e^{x+\frac{1}{x}}$	$\left(1-\frac{1}{x^2}\right)e^{x+\frac{1}{x}}$

$$= \int e^{x+\frac{1}{x}}\,dx + \left[xe^{x+\frac{1}{x}} - \int e^{x+\frac{1}{x}}\,dx\right]$$

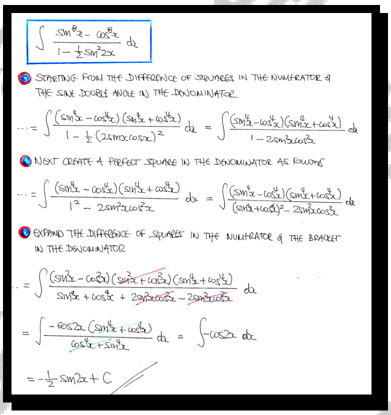
$$= xe^{x+\frac{1}{x}} + C$$

Question 300 (*****)

Use trigonometric identities to find a simplified expression for

$$\int \frac{\sin^8 x-\cos^8 x}{1-\frac{1}{2}\sin^2 2x}\,dx .$$

$\boxed{}$, $\boxed{-\frac{1}{2}\sin 2x+C}$



$$\int \frac{\sin^8 x - \cos^8 x}{1-\frac{1}{2}\sin^2 2x}\,dx$$

STARTING FROM THE DIFFERENCE OF SQUARES IN THE NUMERATOR & THE SINE DOUBLE ANGLE IN THE DENOMINATOR

$$\dots = \int \frac{(\sin^4 x-\cos^4 x)(\sin^4 x+\cos^4 x)}{1-\frac{1}{2}(2\sin x\cos x)^2}\,dx = \int \frac{(\sin^4 x-\cos^4 x)(\sin^4 x+\cos^4 x)}{1-2\sin^2 x\cos^2 x}\,dx$$

NEXT CREATE A PERFECT SQUARE IN THE DENOMINATOR AS FOLLOWS

$$\dots = \int \frac{(\sin^4 x-\cos^4 x)(\sin^4 x+\cos^4 x)}{1^2-2\sin^2 x\cos^2 x}\,dx = \int \frac{(\sin^4 x-\cos^4 x)(\sin^4 x+\cos^4 x)}{(\sin^2 x+\cos^2 x)^2-2\sin^2 x\cos^2 x}\,dx$$

EXPAND THE DIFFERENCE OF SQUARES IN THE NUMERATOR & THE BRACKET IN THE DENOMINATOR

$$\dots = \int \frac{(\sin^2 x-\cos^2 x)(\sin^2 x+\cos^2 x)(\sin^4 x+\cos^4 x)}{\sin^4 x+\cos^4 x+2\sin^2 x\cos^2 x-2\sin^2 x\cos^2 x}\,dx$$

$$= \int \frac{-\cos 2x(\sin^4 x+\cos^4 x)}{\cos^4 x+\sin^4 x}\,dx = \int -\cos 2x\,dx$$

$$= -\frac{1}{2}\sin 2x + C$$

Question 301 (*****)

By using an appropriate substitution followed by trigonometric identities, show that

$$\int_0^{\pi} \frac{x\tan x}{\tan x+\sec x}\,dx = \frac{1}{2}\pi(\pi-2).$$

S -B , proof

- START BY A SUBSTITUTION

$x = \pi - \theta$, $dx = -d\theta$

$x = \pi \mapsto \theta = 0$, $x = 0 \mapsto \theta = \pi$

$\tan(\pi-\theta) = \dfrac{\tan\pi - \tan\theta}{1+\tan\pi\tan\theta} = -\tan\theta$

$\sec(\pi-\theta) = \dfrac{1}{\cos\pi\cos\theta + \sin\pi\sin\theta} = -\dfrac{1}{\cos\theta} = -\sec\theta$

- THUS THE INTEGRAL CAN BE TRANSFORMED TO

$$\int_0^{\pi}\frac{x\tan x}{\sec x+\tan x}\,dx = \int_{\pi}^{0}\frac{(\pi-\theta)(-\tan\theta)}{-\sec\theta-\tan\theta}\,(-d\theta)$$

$$= \int_0^{\pi}\frac{\theta\tan\theta-\pi\tan\theta}{-\sec\theta-\tan\theta}\,d\theta$$

$$= \int_0^{\pi}\frac{\pi\tan\theta-\theta\tan\theta}{\sec\theta+\tan\theta}\,d\theta$$

$$= \int_0^{\pi}\frac{\pi\tan\theta}{\sec\theta+\tan\theta}\,d\theta - \int_0^{\pi}\frac{\theta\tan\theta}{\sec\theta+\tan\theta}\,d\theta$$

- COLLECTING THE RESULTS SO FAR WE HAVE

$$\Rightarrow I = \pi\int_0^{\pi}\frac{\tan\theta}{\sec\theta+\tan\theta}\,d\theta - I \qquad \text{WITH } I = \int_0^{\pi}\frac{x\tan x}{\sec x+\tan x}\,dx$$

$$\Rightarrow 2I = \pi\int_0^{\pi}\frac{\tan\theta}{\sec\theta+\tan\theta}\,d\theta$$

$$\Rightarrow 2I = \pi\int_0^{\pi}\frac{\tan\theta(\sec\theta-\tan\theta)}{(\sec\theta+\tan\theta)(\sec\theta-\tan\theta)}\,d\theta$$

$$\Rightarrow 2I = \pi\int_0^{\pi}\frac{\sec\theta\tan\theta-\tan^2\theta}{\sec^2\theta-\tan^2\theta}\,d\theta$$

$1+\tan^2\theta \equiv \sec^2\theta$

$\sec^2\theta-\tan^2\theta \equiv 1$

$$\Rightarrow 2I = \pi\int_0^{\pi}\sec\theta\tan\theta-\tan^2\theta\,d\theta$$

$$\Rightarrow 2I = \pi\int_0^{\pi}\sec\theta\tan\theta-(\sec^2\theta-1)\,d\theta$$

$$\Rightarrow 2I = \pi\int_0^{\pi}\sec\theta\tan\theta-\sec^2\theta+1\,d\theta$$

$$\Rightarrow 2I = \pi\Big[\sec\theta-\tan\theta+\theta\Big]_0^{\pi}$$

$$\Rightarrow 2I = \pi\Big[(-1-0+\pi)-(1-0+0)\Big]$$

$$\Rightarrow 2I = \pi[\pi-2]$$

$$\Rightarrow I = \tfrac{1}{2}\pi(\pi-2)$$

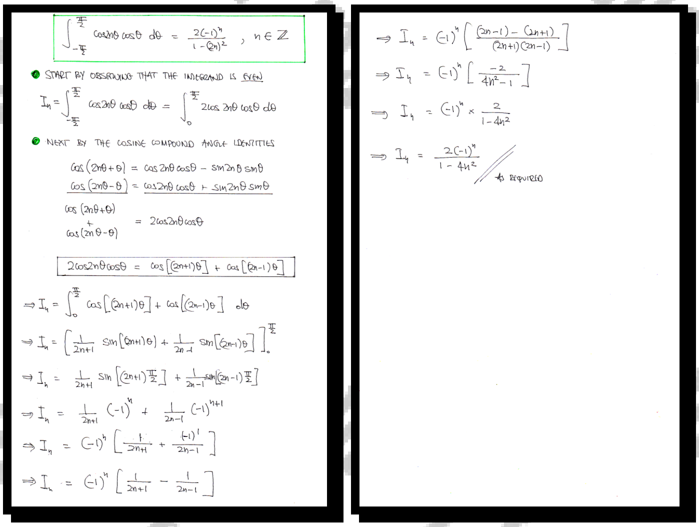
$$\therefore \int_0^{\pi}\frac{x\tan x}{\tan x+\sec x}\,dx = \tfrac{1}{2}\pi(\pi-2)$$

Question 302 (*****)

Show that if n is an integer, then

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \cos(2n\theta)\cos\theta \, d\theta = \frac{2(-1)^n}{1-(2n)^2}.$$

SP N , proof



$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos 2n\theta \cos\theta \, d\theta = \frac{2(-1)^n}{1-(2n)^2}, \quad n \in \mathbb{Z}$$

- START BY OBSERVING THAT THE INTEGRAND IS EVEN

$$I_n = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos 2n\theta \cos\theta \, d\theta = \int_0^{\frac{\pi}{2}} 2\cos 2n\theta \cos\theta \, d\theta$$

- NEXT BY THE COSINE COMPOUND ANGLE IDENTITIES

$$\cos(2n\theta+\theta) = \cos 2n\theta\cos\theta - \sin 2n\theta \sin\theta$$
$$\cos(2n\theta-\theta) = \cos 2n\theta\cos\theta + \sin 2n\theta \sin\theta$$

$$\cos(2n\theta+\theta) + \cos(2n\theta-\theta) = 2\cos 2n\theta \cos\theta$$

$$2\cos 2n\theta\cos\theta = \cos[(2n+1)\theta] + \cos[(2n-1)\theta]$$

$$\Rightarrow I_n = \int_0^{\frac{\pi}{2}} \cos[(2n+1)\theta] + \cos[(2n-1)\theta] \, d\theta$$

$$\Rightarrow I_n = \left[\frac{1}{2n+1}\sin[(2n+1)\theta] + \frac{1}{2n-1}\sin[(2n-1)\theta]\right]_0^{\frac{\pi}{2}}$$

$$\Rightarrow I_n = \frac{1}{2n+1}\sin\left[(2n+1)\frac{\pi}{2}\right] + \frac{1}{2n-1}\sin\left[(2n-1)\frac{\pi}{2}\right]$$

$$\Rightarrow I_n = \frac{1}{2n+1}(-1)^n + \frac{1}{2n-1}(-1)^{n+1}$$

$$\Rightarrow I_n = (-1)^n\left[\frac{1}{2n+1} + \frac{(-1)^1}{2n-1}\right]$$

$$\Rightarrow I_n = (-1)^n\left[\frac{1}{2n+1} - \frac{1}{2n-1}\right]$$

$$\Rightarrow I_n = (-1)^n\left[\frac{(2n-1)-(2n+1)}{(2n+1)(2n-1)}\right]$$

$$\Rightarrow I_n = (-1)^n\left[\frac{-2}{4n^2-1}\right]$$

$$\Rightarrow I_n = (-1)^n \times \frac{2}{1-4n^2}$$

$$\Rightarrow I_n = \frac{2(-1)^n}{1-4n^2}$$

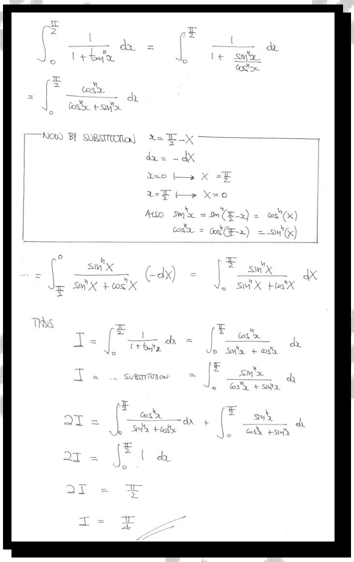
AS REQUIRED

Question 303 (*****)

$$\int_0^{\frac{1}{2}\pi} \frac{1}{1+\tan^n x}\ dx \quad , \ n \in \mathbb{Q} .$$

Find the value of the above integral, for all values of n

$$\boxed{\frac{\pi}{4}}$$



$$\int_0^{\frac{\pi}{2}} \frac{1}{1+\tan^n x}\,dx = \int_0^{\frac{\pi}{2}} \frac{1}{1+\frac{\sin^n x}{\cos^n x}}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos^n x}{\cos^n x+\sin^n x}\,dx$$

NOW BY SUBSTITUTION $x = \frac{\pi}{2} - X$

$dx = -dX$

$x = 0 \longmapsto X = \frac{\pi}{2}$

$x = \frac{\pi}{2} \longmapsto X = 0$

ALSO $\sin^n x = \sin^n\left(\frac{\pi}{2} - X\right) = \cos^n(X)$

$\cos^n x = \cos^n\left(\frac{\pi}{2} - X\right) = \sin^n(X)$

$$\dots = \int_{\frac{\pi}{2}}^{0} \frac{\sin^n X}{\sin^n X + \cos^n X}(-dX) = \int_0^{\frac{\pi}{2}} \frac{\sin^n X}{\sin^n X + \cos^n X}\,dX$$

