INTEGRATION
FINDING AREAS
Question 1  (**)

The diagram above shows the graph of the curve with equation

$$y = e^{x} - 5, \ x \in \mathbb{R}.$$  

Use integration to find the exact area of the finite region bounded by the curve and the coordinate axes.

$$5 \ln 5 - 4 \approx 4.05$$
Question 2 (***)

The figure above shows the graph of the curve with equation

\[ y = \left(2 - e^{2x}\right)^2, \quad x \in \mathbb{R}. \]

The curve touches the \(x\) axis at the point \(P\) and crosses the \(y\) axis at the point \(Q\).

Find the exact area of the finite region bounded by the curve and the line segments \(OP\) and \(OQ\).

\[ \frac{-5}{4} + \ln 4 \]
The diagram above shows the graph of the curve with equation

\[ y = 8 - e^{3x}, \quad x \in \mathbb{R}. \]

The curve meets the \( x \)-axis at the point \( P \) and the \( y \)-axis at the point \( Q \).

Show that the area of the finite region bounded by the curve and the straight line segment \( PQ \) is exactly \( \frac{9}{2} \ln 2 - \frac{7}{3} \).

**Proof**
The figure above shows the graphs of the curves with equations

\[ y = (3 - 2x)^5 \quad \text{and} \quad y = e^{2x-3} - 1. \]

Both curves meet each other at the point with coordinates \( \left(\frac{3}{2}, 0\right) \).

Calculate the area of the shaded region bounded by the two curves and the \( y \) axis giving the answer correct to three decimal places.

\[ \approx 61.775 \]
The figure above shows the graph of the curve with equation
\[ y = 1 + e^{2x}, \quad x \in \mathbb{R}. \]

The point \( C \) has coordinates \((1, 0)\). The point \( B \) lies on the curve so that \( BC \) is parallel to the \( y \) axis. The point \( A \) lies on the \( y \) axis so that \( OABC \) is a rectangle.

The region \( R \) is bounded by the curve, the coordinate axes and the line \( BC \).

The region \( S \) is bounded by the curve, the \( y \) axis and the line \( AB \).

Show that the area of \( R \) is equal to the area of \( S \).
Question 6 \((**+)\)

The figure above shows the graph of the curve with equation

\[ y = 1 + \sin 2x, \quad x \in \mathbb{R}. \]

The point \( P \) lies on the curve where \( x = \frac{\pi}{3} \).

Show that the area of the finite region bounded by the curve, the \( y \) axis and the straight line segment \( OP \) is exactly

\[ \frac{1}{12} \left( 2\pi + 9 - \pi\sqrt{3} \right). \]
The figure above shows the graph of the curve with equation

\[ y = 2 \cos x, \ 0 \leq x \leq \frac{\pi}{2}. \]

Use integration by parts to find the exact area of the finite region bounded by the curve and the \( x \) axis.

\[ \pi - 2 \]
The figure above shows the curve with equation 

\[ f(x) = 15x(x-2)^4, \quad 0 \leq x \leq 2. \]

a) Express \( f(x+2) \) in the form \( Ax^5 + Bx^4 \), where \( A \) and \( B \) are constants.

b) By using part (a), or otherwise find the value of 
\[ \int_0^2 f(x) \, dx. \]

c) Explain geometrically the significance of 
\[ \int_0^2 f(x) \, dx = \int_{-2}^0 f(x+2) \, dx. \]

\[ f(x+2) = 15x^5 + 30x^4, \quad \int_0^2 f(x) \, dx = 32 \]
The figure above shows the curve $C$ with equation

$$y = \frac{1}{2}(e^x - 1), \ x \in \mathbb{R}.$$

The point $P(k,k)$, where $k$ is a positive constant, lies on $C$.

The finite region $R$, shown shaded in the figure, is bounded by $C$, the $x$ axis and the line with equation $x = k$.

Show clearly that the area of $R$ is $\frac{1}{2}k$.

**proof**
Question 10  (***)

The figure above shows the graph of the curve with equation

\[ y = 2e^{\frac{1}{2}x} + 3, \quad x \in \mathbb{R}. \]

The points \( A \) and \( B \) lie on the curve where \( x = 2 \) and \( x = 4 \), respectively.

The finite region \( R \) is bounded by the curve, the \( x \) axis and the lines with equations \( x = 2 \) and \( x = 4 \).

