VARIABLE MASS PROBLEMS

Question 1 (**)

A rocket is moving vertically upwards relative to the surface of the earth. The motion takes place close to the surface of the earth and it is assumed that g is the constant gravitational acceleration.

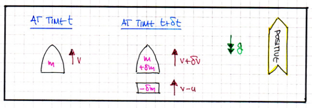
At time t the mass of the rocket is $M(1-kt)$, where M and k are positive constants, and the rocket is moving upwards with speed v.

The rocket expels fuel vertically downwards with speed u relative to the rocket.

Given further that when $t=0$, $v=0$ determine an expression for v in time t, in terms of u, g and k.

$\boxed{}$, $\boxed{v=-gt-u\ln(1-kt)}$

STARTING WITH THE USUAL MOMENTUM/IMPULSE DIAGRAM



BY THE IMPULSE MOMENTUM PRINCIPLE

$$\Rightarrow -mg\times\delta t = \left[(m+\delta m)(v+\delta v)-\delta m(v-u)\right]-mv$$

$$\Rightarrow -mg\,\delta t = \cancel{mv}+m\delta v+\cancel{v\delta m}+\delta m\delta v-\cancel{v\delta m}+u\delta m-\cancel{mv}$$

$$\Rightarrow -mg\,\delta t = m\delta v+u\delta m+\delta m\delta v$$

$$\Rightarrow -mg = m\frac{\delta v}{\delta t}+u\frac{\delta m}{\delta t}+\frac{\delta m\delta v}{\delta t}$$

TAKING LIMITS AND REARRANGING FOR THE ACCELERATION

$$\Rightarrow -mg = m\frac{dv}{dt}+u\frac{dm}{dt}$$

$$\Rightarrow \frac{dv}{dt} = -g-\frac{u}{m}\frac{dm}{dt}$$

USING THE MASS CONSUMPTION RELATIONSHIP

$$\Rightarrow m = M(1-kt)$$

$$\Rightarrow \frac{dm}{dt} = -Mk$$

RETURNING TO THE MAIN O.D.E.

$$\Rightarrow \frac{dv}{dt} = -g-\frac{u}{M(1-kt)}(-Mk)$$

$$\Rightarrow \frac{dv}{dt} = -g+\frac{uk}{1-kt}$$

SOLVING THE O.D.E BY DIRECT INTEGRATION, SUBJECT TO THE CONDITION $t=0, v=0$

$$\Rightarrow \int_{v=0}^{v} 1\,dv = \int_{t=0}^{t} -g+\frac{uk}{1-kt}\,dt$$

$$\Rightarrow [v]_0^v = \left[-gt-u\ln|1-kt|\right]_0^t$$

$$\Rightarrow v-0 = \left[gt+u\ln|1-kt|\right]_t^0$$

$$\Rightarrow v = \cancel{u\ln 1}-(gt+u\ln|1-kt|$$

$$\Rightarrow v = -gt-u\ln|1-kt|$$

Question 2 (***)

A spacecraft is moving in deep space in a straight line with speed $2u$.

At time $t=0$, the mass of the spacecraft is M and at that instant the engines of the spacecraft are fired in a direction opposite to that of the motion of the spacecraft.

Fuel is ejected at a constant mass rate k with speed u relative to the spacecraft.

At time t, the mass of the spacecraft is m and its speed is v.

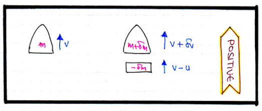
a) Use the impulse momentum principle to show that

$$\frac{dv}{dt}=\frac{uk}{M-kt}.$$

b) Hence determine, in terms of u, the speed of the spacecraft when the mass of the spacecraft is $\frac{1}{3}$ of its initial mass.

, $\boxed{v=(2+\ln 3)u}$

a)



BY THE IMPULSE MOMENTUM PRINCIPLE

$\Rightarrow$ MOMENTUM AFTER − MOMENTUM BEFORE = IMPULSE (OF EXTERNAL FORCES)

$\Rightarrow \left[(m+\delta m)(v+\delta v)-\delta m(v-u)\right]-mv=0$ ← NO EXTERNAL FORCES (DEEP SPACE)

$\Rightarrow \cancel{mv}+m\delta v+\cancel{v\delta m}+\delta m\delta v-\cancel{v\delta m}+u\delta m-\cancel{mv}=0$

$\Rightarrow m\delta v+u\delta m+\delta m\delta v=0$

$\Rightarrow m\frac{\delta v}{\delta t}+u\frac{\delta m}{\delta t}+\frac{\delta m\delta v}{\delta t}=\frac{0}{\delta t}$

TAKING LIMITS & SOLVING THE EQUATION FOR THE ACCELERATION

$\Rightarrow m\frac{dv}{dt}+u\frac{dm}{dt}=0$

$\Rightarrow \boxed{\frac{dv}{dt}=-\frac{u}{m}\frac{dm}{dt}}$

USING THE "FUEL CONSUMPTION RATE" RELATIONSHIP

$\Rightarrow \frac{dm}{dt}=-k$ (CONSTANT)

$\Rightarrow m=M-kt$ (AS $t=0$, $m=M$)

$\Rightarrow \frac{dv}{dt}=-\frac{u}{M-kt}(-k)$

$\Rightarrow \frac{dv}{dt}=\frac{uk}{M-kt}$ AS REQUIRED

b) SOLVING THE O.D.E. BY SEPARATION OF VARIABLES

$\Rightarrow \frac{dv}{dt}=\frac{uk}{M-kt}$

$\Rightarrow \int_{v=2u}^{v}1\,dv=\int_{t=0}^{t}\frac{uk}{M-kt}\,dt$

$\Rightarrow \left[v\right]_{v=2u}^{v=v}=\left[\frac{uk}{-k}\ln|M-kt|\right]_{t=0}^{t=t}$

