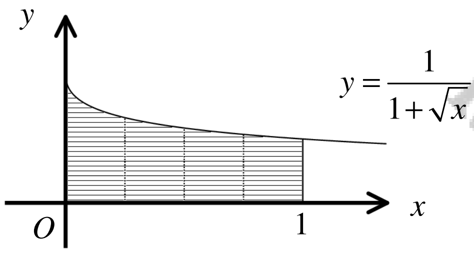
# NUMERICAL INTEGRATION

# THE TRAPEZIUM RULE

**Question 1 (**)**




The figure above shows part of the curve $C$ with equation

$$y = \frac{1}{1+\sqrt{x}}, \quad x \geq 0 .$$

It is required to estimate the area of the shaded region which is bounded by $C$, the coordinate axes and the straight line with equation $x = 1$.

Use the trapezium rule with 4 equally spaced strips to estimate the area of this region, giving the answer correct to 3 decimal places.

$\boxed{\approx 0.635}$

$y = \dfrac{1}{1+\sqrt{x}}$

| $x$ | 0 | 0.25 | 0.5 | 0.75 | 1 |
|---|---|---|---|---|---|
| $y$ | 1 | 0.6667 | 0.5858 | 0.5359 | 0.5 |

AREA $\approx \dfrac{\text{THICKNESS}}{2}\left[\text{FIRST} + \text{LAST} + 2 \times \text{REST}\right]$

AREA $\approx \dfrac{0.25}{2}\left[1 + 0.5 + 2(0.6667 + 0.5858 + 0.5359)\right] \approx 0.635$

**Question 2 (**)**

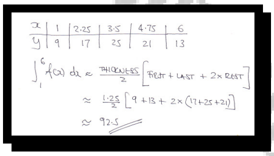
The values of $y$, for a curve with equation $y = f(x)$, have been tabulated below.

| $x$ | 1 | 2.25 | 3.5 | 4.75 | 6 |
|---|---|---|---|---|---|
| $y$ | 9 | 17 | 25 | 21 | 13 |

Use the trapezium rule with all the values from the above table to find an estimate for the integral

$$\int_1^6 f(x)\ dx .$$

$\boxed{\phantom{00}}$, $\boxed{92.5}$



| $x$ | 1 | 2.25 | 3.5 | 4.75 | 6 |
|---|---|---|---|---|---|
| $y$ | 9 | 17 | 25 | 21 | 13 |

$$\int_1^6 f(x)\,dx \approx \frac{\text{THICKNESS}}{2}\left[\text{FIRST} + \text{LAST} + 2\times\text{REST}\right]$$

$$\approx \frac{1.25}{2}\left[9 + 13 + 2\times(17+25+21)\right]$$

$$\approx 92.5$$

**Question 3** (**)

The $y$ values, for the curve with equation $y = \sqrt{x^3 - x}$, have been tabulated below.

| $x$ | 1 | 1.5 | 2 | 2.5 | 3 | 3.5 | 4 |
|---|---|---|---|---|---|---|---|
| $y$ | 0 | 1.369 | 2.449 | 3.623 | | | 7.746 |

**a)** Complete the table.

**b)** Use the trapezium rule with all the values from the table above to find an estimate, correct to 2 decimal places, for the integral

$$\int_1^4 \sqrt{x^3 - x}\ dx .$$

$\boxed{\phantom{0}}$, $\boxed{4.899}$, $\boxed{6.275}$, $\boxed{\approx 11.24}$

(a)

| $x$ | 1 | 1.5 | 2 | 2.5 | 3 | 3.5 | 4 |
|---|---|---|---|---|---|---|---|
| $y$ | 0 | 1.369 | 2.449 | 3.623 | 4.899 | 6.275 | 7.746 |

$y = \sqrt{x^3 - x}$

$$\int_1^4 \sqrt{x^3 - x}\ dx \approx \frac{\text{THICKNESS}}{2}\left[\text{FIRST} + \text{LAST} + 2 \times \text{REST}\right]$$

$$\approx \frac{0.5}{2}\left[0 + 7.746 + 2(1.369 + 2.449 + 3.623 + 4.899 + 6.275)\right]$$

$$\approx 11.24$$

**Question 4** **(**+)**

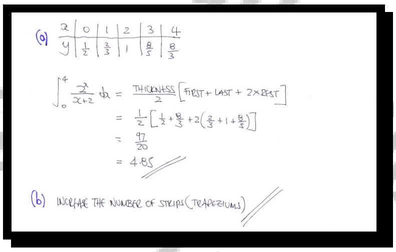
**a)** Use the trapezium rule with five equally spaced ordinates (four strips) to find the value of

$$\int_{0}^{4} \frac{2^{x}}{x+2}\, dx ,$$

giving the answer correct to three significant figures.

**b)** State how a better approximation to the value of the integral can be obtained using the trapezium rule.

$\boxed{\phantom{00}}$ , $\boxed{4.85}$

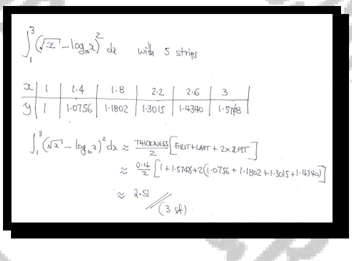


**Question 5** **(**+)**

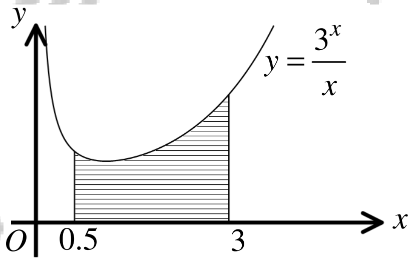
$$I = \int_{1}^{3} \left(\sqrt{x} - \log_{10} x\right)^{2}\, dx .$$

Use the trapezium rule with 5 equally spaced strips to find an estimate for $I$.

$\boxed{\phantom{00}}$ , $\boxed{\approx 2.51}$



**Question 6 (**+)**




The figure above shows part of the curve $C$ with equation

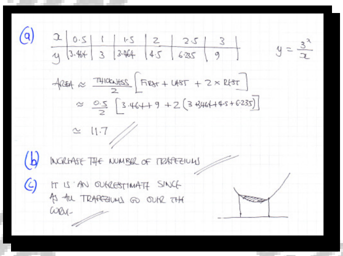
$$y = \frac{3^x}{x}, \quad x \neq 0.$$

**a)** Use the trapezium rule with 5 equally spaced strips to estimate, to three significant figures, the area bounded by $C$, the $x$ axis and the straight lines with equations $x = 0.5$ and $x = 3$.

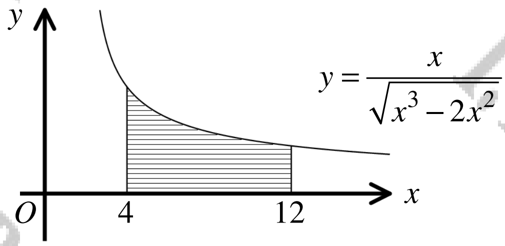
**b)** State how the accuracy of the estimate obtained in part (a) can be improved.

**c)** Explain with the aid of a diagram whether the estimate obtained in part (a) is an underestimate or an overestimate for the actual value for this area.

$\boxed{\phantom{xx}}$, $\boxed{\approx 11.7}$



**Question 7 (**+)**




The figure above shows part of the curve $C$ with equation

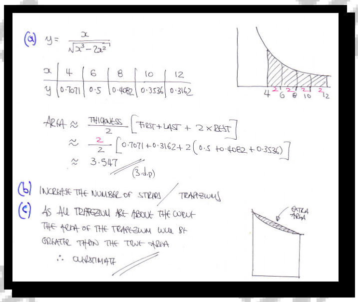
$$y = \frac{x}{\sqrt{x^3 - 2x^2}}.$$

**a)** Use the trapezium rule with 4 equally spaced strips to estimate, to three significant figures, the area bounded by $C$, the $x$ axis and the vertical straight lines with equations $x = 4$ and $x = 12$.

**b)** State how the estimate obtained in part **(a)** can be improved.

**c)** Explain with the aid of a diagram whether the estimate obtained in part **(a)** is an underestimate or an overestimate for the actual value of this area.

$\boxed{\phantom{0}}$, $\boxed{\approx 3.547}$



**Question 8** **(**+)**

$$I = \int_0^{\frac{\pi}{3}} \sqrt{\tan x} \; dx$$

Use the trapezium rule with 4 equally spaced strips to find an estimate for $I$.

$$\boxed{\approx 0.768}$$

$\int_0^{\frac{\pi}{3}} \sqrt{\tan x}\, dx$ with 4 strips

| $x$ | $0$ | $\frac{\pi}{12}$ | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ |
|---|---|---|---|---|---|
| $y$ | $0$ | $0.5176$ | $0.7598$ | $1$ | $1.3161$ |

! MAKE SURE YOU ARE IN RADIANS !

$$\int_0^{\frac{\pi}{3}} \sqrt{\tan x}\, dx \approx \frac{\text{THICKNESS}}{2}\left[\text{FIRST} + \text{LAST} + 2 \times \text{REST}\right]$$

$$\approx \frac{\pi/12}{2}\left[0 + 1.3161 + 2\left[0.5176 + 0.7598 + 1\right]\right]$$

$$\approx 0.768 \quad (3\text{ d.p})$$

**Question 9** **(**+)**

$$I = \int_0^{1} \sqrt{1+\sin x} \; dx$$

Use the trapezium rule with 4 equally spaced strips to estimate the approximate value of $I$, giving the answer correct to 3 decimal places

$$\boxed{\approx 1.202}$$

$y = \sqrt{1+\sin x}$

| $x$ | $0$ | $0.25$ | $0.5$ | $0.75$ | $1$ |
|---|---|---|---|---|---|
| $y$ | $1$ | $1.1169$ | $1.2163$ | $1.2969$ | $1.3570$ |

$$\int_0^{1} \sqrt{1+\sin x}\, dx \approx \frac{\text{THICKNESS}}{2}\left[\text{FIRST} + \text{LAST} + 2(\text{REST})\right]$$

$$\approx \frac{0.25}{2}\left[1 + 1.3570 + 2(1.1169 + 1.2163 + 1.2969)\right]$$

$$\approx 1.202$$

**Question 10** (**+)

$$I = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \text{cosec}\, x \ \ dx$$

Use the trapezium rule with 4 equally spaced strips to find an estimate for $I$ .

$\boxed{\phantom{xx}}$, $\boxed{\approx 1.34}$

| $x$ | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{5\pi}{12}$ | $\frac{\pi}{2}$ |
|---|---|---|---|---|---|
| $y$ | $2$ | $\sqrt{2}$ | $\frac{2}{3}\sqrt{3}$ | $\sqrt{6}-\sqrt{2}$ | $1$ |

$\frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$

$\frac{\pi}{3} \div 4 = \frac{\pi}{12}$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \text{cosec}\, x \, dx = \frac{\text{THICKNESS}}{2}\left[\text{FIRST} + \text{LAST} + 2 \times \text{REST}\right]$$

$$\approx \frac{\pi/12}{2}\left[2 + 1 + 2\left[\cancel{\sqrt{2}} + \tfrac{2}{3}\sqrt{3} + \sqrt{6} - \cancel{\sqrt{2}}\right]\right]$$

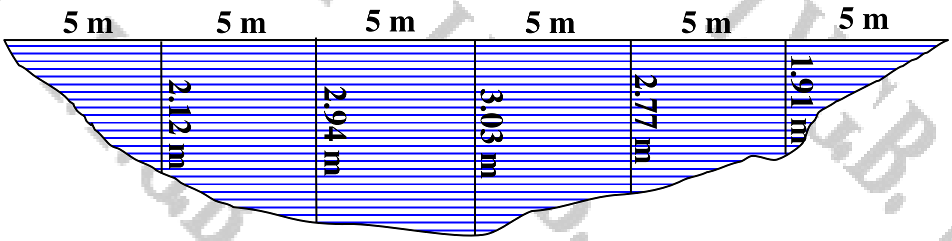
$$\simeq \frac{\pi}{24}\left[3 + \tfrac{4}{3}\sqrt{3} + 2\sqrt{6}\right]$$

$$\simeq 1.33627\ldots$$

$$\simeq 1.34$$

**Question 11** (***)

The figure below shows the cross section of a river.



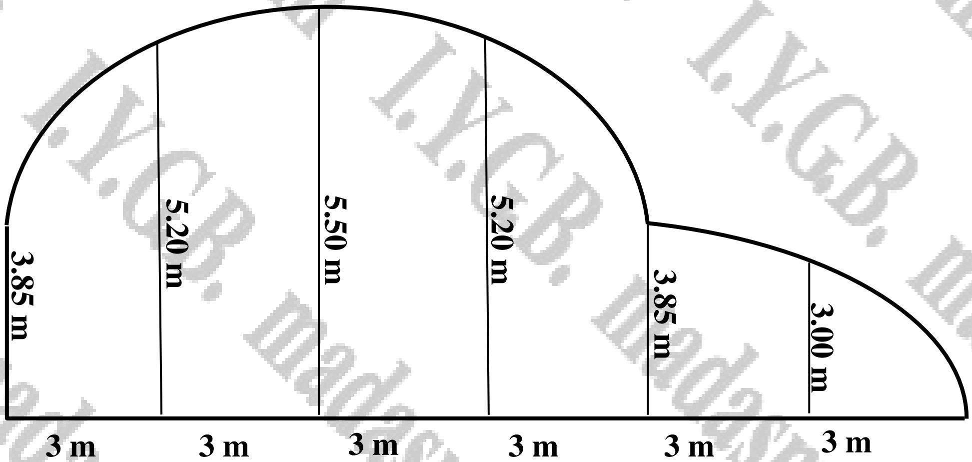

The depth of the river, in metres, from one river bank directly across to the other river bank, is recorded at 5 metre intervals.

