INTEGRATION

by substitution

Question 1

Carry out the following integrations **by substitution** only.

1. $\int 4x(2x-1)^4\,dx = \frac{1}{6}(2x-1)^6 + \frac{1}{5}(2x-1)^5 + C$

2. $\int \frac{2x}{2x+1}\,dx = \frac{1}{2}(2x+1) - \frac{1}{2}\ln|2x+1| + C$

3. $\int x\left(4-x^2\right)^{-\frac{1}{2}}\,dx = -\left(4-x^2\right)^{\frac{1}{2}} + C$

4. $\int \frac{4x}{6x^2-1}\,dx = \frac{1}{3}\ln\left|6x^2-1\right| + C$

5. $\int x(3x-1)^4\,dx = \frac{1}{54}(3x-1)^6 + \frac{1}{45}(3x-1)^5 + C$

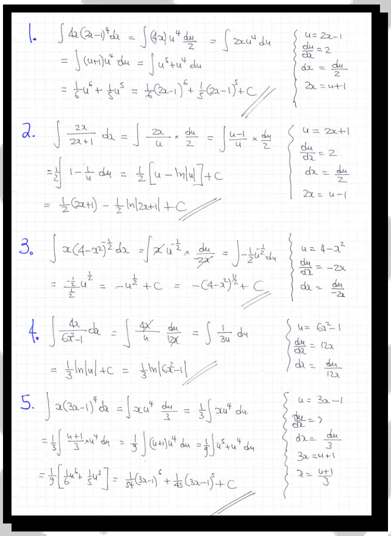
6. $\int \frac{8x}{\sqrt{4x-1}}\,dx = \frac{1}{3}(4x-1)^{\frac{3}{2}} + (4x-1)^{\frac{1}{2}} + C$

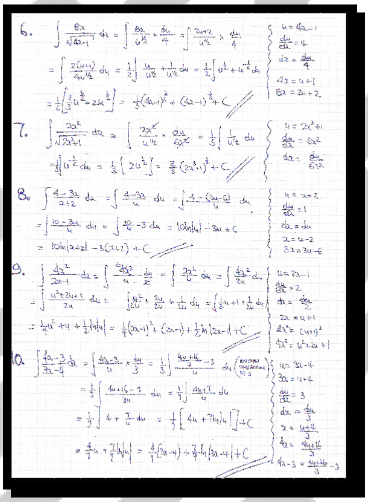
7. $\int \frac{2x^2}{\sqrt{2x^3+1}}\,dx = \frac{2}{3}\left(2x^3+1\right)^{\frac{1}{2}} + C$

8. $\int \frac{4-3x}{x+2}\,dx = 10\ln|x+2| - 3(x+2) + C$

9. $\int \frac{4x^2}{2x-1}\,dx = \frac{1}{4}(2x-1)^2 + (2x-1) + \frac{1}{2}\ln|2x-1| + C$

10. $\int \frac{4x-3}{3x-4}\,dx = \frac{4}{9}(3x-4) + \frac{7}{9}\ln|3x-4| + C$





Question 2

Carry out the following integrations **by substitution** only.

1. $\int 6x(3x-1)^3\,dx = \frac{2}{15}(3x-1)^5 + \frac{1}{6}(3x-1)^4 + C$

2. $\int \frac{5x}{5x-1}\,dx = \frac{1}{5}(5x-1) + \frac{1}{5}\ln|5x-1| + C$

3. $\int 3x\left(x^2+1\right)^{\frac{1}{2}}\,dx = \left(x^2+1\right)^{\frac{3}{2}} + C$

4. $\int \frac{3x^2}{2x^3+1}\,dx = \frac{1}{2}\ln\left|2x^3+1\right| + C$

5. $\int x(2x-1)^5\,dx = \frac{1}{24}(2x-1)^6 + \frac{1}{28}(2x-1)^7 + C$

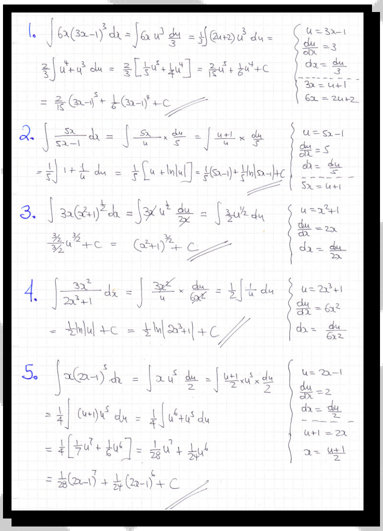
6. $\int \frac{10x}{\sqrt{1-2x}}\,dx = \frac{5}{3}(1-2x)^{\frac{3}{2}} - 5(1-2x)^{\frac{1}{2}} + C$

7. $\int \frac{3x^4}{\sqrt{2x^5+1}}\,dx = \frac{3}{5}\left(2x^5+1\right)^{\frac{1}{2}} + C$

8. $\int \frac{1-x}{1+2x}\,dx = \frac{3}{4}\ln|1+2x| - \frac{1}{4}(1+2x) + C$

9. $\int \frac{6x^2}{2x+3}\,dx = \frac{3}{8}(2x+3)^2 - \frac{9}{2}(2x+3) + \frac{27}{4}\ln|2x+3| + C$

10. $\int \frac{1}{x^{\frac{1}{2}}\sqrt{x^{\frac{1}{2}}-1}}\,dx = 4\sqrt{x^{\frac{1}{2}}-1} + C$



1. $\int 6x(3x-1)^3\,dx = \int 6x\,u^3\,\frac{du}{3} = \frac{1}{3}\int (2u+2)u^3\,du =$

$\frac{2}{3}\int u^4+u^3\,du = \frac{2}{3}\left[\frac{1}{5}u^5+\frac{1}{4}u^4\right] = \frac{2}{15}u^5+\frac{1}{6}u^4+C$

$= \frac{2}{15}(3x-1)^5+\frac{1}{6}(3x-1)^4+C$

$\left\{ u = 3x-1,\ \frac{du}{dx}=3,\ dx=\frac{du}{3},\ 3x = u+1,\ 6x = 2u+2 \right.$

2. $\int \frac{5x}{5x-1}\,dx = \int \frac{5x}{u}\times\frac{du}{5} = \int \frac{u+1}{u}\times\frac{du}{5}$

$= \frac{1}{5}\int 1+\frac{1}{u}\,du = \frac{1}{5}\left[u+\ln|u|\right] = \frac{1}{5}(5x-1)+\frac{1}{5}\ln|5x-1|+C$

$\left\{ u = 5x-1,\ \frac{du}{dx}=5,\ dx=\frac{du}{5},\ 5x = u+1 \right.$

3. $\int 3x(x^2+1)^{\frac{1}{2}}\,dx = \int 3x\,u^{\frac{1}{2}}\,\frac{du}{2x} = \int \frac{3}{2}u^{1/2}\,du$

$\frac{3/2}{3/2}u^{3/2}+C = (x^2+1)^{3/2}+C$

$\left\{ u = x^2+1,\ \frac{du}{dx}=2x,\ dx=\frac{du}{2x} \right.$

4. $\int \frac{3x^2}{2x^3+1}\,dx = \int \frac{3x^2}{u}\times\frac{du}{6x^2} = \frac{1}{2}\int \frac{1}{u}\,du$

$= \frac{1}{2}\ln|u|+C = \frac{1}{2}\ln|2x^3+1|+C$

$\left\{ u = 2x^3+1,\ \frac{du}{dx}=6x^2,\ dx=\frac{du}{6x^2} \right.$

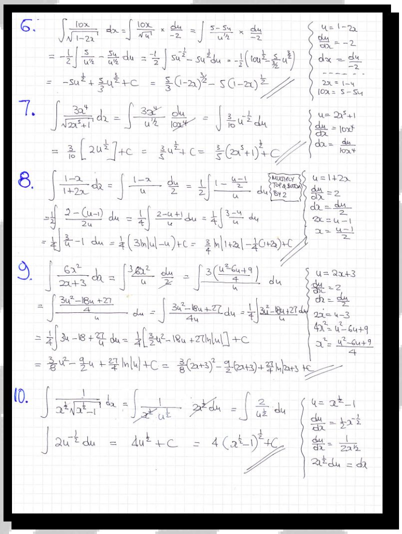
5. $\int x(2x-1)^5\,dx = \int x\,u^5\,\frac{du}{2} = \int \frac{u+1}{2}\times u^5\times\frac{du}{2}$

$= \frac{1}{4}\int (u+1)u^5\,du = \frac{1}{4}\int u^6+u^5\,du$

$= \frac{1}{4}\left[\frac{1}{7}u^7+\frac{1}{6}u^6\right] = \frac{1}{28}u^7+\frac{1}{24}u^6$

$= \frac{1}{28}(2x-1)^7+\frac{1}{24}(2x-1)^6+C$

$\left\{ u = 2x-1,\ \frac{du}{dx}=2,\ dx=\frac{du}{2},\ u+1 = 2x,\ x=\frac{u+1}{2} \right.$



6. $\int \frac{10x}{\sqrt{1-2x}}\,dx = \int \frac{10x}{\sqrt{u}}\times\frac{du}{-2} = \int \frac{5-5u}{u^{1/2}}\times\frac{du}{-2}$

$= -\frac{1}{2}\int \frac{5}{u^{1/2}}-\frac{5u}{u^{1/2}}\,du = -\frac{1}{2}\int 5u^{-\frac{1}{2}}-5u^{\frac{1}{2}}\,du = -\frac{1}{2}\left(10u^{\frac{1}{2}}-\frac{5}{3/2}u^{\frac{3}{2}}\right)$

$= -5u^{\frac{1}{2}}+\frac{5}{3}u^{\frac{3}{2}}+C = \frac{5}{3}(1-2x)^{3/2}-5(1-2x)^{\frac{1}{2}}$

$\left\{ u = 1-2x,\ \frac{du}{dx}=-2,\ dx=\frac{du}{-2},\ 2x = 1-u,\ 10x = 5-5u \right.$

7. $\int \frac{3x^4}{\sqrt{2x^5+1}}\,dx = \int \frac{3x^4}{u^{1/2}}\,\frac{du}{10x^4} = \int \frac{3}{10}u^{-\frac{1}{2}}\,du$

$= \frac{3}{10}\left[2u^{\frac{1}{2}}\right]+C = \frac{3}{5}u^{\frac{1}{2}}+C = \frac{3}{5}(2x^5+1)^{\frac{1}{2}}+C$

$\left\{ u = 2x^5+1,\ \frac{du}{dx}=10x^4,\ dx=\frac{du}{10x^4} \right.$

8. $\int \frac{1-x}{1+2x}\,dx = \int \frac{1-x}{u}\,\frac{du}{2} = \frac{1}{2}\int \frac{1-\frac{u-1}{2}}{u}\,du$ {MULTIPLY TOP & BOTTOM BY 2}

$= \frac{1}{2}\int \frac{2-(u-1)}{2u}\,du = \frac{1}{4}\int \frac{2-u+1}{u}\,du = \frac{1}{4}\int \frac{3-u}{u}\,du$

$= \frac{1}{4}\int \frac{3}{u}-1\,du = \frac{1}{4}\left(3\ln|u|-u\right)+C = \frac{3}{4}\ln|1+2x|-\frac{1}{4}(1+2x)+C$

