MOTION ON AXISYMETRIC SURFACES

Question 1 (***)

A particle of mass m is moving on a smooth axisymmetric surface with equation

$$z = f(r),$$

where r measures the distance from the **vertical** axis of symmetry, and z is the vertical distance along that axis, measured from an arbitrary origin O.

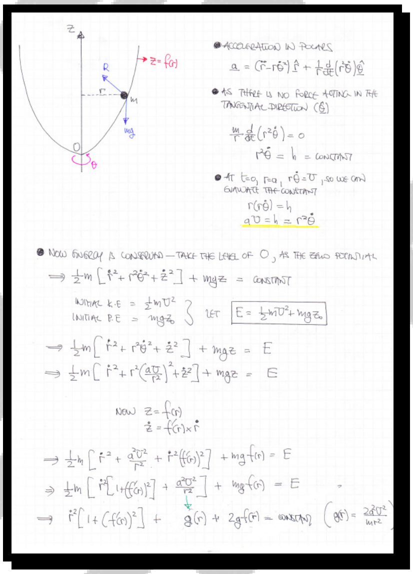
The particle is set in motion at a point on the surface where $r = a$, with horizontal speed U, tangential to the surface.

If air resistance can be ignored, show that

$$\dot{r}^2\left[1+\left[f'(r)\right]^2\right]+2g\,f(r)+g(r)=\text{constant},$$

where g is a function to be found.

$$\boxed{g(r)=\frac{2a^2U^2}{mr^2}}$$



Question 2 (***+)

A particle of mass m is moving on the smooth outer surface of a right circular cone, of semi-vertical angle α, $\alpha < \frac{1}{2}\pi$, whose vertex is uppermost and its axis vertical.

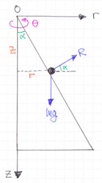
The particle is set in motion at a point on the surface of the cone where the radius is a, with horizontal speed U, tangential to the surface of the cone.

At a general point on the cone with radius is r, the reaction force on the particle is R.

Ignoring air resistance, show that

$$R = m\left[g\tan\alpha - \frac{a^2U^2}{r^3}\right]\cos\alpha .$$

proof



$t=0$
$r=a$
$r\dot{\theta}=U$

- ACCELERATION IN POLARS

$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\hat{\boldsymbol{\theta}}$

$(\hat{r}):\ \boxed{m(\ddot{r} - r\dot{\theta}^2) = R\cos\alpha}$ (I)

$(\hat{\theta}):\ \frac{m}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0$

$r^2\dot{\theta} = h$ (CONSTANT)

$r(r\dot{\theta}) = h$
$aU = h$ } FROM THE INITIAL CONDITIONS

$\boxed{r^2\dot{\theta} = aU}$ (II)

- ALSO IN THE Z DIRECTION

$\boxed{m\ddot{z} = mg - R\sin\alpha}$ (III)

- FINALLY r & z ARE RELATED BY A CONSTRAINT

$\frac{r}{z} = \tan\alpha \implies \boxed{\begin{array}{l} r = z\tan\alpha \\ \dot{r} = \dot{z}\tan\alpha \\ \ddot{r} = \ddot{z}\tan\alpha \end{array}}$ (IV)

- SUBSTITUTE (III) $R = \frac{m}{\sin\alpha}(g - \ddot{z})$ & (II) $\dot{\theta} = \frac{aU}{r^2}$ INTO (I)

$\implies m(\ddot{r} - r\dot{\theta}^2) = R\cos\alpha$

$\implies m\left(\ddot{r} - r\frac{a^2U^2}{r^4}\right) = \frac{m}{\sin\alpha}(g - \ddot{z})\cos\alpha$

$\implies \ddot{r} - \frac{a^2U^2}{r^3} = \cot\alpha\,(g - \ddot{z})$ BY (IV)

$\implies \ddot{r} - \frac{a^2U^2}{r^3} = \cot\alpha\left(g - \frac{\ddot{r}}{\tan\alpha}\right)$

$\implies \ddot{r} - \frac{a^2U^2}{r^3} = g\cot\alpha - \ddot{r}\cot^2\alpha$

$\implies \ddot{r} + \ddot{r}\cot^2\alpha = \frac{a^2U^2}{r^3} + g\cot\alpha$

$\implies \ddot{r}(1 + \cot^2\alpha) = \frac{a^2U^2}{r^3} + g\cot\alpha$

$\implies \ddot{r}\operatorname{cosec}^2\alpha = \frac{a^2U^2}{r^3} + g\cot\alpha$

$\implies \ddot{r} = \left(\frac{a^2U^2}{r^3} + g\cot\alpha\right)\sin^2\alpha$

- RETURNING TO THE REACTION

$\implies R = \frac{m}{\sin\alpha}\left[g - \ddot{z}\right] = \frac{m}{\sin\alpha}\left[g - \frac{\ddot{r}}{\tan\alpha}\right]$

$\implies R = \frac{m}{\sin\alpha}\left[g - \cot\alpha\left[\sin^2\alpha\left[\frac{a^2U^2}{r^3} + g\cot\alpha\right]\right]\right]$

$\implies R = \frac{m}{\sin\alpha}\left[g - \cos\alpha\sin\alpha\left[\frac{a^2U^2}{r^3} + g\cot\alpha\right]\right]$

$\implies R = \frac{m}{\sin\alpha}\left[g - g\cos^2\alpha - \frac{a^2U^2}{r^3}\cos\alpha\sin\alpha\right]$

$\implies R = \frac{m}{\sin\alpha}\left[g(1 - \cos^2\alpha)\right] - \frac{a^2U^2}{r^3}\cos\alpha\sin\alpha\Big]$

$\implies R = \frac{m}{\sin\alpha}\left[g\sin^2\alpha - \frac{a^2U^2}{r^3}\cos\alpha\sin\alpha\right]$

$\implies R = \frac{m\cos\alpha}{\sin\alpha}\left[g\frac{\sin^2\alpha}{\cos\alpha} - \frac{a^2U^2}{r^3}\sin\alpha\right]$

$\implies R = m\cos\alpha\left[g\tan\alpha - \frac{a^2U^2}{r^3}\right]$

AS REQUIRED