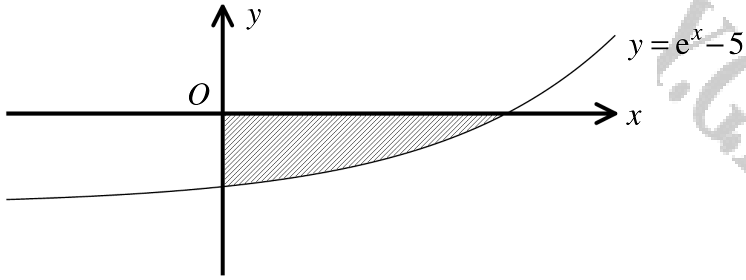
INTEGRATION

FINDING AREAS

Question 1 (**)

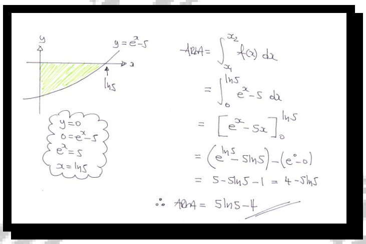


The diagram above shows the graph of the curve with equation

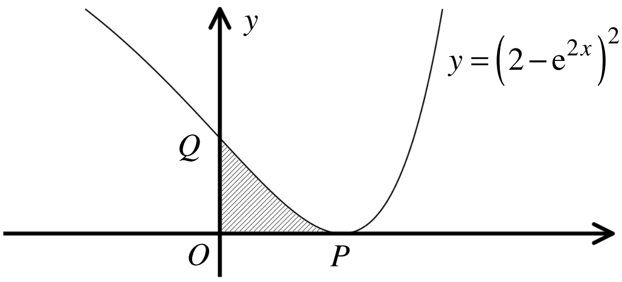
$$y = e^x - 5, \; x \in \mathbb{R}.$$

Use integration to find the exact area of the finite region bounded by the curve and the coordinate axes.

$\boxed{5\ln 5 - 4 \approx 4.05}$



Question 2 **(**+)**



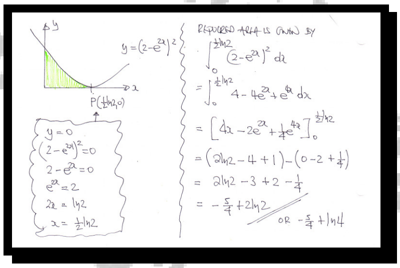
The figure above shows the graph of the curve with equation

$$y = \left(2 - e^{2x}\right)^2, \ x \in \mathbb{R}.$$

The curve touches the x axis at the point P and crosses the y axis at the point Q.

Find the exact area of the finite region bounded by the curve and the line segments OP and OQ.

$$\boxed{-\frac{5}{4} + \ln 4}$$



$y = (2-e^{2x})^2$

$P(\tfrac{1}{2}\ln 2, 0)$

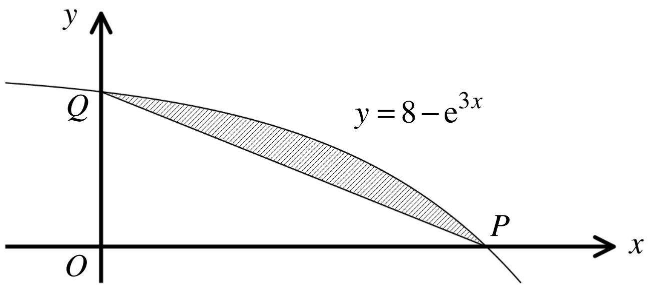
$y = 0$
$(2-e^{2x})^2 = 0$
$2 - e^{2x} = 0$
$e^{2x} = 2$
$2x = \ln 2$
$x = \tfrac{1}{2}\ln 2$

REQUIRED AREA IS GIVEN BY

$$\int_0^{\frac{1}{2}\ln 2} (2-e^{2x})^2 \, dx$$
$$= \int_0^{\frac{1}{2}\ln 2} 4 - 4e^{2x} + e^{4x} \, dx$$
$$= \left[4x - 2e^{2x} + \tfrac{1}{4}e^{4x}\right]_0^{\frac{1}{2}\ln 2}$$
$$= (2\ln 2 - 4 + 1) - (0 - 2 + \tfrac{1}{4})$$
$$= 2\ln 2 - 3 + 2 - \tfrac{1}{4}$$
$$= -\tfrac{5}{4} + 2\ln 2$$

OR $-\tfrac{5}{4} + \ln 4$

Question 3 (+)**



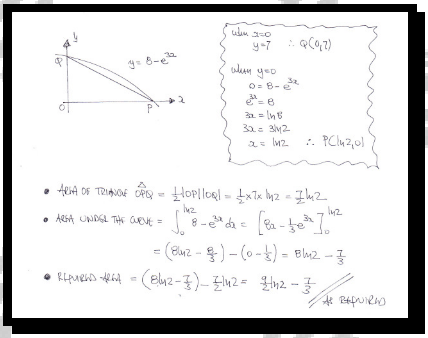
The diagram above shows the graph of the curve with equation

$$y = 8 - e^{3x}, \ x \in \mathbb{R}.$$

The curve meets the x axis at the point P and the y axis at the point Q.

Show that the area of the finite region bounded by the curve and the straight line segment PQ is exactly $\frac{9}{2}\ln 2 - \frac{7}{3}$.

proof

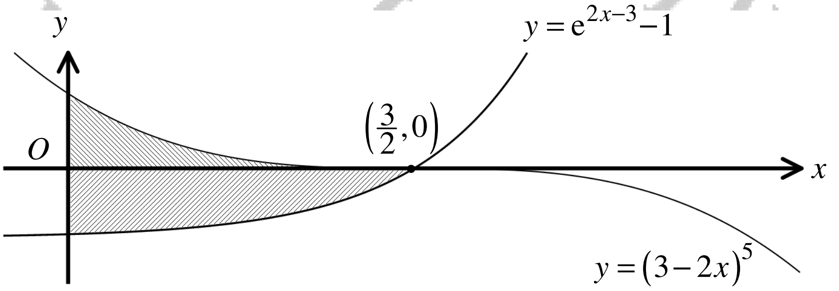


when $x=0$, $y=7$ $\therefore Q(0,7)$

when $y=0$
$0 = 8 - e^{3x}$
$e^{3x} = 8$
$3x = \ln 8$
$3x = 3\ln 2$
$x = \ln 2$ $\therefore P(\ln 2, 0)$

- AREA OF TRIANGLE $\overset{\triangle}{OPQ} = \frac{1}{2}|OP||OQ| = \frac{1}{2} \times 7 \times \ln 2 = \frac{7}{2}\ln 2$
- AREA UNDER THE CURVE $= \int_0^{\ln 2} 8 - e^{3x}\,dx = \left[8x - \frac{1}{3}e^{3x}\right]_0^{\ln 2}$
  $= \left(8\ln 2 - \frac{8}{3}\right) - \left(0 - \frac{1}{3}\right) = 8\ln 2 - \frac{7}{3}$
- REQUIRED AREA $= \left(8\ln 2 - \frac{7}{3}\right) - \frac{7}{2}\ln 2 = \frac{9}{2}\ln 2 - \frac{7}{3}$ // AS REQUIRED

Question 4 (***)



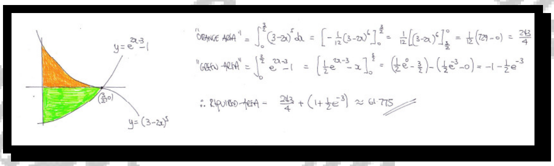
The figure above shows the graphs of the curves with equations

$$y = (3-2x)^5 \quad \text{and} \quad y = e^{2x-3} - 1 .$$

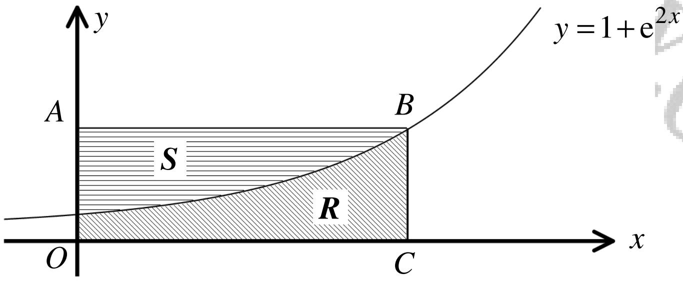
Both curves meet each other at the point with coordinates $\left(\frac{3}{2}, 0\right)$.

Calculate the area of the shaded region bounded by the two curves and the y axis giving the answer correct to three decimal places.

$\boxed{\approx 61.775}$



Question 5 (**+)



The figure above shows the graph of the curve with equation

$$y = 1 + e^{2x}, \ x \in \mathbb{R}.$$

The point C has coordinates $(1,0)$. The point B lies on the curve so that BC is parallel to the y axis. The point A lies on the y axis so that $OABC$ is a rectangle.

The region R is bounded by the curve, the coordinate axes and the line BC.

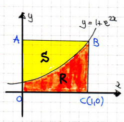
The region S is bounded by the curve, the y axis and the line AB.

Show that the area of R is equal to the area of S.

$\boxed{}$, $\boxed{\text{proof}}$

LOOKING AT THE DIAGRAM

- WHEN $x=1$, $y = 1+e^2$
  I.E $B(1, 1+e^2)$
- AREA OF THE RECTANGLE OABC
  $1 \times (1+e^2) = 1+e^2$



NEXT FIND BY INTEGRATION THE AREA OF R

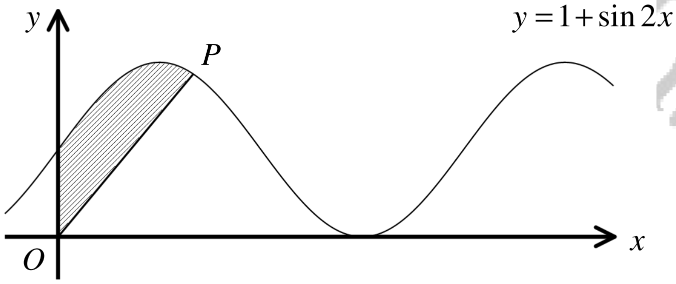
$R = \int_0^1 1+e^{2x} \, dx = \left[x + \frac{1}{2}e^{2x}\right]_0^1 = \left(1 + \frac{1}{2}e^2\right) - \left(0 + \frac{1}{2}\right)$

$= \frac{1}{2} + \frac{1}{2}e^2 = \frac{1}{2}(1+e^2)$

I.E HALF THE AREA OF THE RECTANGLE

$\therefore$ AREA R = AREA S

Question 6 (**+)



The figure above shows the graph of the curve with equation

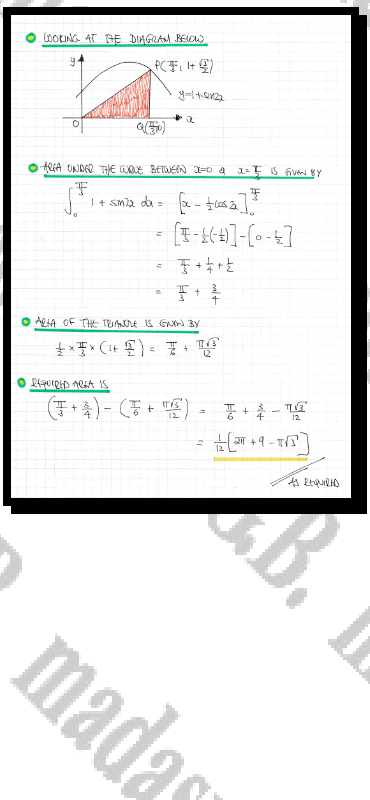
$$y = 1 + \sin 2x, \ x \in \mathbb{R}.$$

The point P lies on the curve where $x = \frac{\pi}{3}$.