$\left\{ u = 1+2x,\ \frac{du}{dx}=2,\ dx=\frac{du}{2},\ 2x = u-1,\ x=\frac{u-1}{2} \right.$

9. $\int \frac{6x^2}{2x+3}\,dx = \int \frac{6x^2}{u}\,\frac{du}{2} = \int \frac{3\left(\frac{u^2-6u+9}{4}\right)}{u}\,du$

$= \int \frac{\frac{3u^2-18u+27}{4}}{u}\,du = \int \frac{3u^2-18u+27}{4u}\,du = \frac{1}{4}\int \frac{3u^2-18u+27}{u}\,du$

$= \frac{1}{4}\int 3u-18+\frac{27}{u}\,du = \frac{1}{4}\left[\frac{3}{2}u^2-18u+27\ln|u|\right]+C$

$= \frac{3}{8}u^2-\frac{9}{2}u+\frac{27}{4}\ln|u|+C = \frac{3}{8}(2x+3)^2-\frac{9}{2}(2x+3)+\frac{27}{4}\ln|2x+3|+C$

$\left\{ u = 2x+3,\ \frac{du}{dx}=2,\ dx=\frac{du}{2},\ 2x = u-3,\ 4x^2 = u^2-6u+9,\ x^2=\frac{u^2-6u+9}{4} \right.$

10. $\int \frac{1}{x^{\frac{1}{2}}\sqrt{x^{\frac{1}{2}}-1}}\,dx = \int \frac{1}{x^{\frac{1}{2}}u^{\frac{1}{2}}}\,2x^{\frac{1}{2}}\,du = \int \frac{2}{u^{\frac{1}{2}}}\,du$

$\int 2u^{-\frac{1}{2}}\,du = 4u^{\frac{1}{2}}+C = 4\left(x^{\frac{1}{2}}-1\right)^{\frac{1}{2}}+C$

$\left\{ u = x^{\frac{1}{2}}-1,\ \frac{du}{dx}=\frac{1}{2}x^{-\frac{1}{2}},\ \frac{du}{dx}=\frac{1}{2x^{1/2}},\ 2x^{\frac{1}{2}}\,du = dx \right.$

Question 3

Carry out the following integrations **by substitution** only.

1. $\int 10x(5x-3)^3\,dx = \frac{2}{25}(5x-3)^5 + \frac{3}{10}(5x-3)^4 + C$

2. $\int \frac{12x}{2x-1}\,dx = 3(2x-1) + 3\ln|2x-1| + C$

3. $\int x\left(x^2-1\right)^{\frac{5}{2}}\,dx = \frac{1}{7}\left(x^2-1\right)^{\frac{7}{2}} + C$

4. $\int \frac{5x^5}{2x^6+7}\,dx = \frac{5}{12}\ln\left|2x^6+7\right| + C$

5. $\int 2x(1-5x)^4\,dx = -\frac{2}{125}(1-5x)^5 + \frac{1}{75}(1-5x)^6 + C$

6. $\int \frac{9x}{\sqrt{4x^2+1}}\,dx = \frac{9}{4}\sqrt{4x^2+1} + C$

7. $\int \frac{3x-1}{\sqrt{4x-1}}\,dx = \frac{1}{8}(4x-1)^{\frac{3}{2}} - \frac{1}{8}(4x-1)^{\frac{1}{2}} + C$

8. $\int \frac{1-2x}{1+3x}\,dx = \frac{5}{9}\ln|1+3x| - \frac{2}{9}(1+3x) + C$

9. $\int \frac{6x^{\frac{1}{2}}}{2x^{\frac{3}{2}}+3}\,dx = 2\ln\left|2x^{\frac{3}{2}}+3\right| + C$

10. $\int \frac{x^{\frac{3}{2}}}{\sqrt{1-3x^{\frac{5}{2}}}}\,dx = -\frac{4}{15}\sqrt{1-3x^{\frac{5}{2}}} + C$

1. $\int 10x(5x-3)^3\,dx = \int (2u+6)u^3 \frac{du}{5} = \frac{1}{5}\int 2u^4 + 6u^3\,du$

$= \frac{1}{5}\left[\frac{2}{5}u^5 + \frac{3}{2}u^4\right] = \frac{2}{25}u^5 + \frac{3}{10}u^4 + C$

$= \frac{2}{25}(5x-3)^5 + \frac{3}{10}(5x-3)^4 + C$

$u = 5x-3$, $\frac{du}{dx} = 5$, $dx = \frac{du}{5}$, $5x = u+3$, $10x = 2u+6$

2. $\int \frac{12x}{2x-1}\,dx = \int \frac{12x}{u}\frac{du}{2} = \int \frac{6x}{u}\,du = \int \frac{3u+3}{u}\,du$

$= \int 3 + \frac{3}{u}\,du = 3u + 3\ln|u| + C = 3(2x-1) + 3\ln|2x-1| + C$

$u = 2x-1$, $\frac{du}{dx} = 2$, $dx = \frac{du}{2}$, $2x = u+1$, $6x = 3u+3$

3. $\int x(x^2-1)^{\frac{5}{2}}\,dx = \int x u^{\frac{5}{2}} \times \frac{du}{2x} = \int \frac{1}{2}u^{\frac{5}{2}}\,du$

$\frac{\frac{1}{2}}{\frac{7}{2}}u^{\frac{7}{2}} + C = \frac{1}{7}(x^2+1)^{\frac{7}{2}} + C$

$u = x^2+1$, $\frac{du}{dx} = 2x$, $dx = \frac{du}{2x}$

4. $\int \frac{5x^5}{2x^6+7}\,dx = \int \frac{5x^5}{u} \times \frac{du}{12x^5} = \frac{5}{12}\int \frac{1}{u}\,du$

$= \frac{5}{12}\ln|u| + C = \frac{5}{12}\ln|2x^6+7| + C$

$u = 2x^6+7$, $\frac{du}{dx} = 12x^5$, $dx = \frac{du}{12x^5}$

5. $\int 2x(1-5x)^4\,dx = \int 2x u^4 \times \frac{du}{-5} = -\frac{2}{5}\int x u^4\,du$

$= -\frac{2}{5}\int \frac{1-u}{5} \times u^4\,du = -\frac{2}{25}\int (1-u)u^4\,du = -\frac{2}{25}\int u^4 - u^5\,du$

$= -\frac{2}{25}\left[\frac{1}{5}u^5 - \frac{1}{6}u^6\right] + C = -\frac{2}{125}u^5 + \frac{1}{75}u^6 + C$

$= \frac{1}{75}(1-5x)^6 - \frac{2}{125}(1-5x)^5 + C$

$u = 1-5x$, $\frac{du}{dx} = -5$, $dx = \frac{du}{-5}$, $5x = 1-u$, $x = \frac{1-u}{5}$

6. $\int \frac{9x}{\sqrt{4x^2+1}}\,dx = \int \frac{9x}{u^{\frac{1}{2}}}\frac{du}{8x} = \frac{9}{8}\int \frac{1}{u^{\frac{1}{2}}}\,du$

$\frac{9}{8}\int u^{-\frac{1}{2}}\,du = \frac{9}{8}\left(2u^{\frac{1}{2}}\right) + C = \frac{9}{4}u^{\frac{1}{2}} + C$

$= \frac{9}{4}\sqrt{4x^2+1} + C$

$u = 4x^2+1$, $\frac{du}{dx} = 8x$, $dx = \frac{du}{8x}$

7. $\int \frac{3x-1}{\sqrt{4x-1}}\,dx = \int \frac{3x-1}{u^{\frac{1}{2}}}\frac{du}{4} = \frac{1}{4}\int \frac{3\left(\frac{u+1}{4}\right)-1}{u^{\frac{1}{2}}}\,du$

MULTIPLY TOP AND BOTTOM OF THE FRACTION BY 4

$= \frac{1}{4}\int \frac{3(u+1)-4}{4u^{\frac{1}{2}}}\,du = \frac{1}{4}\int \frac{3u-1}{4u^{\frac{1}{2}}}\,du = \frac{1}{16}\int \frac{3u}{u^{\frac{1}{2}}} - \frac{1}{u^{\frac{1}{2}}}\,du$

$= \frac{1}{16}\int 3u^{\frac{1}{2}} - u^{-\frac{1}{2}}\,du = \frac{1}{16}\left[2u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\right] = \frac{1}{8}u^{\frac{3}{2}} - \frac{1}{8}u^{\frac{1}{2}} + C$

$= \frac{1}{8}(4x-1)^{\frac{3}{2}} - \frac{1}{8}(4x-1)^{\frac{1}{2}} + C$

$u = 4x-1$, $\frac{du}{dx} = 4$, $dx = \frac{du}{4}$, $4x = u+1$, $x = \frac{u+1}{4}$

8. $\int \frac{1-2x}{1+3x}\,dx = \int \frac{1-2x}{u}\frac{du}{3} = \frac{1}{3}\int \frac{1-2\left(\frac{u-1}{3}\right)}{u}\,du$

MULTIPLY TOP AND BOTTOM OF THE FRACTION BY 3

$= \frac{1}{3}\int \frac{3-2(u-1)}{3u}\,du = \frac{1}{3}\int \frac{3-2u+2}{3u}\,du = \frac{1}{9}\int \frac{5-2u}{u}\,du$

$= \frac{1}{9}\int \frac{5}{u} - 2\,du = \frac{1}{9}\left[5\ln|u| - 2u\right] + C = \frac{5}{9}\ln|u| - \frac{2}{9}u + C$

$= \frac{5}{9}\ln|1+3x| - \frac{2}{9}(1+3x) + C$

$u = 1+3x$, $u-1 = 3x$, $x = \frac{u-1}{3}$, $\frac{du}{dx} = 3$, $dx = \frac{du}{3}$

9. $\int \frac{6x^{\frac{1}{2}}}{2x^{\frac{3}{2}}+3}\,dx = \int \frac{6x^{\frac{1}{2}}}{u}\frac{du}{3x^{\frac{1}{2}}} = \int \frac{2}{u}\,du$

$= 2\ln|u| + C = 2\ln|2x^{\frac{3}{2}}+3| + C$

$u = 2x^{\frac{3}{2}}+3$, $\frac{du}{dx} = 3x^{\frac{1}{2}}$, $dx = \frac{du}{3x^{\frac{1}{2}}}$

10. $\int \frac{x^{\frac{3}{2}}}{\sqrt{1-3x^{\frac{5}{2}}}}\,dx = \int \frac{x^{\frac{3}{2}}}{\sqrt{u}}\frac{2\,du}{-15x^{\frac{3}{2}}}$

$= -\frac{2}{15}\int \frac{1}{\sqrt{u}}\,du = -\frac{2}{15}\int u^{-\frac{1}{2}}\,du$

$= -\frac{2}{15}\left(2u^{\frac{1}{2}}\right) + C = -\frac{4}{15}u^{\frac{1}{2}} + C = -\frac{4}{15}\sqrt{1-3x^{\frac{5}{2}}} + C$

$u = 1-3x^{\frac{5}{2}}$, $\frac{du}{dx} = -\frac{15}{2}x^{\frac{3}{2}}$, $2\,du = -15x^{\frac{3}{2}}\,dx$, $dx = \frac{2\,du}{-15x^{\frac{3}{2}}}$

Question 4

Carry out the following integrations to the answers given, **by using substitution** only.