THUS

$$I = \int_0^{\frac{\pi}{2}} \frac{1}{1+\tan^n x}\,dx = \int_0^{\frac{\pi}{2}} \frac{\cos^n x}{\sin^n x + \cos^n x}\,dx$$

$$I = \dots \text{SUBSTITUTION} = \int_0^{\frac{\pi}{2}} \frac{\sin^n x}{\cos^n x + \sin^n x}\,dx$$

$$2I = \int_0^{\frac{\pi}{2}} \frac{\cos^n x}{\sin^n x + \cos^n x}\,dx + \int_0^{\frac{\pi}{2}} \frac{\sin^n x}{\cos^n x + \sin^n x}\,dx$$

$$2I = \int_0^{\frac{\pi}{2}} 1\,dx$$

$$2I = \frac{\pi}{2}$$

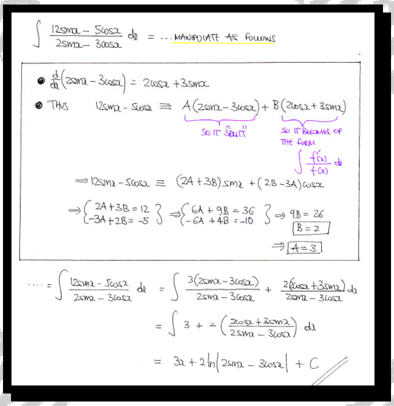
$$I = \frac{\pi}{4}$$

Question 304 (*****)

By suitably rewriting the numerator of the integrand, find a simplified expression for the following integral.

$$\int \frac{12\sin x - 5\cos x}{2\sin x - 3\cos x}\,dx\,.$$

SP-E, $\boxed{3x + 2\ln\left|2\sin x - 3\cos x\right| + C}$

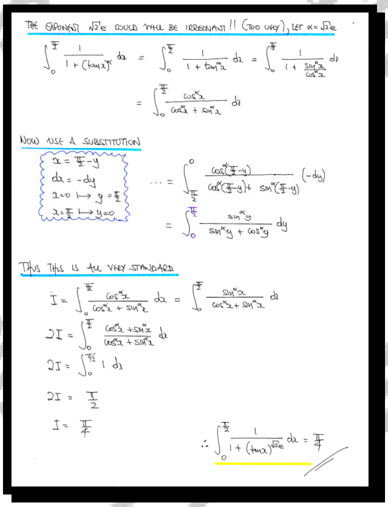


Question 305 (*****)

Find an exact value for the following integral

$$\int_0^{\frac{1}{2}\pi} \frac{1}{1+(\tan x)^{\sqrt{2}\,\mathrm{e}}}\, dx .$$

$\square$, $\boxed{\frac{1}{4}\pi}$



THE EXPONENT $\sqrt{2}e$ COULD WELL BE IRRELEVANT!! (TOO UGLY), LET $\alpha = \sqrt{2}e$

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+(\tan x)^{\alpha}}\,dx = \int_0^{\frac{\pi}{2}} \frac{1}{1+\tan^{\alpha}x}\,dx = \int_0^{\frac{\pi}{2}} \frac{1}{1+\frac{\sin^{\alpha}x}{\cos^{\alpha}x}}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos^{\alpha}x}{\cos^{\alpha}x+\sin^{\alpha}x}\,dx$$

NOW USE A SUBSTITUTION

$x = \frac{\pi}{2} - y$

$dx = -dy$

$x = 0 \mapsto y = \frac{\pi}{2}$

$x = \frac{\pi}{2} \mapsto y = 0$

$$\dots = \int_{\frac{\pi}{2}}^{0} \frac{\cos^{\alpha}(\frac{\pi}{2}-y)}{\cos^{\alpha}(\frac{\pi}{2}-y)+\sin^{\alpha}(\frac{\pi}{2}-y)}\,(-dy)$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin^{\alpha}y}{\sin^{\alpha}y+\cos^{\alpha}y}\,dy$$

THUS THIS IS ALL VERY STANDARD

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos^{\alpha}x}{\cos^{\alpha}x+\sin^{\alpha}x}\,dx = \int_0^{\frac{\pi}{2}} \frac{\sin^{\alpha}x}{\cos^{\alpha}x+\sin^{\alpha}x}\,dx$$

$$2I = \int_0^{\frac{\pi}{2}} \frac{\cos^{\alpha}x+\sin^{\alpha}x}{\cos^{\alpha}x+\sin^{\alpha}x}\,dx$$

$$2I = \int_0^{\pi/2} 1\,dx$$

$$2I = \frac{\pi}{2}$$

$$I = \frac{\pi}{4}$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{1}{1+(\tan x)^{\sqrt{2}e}}\,dx = \frac{\pi}{4}$$

Question 306 (*****)

$$I = \int_{-\infty}^{\infty} \left| x^3 \left(2^{-x^2}\right) \right| \, dx$$

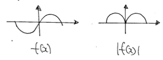
It is given that $I \approx 2$.

Use this fact to estimate the value of $\ln 2$ correct to 1 significant figure.

$\boxed{}$, $\boxed{\ln 2 \approx 0.7}$

- AS $x^3 \times 2^{-x^2}$ IS ODD, THE MODULUS IN THE INTEGRAND WILL BE EVEN

E.g



- HENCE WE HAVE

$$\int_{-\infty}^{\infty} \left| x^3\left(2^{-x^2}\right)\right| dx = 2\int_{0}^{\infty} \left| x^3\left(2^{-x^2}\right)\right| dx = \int_{0}^{\infty} 2x^3\left(2^{-x^2}\right) dx$$

- BY SUBSTITUTION FIRST

$$\dots = \int_{0}^{-\infty} 2x^3\left(2^{u}\right)\left(\frac{du}{-2x}\right)$$

$$= \int_{-\infty}^{0} x^2\left(2^{u}\right) du$$

$$= \int_{-\infty}^{0} -u\left(2^{u}\right) du$$

$$= \int_{0}^{-\infty} u\left(2^{u}\right) du$$

$u = -x^2$

$\frac{du}{dx} = -2x$

$dx = \frac{du}{-2x}$

$x = 0 \longmapsto u = 0$

$x = \infty \longmapsto u = -\infty$

- BY PARTS NEXT

$$\dots = \left[\frac{u\left(2^{u}\right)}{\ln 2}\right]_{0}^{-\infty} - \int_{0}^{-\infty} \frac{2^{u}}{\ln 2} \, du$$

$$= 0 - 0 - \frac{1}{\ln 2}\int_{0}^{-\infty} 2^{u} \, du$$

u	1
$\frac{2^u}{\ln 2}$	2^u

$$= -\frac{1}{\ln 2}\left[\frac{2^{u}}{\ln 2}\right]_{0}^{-\infty}$$

$$= -\frac{1}{\ln 2}\left[0 - \frac{1}{\ln 2}\right]$$

$$= \frac{1}{(\ln 2)^2}$$

- FINALLY WE HAVE

$$\frac{1}{(\ln 2)^2} \approx 2 \ (1\text{ sf})$$

$$(\ln 2)^2 \approx \frac{1}{2} \approx 0.49$$

$$\ln 2 \approx 0.7 \ (1\text{ sf})$$

Question 307 (*****)

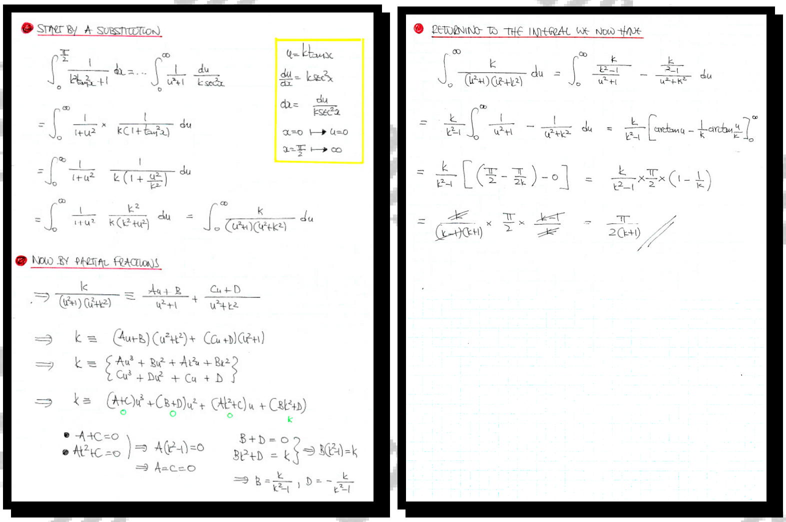
The definite integral I is defined in terms of the constant k, where $k \neq 0$, $k \neq \pm 1$.

$$I = \int_0^{\frac{1}{2}\pi} \frac{1}{1+k^2 \tan^2 x} \, dx .$$

Use appropriate integration techniques to show that

$$I = \frac{\pi}{2(k+1)} .$$

[ ] , [proof]



START BY A SUBSTITUTION

$u = k\tan x$, $\frac{du}{dx} = k\sec^2 x$, $dx = \frac{du}{k\sec^2 x}$, $x=0 \mapsto u=0$, $x=\frac{\pi}{2} \mapsto \infty$

$$\int_0^{\frac{\pi}{2}} \frac{1}{k^2\tan^2 x+1}\,dx = \dots \int_0^{\infty} \frac{1}{u^2+1}\,\frac{du}{k\sec^2 x}$$

$$= \int_0^{\infty} \frac{1}{1+u^2} \times \frac{1}{k(1+\tan^2 x)}\,du$$

$$= \int_0^{\infty} \frac{1}{1+u^2}\,\frac{1}{k\left(1+\frac{u^2}{k^2}\right)}\,du$$

$$= \int_0^{\infty} \frac{1}{1+u^2}\,\frac{k^2}{k(k^2+u^2)}\,du = \int_0^{\infty} \frac{k}{(u^2+1)(u^2+k^2)}\,du$$

NOW BY PARTIAL FRACTIONS

$$\Rightarrow \frac{k}{(u^2+1)(u^2+k^2)} \equiv \frac{Au+B}{u^2+1} + \frac{Cu+D}{u^2+k^2}$$

$$\Rightarrow k \equiv (Au+B)(u^2+k^2) + (Cu+D)(u^2+1)$$

$$\Rightarrow k \equiv \left\{ \begin{matrix} Au^3 + Bu^2 + Ak^2u + Bk^2 \\ Cu^3 + Du^2 + Cu + D \end{matrix} \right\}$$

$$\Rightarrow k \equiv (A+C)u^3 + (B+D)u^2 + (Ak^2+C)u + (Bk^2+D)$$

- $A+C=0$
- $Ak^2+C=0$

$\Rightarrow A(k^2-1)=0 \Rightarrow A=C=0$

$B+D=0$, $Bk^2+D=k$ $\Rightarrow B(k^2-1)=k$

$\Rightarrow B = \frac{k}{k^2-1}$, $D = -\frac{k}{k^2-1}$

RETURNING TO THE INTEGRAL WE NOW HAVE

$$\int_0^{\infty} \frac{k}{(u^2+1)(u^2+k^2)}\,du = \int_0^{\infty} \frac{\frac{k}{k^2-1}}{u^2+1} - \frac{\frac{k}{k^2-1}}{u^2+k^2}\,du$$

$$= \frac{k}{k^2-1}\int_0^{\infty} \frac{1}{u^2+1} - \frac{1}{u^2+k^2}\,du = \frac{k}{k^2-1}\left[\arctan u - \frac{1}{k}\arctan\frac{u}{k}\right]_0^{\infty}$$

$$= \frac{k}{k^2-1}\left[\left(\frac{\pi}{2} - \frac{\pi}{2k}\right) - 0\right] = \frac{k}{k^2-1} \times \frac{\pi}{2} \times \left(1 - \frac{1}{k}\right)$$

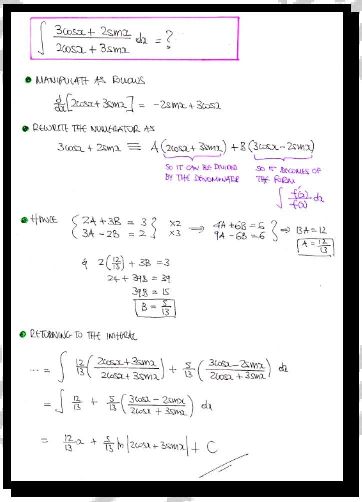
$$= \frac{k}{(k-1)(k+1)} \times \frac{\pi}{2} \times \frac{k-1}{k} = \frac{\pi}{2(k+1)}$$

Question 308 (*****)

By suitably rewriting the numerator of the integrand, find a simplified expression for the following integral.