**a)** Determine the exact area of \( R \).

The points \( C \) and \( D \) lie on the \( y \) axis so that the line segments \( CA \) and \( DB \) are parallel to the \( x \) axis.

The region \( S \) is bounded by the curve, the \( y \) axis and the line segments \( CA \) and \( DB \).

**b)** Determine the exact area of \( S \).

\[
\text{area of } R = 4e^2 - 4e + 6, \quad \text{area of } S = 4e^2
\]
The figure above shows the graph of the curve with equation

\[ y = (x+2)e^{-x}, \quad x \in \mathbb{R}. \]

The finite region \( R \), shown shaded in the figure, is bounded by the curve and the coordinate axes.

Use integration by parts to show that the area of \( R \) is \( e^2 - 3 \).

**proof**
Question 12  (***)

The figure above shows the graph of the curve with equation

\[ y = x\sqrt{1 - 2x}, \quad x \leq \frac{1}{2} \]

Use integration by substitution to find the area of the finite region bounded by the curve and the \( x \) axis.
The figure above shows the graph of the curve with equation

\[ y = \frac{c}{x} \]

where \( c \) is a positive constant.

The points \( A \) and \( B \) lie on the curve here \( x = 3 \) and \( x = 6 \) respectively.

The finite region \( R \) is bounded by the curve, the \( x \) axis and the lines with equations \( x = 3 \) and \( x = 6 \).

a) Given that the area of \( R \) is \( \ln 4096 \), show that \( c = 12 \).

The point \( C \) is such so that \( AC \) is parallel to the \( x \) axis and \( BC \) is parallel to the \( y \) axis. The finite region \( S \) is bounded by the curve and the line segments \( AC \) and \( BC \).

b) Find the volume of the solid produced when \( S \) is complete revolved about the \( x \) axis.

\[ 3\pi \]

[solution overleaf]
The figure above shows the graph of the curve with equation

\[ y = \frac{1}{\sqrt{4x+1}}, \quad x > -\frac{1}{4} \]

The finite region \( R \) is bounded by the curve, the coordinate axes and the line with equation \( x = a \), where \( a \) is a positive constant.

When \( R \) is revolved by \( 2\pi \) radians in the \( x \) axis the resulting solid of revolution has a volume of \( \pi \ln 3 \), in appropriate cubic units.

Determine the area of \( R \).

\[ \text{area} = 4 \]
The figure above shows the graph of the curve with equation

\[ y = \frac{2}{x}, \quad x \in \mathbb{R}, \quad x \neq 0. \]

The points \( A \) and \( B \) lie on the curve. The respective \( x \) coordinates of these points are \( k \) and \( 3k \), where \( k \) is a positive constant. The point \( C \) is such so that \( AC \) is parallel to the \( x \) axis and \( BC \) is parallel to the \( y \) axis.

The finite region \( R \) is bounded by the curve and the line segments \( AC \) and \( BC \).

Show that the area of \( R \) is \( \ln\left(\frac{9}{4e^4}\right) \).

**proof**
Question 16  (***)

The figure above shows the graph of the curve with equation

\[ y = (2x - 1)^5, \quad x \in \mathbb{R}. \]

The curve meets the \( x \) axis at the point \( A \) and the point \( B(1,1) \) lies on the curve.

Show that the area of the finite region \( R \) bounded by the curve, the \( x \) axis and the tangent to the curve at \( B \) is \( \frac{1}{30} \).

**proof**
Question 17 (****)

The figure above shows the graph of the curve \( C \) with equation

\[ y = 9x \sin 3x, \ x \in \mathbb{R}. \]

The straight line \( L \) is the tangent to \( C \) at the point \( P \), whose \( x \) coordinate is \( \frac{\pi}{6} \).

**a)** Show that \( L \) passes through the origin \( O \).

The finite region \( R \) bounded by \( C \) and \( L \).

**b)** Show further that the area of \( R \) is

\[ \frac{1}{8} \left( \pi^2 - 8 \right). \]

**proof**
The figure above shows the graph of the curve with equation

\[ y = 8x - 4x \ln x, \quad 0 < x \leq e^2. \]

The region \( R \) is bounded by the curve, the \( x \) axis and the line with equation \( x = 1 \).