1. $\displaystyle\int_0^{\frac{1}{2}} 8x(2x-1)^4 \, dx = \frac{1}{15}$

2. $\displaystyle\int_2^3 \frac{3x}{3x-5} \, dx = 1 + \frac{10}{3}\ln 2$

3. $\displaystyle\int_0^1 x\left(1-x^2\right)^{\frac{3}{2}} \, dx = \frac{1}{5}$

4. $\displaystyle\int_0^1 \frac{4x}{x^2+1} \, dx = 2\ln 2$

5. $\displaystyle\int_1^3 2x(3x-1)^4 \, dx = \frac{55808}{5}$

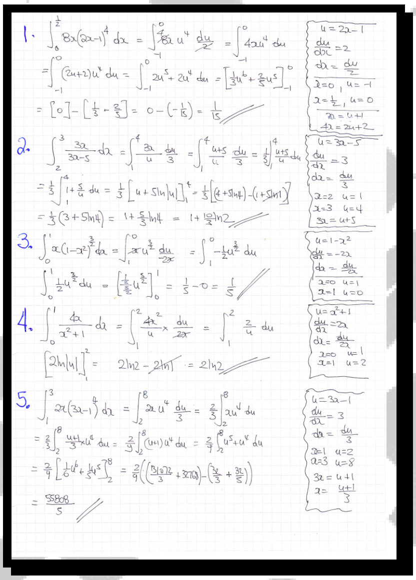
6. $\displaystyle\int_4^8 \frac{6x}{\sqrt{2x-7}} \, dx = 68$

7. $\displaystyle\int_0^1 \frac{x}{\sqrt{9-5x^2}} \, dx = \frac{1}{5}$

8. $\displaystyle\int_0^3 \frac{5-2x}{x+1} \, dx = 14\ln 2 - 6$

9. $\displaystyle\int_0^{\frac{1}{5}} \frac{10x^2}{5x+1} \, dx = \frac{\ln 4 - 1}{25}$

10. $\displaystyle\int_{-\frac{3}{2}}^{-\frac{1}{2}} \frac{5x-2}{2x-5} \, dx = \frac{5}{2} + \frac{21}{4}\ln\left(\frac{3}{4}\right)$



1. $\int_0^{\frac{1}{2}} 8x(2x-1)^4\,dx = \int_{-1}^{0} 8x\,u^4\,\frac{du}{2} = \int_{-1}^{0} 4xu^4\,du$
$= \int_{-1}^{0} (2u+2)u^4\,du = \int_{-1}^{0} 2u^5 + 2u^4\,du = \left[\frac{1}{3}u^6 + \frac{2}{5}u^5\right]_{-1}^{0}$
$= [0] - \left[\frac{1}{3} - \frac{2}{5}\right] = 0 - \left(-\frac{1}{15}\right) = \frac{1}{15}$

$u = 2x-1$, $\frac{du}{dx} = 2$, $dx = \frac{du}{2}$; $x=0,\ u=-1$; $x=\frac{1}{2},\ u=0$; $2x = u+1$; $4x = 2u+2$

2. $\int_2^3 \frac{3x}{3x-5}\,dx = \int_1^4 \frac{3x}{u}\,\frac{du}{3} = \int_1^4 \frac{u+5}{u}\,\frac{du}{3} = \frac{1}{3}\int_1^4 \frac{u+5}{u}\,du$
$= \frac{1}{3}\int_1^4 1 + \frac{5}{u}\,du = \frac{1}{3}\left[u + 5\ln|u|\right]_1^4 = \frac{1}{3}\left[(4+5\ln 4) - (1+5\ln 1)\right]$
$= \frac{1}{3}(3 + 5\ln 4) = 1 + \frac{5}{3}\ln 4 = 1 + \frac{10}{3}\ln 2$

$u = 3x-5$, $\frac{du}{dx} = 3$, $dx = \frac{du}{3}$; $x=2\ \ u=1$; $x=3\ \ u=4$; $3x = u+5$

3. $\int_0^1 x(1-x^2)^{\frac{3}{2}}\,dx = \int_1^0 x u^{\frac{3}{2}}\,\frac{du}{-2x} = \int_1^0 -\frac{1}{2}u^{\frac{3}{2}}\,du$
$\int_0^1 \frac{1}{2}u^{\frac{3}{2}}\,du = \left[\frac{\frac{1}{2}}{\frac{5}{2}}u^{\frac{5}{2}}\right]_0^1 = \frac{1}{5} - 0 = \frac{1}{5}$

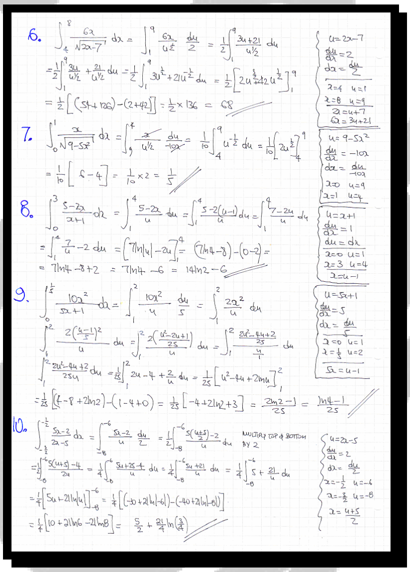
$u = 1-x^2$, $\frac{du}{dx} = -2x$, $dx = \frac{du}{-2x}$; $x=0\ \ u=1$; $x=1\ \ u=0$

4. $\int_0^1 \frac{4x}{x^2+1}\,dx = \int_1^2 \frac{4x}{u} \times \frac{du}{2x} = \int_1^2 \frac{2}{u}\,du$
$\left[2\ln|u|\right]_1^2 = 2\ln 2 - 2\ln 1 = 2\ln 2$

$u = x^2+1$, $\frac{du}{dx} = 2x$, $dx = \frac{du}{2x}$; $x=0\ \ u=1$; $x=1\ \ u=2$

5. $\int_1^3 x(3x-1)^4\,dx = \int_2^8 x u^4\,\frac{du}{3} = \frac{2}{3}\int_2^8 xu^4\,du$
$= \frac{2}{3}\int_2^8 \frac{u+1}{3} \times u^4\,du = \frac{2}{9}\int_2^8 (u+1)u^4\,du = \frac{2}{9}\int_2^8 u^5 + u^4\,du$
$= \frac{2}{9}\left[\frac{1}{6}u^6 + \frac{1}{5}u^5\right]_2^8 = \frac{2}{9}\left(\left(\frac{131072}{3} + 32768\right) - \left(\frac{32}{3} + \frac{32}{5}\right)\right)$
$= \frac{55808}{5}$

$u = 3x-1$, $\frac{du}{dx} = 3$, $dx = \frac{du}{3}$; $x=1\ \ u=2$; $x=3\ \ u=8$; $3x = u+1$; $x = \frac{u+1}{3}$



6. $\int_4^8 \frac{6x}{\sqrt{2x-7}}\,dx = \int_1^9 \frac{6x}{u^{\frac{1}{2}}}\,\frac{du}{2} = \frac{1}{2}\int_1^9 \frac{3u+21}{u^{1/2}}\,du$
$= \frac{1}{2}\int_1^9 \frac{3u}{u^{1/2}} + \frac{21}{u^{1/2}}\,du = \frac{1}{2}\int_1^9 3u^{\frac{1}{2}} + 21u^{-\frac{1}{2}}\,du = \frac{1}{2}\left[2u^{\frac{3}{2}} + 42u^{\frac{1}{2}}\right]_1^9$
$= \frac{1}{2}\left[(54+126) - (2+42)\right] = \frac{1}{2} \times 136 = 68$

$u = 2x-7$, $\frac{du}{dx} = 2$, $dx = \frac{du}{2}$; $x=4\ \ u=1$; $x=8\ \ u=9$; $2x = u+7$; $6x = 3u+21$

7. $\int_0^1 \frac{x}{\sqrt{9-5x^2}}\,dx = \int_9^4 \frac{x}{u^{1/2}}\,\frac{du}{-10x} = \frac{1}{10}\int_4^9 u^{-\frac{1}{2}}\,du = \frac{1}{10}\left[2u^{\frac{1}{2}}\right]_4^9$
$= \frac{1}{10}\left[6 - 4\right] = \frac{1}{10} \times 2 = \frac{1}{5}$

$u = 9-5x^2$, $\frac{du}{dx} = -10x$, $dx = \frac{du}{-10x}$; $x=0\ \ u=9$; $x=1\ \ u=4$

8. $\int_0^3 \frac{5-2x}{x+1}\,dx = \int_1^4 \frac{5-2x}{u}\,du = \int_1^4 \frac{5-2(u-1)}{u}\,du = \int_1^4 \frac{7-2u}{u}\,du$
$= \int_1^4 \frac{7}{u} - 2\,du = \left[7\ln|u| - 2u\right]_1^4 = (7\ln 4 - 8) - (0-2) =$
$= 7\ln 4 - 8 + 2 = 7\ln 4 - 6 = 14\ln 2 - 6$

$u = x+1$, $\frac{du}{dx} = 1$, $du = dx$; $x=0\ \ u=1$; $x=3\ \ u=4$; $x = u-1$

9. $\int_0^{\frac{1}{5}} \frac{10x^2}{5x+1}\,dx = \int_1^2 \frac{10x^2}{u}\,\frac{du}{5} = \int_1^2 \frac{2x^2}{u}\,du$
$\int_1^2 \frac{2\left(\frac{u-1}{5}\right)^2}{u}\,du = \int_1^2 \frac{2\left(\frac{u^2-2u+1}{25}\right)}{u}\,du = \int_1^2 \frac{\frac{2u^2-4u+2}{25}}{\frac{u}{1}}\,du$
$\int_1^2 \frac{2u^2-4u+2}{25u}\,du = \frac{1}{25}\int_1^2 2u - 4 + \frac{2}{u}\,du = \frac{1}{25}\left[u^2 - 4u + 2\ln u\right]_1^2$
$= \frac{1}{25}\left[(4 - 8 + 2\ln 2) - (1 - 4 + 0)\right] = \frac{1}{25}\left[-4 + 2\ln 2 + 3\right] = \frac{2\ln 2 - 1}{25} = \frac{\ln 4 - 1}{25}$

$u = 5x+1$, $\frac{du}{dx} = 5$, $dx = \frac{du}{5}$; $x=0\ \ u=1$; $x=\frac{1}{5}\ \ u=2$; $5x = u-1$

10. $\int_{-\frac{3}{2}}^{-\frac{1}{2}} \frac{5x-2}{2x-5}\,dx = \int_{-8}^{-6} \frac{5x-2}{u}\,\frac{du}{2} = \frac{1}{2}\int_{-8}^{-6} \frac{5\left(\frac{u+5}{2}\right)-2}{u}\,du$ MULTIPLY TOP & BOTTOM BY 2
$= \frac{1}{2}\int_{-8}^{-6} \frac{5(u+5)-4}{2u} = \frac{1}{4}\int_{-8}^{-6} \frac{5u+25-4}{u}\,du = \frac{1}{4}\int_{-8}^{-6} \frac{5u+21}{u}\,du = \frac{1}{4}\int_{-8}^{-6} 5 + \frac{21}{u}\,du$
$= \frac{1}{4}\left[5u + 21\ln|u|\right]_{-8}^{-6} = \frac{1}{4}\left[(-30 + 21\ln|-6|) - (-40 + 21\ln|-8|)\right]$
$= \frac{1}{4}\left[10 + 21\ln 6 - 21\ln 8\right] = \frac{5}{2} + \frac{21}{4}\ln\left(\frac{3}{4}\right)$

$u = 2x-5$, $\frac{du}{dx} = 2$, $dx = \frac{du}{2}$; $x=-\frac{1}{2}\ \ u=-6$; $x=-\frac{3}{2}\ \ u=-8$; $x = \frac{u+5}{2}$

Question 5

Carry out the following integrations to the answers given, **by using substitution** only.