Show that the area of the finite region bounded by the curve, the y axis and the straight line segment OP is exactly

$$\frac{1}{12}\left(2\pi + 9 - \pi\sqrt{3}\right).$$

$\boxed{}$, $\boxed{\text{proof}}$



- LOOKING AT THE DIAGRAM BELOW

$P\left(\frac{\pi}{3}, 1+\frac{\sqrt{3}}{2}\right)$, $y = 1+\sin 2x$, $Q\left(\frac{\pi}{3}, 0\right)$

- AREA UNDER THE CURVE BETWEEN $x=0$ & $x=\frac{\pi}{3}$ IS GIVEN BY

$$\int_0^{\frac{\pi}{3}} 1+\sin 2x \, dx = \left[x - \frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{3}} = \left[\frac{\pi}{3} - \frac{1}{2}\left(-\frac{1}{2}\right)\right] - \left[0 - \frac{1}{2}\right] = \frac{\pi}{3} + \frac{1}{4} + \frac{1}{2} = \frac{\pi}{3} + \frac{3}{4}$$

- AREA OF THE TRIANGLE IS GIVEN BY

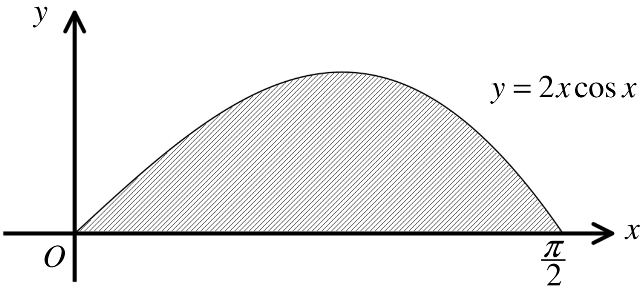
$$\frac{1}{2} \times \frac{\pi}{3} \times \left(1 + \frac{\sqrt{3}}{2}\right) = \frac{\pi}{6} + \frac{\pi\sqrt{3}}{12}$$

- REQUIRED AREA IS

$$\left(\frac{\pi}{3} + \frac{3}{4}\right) - \left(\frac{\pi}{6} + \frac{\pi\sqrt{3}}{12}\right) = \frac{\pi}{6} + \frac{3}{4} - \frac{\pi\sqrt{3}}{12} = \frac{1}{12}\left[2\pi + 9 - \pi\sqrt{3}\right]$$

AS REQUIRED

Question 7 (***)



The figure above shows the graph of the curve with equation

$$y = 2x\cos x\,,\ 0 \le x \le \frac{\pi}{2}\,.$$

Use integration by parts to find the exact area of the finite region bounded by the curve and the x axis.

$\boxed{\pi - 2}$

$A = \int_{x_1}^{x_2} y(x)\,dx = \ldots$ by parts and ignoring limits

$A = \int_0^{\frac{\pi}{2}} 2x\cos x\,dx = 2x\sin x - \int 2\sin x\,dx$

$2x$	2
$\sin x$	$\cos x$

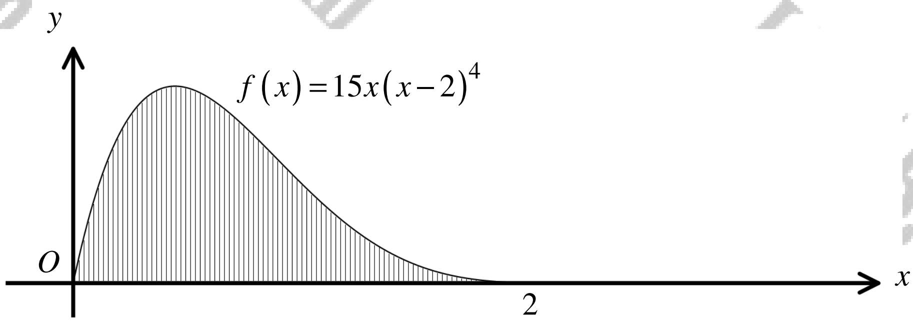
$= 2x\sin x + 2\cos x + C$

REINTRODUCE LIMITS

$A = \left[2x\sin x + 2\cos x\right]_0^{\frac{\pi}{2}} = \left[2\left(\frac{\pi}{2}\right)\sin\frac{\pi}{2} + \cancel{2\cos\frac{\pi}{2}}\right] - \left[0 + 2\cos 0\right]$

$A = \pi - 2$

Question 8 (***)



The figure above shows the curve with equation

$$f(x) = 15x(x-2)^4, \ 0 \le x \le 2.$$

a) Express $f(x+2)$ in the form $Ax^5 + Bx^4$, where A and B are constants.

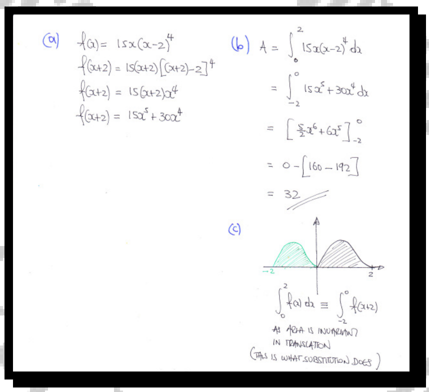
b) By using part (a), or otherwise find the value of

$$\int_0^2 f(x)\ dx.$$

c) Explain geometrically the significance of

$$\int_0^2 f(x)\ dx \equiv \int_{-2}^0 f(x+2)\ dx.$$

$\boxed{f(x+2) = 15x^5 + 30x^4}$, $\boxed{\int_0^2 f(x)\ dx = 32}$



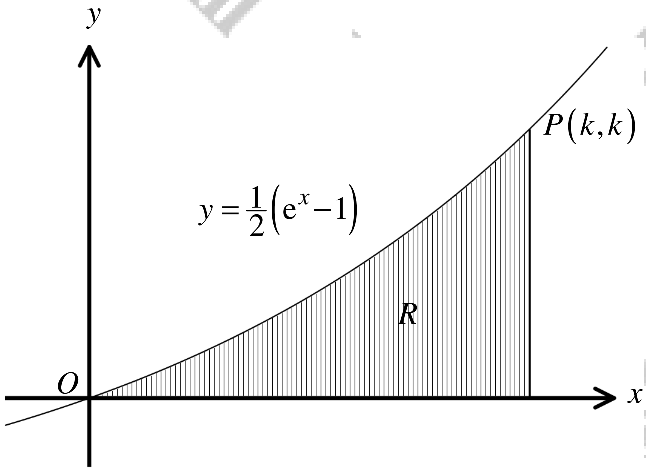
(a) $f(x) = 15x(x-2)^4$
$f(x+2) = 15(x+2)[(x+2)-2]^4$
$f(x+2) = 15(x+2)x^4$
$f(x+2) = 15x^5 + 30x^4$

(b) $A = \int_0^2 15x(x-2)^4\,dx$
$= \int_{-2}^0 15x^5 + 30x^4\,dx$
$= \left[\frac{5}{2}x^6 + 6x^5\right]_{-2}^0$
$= 0 - [160 - 192]$
$= 32$

(c)

$\int_0^2 f(x)\,dx \equiv \int_{-2}^0 f(x+2)$
AS AREA IS INVARIANT
IN TRANSLATION
(THIS IS WHAT SUBSTITUTION DOES)

Question 9 (***)



The figure above shows the curve C with equation

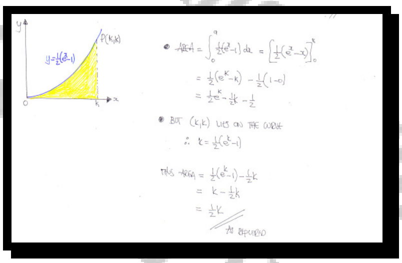
$$y=\frac{1}{2}\left(e^{x}-1\right), \ x\in\mathbb{R}.$$

The point $P(k,k)$, where k is a positive constant, lies on C.

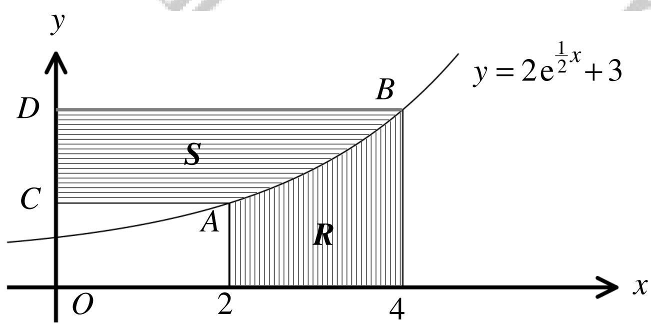
The finite region R, shown shaded in the figure, is bounded by C, the x axis and the line with equation $x=k$.

Show clearly that the area of R is $\frac{1}{2}k$.

$\boxed{\text{proof}}$



Question 10 (***+)



The figure above shows the graph of the curve with equation

$$y = 2e^{\frac{1}{2}x} + 3, \ x \in \mathbb{R}.$$

The points A and B lie on the curve where $x = 2$ and $x = 4$, respectively.

The finite region R is bounded by the curve, the x axis and the lines with equations $x = 2$ and $x = 4$.

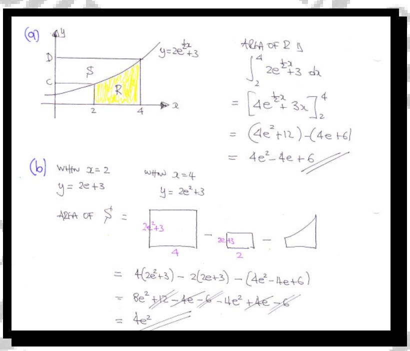
a) Determine the exact area of R.

The points C and D lie on the y axis so that the line segments CA and DB are parallel to the x axis.

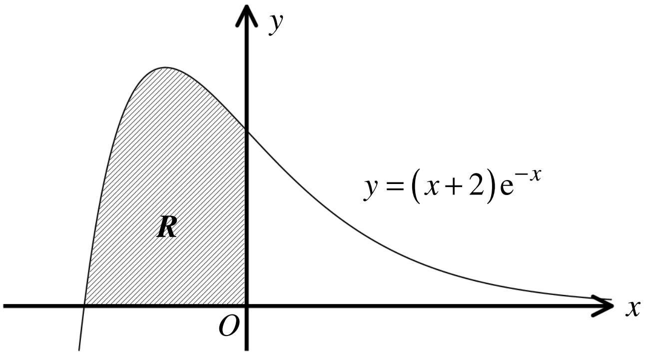
The region S is bounded by the curve, the y axis and the line segments CA and DB.

b) Determine the exact area of S.