$$\int \frac{3\cos x + 2\sin x}{2\cos x + 3\sin x}\,dx\,.$$

$$\boxed{\frac{12}{13}x + \frac{5}{13}\ln\left|2\cos x + 3\sin x\right| + C}$$



$$\int \frac{3\cos x + 2\sin x}{2\cos x + 3\sin x}\,dx = ?$$

- MANIPULATE AS FOLLOWS

$$\frac{d}{dx}\left[2\cos x + 3\sin x\right] = -2\sin x + 3\cos x$$

- REWRITE THE NUMERATOR AS

$$3\cos x + 2\sin x \equiv A\left(2\cos x + 3\sin x\right) + B\left(3\cos x - 2\sin x\right)$$

SO IT CAN BE DIVIDED BY THE DENOMINATOR

SO IT BECOMES OF THE FORM $\int \frac{f'(x)}{f(x)}\,dx$

- HENCE

$$\left.\begin{array}{l} 2A + 3B = 3 \\ 3A - 2B = 2 \end{array}\right\} \begin{array}{l}\times 2 \\ \times 3\end{array} \Rightarrow \left.\begin{array}{l} 4A + 6B = 6 \\ 9A - 6B = 6 \end{array}\right\} \Rightarrow 13A = 12 \quad \boxed{A = \frac{12}{13}}$$

$$\& \quad 2\left(\frac{12}{13}\right) + 3B = 3$$

$$24 + 39B = 39$$

$$39B = 15$$

$$\boxed{B = \frac{5}{13}}$$

- RETURNING TO THE INTEGRAL

$$\dots = \int \frac{12}{13}\left(\frac{2\cos x + 3\sin x}{2\cos x + 3\sin x}\right) + \frac{5}{13}\left(\frac{3\cos x - 2\sin x}{2\cos x + 3\sin x}\right)\,dx$$

$$= \int \frac{12}{13} + \frac{5}{13}\left(\frac{3\cos x - 2\sin x}{2\cos x + 3\sin x}\right)\,dx$$

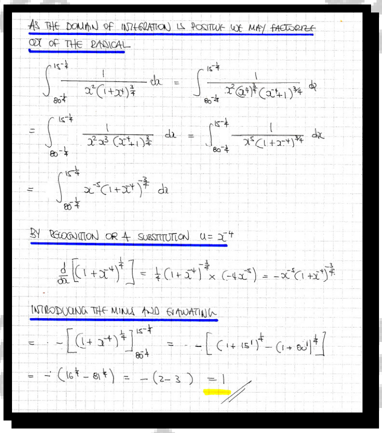
$$= \frac{12}{13}x + \frac{5}{13}\ln\left|2\cos x + 3\sin x\right| + C$$

Question 309 (*****)

Find the value of the following definite integral.

$$\int_{80^{-\frac{1}{4}}}^{15^{-\frac{1}{4}}} \frac{1}{x^2\left(1+x^4\right)^{\frac{3}{4}}}\, dx \ .$$

$\boxed{}$, $\boxed{1}$


AS THE DOMAIN OF INTEGRATION IS POSITIVE WE MAY FACTORIZE OUT OF THE RADICAL

$$\int_{80^{-\frac{1}{4}}}^{15^{-\frac{1}{4}}} \frac{1}{x^2(1+x^4)^{\frac{3}{4}}}\,dx = \int_{80^{-\frac{1}{4}}}^{15^{-\frac{1}{4}}} \frac{1}{x^2(x^4)^{\frac{3}{4}}(x^{-4}+1)^{3/4}}\,dx$$

$$= \int_{80^{-\frac{1}{4}}}^{15^{-\frac{1}{4}}} \frac{1}{x^2x^3(x^{-4}+1)^{\frac{3}{4}}}\,dx = \int_{80^{-\frac{1}{4}}}^{15^{-\frac{1}{4}}} \frac{1}{x^5(1+x^{-4})^{3/4}}\,dx$$

$$= \int_{80^{-\frac{1}{4}}}^{15^{-\frac{1}{4}}} x^{-5}(1+x^{-4})^{-\frac{3}{4}}\,dx$$

BY RECOGNITION OR A SUBSTITUTION $u = x^{-4}$

$$\frac{d}{dx}\left[(1+x^{-4})^{\frac{1}{4}}\right] = \frac{1}{4}(1+x^{-4})^{-\frac{3}{4}} \times (-4x^{-5}) = -x^{-5}(1+x^{-4})^{-\frac{3}{4}}$$

INTRODUCING THE MINUS AND EVALUATING

$$= -\left[(1+x^{-4})^{\frac{1}{4}}\right]_{80^{-\frac{1}{4}}}^{15^{-\frac{1}{4}}} = -\left[(1+15^{1})^{\frac{1}{4}} - (1+80^{1})^{\frac{1}{4}}\right]$$

$$= -\left(16^{\frac{1}{4}} - 81^{\frac{1}{4}}\right) = -(2-3) = 1$$

Question 310 (*****)

Find in exact simplified form the value of

$$\int_0^1 \frac{\sqrt{1-x}}{1-\sqrt{x}} \, dx \, .$$

You may assume that the integral converges.

V, $\square$, $\boxed{\frac{1}{2}(\pi+4)}$

IGNORING THE LIMITS OR ASSUME CONVERGENCE, START BY "RATIONALIZING" THE DENOMINATOR

$$\int_0^1 \frac{\sqrt{1-x}}{1-\sqrt{x}}\,dx = \int_0^1 \frac{\sqrt{1-x}\,(1+\sqrt{x})}{(1-\sqrt{x})(1+\sqrt{x})}\,dx = \int_0^1 \frac{\sqrt{1-x}\,(1+\sqrt{x})}{1-x}\,dx$$

$$= \int_0^1 \frac{1+\sqrt{x}}{\sqrt{1-x}}\,dx$$

NOW USE A SUBSTITUTION ON THE DENOMINATOR

$$u=\sqrt{1-x}, \quad u^2 = 1-x, \quad x = 1-u^2, \quad dx = -2u\,du$$
$$x=0 \mapsto u=1, \quad x=1 \mapsto u=0$$

$$= \int_1^0 \frac{1+\sqrt{1-u^2}}{u}(-2u\,du)$$

$$= \int_1^0 -2-2\sqrt{1-u^2}\,du$$

$$= \int_0^1 2+2\sqrt{1-u^2}\,du$$

$$= \Big[2u\Big]_0^1 + 2\int_0^1 \sqrt{1-u^2}\,du$$

$$= 2 + 2\int_0^1 \sqrt{1-u^2}\,du$$

BY A TRIGONOMETRIC SUBSTITUTION

$$\theta = \arcsin u, \quad \sin\theta = u, \quad du = \cos\theta\,d\theta, \quad \cos\theta = +\sqrt{1-\sin^2\theta}, \quad \cos\theta = \sqrt{1-u^2}$$
$$u=1 \mapsto \theta = \tfrac{\pi}{2}, \quad u=0 \mapsto \theta = 0$$

$$= 2 + 2\int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2\theta}\,(\cos\theta\,d\theta)$$

$$= 2 + 2\int_0^{\pi/2} \cos\theta\cos\theta\,d\theta$$

$$= 2 + 2\int_0^{\pi/2} \cos^2\theta\,d\theta$$

$$= 2 + 2\int_0^{\pi/2} \tfrac{1}{2}+\tfrac{1}{2}\cos 2\theta\,d\theta$$

$$= 2 + \int_0^{\frac{\pi}{2}} 1+\cos 2\theta\,d\theta$$

$$= 2 + \Big[\theta + \tfrac{1}{2}\sin 2\theta\Big]_0^{\pi/2}$$

$$= 2 + \Big[\big(\tfrac{\pi}{2} - \tfrac{1}{2}\sin\pi\big) - (0+0)\Big]$$

$$= 2 + \tfrac{\pi}{2}$$

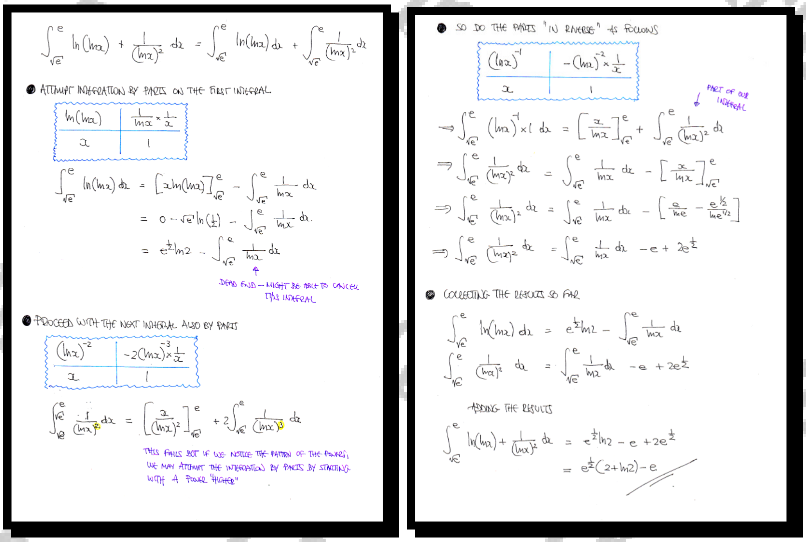
$$= \tfrac{1}{2}(\pi+4)$$

Question 311 (*****)

Find an exact simplified value for

$$\int_{\sqrt{e}}^{e} \ln(\ln x) + \frac{1}{(\ln x)^2} \; dx .$$

$\boxed{}$, $\boxed{e^{\frac{1}{2}}(2+\ln 2) - e}$



Question 312 (*****)

By using appropriate substitutions, or otherwise, show that

$$\int_0^{\frac{1}{2}} \frac{\ln(1+2x)}{1+4x^2}\,dx = \frac{\pi \ln 2}{16}.$$

[illegible], proof

THE ANSWER SUGGESTS A TRIGONOMETRIC SUBSTITUTION

$$1+4x^2 = 1+(2x)^2 = 1+\tan^2\theta = \sec^2\theta$$

- $2x = \tan\theta$ [$\theta = \arctan 2x$ IS THE SUBSTITUTION]
- $2\,dx = \sec^2\theta$
- $x=0 \mapsto \theta=0$
- $x=\frac{1}{2} \mapsto \theta=\frac{\pi}{4}$

TRANSFORMING THE INTEGRAL

$$\int_0^{\frac{1}{2}} \frac{\ln(1+2x)}{1+4x^2}\,dx = \int_0^{\frac{\pi}{4}} \frac{\ln(1+\tan\theta)}{\sec^2\theta}\left(\tfrac{1}{2}\sec^2\theta\,d\theta\right)$$

$$I = \frac{1}{2}\int_0^{\frac{\pi}{4}} \ln(1+\tan\theta)\,d\theta$$

ANOTHER SUBSTITUTION

- $\theta = \frac{\pi}{4}-\phi \Longleftrightarrow \phi = \frac{\pi}{4}-\theta$
- $d\theta = -d\phi \Longleftrightarrow d\phi = -d\theta$
- $\theta=0 \mapsto \phi=\frac{\pi}{4}$
- $\theta=\frac{\pi}{4} \mapsto \phi=0$

THUS WE NOW HAVE

$$I = \frac{1}{2}\int_{\frac{\pi}{4}}^{0} \ln\left(1+\tan\left(\tfrac{\pi}{4}-\phi\right)\right)(-d\phi) = \frac{1}{2}\int_0^{\frac{\pi}{4}} \ln\left[1+\frac{\tan\frac{\pi}{4}-\tan\phi}{1+\tan\frac{\pi}{4}\tan\phi}\right]d\phi$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} \ln\left[1+\frac{1-\tan\phi}{1+\tan\phi}\right]d\phi = \frac{1}{2}\int_0^{\frac{\pi}{4}} \ln\left[\frac{1+\tan\phi+1-\tan\phi}{1+\tan\phi}\right]d\phi$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} \ln\left[\frac{2}{1+\tan\phi}\right]d\phi$$

SPLIT THE LOG, OBSERVING THE DEFINITION OF I

$$\Rightarrow I = \frac{1}{2}\int_0^{\frac{\pi}{4}} \ln(1+\tan\theta)\,d\theta$$

$$\Rightarrow I = \frac{1}{2}\int_0^{\frac{\pi}{4}} \ln 2 - \ln(1+\tan\phi)\,d\phi$$

$$\Rightarrow I = \frac{1}{2}\int_0^{\frac{\pi}{4}} \ln 2\,d\phi - \frac{1}{2}\int_0^{\frac{\pi}{4}} \ln(1+\tan\phi)\,d\phi$$

$$\Rightarrow I = \frac{1}{2}\ln 2 \times \frac{\pi}{4} - I$$

$$\Rightarrow 2I = \frac{\pi\ln 2}{8}$$

$$\Rightarrow I = \frac{\pi\ln 2}{16}$$

$$\therefore \int_0^{\frac{1}{2}} \frac{\ln(1+2x)}{1+4x^2}\,dx = \frac{1}{16}\pi\ln 2$$