1. $\displaystyle\int_0^{\frac{3}{2}} 2x(2x-3)^4 \, dx = \frac{243}{20}$

2. $\displaystyle\int_0^{2} \frac{4x}{4x+1} \, dx = 2-\frac{1}{2}\ln 3$

3. $\displaystyle\int_0^{1} x^2\left(1-x^3\right)^{\frac{9}{2}} \, dx = \frac{2}{33}$

4. $\displaystyle\int_0^{4} \frac{12x}{x^2+9} \, dx = 12\ln\left(\frac{5}{3}\right)$

5. $\displaystyle\int_1^{2} 2x(3x-1)^4 \, dx = \frac{3569}{5}$

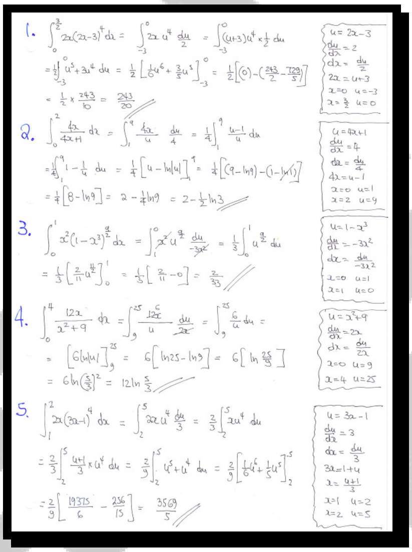
6. $\displaystyle\int_2^{6} \frac{6x}{\sqrt{3x-2}} \, dx = \frac{272}{9}$

7. $\displaystyle\int_0^{1} \frac{x}{\sqrt{16-7x^2}} \, dx = \frac{1}{7}$

8. $\displaystyle\int_5^{6} \frac{1-2x}{x-4} \, dx = -2-7\ln 2$

9. $\displaystyle\int_0^{\frac{1}{3}} \frac{9x^2}{3x+1} \, dx = \frac{\ln 4-1}{6}$

10. $\displaystyle\int_0^{\frac{3}{2}} \frac{2x-3}{2x+3} \, dx = \frac{3}{2}(1-\ln 4)$



1. $\int_0^{\frac{3}{2}} 2x(2x-3)^4\,dx = \int_{-3}^{0} 2x\,u^4\,\frac{du}{2} = \int_{-3}^{0}(u+3)u^4 \times \frac{1}{2}\,du$

$= \frac{1}{2}\int_{-3}^{0} u^5 + 3u^4\,du = \frac{1}{2}\left[\frac{1}{6}u^6 + \frac{3}{5}u^5\right]_{-3}^{0} = \frac{1}{2}\left[(0) - \left(\frac{243}{2} - \frac{729}{5}\right)\right]$

$= \frac{1}{2} \times \frac{243}{10} = \frac{243}{20}$

$u = 2x-3$, $\frac{du}{dx} = 2$, $dx = \frac{du}{2}$, $2x = u+3$, $x=0 \;\; u=-3$, $x=\frac{3}{2} \;\; u=0$

2. $\int_0^2 \frac{4x}{4x+1}\,dx = \int_1^9 \frac{4x}{u}\,\frac{du}{4} = \frac{1}{4}\int_1^9 \frac{u-1}{u}\,du$

$= \frac{1}{4}\int_1^9 1 - \frac{1}{u}\,du = \frac{1}{4}\left[u - \ln|u|\right]_1^9 = \frac{1}{4}\left[(9-\ln 9) - (1 - \ln 1)\right]$

$= \frac{1}{4}\left[8 - \ln 9\right] = 2 - \frac{1}{4}\ln 9 = 2 - \frac{1}{2}\ln 3$

$u = 4x+1$, $\frac{du}{dx} = 4$, $dx = \frac{du}{4}$, $4x = u-1$, $x=0 \;\; u=1$, $x=2 \;\; u=9$

3. $\int_0^1 x^2(1-x^3)^{\frac{9}{2}}\,dx = \int_1^0 x^2 u^{\frac{9}{2}}\,\frac{du}{-3x^2} = \frac{1}{3}\int_0^1 u^{\frac{9}{2}}\,du$

$= \frac{1}{3}\left[\frac{2}{11}u^{\frac{11}{2}}\right]_0^1 = \frac{1}{3}\left[\frac{2}{11} - 0\right] = \frac{2}{33}$

$u = 1-x^3$, $\frac{du}{dx} = -3x^2$, $dx = \frac{du}{-3x^2}$, $x=0 \;\; u=1$, $x=1 \;\; u=0$

4. $\int_0^4 \frac{12x}{x^2+9}\,dx = \int_9^{25} \frac{12x}{u}\,\frac{du}{2x} = \int_9^{25} \frac{6}{u}\,du =$

$= \left[6\ln|u|\right]_9^{25} = 6\left[\ln 25 - \ln 9\right] = 6\left[\ln\frac{25}{9}\right]$

$= 6\ln\left(\frac{5}{3}\right)^2 = 12\ln\frac{5}{3}$

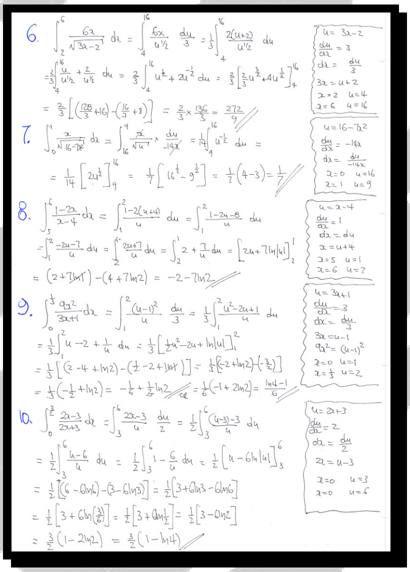
$u = x^2+9$, $\frac{du}{dx} = 2x$, $dx = \frac{du}{2x}$, $x=0 \;\; u=9$, $x=4 \;\; u=25$

5. $\int_1^2 2x(3x-1)^4\,dx = \int_2^5 2x\,u^4\,\frac{du}{3} = \frac{2}{3}\int_2^5 x u^4\,du$

$= \frac{2}{3}\int_2^5 \frac{u+1}{3} \times u^4\,du = \frac{2}{9}\int_2^5 u^5 + u^4\,du = \frac{2}{9}\left[\frac{1}{6}u^6 + \frac{1}{5}u^5\right]_2^5$

$= \frac{2}{9}\left[\frac{19375}{6} - \frac{256}{15}\right] = \frac{3569}{5}$

$u = 3x-1$, $\frac{du}{dx} = 3$, $dx = \frac{du}{3}$, $3x = 1+u$, $x = \frac{u+1}{3}$, $x=1 \;\; u=2$, $x=2 \;\; u=5$



6. $\int_2^6 \frac{6x}{\sqrt{3x-2}}\,dx = \int_4^{16} \frac{6x}{u^{1/2}}\,\frac{du}{3} = \frac{1}{3}\int_4^{16} \frac{2(u+2)}{u^{1/2}}\,du$

$= \frac{2}{3}\int_4^{16} \frac{u}{u^{1/2}} + \frac{2}{u^{1/2}}\,du = \frac{2}{3}\int_4^{16} u^{\frac{1}{2}} + 2u^{-\frac{1}{2}}\,du = \frac{2}{3}\left[\frac{2}{3}u^{\frac{3}{2}} + 4u^{\frac{1}{2}}\right]_4^{16}$

$= \frac{2}{3}\left[\left(\frac{128}{3}+16\right) - \left(\frac{16}{3}+8\right)\right] = \frac{2}{3} \times \frac{136}{3} = \frac{272}{9}$

$u = 3x-2$, $\frac{du}{dx} = 3$, $dx = \frac{du}{3}$, $3x = u+2$, $x=2 \;\; u=4$, $x=6 \;\; u=16$

7. $\int_0^1 \frac{x}{\sqrt{16-7x^2}}\,dx = \int_{16}^{9} \frac{x}{\sqrt{u}} \times \frac{du}{-14x} = \frac{1}{14}\int_9^{16} u^{-\frac{1}{2}}\,du =$

$= \frac{1}{14}\left[2u^{\frac{1}{2}}\right]_9^{16} = \frac{1}{7}\left[16^{\frac{1}{2}} - 9^{\frac{1}{2}}\right] = \frac{1}{7}(4-3) = \frac{1}{7}$

$u = 16-7x^2$, $\frac{du}{dx} = -14x$, $dx = \frac{du}{-14x}$, $x=0 \;\; u=16$, $x=1 \;\; u=9$

8. $\int_5^6 \frac{1-2x}{x-4}\,dx = \int_1^2 \frac{1-2(u+4)}{u}\,du = \int_1^2 \frac{1-2u-8}{u}\,du$

$= \int_1^2 \frac{-2u-7}{u}\,du = \int_2^1 \frac{2u+7}{u}\,du = \int_2^1 2 + \frac{7}{u}\,du = \left[2u + 7\ln|u|\right]_2^1$

$= (2 + 7\ln 1) - (4 + 7\ln 2) = -2 - 7\ln 2$

$u = x-4$, $\frac{du}{dx} = 1$, $dx = du$, $x = u+4$, $x=5 \;\; u=1$, $x=6 \;\; u=2$

9. $\int_0^{\frac{1}{3}} \frac{9x^2}{3x+1}\,dx = \int_1^2 \frac{(u-1)^2}{u}\,\frac{du}{3} = \frac{1}{3}\int_1^2 \frac{u^2-2u+1}{u}\,du$

$= \frac{1}{3}\int_1^2 u - 2 + \frac{1}{u}\,du = \frac{1}{3}\left[\frac{1}{2}u^2 - 2u + \ln|u|\right]_1^2$

$= \frac{1}{3}\left[(2 - 4 + \ln 2) - \left(\frac{1}{2} - 2 + \ln 1\right)\right] = \frac{1}{3}\left[(-2+\ln 2) - \left(-\frac{3}{2}\right)\right]$

$= \frac{1}{3}\left(-\frac{1}{2} + \ln 2\right) = -\frac{1}{6} + \frac{1}{3}\ln 2$ or $= \frac{1}{6}(-1 + 2\ln 2) = \frac{\ln 4 - 1}{6}$

$u = 3x+1$, $\frac{du}{dx} = 3$, $dx = \frac{du}{3}$, $3x = u-1$, $9x^2 = (u-1)^2$, $x=0 \;\; u=1$, $x=\frac{1}{3} \;\; u=2$

10. $\int_0^{\frac{3}{2}} \frac{2x-3}{2x+3}\,dx = \int_3^6 \frac{2x-3}{u}\,\frac{du}{2} = \frac{1}{2}\int_3^6 \frac{(u-3)-3}{u}\,du$

$= \frac{1}{2}\int_3^6 \frac{u-6}{u}\,du = \frac{1}{2}\int_3^6 1 - \frac{6}{u}\,du = \frac{1}{2}\left[u - 6\ln|u|\right]_3^6$

$= \frac{1}{2}\left[(6 - 6\ln 6) - (3 - 6\ln 3)\right] = \frac{1}{2}\left[3 + 6\ln 3 - 6\ln 6\right]$

$= \frac{1}{2}\left[3 + 6\ln\left(\frac{3}{6}\right)\right] = \frac{1}{2}\left[3 + 6\ln\frac{1}{2}\right] = \frac{1}{2}\left[3 - 6\ln 2\right]$

$= \frac{3}{2}(1 - 2\ln 2) = \frac{3}{2}(1 - \ln 4)$

$u = 2x+3$, $\frac{du}{dx} = 2$, $dx = \frac{du}{2}$, $2x = u-3$, $x=0 \;\; u=3$, $x=0 \;\; u=6$

Question 6

Carry out the following integrations.