$\boxed{\text{area of } R = 4e^2 - 4e + 6}$, $\boxed{\text{area of } S = 4e^2}$



Question 11 (***+)



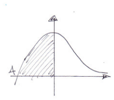
The figure above shows the graph of the curve with equation

$$y=(x+2)e^{-x},\ x\in\mathbb{R}.$$

The finite region R, shown shaded in the figure, is bounded by the curve and the coordinate axes.

Use integration by parts to show that the area of R is e^2-3.

$\boxed{\text{proof}}$



$y=(x+2)e^{-x}$
$y=0$
$x=-2$ BY INSPECTION
$(-2,0)$

$A=\int_{-2}^{0}(x+2)e^{-x}$ = BY PARTS & IGNORING LIMITS

$x+2$	1
$-e^{-x}$	e^{-x}

$-(x+2)e^{-x}-\int -e^{-x}\,dx$

$=-(x+2)e^{-x}+\int e^{-x}\,dx$

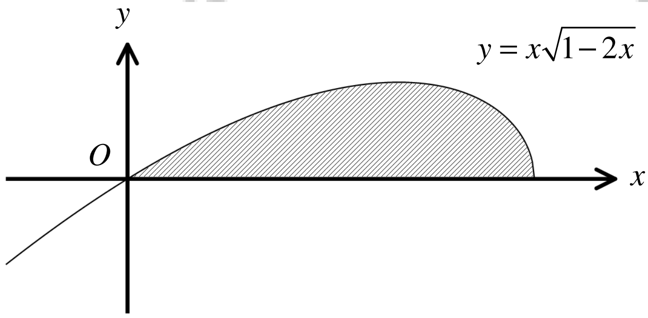
$=-(x+2)e^{-x}-e^{-x}+C$

$\therefore A=\left[-(x+2)e^{-x}-e^{-x}\right]_{-2}^{0}=\left[(x+2)e^{-x}+e^{-x}\right]_{0}^{-2}$

$=(0+e^{2})-(2+1)=e^{2}-3$

AS REQUIRED

Question 12 (***+)

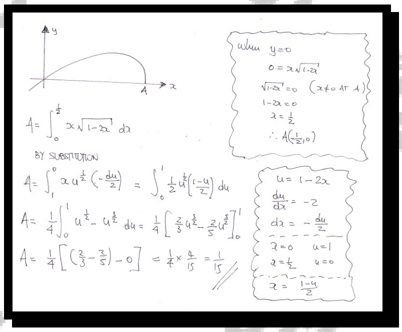


The figure above shows the graph of the curve with equation

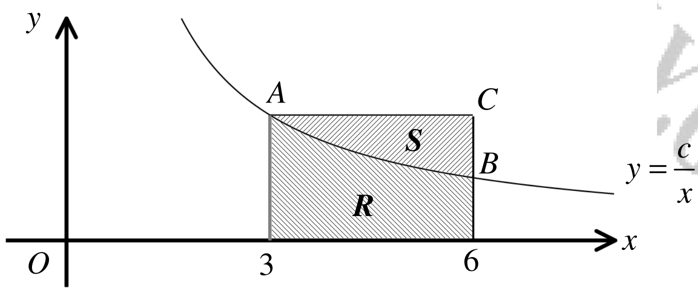
$$y = x\sqrt{1-2x}\ ,\ x \le \frac{1}{2}.$$

Use integration by substitution to find the area of the finite region bounded by the curve and the x axis.

$\boxed{\frac{1}{15}}$



Question 13 (***+)



The figure above shows the graph of the curve with equation

$$y = \frac{c}{x},$$

where c is a positive constant.

The points A and B lie on the curve here $x = 3$ and $x = 6$ respectively.

The finite region R is bounded by the curve, the x axis and the lines with equations $x = 3$ and $x = 6$.

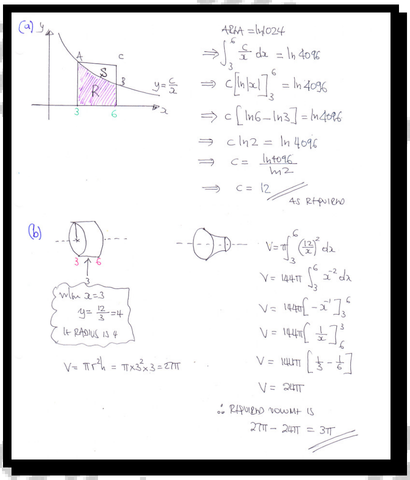
a) Given that the area of R is $\ln 4096$, show that $c = 12$.

The point C is such so that AC is parallel to the x axis and BC is parallel to the y axis. The finite region S is bounded by the curve and the line segments AC and BC.

b) Find the volume of the solid produced when S is complete revolved about the x axis.

$\boxed{3\pi}$

[solution overleaf]



(a)

AREA $= \ln 1024$

$\Rightarrow \int_{3}^{6} \frac{c}{x}\, dx = \ln 4096$

$\Rightarrow c\left[\ln|x|\right]_{3}^{6} = \ln 4096$

$\Rightarrow c\left[\ln 6 - \ln 3\right] = \ln 4096$

$\Rightarrow c \ln 2 = \ln 4096$

$\Rightarrow c = \frac{\ln 4096}{\ln 2}$

$\Rightarrow c = 12$ AS REQUIRED

(b)

WHEN $x=3$
$y = \frac{12}{3} = 4$
IE RADIUS IS 4

$V = \pi r^2 h = \pi \times 3^2 \times 3 = 27\pi$

$V = \pi \int_{3}^{6} \left(\frac{12}{x}\right)^2 dx$

$V = 144\pi \int_{3}^{6} x^{-2}\, dx$

$V = 144\pi\left[-x^{-1}\right]_{3}^{6}$

$V = 144\pi\left[\frac{1}{x}\right]_{6}^{3}$

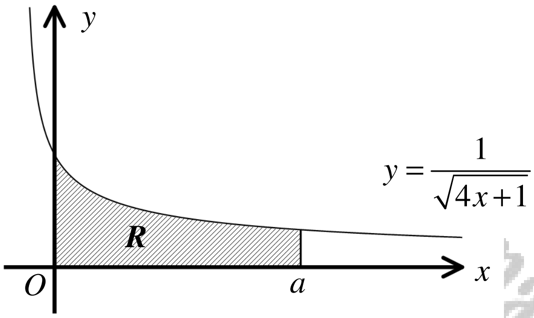
$V = 144\pi\left[\frac{1}{3} - \frac{1}{6}\right]$

$V = 24\pi$

$\therefore$ REQUIRED VOLUME IS

$27\pi - 24\pi = 3\pi$

Question 14 (*+)**



The figure above shows the graph of the curve with equation

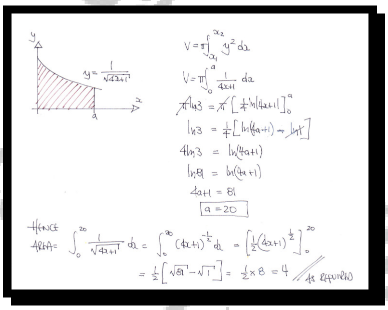
$$y = \frac{1}{\sqrt{4x+1}}, \ x > -\frac{1}{4}.$$

The finite region R is bounded by the curve, the coordinate axes and the line with equation $x = a$, where a is a positive constant.

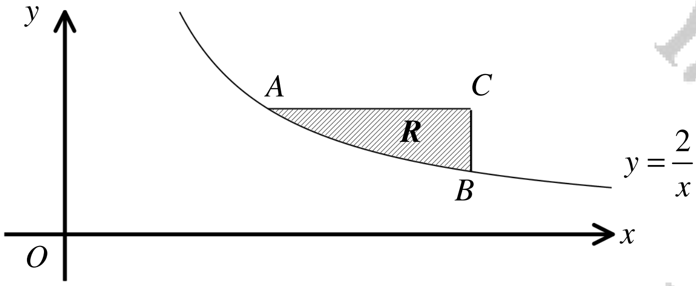
When R is revolved by 2π radians in the x axis the resulting solid of revolution has a volume of $\pi \ln 3$, in appropriate cubic units.

Determine the area of R.

$\boxed{\text{area} = 4}$



Question 15 (***+)



The figure above shows the graph of the curve with equation

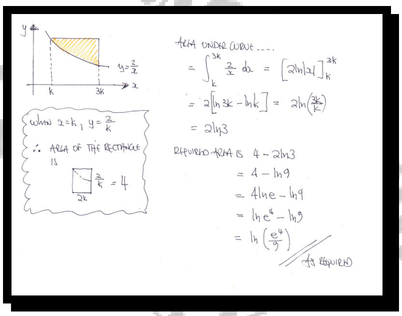
$$y=\frac{2}{x},\ x\in\mathbb{R},\ x\neq 0.$$

The points A and B lie on the curve. The respective x coordinates of these points are k and $3k$, where k is a positive constant. The point C is such so that AC is parallel to the x axis and BC is parallel to the y axis.

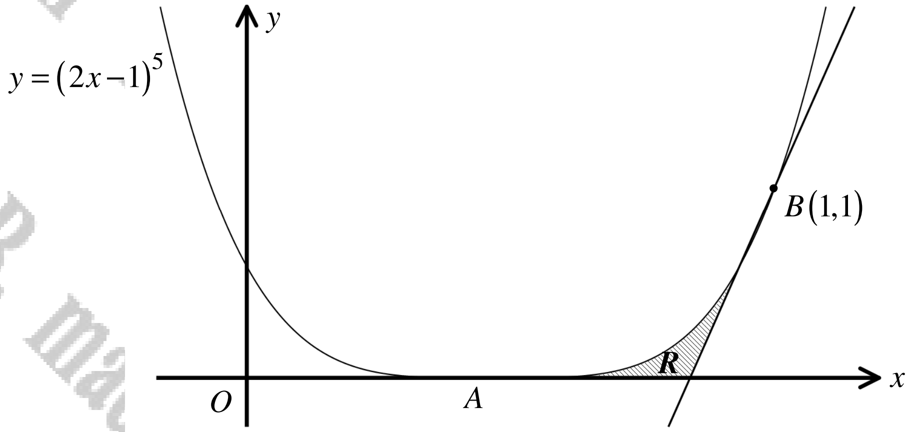
The finite region R is bounded by the curve and the line segments AC and BC.

Show that the area of R is $\ln\left(\frac{1}{9}e^4\right)$.

proof



Question 16 (***+)



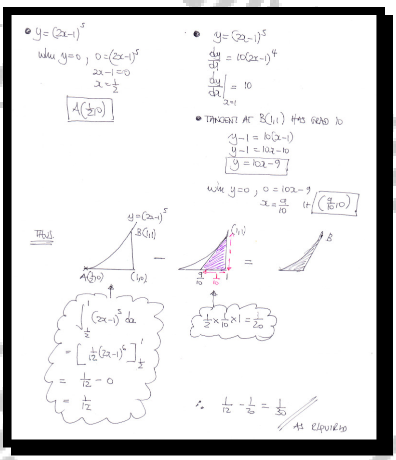
The figure above shows the graph of the curve with equation

$$y=(2x-1)^5,\ x\in\mathbb{R}.$$

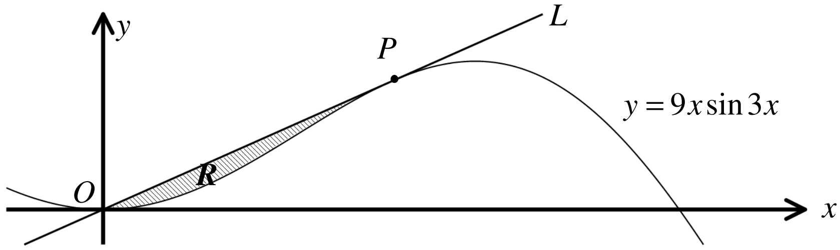
The curve meets the x axis at the point A and the point $B(1,1)$ lies on the curve.