$\Rightarrow v-2u=u\left[\ln|M-kt|\right]_{t=t}^{t=0}$

$\Rightarrow v=2u+u\left[\ln M-\ln|M-kt|\right]$

$\Rightarrow \boxed{v=2u+u\ln\left|\frac{M}{M-kt}\right|}$

FINALLY WE NEED THE TIME WHEN $m=\frac{1}{3}M$

$\Rightarrow m=M-kt$

$\Rightarrow \frac{1}{3}M=M-kt$

$\Rightarrow kt=\frac{2}{3}M$

$\Rightarrow v=2u+u\ln\left|\frac{M}{M-\frac{2}{3}M}\right|$

$\Rightarrow v=2u+u\ln\left|\frac{1}{1/3}\right|$

$\Rightarrow v=2u+u\ln 3$

$\Rightarrow v=(2+\ln 3)u$

Question 3 (***)

The mass m of raindrop, falling through a stationary cloud, increases as it picks up moisture. The raindrop is modelled as a particle falling freely without any resistance. Let m be the mass of the raindrop at time t, and v the speed of the raindrop at time t. When $t = 0$, $v = U$ and $m = m_0$.

The rate of increase of the mass of the raindrop is km, where k is a positive constant.

a) Show clearly that …

i. … $\dfrac{dv}{dt} = g - kv$.

ii. … $v = \dfrac{g}{k} + \left(U - \dfrac{g}{k}\right)e^{-kt}$

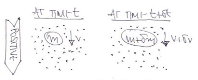
It is further given that the raindrop leaves the cloud when $m = 3m_0$.

b) Show that

$$v = \frac{1}{3k}(Uk + 2g).$$

proof

(a) BY THE IMPULSE MOMENTUM PRINCIPLE



$\Rightarrow mg \times \delta t = (m+\delta m)(v+\delta v) - mv$

$\Rightarrow mg\,\delta t = \cancel{mv} + m\delta v + v\delta m + \delta m\delta v - \cancel{mv}$

$\Rightarrow mg = m\frac{\delta v}{\delta t} + v\frac{\delta m}{\delta t} + \frac{\delta m \delta v}{\delta t}$

$\Rightarrow mg = m\frac{dv}{dt} + v\frac{dm}{dt}$

$\Rightarrow mg = m\frac{dv}{dt} + v(km)$

$\Rightarrow g = \frac{dv}{dt} + kv$

$\Rightarrow \frac{dv}{dt} = g - kv$ // AS REQUIRED

(b) $\frac{1}{g-kv}\,dv = 1\,dt$

$\Rightarrow \int_{v=U}^{v} \frac{-k}{g-kv}\,dv = \int_{t=0}^{t} -k\,dt$

$\Rightarrow \left[\ln|g-kv|\right]_U^v = \left[-kt\right]_0^t$

$\Rightarrow \ln|g-kv| - \ln|g-kU| = -kt$

$\Rightarrow \ln\left|\frac{g-kv}{g-kU}\right| = -kt$

$\Rightarrow \frac{g-kv}{g-kU} = e^{-kt}$

$\Rightarrow g - kv = (g-kU)e^{-kt}$

$\Rightarrow g - (g-kU)e^{-kt} = kv$

$\Rightarrow v = \frac{g}{k} - \frac{1}{k}(g-kU)e^{-kt}$

$\Rightarrow v = \frac{g}{k} + \left(U - \frac{g}{k}\right)e^{-kt}$ // AS REQUIRED

(c) $\frac{dm}{dt} = km$, $t=0$, $m = m_0$

$m = m_0 e^{kt}$

$3m_0 = m_0 e^{kt}$

$3 = e^{kt}$

$\boxed{e^{-kt} = \frac{1}{3}}$

So $v = \frac{g}{k} + \left(U - \frac{g}{k}\right) \times \frac{1}{3}$

$V = \frac{g}{k} + \frac{1}{3}U - \frac{1}{3}\frac{g}{k}$

$V = \frac{1}{3}U + \frac{2g}{3k}$

$V = \frac{1}{3k}(Uk + 2g)$ // AS REQUIRED

Question 4 (***)

A rocket is moving in a straight line in deep space. At time $t = 0$ the mass of the rocket is M and is moving in a straight line with speed 1500 ms^{-1}.

At that instant the engines of the rocket are fired in a direction opposite to that of the motion of the rocket. Fuel is ejected at a constant mass rate λ kg s^{-1} with speed 6000 ms^{-1} relative to the rocket.

At time t, the mass of the rocket is m and its speed is v.

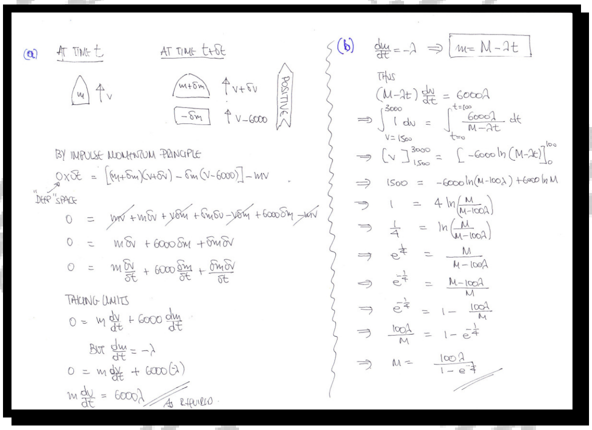
a) Show clearly that

$$m\frac{dv}{dt} = 6000\lambda .$$

When $t = 100$ the rocket is still ejecting fuel and its speed is 3000 ms^{-1}.

b) Express M in terms of λ.

$$\boxed{M = \frac{100\lambda}{1 - e^{-\frac{1}{4}}}}$$


(a) AT TIME t AT TIME $t+\delta t$

m $\uparrow v$ $m+\delta m$ $\uparrow v+\delta v$

$-\delta m$ $\uparrow v-6000$ POSITIVE

BY IMPULSE MOMENTUM PRINCIPLE

$0 \times \delta t = \left[(m+\delta m)(v+\delta v) - \delta m(v-6000)\right] - mv$

"DEEP" SPACE

$0 = \cancel{mv} + m\delta v + \cancel{v\delta m} + \delta m\delta v - \cancel{v\delta m} + 6000\delta m - \cancel{mv}$