1. $\int \frac{x}{\sqrt{x+1}}\,dx = \frac{2}{3}(x+1)^{\frac{3}{2}} - 2(x+1)^{\frac{1}{2}} + C = \frac{2}{3}(x-2)\sqrt{x+1} + C$

2. $\int \frac{2x}{(2x+1)^3}\,dx = -\frac{1}{2}(2x+1)^{-1} + \frac{1}{4}(2x+1)^{-2} + C = -\frac{4x+1}{4(2x+1)^2} + C$

3. $\int \frac{x}{x+1}\,dx = x - \ln|x+1| + C = x + 1 - \ln|x+1| + C$

4. $\int \frac{x}{\sqrt{x-1}}\,dx = \frac{2}{3}(x-1)^{\frac{3}{2}} + 2(x-1)^{\frac{1}{2}} + C = \frac{2}{3}(x+2)\sqrt{x-1} + C$

5. $\int \frac{4x+1}{2x-5}\,dx = 2x + \frac{11}{2}\ln|2x-5| + C$

6. $\int \frac{x^2}{2x-1}\,dx = \frac{1}{16}(2x-1)^2 + \frac{1}{4}(2x-1) + \frac{1}{8}\ln|2x-1| + C = \frac{1}{4}x^2 + \frac{1}{4}x + \frac{1}{8}\ln|2x-1| + C$

7. $\int \frac{2x+1}{2x-1}\,dx = \frac{1}{2}(2x-1) + \ln|2x-1| + C = x + \ln|2x-1| + C$

8. $\int \frac{6x}{\sqrt{2x+3}}\,dx = (2x+3)^{\frac{3}{2}} - 9(2x+3)^{\frac{1}{2}} + C = 2(x-3)\sqrt{2x+3} + C$

9. $\int \frac{3x-1}{2x+3}\,dx = \frac{3}{4}(2x-3) - \frac{11}{4}\ln|2x+3| + C = \frac{3}{2}x - \frac{11}{4}\ln|2x+3| + C$

10. $\int \frac{8x^2}{1-2x}\,dx = -\frac{1}{2}(1-2x)^2 + 2(1-2x) - \ln|1-2x| + C = -2x^2 - 2x - \ln|1-2x| + C$

1. $\int \frac{x}{\sqrt{x+1}}\,dx = \int \frac{u-1}{\sqrt{u}}\,du = \int \frac{u-1}{u^{1/2}}\,du$

$= \int u^{\frac{1}{2}} - u^{-\frac{1}{2}}\,du = \frac{2}{3}u^{\frac{3}{2}} - 2u^{\frac{1}{2}} + C$

$= \frac{2}{3}(x+1)^{\frac{3}{2}} - 2(x+1)^{\frac{1}{2}} + C$

$u = x+1$, $\frac{du}{dx} = 1$, $du = dx$

2. $\int \frac{2x}{(2x+1)^3}\,dx = \int \frac{2x}{u^3} \times \frac{du}{2} = \frac{1}{2}\int \frac{u-1}{u^3}\,du$

$= \frac{1}{2}\int \frac{u}{u^3} - \frac{1}{u^3}\,du = \frac{1}{2}\int u^{-2} - u^{-3}\,du = \frac{1}{2}\left[-u^{-1} + \frac{1}{2}u^{-2}\right]$

$= \frac{1}{2}\left[\frac{1}{2u^2} - \frac{1}{u}\right] + C = \frac{1}{4(2x+1)^2} - \frac{1}{2(2x+1)} + C$

$u = 2x+1$, $\frac{du}{dx} = 2$, $dx = \frac{du}{2}$, $2x = u-1$

3. $\int \frac{x}{x+1}\,dx = \int \frac{x}{u}\,du = \int \frac{u-1}{u}\,du = \int 1 - \frac{1}{u}\,du$

$= u - \ln|u| + C = (x+1) - \ln|x+1| + C$

$u = x+1$, $\frac{du}{dx} = 1$, $dx = du$, $x = u-1$

4. $\int \frac{x}{\sqrt{x+1}}\,dx = \int \frac{x}{\sqrt{u}}\,du = \int \frac{u+1}{u^{1/2}}\,du = \int u^{\frac{1}{2}} + u^{-\frac{1}{2}}\,du$

$= \frac{2}{3}u^{\frac{3}{2}} + 2u^{\frac{1}{2}} + C = \frac{2}{3}(x-1)^{\frac{3}{2}} + 2(x-1)^{\frac{1}{2}} + C$

$u = x-1$, $\frac{du}{dx} = 1$, $dx = du$, $x = u+1$

5. $\int \frac{4x+1}{2x-5}\,dx = \int \frac{4x+1}{u}\,\frac{du}{2} = \frac{1}{2}\int \frac{2u+11}{u}\,du$

$= \frac{1}{2}\int \frac{2u}{u} + \frac{11}{u}\,du = \frac{1}{2}\int 2 + \frac{11}{u}\,du$

$= \frac{1}{2}\left[2u + 11\ln|u|\right] + C = u + \frac{11}{2}\ln|u| + C$

$= (2x-5) + \frac{11}{2}\ln|2x-5| + C$

$u = 2x-5$, $\frac{du}{dx} = 2$, $dx = \frac{du}{2}$, $2x = u+5$, $4x = 2u+10$, $4x+1 = 2u+11$

6. $\int \frac{x^2}{2x-1}\,dx = \int \frac{x^2}{u}\,\frac{du}{2} = \frac{1}{2}\int \frac{\frac{(u+1)^2}{4}}{\frac{u}{1}}\,du$

$= \frac{1}{2}\int \frac{(u+1)^2}{4u}\,du = \frac{1}{8}\int \frac{(u+1)^2}{u}\,du = \frac{1}{8}\int \frac{u^2+2u+1}{u}\,du$

$= \frac{1}{8}\int u + 2 + \frac{1}{u}\,du = \frac{1}{8}\left(\frac{1}{2}u^2 + 2u + \ln|u|\right) + C$

$= \frac{1}{16}u^2 + \frac{1}{4}u + \frac{1}{8}\ln|u| + C = \frac{1}{16}(2x-1)^2 + \frac{1}{4}(2x-1) + \frac{1}{8}\ln|2x-1| + C$

$u = 2x-1$, $\frac{du}{dx} = 2$, $dx = \frac{du}{2}$, $2x = u+1$, $4x^2 = (u+1)^2$, $x^2 = \frac{(u+1)^2}{4}$

7. $\int \frac{2x+1}{2x-1}\,dx = \int \frac{(u+1)+1}{u}\,\frac{du}{2} = \frac{1}{2}\int \frac{u+2}{u}\,du$

$= \frac{1}{2}\int 1 + \frac{2}{u}\,du = \frac{1}{2}\left(u + 2\ln|u|\right) + C = \frac{1}{2}u + \ln|u| + C$

$= \frac{1}{2}(2x-1) + \ln|2x-1| + C$

$u = 2x-1$, $\frac{du}{dx} = 2$, $dx = \frac{du}{2}$, $2x = u+1$

8. $\int \frac{6x}{\sqrt{2x+3}}\,dx = \int \frac{3u-9}{u^{1/2}}\,\frac{du}{2} = \frac{1}{2}\int \frac{3u}{u^{1/2}} - \frac{9}{u^{1/2}}\,du$

$= \frac{1}{2}\int 3u^{\frac{1}{2}} - 9u^{-\frac{1}{2}}\,du = \frac{1}{2}\left[2u^{\frac{3}{2}} - 18u^{\frac{1}{2}}\right] + C$

$= u^{\frac{3}{2}} - 9u^{\frac{1}{2}} + C = (2x+3)^{\frac{3}{2}} - 9(2x+3)^{\frac{1}{2}} + C$

$u = 2x+3$, $\frac{du}{dx} = 2$, $dx = \frac{du}{2}$, $2x = u-3$, $6x = 3u-9$

9. $\int \frac{3x-1}{2x+3}\,dx = \int \frac{3\times\left(\frac{u-3}{2}\right)-1}{u}\,\frac{du}{2} = \frac{1}{2}\int \frac{\frac{3u-9}{2}-1}{u}\,du$

$= \frac{1}{2}\int \frac{\frac{3u-9}{2} - \frac{2}{2}}{u}\,du = \frac{1}{2}\int \frac{\frac{3u-11}{2}}{\frac{u}{1}}\,du = \frac{1}{2}\int \frac{3u-11}{2u}\,du$

$= \frac{1}{4}\int \frac{3u}{u} - \frac{11}{u}\,du = \frac{1}{4}\int 3 - \frac{11}{u}\,du = \frac{1}{4}\left(3u - 11\ln|u| + C\right)$

$= \frac{3}{4}u \quad \frac{11}{4}\ln|2x+3| + C$

$u = 2x+3$, $\frac{du}{dx} = 2$, $dx = \frac{du}{2}$, $2x = u-3$, $x = \frac{u-3}{2}$

10. $\int \frac{8x^2}{1-2x}\,dx = \int \frac{8x^2}{u} \times \frac{du}{-2} = -\int \frac{4x^2}{u}\,du$

$= -\int \frac{(1-u)^2}{u}\,du = -\int \frac{1-2u+u^2}{u}\,du = -\int \frac{1}{u} - 2 + u\,du$

$= \int -\frac{1}{u} + 2 - u\,du = -\ln|u| + 2u - \frac{1}{2}u^2 + C$

$= -\frac{1}{2}(1-2x)^2 + 2(1-2x) - \ln|1-2x| + C$

$u = 1-2x$, $2x = 1-u$, $\frac{du}{dx} = -2$, $dx = \frac{du}{-2}$, $4x^2 = (1-u)^2$

Question 7

Carry out the following integrations **using the substitutions** given.