Show that the area of the finite region R bounded by the curve, the x axis and the tangent to the curve at B is $\frac{1}{30}$.

$\boxed{\text{proof}}$



Question 17 (****)



The figure above shows the graph of the curve C with equation

$$y = 9x\sin 3x, \; x \in \mathbb{R}.$$

The straight line L is the tangent to C at the point P, whose x coordinate is $\frac{\pi}{6}$.

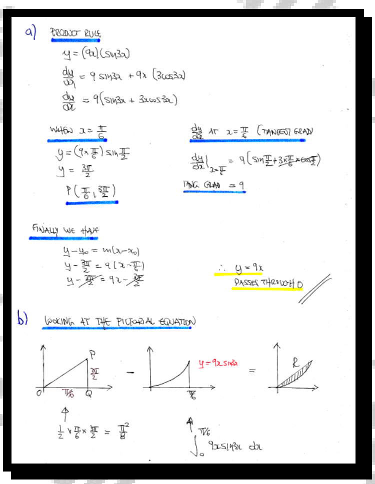
a) Show that L passes through the origin O.

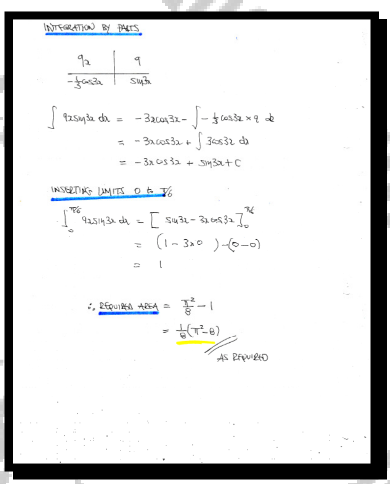
The finite region R bounded by C and L.

b) Show further that the area of R is

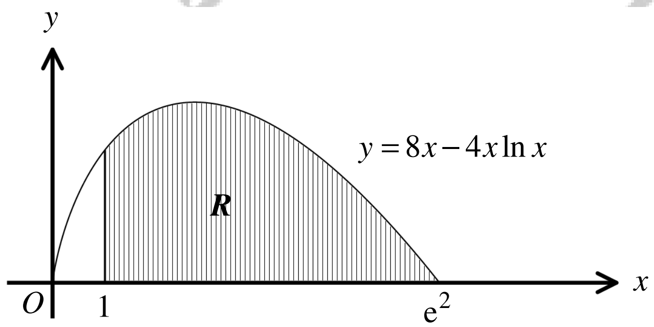
$$\frac{1}{8}\left(\pi^2 - 8\right).$$

$\boxed{}$, $\boxed{\text{proof}}$





Question 18 (****)



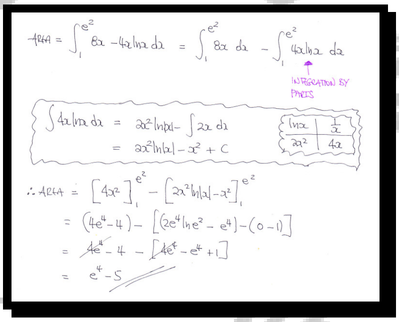
The figure above shows the graph of the curve with equation

$$y = 8x - 4x\ln x\,,\ 0 < x \le e^2 .$$

The region R is bounded by the curve, the x axis and the line with equation $x = 1$.

Determine the exact area of R.

$\boxed{e^4 - 5}$



$$\text{AREA} = \int_1^{e^2} 8x - 4x\ln x \, dx = \int_1^{e^2} 8x \, dx - \int_1^{e^2} 4x\ln x \, dx$$

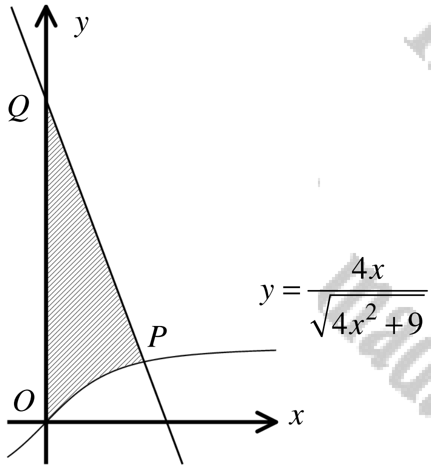
INTEGRATION BY PARTS

$$\int 4x\ln x \, dx = 2x^2\ln|x| - \int 2x \, dx$$
$$= 2x^2\ln|x| - x^2 + C$$

$\ln x$	$\frac{1}{x}$
$2x^2$	$4x$

$$\therefore \text{AREA} = \Big[4x^2\Big]_1^{e^2} - \Big[2x^2\ln|x| - x^2\Big]_1^{e^2}$$
$$= (4e^4 - 4) - \big[(2e^4\ln e^2 - e^4) - (0 - 1)\big]$$
$$= \cancel{4e^4} - 4 - \big[\cancel{4e^4} - e^4 + 1\big]$$
$$= e^4 - 5$$

Question 19 (****)



The figure above shows the graph of the curve with equation

$$y = \frac{4x}{\sqrt{4x^2+9}}, \quad x \in \mathbb{R}.$$

The point P lies on the curve where $x = 2$. The normal to the curve at P meets the y axis at Q.

a) Show that the y coordinate of Q is $\frac{769}{90}$.

b) Find the value of

$$\int_0^2 \frac{4x}{\sqrt{4x^2+9}} \, dx.$$

c) Hence find the area of the finite region bounded by the curve, the y axis and the normal to the curve at P.

$\boxed{}$, $\boxed{2}$, $\boxed{\frac{733}{90} \approx 8.14}$

[solution overleaf]

(a) $y = \frac{4x}{(4x^2+9)^{\frac{1}{2}}}$

$\frac{dy}{dx} = \frac{(4x^2+9)^{\frac{1}{2}} \times 4 - 4x \times \frac{1}{2} \times 8x(4x^2+9)^{-\frac{1}{2}}}{4x^2+9}$

$\left.\frac{dy}{dx}\right|_{x=2} = \frac{5\times4 - 64\times\frac{1}{5}}{25} = \frac{36}{125}$

NORMAL GRADIENT IS $-\frac{125}{36}$

WHEN $x=2$ $y = \frac{4\times2}{5} = \frac{8}{5}$ IE $P(2, \frac{8}{5})$

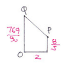
$y - \frac{8}{5} = -\frac{125}{36}(x-2)$

WHEN $x=0$ $y = \frac{125}{18} + \frac{8}{5} = \frac{769}{90}$ AS REQUIRED

(b) $\int_0^2 \frac{4x}{\sqrt{4x^2+9}}\,dx = \ldots$ by recognition or substitution $= \int_0^2 4x(4x^2+9)^{-\frac{1}{2}}$

$= \left[(4x^2+9)^{\frac{1}{2}}\right]_0^2 = 5-3 = 2$

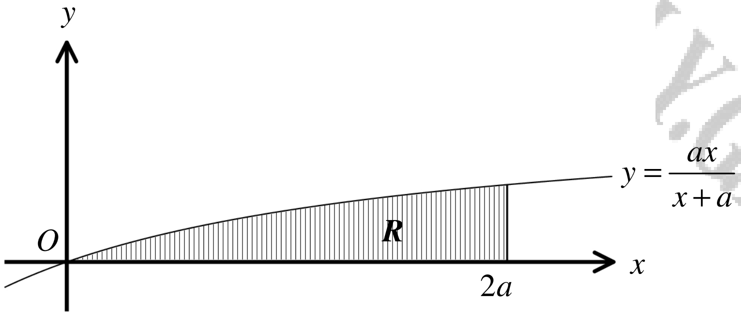
(c)



$= \frac{1}{2}\left(\frac{8}{5}+\frac{769}{30}\right)\times2 = \frac{913}{90}$

$\therefore$ REQUIRED AREA $= \frac{913}{90} - 2 = \frac{733}{90}$

Question 20 (****)



The figure above shows the graph of the curve C with equation

$$y = \frac{ax}{x+a}, \quad x \neq -a, \text{ where } a \text{ is a positive constant.}$$

The curve passes through the origin O.

The finite region R is bounded by C, the x axis and the line with equation $x = 2a$.

By using integration by substitution, or otherwise, show that the area of R is

$$a^2(2 - \ln 3).$$

proof

AREA $= \int_0^{2a} \frac{ax}{x+a}\,dx = \dots$ by SUBSTITUTION

$u = x + a$
$\frac{du}{dx} = 1$
$du = dx$
$x = u - a$
$x = 0,\ u = a$
$x = 2a,\ u = 3a$

$= \int_a^{3a} \frac{a(u-a)}{u}\,du = \int_a^{3a} \frac{au}{u} - \frac{a^2}{u}\,du$

$= \int_a^{3a} a - \frac{a^2}{u}\,du = \left[au - a^2\ln|u|\right]_a^{3a}$

$= \left[3a^2 - a^2\ln(3a)\right] - \left[a^2 - a^2\ln a\right]$

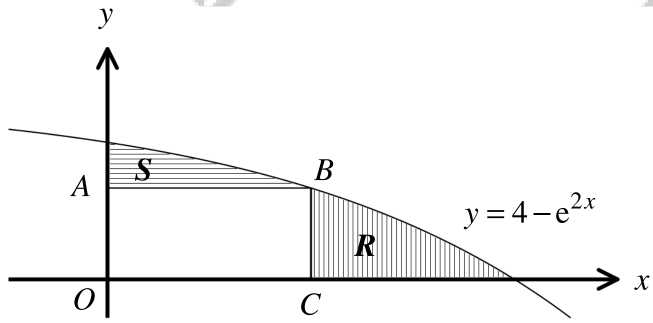
$= 2a^2 + a^2\ln a - a^2\ln(3a)$

$= a^2\left[2 + \ln a - \ln(3a)\right]$

$= a^2\left[2 + \ln\left(\frac{a}{3a}\right)\right] = a^2\left[2 + \ln\left(\frac{1}{3}\right)\right] = a^2\left[2 - \ln 3\right]$

As REQUIRED

Question 21 (****)



The figure above shows the graph of the curve with equation

$$y = 4 - e^{2x}, \ x \in \mathbb{R}.$$

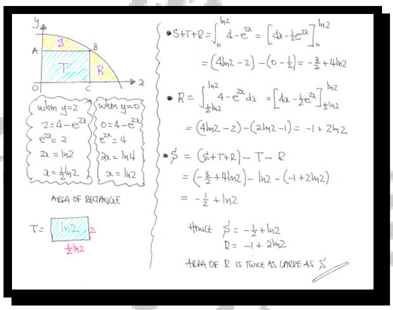
The point B lies on the curve where $y = 2$. The point A lies on the y axis and the point C lies on the x axis so that $OABC$ is a rectangle.