Determine the exact area of \( R \).

\[ e^4 - 5 \]
Question 19     (****)

The figure above shows the graph of the curve with equation

\[ y = \frac{4x}{\sqrt{4x^2 + 9}}, \quad x \in \mathbb{R}. \]

The point \( P \) lies on the curve where \( x = 2 \). The normal to the curve at \( P \) meets the \( y \) axis at \( Q \).

a) Show that the \( y \) coordinate of \( Q \) is \( \frac{769}{90} \).

b) Find the value of

\[ \int_{0}^{2} \frac{4x}{\sqrt{4x^2 + 9}} \, dx. \]

c) Hence find the area of the finite region bounded by the curve, the \( y \) axis and the normal to the curve at \( P \).

\[ 2 \cdot \frac{733}{90} \approx 8.14 \]

[solution overleaf]
Created by T. Madas
The figure above shows the graph of the curve $C$ with equation

$$y = \frac{ax}{x + a}, \quad x \neq -a,$$

where $a$ is a positive constant.

The curve passes through the origin $O$.

The finite region $R$ is bounded by $C$, the $x$ axis and the line with equation $x = 2a$.

By using integration by substitution, or otherwise, show that the area of $R$ is

$$a^2 (2 - \ln 3).$$

**proof**

\[
\begin{align*}
\int_{a}^{2a} \frac{a}{x+a} \, dx &= \left[ a \ln |x+a| \right]_{a}^{2a} \\
&= a \ln 3 - a \ln 2 \\
&= a \ln \left( \frac{3}{2} \right) \\
&= a^2 \ln \left( \frac{2}{3} \right) \\
&= a^2 (2 - \ln 3).
\end{align*}
\]
The figure above shows the graph of the curve with equation

\[ y = 4 - e^{2x}, \quad x \in \mathbb{R}. \]

The point \( B \) lies on the curve where \( y = 2 \). The point \( A \) lies on the \( y \) axis and the point \( C \) lies on the \( x \) axis so that \( OABC \) is a rectangle.

The region \( R \) is bounded by the curve, the \( x \) axis and the line \( BC \).

The region \( S \) is bounded by the curve, the \( y \) axis and the line \( AB \).

Show, by exact calculations, that the area of \( R \) is twice as large as the area of \( S \).

**proof**
The figure above shows the graphs of the curves with equations

\[ y = \cos 2x \quad \text{and} \quad y = \sin x, \quad \text{for} \quad 0 \leq x \leq \pi. \]

The point \( P \) is the first intersection between the graphs for which \( x > 0 \).

a) Show clearly that the \( x \) coordinate of \( P \) is \( \frac{\pi}{6} \).

b) Determine the exact area of \( R \).

\[
\frac{3}{4}(2 - \sqrt{3})
\]
Question 23 (****)

\[ y = \arctan x, \ x \in \mathbb{R}. \]

**a)** By rewriting the above equation in the form \( x = f(y) \), show clearly that

\[ \frac{dy}{dx} = \frac{1}{1 + x^2}. \]

The figure below shows the graphs of the curves with equations

\[ y = \frac{1}{1 + x^2} \quad \text{and} \quad y = \frac{x}{1 + x^2}. \]

The two graphs intersect at the point \( P\left(1, \frac{1}{2}\right) \).

**b)** Find the exact area of the finite region bounded by the two curves and the \( y \) axis, shown shaded in the figure.

\[ \frac{1}{4}(\pi - \ln 4) \]

[solution overleaf]
Question 24  (****)

The figure above shows the graphs of the curves with equations

\[ y = e^{3x} - 1 \quad \text{and} \quad y = 6 + 8e^{-3x}. \]

The curves intersect at the point \( P \).

a) Find the exact coordinates of the point \( P \).

The finite region \( R \) is bounded by the two curves and the \( y \) axis.

b) Determine the exact area of \( R \).

\[(\ln 2, 7)\], \[\text{area} = 7\ln 2\]
Question 25  (****+)

The finite region $R$ is bounded by the curve with equation $y = \ln(x + 3)$, the $y$ axis and the straight line with equation $y = 1$.

Show, with detailed workings, that the area of $R$ is

$$e - 6 + 3\ln 3.$$
Question 26  (***)

The function \( f \) is defined in the largest real domain by

\[
f(x) = (\ln x)^2, \quad x \in (0, \infty).
\]

a) Sketch the graph of \( f(x) \).

The function \( g \) is defined as

\[
g(x) = \ln x, \quad x \in (0, \infty).
\]

b) Determine in exact simplified form the area of the finite region bounded by the graph of \( f \) and the graph of \( g \).