$0 = m\delta v + 6000\delta m + \delta m\delta v$

$0 = m\frac{\delta v}{\delta t} + 6000\frac{\delta m}{\delta t} + \frac{\delta m\delta v}{\delta t}$

TAKING LIMITS

$0 = m\frac{dv}{dt} + 6000\frac{dm}{dt}$

BUT $\frac{dm}{dt} = -\lambda$

$0 = m\frac{dv}{dt} + 6000(-\lambda)$

$m\frac{dv}{dt} = 6000\lambda$ AS REQUIRED

(b) $\frac{dm}{dt} = -\lambda \Rightarrow \boxed{m = M - \lambda t}$

THUS

$(M-\lambda t)\frac{dv}{dt} = 6000\lambda$

$\Rightarrow \int_{v=1500}^{3000} 1 \, dv = \int_{t=0}^{t=100} \frac{6000\lambda}{M-\lambda t} \, dt$

$\Rightarrow \left[v\right]_{1500}^{3000} = \left[-6000\ln(M-\lambda t)\right]_0^{100}$

$\Rightarrow 1500 = -6000\ln(M-100\lambda) + 6000\ln M$

$\Rightarrow 1 = 4\ln\left(\frac{M}{M-100\lambda}\right)$

$\Rightarrow \frac{1}{4} = \ln\left(\frac{M}{M-100\lambda}\right)$

$\Rightarrow e^{\frac{1}{4}} = \frac{M}{M-100\lambda}$

$\Rightarrow e^{-\frac{1}{4}} = \frac{M-100\lambda}{M}$

$\Rightarrow e^{-\frac{1}{4}} = 1 - \frac{100\lambda}{M}$

$\Rightarrow \frac{100\lambda}{M} = 1 - e^{-\frac{1}{4}}$

$\Rightarrow M = \frac{100\lambda}{1-e^{-\frac{1}{4}}}$

Question 5 (***)

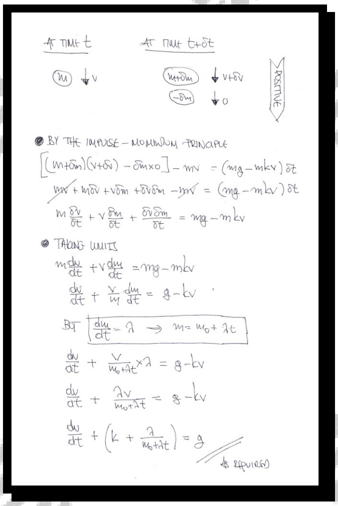
The mass m of raindrop, falling through a stationary cloud, increases as it picks up moisture. Let m be the mass of the raindrop at time t, and v the speed of the raindrop at time t. The mass of the raindrop increases at a constant rate λ, where λ is a positive constant. The raindrop is modelled as a particle falling subject to air resistance of magnitude mkv, where k is a positive constant.

When $t=0$, $m=m_0$.

Show clearly that

$$\frac{dv}{dt}+\left[k+\frac{\lambda}{m_0+\lambda t}\right]v=g\ .$$

$\boxed{\text{proof}}$



AT TIME t AT TIME $t+\delta t$

m $\downarrow v$ $m+\delta m$ $\downarrow v+\delta v$

$-\delta m$ $\downarrow 0$

POSITIVE

- BY THE IMPULSE – MOMENTUM PRINCIPLE

$$\left[(m+\delta m)(v+\delta v)-\delta m\times 0\right]-mv=(mg-mkv)\delta t$$

$$\cancel{mv}+m\delta v+v\delta m+\delta v\delta m-\cancel{mv}=(mg-mkv)\delta t$$

$$m\frac{\delta v}{\delta t}+v\frac{\delta m}{\delta t}+\frac{\delta v\delta m}{\delta t}=mg-mkv$$

- TAKING LIMITS

$$m\frac{dv}{dt}+v\frac{dm}{dt}=mg-mkv$$

$$\frac{dv}{dt}+\frac{v}{m}\frac{dm}{dt}=g-kv$$

BUT $\boxed{\frac{dm}{dt}=\lambda \Rightarrow m=m_0+\lambda t}$

$$\frac{dv}{dt}+\frac{v}{m_0+\lambda t}\times\lambda=g-kv$$

$$\frac{dv}{dt}+\frac{\lambda v}{m_0+\lambda t}=g-kv$$

$$\frac{dv}{dt}+\left(k+\frac{\lambda}{m_0+\lambda t}\right)=g$$

AS REQUIRED

Question 6 (***)

A rocket has initial mass M, which includes the fuel for its flight.

The rocket is initially at rest on the surface of the earth pointing vertically upwards. At time $t=0$ the rocket begins to propel itself by ejecting mass backwards at constant rate λ, and with speed u relative to the rocket.

At time t the speed of the rocket is v.

The rocket is modelled as a particle moving vertically upwards without air resistance.

The motion takes place close to the surface of the earth and it is assumed that g is the constant gravitational acceleration throughout the motion.

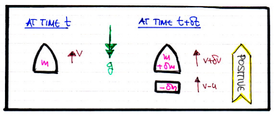
a) Determine an expression, in terms of u, g, λ, M and t, for the acceleration of the rocket and hence deduce that if the rocket lifts off immediately $\lambda > \dfrac{Mg}{u}$.

It is now given that $\lambda = \dfrac{3Mg}{u}$.

b) Find, in terms of u, the speed of the rocket when its mass is $\frac{3}{4}M$.

