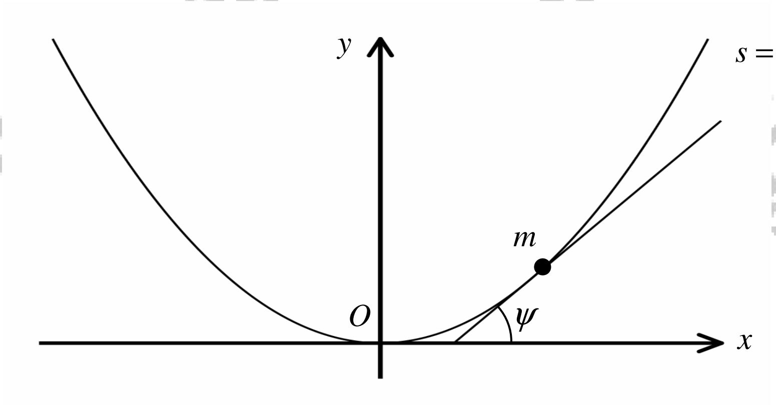
INTRINSIC COORDINATES

Question 1 (**)



A bead of mass m is made to slide along a smooth wire, which is fixed in a vertical plane, and is bent into the shape of an inverted cycloid with intrinsic equation

$$s = 4a\sin\psi ,$$

where a is a positive constant, s is measured from the origin O, and ψ is the angle the tangent to the cycloid makes with the x axis.

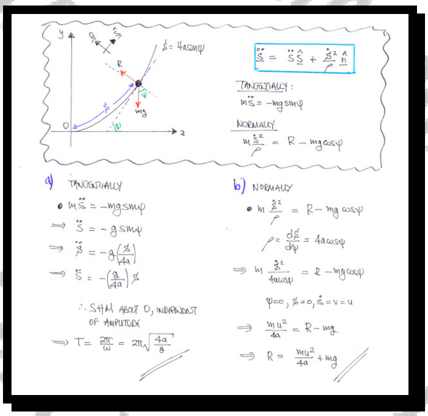
The bead is released from a point on the cycloid where $s \neq 0$.

a) Show that the period of the resulting oscillations is independent of the position from which the bead was released.

The bead passes through the lowest point of the path with speed u.

b) Determine in terms of m, u, a and g, the magnitude of the reaction of the wire on the bead as it passes through the lowest position of the wire

$$\boxed{\tau = 2\pi\sqrt{\frac{4a}{g}}}, \quad \boxed{R = \frac{mu^2}{4a} + mg}$$



Question 2 (**)

A particle moves with constant speed u on the curve with intrinsic equation

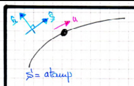
$$s = a\tan\psi\,,\quad 0 \le \psi < \frac{\pi}{2}\,,$$

where a is a positive constant, s is measured from the origin O, and ψ is the angle the tangent to the curve makes with the x axis.

Show that the magnitude of the normal component of the acceleration of the particle, t seconds after starting from the point where $\psi = 0$, is

$$\frac{au^2}{a^2+u^2t^2}\,.$$

M456-B , proof



ACCELERATION IN INTRINSICS

$\underline{a} = \ddot{s}\,\hat{\underline{s}} + \frac{\dot{s}^2}{\rho}\,\hat{\underline{n}}$

- HERE $\dot{s} = u = \text{CONSTANT}$
- $\rho = \frac{ds}{d\psi} = a\sec^2\psi$

THE NORMAL COMPONENT ($\hat{\underline{n}}$) OF THE ACCELERATION IS GIVEN BY

$$\frac{\dot{s}^2}{\rho} = \frac{u^2}{a\sec^2\psi} = \frac{u^2}{a(1+\tan^2\psi)} = \frac{u^2}{a\left(1+\frac{s^2}{a^2}\right)} = \frac{u^2}{a+\frac{s^2}{a}}$$

$$= \frac{au^2}{a^2+s^2}$$

WE NEED TO FULLY ELIMINATE s – THE SPEED $\dot{s}$ ALONG THE CURVE IS CONSTANT

$$\Rightarrow \frac{ds}{dt} = u$$

$$\Rightarrow 1\,ds = u\,dt$$

$$\Rightarrow \int_{s=0}^{s} 1\,ds = \int_{t=0}^{t} u\,dt \longleftarrow \boxed{\begin{array}{l} t=0\,,\ \psi=0 \\ s = a\tan 0 \\ s=0 \end{array}}$$

$$\Rightarrow \big[s\big]_0^s = \big[ut\big]_0^t$$

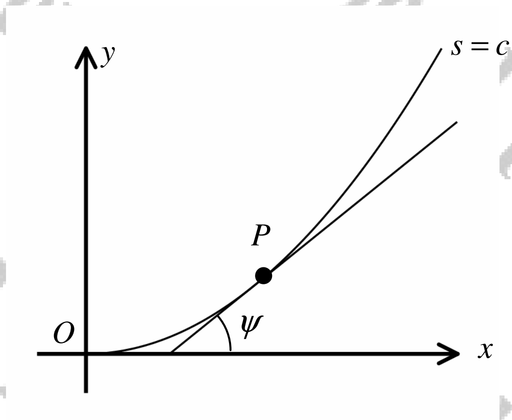
$$\Rightarrow s = ut$$

HENCE THE RESULT FOLLOWS

$$\frac{\dot{s}^2}{\rho} = \dots = \frac{au^2}{a^2+s^2} = \frac{au^2}{a^2+u^2t^2}$$ AS REQUIRED

Created by T. Madas

Question 3 (**)



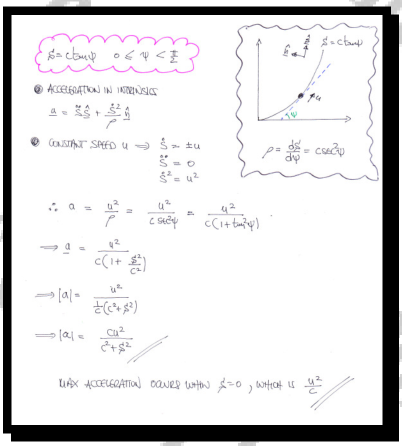
A particle P is moving with constant speed u on the catenary with intrinsic equation

$$s = c\tan\psi\,,\quad 0 \le \psi < \frac{\pi}{2}\,,$$

where c is a positive constant, s is measured from the origin O, and ψ is the angle the tangent to the catenary makes with the x axis.

Find an expression for the magnitude of the acceleration of the particle and hence state the maximum magnitude of its acceleration.

$$\boxed{|a| = \frac{cu^2}{c^2 + s^2}}\,,\quad \boxed{|a|_{\text{max}} = \frac{u^2}{c}}$$



$s = c\tan\psi \quad 0 \leqslant \psi < \frac{\pi}{2}$

- ACCELERATION IN INTRINSICS

$\underline{a} = \ddot{s}\,\hat{\underline{s}} + \frac{\dot{s}^2}{\rho}\,\hat{\underline{n}}$

- CONSTANT SPEED $u \Longrightarrow \dot{s} = \pm u$

$\ddot{s} = 0$

$\dot{s}^2 = u^2$

$\hat{n}$ $\hat{s}$ $s = c\tan\psi$

u ψ

$\rho = \frac{ds}{d\psi} = c\sec^2\psi$

$\therefore\ a = \frac{u^2}{\rho} = \frac{u^2}{c\sec^2\psi} = \frac{u^2}{c(1+\tan^2\psi)}$

$\Longrightarrow \underline{a} = \frac{u^2}{c\left(1+\frac{s^2}{c^2}\right)}$

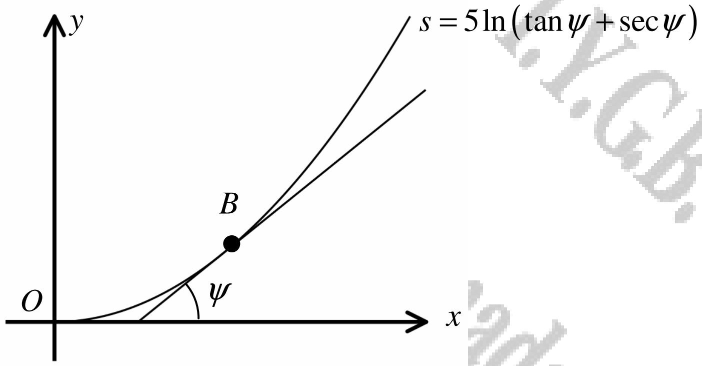
$\Longrightarrow |a| = \frac{u^2}{\frac{1}{c}(c^2+s^2)}$

$\Longrightarrow |a| = \frac{cu^2}{c^2+s^2}$

MAX ACCELERATION OCCURS WHEN $s=0$, WHICH IS $\frac{u^2}{c}$

I.Y.G.B. madasmaths.com

Question 4 (**+)



A bead of mass 0.25 kg is made to slide along a smooth wire, which is fixed in a horizontal plane, and is bent into the shape of a curve with intrinsic equation

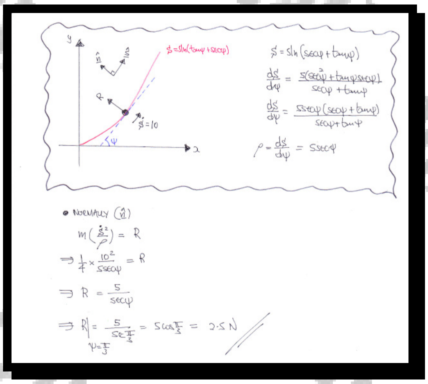
$$s = 5\ln(\tan\psi + \sec\psi), \quad 0 \le \psi < \frac{\pi}{2}$$

where a is a positive constant, s is measured from the origin O, and ψ is the angle the tangent to the curve makes with the x axis.