The region R is bounded by the curve, the x axis and the line BC.

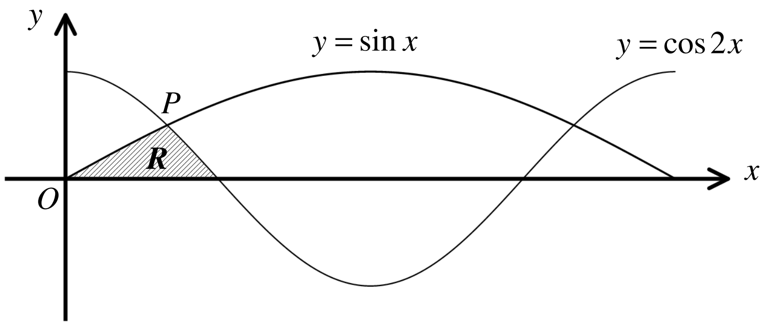
The region S is bounded by the curve, the y axis and the line AB.

Show, by exact calculations, that the area of R is twice as large as the area of S.

$\boxed{\text{proof}}$



Question 22 (****)



The figure above shows the graphs of the curves with equations

$$y = \cos 2x \text{ and } y = \sin x, \text{ for } 0 \le x \le \pi.$$

The point P is the first intersection between the graphs for which $x > 0$.

a) Show clearly that the x coordinate of P is $\frac{\pi}{6}$.

The finite region R, shown shaded in the figure, is bounded by the two curves and the x axis, and includes the point P on its boundary.

b) Determine the exact area of R.

$\boxed{}$, $\boxed{\frac{3}{4}\left(2-\sqrt{3}\right)}$

a) SOLVING SIMULTANEOUSLY WE OBTAIN

$$\left.\begin{array}{l} y = \cos 2x \\ y = \sin x \end{array}\right\} \Rightarrow \begin{array}{l} \cos 2x = \sin x \\ 1 - 2\sin^2 x = \sin x \\ 0 = 2\sin^2 x + \sin x - 1 \\ 0 = (2\sin x - 1)(\sin x + 1) \end{array}$$

$$\sin x = \begin{cases} \frac{1}{2} \\ -1 \end{cases}$$

- $x = \frac{\pi}{6} \pm 2n\pi$
- $x = \frac{5\pi}{6} \pm 2n\pi$

$n = 0, 1, 2, 3, \ldots$

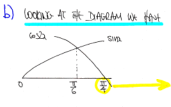
OR

- $x = -\frac{\pi}{2} \pm 2n\pi$
- $x = \frac{3\pi}{2} \pm 2n\pi$

$n = 0, 1, 2, 3, 4, \ldots$

$\therefore$ THE FIRST POSITIVE SOLUTION IS $x = \frac{\pi}{6}$

b) LOOKING AT THE DIAGRAM WE HAVE



$\cos 2x = 0$

$2x = \frac{\pi}{2} \pm 2n\pi$

$2x = \frac{3\pi}{2} \pm 2n\pi$ $\quad n = 0, 1, 2, 3, \ldots$

$x = \frac{\pi}{4} \pm n\pi$

$x = \frac{3\pi}{4} \pm n\pi$

$$\text{AREA} = \int_0^{\pi/6} \sin x \, dx + \int_{\pi/6}^{\pi/4} \cos 2x \, dx$$

$$= \Big[-\cos x\Big]_0^{\pi/6} + \Big[\tfrac{1}{2}\sin 2x\Big]_{\pi/6}^{\pi/4}$$

$$= \left[-\frac{\sqrt{3}}{2} - (-1)\right] + \left[\frac{1}{2} - \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)\right] = -\frac{\sqrt{3}}{2} + 1 + \frac{1}{2} - \frac{\sqrt{3}}{4} = \frac{3}{2} - \frac{3}{4}\sqrt{3}$$

$$\frac{3}{4}(2 - \sqrt{3})$$

Question 23 (****)

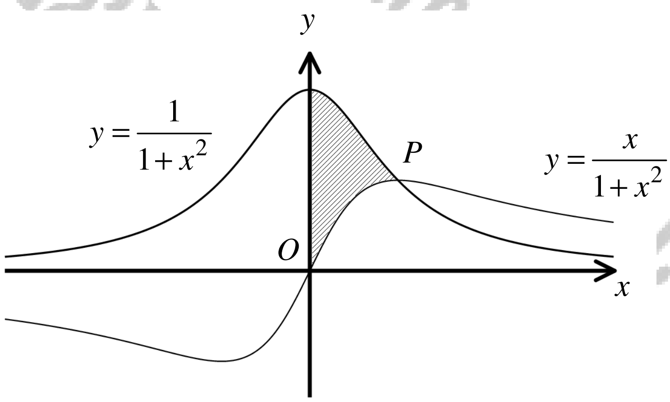
$$y = \arctan x \,, \ x \in \mathbb{R} \,.$$

a) By rewriting the above equation in the form $x = f(y)$, show clearly that

$$\frac{dy}{dx} = \frac{1}{1+x^2} \,.$$

The figure below shows the graphs of the curves with equations

$$y = \frac{1}{1+x^2} \quad \text{and} \quad y = \frac{x}{1+x^2} \,.$$



The two graphs intersect at the point $P\left(1, \frac{1}{2}\right)$.

b) Find the exact area of the finite region bounded by the two curves and the y axis, shown shaded in the figure.

$\boxed{}$, $\boxed{\frac{1}{4}(\pi - \ln 4)}$

[solution overleaf]

a) USING THE SUGGESTION GIVEN

$\Rightarrow y = \arctan x$

$\Rightarrow \tan y = x$

$\Rightarrow x = \tan y$

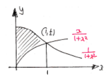
$\Rightarrow \frac{dx}{dy} = \sec^2 y$

$\Rightarrow \frac{dx}{dy} = 1 + \tan^2 y$ BUT $x = \tan y$

$\Rightarrow \frac{dx}{dy} = 1 + x^2$

$\Rightarrow \frac{dy}{dx} = \frac{1}{1+x^2}$ AS REQUIRED

b) LOOKING AT THE DIAGRAM



REQUIRED AREA =

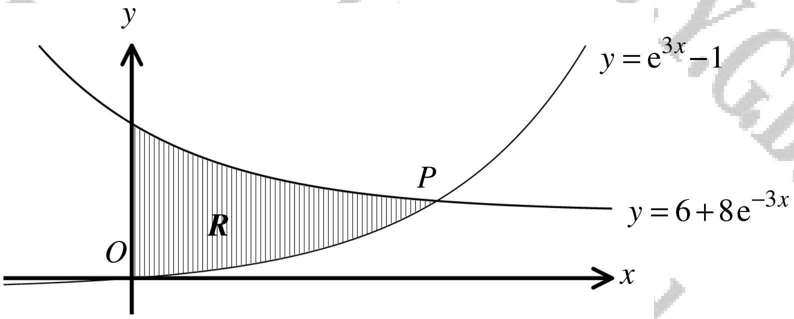
$= \int_0^1 \frac{1}{1+x^2}\,dx - \int_0^1 \frac{x}{1+x^2}\,dx$

$= \left[\arctan x - \frac{1}{2}\ln(1+x^2)\right]_0^1$

$= \left(\frac{\pi}{4} - \frac{1}{2}\ln 2\right) - \left(0 - \cancel{\frac{1}{2}\ln 1}\right)$

$= \frac{1}{4}(\pi - \ln 4)$

Question 24 (****)



The figure above shows the graphs of the curves with equations

$$y = e^{3x} - 1 \quad \text{and} \quad y = 6 + 8e^{-3x}.$$

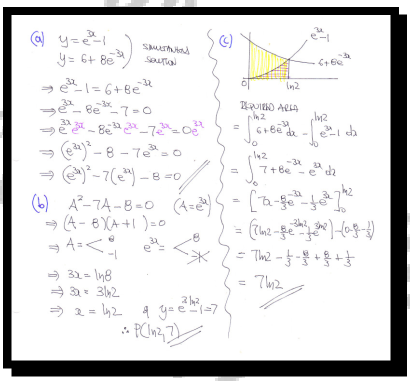
The curves intersect at the point P.

a) Find the exact coordinates of the point P.

The finite region R is bounded by the two curves and the y axis.

b) Determine the exact area of R.

$\boxed{(\ln 2, 7)}$, $\boxed{\text{area} = 7\ln 2}$



Question 25 (****+)

The finite region R is bounded by the curve with equation $y = \ln(x+3)$, the y axis and the straight line with equation $y = 1$.

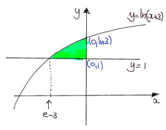
Show, with detailed workings, that the area of R is

$$e - 6 + 3\ln 3.$$

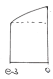
$\boxed{}$, $\boxed{\text{proof}}$

LOOKING AT THE DIAGRAM

- $\ln(x+3) = 1$
  $x+3 = e$
  $x = e-3$



NEXT WE FIND THE AREA UNDER THE CURVE BETWEEN $e-3$ & 0



$$= \int_{e-3}^{0} \ln(x+3)\,dx = \dots \text{ BY PARTS}$$

$\ln(x+3)$	$\frac{1}{x+3}$
x	1

$$= \Big[x\ln(x+3)\Big]_{e-3}^{0} - \int_{e-3}^{0} \frac{x}{x+3}\,dx$$

MANIPULATE, LONG-DIVIDE OR ANOTHER SUBSTITUTION

$$= \Big[x\ln(x+3)\Big]_{e-3}^{0} - \int_{e-3}^{0} \frac{(x+3)-3}{x+3}\,dx$$

$$= \Big[x\ln(x+3)\Big]_{e-3}^{0} - \int_{e-3}^{0} 1 - \frac{3}{x+3}\,dx$$

$$= \Big[x\ln(x+3) - x + 3\ln(x+3)\Big]_{e-3}^{0}$$

$$= (0 - 0 + 3\ln 3) - \big[(e-3)\ln e - (e-3) + 3\ln e\big]$$

$$= 3\ln 3 - \big[\cancel{(e-3)} - \cancel{(e-3)} + 3\big]$$

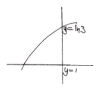
$$= -3 + 3\ln 3$$

FINALLY THE AREA OF THE RECTANGLE MEASURING $|e-3|$ BY 1 I.E $3-e$

$\therefore$ REQUIRED AREA IS $(-3+3\ln 3) - (3-e) = \underline{e + 3\ln 3 - 6}$

ALTERNATIVE BY "y" INTEGRATION

- $y = \ln(x+3)$
  $e^y = x+3$
  $x = e^y - 3$



- AREA $= \int_{y_1}^{y_2} x(y)\,dy$

$$= \int_{1}^{\ln 3} (e^y - 3)\,dy$$

$$= \Big[e^y - 3y\Big]_{1}^{\ln 3}$$

$$= (e^{\ln 3} - 3\ln 3) - (e - 3)$$

$$= 3 - 3\ln 3 - e + 3$$

$$= 6 - 3\ln 3 - e$$

BUT THIS IS "BELOW THE y AXIS" SO NEGATIVE

$\therefore$ AREA $= |6 - 3\ln 3 - e| = \underline{e + 3\ln 3 - 6}$

AS BEFORE

Question 26 (****+)

The function f is defined in the largest real domain by

$$f(x) \equiv (\ln x)^2, \quad x \in (0,\infty).$$

a) Sketch the graph of $f(x)$.