Estimate the cross sectional area of the river, by using the trapezium rule with all the measurements provided in the above figure.

$\boxed{\phantom{00}}$, $\boxed{\approx 63.85 \text{ m}^2}$

$$\text{AREA} \simeq \frac{\text{THICKNESS}}{2}\left[\text{FIRST} + \text{LAST} + 2 \times \text{REST}\right]$$
$$\simeq \frac{5}{2}\left[0 + 0 + 2(2.12 + 2.94 + 3.03 + 2.77 + 1.91)\right]$$
$$\simeq 63.85$$

**Question 12** (***)




The figure above shows the cross section of a tunnel.

The height of the tunnel, in metres, from one end directly across to the other end, is recorded at $3$ metre intervals.

Use the trapezium rule to estimate the cross sectional area of the tunnel.

$\boxed{\approx 74 \text{ m}^2}$

$$\text{AREA} \simeq \frac{\text{THICKNESS}}{2}\left[\text{FIRST} + \text{LAST} + 2 \times \text{REST}\right]$$
$$\simeq \frac{3}{2}\left[3.85 + 0 + 2(5.20 + 5.50 + 5.20 + 3.85 + 3.00)\right]$$
$$\simeq 74.0$$

**Question 13** (***+)

**a)** Use the trapezium rule with 4 equally spaced strips to find an estimate for

$$\int_0^1 e^{-x^2}\, dx.$$

**b)** Use the answer of part **(a)** to find an estimate for

$$\int_0^1 e^{-x^2+3}\, dx.$$

$\boxed{\phantom{00}}$ , $\boxed{\approx 0.743}$ , $\boxed{\approx 14.92}$

(a)

| $x$ | 0 | 0.25 | 0.5 | 0.75 | 1 |
|---|---|---|---|---|---|
| $y$ | 1 | 0.9394 | 0.7788 | 0.5698 | 0.3679 |

$$\int_0^1 e^{-x^2}\,dx \simeq \frac{\text{THICKNESS}}{2}\left[\text{FIRST} + \text{LAST} + 2\times \text{REST}\right]$$
$$\simeq \frac{0.25}{2}\left[1 + 0.3679 + 2(0.9394 + 0.7788 + 0.5698)\right]$$
$$\simeq 0.74299\ldots \simeq 0.743$$

(b) $$\int_0^1 e^{-x^2+3}\,dx = \int_0^1 e^{-x^2}\times e^3\,dx = e^3\int_0^1 e^{-x^2}\,dx$$
$$\approx e^3 \times 0.74299 \simeq 14.92$$

**Question 14** (***+)

**a)** Use the trapezium rule with 5 equally spaced ordinates to estimate the value of the following integral.

$$\int_{2}^{18} \ln\left[\frac{2}{\sqrt{4+\sqrt{x}}}\right] dx .$$

**b)** Use the answer of part **(a)** to estimate the value of

$$\int_{2}^{18} \ln\left(4+\sqrt{x}\right) dx .$$

$\boxed{\phantom{xxxx}}$, $\boxed{\approx -4.467}$, $\boxed{\approx 31.1}$

a) START BY FILLING A TABLE

| $x$ | 2 | 6 | 10 | 14 | 18 |
|---|---|---|---|---|---|
| $\ln\left[\frac{2}{\sqrt{4+\sqrt{x}}}\right]$ | $-0.1514$ | $-0.2389$ | $-0.2913$ | $-0.3302$ | $-0.3615$ |
| | FIRST | REST | REST | REST | LAST |

BY THE TRAPEZIUM RULE

$$\int_{2}^{18} \ln\left[\frac{2}{\sqrt{4+\sqrt{x}}}\right] dx \simeq \frac{\text{THICKNESS}}{2}\left[\text{FIRST} + \text{LAST} + 2\sum(\text{REST})\right]$$

$$\simeq \frac{4}{2}\left[-0.1514 - 0.3615 + 2(-0.2389 - 0.2913 - 0.3302)\right]$$

$$\simeq -4.467$$

b) PROCEED AS FOLLOWS

$$\Rightarrow \int_{2}^{18} \ln\left[\frac{2}{\sqrt{4+\sqrt{x}}}\right] dx = \int_{2}^{18} \ln 2 - \ln(4+\sqrt{x})^{\frac{1}{2}}\, dx$$

$$\Rightarrow -4.467 = (\ln 2)\int_{2}^{18} 1\, dx - \int_{2}^{18} \ln(4+\sqrt{x})^{\frac{1}{2}}\, dx$$

$$\Rightarrow -4.467 = (\ln 2)\left[x\right]_{2}^{18} - \frac{1}{2}\int_{2}^{18} \ln(4+\sqrt{x})\, dx$$

$$\Rightarrow -4.467 = 16\ln 2 - \frac{1}{2}\int_{2}^{18} \ln(4+\sqrt{x})\, dx$$

$$\Rightarrow \frac{1}{2}\int_{2}^{18} \ln(4+\sqrt{x})\, dx = 4.467 + 16\ln 2$$

$$\Rightarrow \int_{2}^{18} \ln(4+\sqrt{x})\, dx = 2(4.467 + 16\ln 2)$$

$$\Rightarrow \int_{2}^{18} \ln(4+\sqrt{x})\, dx \approx 31.1$$

**Question 15** (***+)

**a)** Use the trapezium rule with 5 equally spaced ordinates to estimate the value of the following integral.

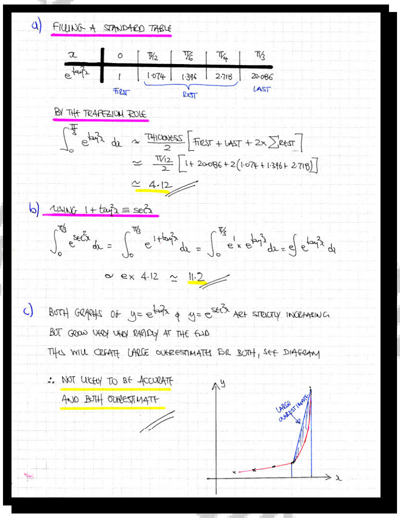
$$\int_0^{\frac{1}{3}\pi} e^{\tan^2 x} \, dx \, .$$

**b)** Use the answer of part **(a)** to estimate the value of

$$\int_0^{\frac{1}{3}\pi} e^{\sec^2 x} \, dx \, .$$

**c)** Discuss briefly whether the estimates of the previous parts of the question are likely to be accurate, stating further whether they are overestimates or underestimates to the true values of these integrals.

$\boxed{\phantom{xxxx}}$, $\boxed{\approx 4.12}$, $\boxed{\approx 11.2}$



a) FILLING A STANDARD TABLE

| $x$ | $0$ | $\pi/12$ | $\pi/6$ | $\pi/4$ | $\pi/3$ |
|---|---|---|---|---|---|
| $e^{\tan^2 x}$ | 1 | 1.074 | 1.396 | 2.718 | 20.086 |
| | FIRST | REST | | | LAST |

BY THE TRAPEZIUM RULE

$$\int_0^{\frac{\pi}{3}} e^{\tan^2 x}\,dx \approx \frac{\text{THICKNESS}}{2}\left[\text{FIRST} + \text{LAST} + 2 \times \sum \text{REST}\right]$$

$$\approx \frac{\pi/12}{2}\left[1 + 20.086 + 2(1.074 + 1.396 + 2.718)\right]$$

$$\simeq 4.12$$

b) USING $1 + \tan^2 x \equiv \sec^2 x$

$$\int_0^{\pi/3} e^{\sec^2 x}\,dx = \int_0^{\pi/3} e^{1+\tan^2 x}\,dx = \int_0^{\pi/3} e^1 \times e^{\tan^2 x}\,dx = e\int e^{\tan^2 x}\,dx$$

$$\approx e \times 4.12 \simeq 11.2$$

c) BOTH GRAPHS OF $y = e^{\tan^2 x}$ & $y = e^{\sec^2 x}$ ARE STRICTLY INCREASING

BUT GROW VERY VERY RAPIDLY AT THE END

THIS WILL CREATE LARGE OVERESTIMATES FOR BOTH, SEE DIAGRAM

$\therefore$ NOT LIKELY TO BE ACCURATE

AND BOTH OVERESTIMATE

**Question 16** (****)

**a)** Use the trapezium rule with $4$ equally spaced strips to find an estimate for

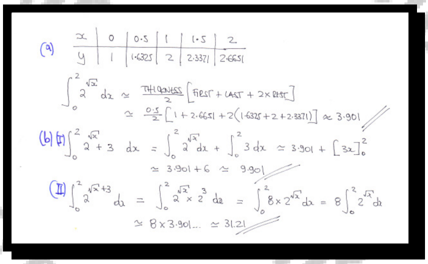
$$\int_0^2 2^{\sqrt{x}}\ dx .$$

**b)** Use the answer of part (**a**) to find estimates for …

**i.** … $\displaystyle\int_0^2 2^{\sqrt{x}} + 3\ dx .$

**ii.** … $\displaystyle\int_0^2 2^{\sqrt{x}+3}\ dx .$

$\boxed{\text{[illegible]}}$ , $\boxed{\approx 3.901}$ , $\boxed{\approx 9.901}$ , $\boxed{\approx 31.21}$



(a)

| $x$ | 0 | 0.5 | 1 | 1.5 | 2 |
|---|---|---|---|---|---|
| $y$ | 1 | 1.6325 | 2 | 2.3371 | 2.6651 |

$\displaystyle\int_0^2 2^{\sqrt{x}}\,dx \simeq \frac{\text{THICKNESS}}{2}\left[\text{FIRST} + \text{LAST} + 2\times\text{REST}\right]$

$\simeq \frac{0.5}{2}\left[1 + 2.6651 + 2(1.6325 + 2 + 2.3371)\right] \simeq 3.901$

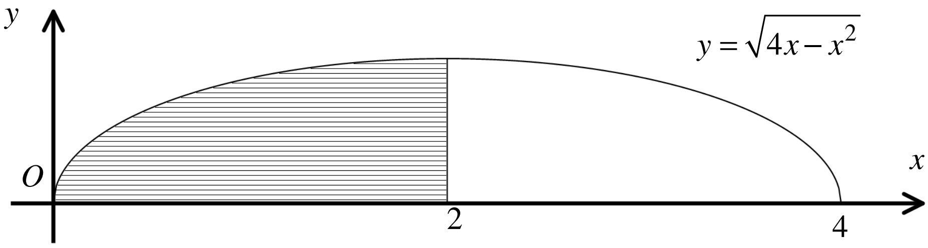
(b) (I) $\displaystyle\int_0^2 2^{\sqrt{x}} + 3\ dx = \int_0^2 2^{\sqrt{x}}\,dx + \int_0^2 3\,dx \simeq 3.901 + \left[3x\right]_0^2$

$\simeq 3.901 + 6 \simeq 9.901$

(II) $\displaystyle\int_0^2 2^{\sqrt{x}+3}\,dx = \int_0^2 2^{\sqrt{x}}\times 2^3\,dx = \int_0^2 8\times 2^{\sqrt{x}}\,dx = 8\int_0^2 2^{\sqrt{x}}\,dx$

$\simeq 8\times 3.901\ldots \simeq 31.21$

**Question 17** (****)




The figure above shows part of the curve $C$ with equation

$$y=\sqrt{4x-x^2}\,.$$

**a)** Use the trapezium rule with 5 equally spaced trapeziums to estimate, to three significant figures, the area bounded by $C$, the $x$ axis and the vertical straight line with equation $x=2$.

**b)** Hence find an estimate for

$$\int_0^2 3+\sqrt{4x-x^2}\;dx\,.$$

**a)** State, with justification, whether the answer of part **(a)** will increase or decrease if more than 5 trapeziums are used.

$\boxed{\phantom{0}}$, $\boxed{\approx 3.04}$, $\boxed{\approx 9.04}$

a)

| $x$ | 0 | 0.4 | 0.8 | 1.2 | 1.6 | 2 |
|---|---|---|---|---|---|---|
| $y$ | 0 | 1.2 | 1.6 | 1.8330 | 1.9596 | 2 |

AREA $\approx \frac{\text{THICKNESS}}{2}\left[\text{FIRST} + \text{LAST} + 2\times\text{REST}\right]$

AREA $\approx \frac{0.4}{2}\left[0+2+2(1.2+1.6+1.8330+1.9596)\right]$

AREA $\approx$ 3.03704...