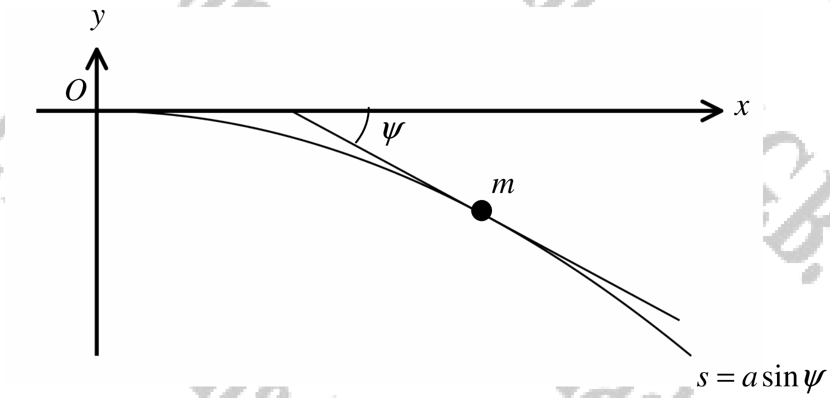
The bead is moving along the wire with constant speed 10 ms^{-1}.

Determine the magnitude of the reaction of the wire on the bead, when $\psi = \frac{\pi}{3}$.

$\boxed{R = 2.5 \text{ N}}$



Question 5 (***)



The figure above shows a particle of mass m, which is free to slide along a smooth surface, whose vertical cross section is the curve C with intrinsic equation

$$s = a\sin\psi, \quad 0 \le \psi < \frac{\pi}{2}$$

where a is a positive constant.

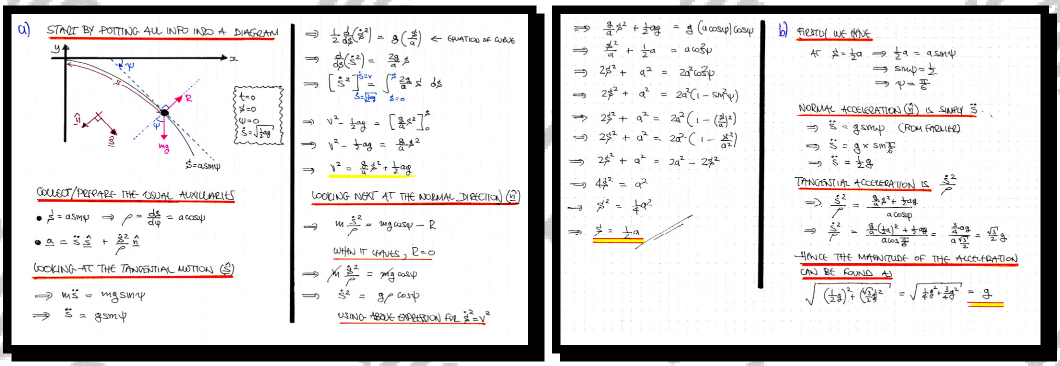
The arclength s is measured from the origin O, and the angle ψ is the angle the tangent to C makes with the positive x axis, as shown in the figure above.

The particle is projected from O with speed $\sqrt{\frac{1}{2}ag}$ and leaves the surface at the point P.

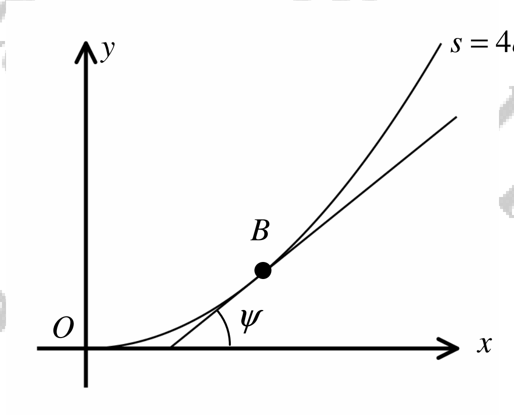
a) Find the value of s at P.

b) Determine the magnitude of the acceleration at P.

$\boxed{s = \frac{1}{2}a}$, $\boxed{\text{acceleration} = g}$



Question 6 (***)



A bead of mass m is made to slide along a smooth wire, which is fixed in a vertical plane, and is bent into the shape of an inverted cycloid with intrinsic equation

$$s = 4a\sin\psi, \quad 0 \le \psi < \frac{\pi}{2}$$

where a is a positive constant, s is measured from the origin O, and ψ is the angle the tangent to the cycloid makes with the x axis.

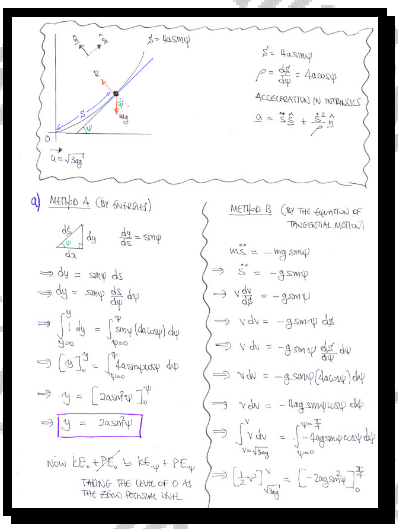
The bead is projected from O with tangential speed $\sqrt{3ag}$.

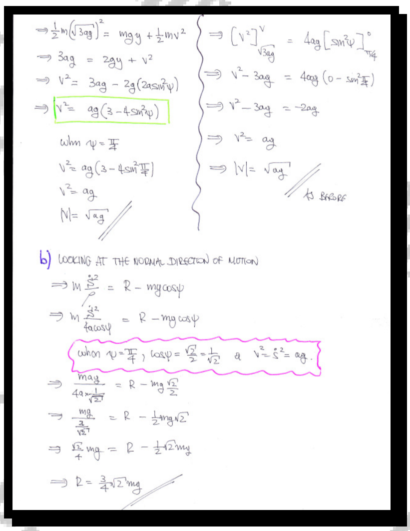
Show that when $\psi = \frac{\pi}{4}$, …

a) … the speed of the bead is $\sqrt{ag}$.

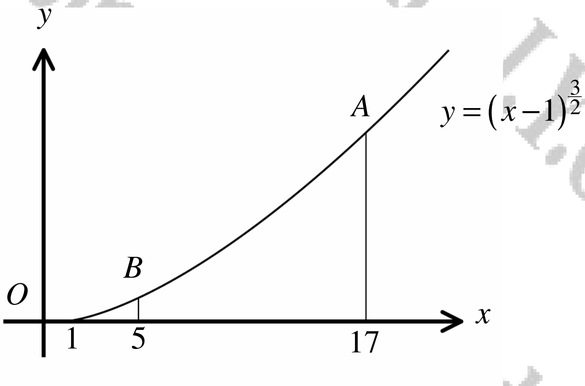
b) … the magnitude of the reaction of the wire on the bead is $\frac{3}{4}\sqrt{2}\,mg$.

proof





Question 7 (*+)**



A rollercoaster car, of mass 200 kg, is constrained to move along a rail path with Cartesian equation

$$y=(x-1)^{\frac{3}{2}}.$$

The car comes to instantaneous rest at the point A, where $x=17$, and immediately begins to freely slide downwards towards the origin O, as shown in the figure above. The point B, lies on the same rail path, where $x=5$. The car is modelled as a particle moving along a smooth rail path without any air resistance.