AS REQUIRED

Question 313 (*****)

Use appropriate integration techniques to show that

$$\int_0^1 4x\arctan x \ dx = \pi - 2.$$

proof

$$\int_0^1 4x\arctan x = \pi - 2$$

- BY SUBSTITUTION

$x = \tan\theta$
$\theta = \arctan x$
$\frac{dx}{d\theta} = \sec^2\theta \implies dx = \sec^2\theta \, d\theta$
$x = 0 \mapsto \theta = 0$
$x = 1 \mapsto \theta = \frac{\pi}{4}$

$$\ldots = \int_0^{\frac{\pi}{4}} (4\tan\theta)\arctan(\tan\theta) \times \sec^2\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{4}} 4\theta\tan\theta\sec^2\theta \ d\theta$$

- INTEGRATION BY PARTS NEXT

θ	1
$2\tan^2\theta$	$4\tan\theta\sec^2\theta$

$$= \left[2\theta\tan^2\theta\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} 2\tan^2\theta \ d\theta$$

$$= \left(\frac{\pi}{2} - 0\right) - \int_0^{\frac{\pi}{4}} 2(\sec^2\theta - 1)\,d\theta$$

$$= \frac{\pi}{2} - \int_0^{\frac{\pi}{4}} 2\sec^2\theta - 2 \ d\theta$$

$$= \frac{\pi}{2} - \left[2\tan\theta - 2\theta\right]_0^{\pi/4} = \frac{\pi}{2} - \left[\left(2 - \frac{\pi}{2}\right) - 0\right]$$

$$= \pi - 2$$

Question 314 (*****)

$$I = \int_0^{\frac{1}{2}} \frac{\ln(1+2x)}{1+4x^2} \, dx .$$

a) Use an appropriate trigonometric substitution to show that

$$I = \int_0^{\frac{\pi}{4}} \frac{1}{2}\ln\sqrt{2} + \frac{1}{2}\ln\left[\frac{\cos\left(\theta - \frac{1}{4}\pi\right)}{\cos\theta}\right] d\theta .$$

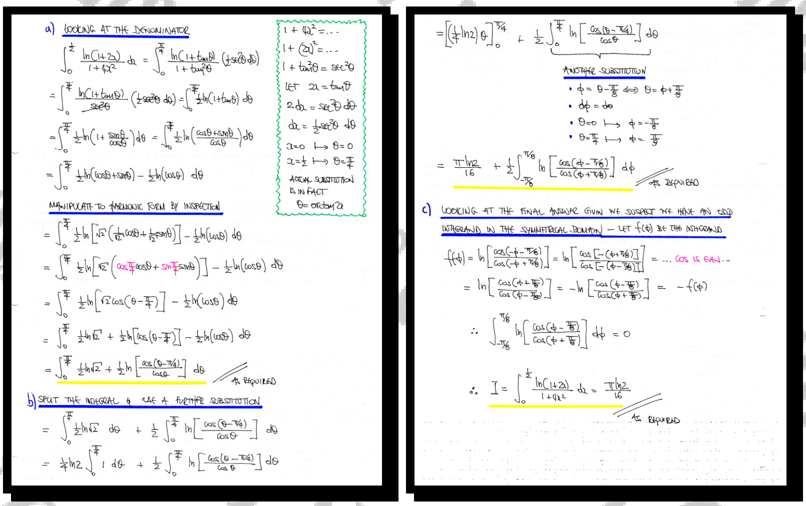
b) Show further that

$$I = \frac{\pi \ln 2}{16} + \int_{-\frac{\pi}{8}}^{\frac{\pi}{8}} \frac{1}{2}\ln\left[\frac{\cos\left(\varphi - \frac{1}{8}\pi\right)}{\cos\left(\varphi + \frac{1}{8}\pi\right)}\right] d\varphi .$$

c) Deduce that

$$I = \frac{\pi \ln 2}{16} .$$

V, , proof



Question 315 (*****)

$$I = \int \frac{2x+1}{\sqrt{x-1}} \, dx .$$

a) Show that

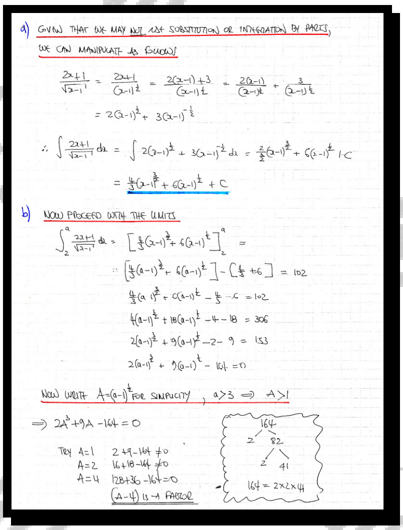
$$I = \frac{4}{3}(x-1)^{\frac{3}{2}} + 6(x-1)^{\frac{1}{2}} + \text{constant} .$$

You may not use any substitution or integration by parts.

b) Determine the value of a, given that

$$\int_2^a \frac{2x+1}{\sqrt{x-1}} \, dx = 102 .$$

$\boxed{}$, $\boxed{a = 17}$



a) GIVEN THAT WE MAY NOT USE SUBSTITUTION OR INTEGRATION BY PARTS, WE CAN MANIPULATE AS FOLLOWS

$$\frac{2x+1}{\sqrt{x-1}} = \frac{2x+1}{(x-1)^{\frac{1}{2}}} = \frac{2(x-1)+3}{(x-1)^{\frac{1}{2}}} = \frac{2(x-1)}{(x-1)^{\frac{1}{2}}} + \frac{3}{(x-1)^{\frac{1}{2}}}$$

$$= 2(x-1)^{\frac{1}{2}} + 3(x-1)^{-\frac{1}{2}}$$

$$\therefore \int \frac{2x+1}{\sqrt{x-1}} \, dx = \int 2(x-1)^{\frac{1}{2}} + 3(x-1)^{-\frac{1}{2}} \, dx = \frac{2}{\frac{3}{2}}(x-1)^{\frac{3}{2}} + 6(x-1)^{\frac{1}{2}} + C$$

$$= \frac{4}{3}(x-1)^{\frac{3}{2}} + 6(x-1)^{\frac{1}{2}} + C$$

b) NOW PROCEED WITH THE LIMITS

$$\int_2^a \frac{2x+1}{\sqrt{x-1}} \, dx = \left[\frac{4}{3}(x-1)^{\frac{3}{2}} + 6(x-1)^{\frac{1}{2}}\right]_2^a =$$

$$\therefore \left[\frac{4}{3}(a-1)^{\frac{3}{2}} + 6(a-1)^{\frac{1}{2}}\right] - \left[\frac{4}{3} + 6\right] = 102$$

$$\frac{4}{3}(a-1)^{\frac{3}{2}} + 6(a-1)^{\frac{1}{2}} - \frac{4}{3} - 6 = 102$$

$$4(a-1)^{\frac{3}{2}} + 18(a-1)^{\frac{1}{2}} - 4 - 18 = 306$$

$$2(a-1)^{\frac{3}{2}} + 9(a-1)^{\frac{1}{2}} - 2 - 9 = 153$$

$$2(a-1)^{\frac{3}{2}} + 9(a-1)^{\frac{1}{2}} - 164 = 0$$

NOW WRITE $A = (a-1)^{\frac{1}{2}}$ FOR SIMPLICITY, $a > 3 \Rightarrow A > 1$

$$\Rightarrow 2A^3 + 9A - 164 = 0$$

TRY $A=1$ $\quad 2+9-164 \neq 0$

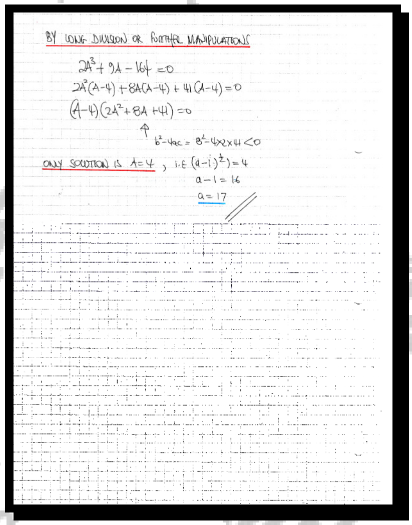
$A=2$ $\quad 16+18-164 \neq 0$

$A=4$ $\quad 128+36-164=0$

$(A-4)$ IS A FACTOR

164 → 2, 82 → 2, 41

$164 = 2 \times 2 \times 41$



BY LONG DIVISION OR FURTHER MANIPULATIONS

$$2A^3 + 9A - 164 = 0$$

$$2A^2(A-4) + 8A(A-4) + 41(A-4) = 0$$

$$(A-4)(2A^2 + 8A + 41) = 0$$

$b^2 - 4ac = 8^2 - 4 \times 2 \times 41 < 0$

ONLY SOLUTION IS $A = 4$, i.e. $(a-1)^{\frac{1}{2}} = 4$

$$a - 1 = 16$$

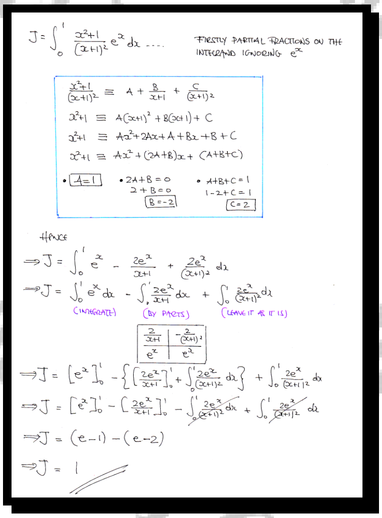
$$a = 17$$

Question 316 (*****)

$$J = \int_0^1 \frac{\left(x^2+1\right)\mathrm{e}^x}{\left(x+1\right)^2}\, dx .$$

Show that $J = 1$

[ ] , [proof]



$J = \int_0^1 \frac{x^2+1}{(x+1)^2} e^x \, dx \ldots$ FIRSTLY PARTIAL FRACTIONS ON THE INTEGRAND IGNORING e^x

$\frac{x^2+1}{(x+1)^2} \equiv A + \frac{B}{x+1} + \frac{C}{(x+1)^2}$

$x^2+1 \equiv A(x+1)^2 + B(x+1) + C$

$x^2+1 \equiv Ax^2 + 2Ax + A + Bx + B + C$

$x^2+1 \equiv Ax^2 + (2A+B)x + (A+B+C)$

- $A=1$
- $2A+B=0$, $2+B=0$, $B=-2$
- $A+B+C=1$, $1-2+C=1$, $C=2$

HENCE

$\Rightarrow J = \int_0^1 e^x - \frac{2e^x}{x+1} + \frac{2e^x}{(x+1)^2} \, dx$

$\Rightarrow J = \int_0^1 e^x \, dx - \int_0^1 \frac{2e^x}{x+1} \, dx + \int_0^1 \frac{2e^x}{(x+1)^2} \, dx$

(INTEGRATE) (BY PARTS) (LEAVE IT AS IT IS)

$\frac{2}{x+1}$	$-\frac{2}{(x+1)^2}$
e^x	e^x

$\Rightarrow J = \left[e^x\right]_0^1 - \left\{\left[\frac{2e^x}{x+1}\right]_0^1 + \int_0^1 \frac{2e^x}{(x+1)^2} \, dx\right\} + \int_0^1 \frac{2e^x}{(x+1)^2} \, dx$

$\Rightarrow J = \left[e^x\right]_0^1 - \left[\frac{2e^x}{x+1}\right]_0^1 - \int_0^1 \frac{2e^x}{(x+1)^2} \, dx + \int_0^1 \frac{2e^x}{(x+1)^2} \, dx$

$\Rightarrow J = (e-1) - (e-2)$

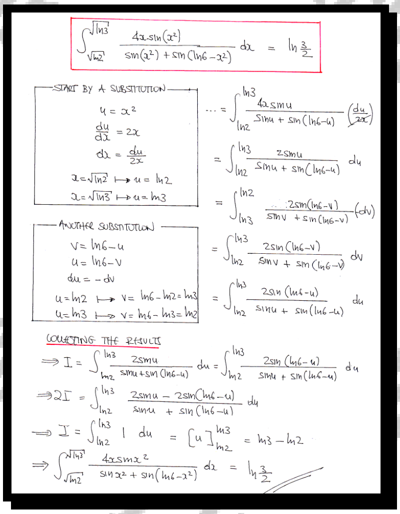
$\Rightarrow J = 1$

Question 317 (*****)