AREA $\approx$ 3.04 (3 s.f.)

b) $\int_0^2 3+\sqrt{4x-x^2}\,dx$ = REPRESENT THE SAME CURVE TRANSLATED "UP" BY 3

I.E ≈ 3.04, 3, 2

∴ APPROXIMATE VALUE $6+3.04\approx 9.04$

c) THE SHAPE OF THE CURVE IS SUCH SO THAT THE TRAPEZIUMS "STAY UNDER"

THE TRAPEZIUM RULE UNDERESTIMATES, SO INCREASING THE NUMBER OF TRAPEZIUMS, WILL INCREASE THE VALUE

**Question 18** (****)

**a)** **Use the trapezium rule** with 4 equally spaced strips to find an estimate for

$$\int_0^{\frac{\pi}{3}} \cos^2 x \ dx .$$

**b)** **Use the answer of part (a)** to find an estimate for

$$\int_0^{\frac{\pi}{3}} \sin^2 x \ dx .$$

$\boxed{\phantom{0.000}}$ , $\boxed{\approx 0.735}$ , $\boxed{\approx 0.312}$

(a)

| $x$ | $0$ | $\frac{\pi}{12}$ | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ |
|---|---|---|---|---|---|
| $y$ | $1$ | $\frac{2+\sqrt{3}}{4}$ | $\frac{3}{4}$ | $\frac{1}{2}$ | $\frac{1}{4}$ |

$$\int_0^{\frac{\pi}{3}} \cos^2 x \, dx \simeq \frac{\text{THICKNESS}}{2}\left[\text{FIRST} + \text{LAST} + 2 \times \text{REST}\right]$$

$$\simeq \frac{\pi/12}{2}\left[1 + \frac{1}{4} + 2\left(\frac{2+\sqrt{3}}{4} + \frac{3}{4} + \frac{1}{2}\right)\right] \simeq 0.735$$

(b)

$$\int_0^{\frac{\pi}{3}} \sin^2 x \, dx = \int_0^{\frac{\pi}{3}} 1 - \cos^2 x \, dx = \int_0^{\frac{\pi}{3}} 1 \, dx - \int_0^{\frac{\pi}{3}} \cos^2 x \, dx$$

$$= \left[x\right]_0^{\frac{\pi}{3}} - \int_0^{\frac{\pi}{3}} \cos^2 x \, dx \approx \frac{\pi}{3} - 0.735\ldots \simeq 0.312$$

**Question 19** **(****)**

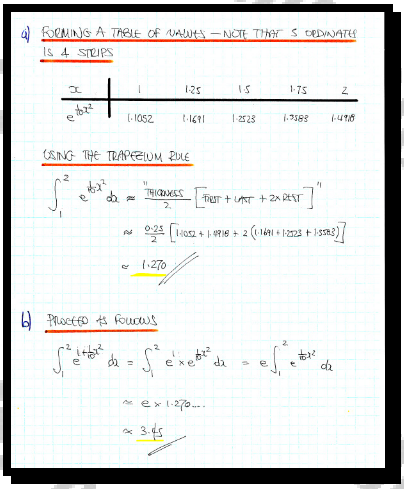
**a)** Find an estimate for the following integral, by using the trapezium rule with 5 equally spaced **ordinates.** to for

$$\int_1^2 e^{\frac{1}{10}x^2}\, dx .$$

**b)** Use the answer of part **(a)** to find estimates for

$$\int_1^2 e^{1+\frac{1}{10}x^2}\, dx .$$

$\boxed{\phantom{000}}$ , $\boxed{\approx 1.270}$ , $\boxed{\approx 3.45}$



a) FORMING A TABLE OF VALUES – NOTE THAT 5 ORDINATES IS 4 STRIPS

| $x$ | 1 | 1.25 | 1.5 | 1.75 | 2 |
|---|---|---|---|---|---|
| $e^{\frac{1}{10}x^2}$ | 1.1052 | 1.1691 | 1.2523 | 1.3583 | 1.4918 |

USING THE TRAPEZIUM RULE

$$\int_1^2 e^{\frac{1}{10}x^2}\, dx \approx \text{"}\frac{\text{THICKNESS}}{2}\left[\text{FIRST} + \text{LAST} + 2\times\text{REST}\right]\text{"}$$

$$\approx \frac{0.25}{2}\left[1.1052 + 1.4918 + 2(1.1691 + 1.2523 + 1.3583)\right]$$

$$\approx 1.270$$

b) PROCEED AS FOLLOWS

$$\int_1^2 e^{1+\frac{1}{10}x^2}\, dx = \int_1^2 e^1 \times e^{\frac{1}{10}x^2}\, dx = e\int_1^2 e^{\frac{1}{10}x^2}\, dx$$

$$\approx e \times 1.270\ldots$$

$$\approx 3.45$$

**Question 20** (****)

a) **Use the trapezium rule** with 6 equally spaced strips to find an estimate, correct to 3 decimal places, for

$$\int_0^{1.2} \sin^2 x \, dx .$$

b) **Use the answer of part (a)** to find an estimate for

$$\int_0^{1.2} \cos 2x \, dx .$$

c) **Use the answer of part (b)** to find an estimate for

$$\int_0^{1.2} \left[\cos^4 x - \sin^4 x\right] dx .$$

$\boxed{\phantom{0.000}}$ , $\boxed{\approx 0.433}$ , $\boxed{\approx 0.334}$ , $\boxed{\approx 0.334}$

a) FORMING A TABLE OF VALUES

| $x$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 | 1.2 |
|---|---|---|---|---|---|---|---|
| $y = \sin^2 x$ | 0 | 0.0395 | 0.1516 | 0.3188 | 0.5146 | 0.7081 | 0.8687 |

USING THE TRAPEZIUM RULE

$$\int_0^{1.2} \sin^2 x \, dx \simeq \text{"}\frac{\text{THICKNESS}}{2}\left[\text{FIRST} + \text{LAST} + 2 \times \text{REST}\right]\text{"}$$

$$\simeq \frac{0.2}{2}\left[0 + 0.8687 + 2(0.0395 + 0.1516 + 0.3188 + 0.5146 + 0.7081)\right]$$

$$\simeq 0.433$$

b) $$\int_0^{1.2} \cos 2x \, dx = \int_0^{1.2} 1 - 2\sin^2 x \, dx$$

$$= \int_0^{1.2} 1 \, dx - 2\int_0^{1.2} \sin^2 x \, dx$$

$$\left[x\right]_0^{1.2} - 2 \times 0.433$$

$$\approx (1.2 - 0) - 0.866$$

$$\simeq 0.334$$

c) $$\int_0^{1.2} \cos^4 x - \sin^4 x \, dx = \int (\cos^2 x)^2 - (\sin^2 x)^2 \, dx$$

$$= \int_0^{1.2} (\cos^2 x - \sin^2 x)\cancel{(\cos^2 x + \sin^2 x)} \, dx$$

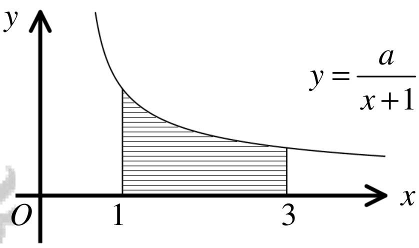
$\cos^4 x - \sin^4 x = (\cos^2 x)^2 - (\sin^2 x)^2 = \ldots$ difference of squares

$$= \int \cos^2 x - \sin^2 x \, dx$$

$$= \int \cos 2x \, dx$$

$$= 0.334$$

**Question 21** (****+)




The figure above shows part of the curve $C$ with equation

$$y = \frac{a}{x+1},$$

where $a$ is a positive integer.

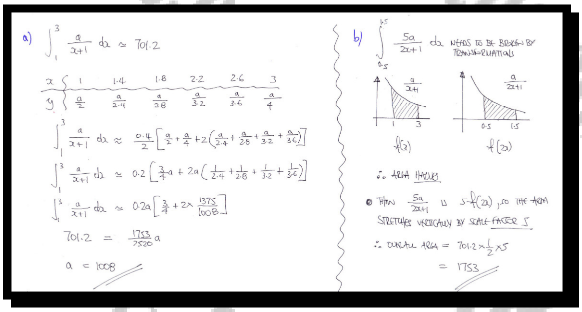
When the trapezium rule with 5 equally spaced strips is used, the area bounded by $C$, the $x$ axis and the vertical straight lines with equations $x = 1$ and $x = 3$, is approximated to $701.2$ square units.

**a)** Determine the value of $a$.

**b)** By considering suitable graph transformation, find an approximate value of

$$\int_{0.5}^{1.5} \frac{5a}{2x+1}\, dx .$$

$\boxed{\phantom{x}}$, $\boxed{a = 1008}$, $\boxed{\approx 1753}$



a) $\int_1^3 \frac{a}{x+1}\,dx \simeq 701.2$

| $x$ | 1 | 1.4 | 1.8 | 2.2 | 2.6 | 3 |
|---|---|---|---|---|---|---|
| $y$ | $\frac{a}{2}$ | $\frac{a}{2.4}$ | $\frac{a}{2.8}$ | $\frac{a}{3.2}$ | $\frac{a}{3.6}$ | $\frac{a}{4}$ |

$\int_1^3 \frac{a}{x+1}\,dx \approx \frac{0.4}{2}\left[\frac{a}{2} + \frac{a}{4} + 2\left(\frac{a}{2.4} + \frac{a}{2.8} + \frac{a}{3.2} + \frac{a}{3.6}\right)\right]$

$\int_1^3 \frac{a}{x+1}\,dx \simeq 0.2\left[\frac{3}{4}a + 2a\left(\frac{1}{2.4} + \frac{1}{2.8} + \frac{1}{3.2} + \frac{1}{3.6}\right)\right]$

$\int_1^3 \frac{a}{x+1}\,dx \simeq 0.2a\left[\frac{3}{4} + 2 \times \frac{1375}{1008}\right]$

$701.2 = \frac{1753}{2520}a$

$a = 1008$

b) $\int_{0.5}^{1.5} \frac{5a}{2x+1}\,dx$ NEEDS TO BE BROKEN BY TRANSFORMATIONS



$\therefore$ AREA HALVES

• THEN $\frac{5a}{2x+1}$ IS $5f(2x)$, SO THE AREA STRETCHES VERTICALLY BY SCALE FACTOR 5

$\therefore$ OVERALL AREA $= 701.2 \times \frac{1}{2} \times 5$

$= 1753$

**Question 22** (*****)

The trapezium rule with $n$ equally spaced intervals is to be used to estimate the value of the following integral

$$\int_0^1 2^x \, dx .$$

Show that the value of this estimate is given by

$$\frac{1}{2n}\left[\frac{2^{\frac{1}{n}}+1}{2^{\frac{1}{n}}-1}\right].$$

$\boxed{\phantom{SP}}$, $\boxed{\text{proof}}$

| $x$ | $0$ | $\frac{1}{n}$ | $\frac{2}{n}$ | $\frac{3}{n}$ | $\dots\dots$ | $\frac{n-1}{n}$ | $\frac{n}{n}=1$ |
|---|---|---|---|---|---|---|---|
| $y$ | $2^0$ | $2^{\frac{1}{n}}$ | $2^{\frac{2}{n}}$ | $2^{\frac{3}{n}}$ | $\dots\dots$ | $2^{\frac{n-1}{n}}$ | $2^1$ |

$$\int_0^1 2^x \, dx \approx \frac{\text{"THICKNESS"}}{2}\left[\text{FIRST} + \text{LAST} + 2\times\text{REST}\right]$$

$$\approx \frac{\frac{1}{n}}{2}\left[2^0+2^1+2\big(\underbrace{2^{\frac{1}{n}}+2^{\frac{2}{n}}+2^{\frac{3}{n}}+\dots+2^{\frac{n-1}{n}}}_{n-1 \text{ TERMS}}\big)\right]$$

$$\approx \frac{1}{2n}\left[1+2+2\left[\left(2^{\frac{1}{n}}\right)+\left(2^{\frac{1}{n}}\right)^2+\left(2^{\frac{1}{n}}\right)^3+\dots+\left(2^{\frac{1}{n}}\right)^{n-1}\right]\right]$$

$$\approx \frac{1}{2n}\left[3+2\times\frac{2^{\frac{1}{n}}\left[\left(2^{\frac{1}{n}}\right)^{n-1}-1\right]}{2^{\frac{1}{n}}-1}\right]$$

SUMMATION OF A G.P.

$$S_N = \frac{a(r^N-1)}{r-1}$$

$$\approx \frac{1}{2n}\left[3+2\times\frac{\left(2^{\frac{1}{n}}\right)^n-2^{\frac{1}{n}}}{2^{\frac{1}{n}}-1}\right]$$

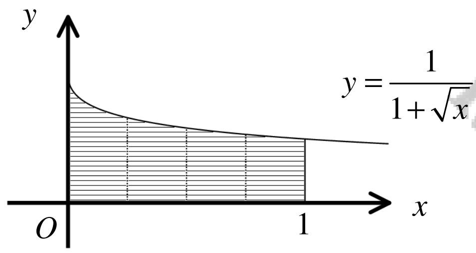
$$\approx \frac{1}{2n}\left[3+2\times\frac{2-2^{\frac{1}{n}}}{2^{\frac{1}{n}}-1}\right]$$

$$\approx \frac{1}{2n}\left[\frac{3\times2^{\frac{1}{n}}-3+4-2\times2^{\frac{1}{n}}}{2^{\frac{1}{n}}-1}\right]$$

$$\approx \frac{1}{2n}\left[\frac{2^{\frac{1}{n}}+1}{2^{\frac{1}{n}}-1}\right]$$

# SIMPSON'S RULE

**Question 1** (**)




The figure above shows part of the curve $C$ with equation

$$y = \frac{1}{1+\sqrt{x}}, \quad x \geq 0 .$$

It is required to estimate the area of the shaded region bounded by $C$, the coordinate axes and the straight line with equation $x = 1$.

Use Simpson's rule with 4 equally spaced strips to estimate the area of this region, giving the answer correct to 3 decimal places.