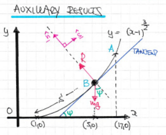
As the car passes through B, calculate ...

a) ... the magnitude of the acceleration of the car as it passes through B.

b) ... the magnitude of the normal reaction exerted by the rails onto the car

$\boxed{}$, $\boxed{a \approx 16.0 \text{ ms}^{-2}}$, $\boxed{R \approx 3223 \text{ N}}$

a) STARTING WITH A DIAGRAM AND PREPARING SOME AUXILIARY RESULTS



$\frac{dy}{dx}=\frac{3}{2}(x-1)^{\frac{1}{2}}$

$\tan\psi=\frac{3}{2}(x-1)^{\frac{1}{2}}$

AT POINT B, $x=5$

$\tan\psi=3$

BY ENERGIES, TAKING THE LEVEL OF THE x AXIS AS THE ZERO POTENTIAL LEVEL

$y_A=(17-1)^{\frac{3}{2}}=64$

$y_B=(5-1)^{\frac{3}{2}}=8$

$\cancel{KE_A}+P.E_A = KE_B + P.E_B$

$\cancel{m}gy_A=\frac{1}{2}\cancel{m}v^2+\cancel{m}gy_B$

$2g\times 64 = v^2 + 2g\times 8$

$v^2 = 128g - 16g$

$v^2 = 112g$

NEXT THE RADIUS OF CURVATURE ρ, IN CARTESIAN

$$\rho=\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}=\frac{\left[1+\left[\frac{3}{2}(x-1)^{\frac{1}{2}}\right]^2\right]^{\frac{3}{2}}}{\frac{3}{4}(x-1)^{-\frac{1}{2}}}$$

$$\rho_B=\frac{\left[1+\left(\frac{3}{2}\times 4^{\frac{1}{2}}\right)^2\right]^{\frac{3}{2}}}{\frac{3}{4}\times 4^{-\frac{1}{2}}}=\frac{10\sqrt{10}}{\frac{3}{8}}=\frac{80}{3}\sqrt{10}$$

ACCELERATION IN INTRINSICS $\underline{a}=\ddot{s}\,\underline{\hat{s}}+\frac{\dot{s}^2}{\rho}\,\underline{\hat{n}}$

• TANGENTIALLY

$m\ddot{s}=-mg\sin\psi$

$\ddot{s}=-g\sin\psi$

$\ddot{s}_B=-g\left(\frac{3}{\sqrt{10}}\right)$

$|\ddot{s}_B|=\frac{3g}{\sqrt{10}}$

• NORMALLY

$\frac{\dot{s}^2}{\rho}=\frac{v^2}{\rho}$

$\left[\frac{\dot{s}^2}{\rho}\right]_B=\frac{112g}{\frac{80}{3}\sqrt{10}}$

$\left[\frac{\dot{s}^2}{\rho}\right]_B=\frac{21g}{5\sqrt{10}}$

MAGNITUDE OF ACCELERATION AT B

$$\sqrt{|\ddot{s}|_B^2+\left[\frac{\dot{s}^2}{\rho}\right]_B^2}=\frac{3g}{\sqrt{10}}\sqrt{1^2+\left(\frac{7}{5}\right)^2}$$

$$=\frac{3g}{5}\sqrt{\frac{74}{10}}$$

$$\approx 16.0 \text{ ms}^{-2}$$

b) LOOKING AT THE MOTION, NORMALLY

$\Rightarrow m\left(\frac{v^2}{\rho}\right)=R-mg\cos\psi$

$\Rightarrow R=\frac{mv^2}{\rho}+mg\cos\psi$

AT B WE HAVE $\rho=\frac{80}{3}\sqrt{10}$, $v^2=112g$, $\cos\psi=\frac{1}{\sqrt{10}}$

$\Rightarrow R=\frac{200\times 112g}{\frac{80}{3}\sqrt{10}}+200g\times\frac{1}{\sqrt{10}}$

$\Rightarrow R=\frac{840g}{\sqrt{10}}+\frac{200g}{\sqrt{10}}$

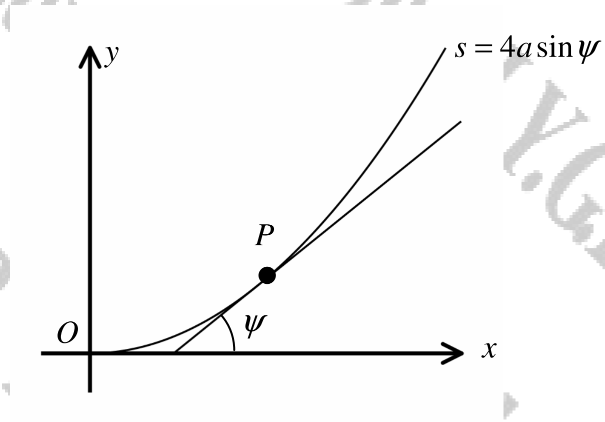
$\Rightarrow R=\frac{1040g}{\sqrt{10}}$

$\Rightarrow R=104\sqrt{10}\,g$

$\Rightarrow R=3222.993391\ldots$

$\Rightarrow R\approx 3223$ N

Question 8 (*+)**



A bead of mass m is made to slide along a smooth wire, which is fixed in a horizontal plane, and is bent into the shape of a curve with intrinsic equation

$$s = 4a\sin\psi\,, \quad 0 \le \psi < \frac{\pi}{2}$$

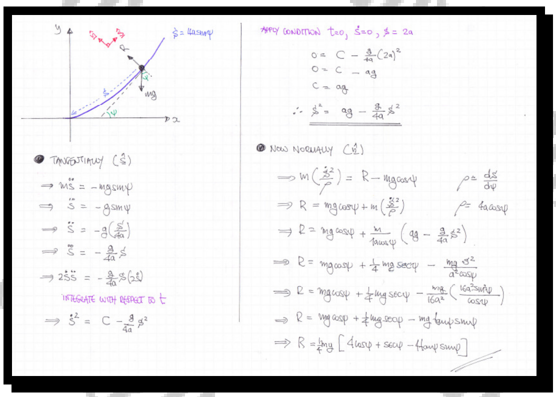
where a is a positive constant, s is measured from the origin O, and ψ is the angle the tangent to the curve makes with the x axis.

The bead is released from rest from a point on the wire where $s = 2a$.

Show the magnitude of the reaction of the wire on the bead, R, is given by

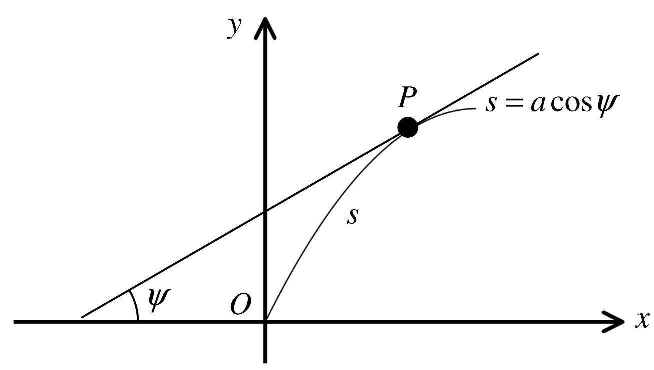
$$R = \frac{1}{4}mg\left[4\cos\psi + \sec\psi - 4\tan\psi\sin\psi\right].$$

proof



I.Y.G.B. madasmaths.com

Question 9 (****)



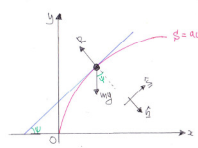
The figure above shows a bead of mass m, modelled as a particle P, sliding freely along a smooth wire bent into the shape of the curve with intrinsic equation

$$s = a\cos\psi, \quad 0 \le \psi \le \frac{\pi}{2},$$

where a is a positive constant, s is measured from the origin O, and ψ is the angle the tangent to the curve makes with the x axis.

Given that the bead was projected from the highest point on the wire with tangential speed $\sqrt{\frac{1}{2}ag}$, determine a simplified expression for the magnitude of the normal reaction between the bead and the wire, as the bead reaches O.

$$\boxed{\frac{1}{2}(\pi+1)mg}$$



ACCELERATION IN INTRINSICS

$\underline{a} = \ddot{s}\,\hat{\underline{s}} + \frac{\dot{s}^2}{\rho}\,\hat{\underline{n}}$ & $\rho = \frac{ds}{d\psi}$

LOOKING IN THE TANGENTIAL DIRECTION $(\hat{\underline{s}})$

$\Rightarrow m\ddot{s} = -mg\sin\psi$

$\Rightarrow \ddot{s} = -g\sin\psi$

$\Rightarrow \frac{1}{2}\frac{d}{ds}(\dot{s}^2) = -g\sin\psi$

$\Rightarrow \dot{s}^2 = \int -2g\sin\psi\, ds$

$\Rightarrow \dot{s}^2 = \int -2g\sin\psi\,\frac{ds}{d\psi}\,d\psi$

$\Rightarrow \dot{s}^2 = \int 2g\sin\psi(-a\sin\psi)\,d\psi$

$\Rightarrow \dot{s}^2 = \int 2ag\sin^2\psi\, d\psi$

INCLUDING LIMITS

$\Rightarrow \left[\dot{s}^2\right]_{\dot{s}=\sqrt{\frac{1}{2}ag}}^{\dot{s}} = \int_{\psi=0}^{\psi=\frac{\pi}{2}} 2ag\left(\frac{1}{2}-\frac{1}{2}\cos 2\psi\right)d\psi$

$\Rightarrow \dot{s}^2 - \frac{1}{2}ag = \int_{\psi=0}^{\frac{\pi}{2}} ag(1-\cos 2\psi)\,d\psi$

$\Rightarrow \dot{s}^2 - \frac{1}{2}ag = ag\left[\psi - \frac{1}{4}\sin 2\psi\right]_0^{\frac{\pi}{2}}$

$\Rightarrow \dot{s}^2 - \frac{1}{2}ag = \frac{\pi ag}{2}$

$\Rightarrow \dot{s}^2 = \frac{1}{2}\pi ag + \frac{1}{2}ag$

$\Rightarrow \dot{s}^2 = \frac{1}{2}ag(\pi+1)$

NOW NORMALLY $(\hat{\underline{n}})$ (WHEN $\psi = \frac{\pi}{2}$)

$\Rightarrow m\frac{\dot{s}^2}{\rho} = mg\cos\psi - R$

$\Rightarrow R = mg\cos\psi - \frac{m\dot{s}^2}{\rho}$

$\Rightarrow R = mg\cos\psi - \frac{m\left[\frac{1}{2}ag(\pi+1)\right]}{-a\sin\psi}$