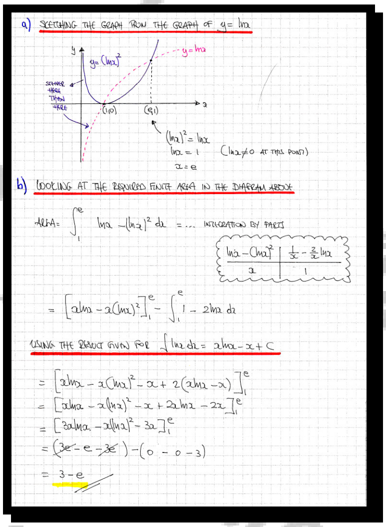
The function g is defined as

$$g(x) \equiv \ln x, \quad x \in (0,\infty).$$

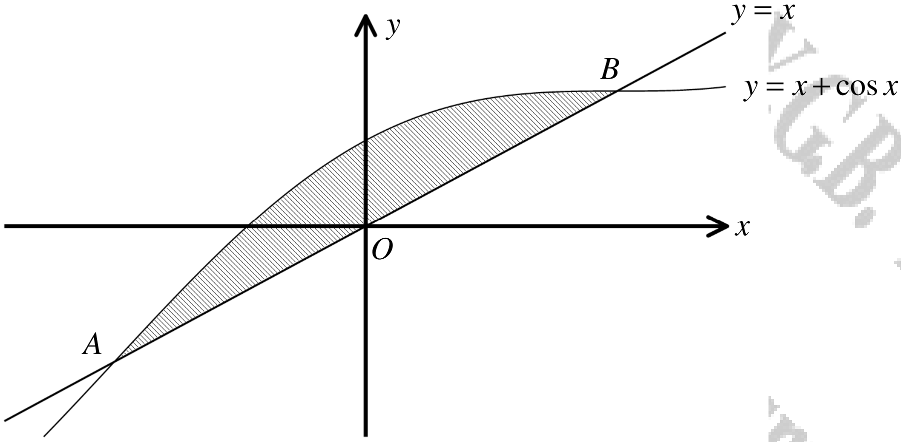
b) Determine in exact simplified form the area of the finite region bounded by the graph of f and the graph of g.

You may assume that $\int \ln x \, dx = x\ln x - x + \text{constant}$.

$\boxed{}$, $\boxed{3-e}$



Question 27 (*****)



The figure above shows the graph of the curve C with equation

$$y = x + \cos x, \ x \in \mathbb{R}$$

and the straight line L with equation

$$y = x, \ x \in \mathbb{R}.$$

Show that the area of the finite region bounded by C and L is 2 square units.

$\boxed{}$, $\boxed{\text{proof}}$

FIND THE x CO-ORDINATES OF THE POINTS OF INTERSECTION A & B

$$\left.\begin{array}{l} y = x \\ y = x + \cos x \end{array}\right\} \Rightarrow x = x + \cos x$$
$$\Rightarrow \cos x = 0$$
$$\Rightarrow x = \left\langle \begin{array}{l} -\pi/2 \\ \pi/2 \end{array} \right. \quad \text{(BY INSPECTION)}$$

HENCE THE REQUIRED AREA IS GIVEN BY THE INTEGRAL

$$\int_{-\pi/2}^{\pi/2} (x + \cos x) - x \, dx = \int_{-\pi/2}^{\pi/2} \cos x \, dx$$
$$= 2\int_{0}^{\pi/2} \cos x \, dx$$
$$= \Big[2\sin x \Big]_{0}^{\pi/2}$$
$$= 2\sin\frac{\pi}{2} - \cancel{2\sin 0}$$
$$= 2$$

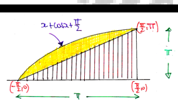
AS REQUIRED

ALTERNATIVE METHOD

AFTER OBTAINING THE CO-ORDINATES OF A & B AS

$$A\left(-\frac{\pi}{2}, -\frac{\pi}{2}\right) \ \& \ B\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$$

TRANSLATE BOTH GRAPHS UPWARDS BY $\pi/2$ IE $y = x + \cos x + \frac{\pi}{2}$, $y = x + \frac{\pi}{2}$



THE AREA UNDER THE CURVE $x + \cos x + \frac{\pi}{2}$ IS

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cancel{x} + \cos x + \frac{\pi}{2} \, dx = \int_{0}^{\pi} 2\cos x + \pi \, dx$$

ODD EVEN

$$\Big[2\sin x + \pi x \Big]_{0}^{\pi/2} = \left(2 + \tfrac{1}{2}\pi^2\right) - (0) = 2 + \frac{\pi^2}{2}$$

AREA OF THE TRIANGLE

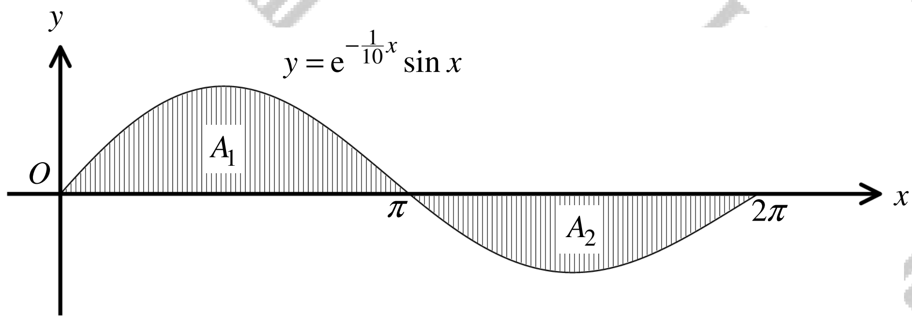
$$\frac{1}{2} \times \pi \times \pi = \frac{1}{2}\pi^2$$

HENCE THE REQUIRED AREA IS

$$2 + \frac{\pi^2}{2} - \frac{1}{2}\pi^2 = 2$$

AS BEFORE

Question 28 (*****)



The figure above shows the curve with equation

$$y = e^{-\frac{1}{10}x}\sin x, \quad 0 \le x \le 2\pi.$$

The curve meets the x axis at the origin and at the points $(\pi, 0)$ and $(2\pi, 0)$.

The finite region bounded by the curve for $0 \le x \le \pi$ and the x axis is denoted by A_1, and similarly A_2 denotes the finite region bounded by the curve for $\pi \le x \le 2\pi$ and the x axis.

By considering a suitable translation of the curve determine with justification the ratio of the areas of A_1 and A_2.

$$\boxed{\frac{A_1}{A_2} = e^{-\frac{\pi}{10}}}$$

STARTING FROM A_2

$$A_2 = \int_{\pi}^{2\pi} e^{-\frac{1}{10}x}\sin x \, dx$$

TRANSLATE "LEFT" BY π, BY THE SUBSTITUTION

$x = X + \pi$
$\frac{dx}{dX} = 1$
$dx = dX$
• $x = \pi \longmapsto X = 0$
• $x = 2\pi \longmapsto X = \pi$

$$A_2 = \int_0^{\pi} e^{-\frac{1}{10}(X+\pi)}\sin(X+\pi)\, dX$$

$$A_2 = \int_0^{\pi} e^{-\frac{1}{10}X} e^{-\frac{1}{10}\pi}\sin(X+\pi)\, dX$$

USING THE COMPOUND ANGLE FORMULA

$$\sin(X+\pi) = \sin X\cos\pi + \cancel{\cos X\sin\pi} = -\sin X$$

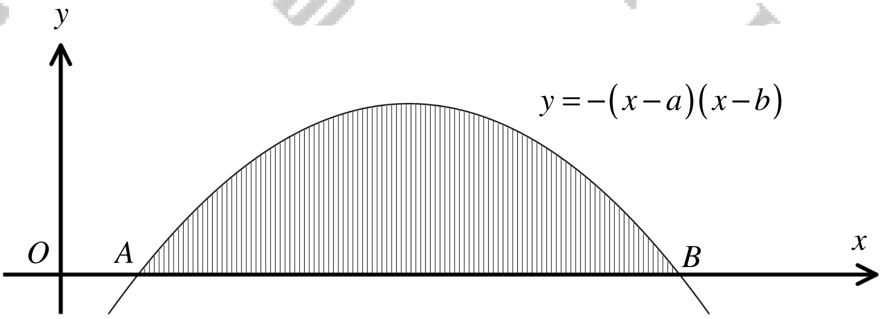
$$A_2 = \int_0^{\pi} e^{-\frac{1}{10}X} e^{-\frac{1}{10}\pi}(-\sin X)\, dX$$

$$A_2 = -e^{-\frac{1}{10}\pi}\int_0^{\pi} e^{-\frac{1}{10}X}\sin X\, dX$$

IGNORING THE MINUS AS THIS MERELY INDICATES THAT THE AREA IS BELOW THE x AXIS AND RELABELING $X \longmapsto x$

$$\frac{A_2}{A_1} = \left| \frac{-e^{-\frac{1}{10}\pi}\int_0^{\pi} e^{-\frac{1}{10}x}\sin x\, dx}{\int_0^{\pi} e^{-\frac{1}{10}x}\sin x\, dx} \right| = e^{-\frac{1}{10}\pi}$$

Question 29 (*****)



The figure above shows the parabola with equation

$$y = -(x-a)(x-b), \; b > a > 0.$$

The curve meets the x axis at the points A and B.

a) Show that the area of the finite region R, bounded by the parabola and the x axis is

$$\tfrac{1}{6}(b-a)^3.$$

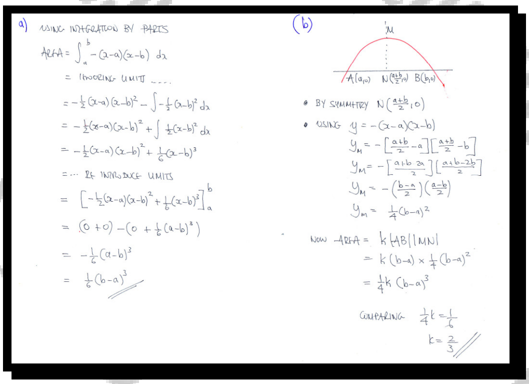
The midpoint of AB is N. The point M is the maximum point of the parabola.

b) Show clearly that the area of R is given by

$$k|AB||MN|,$$

where k is a constant to be found.