1. $$\int x\sqrt{1-x}\,dx = \frac{2}{5}(1-x)^{\frac{5}{2}} - \frac{2}{3}(1-x)^{\frac{3}{2}} + C$$ Use $u = 1-x$, or $u = \sqrt{1-x}$

2. $$\int \frac{6x}{\sqrt{2x+1}}\,dx = (2x+1)^{\frac{3}{2}} - 3(2x+1)^{\frac{1}{2}} + C$$ Use $u = 2x+1$, or $u = \sqrt{2x+1}$

3. $$\int \cos^3 x\,dx = \sin x - \frac{1}{3}\sin^3 x + C$$ Use $u = \sin x$

4. $$\int \sec^4 x\,dx = \tan x + \frac{1}{3}\tan^3 x + C$$ Use $u = \tan x$

5. $$\int \frac{1}{\sqrt{x}(x-4)}\,dx = \frac{1}{2}\ln\left|\frac{\sqrt{x}-2}{\sqrt{x}+2}\right| + C$$ Use $u = \sqrt{x}$

6. $$\int \frac{\sqrt{x^2+9}}{x}\,dx = \sqrt{x^2+9} + \frac{3}{2}\ln\left|\frac{\sqrt{x^2+9}-3}{\sqrt{x^2+9}+3}\right| + C$$ Use $u = \sqrt{x^2+9}$

7. $$\int \frac{1+\cos x}{\sin x}\,dx = \ln\left|\cos x - 1\right| + C$$ Use $u = \cos x$

8. $$\int \frac{1}{1+\sqrt{x-2}}\,dx = 2\sqrt{x-2} + 2\ln\left(1+\sqrt{x-2}\right) + C$$ Use $u = \sqrt{x-2}$

9. $$\int \sec^2 x \tan x\sqrt{1+\tan x}\,dx = \frac{2}{15}(3\tan x - 2)(1+\tan x)^{\frac{3}{2}} + C$$ Use $u = \sqrt{1+\tan x}$

10. $$\int \frac{9}{\sqrt{x}(9x-1)}\,dx = 3\ln\left|\frac{3\sqrt{x}-1}{3\sqrt{x}+1}\right| + C$$ Use $u = \sqrt{x}$

1. $\int x\sqrt{1-x}\,dx = \int x\sqrt{u}\,(-du)$

$= \int (1-u)u^{\frac{1}{2}}(-du) = \int (u-1)u^{\frac{1}{2}}\,du = \int u^{\frac{3}{2}} - u^{\frac{1}{2}}\,du$

$= \frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}} + C = \frac{2}{5}(1-x)^{\frac{5}{2}} - \frac{2}{3}(1-x)^{\frac{3}{2}} + C$

$u = 1-x$, $\frac{du}{dx} = -1$, $dx = -du$, $x = 1-u$

ALTERNATIVE

$\int x\sqrt{1-x}\,dx = \int xu\,(-2u\,du) = \int -2xu^2\,du$

$= \int -2(1-u^2)u^2\,du = \int -2u^2 + 2u^4\,du$

$= -\frac{2}{3}u^3 + \frac{2}{5}u^5 + C = \frac{2}{5}(1-x)^{\frac{5}{2}} - \frac{2}{3}(1-x)^{\frac{3}{2}} + C$ AS BEFORE

$u = \sqrt{1-x}$, $u^2 = 1-x$, $2u\frac{du}{dx} = -1$, $2u\,du = -dx$, $dx = -2u\,du$, $x = 1-u^2$

2. $\int \frac{6x}{\sqrt{2x+1}}\,dx = \int \frac{6x}{u^{\frac{1}{2}}}\,\frac{du}{2} = \int \frac{3u-3}{2u^{\frac{1}{2}}}\,du$

$= \int \frac{3}{2}u^{\frac{1}{2}} - \frac{3}{2}u^{-\frac{1}{2}}\,du = u^{\frac{3}{2}} - 3u^{\frac{1}{2}} + C$

$= (2x+1)^{\frac{3}{2}} - 3(2x+1)^{\frac{1}{2}} + C$

$u = 2x+1$, $\frac{du}{dx} = 2$, $dx = \frac{du}{2}$, $2x = u-1$, $6x = 3u-3$

ALTERNATIVE

$\int \frac{6x}{\sqrt{2x+1}}\,dx = \int \frac{6x}{u}\,u\,du = \int 6x\,du$

$= \int 3u^2 - 3\,du = u^3 - 3u + C$

$= (2x+1)^{\frac{3}{2}} - 3(2x+1)^{\frac{1}{2}} + C$

$u = \sqrt{2x+1}$, $u^2 = 2x+1$, $2u\frac{du}{dx} = 2$, $dx = u\,dx$, $2x = u^2-1$, $6x = 3u^2-3$

3. $\int \cos^3 x\,dx = \int \cos^3 x\,\frac{du}{\cos x} = \int \cos^2 x\,du$

$= \int 1 - \sin^2 x\,du = \int 1 - u^2\,du = u - \frac{1}{3}u^3 + C$

$= \sin x - \frac{1}{3}\sin^3 x + C$

$u = \sin x$, $\frac{du}{dx} = \cos x$, $dx = \frac{du}{\cos x}$

4. $\int \sec^4 x\,dx = \int \sec^4 x\,\frac{du}{\sec^2 x} = \int \sec^2 x\,du$

$= \int 1 + \tan^2 x\,du = \int 1 + u^2\,du = u + \frac{1}{3}u^3 + C$

$= \tan x + \frac{1}{3}\tan^3 x + C$

$u = \tan x$, $\frac{du}{dx} = \sec^2 x$, $dx = \frac{du}{\sec^2 x}$

5. $\int \frac{1}{\sqrt{x}(x-4)}\,dx = \int \frac{1}{u(u^2-4)}\,2u\,du = \int \frac{2}{u^2-4}\,du$

BY PARTIAL FRACTIONS SINCE $= \int \frac{2}{(u-2)(2+2)}\,du$

$\frac{2}{(u-2)(u+2)} \equiv \frac{A}{u-2} + \frac{B}{u+2}$

$2 \equiv A(u+2) + B(u-2)$

If $u=2$ $4A=2 \Rightarrow A=\frac{1}{2}$

If $u=-2$ $-4B=2 \Rightarrow B=-\frac{1}{2}$

$= \int \frac{\frac{1}{2}}{u-2} - \frac{\frac{1}{2}}{u+2}\,du$

$= \frac{1}{2}\ln|u-2| - \frac{1}{2}\ln|u+2| = \frac{1}{2}\ln\left|\frac{u-2}{u+2}\right|$

$= \frac{1}{2}\ln\left|\frac{\sqrt{x}-2}{\sqrt{x}+2}\right| + C$

$u = \sqrt{x}$, $u^2 = x$, $2u\frac{du}{dx} = 1$, $dx = 2u\,du$

6. $\int \frac{\sqrt{x^2+9}}{x}\,dx = \int \frac{u}{x}\,\frac{u}{x}\,du = \int \frac{u^2}{x^2}\,du$

$\int \frac{u^2}{u^2-9}\,du = \int \frac{(u^2-9)+9}{(u^2-9)}\,du = \int 1 + \frac{9}{u^2-9}\,du$

$\int 1 + \frac{9}{(u+3)(u-3)}\,du =$

PARTIAL FRACTIONS

$\frac{9}{(u+3)(u-3)} \equiv \frac{A}{u+3} + \frac{B}{u-3}$

$9 \equiv A(u-3) + B(u+3)$

If $x=3$ $9 = 6B \Rightarrow B = 3/2$

If $x=-3$ $9 = -6A \Rightarrow A = -3/2$

$= \int 1 + \frac{3/2}{u-3} - \frac{3/2}{u+3}\,du = u + \frac{3}{2}\ln|u-3| - \frac{3}{2}\ln|u+3| + C$

$= u + \frac{3}{2}\ln\left|\frac{u-3}{u+3}\right| + C = \sqrt{x^2+9} + \frac{3}{2}\ln\left|\frac{\sqrt{x^2+9}-3}{\sqrt{x^2+9}+3}\right| + C$

$u = \sqrt{x^2+9}$, $u^2 = x^2+9$, $2u\frac{du}{dx} = 2x$, $\frac{u}{x}du = dx$, $x^2 = u^2-9$

7. $\int \frac{1+\cos x}{\sin x}\,dx = \int \frac{1+u}{\sin x}\,\frac{du}{-\sin x} = \int -\frac{1+u}{\sin^2 x}\,du$

$= \int -\frac{1+u}{1-\cos^2 x}\,du = \int -\frac{1+u}{1-u^2}\,du$

$= \int -\frac{1+u}{(1-u)(1+u)}\,du = \int \frac{1}{u-1}\,du = \ln|u-1| + C$

$= \ln|\cos x - 1| + C$

$u = \cos x$, $\frac{du}{dx} = -\sin x$, $dx = \frac{du}{-\sin x}$

8. $\int \frac{1}{1+\sqrt{x-2}}\,dx = \int \frac{1}{1+u} \times 2u\,du$

$= \int \frac{2u}{u+1}\,du = \int \frac{2(u+1)-2}{u+1}\,du = \int 2 - \frac{2}{u+1}\,du$

$= 2u + 2\ln|u+1| + C = 2\sqrt{x-2} + 2\ln|1+\sqrt{x-2}| + C$

$u = \sqrt{x-2}$, $u^2 = x-2$, $2u\frac{du}{dx} = 1$, $2u\,du = dx$

ALTERNATIVE

$\int \frac{1}{1+\sqrt{x-2}}\,dx = \int \frac{1}{u}\,2(u-1)\,du = \int \frac{2u-2}{u}\,du$

$= \int 2 - \frac{2}{u}\,du = 2u - 2\ln|u|$

$= 2(1+\sqrt{x-2}) - 2\ln|1+\sqrt{x-2}| + C$

$= 2 + 2\sqrt{x-2} - 2\ln|1+\sqrt{x-2}| + C$ AS BEFORE

$u = 1+\sqrt{x-2}$, $u-1 = \sqrt{x-2}$, $(u-1)^2 = x-2$, $x = 2+(u-1)^2$, $\frac{dx}{du} = 2(u-1)$, $dx = 2(u-1)\,du$

9. $\int \sec^2 x\tan x\sqrt{1+\tan x}\,dx$

$= \int \sec^2 x\tan x \times u\,\frac{2u}{\sec^2 x}\,du = \int 2u^2\tan x\,du$

$= \int 2u^2(u^2-1)\,du = \int 2u^4 - 2u^2\,du$

$= \frac{2}{5}u^5 - \frac{2}{3}u^3 + C = \frac{2}{5}(1+\tan x)^{\frac{5}{2}} - \frac{2}{3}(1+\tan x)^{\frac{3}{2}} + C$

OR $= \frac{2}{15}(1+\tan x)^{\frac{3}{2}}\left[3(1+\tan x) - 5\right] + C$

$= \frac{2}{15}(3\tan x - 2)(1+\tan x)^{\frac{3}{2}} + C$

$u = \sqrt{1+\tan x}$, $u^2 = 1+\tan x$, $2u\frac{du}{dx} = \sec^2 x$, $2u\,du = \sec^2 x\,dx$, $dx = \frac{2u}{\sec^2 x}\,du$, $\tan x = u^2-1$

10. $\int \frac{9}{\sqrt{x}(9x-1)}\,dx = \int \frac{9}{u(9u^2-1)}\,2u\,du = \int \frac{18}{9u^2-1}\,du$

$= \int \frac{18}{(3u-1)(3u+1)}\,du$

BY PARTIAL FRACTIONS

$\frac{18}{(3u-1)(3u+1)} \equiv \frac{A}{3u-1} + \frac{B}{3u+1}$

$18 \equiv A(3u+1) + B(3u-1)$

- if $u=\frac{1}{3}$ $2A = 18 \Rightarrow A = 9$
- if $u=-\frac{1}{2}$ $-2B = 18 \Rightarrow B = -9$

$= \int \frac{9}{3u-1} - \frac{9}{3u+1}\,du = 3\ln|3u-1| - 3\ln|3u+1| = 3\ln\left|\frac{3u-1}{3u+1}\right| + C$

$= 3\ln\left|\frac{3\sqrt{x}-1}{3\sqrt{x}+1}\right| + C$

$u = \sqrt{x}$, $u^2 = x$, $2u\frac{du}{dx} = 1$, $2u\,du = dx$

Created by T. Madas

Question 8

Carry out the following integrations.