$\boxed{\phantom{xx}}$, $\boxed{0.623 - 0.624}$

$\boxed{\text{AREA} = \int_0^1 \frac{1}{1+\sqrt{x}}\,dx}$

FORM A TABLE OF VALUES FOR THE INTEGRAND

| $x$ | 0 | 0.25 | 0.5 | 0.75 | 1 |
|---|---|---|---|---|---|
| $y$ | 1 | $\frac{2}{3}$ | $2-\sqrt{2}$ | $4-2\sqrt{3}$ | $\frac{1}{2}$ |
| | FIRST | ODD | EVEN | ODD | LAST |

USING SIMPSON'S FORMULA

$\Rightarrow \text{AREA} \approx \frac{\text{"THICKNESS"}}{3}\left[\text{FIRST} + \text{LAST} + 4\times\text{ODDS} + 2\times\text{EVENS}\right]$

$\Rightarrow \text{AREA} \approx \frac{0.25}{3}\left[1 + \frac{1}{2} + 4\left(\frac{2}{3} + 4 - 2\sqrt{3}\right) + 2\times\left(2-\sqrt{2}\right)\right]$

$\Rightarrow \text{AREA} \simeq 0.623486\ldots$

$\Rightarrow \text{AREA} \simeq 0.623$ // 3. d.p.

**Question 2** (**)

The values of $y$ for the curve with equation $y = \dfrac{1}{\sqrt{x^3+1}}$ have been tabulated below.

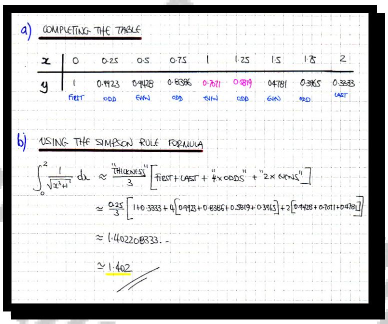
| $x$ | 0 | 0.25 | 0.5 | 0.75 | 1 | 1.25 | 1.5 | 1.75 | 2 |
|---|---|---|---|---|---|---|---|---|---|
| $y$ | 1 | 0.9923 | 0.9428 | 0.8386 | | | 0.4781 | 0.3965 | 0.3333 |

**a)** Complete the table.

**b)** Use Simpson's rule with all the values from the table to find an estimate to 3 decimal places for the integral

$$\int_0^2 \frac{1}{\sqrt{x^3+1}}\,dx\,.$$

$\boxed{\phantom{0.7071}}$ , $\boxed{0.7071}$ , $\boxed{0.5819}$ , $\boxed{1.402}$



a) COMPLETING THE TABLE

| $x$ | 0 | 0.25 | 0.5 | 0.75 | 1 | 1.25 | 1.5 | 1.75 | 2 |
|---|---|---|---|---|---|---|---|---|---|
| $y$ | 1 | 0.9923 | 0.9428 | 0.8386 | 0.7071 | 0.5819 | 0.4781 | 0.3965 | 0.3333 |
| | FIRST | ODD | EVEN | ODD | EVEN | ODD | EVEN | ODD | LAST |

b) USING THE SIMPSON RULE FORMULA

$$\int_0^2 \frac{1}{\sqrt{x^3+1}}\,dx \approx \frac{\text{"THICKNESS"}}{3}\left[\text{FIRST} + \text{LAST} + \text{"}4\times\text{ODDS"} + \text{"}2\times\text{EVENS"}\right]$$

$$\approx \frac{0.25}{3}\left[1 + 0.3333 + 4\left[0.9923 + 0.8386 + 0.5819 + 0.3965\right] + 2\left[0.9428 + 0.7071 + 0.4781\right]\right]$$

$$\approx 1.402208333\ldots$$

$$\approx 1.402$$

**Question 3** (**)

$$I = \int_0^2 \frac{1}{x^3+1}\,dx .$$

Use 3 equally spaced ordinates, to estimate the value of $I$ …

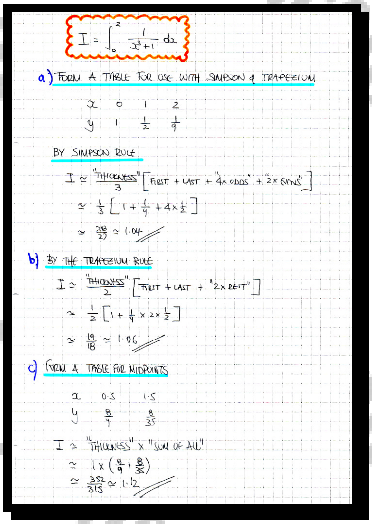
**a)** … by Simpson's rule.

**b)** … by the trapezium rule.

**c)** … by the mid-ordinate rule.

All steps in the calculations must be shown.

$\boxed{\phantom{0000}}$ , $\boxed{1.04}$ , $\boxed{1.06}$ , $\boxed{1.12}$



$\boxed{I = \int_0^2 \frac{1}{x^3+1}\,dx}$

a) FORM A TABLE FOR USE WITH SIMPSON & TRAPEZIUM

| | | | |
|---|---|---|---|
| $x$ | 0 | 1 | 2 |
| $y$ | 1 | $\frac{1}{2}$ | $\frac{1}{9}$ |

BY SIMPSON RULE

$I \simeq \frac{\text{"THICKNESS"}}{3}\left[\text{FIRST} + \text{LAST} + \text{"4} \times \text{ODDS"} + \text{"2} \times \text{EVENS"}\right]$

$\simeq \frac{1}{3}\left[1 + \frac{1}{9} + 4 \times \frac{1}{2}\right]$

$\simeq \frac{28}{27} \simeq 1.04$

b) BY THE TRAPEZIUM RULE

$I \simeq \frac{\text{"THICKNESS"}}{2}\left[\text{FIRST} + \text{LAST} + \text{"2} \times \text{REST"}\right]$

$\simeq \frac{1}{2}\left[1 + \frac{1}{9} \times 2 \times \frac{1}{2}\right]$

$\simeq \frac{19}{18} \simeq 1.06$

c) FORM A TABLE FOR MIDPOINTS

| | | |
|---|---|---|
| $x$ | 0.5 | 1.5 |
| $y$ | $\frac{8}{9}$ | $\frac{8}{35}$ |

$I \simeq \text{"THICKNESS"} \times \text{"SUM OF ALL"}$

$\simeq 1 \times \left(\frac{8}{9} + \frac{8}{35}\right)$

$\simeq \frac{352}{315} \simeq 1.12$

**Question 4** (**)

$$I = \int_{1}^{2.5} \sqrt{x^3+1} \; dx .$$

Use Simpson's rule with 6 equally spaced strips, to estimate the value of $I$.

All steps in the calculation must be shown and the final answer must be correct to 3 significant figures.

$\boxed{\phantom{xx}}$, $\boxed{3.89}$

FORMING A TABLE OF VALUES FOR THE INTEGRAND, USING 7 ORDINATES (6 STRIPS)

| $x$ | 1 | 1.25 | 1.50 | 1.75 | 2 | 2.25 | 2.5 |
|---|---|---|---|---|---|---|---|
| $\sqrt{x^3+1}$ | 1.4142 | 1.7185 | 2.0917 | 2.5218 | 3 | 3.5200 | 4.0774 |
| | FIRST | ODD | EVEN | ODD | EVEN | ODD | LAST |

USING SIMPSON'S RULE

$$\int_{1}^{2.5} \sqrt{x^3+1}\,dx \approx \frac{\text{"THICKNESS"}}{3}\left[\text{FIRST} + \text{LAST} + 2 \times \text{EVEN} + 4 \times \text{ODDS}\right]$$

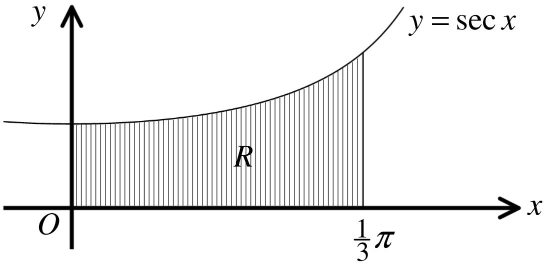
$$\approx \frac{0.25}{3}\left[1.4142 + 4.0774 + 2(2.0917 + 3) + 4(1.7185 + 2.5218 + 3.5200)\right]$$

$$\approx \frac{1}{12} \times 46.7162\ldots$$

$$\simeq 3.8930\ldots$$

$$\simeq 3.89 \quad \text{3 s.f.}$$

**Question 5** (**)



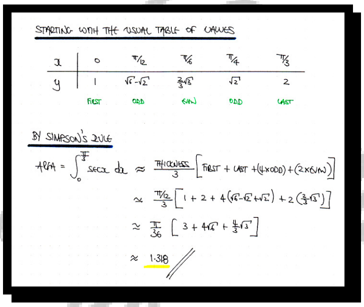

The figure above shows part of the curve with equation $y = \sec x$.

The region $R$, shown shaded in the figure, is bounded by the curve, the $x$ axis and the straight line with equation $x = \frac{1}{3}\pi$.

Use Simpson's rule with 4 equally spaced intervals to estimate the area of $R$.

[*The answer must be supported with detailed calculations.*]

$\boxed{\phantom{000}}$, $\boxed{1.318}$


STARTING WITH THE USUAL TABLE OF VALUES

| $x$ | $0$ | $\pi/12$ | $\pi/6$ | $\pi/4$ | $\pi/3$ |
|---|---|---|---|---|---|
| $y$ | $1$ | $\sqrt{6}-\sqrt{2}$ | $\frac{2}{3}\sqrt{3}$ | $\sqrt{2}$ | $2$ |
| | FIRST | ODD | EVEN | ODD | LAST |

BY SIMPSON'S RULE

$$\text{AREA} = \int_0^{\frac{\pi}{3}} \sec x \, dx \approx \frac{\text{THICKNESS}}{3}\left[\text{FIRST} + \text{LAST} + (4 \times \text{ODD}) + (2 \times \text{EVEN})\right]$$

$$\approx \frac{\pi/12}{3}\left[1 + 2 + 4\left(\sqrt{6}-\sqrt{2}+\sqrt{2}\right) + 2\left(\tfrac{2}{3}\sqrt{3}\right)\right]$$

$$\approx \frac{\pi}{36}\left[3 + 4\sqrt{6} + \tfrac{4}{3}\sqrt{3}\right]$$

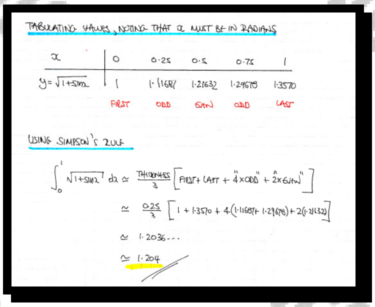
$$\approx 1.318$$

**Question 6** (**)

$$I = \int_0^1 \sqrt{1+\sin x}\; dx$$

Use Simpson's rule with 4 equally spaced strips to estimate the approximate value of $I$, giving the answer correct to 3 decimal places

$\boxed{\phantom{0000}}$, $\boxed{\approx 1.202}$


TABULATING VALUES, NOTING THAT $x$ MUST BE IN RADIANS

| $x$ | 0 | 0.25 | 0.5 | 0.75 | 1 |
|---|---|---|---|---|---|
| $y=\sqrt{1+\sin x}$ | 1 | 1.11687 | 1.21632 | 1.29678 | 1.3570 |
| | FIRST | ODD | EVEN | ODD | LAST |

USING SIMPSON'S RULE

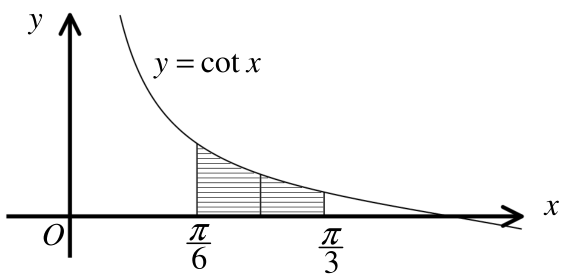
$$\int_0^1 \sqrt{1+\sin x}\,dx \simeq \frac{\text{THICKNESS}}{3}\left[\text{FIRST}+\text{LAST}+4\times\text{"ODD"}+2\times\text{"EVEN"}\right]$$

$$\simeq \frac{0.25}{3}\left[1+1.3570+4(1.11687+1.29678)+2(1.21632)\right]$$

$$\simeq 1.2036\ldots$$

$$\simeq 1.204$$

**Question 7** **(**)**




The figure above shows part of the curve with equation $y = \cot x$.

The region $R$, shown shaded in the figure, is bounded by the curve, the $x$ axis and the straight lines with equations

$$x = \frac{\pi}{6} \qquad \text{and} \qquad x = \frac{\pi}{3}.$$

Use Simpson's rule with $3$ equally spaced ordinates to estimate the area of $R$.

[*The answer must be supported with detailed calculations.*]

$\boxed{0.551}$

FORMING THE USUAL TABLE

| $x$ | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ |
|---|---|---|---|
| $y$ | $\sqrt{3}$ | $1$ | $\frac{1}{\sqrt{3}}$ |
| | (FIRST) | (ODD) | (LAST) |

BY SIMPSON'S RULE NOTING THAT THERE ARE NO "EVENS"

$$\text{AREA} \simeq \frac{\text{THICKNESS}}{3}\left[\text{FIRST} + \text{LAST} + (4 \times \text{ODDS}) + (2 \times \text{EVENS})\right]$$

$$\simeq \frac{\pi/12}{3}\left[\sqrt{3} + \frac{1}{\sqrt{3}} + 4 \times 1\right]$$

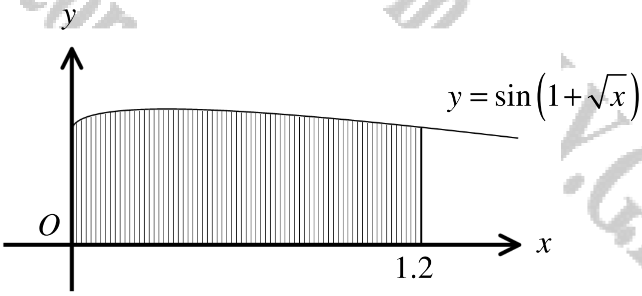
$$\simeq \frac{\pi}{36}\left[4 + \frac{4}{3}\sqrt{3}\right]$$

$$\simeq \frac{\pi}{9}\left[1 + \frac{\sqrt{3}}{3}\right]$$

$$\simeq \frac{\pi}{27}\left[3 + \sqrt{3}\right]$$

$$\simeq 0.551$$

**Question 8** (**+)




The figure above shows part of the curve $C$ with equation

$$y=\sin\left(1+\sqrt{x}\right), \quad x \geq 0 .$$

It is required to estimate the volume of the solid of revolution, when the area of the shaded region bounded by $C$, the coordinate axes and the straight line with equation $x=1.2$ is fully revolved about the $x$ axis.