$\boxed{\text{SP}}$, $\boxed{k = \frac{2}{3}}$



a) USING INTEGRATION BY PARTS

$\text{AREA} = \int_a^b -(x-a)(x-b)\,dx$

$= \text{IGNORING LIMITS} \dots$

$= -\frac{1}{2}(x-a)(x-b)^2 - \int -\frac{1}{2}(x-b)^2\,dx$

$= -\frac{1}{2}(x-a)(x-b)^2 + \int \frac{1}{2}(x-b)^2\,dx$

$= -\frac{1}{2}(x-a)(x-b)^2 + \frac{1}{6}(x-b)^3$

$= \dots$ RE INTRODUCE LIMITS

$= \left[-\frac{1}{2}(x-a)(x-b)^2 + \frac{1}{6}(x-b)^3\right]_a^b$

$= (0+0) - \left(0 + \frac{1}{6}(a-b)^3\right)$

$= -\frac{1}{6}(a-b)^3$

$= \frac{1}{6}(b-a)^3$

(b)

- BY SYMMETRY $N\left(\frac{a+b}{2}, 0\right)$
- USING $y = -(x-a)(x-b)$

$y_M = -\left[\frac{a+b}{2} - a\right]\left[\frac{a+b}{2} - b\right]$

$y_M = -\left[\frac{a+b-2a}{2}\right]\left[\frac{a+b-2b}{2}\right]$

$y_M = -\left(\frac{b-a}{2}\right)\left(\frac{a-b}{2}\right)$

$y_M = \frac{1}{4}(b-a)^2$

NOW AREA $= k|AB||MN|$

$= k(b-a) \times \frac{1}{4}(b-a)^2$

$= \frac{1}{4}k(b-a)^3$

COMPARING $\frac{1}{4}k = \frac{1}{6}$

$k = \frac{2}{3}$

Question 30 (*****)

Determine, in exact simplified form, the area of the finite region bounded by the curves with equations

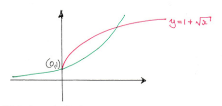
$$y = 1 + \sqrt{x}, \; x \in \mathbb{R}, \; x \geq 0.$$

$$y = 4^{\frac{x}{9}}, \; x \in \mathbb{R}.$$

$\boxed{}$, $\boxed{27\left(1 - \frac{1}{\ln 4}\right)}$

$\boxed{y = 1 + \sqrt{x}, \; x \geq 0}$ $\boxed{y = 4^{\frac{x}{9}}, \; x \in \mathbb{R}}$

- NEED THE INTERSECTIONS OF THE TWO GRAPHS



- BY INSPECTION $(0,1)$ IS ONE SUCH POINT AND FROM THE SKETCH IT IS EVIDENT THAT THE TWO GRAPHS INTERSECT ONCE MORE — BY TRYING A FEW INTEGER VALUES WE SEE THAT THE REQUIRED POINT IS $(9,4)$

- HENCE THE REQUIRED AREA IS GIVEN BY

$$A = \int_0^9 1 + \sqrt{x} - 4^{\frac{x}{9}} \, dx = \int_0^9 1 + x^{\frac{1}{2}} - 4^{\frac{1}{9}x} \, dx$$

$$= \left[x + \frac{2}{3}x^{\frac{3}{2}} - 4^{\frac{1}{9}x} \times \frac{1}{\ln 4} \times \frac{1}{\frac{1}{9}} \right]_0^9$$

$$= \left[x + \frac{2}{3}x^{\frac{3}{2}} - \frac{9}{\ln 4} \times 4^{\frac{1}{9}x} \right]_0^9$$

$$= \left(9 + \frac{2}{3} \times 9^{\frac{3}{2}} - \frac{9}{\ln 4} \times 4 \right) - \left(0 + 0 - \frac{9}{\ln 4} \times 1 \right)$$

$$= 9 + 18 - \frac{36}{\ln 4} + \frac{9}{\ln 4}$$

$$= 27 - \frac{27}{\ln 4}$$

$$= 27\left(1 - \frac{1}{\ln 4}\right) //$$

Question 31 (*****)

The straight line L with equation $y = kx$, where k is a positive constant, meets the curve C with equation $y = x\mathrm{e}^{-2x}$, at the point P.

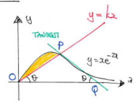
The tangent to C at P meets the x axis at the point Q.

Given that $|OP| = |PQ|$, find in exact simplified form the area of the **finite** region bounded by C and L.

$\boxed{}$, $\boxed{\text{area} = \frac{1}{2}\left(1 - 5\mathrm{e}^{-2}\right)}$

- START WITH A BRIEF SKETCH OF THE CURVE

$y = xe^{-2x}$ $\quad x=0,\ y=0$; $y>0$ if $x>0$; As $x \to \infty$, $y \to 0$



- FIND THE CO-ORDINATES OF P

$$\left.\begin{array}{l} y = xe^{-2x} \\ y = kx \end{array}\right\} \Rightarrow xe^{-2x} = kx$$
$$\Rightarrow e^{-2x} = k \quad (x \neq 0)$$
$$\Rightarrow -2x = \ln k$$
$$\Rightarrow x = -\tfrac{1}{2}\ln k$$
$$\therefore P\left(-\tfrac{1}{2}\ln k,\ -\tfrac{1}{2}k\ln k\right)$$

- FIND THE GRADIENT OF THE TANGENT AT P

$$\frac{dy}{dx} = e^{-2x} - 2xe^{-2x} = e^{-2x}(1-2x)$$
$$\left.\frac{dy}{dx}\right|_{x=P} = k\left[1 - 2\left(-\tfrac{1}{2}\ln k\right)\right] = k\left[1 + \ln k\right]$$

- AS THE TANGENT & THE LINE $y = kx$ HAVE THE SAME INCLINATION TO THE x AXIS, THEIR GRADIENTS HAVE THE SAME MAGNITUDE AND OPPOSITE SIGNS, HENCE

$$k\left[1 + \ln k\right] = -k$$
$$1 + \ln k = -1 \quad (k \neq 0)$$
$$\Rightarrow \ln k = -2$$

INVERTING

$$\Rightarrow k = e^{-2}$$

- FINALLY THE REQUIRED AREA CAN BE FOUND BY INTEGRATION

IF $k = e^{-2}$ $\quad P(1, e^{-2})$

$$\text{AREA} = \int_0^1 xe^{-2x} - e^{-2}x \, dx$$

BY PARTS

x	1
$-\frac{1}{2}e^{-2x}$	e^{-2x}

$$= \left[-\tfrac{1}{2}xe^{-2x}\right]_0^1 + \int_0^1 \tfrac{1}{2}e^{-2x}\,dx - \left[\tfrac{1}{2}e^{-2}x^2\right]_0^1$$
$$= \left[-\tfrac{1}{2}xe^{-2x} - \tfrac{1}{4}e^{-2x} - \tfrac{1}{2}e^{-2}x^2\right]_0^1$$
$$= \tfrac{1}{4}\left[2xe^{-2x} + e^{-2x} + 2e^{-2}x^2\right]_1^0$$
$$= \tfrac{1}{4}\left[(0+1+0) - (2e^{-2} + e^{-2} + 2e^{-2})\right]$$
$$= \tfrac{1}{4}\left[1 - 5e^{-2}\right]$$

Question 32 (*****)

Two curves are defined in the largest possible real number domain and have equations

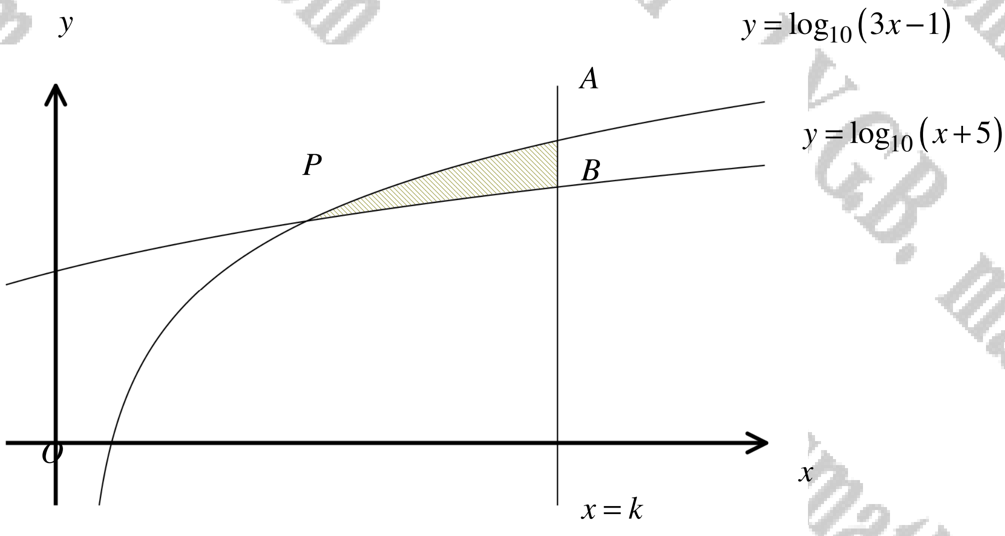
$$y^2 = \frac{4(4-x)}{x} \qquad \text{and} \qquad x^2 = \frac{4(4-y)}{y}.$$

a) Show that the two curves have one, and only one, common point which is also a point of common tangency.

b) Find the exact value of the area enclosed by the common tangent to the curves, and either of the two curves.

$\boxed{}$, $\boxed{2(\pi-3)}$

Question 33 (*****)



The figure above shows the graphs of the curves with equations

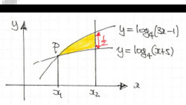
$$y = \log_{10}(3x-1), \quad \text{and} \quad y = \log_{10}(x+5).$$

The two curves intersect at the point P

The straight line with equation $x = k$, $k > 3$, meets the graph of $y = \log_{10}(3x-1)$ at the point A and the graph of $y = \log_{10}(x+5)$ at the point B, so that $|AB| = \frac{1}{2}$.

Determine the value of the area of the finite region, bounded by the two curves and the straight line $x = k$, shown shaded in the above figure.

$\boxed{\text{area} = \frac{8}{3}}$



- FIRSTLY FIND THE x CO-ORDINATE OF P BY INSPECTION

$\Rightarrow \log_4(3x-1) = \log_4(x+5)$

$\Rightarrow 3x-1 = x+5$

$\Rightarrow 2x = 6$

$\Rightarrow x = 3$ (y IS NOT NEEDED)

- NEXT WE FIND x_2

$\Rightarrow \log_4(3x-1) - \log_4(x+5) = \frac{1}{2}$

$\Rightarrow \log_4\left(\frac{3x-1}{x+5}\right) = \frac{1}{2}$

$\Rightarrow \frac{3x-1}{x+5} = 4^{\frac{1}{2}}$

$\Rightarrow \frac{3x-1}{x+5} = 2$

$\Rightarrow 3x-1 = 2x+10$

$\Rightarrow x = 11$ (y IS NOT NEEDED)

- NEXT WE FIND THE REQUIRED AREA BY INTEGRATION

$\Rightarrow \text{AREA} = \int_3^{11} \log_4(3x-1)\,dx - \int_3^{11} \log_4(x+5)\,dx$

$\Rightarrow \text{AREA} = \int_3^{11} \frac{\ln(3x-1)}{\ln 4}\,dx - \int_3^{11} \frac{\ln(x+5)}{\ln 4}\,dx$

- APPLY SUBSTITUTIONS FOR SIMPLICITY BEFORE I.B.P.