You may assume that

\[
\int \ln x \, dx = x \ln x - x + \text{constant}.
\]
The figure above shows the graph of the curve \( C \) with equation
\[
y = x + \cos x, \; x \in \mathbb{R}
\]
and the straight line \( L \) with equation
\[
y = x, \; x \in \mathbb{R}.
\]
Show that the area of the finite region bounded by \( C \) and \( L \) is 2 square units.
The figure above shows the curve with equation

\[ y = e^{-\frac{1}{10}x} \sin x, \quad 0 \leq x \leq 2\pi. \]

The curve meets the \( x \) axis at the origin and at the points \((\pi, 0)\) and \((2\pi, 0)\).

The finite region bounded by the curve for \( 0 \leq x \leq \pi \) and the \( x \) axis is denoted by \( A_1 \), and similarly \( A_2 \) denotes the finite region bounded by the curve for \( \pi \leq x \leq 2\pi \) and the \( x \) axis.

By considering a suitable translation of the curve determine with justification the ratio of the areas of \( A_1 \) and \( A_2 \).

\[
\frac{A_1}{A_2} = e^{-\frac{\pi}{10}}
\]
The figure above shows the parabola with equation

\[ y = -(x-a)(x-b), \quad b > a > 0. \]

The curve meets the \( x \)-axis at the points \( A \) and \( B \).

a) Show that the area of the finite region \( R \), bounded by the parabola and the \( x \)-axis is

\[ \frac{1}{6}(b-a)^3. \]

The midpoint of \( AB \) is \( N \). The point \( M \) is the maximum point of the parabola.

b) Show clearly that the area of \( R \) is given by

\[ k |AB||MN|, \]

where \( k \) is a constant to be found.

\[ k = \frac{2}{3} \]
Question 30  

Determine, in exact simplified form, the area of the finite region bounded by the curves with equations

\[ y = 1 + \sqrt{x}, \quad x \in \mathbb{R}, \quad x \geq 0. \]

\[ y = 4^{\frac{x}{2}}, \quad x \in \mathbb{R}. \]

\[ 27 \left( 1 - \frac{1}{\ln 4} \right) \]
Question 31

The straight line $L$ with equation $y = kx$, where $k$ is a positive constant, meets the curve $C$ with equation $y = xe^{-2x}$, at the point $P$.

The tangent to $C$ at $P$ meets the $x$ axis at the point $Q$.

Given that $|OP| = |PO|$, find in exact simplified form the area of the finite region bounded by $C$ and $L$.

\[
\text{area} = \frac{1}{2} \left(1 - 5e^{-2}\right)
\]
Question 32

Two curves are defined in the largest possible real number domain and have equations

\[ y^2 = \frac{4(4-x)}{x} \quad \text{and} \quad x^2 = \frac{4(4-y)}{y}. \]

a) Show that the two curves have one, and only one, common point which is also a point of common tangency.

b) Find the exact value of the area enclosed by the common tangent to the curves, and either of the two curves.

\[ \int \left( \frac{4}{x} - x \right) \, dx \]
Question 33  (*****)

The figure above shows the graphs of the curves with equations

\[ y = \log_{10}(3x-1), \quad \text{and} \quad y = \log_{10}(x+5). \]

The two curves intersect at the point \( P \)

The straight line with equation \( x = k \), \( k > 3 \), meets the graph of \( y = \log_{10}(3x-1) \) at the point \( A \) and the graph of \( y = \log_{10}(x+5) \) at the point \( B \), so that \( |AB| = \frac{1}{2} \).

Determine the value of the area of the finite region, bounded by the two curves and the straight line \( x = k \), shown shaded in the above figure.

\[
\text{area} = \frac{8}{3}
\]
Question 34     (*****)

The curve \( C \) has equation

\[
y + 2 = \left[ \ln(4x + 1) \right]^2, \quad x \in \mathbb{R}, \quad x \geq -\frac{1}{4}
\]

Sketch the graph of \( C \) and hence determine, in exact simplified form, the area of the finite region bounded by \( C \), for which \( x \geq 0 \), and the coordinate axes.

\[
\text{Area} = \frac{1}{2} \left( \sqrt{2} - 1 \right) e^{\sqrt{2}}
\]
The figure above shows the curve with equation

$$2x^2 + 2xy + y^2 = 50.$$ 

Determine the area of the finite region bounded by the $x$-axis and the part of the curve for which $y \geq 0$. 

$$25\pi$$