$\Rightarrow R = mg\cos\psi + \frac{1}{2}mg\,\frac{\pi+1}{\sin\psi}$

WHEN $\psi = \frac{\pi}{2}$

$\Rightarrow R = 0 + \frac{1}{2}mg\,\frac{\pi+1}{1}$

$\Rightarrow R = \frac{1}{2}(\pi+1)mg$ //

IN THE DIRECTION SHOWN

Question 10 (****)

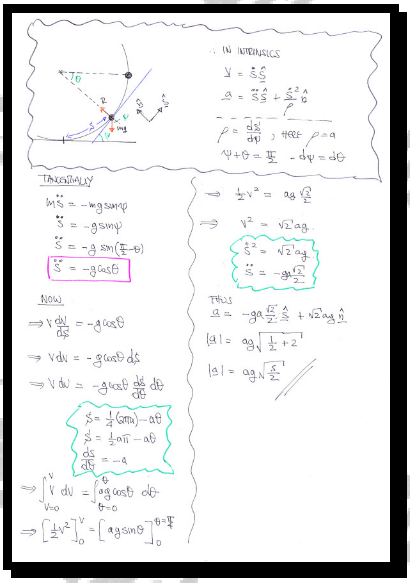
A bead B is free to slide along a smooth wire which is bent into a circle of radius a, with centre at C. The wire is fixed in a vertical plane.

The bead is held at a point A, where CA is horizontal, and released from rest.

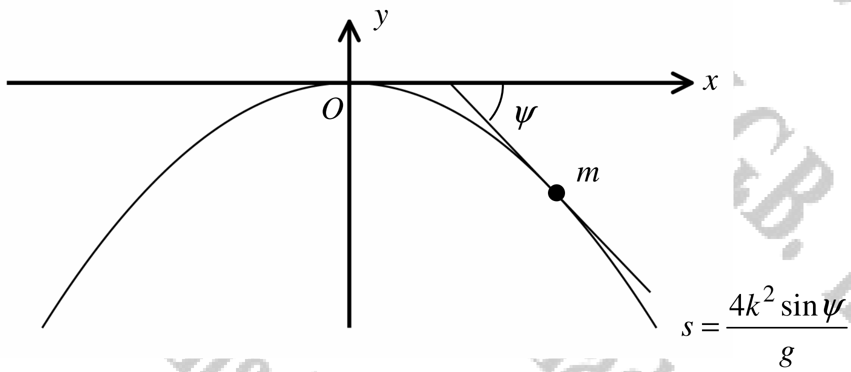
When the angle ACB is θ the bead has speed v.

Use intrinsic coordinates (s, ψ) to find the magnitude of the acceleration of the bead when $\theta = \frac{\pi}{4}$.

$$\boxed{|a| = ag\sqrt{\frac{5}{2}}}$$



Question 11 (****)



A bead of mass m is made to slide along a smooth wire, which is fixed in a vertical plane, and is bent into the shape cycloidal arch with intrinsic equation

$$s = \frac{4k^2 \sin\psi}{g},$$

where k is a positive constant.

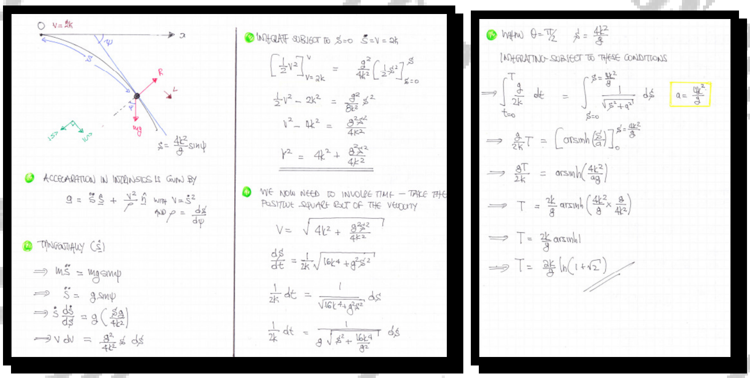
The arclength s is measured from the origin O, and the angle ψ is the angle the tangent to the cycloid makes with the positive x axis as shown in the figure above.

The bead passes through the highest point of the cycloidal arch O, at time $t = 0$, with speed $2k$. The particle passes through the point where $\theta = \frac{1}{2}\pi$, when $t = T$

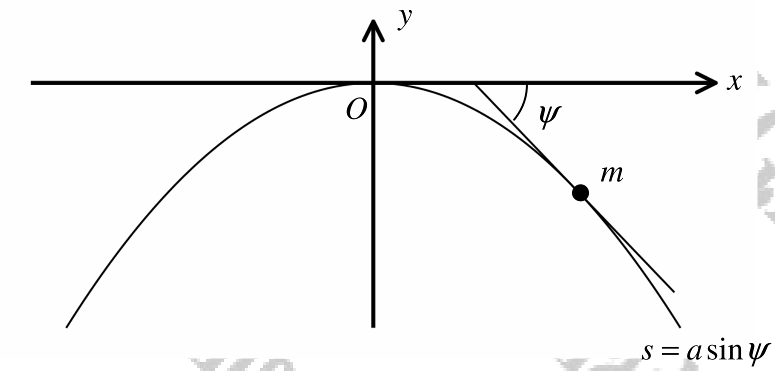
Show that

$$T = \frac{2k}{g}\ln\left(1+\sqrt{2}\right).$$

proof



Question 12 (****)



A bead of mass m is made to slide along a smooth wire, which is fixed in a vertical plane, and is bent into the shape cycloidal arch with intrinsic equation

$$s = a\sin\psi,$$

where a is a positive constant.

The arclength s is measured from the origin O, and the angle ψ is the angle the tangent to the cycloid makes with the positive x axis as shown in the figure above.

The bead passes through the highest point of the cycloidal arch O, with speed $\sqrt{\frac{1}{2}ag}$.

When the particle has travelled a distance s, its speed is v and the normal reaction from the wire to the bead is R.

a) Show, with a detailed method, that …

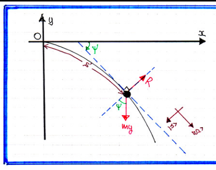
i. … $v^2 = \dfrac{g}{2a}\left[2s^2 + a^2\right]$.

ii. … $R = \dfrac{mg}{2\cos\psi}\left[1 - 4\sin^2\psi\right]$.

b) Find the distance travelled by the bead by the time $R = 0$.

$\boxed{}$, $\boxed{d = \frac{1}{2}a}$

[solution overleaf]



$s = a\sin\psi$

$\rho = \frac{ds}{d\psi} = a\cos\psi$

ACCELERATION

$\underline{a} = \ddot{s}\,\hat{s} + \frac{\dot{s}^2}{\rho}\,\hat{n}$

a) I) LOOKING AT THE TANGENTIAL DIRECTION

$\Rightarrow m\ddot{s} = mg\sin\psi$

$\Rightarrow \ddot{s} = g\sin\psi$

$\Rightarrow \frac{1}{2}\frac{d}{ds}(\dot{s}^2) = g\sin\psi$

INTEGRATE SUBJECT TO CONDITIONS

$\Rightarrow \frac{1}{2}\left[\dot{s}^2\right]_{\dot{s}=\sqrt{\frac{1}{2}ag}}^{\dot{s}} = \int_{s=0}^{s} g\sin\psi \, ds$

$\Rightarrow \frac{1}{2}\left[\dot{s}^2 - \frac{1}{2}ag\right] = \int_{s=0}^{s=s} g\left(\frac{s}{a}\right) ds$

$\Rightarrow \frac{1}{2}\dot{s}^2 - \frac{1}{4}ag = \left[\frac{g}{2a}s^2\right]_{s=0}^{s=s}$

$\Rightarrow \frac{1}{2}\dot{s}^2 - \frac{1}{4}ag = \frac{g}{2a}s^2$

$\Rightarrow \dot{s}^2 - \frac{1}{2}ag = \frac{g}{a}s^2$

$\Rightarrow \dot{s}^2 = \frac{1}{2}ag + \frac{g}{a}s^2$

$\Rightarrow v^2 = \frac{g}{2a}\left[2s^2 + a^2\right]$ // AS REQUIRED

ALTERNATIVE

$\Rightarrow m\ddot{s} = mg\sin\psi$

$\Rightarrow \ddot{s} = g\sin\psi$

$\Rightarrow v\frac{dv}{ds} = g\sin\psi$

$\Rightarrow v\,dv = g\sin\psi\,ds$

$\Rightarrow v\,dv = g\sin\psi\,\frac{ds}{d\psi}\,d\psi$

$\Rightarrow v\,dv = g\sin\psi\,(a\cos\psi)\,d\psi$

$\Rightarrow v\,dv = ag\sin\psi\cos\psi\,d\psi$

$\Rightarrow \int_{v=\sqrt{\frac{1}{2}ag}}^{v} v\,dv = \int_{\psi=0}^{\psi} ag\sin\psi\cos\psi\,d\psi$