$u = 3x-1$	$v = x+5$
$\frac{du}{dx} = 3$	$\frac{dv}{dx} = 1$
$dx = \frac{du}{3}$	$dv = dx$
$x = 3 \mapsto u = 8$	$x = 3 \mapsto v = 8$
$x = 11 \mapsto u = 32$	$x = 11 \mapsto v = 16$

$\Rightarrow \text{AREA} = \int_8^{32} \frac{\ln u}{\ln 4}\,\frac{du}{3} - \int_8^{16} \frac{\ln v}{\ln 4}\,dv$

- NOW INTEGRATION BY PARTS OR USE STANDARD RESULT

$\int \ln x\,dx = x\ln x - x + C$

$\Rightarrow \text{AREA} = \frac{1}{3\ln 4}\Big[u\ln u - u\Big]_8^{32} - \frac{1}{\ln 4}\Big[v\ln v - v\Big]_8^{16}$

$\Rightarrow \text{AREA} = \frac{1}{3\ln 4}\Big[(32\ln 32 - 32) - (8\ln 8 - 8)\Big] - \frac{1}{\ln 4}\Big[(16\ln 16 - 16) - (8\ln 8 - 8)\Big]$

$\Rightarrow \text{AREA} = \frac{1}{6\ln 2}\Big[32\ln 32 - 8\ln 8 - 24\Big] - \frac{1}{2\ln 2}\Big[16\ln 16 - 8\ln 8 - 8\Big]$

$\Rightarrow \text{AREA} = \frac{1}{6\ln 2}\Big[160\ln 2 - 24\ln 2 - 24\Big] - \frac{1}{2\ln 2}\Big[64\ln 2 - 24\ln 2 - 8\Big]$

$\Rightarrow \text{AREA} = \frac{1}{6\ln 2}\Big[136\ln 2 - 24\Big] - \frac{1}{2\ln 2}\Big[40\ln 2 - 8\Big]$

$\Rightarrow \text{AREA} = \frac{1}{3\ln 2}\Big[68\ln 2 - 12\Big] - \frac{1}{\ln 2}\Big[20\ln 2 - 4\Big]$

$\Rightarrow \text{AREA} = \frac{1}{3\ln 2}\Big[68\ln 2 \cancel{-12} - 60\ln 2 \cancel{+12}\Big]$

$\Rightarrow \text{AREA} = \frac{8\ln 2}{3\ln 2}$

$\Rightarrow \text{AREA} = \frac{8}{3}$

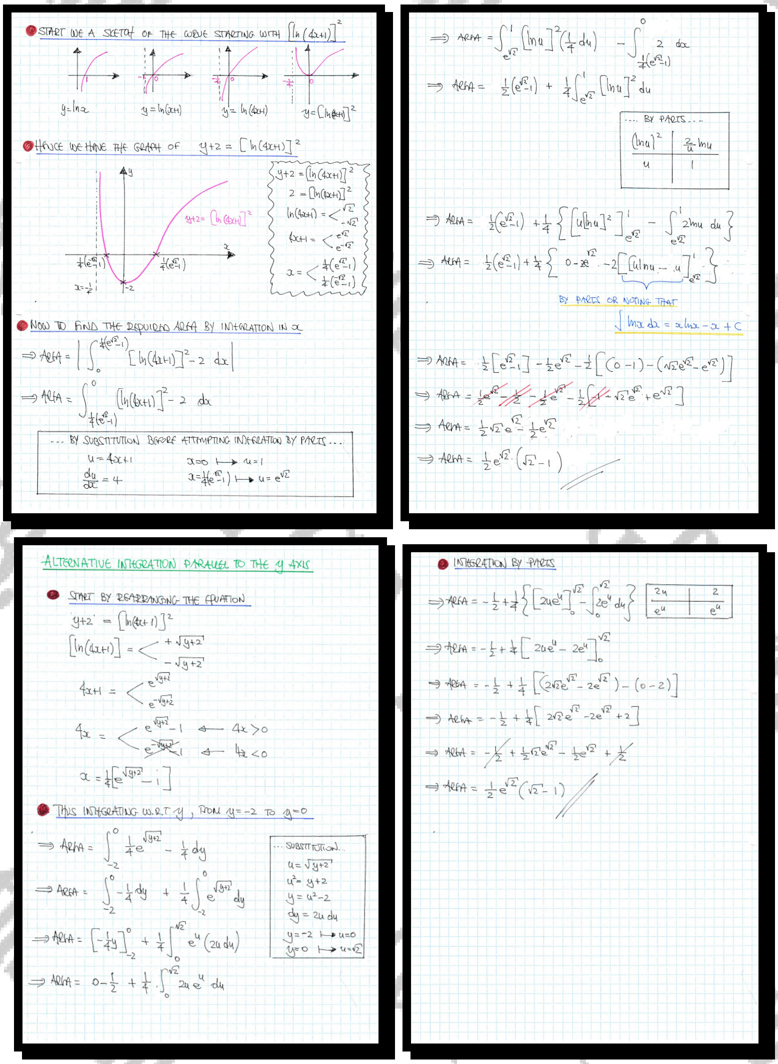
Question 34 (*****)

The curve C has equation

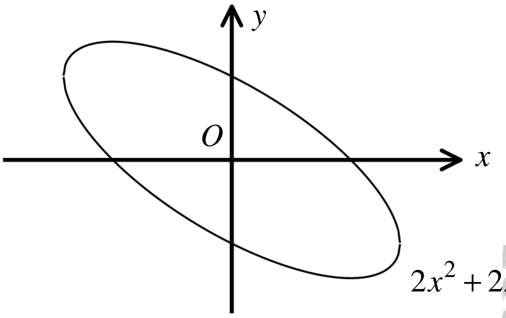
$$y+2=\left[\ln(4x+1)\right]^2, \quad x\in\mathbb{R}, \quad x\geq -\frac{1}{4}.$$

Sketch the graph of C and hence determine, in exact simplified form, the area of the finite region bounded by C, for which $x\geq 0$, and the coordinate axes.

$\boxed{}$, $\boxed{\text{area} = \frac{1}{2}\left(\sqrt{2}-1\right)e^{\sqrt{2}}}$



Question 35 (*****)



$$2x^2 + 2xy + y^2 = 50$$

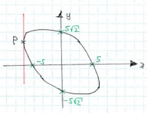
The figure above shows the curve with equation

$$2x^2 + 2xy + y^2 = 50.$$

Determine the area of the finite region bounded by the x axis and the part of the curve for which $y \geq 0$.

$\boxed{}$, $\boxed{25\pi}$

- FIRSTLY PRODUCE A SKETCH WITH x & y INTERCEPTS



- NEXT FIND THE x CO-ORDINATE OF THE POINT P (VERTICAL TANGENT)

$$2x^2 + 2xy + y^2 = 50$$
$$\frac{d}{dx}\left(2x^2 + 2xy + y^2\right) = 0$$
$$4x + 2y + 2x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$
$$2(y+2x) = -2(x+y)\frac{dy}{dx}$$
$$\frac{dy}{dx} = -\frac{y+2x}{x+y} \leftarrow \text{INFINITE GRADIENT} \Longrightarrow \text{DENOMINATOR ZERO}$$

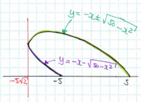
- SOLVING SIMULTANEOUSLY WITH THE EQUATION OF THE CURVE WE OBTAIN

$$y + x = 0 \qquad 2x^2 + 2x(-x) + (-x)^2 = 50$$
$$y = -x \qquad x^2 = 50$$
$$x = \pm 5\sqrt{2}$$
$$\therefore P(-5\sqrt{2}, 5\sqrt{2})$$

- NEXT REARRANGE THE EQUATION OF THE CURVE, IN THE FORM $y = f(x)$

$$\Longrightarrow y^2 + 2xy + 2x^2 = 50$$
$$\Longrightarrow (y+x)^2 - x^2 + 2x^2 = 50$$
$$\Longrightarrow (y+x)^2 = 50 - x^2$$
$$\Longrightarrow y + x = \pm\sqrt{50 - x^2}$$
$$\Longrightarrow y = -x \pm \sqrt{50 - x^2}$$

- THUS WE NOW HAVE THE "PICTURE" OPPOSITE – IN BOTH CASES WE HAVE TO INTEGRATE $\sqrt{50-x^2}$, SO PREPARE THIS PART FIRST



$$\int_{x_1}^{x_2} \sqrt{50 - x^2}\,dx = \dots \text{ BY SUBSTITUTION} \qquad \boxed{\begin{array}{l} x = \sqrt{50}\sin\theta \\ dx = \sqrt{50}\cos\theta \end{array}}$$
$$= \int_{\theta_1}^{\theta_2} \sqrt{50 - 50\sin^2\theta}\left(\sqrt{50}\cos\theta\,d\theta\right)$$
$$= \int_{\theta_1}^{\theta_2} \sqrt{50(1-\sin^2\theta)}\,\sqrt{50}\cos\theta\,d\theta$$
$$= \int_{\theta_1}^{\theta_2} 50\cos^2\theta\,d\theta = \int_{\theta_1}^{\theta_2} 25 + 25\cos 2\theta\,d\theta$$
$$= \left[25\theta + \frac{25}{2}\sin 2\theta\right]_{\theta_1}^{\theta_2}$$

- HENCE THE REQUIRED AREA CAN BE FOUND

$$\text{◠} - \text{◺} = \int_{-5\sqrt{2}}^{5} -x + \sqrt{50-x^2}\,dx - \int_{-5\sqrt{2}}^{-5} -x - \sqrt{50-x^2}\,dx$$
$$= \int_{-5\sqrt{2}}^{5} -x\,dx + \int_{-5\sqrt{2}}^{-5} x\,dx + \int_{-5\sqrt{2}}^{5} \sqrt{50-x^2}\,dx + \int_{-5\sqrt{2}}^{-5} \sqrt{50-x^2}\,dx$$

- CHANGING THE LIMITS IN THE SUBSTITUTION INTEGRALS

$$\boxed{\begin{array}{l} x = 5\sqrt{2}\sin\theta \\ x = 5 \longmapsto \theta = \frac{\pi}{4} \\ x = -5 \longmapsto \theta = -\frac{\pi}{4} \\ x = -5\sqrt{2} \longmapsto \theta = -\frac{\pi}{2} \end{array}}$$

$$= \left[-\frac{1}{2}x^2\right]_{-5\sqrt{2}}^{5} + \left[\frac{1}{2}x^2\right]_{-5\sqrt{2}}^{-5} + \left[25\theta + \frac{25}{2}\sin 2\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{4}} + \left[25\theta + \frac{25}{2}\sin 2\theta\right]_{-\frac{\pi}{2}}^{-\frac{\pi}{4}}$$
$$= \left[\cancel{-\frac{25}{2}} + \cancel{\frac{50}{2}}\right] + \left[\cancel{\frac{25}{2}} - \cancel{\frac{50}{2}}\right] + \left[\left(\cancel{\frac{25\pi}{4}} + \cancel{\frac{25}{2}}\right) - \left(-\frac{25\pi}{2} + 0\right)\right]$$
$$+ \left[\left(\cancel{-\frac{25\pi}{4}} - \cancel{\frac{25}{2}}\right) - \left(-\frac{25\pi}{2} + 0\right)\right]$$

$$= 25\pi$$