Use Simpson's rule with $7$ equally spaced ordinates to find an approximation for the volume of this solid.

[*The answer must be supported with detailed calculations*.]

$\boxed{3.42}$

STARTING BY SETTING AN EXPRESSION FOR THE INTEGRAL TO BE APPROXIMATED

$$V=\pi\int_{x_1}^{x_2}(y(x))^2\,dx=\pi\int_0^{1.2}\sin^2(1+\sqrt{x})\,dx$$

SETTING UP A TABLE OF VALUES WITH 7 ORDINATES (6 STRIPS)

| $x$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 | 1.2 |
|---|---|---|---|---|---|---|---|
| $y$ | 0.7081 | 0.9848 | 0.9962 | 0.9590 | 0.8989 | 0.8268 | 0.7491 |
| | FIRST | ODD | EVEN | ODD | EVEN | ODD | LAST |

BY THE SIMPSON FORMULA, AND "RE-INTRODUCING" $\pi$ IN THE CALCULATION

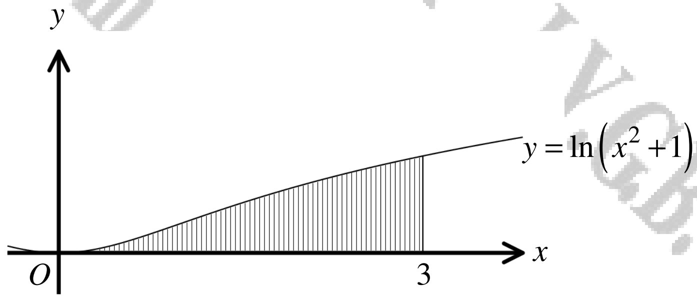
$$\Rightarrow V \simeq \pi \times \frac{\text{"THICKNESS"}}{3}\Big[\text{FIRST} + \text{LAST} + (2\times\text{EVEN}) + (4\times\text{ODD})\Big]$$

$$\Rightarrow V \simeq \pi \times \frac{0.2}{3}\Big[0.7081+0.7491+4(0.9848+0.9590+0.8268)+2(0.9962+0.8989)\Big]$$

$$\Rightarrow V \simeq \frac{\pi}{15}\times 16.3298$$

$$\Rightarrow V \approx 3.420$$

**Question 9** (**+)




The figure above shows part of the curve with equation

$$y = \ln\left(x^2 + 4\right).$$

The region $R$, shown shaded in the figure, is bounded by the curve, the $x$ axis and the straight line with equation $x = 3$.

**a)** Use Simpson's rule with $7$ equally spaced ordinates to estimate the area of $R$.

[*The answer must be supported with detailed calculations.*]

**b)** Deduce an estimate for the value of

$$\int_0^3 \ln\left(\tfrac{1}{4}x^2 + 1\right)\, dx .$$

[illegible], $\boxed{5.626}$, $\boxed{1.47}$

a) SETTING UP A TABLE OF VALUES WITH GAP OF 0.5

| $x$ | 0 | 0.5 | 1 | 1.5 | 2 | 2.5 | 3 |
|---|---|---|---|---|---|---|---|
| $y$ | $\ln 4$ | $\ln 4.25$ | $\ln 5$ | $\ln 6.25$ | $\ln 8$ | $\ln 10.25$ | $\ln 13$ |
| | FIRST | ODD | EVEN | ODD | EVEN | ODD | LAST |

BY SIMPSON RULE

$$\text{AREA} \approx \frac{\text{THICKNESS}}{3}\left[\text{FIRST} + \text{LAST} + (2\times\text{EVEN}) + (4\times\text{ODD})\right]$$

$$\approx \frac{0.5}{3}\left[\ln 4 + \ln 13 + 4\left[\ln 4.25 + \ln 6.25 + \ln 10.25\right] + 2\left[\ln 5 + \ln 8\right]\right]$$

$$\approx \frac{1}{6} \times 33.7561152 4\ldots$$

$$\approx 5.626$$

b) USING THE ANSWER FROM PART (a)

$$\int_0^3 \ln\left(\tfrac{1}{4}x^2+1\right) dx = \int_0^3 \ln\left[\tfrac{1}{4}(x^2+4)\right] dx = \ln$$

$$= \int_0^3 \ln\tfrac{1}{4} + \ln(x^2+4)\, dx$$

$$= \int_0^3 \ln(x^2+4)\, dx + \int_0^3 \ln\tfrac{1}{4}\, dx$$

$$\approx 5.626 - \ln 4 \int_0^3 1\, dx$$

$$\approx 5.626 - (\ln 4)\left[x\right]_0^3$$

$$\approx 5.626 - 3\ln 4$$

$$\approx 1.47$$

**Question 10** (**+)

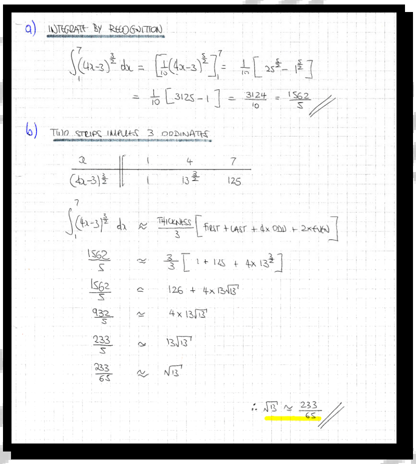
**a)** Find the exact value of the following integral

$$\int_1^7 (4x-3)^{\frac{3}{2}} \, dx, \; x \geq 0 .$$

**b)** Use Simpson's rule with 2 strips and the answer of part **(a)** to show that

$$\sqrt{13} \approx \frac{233}{65} .$$

$\boxed{\phantom{0}}$ , $\boxed{\dfrac{1562}{5}}$



a) INTEGRATE BY RECOGNITION

$$\int_1^7 (4x-3)^{\frac{3}{2}} \, dx = \left[\frac{1}{10}(4x-3)^{\frac{5}{2}}\right]_1^7 = \frac{1}{10}\left[25^{\frac{5}{2}} - 1^{\frac{5}{2}}\right]$$

$$= \frac{1}{10}\left[3125 - 1\right] = \frac{3124}{10} = \frac{1562}{5}$$

b) TWO STRIPS IMPLIES 3 ORDINATES

| $x$ | 1 | 4 | 7 |
|---|---|---|---|
| $(4x-3)^{\frac{3}{2}}$ | 1 | $13^{\frac{3}{2}}$ | 125 |

$$\int_1^7 (4x-3)^{\frac{3}{2}} \, dx \approx \frac{\text{THICKNESS}}{3}\left[\text{FIRST} + \text{LAST} + 4 \times \text{ODD} + 2 \times \text{EVEN}\right]$$

$$\frac{1562}{5} \approx \frac{3}{3}\left[1 + 125 + 4 \times 13^{\frac{3}{2}}\right]$$

$$\frac{1562}{5} \approx 126 + 4 \times 13\sqrt{13}$$

$$\frac{932}{5} \approx 4 \times 13\sqrt{13}$$

$$\frac{233}{5} \approx 13\sqrt{13}$$

$$\frac{233}{65} \approx \sqrt{13}$$

$$\therefore \sqrt{13} \approx \frac{233}{65}$$

**Question 11** (**+)

The values of $y$ for the curve $C$ with equation $y = f(x)$ have been tabulated below.

| $x$ | $-3$ | $-1$ | $1$ | $3$ | $5$ |
|---|---|---|---|---|---|
| $y$ | $6$ | $12$ | $18$ | $25$ | $a$ |

The average value of $f(x)$ in the interval $(-3,5)$ is $17$.

Use Simpson's rule with all the values from the table to find an estimate for the value of the constant $a$.

$\boxed{a = 14}$

| $x$ | $-3$ | $-1$ | $1$ | $3$ | $5$ |
|---|---|---|---|---|---|
| $y=f(x)$ | $6$ | $12$ | $18$ | $25$ | $a$ |
| | FIRST | ODD | EVEN | ODD | LAST |

USING SIMPSON'S RULE

$$\int_{-3}^{5} f(x)\,dx \approx \frac{\text{"THICKNESS"}}{3}\left[\text{FIRST} + \text{LAST} + 4\times\text{ODDS} + 2\times\text{EVENS}\right]$$

$$\approx \frac{2}{3}\left[6 + a + 4(12+25) + 2\times 18\right]$$

$$\approx \frac{2}{3}\left[a + 190\right]$$

$$\approx \frac{2}{3}a + \frac{380}{3}$$

NOW THE AVERAGE VALUE OF THE FUNCTION IS 17

$$\Rightarrow \frac{\int_{-3}^{5} f(x)\,dx}{5-(-3)} = 17$$

$$\Rightarrow \frac{\frac{2}{3}a + \frac{380}{3}}{8} = 17$$

$$\Rightarrow \frac{2}{3}a + \frac{380}{3} = 136$$

$$\Rightarrow 2a + 380 = 408$$

$$\Rightarrow 2a = 28$$

$$\Rightarrow a = 14$$

**Question 12** (**+)

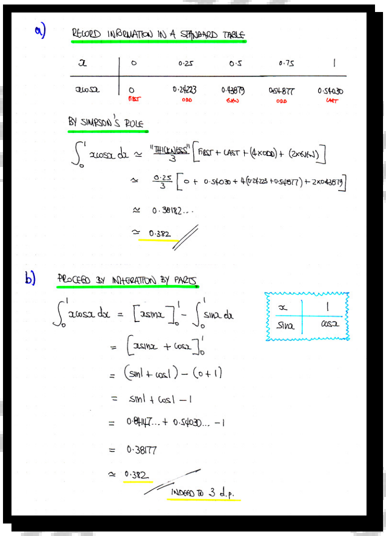
$$I = \int_0^1 x\cos x \; dx .$$

**a)** Use Simpson's rule with $4$ equally spaced strips to estimate the value of $I$ .

All steps in the calculation must be shown and the final answer must be correct to $3$ decimal places.

**b)** Use integration by parts to show that the value of $I$ found in part **(a)** is indeed correct to three decimal places.

$\boxed{0.382}$



a) RECORD INFORMATION IN A STANDARD TABLE

| $x$ | 0 | 0.25 | 0.5 | 0.75 | 1 |
|---|---|---|---|---|---|
| $x\cos x$ | 0 | 0.24223 | 0.43879 | 0.54877 | 0.54030 |
| | FIRST | ODD | EVEN | ODD | LAST |

BY SIMPSON'S RULE

$$\int_0^1 x\cos x\,dx \simeq \frac{\text{"THICKNESS"}}{3}\left[\text{FIRST} + \text{LAST} + (4\times\text{ODD}) + (2\times\text{EVEN})\right]$$

$$\simeq \frac{0.25}{3}\left[0 + 0.54030 + 4(0.24223 + 0.54877) + 2\times 0.43879\right]$$

$$\simeq 0.38182\ldots$$

$$\simeq 0.382$$

b) PROCEED BY INTEGRATION BY PARTS

| $x$ | $1$ |
|---|---|
| $\sin x$ | $\cos x$ |

$$\int_0^1 x\cos x\,dx = \Big[x\sin x\Big]_0^1 - \int_0^1 \sin x\,dx$$

$$= \Big[x\sin x + \cos x\Big]_0^1$$

$$= (\sin 1 + \cos 1) - (0 + 1)$$

$$= \sin 1 + \cos 1 - 1$$

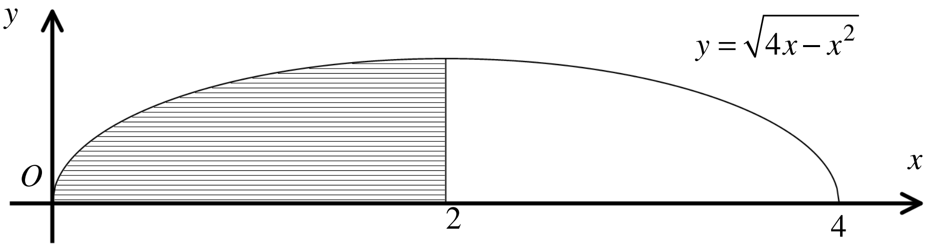
$$= 0.84147\ldots + 0.54030\ldots - 1$$

$$= 0.38177$$

$$\simeq 0.382$$

INDEED TO 3 d.p.