$\boxed{}$, $\boxed{\dfrac{dv}{dt} = \dfrac{u\lambda}{M-\lambda t} - g}$, $\boxed{v = u\left[-\dfrac{1}{12} + \ln\dfrac{4}{3}\right]}$

a) STARTING WITH THE USUAL MOMENTUM / IMPULSE DIAGRAM



BY THE IMPULSE MOMENTUM PRINCIPLE

$\Rightarrow -mg\,\delta t = \left[(m+\delta m)(v+\delta v) + (-\delta m)(v-u)\right] - \left[mv\right]$

$\Rightarrow -mg\,\delta t = \cancel{mv} + m\delta v + \cancel{v\delta m} + \delta m\delta v - \cancel{v\delta m} + u\delta m - \cancel{mv}$

$\Rightarrow -mg\,\delta t = m\delta v + u\delta m + \delta m\delta v$

$\Rightarrow -mg = m\dfrac{\delta v}{\delta t} + u\dfrac{\delta m}{\delta t} + \dfrac{\delta m\,\delta v}{\delta t}$

TAKING LIMITS AND REARRANGE FOR THE ACCELERATION

$\Rightarrow -mg = m\dfrac{dv}{dt} + u\dfrac{dm}{dt}$

$\Rightarrow \dfrac{dv}{dt} = -g - \dfrac{u}{m}\dfrac{dm}{dt}$

NEXT, AS THE EJECTION RATE OF THE FUEL IS CONSTANT

$\Rightarrow \dfrac{dm}{dt} = -\lambda$, SUBJECT TO $t=0$, $m=M$

$\Rightarrow m = M - \lambda t$

COMBINING THE LAST TWO EXPRESSIONS

$\Rightarrow \dfrac{dv}{dt} = -g - \dfrac{u}{M-\lambda t}(-\lambda)$

$\Rightarrow \dfrac{dv}{dt} = \dfrac{u\lambda}{M-\lambda t} - g$

FINALLY FOR IMMEDIATE LIFT OFF $\frac{dv}{dt} > 0$, AT $t=0$

$\Rightarrow \dfrac{u\lambda}{M} - g > 0$

$\Rightarrow u\lambda - gM > 0$

$\Rightarrow u\lambda > Mg$

$\Rightarrow \lambda > \dfrac{Mg}{u}$ AS REQUIRED

b) SOLVING THE O.D.E BY DIRECT INTEGRATION – FIRST WE REQUIRE TO FIND THE TIME WHEN $m = \frac{3}{4}M$

$\Rightarrow m = M - \lambda t$

$\Rightarrow \frac{3}{4}M = M - \left(\frac{3Mg}{u}\right)t$

$\Rightarrow \frac{3Mg}{u}t = \frac{1}{4}M$

$\Rightarrow t = \dfrac{u}{12g}$

SOLVING THE O.D.E, SUBJECT TO $t=0$, $v=0$,

$\Rightarrow 1\,dv = \left(\dfrac{u\lambda}{M-\lambda t} - g\right)dt$

$\Rightarrow 1\,dv = \left(\dfrac{u\,\frac{3Mg}{u}}{M - \frac{3Mg}{u}t} - g\right)dt$

$\Rightarrow 1\,dv = \left(\dfrac{3Mg}{M - \frac{3Mgt}{u}} - g\right)dt$

$\Rightarrow 1\,dv = \left(\dfrac{3g}{1 - \frac{3gt}{u}} - g\right)dt$

$\Rightarrow 1\,dv = \left(\dfrac{3ug}{u-3gt} - g\right)dt$

$\Rightarrow \displaystyle\int_{v=0}^{v} 1\,dv = \int_{t=0}^{t=\frac{u}{12g}} \frac{3ug}{u-3gt} - g\,dt$

$\Rightarrow \left[v\right]_0^v = \left[\dfrac{3ug}{-3g}\ln|u-3gt| - gt\right]_{t=0}^{t=\frac{u}{12g}}$

$\Rightarrow v = \left[-u\ln|u-3gt| - gt\right]_{t=0}^{t=\frac{u}{12g}}$

$\Rightarrow v = \left[u\ln|u-3gt| + gt\right]_{t=\frac{u}{12g}}^{t=0}$

$\Rightarrow v = \left[u\ln u\right] - \left[u\ln\left|u - 3g\left(\frac{u}{12g}\right)\right| + g\left(\frac{u}{12g}\right)\right]$

$\Rightarrow v = u\ln u - u\ln\left(\frac{3}{4}u\right) - \frac{1}{12}u$

$\Rightarrow v = u\ln\left(\dfrac{u}{\frac{3}{4}u}\right) - \frac{1}{12}u$

$\Rightarrow v = u\ln\frac{4}{3} - \frac{1}{12}u$

$\Rightarrow v = \left(\ln\frac{4}{3} - \frac{1}{12}\right)u$

Question 7 (***+)

A rocket, of initial mass M, propels itself forward by ejecting burned fuel.

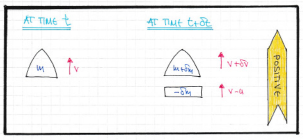
The initial speed of the rocket is U.

The burned fuel is ejected with constant speed u, relative to the rocket, in an opposite direction to that of the rocket's motion.

When all the fuel has been consumed, the mass of the rocket is $\frac{1}{4}M$.

By modelling the rocket as a particle and further assuming that there are no external forces acting on the rocket, determine, in terms of u and U, the speed of the rocket when all its fuel has been consumed.

$\boxed{}$, $\boxed{v = U + u\ln 4}$



BY THE IMPULSE-MOMENTUM PRINCIPLE, NOTING FURTHER THAT THERE ARE NO EXTERNAL FORCES

$$\Rightarrow 0 = \left[(m+\delta m)(v+\delta v) - \delta m(v-u)\right] - mv$$

$$\Rightarrow 0 = \cancel{mv} + m\delta v + \cancel{v\delta m} + \delta m\delta v - \cancel{v\delta m} + u\delta m - \cancel{mv}$$

$$\Rightarrow 0 = m\frac{\delta v}{\delta m} + \frac{\delta m\delta v}{\delta m} + u\frac{\delta m}{\delta m}$$

TAKING LIMITS, WE OBTAIN

$$\Rightarrow m\frac{dv}{dm} + u = 0$$

SOLVE THE O.D.E, SUBJECT TO THE INITIAL CONDITIONS

$$\Rightarrow m\frac{dv}{dm} = -u$$

$$\Rightarrow 1\,dv = -\frac{u}{m}\,dm$$

$$\Rightarrow \int_{v=U}^{v} 1\,dv = \int_{m=M}^{\frac{1}{4}M} -\frac{u}{m}\,dm$$

$$\Rightarrow \left[v\right]_{U}^{v} = \left[-u\ln m\right]_{M}^{\frac{1}{4}M}$$

$$\Rightarrow v - U = \left[u\ln m\right]_{\frac{1}{4}M}^{M}$$

$$\Rightarrow v - U = u\left[\ln M - \ln\tfrac{1}{4}M\right]$$

$$\Rightarrow v - U = u\ln\left[\frac{\cancel{M}}{\frac{1}{4}\cancel{M}}\right]$$

$$\Rightarrow v - U = u\ln 4$$

$$\Rightarrow v = U + u\ln 4$$

Question 8 (***+)

A raindrop absorbs water as it falls vertically under gravity through a cloud. In this model the cloud is assumed to consist of stationary water particles.