By using an appropriate substitution or substitutions, show that

$$\int_{\sqrt{\ln 2}}^{\sqrt{\ln 3}} \frac{4x\sin\left(x^2\right)}{\sin\left(x^2\right)+\sin\left(\ln 6 - x^2\right)}\,dx = \ln\left(\frac{3}{2}\right).$$

proof



$$\int_{\sqrt{\ln 2}}^{\sqrt{\ln 3}} \frac{4x\sin(x^2)}{\sin(x^2)+\sin(\ln 6 - x^2)}\,dx = \ln\frac{3}{2}$$

START BY A SUBSTITUTION

$u = x^2$

$\frac{du}{dx} = 2x$

$dx = \frac{du}{2x}$

$x = \sqrt{\ln 2} \mapsto u = \ln 2$

$x = \sqrt{\ln 3} \mapsto u = \ln 3$

$$\dots = \int_{\ln 2}^{\ln 3} \frac{4x\sin u}{\sin u + \sin(\ln 6 - u)}\left(\frac{du}{2x}\right)$$

$$= \int_{\ln 2}^{\ln 3} \frac{2\sin u}{\sin u + \sin(\ln 6 - u)}\,du$$

$$= \int_{\ln 3}^{\ln 2} \frac{2\sin(\ln 6 - v)}{\sin v + \sin(\ln 6 - v)}(-dv)$$

$$= \int_{\ln 2}^{\ln 3} \frac{2\sin(\ln 6 - v)}{\sin v + \sin(\ln 6 - v)}\,dv$$

$$= \int_{\ln 2}^{\ln 3} \frac{2\sin(\ln 6 - u)}{\sin u + \sin(\ln 6 - u)}\,du$$

ANOTHER SUBSTITUTION

$v = \ln 6 - u$

$u = \ln 6 - v$

$du = -dv$

$u = \ln 2 \mapsto v = \ln 6 - \ln 2 = \ln 3$

$u = \ln 3 \mapsto v = \ln 6 - \ln 3 = \ln 2$

COLLECTING THE RESULTS

$$\Rightarrow I = \int_{\ln 2}^{\ln 3} \frac{2\sin u}{\sin u + \sin(\ln 6 - u)}\,du = \int_{\ln 2}^{\ln 3} \frac{2\sin(\ln 6 - u)}{\sin u + \sin(\ln 6 - u)}\,du$$

$$\Rightarrow 2I = \int_{\ln 2}^{\ln 3} \frac{2\sin u - 2\sin(\ln 6 - u)}{\sin u + \sin(\ln 6 - u)}\,du$$

$$\Rightarrow I = \int_{\ln 2}^{\ln 3} 1\,du = [u]_{\ln 2}^{\ln 3} = \ln 3 - \ln 2$$

$$\Rightarrow \int_{\sqrt{\ln 2}}^{\sqrt{\ln 3}} \frac{4x\sin x^2}{\sin x^2 + \sin(\ln 6 - x^2)}\,dx = \ln\frac{3}{2}$$

Question 318 (*****)

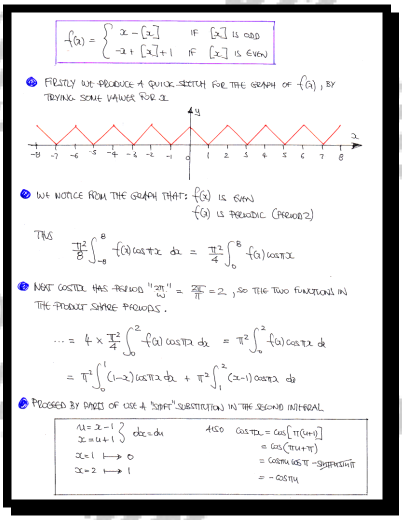
$$f(x)=\begin{cases} x-[x] & x\in\mathbb{R},\ [x]=2k+1,\ k\in\mathbb{Z} \\ -x+[x]+1 & x\in\mathbb{R}\ \ [x]=2k,\quad k\in\mathbb{Z} \end{cases},$$

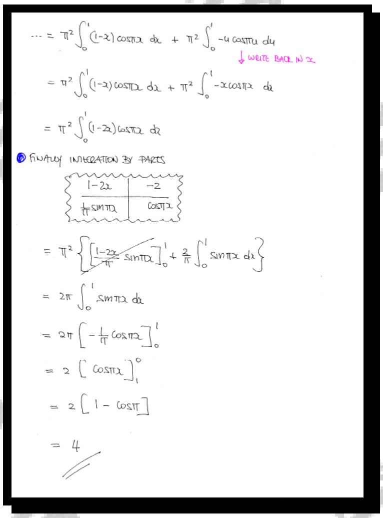
where $[x]$ is defined as the greatest integer less or equal to x.

Find the value of

$$\frac{\pi^2}{8}\int_{-8}^{8} f(x)\cos(\pi x)\ dx.$$

$\boxed{}$, $\boxed{4}$



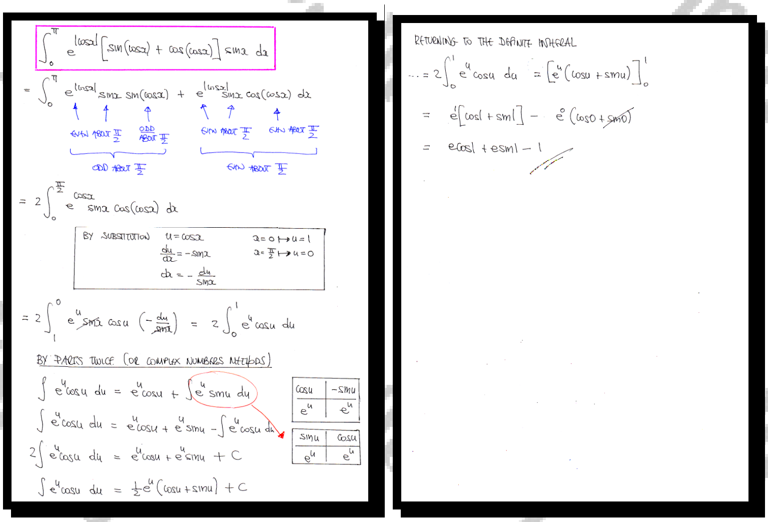


Question 319 (*****)

By using symmetry arguments, find the exact value of the following integral

$$\int_0^{\pi} e^{|\cos x|}\left[\sin(\cos x)+\cos(\cos x)\right]\sin x \ dx \ .$$

$\boxed{}$, $\boxed{e(\cos 1+\sin 1)-1}$


$$\int_0^{\pi} e^{|\cos x|}\left[\sin(\cos x)+\cos(\cos x)\right]\sin x \, dx$$

$$= \int_0^{\pi} e^{|\cos x|}\sin x \sin(\cos x) + e^{|\cos x|}\sin x\cos(\cos x)\, dx$$

even about $\frac{\pi}{2}$, even about $\frac{\pi}{2}$, odd about $\frac{\pi}{2}$ → odd about $\frac{\pi}{2}$; even about $\frac{\pi}{2}$, even about $\frac{\pi}{2}$, even about $\frac{\pi}{2}$ → even about $\frac{\pi}{2}$

$$= 2\int_0^{\frac{\pi}{2}} e^{\cos x}\sin x\cos(\cos x)\, dx$$

By substitution: $u=\cos x$, $\frac{du}{dx}=-\sin x$, $dx=-\frac{du}{\sin x}$; $x=0 \mapsto u=1$, $x=\frac{\pi}{2}\mapsto u=0$

$$= 2\int_1^0 e^u \sin x\cos u\left(-\frac{du}{\sin x}\right) = 2\int_0^1 e^u\cos u\, du$$

By parts twice (or complex numbers methods)

$$\int e^u\cos u\, du = e^u\cos u + \int e^u\sin u\, du$$

$$\int e^u\cos u\, du = e^u\cos u + e^u\sin u - \int e^u\cos u\, du$$

$$2\int e^u\cos u\, du = e^u\cos u + e^u\sin u + C$$

$$\int e^u\cos u\, du = \tfrac{1}{2}e^u(\cos u+\sin u)+C$$

Returning to the definite integral

$$\ldots = 2\int_0^1 e^u\cos u\, du = \left[e^u(\cos u+\sin u)\right]_0^1$$

$$= e^1[\cos 1+\sin 1] - e^0(\cos 0+\sin 0)$$

$$= e\cos 1 + e\sin 1 - 1$$

Question 320 (*****)

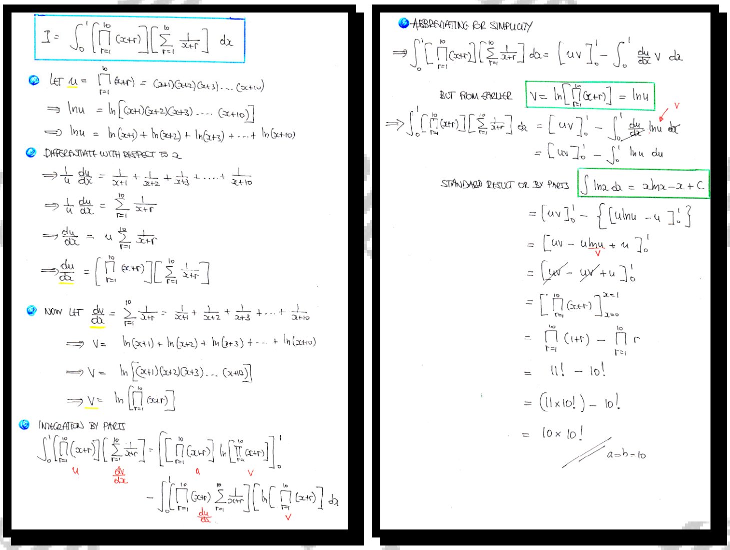
$$I = \int_0^1 \left[\prod_{r=1}^{10} (x+r) \right] \left[\sum_{r=1}^{10} \left(\frac{1}{x+r} \right) \right] dx .$$

Show by a detailed method that

$$I = a \times b!,$$

where a and b are positive integers to be found.

[ ] , $\boxed{a = b = 10}$



$I = \int_0^1 \left[\prod_{r=1}^{10}(x+r)\right]\left[\sum_{r=1}^{10}\frac{1}{x+r}\right] dx$

LET $u = \prod_{r=1}^{10}(x+r) = (x+1)(x+2)(x+3)\dots(x+10)$

$\Rightarrow \ln u = \ln\left[(x+1)(x+2)(x+3)\dots(x+10)\right]$

$\Rightarrow \ln u = \ln(x+1) + \ln(x+2) + \ln(x+3) + \dots + \ln(x+10)$

DIFFERENTIATE WITH RESPECT TO x

$\Rightarrow \frac{1}{u}\frac{du}{dx} = \frac{1}{x+1} + \frac{1}{x+2} + \frac{1}{x+3} + \dots + \frac{1}{x+10}$

$\Rightarrow \frac{1}{u}\frac{du}{dx} = \sum_{r=1}^{10}\frac{1}{x+r}$

$\Rightarrow \frac{du}{dx} = u\sum_{r=1}^{10}\frac{1}{x+r}$

$\Rightarrow \frac{du}{dx} = \left[\prod_{r=1}^{10}(x+r)\right]\left[\sum_{r=1}^{10}\frac{1}{x+r}\right]$

NOW LET $\frac{dv}{dx} = \sum_{r=1}^{10}\frac{1}{x+r} = \frac{1}{x+1} + \frac{1}{x+2} + \frac{1}{x+3} + \dots + \frac{1}{x+10}$

$\Rightarrow v = \ln(x+1) + \ln(x+2) + \ln(x+3) + \dots + \ln(x+10)$

$\Rightarrow v = \ln\left[(x+1)(x+2)(x+3)\dots(x+10)\right]$

$\Rightarrow v = \ln\left[\prod_{r=1}^{10}(x+r)\right]$

INTEGRATION BY PARTS

$\int_0^1 \left[\prod_{r=1}^{10}(x+r)\right]\left[\sum_{r=1}^{10}\frac{1}{x+r}\right] = \left[\left[\prod_{r=1}^{10}(x+r)\right]\ln\left[\prod_{r=1}^{10}(x+r)\right]\right]_0^1 - \int_0^1 \left[\prod_{r=1}^{10}(x+r)\sum_{r=1}^{10}\frac{1}{x+r}\right]\left[\ln\left[\prod_{r=1}^{10}(x+r)\right]\right] dx$

ABBREVIATING FOR SIMPLICITY

$\Rightarrow \int_0^1 \left[\prod_{r=1}^{10}(x+r)\right]\left[\sum_{r=1}^{10}\frac{1}{x+r}\right] dx = \left[uv\right]_0^1 - \int_0^1 \frac{du}{dx} v \, dx$

BUT FROM EARLIER $v = \ln\left[\prod_{r=1}^{10}(x+r)\right] = \ln u$

$\Rightarrow \int_0^1 \left[\prod_{r=1}^{10}(x+r)\right]\left[\sum_{r=1}^{10}\frac{1}{x+r}\right] dx = \left[uv\right]_0^1 - \int_0^1 \frac{du}{dx} \ln u \, dx$

$= \left[uv\right]_0^1 - \int_0^1 \ln u \, du$

STANDARD RESULT OR BY PARTS $\int \ln x \, dx = x\ln x - x + C$

$= \left[uv\right]_0^1 - \left\{\left[u\ln u - u\right]_0^1\right\}$

$= \left[uv - u\ln u + u\right]_0^1$

$= \left[uv - uv + u\right]_0^1$

$= \left[\prod_{r=1}^{10}(x+r)\right]_{x=0}^{x=1}$

$= \prod_{r=1}^{10}(1+r) - \prod_{r=1}^{10} r$

$= 11! - 10!$

$= (11 \times 10!) - 10!$

$= 10 \times 10!$

$a = b = 10$

Question 321 (*****)

$$I = \int \sqrt{\tan x}\ dx .$$

a) Use a suitable substitution to show that

$$I = \int \frac{1+\frac{1}{u^2}}{\left(u-\frac{1}{u}\right)^2+2}\,dx + I = \int \frac{1-\frac{1}{u^2}}{\left(u+\frac{1}{u}\right)^2-2}\,dx .$$

b) By using a further substitution in each of the integrals of part **(a)** find a simplified expression for I, in terms of x.