1. $\displaystyle\int \frac{1}{2+\sqrt{x}}\,dx = 2\sqrt{x} - 4\ln\left|2+\sqrt{x}\right| + C$

2. $\displaystyle\int \frac{x^2}{1-2x}\,dx = -\frac{1}{4}x(x+1) - \frac{1}{8}\ln|1-2x| = -\frac{1}{16}(1-2x)^2 + \frac{1}{4}(1-2x) - \frac{1}{8}\ln|1-2x| + C$

3. $\displaystyle\int \frac{3x^2+2}{4x+1}\,dx = \tfrac{1}{64}\left(24x^2 - 12x + 35\ln|4x+1|\right) + C = \tfrac{3}{128}(4x+1)^2 - \tfrac{3}{32}(4x+1) + \frac{35}{64}\ln|4x+1|$

4. $\displaystyle\int \frac{4-3x}{2x+1}\,dx = -\frac{3}{2}x + \frac{11}{4}\ln|2x+1| + C$

5. $\displaystyle\int \frac{x+1}{x-5}\,dx = x + 6\ln|x-5| + C$

6. $\displaystyle\int \frac{x^2}{x-2}\,dx = \frac{1}{2}x^2 + 2x + 4\ln|x-2| + C$

7. $\displaystyle\int \frac{1}{2+\sqrt{x-1}}\,dx = 2\sqrt{x-1} - 4\ln\left|\sqrt{x-1}+2\right| + C$

8. $\displaystyle\int \frac{x+4}{x-4}\,dx = x + 8\ln|x-4| + C$

Question 9

Carry out the following integrations.

1. $\int x\sqrt{x+1}\ dx = \frac{2}{15}(3x+2)\sqrt{(x+1)^3} + C$

2. $\int \frac{x+1}{\sqrt[3]{x^2+2x+3}}\ dx = \frac{3}{4}\sqrt[3]{(x^2+2x+3)^2} + C$

3. $\int \frac{3x^3+5x}{x^2+1}\ dx = \frac{3}{2}x^2 + \ln\left|x^2+1\right| + C$

4. $\int \frac{2x+1}{3x-1}\ dx = \frac{2}{3}x + \frac{5}{9}\ln\left|3x-1\right| + C$

5. $\int \frac{1}{(x-1)\sqrt{x^2-1}}\ dx = -\sqrt{\frac{x+1}{x-1}} + C$, use $x-1 = \frac{1}{u}$

6. $\int \frac{4x^3\sqrt{x^4+1}}{1+\sqrt{x^4+1}}\ dx = x^4 - 2\sqrt{x^4+1} + 2\ln\left|1+\sqrt{x^4+1}\right| + C$

Question 10

Carry out the following integrations to the answers given.

1. $\displaystyle\int_0^{\frac{1}{2}} \frac{x}{(2-x)^2}\, dx = \frac{1}{3} + \ln\left(\frac{3}{4}\right)$

2. $\displaystyle\int_1^{2} \frac{x}{(2x-1)^2}\, dx = \frac{2+\ln 27}{12}$

3. $\displaystyle\int_0^{2} \frac{x+2}{\sqrt{4x+1}}\, dx = \frac{17}{6}$

4. $\displaystyle\int_0^{36} \frac{1}{\sqrt{x}\left(\sqrt{x}+2\right)}\, dx = \ln 16$

5. $\displaystyle\int_{-6}^{\frac{3}{2}} \frac{x}{\sqrt{4-2x}}\, dx = -\frac{9}{2}$

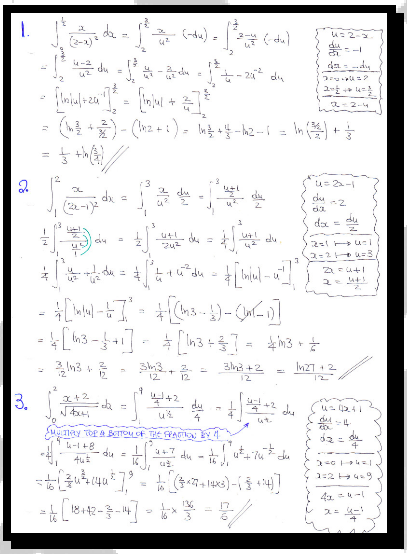
6. $\displaystyle\int_1^{5} \frac{x+1}{(2x-1)^{\frac{3}{2}}}\, dx = 2$

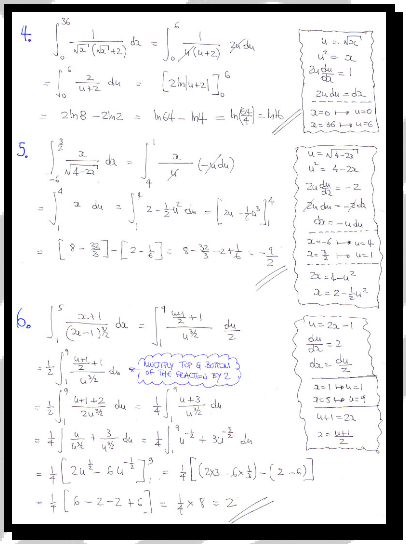
7. $\displaystyle\int_{\frac{2}{3}}^{1} \frac{x}{2x-1}\, dx = \frac{1}{6} + \frac{1}{4}\ln 3$

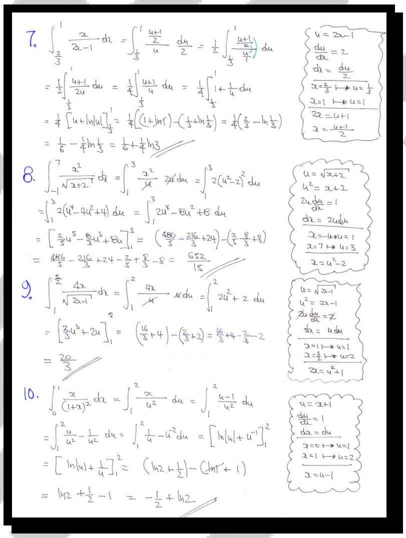
8. $\displaystyle\int_{-1}^{7} \frac{x^2}{\sqrt{x+2}}\, dx = \frac{652}{15}$

9. $\displaystyle\int_1^{\frac{5}{2}} \frac{4x}{\sqrt{2x-1}}\, dx = \frac{20}{3}$

10. $\displaystyle\int_0^{1} \frac{x}{(1+x)^2}\, dx = -\frac{1}{2} + \ln 2$







Question 11

Carry out the following integrations to the answers given.

1. $\displaystyle\int_0^3 x\sqrt{x+1}\ dx = \frac{116}{15}$

2. $\displaystyle\int_0^2 \frac{6x^3}{\sqrt{x^2+1}}\ dx = 4\left(1+\sqrt{5}\right)$

3. $\displaystyle\int_{-1}^0 \frac{x^2}{1-x}\ dx = -\frac{1}{2}+\ln 2$

4. $\displaystyle\int_0^{100} \frac{1}{20-\sqrt{x}}\ dx = 40\ln 2 - 20$

5. $\displaystyle\int_0^{\frac{1}{4}} 2x\sqrt{1-4x}\ dx = \frac{1}{30}$

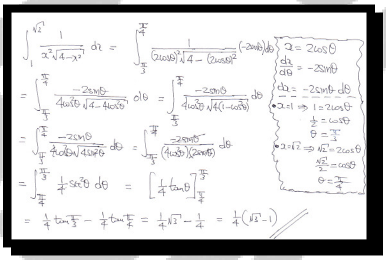
6. $\displaystyle\int_0^{\frac{\pi}{2}} \sin x\cos x\left(1+\sin x\right)^5\ dx = \frac{107}{14}$

7. $\displaystyle\int_2^5 \frac{x^2}{\sqrt{x-1}}\ dx = \frac{356}{15}$

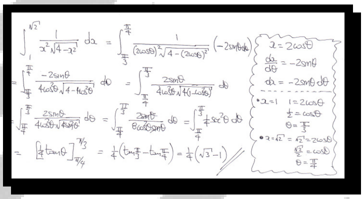
Question 12

Carry out the following integrations to the answers given.

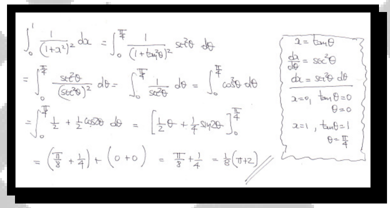
1. $\displaystyle\int_0^{\sqrt{2}} \frac{x^2}{\sqrt{4-x^2}}\, dx = \frac{\pi}{2} - 1$, use $x = 2\sin\theta$



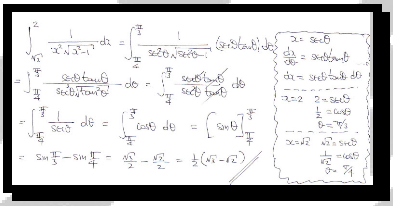
2. $\displaystyle\int_1^{\sqrt{2}} \frac{1}{x^2\sqrt{4-x^2}}\, dx = \frac{1}{4}\left(\sqrt{3}-1\right)$, use $x = 2\cos\theta$



3. $\displaystyle\int_0^{1} \frac{1}{\left(1+x^2\right)^2}\, dx = \frac{1}{8}(\pi+2)$, use $x = \tan\theta$



4. $\displaystyle\int_{\sqrt{2}}^{2} \frac{1}{x^2\sqrt{x^2-1}}\,dx = \frac{1}{2}\left(\sqrt{3}-\sqrt{2}\right)$, use $x = \sec\theta$



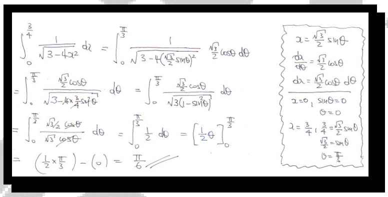
$$\int_{\sqrt{2}}^{2} \frac{1}{x^2\sqrt{x^2-1}}\,dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\sec^2\theta\sqrt{\sec^2\theta-1}}(\sec\theta\tan\theta)\,d\theta$$
$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec\theta\tan\theta}{\sec^2\theta\sqrt{\tan^2\theta}}\,d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec\theta\tan\theta}{\sec^2\theta\tan\theta}\,d\theta$$
$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\sec\theta}\,d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cos\theta\,d\theta = \Big[\sin\theta\Big]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$
$$= \sin\frac{\pi}{3} - \sin\frac{\pi}{4} = \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} = \frac{1}{2}\left(\sqrt{3}-\sqrt{2}\right)$$