At time t, the mass of the raindrop is m and its speed is v. You may assume that the only force acting on the raindrop is its weight.

The raindrop starts from rest at $t = 0$.

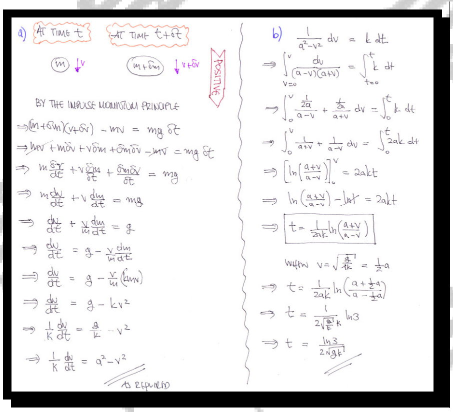
a) Given further that $\frac{dm}{dt} = kmv$, where k is a positive constant, show by the momentum impulse principle that

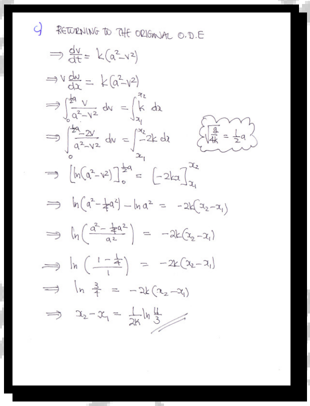
$$\frac{dv}{dt} = k\left(a^2 - v^2\right), \text{ where } a^2 = \frac{g}{k}.$$

b) Find an expression for the time, in terms of g and k, taken for the raindrop to reach a speed of $\sqrt{\frac{g}{4k}}$.

c) Determine the distance covered by the raindrop in accelerating from rest to a speed of $\sqrt{\frac{g}{4k}}$.

$\boxed{t = \frac{\ln 3}{2\sqrt{gk}}}$, $\boxed{d = \frac{1}{2k}\ln\left(\frac{4}{3}\right)}$





Question 9 (***+)

A vehicle with a driver is moving in a straight line by ejecting propellant backwards.

At time t, the vehicle is moving with speed v and has mass m. The propellant is ejected backwards at the constant rate k, with constant speed u relative to the vehicle.

The mass of the vehicle and the driver is M, and are modelled as a particle moving with any resistance.

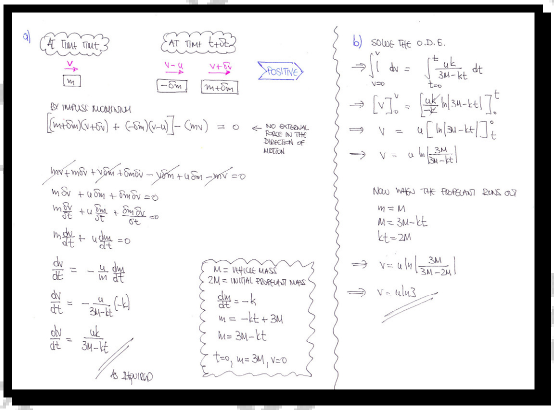
The vehicle starts from rest loaded with propellant of mass $2M$.

a) Show that the acceleration of the vehicle at time t is

$$\frac{uk}{3M-kt}.$$

b) Find the speed of the vehicle when the propellant runs out.

$\boxed{v = u\ln 3}$



Question 10 (***+)

A rocket has initial mass M, which includes the fuel for its flight. It is initially at rest on the surface of the earth pointing vertically upwards. At time $t=0$ the rocket begins to propel itself by ejecting mass backwards at constant rate and with speed u relative to the rocket.

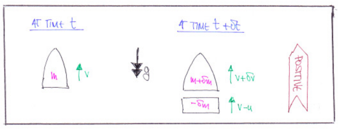
At time t the speed of the rocket is v.

The initial mass of the fuel is $\frac{1}{2}M$ and this fuel mass is all used up after time T.

The rocket is modelled as a particle moving without air resistance. The motion takes place close to the surface of the earth and it is assumed that g is the constant gravitational acceleration throughout the motion.

Determine, in terms of u, g and T, the speed of the rocket at the instant when its fuel is all used up.

$$\boxed{v = u\ln 2 - gT}$$



- BY THE IMPULSE – MOMENTUM PRINCIPLE

$$\Rightarrow -mg\times\delta t = \left[(m+\delta m)(v+\delta v) - \delta m(v-u)\right] - mv$$
$$\Rightarrow -mg\,\delta t = \cancel{mv} + m\delta v + \cancel{v\delta m} + \delta m\delta v - \cancel{v\delta m} + u\delta m - \cancel{mv}$$
$$\Rightarrow -mg\,\delta t = m\delta v + u\delta m + \delta m\delta v$$
$$\Rightarrow -mg = m\frac{\delta v}{\delta t} + u\frac{\delta m}{\delta t} + \frac{\delta m\delta v}{\delta t}$$