You may assume without proof that

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\left[\frac{x}{a}\right] + \text{constant} .$$

$$\boxed{},\ \boxed{\frac{1}{\sqrt{2}}\arctan\left[\frac{\tan x - 1}{\sqrt{2\tan x}}\right] + \frac{1}{2\sqrt{2}}\ln\left[\frac{\tan x - \sqrt{2\tan x}+1}{\tan x + \sqrt{2\tan x}+1}\right] + C}$$

a) $\int \sqrt{\tan x}\,dx$ ... BY SUBSTITUTION

$u = \sqrt{\tan x}$

$u^2 = \tan x$

$2u\,du = \sec^2 x\,dx$

$2u\,du = (1+\tan^2 x)\,dx$

$2u\,du = (1+u^4)\,dx$

$dx = \frac{2u}{1+u^4}\,du$

$$= \int u\left(\frac{2u}{1+u^4}\right)du = \int \frac{2u^2}{1+u^4}\,du$$

$$= \int \frac{2}{\frac{1}{u^2}+u^2}\,du = \int \frac{1+\frac{1}{u^2}+1-\frac{1}{u^2}}{\frac{1}{u^2}+u^2}\,du$$

$$= \int \frac{1+\frac{1}{u^2}}{\frac{1}{u^2}+u^2}\,du + \int \frac{1-\frac{1}{u^2}}{\frac{1}{u^2}+u^2}\,du$$

$$= \int \frac{1+\frac{1}{u^2}}{\left(u-\frac{1}{u}\right)^2+2}\,du + \int \frac{1-\frac{1}{u^2}}{\left(u+\frac{1}{u}\right)^2-2}\,du$$

b) NOW ANOTHER SUBSTITUTION FOR EACH INTEGRAL

$v = u - \frac{1}{u}$, $dv = \left(1+\frac{1}{u^2}\right)du$

$w = u + \frac{1}{u}$, $dw = \left(1-\frac{1}{u^2}\right)du$

$$\dots = \int \frac{1+\frac{1}{u^2}}{v^2+2}\,\frac{dv}{1+\frac{1}{u^2}} + \int \frac{1-\frac{1}{u^2}}{w^2-2}\,\frac{dw}{1-\frac{1}{u^2}}$$

$$= \int \frac{1}{v^2+2}\,dv + \int \frac{1}{w^2-2}\,dw$$

$$= \frac{1}{\sqrt{2}}\arctan\frac{v}{\sqrt{2}} + \int \frac{1}{(w-\sqrt{2})(w+\sqrt{2})}\,dw$$

PARTIAL FRACTIONS BY COVER UP

$$= \frac{1}{\sqrt{2}}\arctan\left(\frac{u}{\sqrt{2}} - \frac{1}{\sqrt{2}u}\right) + \int \frac{\frac{1}{2\sqrt{2}}}{w-\sqrt{2}} - \frac{\frac{1}{2\sqrt{2}}}{w+\sqrt{2}}\,dw$$

$$= \frac{1}{\sqrt{2}}\arctan\left(\frac{u^2-1}{\sqrt{2}u}\right) + \frac{1}{2\sqrt{2}}\int \frac{1}{w-\sqrt{2}} - \frac{1}{w+\sqrt{2}}\,dw$$

$$= \frac{1}{\sqrt{2}}\arctan\left[\frac{u^2-1}{\sqrt{2}u}\right] + \frac{1}{2\sqrt{2}}\ln\left[\frac{w-\sqrt{2}}{w+\sqrt{2}}\right] + C$$

$$= \frac{1}{\sqrt{2}}\arctan\left[\frac{u^2-1}{\sqrt{2}u}\right] + \frac{1}{2\sqrt{2}}\ln\left[\frac{1+\frac{1}{u}-\sqrt{2}}{1+\frac{1}{u}+\sqrt{2}}\right] + C$$

$$= \frac{1}{\sqrt{2}}\arctan\left[\frac{u^2-1}{\sqrt{2}u}\right] + \frac{1}{2\sqrt{2}}\ln\left[\frac{u+1-\sqrt{2}u}{u+1+\sqrt{2}u}\right] + C$$

FINALLY REVERSING THE FINAL SUBSTITUTION WE OBTAIN

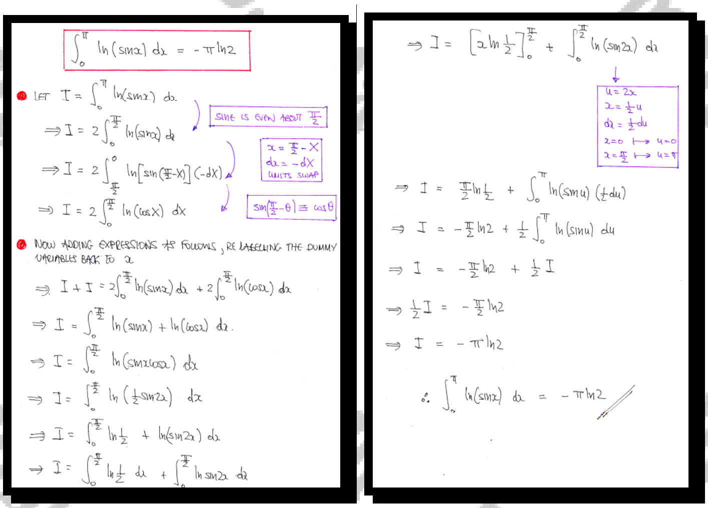
$$= \frac{1}{\sqrt{2}}\arctan\left[\frac{\tan x - 1}{\sqrt{2\tan x}}\right] + \frac{1}{2\sqrt{2}}\ln\left[\frac{\tan x - \sqrt{2\tan x}+1}{\tan x + \sqrt{2\tan x}+1}\right] + C$$

Question 322 (*****)

By using an appropriate substitution or substitutions, show that

$$\int_0^{\pi} \ln(\sin x)\ dx = -\pi \ln 2 .$$

proof



$\int_0^{\pi} \ln(\sin x)\, dx = -\pi \ln 2$

- LET $I = \int_0^{\pi} \ln(\sin x)\, dx$

$\Rightarrow I = 2\int_0^{\frac{\pi}{2}} \ln(\sin x)\, dx$ (SINE IS EVEN ABOUT $\frac{\pi}{2}$)

$\Rightarrow I = 2\int_{\frac{\pi}{2}}^{0} \ln\left[\sin\left(\frac{\pi}{2}-X\right)\right](-dX)$ ($x = \frac{\pi}{2} - X$, $dx = -dX$, LIMITS SWAP)

$\Rightarrow I = 2\int_0^{\frac{\pi}{2}} \ln(\cos X)\, dX$ ($\sin\left(\frac{\pi}{2}-\theta\right) \equiv \cos\theta$)

- NOW ADDING EXPRESSIONS AS FOLLOWS, RE LABELLING THE DUMMY VARIABLES BACK TO x

$\Rightarrow I + I = 2\int_0^{\frac{\pi}{2}} \ln(\sin x)\, dx + 2\int_0^{\frac{\pi}{2}} \ln(\cos x)\, dx$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \ln(\sin x) + \ln(\cos x)\, dx$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \ln(\sin x \cos x)\, dx$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \ln\left(\frac{1}{2}\sin 2x\right)\, dx$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \ln\frac{1}{2} + \ln(\sin 2x)\, dx$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \ln\frac{1}{2}\, dx + \int_0^{\frac{\pi}{2}} \ln \sin 2x\, dx$

$\Rightarrow I = \left[x \ln\frac{1}{2}\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \ln(\sin 2x)\, dx$

($u = 2x$, $x = \frac{1}{2}u$, $dx = \frac{1}{2}du$, $x = 0 \mapsto u = 0$, $x = \frac{\pi}{2} \mapsto u = \pi$)

$\Rightarrow I = \frac{\pi}{2}\ln\frac{1}{2} + \int_0^{\pi} \ln(\sin u)\left(\frac{1}{2}du\right)$

$\Rightarrow I = -\frac{\pi}{2}\ln 2 + \frac{1}{2}\int_0^{\pi} \ln(\sin u)\, du$

$\Rightarrow I = -\frac{\pi}{2}\ln 2 + \frac{1}{2}I$

$\Rightarrow \frac{1}{2}I = -\frac{\pi}{2}\ln 2$

$\Rightarrow I = -\pi \ln 2$

$\therefore \int_0^{\pi} \ln(\sin x)\, dx = -\pi \ln 2$

Question 323 (*****)

It is given that

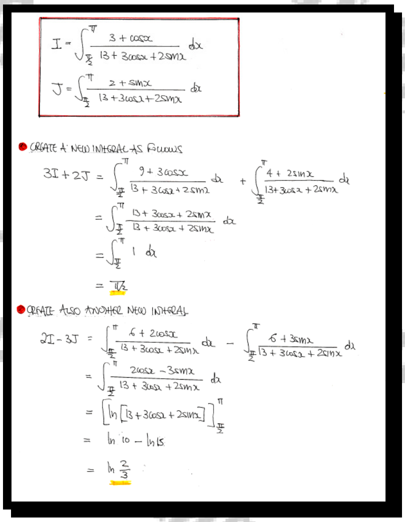
$$I=\int_{\frac{1}{2}\pi}^{\pi}\frac{3+\cos x}{13+3\cos x+2\sin x}\,dx \quad \text{and} \quad J=\int_{\frac{1}{2}\pi}^{\pi}\frac{2+\sin x}{13+3\cos x+2\sin x}\,dx .$$

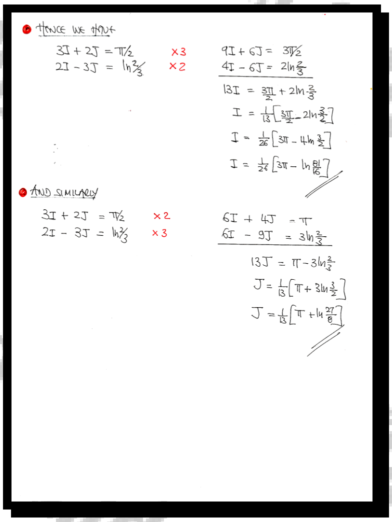
By considering two linear combinations in I and J, show that

$$I=\frac{1}{26}\left[3\pi-\ln\left(\frac{81}{16}\right)\right],$$

and find a similar expression for J.