$\Rightarrow \left[\frac{1}{2}v^2\right]_{v=\sqrt{\frac{1}{2}ag}}^{v} = \left[\frac{1}{2}ag\sin^2\psi\right]_{\psi=0}^{\psi}$

$\Rightarrow \frac{1}{2}v^2 - \frac{1}{4}ag = \frac{1}{2}ag\sin^2\psi - 0$

$\Rightarrow v^2 - \frac{1}{2}ag = ag\sin^2\psi$

$\Rightarrow v^2 = ag\sin^2\psi + \frac{1}{2}ag$

$\Rightarrow v^2 = ag\left(\frac{s^2}{a^2}\right) + \frac{1}{2}ag$

$\Rightarrow v^2 = \frac{g}{a}s^2 + \frac{1}{2}ag$

$\Rightarrow v^2 = \frac{g}{2a}\left[2s^2 + a^2\right]$

(AS BEFORE)

ALTERNATIVE VARIATION BY ENERGIES, TAKING THE LEVEL OF THE x AXIS AS THE ZERO POTENTIAL LEVEL

$\Rightarrow \frac{dy}{ds} = \sin\psi$

$\Rightarrow \int 1\,dy = \sin\psi\,ds$

$\Rightarrow \int 1\,dy = \sin\psi\,\frac{ds}{d\psi}\,d\psi$

$\Rightarrow \int 1\,dy = \sin\psi\,(a\cos\psi)\,d\psi$

$\Rightarrow \int 1\,dy = a\sin\psi\cos\psi\,d\psi$

$\Rightarrow \int_{y=0}^{y} 1\,dy = \int_{\psi=0}^{\psi} a\sin\psi\cos\psi\,d\psi$

$\Rightarrow \left[y\right]_0^y = \left[\frac{1}{2}a\sin^2\psi\right]_0^{\psi}$

$\Rightarrow \boxed{y = \frac{1}{2}a\sin^2\psi}$

(WHERE y IS MEASURED "DOWNWARDS")

NOW $\quad K.E_0 + \cancel{P.E_0} = K.E_\psi + P.E_\psi$

$\Rightarrow \frac{1}{2}mu^2 + 0 = \frac{1}{2}mv^2 - mgy$

$\Rightarrow u^2 = v^2 - 2gy$

$\Rightarrow v^2 = u^2 + 2gy$

$\Rightarrow v^2 = \left(\sqrt{\frac{1}{2}ag}\right)^2 + 2g\left(\frac{1}{2}a\sin^2\psi\right)$

$\Rightarrow v^2 = \frac{1}{2}ag + ag\sin^2\psi$

$\Rightarrow v^2 = \frac{1}{2}ag + ag\left(\frac{s^2}{a^2}\right)$

$\Rightarrow v^2 = \frac{g}{a}s^2 + \frac{1}{2}ag$

$\Rightarrow v^2 = \frac{g}{2a}\left(2s^2 + a^2\right)$ // AS BEFORE

a) II) LOOKING AT THE EQUATION OF MOTION IN THE NORMAL DIRECTION ($\hat{n}$)

$\Rightarrow m\frac{\dot{s}^2}{\rho} = mg\cos\psi - R$

$\Rightarrow R = mg\cos\psi - \frac{m\dot{s}^2}{\rho}$

$\Rightarrow R = mg\left[\cos\psi - \frac{\dot{s}^2}{g\rho}\right]$

$\Rightarrow R = mg\left[\cos\psi - \frac{1}{\cancel{g}(a\cos\psi)} \times \frac{\cancel{g}}{2a}\left(2s^2 + a^2\right)\right]$

$\Rightarrow R = mg\left[\cos\psi - \frac{2s^2 + a^2}{2a^2\cos\psi}\right]$

$\Rightarrow R = \frac{mg}{2\cos\psi}\left[2\cos^2\psi - \frac{2s^2 + a^2}{a^2}\right]$

$\Rightarrow R = \frac{mg}{2\cos\psi}\left[2(1 - \sin^2\psi) - 2\left(\frac{s^2}{a^2}\right) - 1\right]$

$\Rightarrow R = \frac{mg}{2\cos\psi}\left[2 - 2\sin^2\psi - 2\sin^2\psi - 1\right]$

$\Rightarrow R = \frac{mg}{2\cos\psi}\left[1 - 4\sin^2\psi\right]$ // AS REQUIRED

c) FINALLY IF $R = 0$

- $1 - 4\sin^2\psi = 0$

  $\sin^2\psi = \frac{1}{4}$

  $\sin\psi = +\frac{1}{2}$

  $\left(\psi = \frac{\pi}{6}\right)$

- $s = a\sin\psi$

  $s = \frac{1}{2}a$

ALTERNATIVE IF $R = 0$

$\Rightarrow 1 - 4\sin^2\psi = 0$

$\Rightarrow 1 - 4\left(\frac{s}{a}\right)^2 = 0$

$\Rightarrow 1 - \frac{4s^2}{a^2} = 0$

$\Rightarrow a^2 - 4s^2 = 0$

$\Rightarrow (a - 2s)(a + 2s)$

$\Rightarrow s = \frac{1}{2}a$ or $\cancel{-\frac{1}{2}a}$

Question 13 (****)

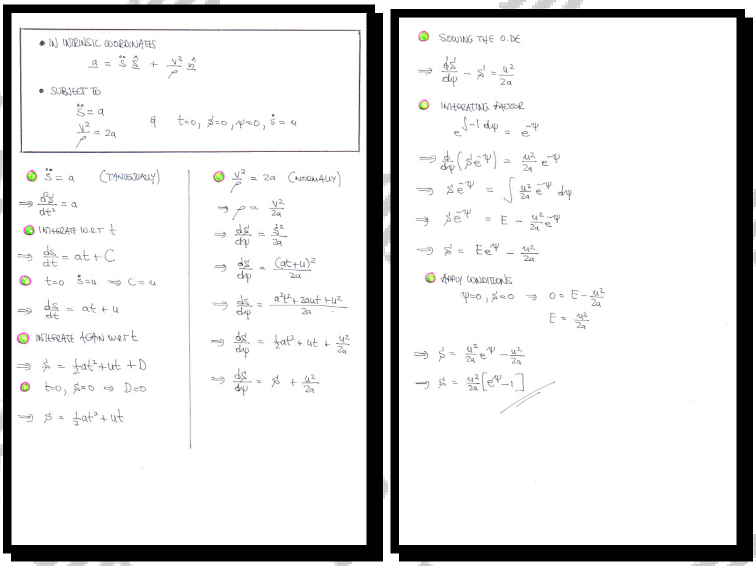
A particle is constrained to move on a curve with intrinsic equation $s = f(\psi)$.

It is moving in such a way so that it has constant tangential acceleration of magnitude a, and constant radial acceleration of magnitude $2a$, where a is a positive constant.

When $t = 0$, $s = 0$, $\psi = 0$ and the particle is moving with speed u.

Find the intrinsic equation of the curve on which the particle is moving.

$$\boxed{s = \frac{u^2}{2a}\left[e^{\psi} - 1\right]}$$



- IN INTRINSIC COORDINATES

$$\underline{a} = \ddot{s}\,\hat{\underline{s}} + \frac{v^2}{\rho}\,\hat{\underline{n}}$$

- SUBJECT TO

$$\ddot{s} = a \qquad \& \qquad t=0,\ s=0,\ \psi=0,\ \dot{s}=u$$
$$\frac{v^2}{\rho} = 2a$$

$\ddot{s} = a$ (TANGENTIALLY)

$\Rightarrow \frac{d^2s}{dt^2} = a$

INTEGRATE W.R.T t

$\Rightarrow \frac{ds}{dt} = at + C$

$t=0 \quad \dot{s}=u \Rightarrow C = u$

$\Rightarrow \frac{ds}{dt} = at + u$

INTEGRATE AGAIN W.R.T t

$\Rightarrow s = \frac{1}{2}at^2 + ut + D$

$t=0,\ s=0 \Rightarrow D=0$

$\Rightarrow s = \frac{1}{2}at^2 + ut$

$\frac{v^2}{\rho} = 2a$ (NORMALLY)

$\Rightarrow \rho = \frac{v^2}{2a}$

$\Rightarrow \frac{ds}{d\psi} = \frac{\dot{s}^2}{2a}$

$\Rightarrow \frac{ds}{d\psi} = \frac{(at+u)^2}{2a}$

$\Rightarrow \frac{ds}{d\psi} = \frac{a^2t^2 + 2aut + u^2}{2a}$

$\Rightarrow \frac{ds}{d\psi} = \frac{1}{2}at^2 + ut + \frac{u^2}{2a}$

$\Rightarrow \frac{ds}{d\psi} = s + \frac{u^2}{2a}$

SOLVING THE O.DE

$\Rightarrow \frac{ds}{d\psi} - s = \frac{u^2}{2a}$

INTEGRATING FACTOR

$$e^{\int -1\,d\psi} = e^{-\psi}$$

$\Rightarrow \frac{d}{d\psi}\left(s e^{-\psi}\right) = \frac{u^2}{2a}e^{-\psi}$

$\Rightarrow s e^{-\psi} = \int \frac{u^2}{2a}e^{-\psi}\,d\psi$

$\Rightarrow s e^{-\psi} = E - \frac{u^2}{2a}e^{-\psi}$

$\Rightarrow s = Ee^{\psi} - \frac{u^2}{2a}$

APPLY CONDITIONS

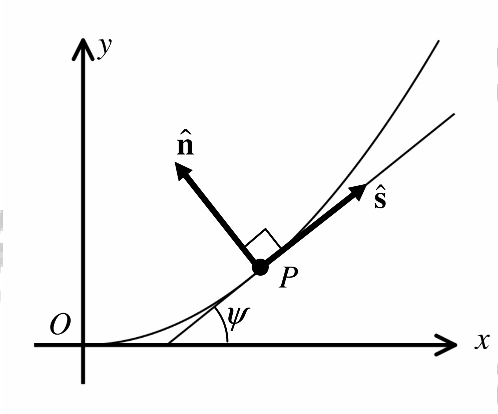
$\psi=0,\ s=0 \Rightarrow 0 = E - \frac{u^2}{2a}$

$E = \frac{u^2}{2a}$

$\Rightarrow s = \frac{u^2}{2a}e^{\psi} - \frac{u^2}{2a}$

$\Rightarrow s = \frac{u^2}{2a}\left[e^{\psi} - 1\right]$

Question 14 (****)



A particle P is constrained to move on a path, so that its distance travelled along that path, measured from the origin O, is denoted by s. The angle the tangent to P at any position on the path makes with the x axis is denoted by ψ.