- TAKING LIMITS WE OBTAIN

$$\boxed{m\frac{dv}{dt} + u\frac{dm}{dt} = -mg} \quad (\mathrm{I})$$

- NOW THE MASS OF THE FUEL IS $\frac{1}{2}M$ & IS BURNED AT CONSTANT RATE λ IN TIME T

$$\text{i.e. } \frac{\frac{1}{2}M}{T} = \lambda \quad \underline{\text{OR}} \quad \lambda = \frac{M}{2T}$$

$$\text{HENCE } \boxed{\frac{dm}{dt} = -\lambda = \frac{-M}{2T}} \quad (\mathrm{II})$$

$$\boxed{m = M - \frac{Mt}{2T}} \quad (\mathrm{III})$$

- SIMPLIFYING (I), THEN SUBSTITUTING (II) & (III)

$$\Rightarrow \frac{dv}{dt} = -\frac{u}{m}\frac{dm}{dt} - g$$
$$\Rightarrow \frac{dv}{dt} = -\frac{u}{M-\frac{Mt}{2T}}\left(-\frac{M}{2T}\right) - g$$
$$\Rightarrow \frac{dv}{dt} = \frac{2uT}{2MT-Mt}\times\frac{M}{2T} - g$$
$$\Rightarrow \frac{dv}{dt} = \frac{Mu}{2MT-Mt} - g$$
$$\Rightarrow \frac{dv}{dt} = \frac{u}{2T-t} - g$$

- SEPARATING VARIABLES & INTEGRATING SUBJECT TO $t=0 \quad v=0$, $t=T \quad v=V$

$$\Rightarrow \int_{v=0}^{V} 1\,dv = \int_{t=0}^{T} \frac{u}{2T-t} - g\ dt$$
$$\Rightarrow \left[v\right]_0^V = \left[-u\ln|2T-t| - gt\right]_0^T$$
$$\Rightarrow V - 0 = \left[u\ln(2T-t) + gt\right]_T^0$$
$$\Rightarrow V = (u\ln 2T + 0) - (u\ln T + gT)$$
$$\Rightarrow V = u\ln 2T - u\ln T - gT$$
$$\Rightarrow V = u\ln\frac{2T}{T} - gT$$
$$\Rightarrow V = u\ln 2 - gT$$

Question 11 (****)

A hailstone whose shape remains spherical at all times is falling under gravity through a stationary cloud. It is further assumed that air resistance to the motion of the hailstone is negligible.

The mass of the hailstone increases, as it picks moisture from the still cloud, so that the radius r of the hailstone satisfies

$$\frac{dr}{dt} = kr,$$

where k is a positive constant.

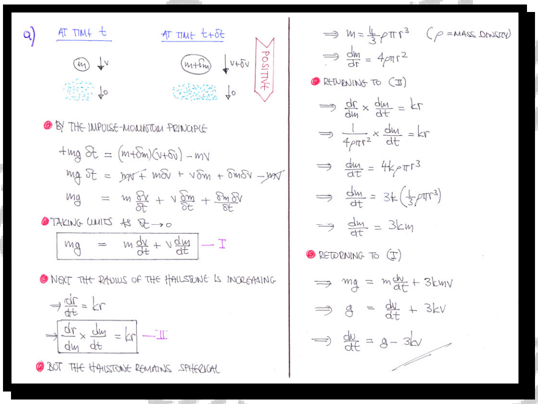
At time t, the speed of the hailstone is v.

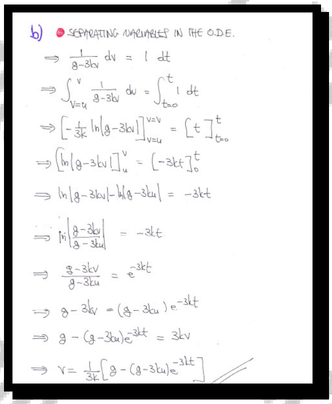
a) Use the momentum impulse principle to show that the acceleration of the falling hailstone is

$$g - 3kv.$$

b) Given further that when $t = 0$ the hailstone has speed u, find an expression for v in terms of g, k, u and t.

$$\boxed{v = \frac{1}{3k}\left[g - \left(g - 3ku\right)\right]e^{-3kt}}$$





Question 12 (****)

A particle P, whose initial mass is M, is projected vertically upwards from the ground at time $t = 0$ with speed $\frac{g}{k}$, where k is a positive constant.

As P moves upwards it gains mass by picking up small droplets of moisture from the atmosphere. The droplets are assumed to be at rest before they are picked up. It is further assumed that during the motion the acceleration due to gravity is constant.

At time t the speed of P is v and its mass is $M e^{kt}$.

Show that when the particle reaches its highest point its mass is $2M$

$\boxed{\text{proof}}$

AT TIME t — AT TIME $t+\delta t$

m ↑ v — $m+\delta m$ ↑ $v+\delta v$ — POSITIVE

- BY THE IMPULSE MOMENTUM PRINCIPLE

$$\Rightarrow -mg\,\delta t = (m+\delta m)(v+\delta v) - mv$$
$$\Rightarrow -mg\,\delta t = \cancel{mv} + m\delta v + v\delta m + \delta m\delta v - \cancel{mv}$$
$$\Rightarrow -mg = m\frac{\delta v}{\delta t} + v\frac{\delta m}{\delta t} + \frac{\delta m\delta v}{\delta t}$$

- TAKING LIMITS

$$\Rightarrow -mg = m\frac{dv}{dt} + v\frac{dm}{dt}$$
$$\Rightarrow -g = \frac{dv}{dt} + \frac{v}{m}\frac{dm}{dt}$$
$$\Rightarrow \frac{dv}{dt} = -\frac{v}{m}\frac{dm}{dt} - g$$
$$\Rightarrow \frac{dv}{dt} = -\frac{v}{m}(km) - g$$
$$\Rightarrow \boxed{\frac{dv}{dt} = -kv - g}$$

$$m = Me^{kt}$$
$$\frac{dm}{dt} = Mke^{kt}$$
$$\frac{dm}{dt} = km$$

- SEPARATE VARIABLES AND INTEGRATE SUBJECT TO $t=0$, $v = \frac{g}{k}$, LOOKING FOR THE TIME WHEN $v=0$

$$\Rightarrow \int_{v=\frac{g}{k}}^{0} \frac{1}{kv+g}\,dv = \int_{t=0}^{t} -1\,dt$$
$$\Rightarrow \left[\frac{1}{k}\ln|kv+g|\right]_{\frac{g}{k}}^{0} = \left[-t\right]_{0}^{t}$$
$$\Rightarrow \frac{1}{k}\ln g - \frac{1}{k}\ln 2g = -t$$
$$\Rightarrow \frac{1}{k}\ln\left(\frac{2g}{g}\right) = t$$
$$\Rightarrow t = \frac{1}{k}\ln 2$$

& SINCE $m = Me^{kt} = Me^{k\left(\frac{1}{k}\ln 2\right)} = Me^{\ln 2} = 2M$ AS REQUIRED

Question 13 (****)

A jet fuel propelled car is moving in a straight line on level horizontal ground.