$\boxed{}$, $\boxed{I=\frac{1}{13}\left[\pi+\ln\left(\frac{27}{8}\right)\right]}$



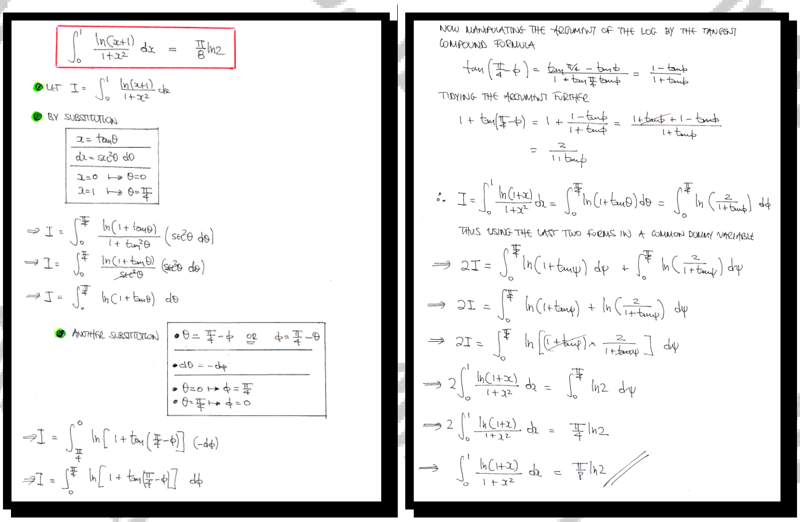


Question 324 (*****)

By using an appropriate substitution or substitutions, show that

$$\int_0^1 \frac{\ln(x+1)}{1+x^2}\,dx = \frac{\pi \ln 2}{8}.$$

S-G, proof



$$\int_0^1 \frac{\ln(x+1)}{1+x^2}\,dx = \frac{\pi}{8}\ln 2$$

- LET $I = \int_0^1 \frac{\ln(x+1)}{1+x^2}\,dx$
- BY SUBSTITUTION

$x = \tan\theta$
$dx = \sec^2\theta\,d\theta$
$x=0 \mapsto \theta = 0$
$x=1 \mapsto \theta = \frac{\pi}{4}$

$\Rightarrow I = \int_0^{\frac{\pi}{4}} \frac{\ln(1+\tan\theta)}{1+\tan^2\theta}(\sec^2\theta\,d\theta)$

$\Rightarrow I = \int_0^{\frac{\pi}{4}} \frac{\ln(1+\tan\theta)}{\sec^2\theta}(\sec^2\theta\,d\theta)$

$\Rightarrow I = \int_0^{\frac{\pi}{4}} \ln(1+\tan\theta)\,d\theta$

- ANOTHER SUBSTITUTION

$\theta = \frac{\pi}{4} - \phi$ OR $\phi = \frac{\pi}{4} - \theta$
$d\theta = -d\phi$
$\theta = 0 \mapsto \phi = \frac{\pi}{4}$
$\theta = \frac{\pi}{4} \mapsto \phi = 0$

$\Rightarrow I = \int_{\frac{\pi}{4}}^0 \ln\left[1+\tan\left(\frac{\pi}{4}-\phi\right)\right](-d\phi)$

$\Rightarrow I = \int_0^{\frac{\pi}{4}} \ln\left[1+\tan\left(\frac{\pi}{4}-\phi\right)\right]d\phi$

NOW MANIPULATING THE ARGUMENT OF THE LOG BY THE TANGENT COMPOUND FORMULA

$\tan\left(\frac{\pi}{4}-\phi\right) = \frac{\tan\frac{\pi}{4} - \tan\phi}{1+\tan\frac{\pi}{4}\tan\phi} = \frac{1-\tan\phi}{1+\tan\phi}$

TIDYING THE ARGUMENT FURTHER

$1+\tan\left(\frac{\pi}{4}-\phi\right) = 1 + \frac{1-\tan\phi}{1+\tan\phi} = \frac{1+\tan\phi+1-\tan\phi}{1+\tan\phi}$

$= \frac{2}{1+\tan\phi}$

$\therefore I = \int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx = \int_0^{\frac{\pi}{4}} \ln(1+\tan\theta)\,d\theta = \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan\phi}\right)d\phi$

THUS USING THE LAST TWO FORMS IN A COMMON DUMMY VARIABLE

$\Rightarrow 2I = \int_0^{\frac{\pi}{4}} \ln(1+\tan\psi)\,d\psi + \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan\psi}\right)d\psi$

$\Rightarrow 2I = \int_0^{\frac{\pi}{4}} \ln(1+\tan\psi) + \ln\left(\frac{2}{1+\tan\psi}\right)d\psi$

$\Rightarrow 2I = \int_0^{\frac{\pi}{4}} \ln\left[(1+\tan\psi)\times\frac{2}{1+\tan\psi}\right]d\psi$

$\Rightarrow 2\int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx = \int_0^{\frac{\pi}{4}} \ln 2\,d\psi$

$\Rightarrow 2\int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx = \frac{\pi}{4}\ln 2$

$\Rightarrow \int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx = \frac{\pi}{8}\ln 2$

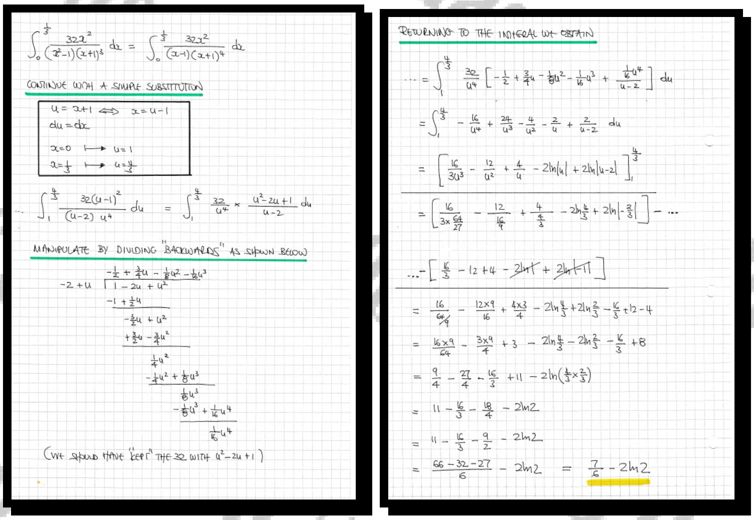
Question 325 (*****)

It is given that

$$I = \int_0^{\frac{1}{3}} \frac{32x^2}{\left(x^2-1\right)\left(x+1\right)^3}\, dx .$$

Show that $I = \frac{7}{6} - 2\ln 2$.

[ ], [proof]



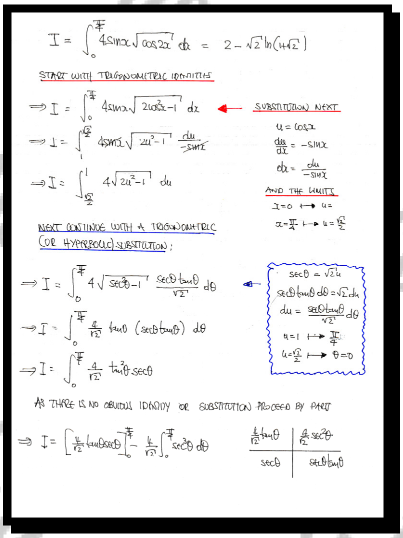
Question 326 (*****)

$$I = \int_0^{\frac{1}{2}\pi} 4\sin x\sqrt{\cos 2x}\ dx .$$

By using an appropriate substitution or substitutions, show that

$$I = 2 - \sqrt{2}\ln\left(1+\sqrt{2}\right).$$

[ ], [proof]



$I = \int_0^{\frac{\pi}{4}} 4\sin x\sqrt{\cos 2x}\ dx = 2 - \sqrt{2}\ln(1+\sqrt{2})$

START WITH TRIGONOMETRIC IDENTITIES

$\Rightarrow I = \int_0^{\frac{\pi}{4}} 4\sin x\sqrt{2\cos^2 x - 1}\ dx$ ← SUBSTITUTION NEXT

$u = \cos x$

$\frac{du}{dx} = -\sin x$

$dx = \frac{du}{-\sin x}$

AND THE LIMITS

$x = 0 \mapsto u =$

$x = \frac{\pi}{4} \mapsto u = \frac{\sqrt{2}}{2}$

$\Rightarrow I = \int_1^{\frac{\sqrt{2}}{2}} 4\sin x\sqrt{2u^2 - 1}\ \frac{du}{-\sin x}$

$\Rightarrow I = \int_{\frac{\sqrt{2}}{2}}^1 4\sqrt{2u^2-1}\ du$

NEXT CONTINUE WITH A TRIGONOMETRIC (OR HYPERBOLIC) SUBSTITUTION;

$\sec\theta = \sqrt{2}u$

$\sec\theta\tan\theta\, d\theta = \sqrt{2}\, du$

$du = \frac{\sec\theta\tan\theta}{\sqrt{2}}\, d\theta$

$u = 1 \mapsto \frac{\pi}{4}$

$u = \frac{\sqrt{2}}{2} \mapsto \theta = 0$

$\Rightarrow I = \int_0^{\frac{\pi}{4}} 4\sqrt{\sec^2\theta - 1}\ \frac{\sec\theta\tan\theta}{\sqrt{2}}\, d\theta$

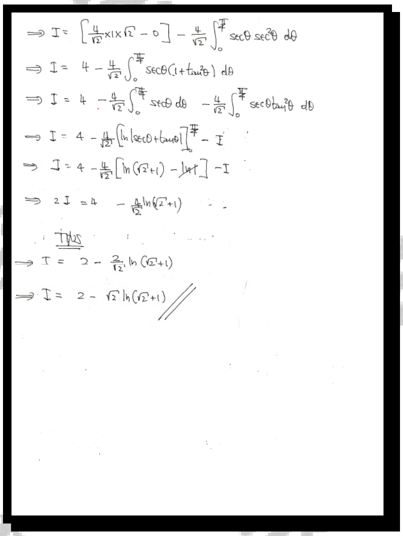
$\Rightarrow I = \int_0^{\frac{\pi}{4}} \frac{4}{\sqrt{2}}\tan\theta\,(\sec\theta\tan\theta)\, d\theta$

$\Rightarrow I = \int_0^{\frac{\pi}{4}} \frac{4}{\sqrt{2}}\tan^2\theta\sec\theta$

AS THERE IS NO OBVIOUS IDENTITY OR SUBSTITUTION PROCEED BY PARTS

$\frac{4}{\sqrt{2}}\tan\theta$	$\frac{4}{\sqrt{2}}\sec^2\theta$
$\sec\theta$	$\sec\theta\tan\theta$

$\Rightarrow I = \left[\frac{4}{\sqrt{2}}\tan\theta\sec\theta\right]_0^{\frac{\pi}{4}} - \frac{4}{\sqrt{2}}\int_0^{\frac{\pi}{4}} \sec^3\theta\, d\theta$



$\Rightarrow I = \left[\frac{4}{\sqrt{2}} \times 1 \times \sqrt{2} - 0\right] - \frac{4}{\sqrt{2}}\int_0^{\frac{\pi}{4}} \sec\theta\sec^2\theta\, d\theta$

$\Rightarrow I = 4 - \frac{4}{\sqrt{2}}\int_0^{\frac{\pi}{4}} \sec\theta(1+\tan^2\theta)\, d\theta$

$\Rightarrow I = 4 - \frac{4}{\sqrt{2}}\int_0^{\frac{\pi}{4}} \sec\theta\, d\theta - \frac{4}{\sqrt{2}}\int_0^{\frac{\pi}{4}} \sec\theta\tan^2\theta\, d\theta$

$\Rightarrow I = 4 - \frac{4}{\sqrt{2}}\left[\ln|\sec\theta + \tan\theta|\right]_0^{\frac{\pi}{4}} - I$

$\Rightarrow I = 4 - \frac{4}{\sqrt{2}}\left[\ln(\sqrt{2}+1) - \ln 1\right] - I$

$\Rightarrow 2I = 4 - \frac{4}{\sqrt{2}}\ln(\sqrt{2}+1)$

THUS

$\Rightarrow I = 2 - \frac{2}{\sqrt{2}}\ln(\sqrt{2}+1)$

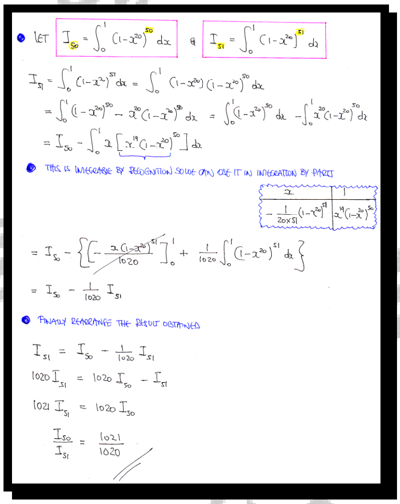
$\Rightarrow I = 2 - \sqrt{2}\ln(\sqrt{2}+1)$ //

Question 327 (*****)

Find as an exact fraction the value of I,

$$I = \frac{\displaystyle\int_0^1 \left(1-x^{20}\right)^{50}\ dx}{\displaystyle\int_0^1 \left(1-x^{20}\right)^{51}\ dx}.$$