**Question 13 (**+)**




The figure above shows part of the curve $C$ with equation

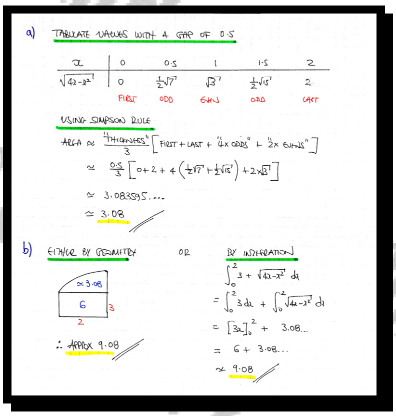
$$y=\sqrt{4x-x^2}\,.$$

**a)** Use Simpson's rule with 4 equally strips to estimate, to three significant figures, the area bounded by $C$, the $x$ axis and the vertical straight line with equation $x=2$.

**b)** Hence find an estimate for

$$\int_0^2 3+\sqrt{4x-x^2}\;dx\,.$$

$\boxed{\phantom{0}}$ , $\boxed{\approx 3.08}$ , $\boxed{\approx 9.08}$



a) TABULATE VALUES WITH A GAP OF 0.5

| $x$ | 0 | 0.5 | 1 | 1.5 | 2 |
|---|---|---|---|---|---|
| $\sqrt{4x-x^2}$ | 0 | $\frac{1}{2}\sqrt{7}$ | $\sqrt{3}$ | $\frac{1}{2}\sqrt{15}$ | 2 |
| | FIRST | ODD | EVEN | ODD | LAST |

USING SIMPSON RULE

$$\text{AREA} \approx \frac{\text{"THICKNESS"}}{3}\left[\text{FIRST} + \text{LAST} + \text{"}4\times\text{ODDS"} + \text{"}2\times\text{EVENS"}\right]$$

$$\approx \frac{0.5}{3}\left[0+2+4\left(\tfrac{1}{2}\sqrt{7}+\tfrac{1}{2}\sqrt{15}\right)+2\times\sqrt{3}\right]$$

$$\approx 3.083595\ldots$$

$$\approx 3.08$$

b) EITHER BY GEOMETRY OR BY INTEGRATION

$\approx 3.08$

6

3

2

$\therefore$ APPROX 9.08

$$\int_0^2 3+\sqrt{4x-x^2}\,dx$$

$$=\int_0^2 3\,dx+\int_0^2\sqrt{4x-x^2}\,dx$$

$$=\left[3x\right]_0^2+\ 3.08\ldots$$

$$=6+3.08\ldots$$

$$\approx 9.08$$

**Question 14** (**+)

**a)** Use Simpson's rule with 5 equally spaced ordinates to estimate the value of

$$\int_0^{\frac{1}{3}\pi} e^{\sec^2 x}\, dx .$$

**b)** Use the answer of part **(a)** to estimate the value of

$$\int_0^{\frac{1}{3}\pi} e^{\tan^2 x}\, dx .$$

**c)** Explain whether the estimates of the previous parts of the question are likely to be accurate.

$\boxed{\phantom{xx}}$, $\boxed{\approx 9.26}$, $\boxed{\approx 3.41}$

a) FILLING A STANDARD TABLE

| $x$ | $0$ | $\frac{\pi}{12}$ | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ |
|---|---|---|---|---|---|
| $y = e^{\sec^2 x}$ | 2.718 | 2.921 | 3.794 | 7.389 | 54.598 |
| | FIRST | ODD | EVEN | ODD | LAST |

BY SIMPSON'S RULE

$$\int_0^{\pi/3} e^{\sec^2 x}\, dx \simeq \frac{\text{THICKNESS}}{3}\left[\text{FIRST} + \text{LAST} + 4\times\text{ODD} + 2\times\text{EVEN}\right]$$

$$\simeq \frac{\pi/12}{3}\left[2.718 + 54.598 + 4(2.921 + 7.389) + 2\times 3.794\right]$$

$$\simeq 9.26$$

b) USING $1 + \tan^2 x = \sec^2 x$

$$\int_0^{\pi/3} e^{\tan^2 x}\, dx = \int_0^{\pi/3} e^{\sec^2 x - 1}\, dx = \int_0^{\pi/3} e^{\sec^2 x} \times e^{-1}\, dx = \frac{1}{e}\int_0^{\pi/3} e^{\sec^2 x}\, dx$$

$$= \frac{1}{e} \times 9.26\ldots \simeq 3.41$$

c) THE GRAPH OF $y = e^{\sec^2 x}$ IS STRICTLY INCREASING AND AS WE GET CLOSE TO $\frac{\pi}{3}$ VERY RAPIDLY

THEREFORE THE ESTIMATES ARE LIKELY TO BE INACCURATE

**Question 15** (**+)

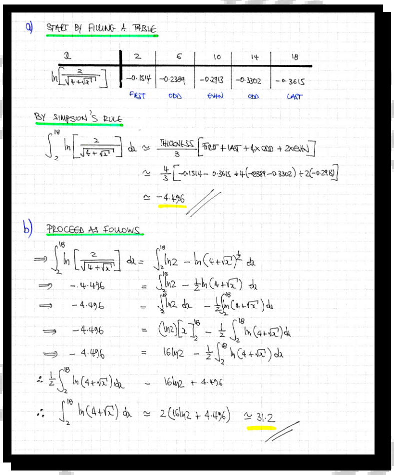
**a)** Use Simpson's rule with 5 equally spaced ordinates to estimate the value of

$$\int_{2}^{18} \ln\left[\frac{2}{\sqrt{4+\sqrt{x}}}\right]\,dx\,.$$

**b)** Use the answer of part **(a)** to estimate the value of

$$\int_{2}^{18} \ln\left(4+\sqrt{x}\right)\,dx\,.$$

$\boxed{\phantom{xx}}$ , $\boxed{\approx -4.496}$ , $\boxed{\approx 31.2}$



a) START BY FILLING A TABLE

| $x$ | 2 | 6 | 10 | 14 | 18 |
|---|---|---|---|---|---|
| $\ln\left[\frac{2}{\sqrt{4+\sqrt{x}}}\right]$ | $-0.1514$ | $-0.2389$ | $-0.2913$ | $-0.3302$ | $-0.3615$ |
| | FIRST | ODD | EVEN | ODD | LAST |

BY SIMPSON'S RULE

$$\int_{2}^{18} \ln\left[\frac{2}{\sqrt{4+\sqrt{x}}}\right]dx \simeq \frac{\text{THICKNESS}}{3}\left[\text{FIRST}+\text{LAST}+4\times\text{ODD}+2\times\text{EVEN}\right]$$

$$\simeq \frac{4}{3}\left[-0.1514-0.3615+4(-0.2389-0.3302)+2(-0.2913)\right]$$

$$\simeq -4.496$$

b) PROCEED AS FOLLOWS

$$\Longrightarrow \int_{2}^{18} \ln\left[\frac{2}{\sqrt{4+\sqrt{x}}}\right]dx = \int_{2}^{18} \ln 2 - \ln\left(4+\sqrt{x}\right)^{\frac{1}{2}}\,dx$$

$$\Longrightarrow -4.496 = \int_{2}^{18} \ln 2 - \tfrac{1}{2}\ln\left(4+\sqrt{x}\right)\,dx$$

$$\Longrightarrow -4.496 = \int_{2}^{18} \ln 2\,dx - \tfrac{1}{2}\int_{2}^{18}\ln\left(4+\sqrt{x}\right)dx$$

$$\Longrightarrow -4.496 = (\ln 2)\left[x\right]_{2}^{18} - \tfrac{1}{2}\int_{2}^{18}\ln\left(4+\sqrt{x}\right)dx$$

$$\Longrightarrow -4.496 = 16\ln 2 - \tfrac{1}{2}\int_{2}^{18}\ln\left(4+\sqrt{x}\right)dx$$

$$\therefore \tfrac{1}{2}\int_{2}^{18}\ln\left(4+\sqrt{x}\right)dx = 16\ln 2 + 4.496$$

$$\therefore \int_{2}^{18}\ln\left(4+\sqrt{x}\right)dx \simeq 2\left(16\ln 2 + 4.496\right) \simeq 31.2$$

**Question 16** (**+)

a) **Use Simpson's rule** with $4$ equally spaced strips to find an estimate for

$$\int_0^{\frac{\pi}{3}} \cos^2 x \, dx .$$

b) **Use the answer of part (a)** to find an estimate for

$$\int_0^{\frac{\pi}{3}} \sin^2 x \, dx .$$

$\boxed{\phantom{0.000}}$ , $\boxed{\approx 0.740}$ , $\boxed{\approx 0.307}$

a) FORMING A TABLE OF VALUES

| $x$ | $0$ | $\frac{\pi}{12}$ | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ |
|---|---|---|---|---|---|
| $\cos^2 x$ | $1$ | $\frac{2+\sqrt{3}}{4}$ | $\frac{3}{4}$ | $\frac{1}{2}$ | $\frac{1}{4}$ |
| | FIRST | ODD | EVEN | ODD | LAST |

BY SIMPSON RULE

$$\int_0^{\frac{\pi}{3}} \cos^2 x \, dx \simeq \frac{\text{"THICKNESS"}}{3}\left[\text{FIRST} + \text{LAST} + 4 \times \text{ODDS} + 2 \times \text{EVENS}\right]$$

$$\simeq \frac{\pi/12}{3}\left[1 + \frac{1}{4} + 4\left[\frac{2+\sqrt{3}}{4} + \frac{1}{2}\right] + 2 \times \frac{3}{4}\right]$$

$$\simeq \frac{\pi}{36}\left[1 + \frac{1}{4} + 2 + \sqrt{3} + 2 + \frac{3}{2}\right]$$

$$\simeq \frac{\pi}{36} \times \frac{27+4\sqrt{3}}{4}$$

$$\simeq 0.7401985696\ldots$$

$$\simeq 0.740$$

b) USING $\cos^2 x + \sin^2 x \equiv 1$

$$\int_0^{\frac{\pi}{3}} \sin^2 x \, dx = \int_0^{\frac{\pi}{3}} 1 - \cos^2 x \, dx = \int_0^{\frac{\pi}{3}} 1 \, dx - \int_0^{\frac{\pi}{3}} \cos^2 x \, dx$$

USING THE APPROXIMATION OF PART (a)

$$\simeq \left[x\right]_0^{\frac{\pi}{3}} - 0.740198\ldots$$

$$\simeq \frac{\pi}{3} - 0.740198\ldots$$

$$\simeq 0.30699\ldots$$

$$\simeq 0.307$$

**Question 17** (**+)

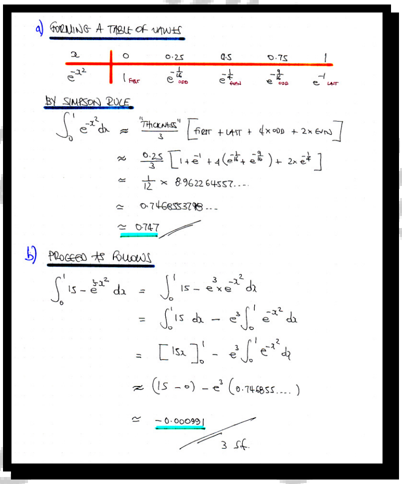
**a)** Use Simpson's rule with $4$ equally spaced strips to find an estimate for

$$\int_0^1 e^{-x^2} \, dx .$$

**b)** Use the answer of part **(a)** to find an estimate for

$$\int_0^1 15 - e^{3-x^2} \, dx .$$

$\boxed{\phantom{000}}$ , $\boxed{\approx 0.747}$ , $\boxed{\approx 0.000991}$



a) FORMING A TABLE OF VALUES

| $x$ | $0$ | $0.25$ | $0.5$ | $0.75$ | $1$ |
|---|---|---|---|---|---|
| $e^{-x^2}$ | $1$ FIRST | $e^{-\frac{1}{16}}$ ODD | $e^{-\frac{1}{4}}$ EVEN | $e^{-\frac{9}{16}}$ ODD | $e^{-1}$ LAST |

BY SIMPSON RULE

$$\int_0^1 e^{-x^2} dx \approx \frac{\text{"THICKNESS"}}{3}\left[\text{FIRST} + \text{LAST} + 4\times\text{ODD} + 2\times\text{EVEN}\right]$$

$$\approx \frac{0.25}{3}\left[1+e^{-1}+4\left(e^{-\frac{1}{16}}+e^{-\frac{9}{16}}\right)+2\times e^{-\frac{1}{4}}\right]$$

$$\simeq \frac{1}{12}\times 8.962264557\ldots$$

$$\simeq 0.7468553798\ldots$$

$$\simeq 0.747$$

b) PROCEED AS FOLLOWS

$$\int_0^1 15 - e^{3-x^2} dx = \int_0^1 15 - e^3 \times e^{-x^2} dx$$

$$= \int_0^1 15 \, dx - e^3 \int_0^1 e^{-x^2} dx$$

$$= \left[15x\right]_0^1 - e^3 \int_0^1 e^{-x^2} dx$$

$$\approx (15-0) - e^3(0.746855\ldots)$$

$$\simeq -0.000991$$

3 s.f.