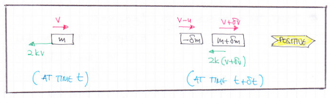
The car propels itself forward by ejecting burned fuel backwards at constant rate k, with speed u relative to the car, where k and u are positive constants.

At time t, the car experiences resistance to its motion of magnitude $2kv$, where v is the speed of the car at that instant.

At time $t = 0$, the car starts from rest with half its mass consisting of fuel.

Show that at the instant when all the fuel has been used up, $v = \frac{3}{8}u$.

proof



- BY THE IMPULSE MOMENTUM PRINCIPLE

$$\Rightarrow -2k(v+\delta v)\delta t = \left[(m+\delta m)(v+\delta v) - \delta m(v-u)\right] - mv$$

$$\Rightarrow -2kv\delta t - 2k\delta v\delta t = \cancel{mv} + m\delta v + \cancel{v\delta m} + \delta m\delta v - \cancel{v\delta m} + u\delta m - \cancel{mv}$$

$$\Rightarrow -2kv - 2k\delta v = m\frac{\delta v}{\delta t} + u\frac{\delta m}{\delta t} + \frac{\delta m\delta v}{\delta t}$$

- TAKING LIMITS GIVES

$$\boxed{-2kv = m\frac{dv}{dt} + u\frac{dm}{dt}}$$

- NOW IT IS GIVEN THAT $\frac{dm}{dt} = -k$ (CONSTANT), $t=0$ $m = M$ (SAY)

$\therefore$ BY INSPECTION OR SOLVING THE MINI P.D.E

$$\boxed{m = M - kt}$$

- THUS THE EQUATION OF MOTION GIVES

$$\Rightarrow -2kv = (M-kt)\frac{dv}{dt} + u(-k)$$

$$\Rightarrow uk - 2kv = (M-kt)\frac{dv}{dt}$$

$$\Rightarrow \boxed{\frac{dv}{dt} = \frac{k(u-2v)}{M-kt}}$$

- SOLVING THE O.D.E SUBJECT TO $t=0,\ v=0$

& REQUIRE v WHEN $m = \frac{1}{2}M$

ie $m = M - kt$

$\frac{1}{2}M = M - kt$

$kt = \frac{1}{2}M$

$t = \frac{M}{2k}$

- SEPARATING VARIABLES

$$\Rightarrow \int_{v=0}^{v} \frac{1}{u-2v}\,dv = \int_{t=0}^{t=M/2k} \frac{k}{M-kt}\,dt$$

$$\Rightarrow \left[-\frac{1}{2}\ln|u-2v|\right]_{v=0}^{v} = -\left[\ln|M-kt|\right]_{t=0}^{t=\frac{M}{2k}}$$

$$\Rightarrow -\frac{1}{2}\ln|u-2v| + \frac{1}{2}\ln u = -\ln\left|M - \frac{M}{2}\right| + \ln M$$

$$\Rightarrow \frac{1}{2}\ln\left|\frac{u}{u-2v}\right| = \ln\left|\frac{M}{M/2}\right|$$

$$\Rightarrow \frac{1}{2}\ln\left|\frac{u}{u-2v}\right| = \ln 2$$

$$\Rightarrow \ln\left|\frac{u}{u-2v}\right| = \ln 4$$

$$\Rightarrow \frac{u}{u-2v} = 4$$

$$\Rightarrow \frac{u}{4} = u - 2v$$

$$\Rightarrow 2v = \frac{3}{4}u$$

$$\Rightarrow v = \frac{3}{8}u$$

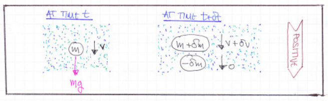
Question 14 (****)

A raindrop falls from rest at time $t = 0$, through still air. At time t the raindrop has speed v and mass $M\,\mathrm{e}^{kt}$, where M and k are positive constants.

The only force acting on the raindrop is its weight, $Mg\,\mathrm{e}^{kt}$, where g is the constant gravitational acceleration.

Determine the time it takes the raindrop, and the distance it covers, until the instant that its speed is half of its terminal speed.

$$\boxed{T = \frac{\ln 2}{k}}, \quad \boxed{d = \frac{g(\ln 4 - 1)}{2k^2}}$$



- BY THE IMPULSE MOMENTUM PRINCIPLE

$\Rightarrow mg\,\delta t = \left[(m+\delta m)(v+\delta v) - \delta m \times 0\right] - mv$

$\Rightarrow mg\,\delta t = \cancel{mv} + m\delta v + v\delta m + \delta m\delta v - \cancel{mv}$

$\Rightarrow mg = m\frac{\delta v}{\delta t} + v\frac{\delta m}{\delta t} + \frac{\delta m\,\delta v}{\delta t}$

- TAKING LIMITS TO OBTAIN THE EQUATION OF MOTION

$\Rightarrow mg = m\frac{dv}{dt} + v\frac{dm}{dt}$

$\Rightarrow mg - v\frac{dm}{dt} = m\frac{dv}{dt}$

$\Rightarrow \boxed{\frac{dv}{dt} = g - \frac{v}{m}\frac{dm}{dt}}$

- NOW $m = Me^{kt}$

$\frac{dm}{dt} = Mke^{kt}$

$\frac{dm}{dt} = km$

$\Rightarrow \frac{dv}{dt} = g - \frac{v}{m}(km)$

$\Rightarrow \boxed{\frac{dv}{dt} = g - kv}$

- TERMINAL SPEED BY INSPECTION $\Rightarrow \frac{dv}{dt} = 0$

$\Rightarrow g - kv = 0$

$\Rightarrow v = \frac{g}{k}$ ← TERMINAL SPEED

- INTEGRATING EACH O.D.E SUBJECT TO THE INITIAL CONDITIONS $t=0, v=0, x=0$ & REQUIRING TO FIND x & t WHEN $v = \frac{g}{2k}$