$\boxed{}$, $\boxed{\dfrac{1021}{1020}}$


LET $I_{50} = \int_0^1 (1-x^{20})^{50}\,dx$ & $I_{51} = \int_0^1 (1-x^{20})^{51}\,dx$

$I_{51} = \int_0^1 (1-x^{20})^{51}\,dx = \int_0^1 (1-x^{20})(1-x^{20})^{50}\,dx$

$= \int_0^1 (1-x^{20})^{50} - x^{20}(1-x^{20})^{50}\,dx = \int_0^1 (1-x^{20})^{50}\,dx - \int_0^1 x^{20}(1-x^{20})^{50}\,dx$

$= I_{50} - \int_0^1 x\left[x^{19}(1-x^{20})^{50}\right]dx$

THIS IS INTEGRABLE BY RECOGNITION SO WE CAN USE IT IN INTEGRATION BY PARTS

$= I_{50} - \left\{\left[-\frac{x(1-x^{20})^{51}}{1020}\right]_0^1 + \frac{1}{1020}\int_0^1 (1-x^{20})^{51}\,dx\right\}$

$= I_{50} - \frac{1}{1020} I_{51}$

FINALLY REARRANGE THE RESULT OBTAINED

$I_{51} = I_{50} - \frac{1}{1020} I_{51}$

$1020 I_{51} = 1020 I_{50} - I_{51}$

$1021 I_{51} = 1020 I_{50}$

$\frac{I_{50}}{I_{51}} = \frac{1021}{1020}$

Question 328 (*****)

The integral I is defined as

$$I = \int_0^{\pi} \frac{\sin^2 x}{1+\cos^2 x} \, dx .$$

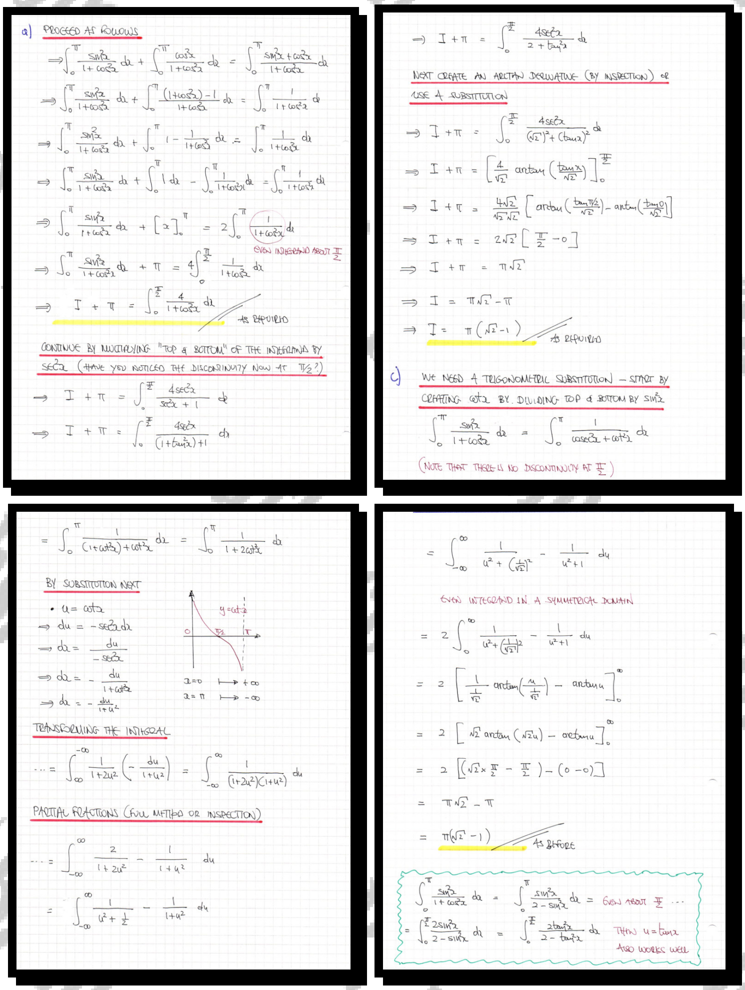
a) Show by a detailed method that

$$I + \pi = \int_o^{\frac{1}{2}\pi} \frac{4}{1+\cos^2 x} \, dx .$$

b) Hence, find the value of I in exact simplified form.

c) Verify the answer obtained in part **(b)** by an alternative method by first writing the integrand of I as a function of $\cot^2 x$.

[ ] , $I = \pi\left(\sqrt{2}-1\right)$

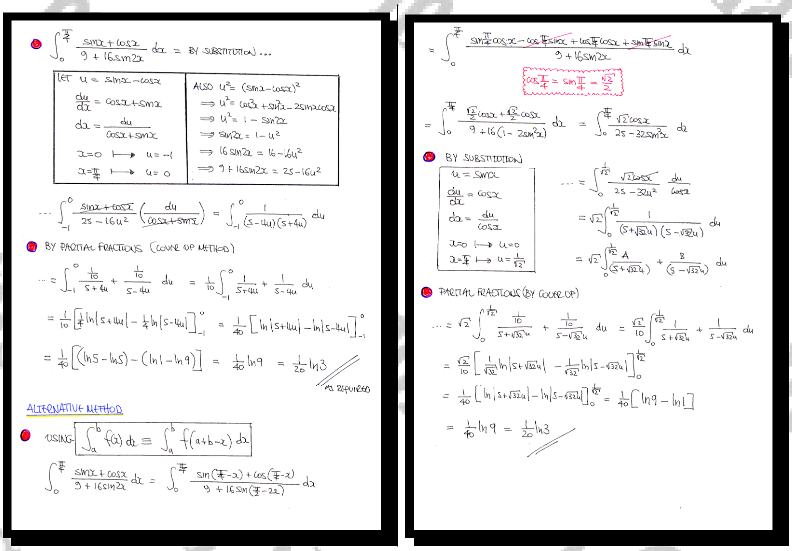


Question 329 (*****)

By using an appropriate substitution or substitutions, followed by partial fractions show that

$$\int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{9 + 16\sin 2x}\, dx = \frac{\ln 3}{20}.$$

, proof



Question 330 (*****)

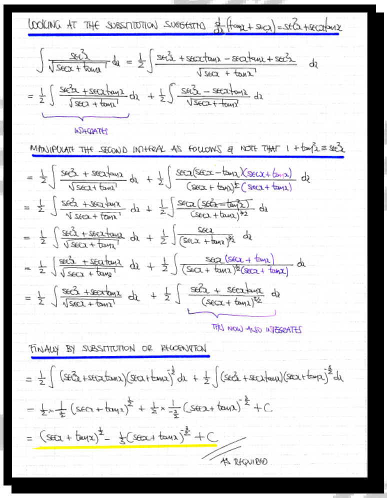
$$I = \int \frac{\sec^2 x}{\sqrt{\sec x + \tan x}}\, dx .$$

Without using a verification approach, show that

$$I = (\sec x + \tan x)^{\frac{1}{2}} - \frac{1}{3}(\sec x + \tan x)^{-\frac{3}{2}} + \text{ constant} .$$

You may consider the substitution $u = \sec x + \tan x$ useful at some stages in the manipulation of the integrand.

□, proof



LOOKING AT THE SUBSTITUTION SUGGESTED $\frac{d}{dx}(\tan x + \sec x) = \sec^2 x + \sec x \tan x$

$$\int \frac{\sec^2 x}{\sqrt{\sec x + \tan x}}\, dx = \frac{1}{2}\int \frac{\sec^2 x + \sec x \tan x - \sec x \tan x + \sec^2 x}{\sqrt{\sec x + \tan x}}\, dx$$

$$= \frac{1}{2}\int \frac{\sec^2 x + \sec x \tan x}{\sqrt{\sec x + \tan x}}\, dx + \frac{1}{2}\int \frac{\sec^2 x - \sec x \tan x}{\sqrt{\sec x + \tan x}}\, dx$$

INTEGRATES

MANIPULATE THE SECOND INTEGRAL AS FOLLOWS & NOTE THAT $1 + \tan^2 x \equiv \sec^2 x$

$$= \frac{1}{2}\int \frac{\sec^2 x + \sec x \tan x}{\sqrt{\sec x + \tan x}}\, dx + \frac{1}{2}\int \frac{\sec x(\sec x - \tan x)(\sec x + \tan x)}{(\sec x + \tan x)^{\frac{1}{2}}(\sec x + \tan x)}\, dx$$

$$= \frac{1}{2}\int \frac{\sec^2 x + \sec x \tan x}{\sqrt{\sec x + \tan x}}\, dx + \frac{1}{2}\int \frac{\sec x(\sec^2 x - \tan^2 x)}{(\sec x + \tan x)^{3/2}}\, dx$$

$$= \frac{1}{2}\int \frac{\sec^2 x + \sec x \tan x}{\sqrt{\sec x + \tan x}}\, dx + \frac{1}{2}\int \frac{\sec x}{(\sec x + \tan x)^{3/2}}\, dx$$

$$= \frac{1}{2}\int \frac{\sec^2 x + \sec x \tan x}{\sqrt{\sec x + \tan x}}\, dx + \frac{1}{2}\int \frac{\sec x(\sec x + \tan x)}{(\sec x + \tan x)^{3/2}(\sec x + \tan x)}\, dx$$

$$= \frac{1}{2}\int \frac{\sec^2 x + \sec x \tan x}{\sqrt{\sec x + \tan x}}\, dx + \frac{1}{2}\int \frac{\sec^2 x + \sec x \tan x}{(\sec x + \tan x)^{5/2}}\, dx$$

THIS NOW ALSO INTEGRATES

FINALLY BY SUBSTITUTION OR RECOGNITION

$$= \frac{1}{2}\int (\sec^2 x + \sec x \tan x)(\sec x + \tan x)^{-\frac{1}{2}}\, dx + \frac{1}{2}\int (\sec^2 x + \sec x \tan x)(\sec x + \tan x)^{-\frac{5}{2}}\, dx$$

$$= \frac{1}{2} \times \frac{1}{\frac{1}{2}}(\sec x + \tan x)^{\frac{1}{2}} + \frac{1}{2} \times \frac{1}{-\frac{3}{2}}(\sec x + \tan x)^{-\frac{3}{2}} + C$$

$$= (\sec x + \tan x)^{\frac{1}{2}} - \frac{1}{3}(\sec x + \tan x)^{-\frac{3}{2}} + C$$

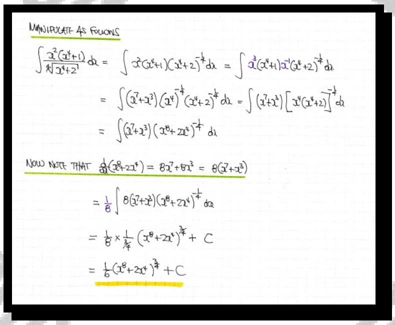
AS REQUIRED

Question 331 (*****)

Use an appropriate integration method to determine an antiderivative for the following indefinite integral.

$$\int \frac{x^2\left(x^4+1\right)}{\sqrt[4]{x^4+1}}\,dx \ .$$

V, $\square$, $\boxed{\frac{1}{6}\left(x^8+2x^4\right)^{\frac{1}{2}}+C}$



MANIPULATE AS FOLLOWS

$$\int \frac{x^2(x^4+1)}{\sqrt[4]{x^4+2}}\,dx = \int x^2(x^4+1)(x^4+2)^{-\frac{1}{4}}\,dx = \int x^3(x^4+1)x^{-1}(x^4+2)^{-\frac{1}{4}}\,dx$$

$$= \int (x^7+x^3)(x^4)^{-\frac{1}{4}}(x^4+2)^{-\frac{1}{4}}\,dx = \int (x^7+x^3)\left[x^4(x^4+2)\right]^{-\frac{1}{4}}\,dx$$

$$= \int (x^7+x^3)(x^8+2x^4)^{-\frac{1}{4}}\,dx$$

NOW NOTE THAT $\frac{d}{dx}(x^8+2x^4) = 8x^7+8x^3 = 8(x^7+x^3)$

$$= \frac{1}{8}\int 8(x^7+x^3)(x^8+2x^4)^{-\frac{1}{4}}\,dx$$

$$= \frac{1}{8}\times\frac{1}{\frac{3}{4}}(x^8+2x^4)^{\frac{3}{4}} + C$$

$$= \frac{1}{6}(x^8+2x^4)^{\frac{3}{4}} + C$$

Question 332 (*****)

Use partial fractions followed by integration by parts to show that

$$\int_0^{\infty}\left[\frac{x^2+3x+3}{(x+1)^3}\right]\mathrm{e}^{-x}\sin x \; dx = \frac{1}{2}.$$

, proof