**Question 18** (***+)

a) **Use the Simpson's rule** with 6 equally spaced strips to find an estimate, correct to 3 decimal places, for

$$\int_0^{1.2} \cos^2 x \; dx .$$

b) **Use the answer of part (a)** to find an estimate for

$$\int_0^{1.2} \cos 2x \; dx .$$

c) **Use the answer of part (b)** to find an estimate for

$$\int_0^{1.2} \left[\cos^4 x - \sin^4 x\right] \; dx .$$

$\boxed{\phantom{0000}}$ , $\boxed{\approx 0.769}$ , $\boxed{\approx 0.337}$ , $\boxed{\approx -0.337}$

a) START BY FORMING A TABLE OF VALUES

| $x$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 | 1.2 |
|---|---|---|---|---|---|---|---|
| $y=\cos^2 x$ | 1 | 0.9605 | 0.8484 | 0.6812 | 0.4854 | 0.2919 | 0.1313 |
| | FIRST | ODD | EVEN | ODD | EVEN | ODD | LAST |

BY SIMPSON RULE

$$\int_0^{1.2} \cos^2 x \, dx \simeq \frac{\text{THICKNESS}}{3}\left[\text{FIRST} + \text{LAST} + 4\times\text{ODD} + 2\times\text{EVEN}\right]$$

$$\simeq \frac{0.2}{3}\left[1 + 0.1313 + 4(0.9605 + 0.6812 + 0.2913) + 2(0.8484 + 0.4854)\right]$$

$$\simeq 0.7687\ldots$$

$$\simeq 0.769$$

b) USING THE DOUBLE ANGLE IDENTITY $\cos 2x \equiv 2\cos^2 x - 1$

$$\int_0^{1.2} \cos 2x \, dx = \int_0^{1.2} 2\cos^2 x - 1 \, dx$$

$$= 2\int_0^{1.2} \cos^2 x \, dx - \int_0^{1.2} 1 \, dx$$

$$= 2 \times 0.769\ldots - \left[x\right]_0^{1.2}$$

$$= 1.537\ldots - 1.2$$

$$= 0.337$$

c) USING MORE IDENTITIES

$$\sin^4 x - \cos^4 x = (\sin^2 x)^2 - (\cos^2 x)^2$$

$$= (\sin^2 x - \cos^2 x)(\sin^2 x + \cos^2 x)$$

$$= (\sin^2 x - \cos^2 x) \times 1$$

$$= -(\cos^2 x - \sin^2 x)$$

$$= -\cos 2x$$

HENCE WE SIMPLY HAVE

$$\int_0^{1.2} \sin^4 x - \cos^4 x \, dx = \int_0^{1.2} -\cos 2x \, dx$$

$$= -\int_0^{1.2} \cos 2x \, dx$$

$$= -(0.337)$$

$$= -0.337$$

**Question 19** (****)

$$I = \int_0^8 2^x \; dx$$

**a)** Use Simpson's rule with 8 equally spaced intervals to verify that

$$I \approx \frac{1105}{3}.$$

[The answer must be supported with detailed calculations.]

**b)** Find the exact value of $I$, by writing $2^x = e^{x\ln 2}$.

**c)** Hence show that

$$\ln 2 \approx \frac{9}{13}.$$

$\boxed{\phantom{xxx}}$, $\boxed{I = \dfrac{255}{\ln 2}}$

a) FORMING A TABLE OF VALUES

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| $2^x$ | 1 | 2 | 4 | 8 | 16 | 32 | 64 | 128 | 256 |
| | FIRST | ODD | EVEN | ODD | EVEN | ODD | EVEN | ODD | LAST |

BY SIMPSON RULE

$$\int_0^8 2^x \, dx \approx \frac{\text{"THICKNESS"}}{3}\left[\text{FIRST} + \text{LAST} + 4 \times \text{ODD} + 2 \times \text{EVEN}\right]$$

$$\approx \frac{1}{3}\left[1 + 256 + 4(2+8+32+128) + 2(4+16+64)\right]$$

$$\approx \frac{1}{3} \times 1105$$

$$\approx \frac{1105}{3}$$ AS REQUIRED

b) INTEGRATING DIRECTLY

$$I = \int_0^8 2^x \, dx = \int_0^8 e^{x\ln 2} \, dx = \left[\frac{1}{\ln 2} e^{x\ln 2}\right]_0^8$$

$$= \frac{1}{\ln 2}\left[2^x\right]_0^8 = \frac{1}{\ln 2}\left[256 - 1\right] = \frac{255}{\ln 2}$$

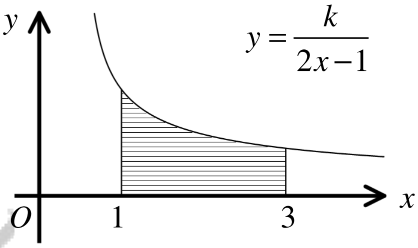
c) COMBINING RESULTS WE DEDUCE

$$\Rightarrow I = \frac{255}{\ln 2} \simeq \frac{1105}{3}$$

$$\Rightarrow \ln 2 \approx \frac{3 \times 255}{1105}$$

$$\Rightarrow \ln 2 \approx \frac{9}{13}$$ AS REQUIRED

**Question 20** (****+)




The figure above shows part of the curve $C$ with equation

$$y = \frac{k}{2x-1},$$

where $k$ is a positive constant.

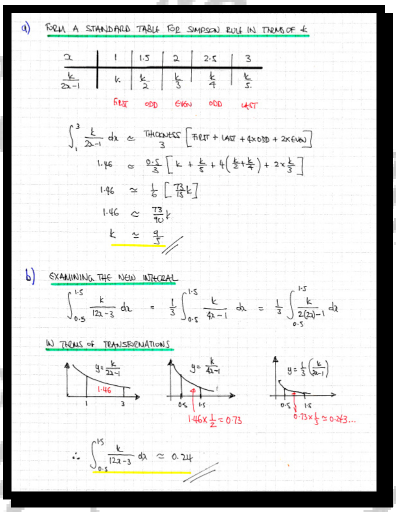
When Simpson's rule with 4 equally spaced strips is used, the area bounded by $C$, the $x$ axis and the vertical straight lines with equations $x = 1$ and $x = 3$, is approximated to 30 square units.

**a)** Determine the value of $k$.

**b)** By considering suitable graph transformation, find an approximate value of

$$\int_{0.5}^{1.5} \frac{k}{12x-3}\, dx.$$

$\boxed{\phantom{P3}}$, $\boxed{k = \dfrac{9}{5}}$, $\boxed{\approx 0.24}$


a) FORM A STANDARD TABLE FOR SIMPSON RULE IN TERMS OF $k$

| $x$ | 1 | 1.5 | 2 | 2.5 | 3 |
|---|---|---|---|---|---|
| $\frac{k}{2x-1}$ | $k$ | $\frac{k}{2}$ | $\frac{k}{3}$ | $\frac{k}{4}$ | $\frac{k}{5}$ |
| | FIRST | ODD | EVEN | ODD | LAST |

$\int_1^3 \frac{k}{2x-1}\,dx \simeq \frac{\text{THICKNESS}}{3}\left[\text{FIRST} + \text{LAST} + 4\times\text{ODD} + 2\times\text{EVEN}\right]$

$1.46 \simeq \frac{0.5}{3}\left[k + \frac{k}{5} + 4\left(\frac{k}{2}+\frac{k}{4}\right) + 2\times\frac{k}{3}\right]$

$1.46 \simeq \frac{1}{6}\left[\frac{73}{15}k\right]$

$1.46 \simeq \frac{73}{90}k$

$k \simeq \frac{9}{5}$

b) EXAMINING THE NEW INTEGRAL

$\int_{0.5}^{1.5} \frac{k}{12x-3}\,dx = \frac{1}{3}\int_{0.5}^{1.5} \frac{k}{4x-1}\,dx = \frac{1}{3}\int_{0.5}^{1.5} \frac{k}{2(2x)-1}\,dx$

IN TERMS OF TRANSFORMATIONS

$y = \frac{k}{2x-1}$ 1.46 1 3

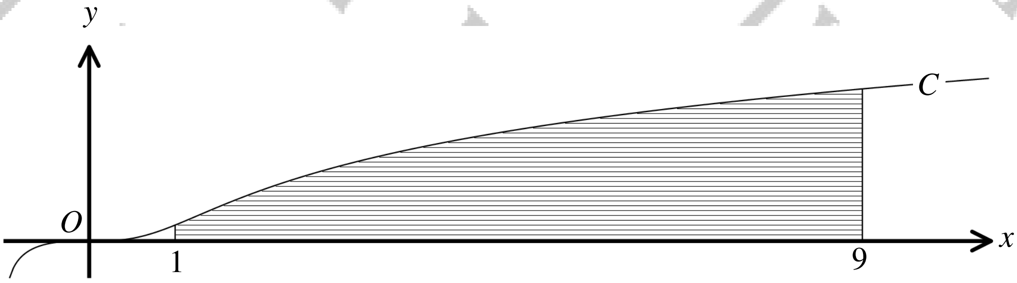
$y = \frac{k}{4x-1}$ 0.5 1.5 $1.46 \times \frac{1}{2} = 0.73$

$y = \frac{1}{3}\left(\frac{k}{2x-1}\right)$ 0.5 1.5 $0.73 \times \frac{1}{3} \simeq 0.243\ldots$

$\therefore \int_{0.5}^{1.5} \frac{k}{12x-3}\,dx \simeq 0.24$

# MID-ORDINATE RULE

**Question 1** (**)




The figure above shows part of the curve $C$ with equation

$$y = \ln\left(1+x^3\right), \quad x > -1 .$$

The area of the shaded region bounded by $C$, the $x$ axis and the straight lines with equations $x=1$ and $x=9$ is to be estimated by the mid-ordinate rule using $4$ equally spaced strips.

Find an estimate for the area of this region.

All steps in the calculation must be shown and the final answer must be correct to 3 significant figures.

$\boxed{\phantom{000}}$, $\boxed{36.0}$

DRAWING A TABLE OF VALUES BASED ON MIDPOINTS

1 2 3 4 5 6 7 8 9

| $x$ | $2$ | $4$ | $6$ | $8$ |
|---|---|---|---|---|
| $\ln(1+x^3)$ | $\ln 9$ | $\ln 65$ | $\ln 217$ | $\ln 513$ |

USING THE MID-ORDINATE RULE

AREA $\simeq$ (THICKNESS) $\times$ (SUM OF ALL)

$\simeq 2 \times [\ln 9 + \ln 65 + \ln 217 + \ln 513)$

$\simeq 35.98357\ldots$

$\simeq 36.0$ 3 sf

**Question 2** (**)

$$I = \int_0^2 3^x \ dx$$

Use the mid-ordinate rule with 4 strips of equal width to obtain an estimate for $I$.

All steps in the calculation must be recorded and the final answer must be correct to three significant figures.

$\boxed{\phantom{000}}$, $\boxed{7.19}$

DRAW A TABLE OF VALUES BASED ON MIDPOINTS

0.5

0 0.25 0.5 0.75 1 1.25 1.5 1.75 2

| $x$ | 0.25 | 0.75 | 1.25 | 1.75 |
|---|---|---|---|---|
| $3^x$ | $3^{0.25}$ | $3^{0.75}$ | $3^{1.25}$ | $3^{1.75}$ |

USING THE MID-ORDINATE RULE FORMULA

$$\int_0^2 3^x \, dx \approx (\text{THICKNESS}) \times (\text{SUM OF ALL})$$

$$\approx 0.5 \times \left[ 3^{0.25} + 3^{0.75} + 3^{1.25} + 3^{1.75} \right]$$

$$\approx 7.191162\ldots$$

$$\approx 7.19 \quad \text{3 sf}$$

**Question 3** (**+)

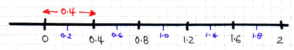
$$I = \int_0^2 \arctan x \ dx .$$

Use the mid-ordinate rule with $6$ equally spaced ordinates to find an estimate for $I$ .

All steps in the calculation must be shown and the final answer must be correct to 3 decimal places.

$\boxed{\phantom{xxxx}}$ , $\boxed{1.415}$

DRAWING UP A TABLE BASED ON MIDPOINTS




| $x$ | 0.2 | 0.6 | 1.0 | 1.4 | 1.8 |
|---|---|---|---|---|---|
| $\arctan x$ | 0.19740 | 0.54042 | 0.78540 | 0.95055 | 1.06370 |

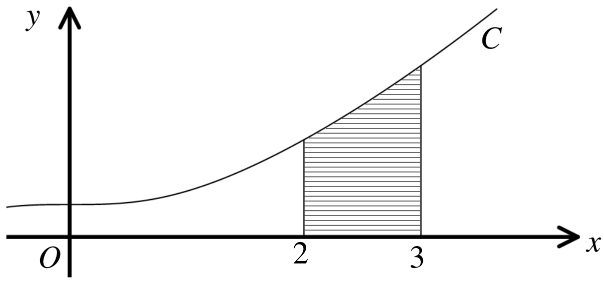
[NOTE THAT $x$ MUST BE IN RADIANS]

USING THE MID-ORDINATE RULE

$$\int_0^2 \arctan x \, dx \simeq (\text{THICKNESS}) \times (\text{SUM OF ALL})$$
$$\simeq 0.4 \times [0.19740 + 0.54042 + \dots + 1.06370]$$
$$\simeq 1.414988\dots$$
$$\simeq 1.415$$

NOTE THAT 6 EQUALLY SPACED ORDINATES IS THE SAME AS 5 EQUALLY SPACED STRIPS

**Question 4** (**+)




The figure above shows part of the curve $C$ with equation

$$y = \sqrt{1 + x^3}, \quad x \geq -1.$$

The shaded region is bounded by $C$, the $x$ axis and the straight lines with equations $x = 2$ and $x = 3$ is to be estimated by the mid-ordinate rule using 5 equally spaced ordinates.

Calculate, correct to 2 decimal places the area of this region.

[The answer must be supported with a detailed method.]