$x = \sec\theta$, $\frac{dx}{d\theta} = \sec\theta\tan\theta$, $dx = \sec\theta\tan\theta\,d\theta$

$x=2$: $2 = \sec\theta$, $\frac{1}{2} = \cos\theta$, $\theta = \pi/3$

$x=\sqrt{2}$: $\sqrt{2} = \sec\theta$, $\frac{1}{\sqrt{2}} = \cos\theta$, $\theta = \pi/4$

5. $\displaystyle\int_{0}^{\frac{3}{4}} \frac{1}{\sqrt{3-4x^2}}\,dx = \frac{\pi}{6}$, use $x = \frac{\sqrt{3}}{2}\sin\theta$



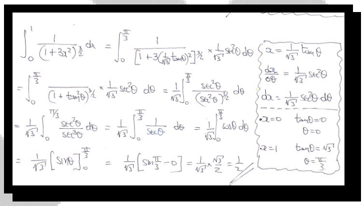
$$\int_0^{\frac{3}{4}} \frac{1}{\sqrt{3-4x^2}}\,dx = \int_0^{\frac{\pi}{3}} \frac{1}{\sqrt{3-4\left(\frac{\sqrt{3}}{2}\sin\theta\right)^2}}\,\frac{\sqrt{3}}{2}\cos\theta\,d\theta$$
$$= \int_0^{\frac{\pi}{3}} \frac{\frac{\sqrt{3}}{2}\cos\theta}{\sqrt{3-4\times\frac{3}{4}\sin^2\theta}}\,d\theta = \int_0^{\frac{\pi}{3}} \frac{\frac{\sqrt{3}}{2}\cos\theta}{\sqrt{3(1-\sin^2\theta)}}\,d\theta$$
$$= \int_0^{\frac{\pi}{3}} \frac{\sqrt{3}/2\,\cos\theta}{\sqrt{3}\cos\theta}\,d\theta = \int_0^{\frac{\pi}{3}} \frac{1}{2}\,d\theta = \left[\frac{1}{2}\theta\right]_0^{\frac{\pi}{3}}$$
$$= \left(\frac{1}{2}\times\frac{\pi}{3}\right) - (0) = \frac{\pi}{6}$$

$x = \frac{\sqrt{3}}{2}\sin\theta$, $\frac{dx}{d\theta} = \frac{\sqrt{3}}{2}\cos\theta$, $dx = \frac{\sqrt{3}}{2}\cos\theta\,d\theta$

$x=0$, $\sin\theta = 0$, $\theta = 0$

$x=\frac{3}{4}$, $\frac{3}{4} = \frac{\sqrt{3}}{2}\sin\theta$, $\frac{\sqrt{3}}{2} = \sin\theta$, $\theta = \frac{\pi}{3}$

6. $\displaystyle\int_{0}^{1} \frac{1}{\left(1+3x^2\right)^{\frac{3}{2}}}\,dx = \frac{1}{2}$, use $x = \frac{1}{\sqrt{3}}\tan\theta$



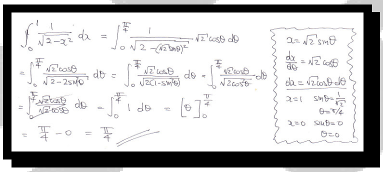
$$\int_0^1 \frac{1}{(1+3x^2)^{\frac{3}{2}}}\,dx = \int_0^{\frac{\pi}{3}} \frac{1}{\left[1+3\left(\frac{1}{\sqrt{3}}\tan\theta\right)^2\right]^{3/2}} \times \frac{1}{\sqrt{3}}\sec^2\theta\,d\theta$$
$$= \int_0^{\frac{\pi}{3}} \frac{1}{(1+\tan^2\theta)^{3/2}} \times \frac{1}{\sqrt{3}}\sec^2\theta\,d\theta = \frac{1}{\sqrt{3}}\int_0^{\frac{\pi}{3}} \frac{\sec^2\theta}{(\sec^2\theta)^{3/2}}\,d\theta$$
$$= \frac{1}{\sqrt{3}}\int_0^{\pi/3} \frac{\sec^2\theta}{\sec^3\theta}\,d\theta = \frac{1}{\sqrt{3}}\int_0^{\frac{\pi}{3}} \frac{1}{\sec\theta}\,d\theta = \frac{1}{\sqrt{3}}\int_0^{\frac{\pi}{3}} \cos\theta\,d\theta$$
$$= \frac{1}{\sqrt{3}}\Big[\sin\theta\Big]_0^{\frac{\pi}{3}} = \frac{1}{\sqrt{3}}\left[\sin\frac{\pi}{3} - 0\right] = \frac{1}{\sqrt{3}}\times\frac{\sqrt{3}}{2} = \frac{1}{2}$$

$x = \frac{1}{\sqrt{3}}\tan\theta$, $\frac{dx}{d\theta} = \frac{1}{\sqrt{3}}\sec^2\theta$, $dx = \frac{1}{\sqrt{3}}\sec^2\theta\,d\theta$

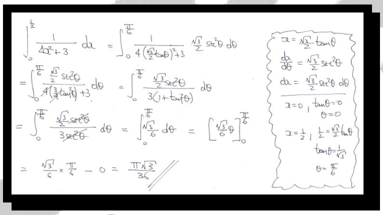
• $x=0$: $\tan\theta = 0$, $\theta = 0$

• $x=1$: $\tan\theta = \sqrt{3}$, $\theta = \frac{\pi}{3}$

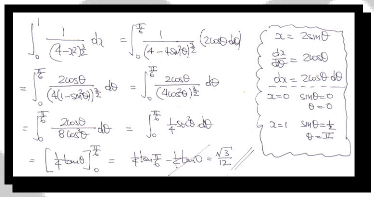
7. $\displaystyle\int_0^1 \frac{1}{\sqrt{2-x^2}}\ dx = \frac{\pi}{4}$, use $x = \sqrt{2}\sin\theta$



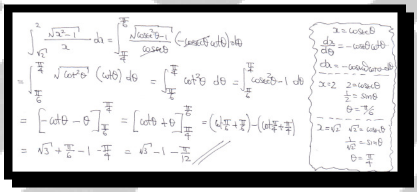
8. $\displaystyle\int_0^{\frac{1}{2}} \frac{1}{4x^2+3}\ dx = \frac{\pi\sqrt{3}}{36}$, use $x = \frac{\sqrt{3}}{2}\tan\theta$



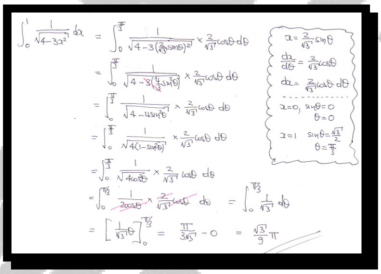
9. $\displaystyle\int_0^1 \frac{1}{\left(4-x^2\right)^{\frac{3}{2}}}\ dx = \frac{\sqrt{3}}{12}$, use $x = 2\sin\theta$



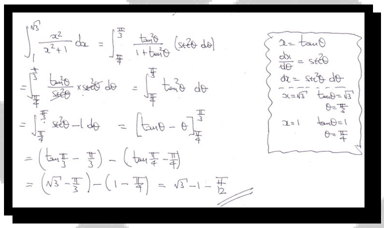
10. $\displaystyle\int_{\sqrt{2}}^{2} \frac{\sqrt{x^2-1}}{x}\ dx = \sqrt{3}-1-\frac{\pi}{12}$, use $x = \operatorname{cosec}\theta$



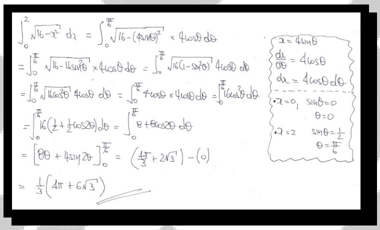
11. $\displaystyle\int_{0}^{1} \frac{1}{\sqrt{4-3x^2}}\ dx = \frac{\pi\sqrt{3}}{9}$, use $x = \frac{2}{\sqrt{3}}\sin\theta$



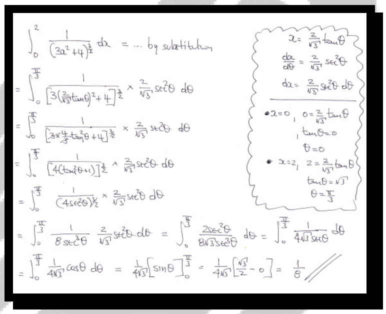
12. $\displaystyle\int_{1}^{\sqrt{3}} \frac{x^2}{x^2+1}\ dx = \sqrt{3}-1-\frac{\pi}{12}$, use $x = \tan\theta$



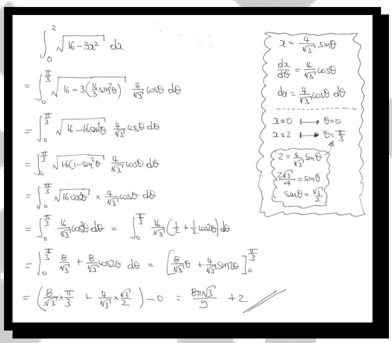
13. $\displaystyle\int_0^2 \sqrt{16-x^2}\ dx = \frac{1}{3}\left(4\pi + 6\sqrt{3}\right)$, use $x = 4\sin\theta$



14. $\displaystyle\int_0^2 \frac{1}{\left(3x^2+4\right)^{\frac{3}{2}}}\ dx = \frac{1}{8}$, use $x = \frac{2}{\sqrt{3}}\tan\theta$



15. $\displaystyle\int_0^2 \sqrt{16-3x^2}\ dx = \frac{8\pi\sqrt{3}}{9} + 2$, use $x = \frac{4}{\sqrt{3}}\sin\theta$



16. $\displaystyle\int_0^3 \frac{27}{\left(9+x^2\right)^2}\,dx = \frac{\pi}{8}+\frac{1}{4}$, use $x = 3\tan\theta$

$$\int_0^3 \frac{27}{(9+x^2)^2}\,dx = \ldots = \int_0^{\frac{\pi}{4}} \frac{27}{(9+9\tan^2\theta)^2}\left(3\sec^2\theta\,d\theta\right)$$

$$= \int_0^{\frac{\pi}{4}} \frac{81\sec^2\theta}{\left[9(1+\tan^2\theta)\right]^2}\,d\theta = \int_0^{\frac{\pi}{4}} \frac{81\sec^2\theta}{81(\sec^2\theta)^2}\,d\theta = \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{\sec^4\theta}\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \cos^2\theta\,d\theta = \int_0^{\pi/4} \frac{1}{2}+\frac{1}{2}\cos 2\theta\,d\theta$$

$$= \left[\frac{1}{2}\theta+\frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{4}} = \left(\frac{\pi}{8}+\frac{1}{4}\right)-(0+0)$$

$$= \frac{\pi}{8}+\frac{1}{4}$$

$x = 3\tan\theta$

$\frac{dx}{d\theta} = 3\sec^2\theta$

$dx = 3\sec^2\theta\,d\theta$

$x=0,\ \theta=0$

$x=3,\ \theta=\frac{\pi}{4}$