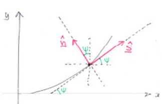
The unit vector along the tangent to the path at P is denoted by $\hat{\mathbf{s}}$ and the unit vector along the normal to the path at P, directed towards the centre of curvature of the path is denoted by $\hat{\mathbf{n}}$, as shown in the figure above.

Show that the acceleration of the particle is

$$\ddot{s}\hat{\mathbf{s}} + \frac{\dot{s}^2}{\rho}\hat{\mathbf{n}},$$

where ρ denotes the radius of curvature.

proof



$\hat{\underline{s}} = (1\cos\psi)\underline{i} + (1\sin\psi)\underline{j}$
$\hat{\underline{n}} = (-1\sin\psi)\underline{i} + (1\cos\psi)\underline{j}$

$\hat{\underline{s}} = (\cos\psi, \sin\psi)$
$\hat{\underline{n}} = (-\sin\psi, \cos\psi)$

$\frac{d\hat{\underline{s}}}{dt} = \frac{d}{dt}(\cos\psi, \sin\psi) = (-\dot{\psi}\sin\psi, \dot{\psi}\cos\psi)$
$= \dot{\psi}(-\sin\psi, \cos\psi) = \dot{\psi}\hat{\underline{n}}$

NOW THE VELOCITY VECTOR HAS MAGNITUDE (SPEED) V & IS AT A TANGENTIAL DIRECTION

$\underline{V} = V\hat{\underline{s}} = \dot{s}\hat{\underline{s}}$ SINCE $V = \dot{s}$

$\Rightarrow \frac{d\underline{v}}{dt} = \frac{d}{dt}(V\hat{\underline{s}}) = \frac{dv}{dt}\hat{\underline{s}} + v\frac{d}{dt}(\hat{\underline{s}})$

$\Rightarrow \frac{d\underline{v}}{dt} = \frac{d\dot{s}}{dt}\hat{\underline{s}} + v\dot{\psi}\hat{\underline{n}}$

$\Rightarrow \frac{d\underline{v}}{dt} = \ddot{s}\hat{\underline{s}} + \dot{s}\dot{\psi}\hat{\underline{n}}$

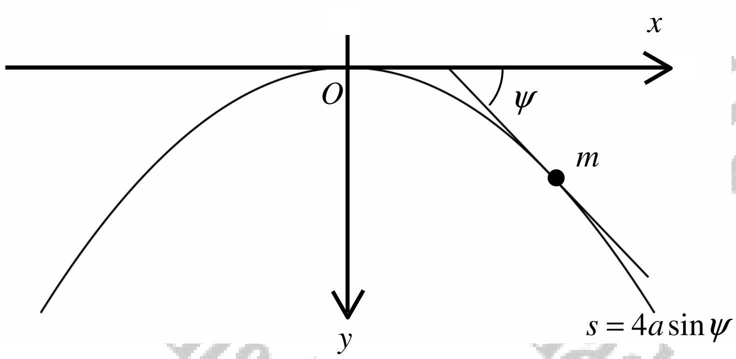
$\Rightarrow \frac{d\underline{v}}{dt} = \ddot{s}\hat{\underline{s}} + \dot{s}\frac{d\psi}{dt}\hat{\underline{n}}$

$\Rightarrow \frac{d\underline{v}}{dt} = \ddot{s}\hat{\underline{s}} + \dot{s}\frac{d\psi}{ds}\times\frac{ds}{dt}\hat{\underline{n}}$

$\Rightarrow \frac{d\underline{v}}{dt} = \ddot{s}\hat{\underline{s}} + \dot{s}\frac{1}{\rho}\dot{s}\hat{\underline{n}}$

$\Rightarrow \frac{d\underline{v}}{dt} = \ddot{s}\hat{\underline{s}} + \frac{\dot{s}^2}{\rho}\hat{\underline{n}}$

Question 15 (****)



A section of thin flexible gutter type tubing, with a smooth groove running along its length is bend into the shape of a cycloid with intrinsic equation

$$s = 4a\sin\psi\,,\ 0 \le \psi < \frac{\pi}{2}\,,$$

where a is a positive constant.

The cycloidal tubing is fixed in a vertical plane with its vertex coinciding with a Cartesian origin O, where the directions of x and y increasing are measured as shown the above figure.

The arclength s is measured from O, and the angle ψ is the angle the tangent to the cycloidal tubing makes with the positive x axis also shown in the figure above.

A particle of mass m is placed in the groove of the tubing at O.

The particle is slightly disturbed and begins to travel down the rod, where the groove keeps the particle from falling to either side of the tubing.

a) Show that while the particle is still in contact with the tubing

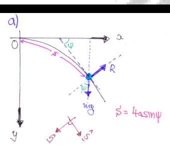
$$y = 2a\sin^2\psi\,.$$

b) Show further than the particle leaves the tubing when $y = a$.

proof

[solution overleaf]

a)



$\sin\psi = dy/ds$
$\cos\psi = dx/ds$
$\tan\psi = dy/ds$

FROM THE STANDARD TRIANGLE

$$\frac{dy}{ds} = \sin\psi$$
$$1\,dy = \sin\psi\,ds$$
$$1\,dy = \sin\psi\,\frac{ds}{d\psi}\,d\psi$$
$$1\,dy = \sin\psi\,(4a\cos\psi)\,d\psi$$
$$\int_{y=0}^{y} 1\,dy = \int_{\psi=0}^{\psi} 4a\sin\psi\cos\psi\,d\psi$$
$$[y]_0^y = [2a\sin^2\psi]_0^\psi$$
$$y = 2a\sin^2\psi$$

b) METHOD A (BY ENERGIES)

$$KE_0 + PE_0 = KE_\psi + P.E_\psi$$

(TAKING THE x AXIS AS THE ZERO POTENTIAL LEVEL)

$$\Rightarrow 0 + 0 = \tfrac{1}{2}mv^2 - mgh$$
$$\Rightarrow v^2 = 2gh$$
$$\Rightarrow v^2 = 2g(2a\sin^2\psi)$$
$$\Rightarrow v^2 = 4ag\sin^2\psi$$

METHOD B (BY THE EQUATION OF MOTION)

ACCELERATION IN INTRINSICS IS GIVEN BY

$$\boxed{\underline{a} = \ddot{s}\,\underline{\hat{s}} + \frac{v^2}{\rho}\,\underline{\hat{n}}}$$

LOOKING AT TANGENTIAL DIRECTION

$$\cancel{m}\ddot{s} = \cancel{m}g\sin\psi$$
$$\frac{d\dot{s}}{dt} = g\sin\psi$$
$$\frac{d\dot{s}}{ds}\frac{ds}{dt} = g\left(\frac{s}{4a}\right)$$
$$\frac{d\dot{s}}{ds}\dot{s} = \frac{g}{4a}s$$

VARIATION

$$\ddot{s} = g\sin\psi$$
$$\ddot{s} = \frac{g}{4a}s$$
$$2\dot{s}\ddot{s} = \frac{g}{2a}s\dot{s}$$

INTEGRATE w.r.t t

$$\dot{s}^2 = \frac{g}{4a}s^2 + C$$

$$\Rightarrow \int_{\dot{s}=0}^{\dot{s}} \dot{s}\,d\dot{s} = \int_{s=0}^{s} \frac{g}{4a}s\,ds$$
$$\Rightarrow \tfrac{1}{2}\dot{s}^2 = \tfrac{1}{2}\left(\frac{g}{4a}\right)s^2$$
$$\Rightarrow \dot{s}^2 = \frac{g}{4a}s^2$$
$$\Rightarrow \dot{s}^2 = \frac{g}{4a}(4a\sin\psi)^2$$
$$\Rightarrow v^2 = 4ag\sin^2\psi$$