$\boxed{\phantom{0000}}$, $\boxed{4.10}$

DRAWING A TABLE BASED ON MIDPOINTS

0.25

2   2.125   2.25   2.325   2.5   2.625   2.75   2.875   3

| $x$ | 2.125 | 2.375 | 2.625 | 2.875 |
|---|---|---|---|---|
| $\sqrt{1+x^3}$ | 3.25510 | 3.79427 | 4.36897 | 4.97631 |

USING THE MIDORDINATE RULE FORMULA

AREA $\simeq$ (THICKNESS) $\times$ (SUM OF ALL)

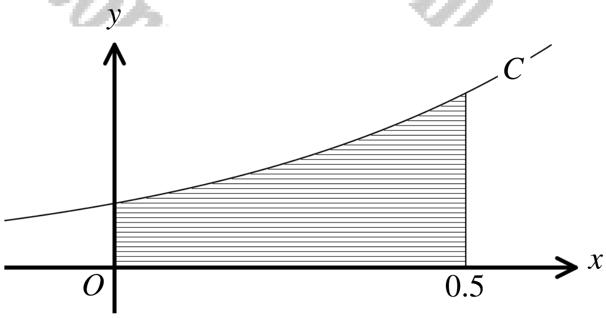
AREA $\simeq 0.25 \times 16.39465$

AREA $\simeq 4.09866\ldots$

AREA $\simeq 4.10$

NOTE THAT 5 EQUALLY SPACED ORDINATES IS THE SAME AS 4 EQUALLY SPACED STRIPS

**Question 5** **(**+)**




The figure above shows part of the curve $C$ with equation

$$y = e^{2x+1}, \ x \in \mathbb{R}.$$

Use the mid-ordinate rule with $6$ equally spaced ordinates to estimate the area of the shaded region bounded by $C$, the $x$ axis and the straight line with equation $x = 0.5$.

Give the answer correct to $2$ decimal places.

[The answer must be supported with detailed calculations.]

$\boxed{\phantom{00}}$, $\boxed{2.33}$

FORMING A TABLE OF VALUES BASED ON MIDPOINTS

| $x$ | 0.05 | 0.15 | 0.25 | 0.35 | 0.45 |
|---|---|---|---|---|---|
| $e^{2x+1}$ | $e^{1.1}$ | $e^{1.3}$ | $e^{1.5}$ | $e^{1.7}$ | $e^{1.9}$ |

BY THE MID-ORDINATE RULE

AREA $\simeq$ "THICKNESS" $\times$ "SUM OF ALL"

$\simeq 0.1 \times \left(e^{1.1} + e^{1.3} + e^{1.5} + e^{1.7} + e^{1.9}\right)$

$\simeq 0.1 \times 23.3149936$

$\simeq 2.33$ (2 d.p)

**Question 6** **(**+)**

Use the mid-ordinate rule with $4$ strips of equal width to find an estimate for

$$\int_{0.6}^{1} \sin\sqrt{3x}\ \, dx\,,$$

giving the final answer correct to five decimal places.

All steps in the calculations must be recorded.

$\boxed{\phantom{0.39751}}$ , $\boxed{0.39751}$

FORMING A TABLE OF VALUES BASED ON MIDPOINTS

| $x$ | 0.65 | 0.75 | 0.85 | 0.95 |
|---|---|---|---|---|
| $\sin\sqrt{3x}$ | 0.984836 | 0.997495 | 0.999660 | 0.993117 |

BY THE MIDORDINATE RULE

$$\int_{0.6}^{1} \sin\sqrt{3x}\, dx \simeq \text{"THICKNESS"} \times \text{"SUM OF ALL"}$$

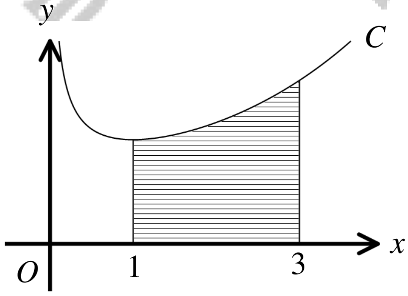
$$\simeq 0.1 \times [0.984836 + 0.997495 + 0.999660 + 0.993117]$$

$$\simeq 0.1 \times 3.975108\ldots$$

$$\simeq 0.39751$$

5 d.p

**Question 7** (***)




The figure above shows part of the curve $C$ with equation

$$y = \sqrt{\frac{e^x}{5x}}, \quad x > 0.$$

The area of the shaded region bounded by $C$, the $x$ axis and the straight lines with equations $x = 1$ and $x = 3$ is to be estimated by the mid-ordinate rule using 5 equally spaced strips.

Find, correct to 3 decimal places, the area of this region.

[The answer must be supported with detailed calculations.]

$\boxed{\phantom{000}}$, $\boxed{1.773}$

FORMING A TABLE OF VALUES BASED ON MIDPOINTS

1, 1.4, 1.8, 2.2, 2.6, 3

| $x$ | 1.2 | 1.6 | 2 | 2.2 | 2.8 |
|---|---|---|---|---|---|
| $\sqrt{\frac{e^x}{5x}}$ | 0.74388 | 0.78685 | 0.85960 | 0.95844 | 1.0838 |

BY THE MID-ORDINATE RULE

$$\int_1^3 \sqrt{\frac{e^x}{5x}}\,dx \simeq \text{"THICKNESS"} \times \text{"SUM OF ALL"}$$

$$\simeq 0.4 \times [0.74388 + 0.78685 + 0.85960 + 0.95844 + 1.0838]$$

$$\simeq 0.4 \times 4.43257$$

$$\simeq 1.773$$

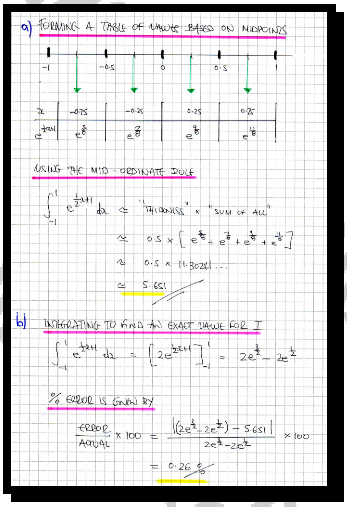
**Question 8** (***)

$$I = \int_{-1}^{1} e^{\frac{1}{2}x+1} \, dx$$

**a)** Use the mid-ordinate rule with 5 ordinates to find an estimate for $I$, giving the final answer correct to 3 decimal places.

**b)** Calculate the percentage error in the estimate of part **(a)**.

$\boxed{5.651}$, $\boxed{0.26\%}$



a) FORMING A TABLE OF VALUES BASED ON MIDPOINTS

| $x$ | $-0.75$ | $-0.25$ | $0.25$ | $0.75$ |
|---|---|---|---|---|
| $e^{\frac{1}{2}x+1}$ | $e^{\frac{5}{8}}$ | $e^{\frac{7}{8}}$ | $e^{\frac{9}{8}}$ | $e^{\frac{11}{8}}$ |

USING THE MID-ORDINATE RULE

$$\int_{-1}^{1} e^{\frac{1}{2}x+1}\,dx \simeq \text{"THICKNESS"} \times \text{"SUM OF ALL"}$$
$$\simeq 0.5 \times \left[e^{\frac{5}{8}} + e^{\frac{7}{8}} + e^{\frac{9}{8}} + e^{\frac{11}{8}}\right]$$
$$\simeq 0.5 \times 11.30241\ldots$$
$$\simeq 5.651$$

b) INTEGRATING TO FIND AN EXACT VALUE FOR $I$

$$\int_{-1}^{1} e^{\frac{1}{2}x+1}\,dx = \left[2e^{\frac{1}{2}x+1}\right]_{-1}^{1} = 2e^{\frac{3}{2}} - 2e^{\frac{1}{2}}$$

% ERROR IS GIVEN BY

$$\frac{\text{ERROR}}{\text{ACTUAL}} \times 100 = \frac{\left|\left(2e^{\frac{3}{2}} - 2e^{\frac{1}{2}}\right) - 5.651\right|}{2e^{\frac{3}{2}} - 2e^{\frac{1}{2}}} \times 100$$
$$= 0.26\,\%$$

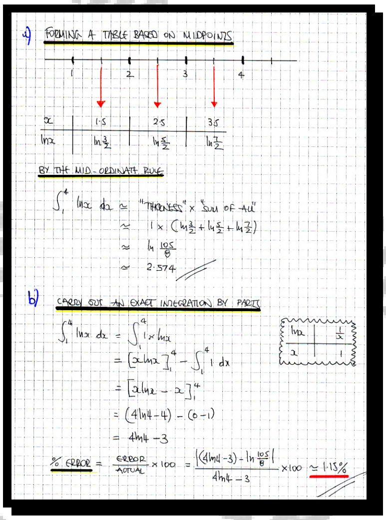
**Question 9** (***)

$$I = \int_1^4 \ln x \ dx$$

**a)** Use the mid-ordinate rule with $3$ equally spaced strips to estimate the value of $I$, giving the final answer correct to $3$ decimal places.

**b)** Calculate the percentage error in the estimate of part **(a)**.

$\boxed{\phantom{xx}}$, $\boxed{2.574}$, $\boxed{1.15\%}$



a) FORMING A TABLE BASED ON MIDPOINTS

| $x$ | $1.5$ | $2.5$ | $3.5$ |
|---|---|---|---|
| $\ln x$ | $\ln\frac{3}{2}$ | $\ln\frac{5}{2}$ | $\ln\frac{7}{2}$ |

BY THE MID-ORDINATE RULE

$$\int_1^4 \ln x \ dx \simeq \text{"THICKNESS"} \times \text{"SUM OF ALL"}$$
$$\simeq 1 \times \left(\ln\tfrac{3}{2} + \ln\tfrac{5}{2} + \ln\tfrac{7}{2}\right)$$
$$\simeq \ln\frac{105}{8}$$
$$\simeq 2.574$$

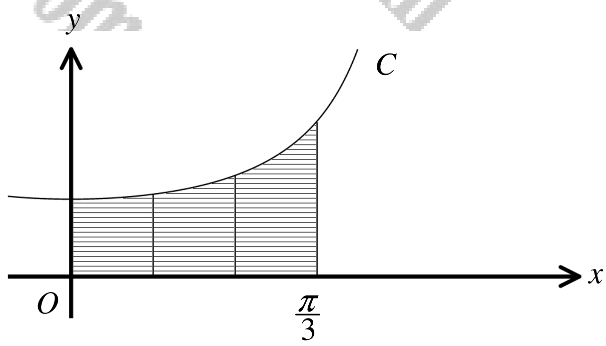
b) CARRY OUT AN EXACT INTEGRATION BY PARTS

| $\ln x$ | $\frac{1}{x}$ |
|---|---|
| $x$ | $1$ |

$$\int_1^4 \ln x \ dx = \int_1^4 1 \times \ln x$$
$$= \left[x\ln x\right]_1^4 - \int_1^4 1 \ dx$$
$$= \left[x\ln x - x\right]_1^4$$
$$= (4\ln 4 - 4) - (0 - 1)$$
$$= 4\ln 4 - 3$$

$$\text{\% ERROR} = \frac{\text{ERROR}}{\text{ACTUAL}} \times 100 = \frac{\left|(4\ln 4 - 3) - \ln\frac{105}{8}\right|}{4\ln 4 - 3} \times 100 \simeq 1.15\%$$

**Question 10** (***+)




The figure above shows part of the curve $C$ with equation

$$y = \sec x, \quad 0 \le x \le \frac{1}{3}\pi .$$

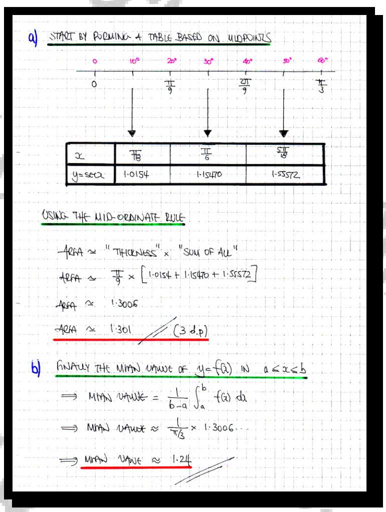
The shaded region bounded by $C$, the coordinate axes and the vertical straight line with equation $x = \frac{1}{3}\pi$ is to be estimated by the mid-ordinate rule using 3 equally spaced strips.

**a)** Find, correct to 3 decimal places, the area of this region.

*The answer must be supported with detailed calculations.*

**b)** Hence estimate the mean value of $y = \sec x$ in the interval $0 \le x \le \frac{1}{3}\pi$.

$\boxed{\phantom{0000}}$, $\boxed{1.301}$, $\boxed{1.24}$



a) START BY FORMING A TABLE BASED ON MIDPOINTS

0, 10°, 20°, 30°, 40°, 50°, 60°

$0$, $\frac{\pi}{9}$, $\frac{2\pi}{9}$, $\frac{\pi}{3}$

| $x$ | $\frac{\pi}{18}$ | $\frac{\pi}{6}$ | $\frac{5\pi}{18}$ |
|---|---|---|---|
| $y=\sec x$ | 1.0154 | 1.15470 | 1.55572 |

USING THE MID-ORDINATE RULE

AREA $\simeq$ "THICKNESS" $\times$ "SUM OF ALL"

AREA $\simeq \frac{\pi}{9} \times \left[1.0154 + 1.15470 + 1.55572\right]$

AREA $\simeq$ 1.3006

AREA $\simeq$ 1.301 // (3 d.p)

b) FINALLY THE MEAN VALUE OF $y = f(x)$ IN $a \le x \le b$

$\Rightarrow$ MEAN VALUE $= \frac{1}{b-a}\int_a^b f(x)\,dx$

$\Rightarrow$ MEAN VALUE $\approx \frac{1}{\pi/3} \times 1.3006\ldots$

$\Rightarrow$ MEAN VALUE $\approx$ 1.24 //