

Created by T. Madas

MOTION ON AXISYMETRIC SURFACES

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Question 1 (*)**

A particle of mass m is moving on a smooth axisymmetric surface with equation

$$z = f(r),$$

where r measures the distance from the **vertical** axis of symmetry, and z is the vertical distance along that axis, measured from an arbitrary origin O .

The particle is set in motion at a point on the surface where $r = a$, with horizontal speed U , tangential to the surface.

If air resistance can be ignored, show that

$$\dot{r}^2 \left[1 + [f'(r)]^2 \right] + 2g f(r) + g(r) = \text{constant},$$

where g is a function to be found.

$$g(r) = \frac{2a^2 U^2}{mr^2}$$

The diagram shows a particle of mass m on a surface $z = f(r)$. The vertical axis is z and the horizontal axis is r . The particle is at a point where $r = a$ and $z = f(a)$. The velocity vector is horizontal and tangential to the surface, with magnitude U . The forces acting on the particle are gravity mg acting vertically downwards and a normal reaction force R acting perpendicular to the surface.

ACCELERATION IN POLARS
 $a = (\ddot{r} - r\dot{\theta}^2)\hat{r} + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\hat{\theta}$

AS THERE IS NO FORCE ACTING IN THE TANGENTIAL DIRECTION ($\hat{\theta}$)
 $\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0$
 $r^2\dot{\theta} = h = \text{CONSTANT}$

At any point, $r\dot{\theta} = U$, so we can calculate the constant
 $r\dot{\theta} = h$
 $aU = h \Rightarrow r^2\dot{\theta} = aU$

Now energy is conserved - TAKE THE LEVEL OF O , AS THE ZERO POTENTIAL
 $\Rightarrow \frac{1}{2}m[\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2] + mgz = \text{CONSTANT}$
 where $KE = \frac{1}{2}mU^2$
 initial PE = mgz } LET $E = \frac{1}{2}mU^2 + mgz$

$\Rightarrow \frac{1}{2}m[\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2] + mgz = E$
 $\Rightarrow \frac{1}{2}m[\dot{r}^2 + r^2(\frac{aU}{r})^2 + \dot{z}^2] + mgz = E$

Now $z = f(r)$
 $\dot{z} = f'(r)\dot{r}$

$\Rightarrow \frac{1}{2}m[\dot{r}^2 + \frac{a^2 U^2}{r^2} + f'(r)^2 \dot{r}^2] + mgf(r) = E$
 $\Rightarrow \frac{1}{2}m[\dot{r}^2(1 + f'(r)^2) + \frac{a^2 U^2}{r^2}] + mgf(r) = E$
 $\Rightarrow \dot{r}^2[1 + f'(r)^2] + \frac{2a^2 U^2}{r^2} + 2gf(r) = \frac{2E}{m}$ (8M)

Question 2 (*)**

A particle of mass m is moving on the smooth outer surface of a right circular cone, of semi-vertical angle α , $\alpha < \frac{1}{2}\pi$, whose vertex is uppermost and its axis vertical.

The particle is set in motion at a point on the surface of the cone where the radius is a , with horizontal speed U , tangential to the surface of the cone.

At a general point on the cone with radius is r , the reaction force on the particle is R .

Ignoring air resistance, show that

$$R = m \left[g \tan \alpha - \frac{a^2 U^2}{r^3} \right] \cos \alpha .$$

proof

Diagram: A right circular cone with semi-vertical angle α . A particle of mass m is on the surface at a distance r from the vertical axis. The radius of the cone at the top is a . The particle has a horizontal speed U tangential to the surface. Forces acting on the particle are the reaction force R perpendicular to the surface and the weight mg acting vertically downwards.

Left Column of Work:

- **ACCELERATION IN RADIAL**
 $a = (\ddot{r} - r\dot{\theta}^2) \hat{r} + \frac{1}{r} \frac{d}{dt}(r^2\dot{\theta}) \hat{\theta}$
 (i): $m(\ddot{r} - r\dot{\theta}^2) = R \cos \alpha$ (ii)
 (ii): $m \frac{d}{dt}(r^2\dot{\theta}) = 0$
 $r\dot{\theta} = h$ (constant)
 $r^2\dot{\theta} = h$ from the initial conditions
 $aU = h$
 $r\dot{\theta} = aU$ (iii)

Right Column of Work:

- **ALSO IN THE Z DIRECTION**
 $m\ddot{z} = mg - R \sin \alpha$ (iv)
- **THINK! (r & z ARE RELATED BY A CONSTANT)**
 $\frac{dz}{dt} = b \sin \alpha \Rightarrow r = z \tan \alpha$
 $\Rightarrow \dot{r} = b \tan \alpha$
 $\Rightarrow \ddot{r} = \frac{b}{z} \tan \alpha$ (v)
- **SUBSTITUTE (iii) $R = \frac{m}{\sin \alpha} (g - \ddot{z})$ in (ii) $\hat{\theta} = \frac{aU}{r^2} \hat{\theta}$ into (ii)**
 $\Rightarrow m(\ddot{r} - r\dot{\theta}^2) = \frac{m}{\sin \alpha} (g - \ddot{z})$
 $\Rightarrow m(\ddot{r} - r \frac{a^2 U^2}{r^3}) = \frac{m}{\sin \alpha} (g - \ddot{z}) \cos \alpha$
 $\Rightarrow \ddot{r} - \frac{a^2 U^2}{r^2} = \cot \alpha (g - \ddot{z})$ by (v)
 $\Rightarrow \ddot{r} - \frac{a^2 U^2}{r^2} = g \cot \alpha - \ddot{r} \cot \alpha$
 $\Rightarrow \ddot{r} + \ddot{r} \cot \alpha = \frac{a^2 U^2}{r^2} + g \cot \alpha$
 $\Rightarrow \ddot{r} (1 + \cot \alpha) = \frac{a^2 U^2}{r^2} + g \cot \alpha$
- **REWORKING THE FRACTION**
 $\Rightarrow \ddot{r} \csc \alpha = \frac{a^2 U^2}{r^2} + g \cot \alpha$
 $\Rightarrow \ddot{r} = \left(\frac{a^2 U^2}{r^2} + g \cot \alpha \right) \sin \alpha$
 $\Rightarrow R = \frac{m}{\sin \alpha} (g - \ddot{z}) = \frac{m}{\sin \alpha} \left[g - \frac{\ddot{r}}{\tan \alpha} \right]$
 $\Rightarrow R = \frac{m}{\sin \alpha} \left[g - \cot \alpha \left(\frac{a^2 U^2}{r^2} + g \cot \alpha \right) \right]$
 $\Rightarrow R = \frac{m}{\sin \alpha} \left[g - g \cot^2 \alpha - \frac{a^2 U^2}{r^2} \csc \alpha \sin \alpha \right]$
 $\Rightarrow R = \frac{m}{\sin \alpha} \left[g(1 - \cot^2 \alpha) - \frac{a^2 U^2}{r^2} \csc \alpha \sin \alpha \right]$
 $\Rightarrow R = \frac{m \csc \alpha}{\sin \alpha} \left[g \frac{\sin^2 \alpha}{\cos \alpha} - \frac{a^2 U^2}{r^2} \sin \alpha \right]$
 $\Rightarrow R = m \csc \alpha \left[g \tan \alpha - \frac{a^2 U^2}{r^3} \right]$

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