APPLY CONDITION; $s=0$, $\dot{s}=0$, $C=0$

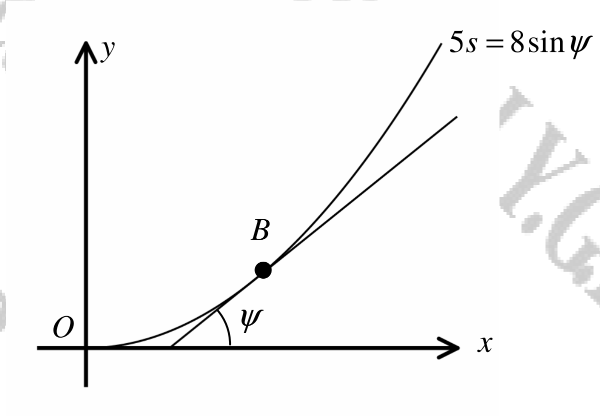
$$\dot{s}^2 = \frac{g}{4a}s^2$$

& MERGES

NOW LOOKING AT NORMAL DIRECTION WHEN THE PARTICLE LEAVES

$$\Rightarrow m\frac{v^2}{\rho} = mg\cos\psi - \cancel{R}$$
$$\Rightarrow \frac{v^2}{\rho} = g\cos\psi$$
$$\Rightarrow \frac{4ag\sin^2\psi}{4a\cos\psi} = g\cos\psi$$
$$\Rightarrow \sin^2\psi = \cos^2\psi$$
$$\Rightarrow \tan^2\psi = 1$$
$$\Rightarrow \psi = \frac{\pi}{4} \quad \text{(IGNORE MINUS)}$$
$$\therefore y = 2a\sin^2\left(\frac{\pi}{4}\right) = 2a\left(\frac{\sqrt{2}}{2}\right)^2 = a$$

Question 16 (****)



A bead B is made to slide along a smooth wire, which is fixed in a vertical plane, and is bent into the shape of an inverted cycloid with intrinsic equation

$$5s = 8\sin\psi, \quad 0 \le \psi < \frac{\pi}{2},$$

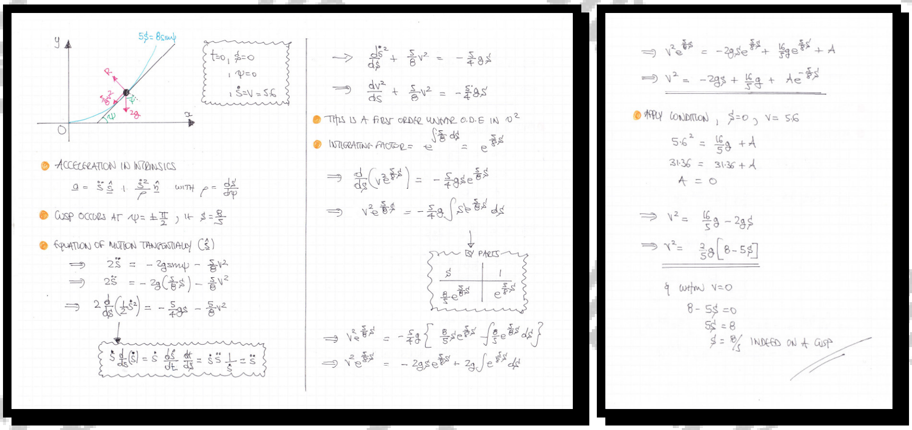
where s is measured from the origin O, and ψ is the angle the tangent to the cycloid makes with the x axis.

The bead is projected from O with tangential speed 5.6 ms^{-1}.

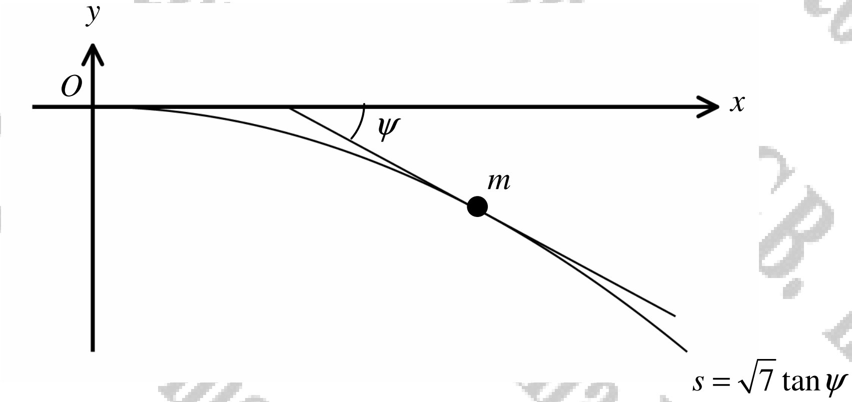
Use intrinsic coordinates to find an expression, in terms of s and g, for the speed of the bead and hence show that the bead comes to rest at a cusp.

You may not consider energy conservation in this question.

$$\boxed{v^2 = \frac{1}{2}g[8-5s]}$$



Question 17 (****)



The figure above shows a particle of mass m, which is free to slide along a smooth surface, whose vertical cross section is the curve C with intrinsic equation

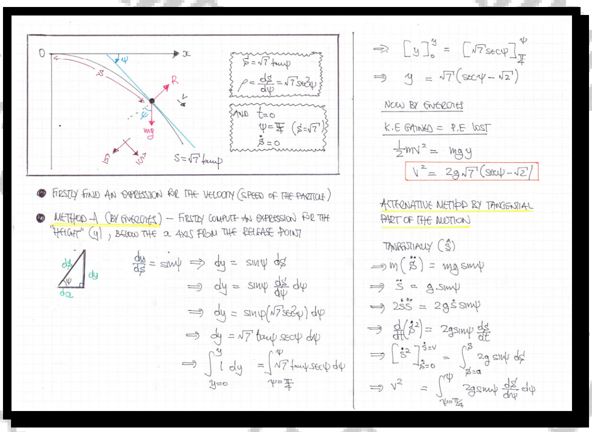
$$s=\sqrt{7}\tan\psi\,,\ 0\le\psi<\frac{\pi}{2}.$$

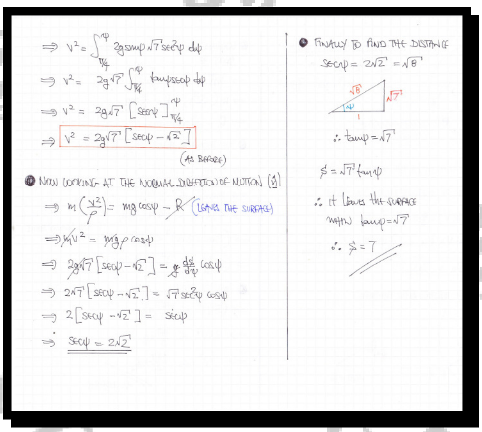
The arclength s is measured from the origin O, and the angle ψ is the angle the tangent to C makes with the positive x axis, as shown in the figure above.

The particle is released from rest from a point A on the surface, where $\psi=\frac{1}{4}\pi$, and leaves the surface at the point B.

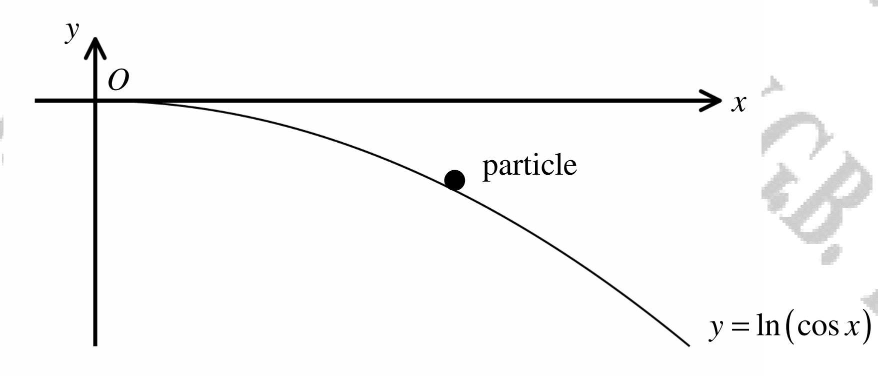
Determine the distance AB along the curved surface.

$\boxed{d=7}$





Question 18 (*****)



The figure above shows a particle which is free to slide along a smooth surface, whose vertical cross section is the curve C with equation

$$y = \ln(\cos x),\ 0 \le x < \frac{\pi}{2}.$$

The particle is projected from O with speed $\sqrt{\frac{1}{3}g}$ tangential to C and leaves the surface at the point P.

Show that the distance OP along C is $\text{arcosh}\left(e^{\frac{1}{3}}\right)$.

$\boxed{\text{proof}}$

$\boxed{y = \ln(\cos x) \quad 0 \le x < \frac{\pi}{2}}$

- START BY OBTAINING AN INTRINSIC EQUATION OF THE CURVE, WHERE s IS MEASURED FROM $(0,0)$

$\frac{dy}{dx} = \tan\psi \qquad \frac{dy}{dx} = \frac{1}{\cos x}(-\sin x) = -\tan x = \tan(-x)$

$\boxed{\psi = -x}$

$s = \int_{x=0}^{x} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_{x=0}^{x} \sqrt{1 + \tan^2 x}\, dx = \int_{x=0}^{x} \sec x\, dx$

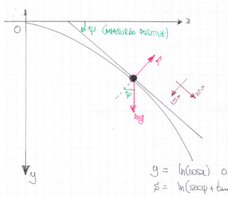
$= \Big[\ln|\sec x + \tan x|\Big]_0^x = \ln|\sec x + \tan x| - \cancel{\ln 1}$

$\therefore\ s = \ln(\sec x + \tan x) \qquad 0 \le x < \frac{\pi}{2}$

$s = \ln(\sec\psi - \tan\psi) \qquad -\frac{\pi}{2} < \psi \le 0$ IN DIAGRAM BELOW x AXIS

$s = \ln(\sec\psi + \tan\psi) \qquad 0 \le \psi < \frac{\pi}{2}$ OR DEAL WITH ψ AS ANGLE WITH $-x$ AXIS

- NEXT START WITH A GOOD DIAGRAM



$\rho = \frac{ds}{d\psi}$

$\rho = \frac{\sec^2\psi + \sec\psi\tan\psi}{\sec\psi + \tan\psi}$

$\rho = \frac{\sec\psi(\cancel{\sec\psi + \tan\psi})}{\cancel{\sec\psi + \tan\psi}}$

$\rho = \frac{ds}{d\psi} = \sec\psi$

$y = \ln(\cos x) \quad 0 \le x < \frac{\pi}{2}$

$s = \ln(\sec\psi + \tan\psi) \quad 0 \le \psi < \frac{\pi}{2}$

- ACCELERATION IN INTRINSICS IS GIVEN BY

$\underline{a} = \ddot{s}\hat{\underline{s}} + \frac{\dot{s}^2}{\rho}\hat{\underline{n}} = \dot{v}\hat{\underline{s}} + \frac{v^2}{\rho}\hat{\underline{n}}$

- IN THE TANGENTIAL DIRECTION $(\hat{\underline{s}})$

$\Rightarrow \cancel{m}\ddot{s} = \cancel{m}g\sin\psi$

$\Rightarrow v\frac{dv}{ds} = g\sin\psi$

$\Rightarrow v\, dv = g\sin\psi\, ds$

$\Rightarrow v\, dv = g\sin\psi\left(\frac{ds}{d\psi}\right)d\psi$

$\Rightarrow v\, dv = g\sin\psi\sec\psi\, d\psi$

$\Rightarrow \int_{v=\sqrt{\frac{g}{3}}}^{v} v\, dv = \int_{\psi=0}^{\psi} g\tan\psi\, d\psi$

$\Rightarrow \left[\frac{1}{2}v^2\right]_{\sqrt{\frac{g}{3}}}^{v} = \Big[g\ln|\sec\psi|\Big]_0^{\psi}$

$\Rightarrow \frac{1}{2}v^2 - \frac{1}{6}g = g\ln|\sec\psi| - \cancel{g\ln 1}$

$\Rightarrow v^2 - \frac{1}{3}g = 2g\ln(\sec\psi)$

$\Rightarrow v^2 = \frac{1}{3}g + 2g\ln(\sec\psi)$

- LOOKING AT THE NORMAL DIRECTION $(\hat{\underline{n}})$

$\Rightarrow m\frac{v^2}{\rho} = mg\cos\psi - \cancel{R}$ (LEAVES THE SURFACE

$\Rightarrow \cancel{m}v^2 = \cancel{m}g\rho\cos\psi$

$\Rightarrow \frac{1}{3}g + 2g\ln(\sec\psi) = g\rho\cos\psi$

$\Rightarrow \frac{1}{3} + 2\ln(\sec\psi) = \rho\cos\psi$

$\Rightarrow \frac{1}{3} + 2\ln(\sec\psi) = (\sec\psi)\cos\psi$

$\Rightarrow \frac{1}{3} + 2\ln(\sec\psi) = 1$

$\Rightarrow 2\ln(\sec\psi) = \frac{2}{3}$

$\Rightarrow \ln(\sec\psi) = \frac{1}{3}$

$\Rightarrow \sec\psi = e^{\frac{1}{3}}$

BUT $1 + \tan^2\psi \equiv \sec^2\psi$

$\tan\psi = +\sqrt{\sec^2\psi - 1}$

$\tan\psi = \sqrt{e^{\frac{2}{3}} - 1}$

- RETURNING TO THE INTRINSIC

$s = \ln(\sec\psi + \tan\psi)$

$s = \ln\left[e^{\frac{1}{3}} + \sqrt{e^{\frac{2}{3}} - 1}\right]$

$s = \text{arcosh}\left(e^{\frac{1}{3}}\right)$

Question 19 (*****)

A right prism is fixed so that its axis is horizontal.

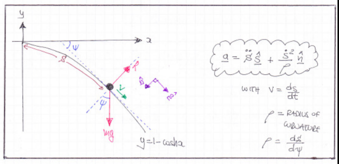
A particle is placed on the highest point of the outer smooth surface of the prism whose cross section has equation

$$y = 1 - \cosh x, \quad x \geq 0 .$$

The particle is slightly disturbed and begins to move in a path along the cross section of the outer surface of the prism whose equation is given above.

Determine the distance the particle travels until the instant it leaves the surface.

$\boxed{d = \sqrt{3}}$



$\underline{a} = \ddot{s}\,\hat{\underline{s}} + \dfrac{\dot{s}^2}{\rho}\,\hat{\underline{n}}$

WITH $v = \dfrac{ds}{dt}$

ρ = RADIUS OF CURVATURE

$\rho = \dfrac{ds}{d\psi}$

- FIRSTLY OBTAIN AN INTRINSIC EQUATION OF THE CURVE, IN THE FORM $s = f(\psi)$

$\Rightarrow y = 1 - \cosh x$

$\Rightarrow \dfrac{dy}{dx} = -\sinh x$

$\Rightarrow \left(\dfrac{dy}{dx}\right)^2 + 1 = 1 + \sinh^2 x = \cosh^2 x$

$\Rightarrow s = \displaystyle\int_{x=0}^{x} \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}\, dx$

$\Rightarrow s = \displaystyle\int_{x=0}^{x} \cosh x \, dx$

$\Rightarrow s = \left[\sinh x\right]_0^x$

$\boxed{s = \sinh x}$

- NOW $\dfrac{dy}{dx} = -\tan\psi$ & $\dfrac{dy}{dx} = -\sinh x$

$\therefore \sinh x = \tan\psi$

$\therefore s = \tan\psi$

- NOW LOOKING AT THE TANGENTIAL DIRECTION $(\hat{\underline{s}})$

$\Rightarrow \cancel{m}\ddot{s} = \cancel{m}g\sin\psi$

$\Rightarrow \dfrac{d}{ds}\left(\dfrac{1}{2}\dot{s}^2\right) = g\sin\psi$

$\dfrac{d}{ds}\left(\dfrac{1}{2}\dot{s}^2\right) = \dot{s}\dfrac{d\dot{s}}{ds} = \dot{s}\dfrac{d\dot{s}}{dt}\dfrac{dt}{ds} = \cancel{\dot{s}}\,\ddot{s}\,\dfrac{1}{\cancel{\dot{s}}}$

$\Rightarrow \dfrac{1}{2}\dot{s}^2 = \displaystyle\int g\sin\psi \, ds$

$\Rightarrow \dfrac{1}{2}\dot{s}^2 = \displaystyle\int g\sin\psi \dfrac{ds}{d\psi}\, d\psi$

$s = \tan\psi$

$\rho = \dfrac{ds}{d\psi} = \sec^2\psi$

$\Rightarrow \dfrac{1}{2}\dot{s}^2 = \displaystyle\int g\sin\psi\sec^2\psi \, d\psi$

$\Rightarrow \dot{s}^2 = 2g\displaystyle\int \tan\psi\sec\psi \, d\psi$

$\Rightarrow \dot{s}^2 = 2g\sec\psi + C$

$t=0,\ s=0,\ \psi=0,\ \dot{s}=v=0$

$0 = 2g + C$

$C = -2g$

$\Rightarrow \dot{s}^2 = v^2 = 2g\sec\psi - 2g$

$\therefore v^2 = 2g(\sec\psi - 1)$

- NOW LOOKING AT THE NORMAL DIRECTION $(\hat{\underline{n}})$

$\Rightarrow m\left(\dfrac{v^2}{\rho}\right) = mg\cos\psi - R$

WHEN IT LEAVES THE SURFACE $R=0$

$\Rightarrow \dfrac{mv^2}{\rho} = mg\cos\psi$

$\Rightarrow \dfrac{v^2}{\rho} = g\cos\psi$

$\Rightarrow v^2 = g\rho\cos\psi$

$\Rightarrow 2\cancel{g}(\sec\psi - 1) = \cancel{g}\left(\dfrac{ds}{d\psi}\right)\cos\psi$

$\Rightarrow 2\sec\psi - 2 = \sec^2\psi\cos\psi$

$\Rightarrow 2\sec\psi - 2 = \sec\psi$

$\Rightarrow \sec\psi = 2 \quad (\text{OR } \cos\psi = \tfrac{1}{2})$

$\Rightarrow \psi = \pi/3$

BUT $s = \tan\psi$

$s = \tan\left(\dfrac{\pi}{3}\right)$

$s = \sqrt{3}$

AS REQUIRED