$\Rightarrow \frac{dv}{dt} = g - kv$

$\Rightarrow \int_0^{\frac{g}{2k}} \frac{1}{g-kv}\,dv = \int_0^t 1\,dt$

$\Rightarrow \int_0^{g/2k} \frac{-k}{g-kv}\,dv = \int_0^t -k\,dt$

$\Rightarrow \left[\ln(g-kv)\right]_0^{\frac{g}{2k}} = \left[-kt\right]_0^t$

$\Rightarrow \ln\left(g - \frac{g}{2}\right) - \ln g = -kt - 0$

$\Rightarrow \ln\left(\frac{1}{2}g\right) - \ln g = -kt$

$\Rightarrow \ln\left(\frac{\frac{1}{2}g}{g}\right) = -kt$

$\Rightarrow \ln\frac{1}{2} = -kt$

$\Rightarrow -\ln 2 = -kt$

$\Rightarrow t = \frac{1}{k}\ln 2$

$\Rightarrow \frac{dv}{dt} = g - kv$

$\Rightarrow v\frac{dv}{dx} = g - kv$

$\Rightarrow \frac{v}{g-kv}\,dv = 1\,dx$

$\Rightarrow \frac{-kv}{g-kv}\,dv = -k\,dx$

$\Rightarrow \frac{(g-kv)-g}{g-kv}\,dv = -k\,dx$

$\Rightarrow \left(1 - \frac{g}{g-kv}\right)dv = -k\,dx$

$\Rightarrow \int_{v=0}^{v=g/2k} 1 - \frac{g}{g-kv}\,dv = \int_{x=0}^{x} -k\,x$

$\Rightarrow \left[v + \frac{g}{k}\ln|g-kv|\right]_0^{\frac{g}{2k}} = \left[-kx\right]_0^x$

$\Rightarrow \left(\frac{g}{2k} + \frac{g}{k}\ln\left|g - \frac{1}{2}g\right|\right) - \left(\frac{g}{k}\ln g\right) = -kx$

$\Rightarrow \frac{g}{2k} + \frac{g}{k}\ln\left(\frac{\frac{1}{2}g}{g}\right) = -kx$

$\Rightarrow \frac{g}{2k} + \frac{g}{k}\ln\frac{1}{2} = -kx$

$\Rightarrow \frac{g}{2k}(1 - 2\ln 2) = -kx$

$\Rightarrow x = \frac{g}{2k^2}(2\ln 2 - 1)$

$\Rightarrow x = \frac{g}{2k^2}(\ln 4 - 1)$

Question 15 (****)

A rocket has initial mass $2M$, which includes the mass of the fuel for its flight, M.

At time $t = 0$ the rocket is at rest above the surface of the earth pointing vertically downwards when it begins to propel itself by ejecting mass backwards at constant rate $0.5M$, with speed u relative to the rocket.

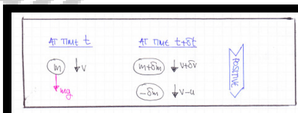
The rocket is modelled as a particle moving without air resistance.

The motion takes place close to the surface of the earth and it is assumed that g is the constant gravitational acceleration throughout the motion.

Determine, in terms of u and g the distance covered by the rocket by the time all its fuel has been used up.

You may assume that the rocket has not reached the Earth's surface by that instant.

$$\boxed{v = u\ln 2 - gT}$$



- IMPULSE = CHANGE IN MOMENTUM

$\Rightarrow mg\,\delta t = \left[(m+\delta m)(v+\delta v) - \delta m(v-u)\right] - mv$

$\Rightarrow mg\,\delta t = \cancel{mv} + m\delta v + \cancel{v\delta m} + \delta m\delta v - \cancel{v\delta m} + u\delta m - \cancel{mv}$

$\Rightarrow mg\,\delta t = m\delta v + u\delta m + \delta m\delta v$

$\Rightarrow mg = m\frac{\delta v}{\delta t} + u\frac{\delta m}{\delta t} + \frac{\delta m\delta v}{\delta t}$

- TAKING LIMITS WE OBTAIN

$\Rightarrow mg = m\frac{dv}{dt} + u\frac{dm}{dt}$

$\Rightarrow g = \frac{dv}{dt} + \frac{u}{m}\frac{dm}{dt}$

$\Rightarrow \boxed{\frac{dv}{dt} = g - \frac{u}{m}\frac{dm}{dt}}$

- NOW $\frac{dm}{dt} = -\frac{1}{2}M$, $t=0$ $m = 2M$

BY INSPECTION OR SOLVING AN ODE. $m = 2M - \frac{1}{2}Mt$

- THE EQUATION OF MOTION BECOMES

$\Rightarrow \frac{dv}{dt} = g - \frac{u}{2M - \frac{1}{2}Mt}\left(-\frac{1}{2}M\right)$

$\Rightarrow \frac{dv}{dt} = g + \frac{u}{1-\frac{1}{4}t}$

$\Rightarrow \frac{dv}{dt} = g + \frac{4u}{4-t}$

- INTEGRATING W.R.T t

$\Rightarrow v = gt - 4u\ln(4-t) + A$

- WHEN $t=0$ $v=0$

$0 = 0 - 4u\ln 4 + A$

$A = 8u\ln 2$

$\Rightarrow v = gt - 4u\ln(4-t) + 8u\ln 2$

$\Rightarrow \frac{dx}{dt} = gt - 4u\ln(4-t) + 8u\ln 2$

- INTEGRATE W.R.T t AGAIN

$\Rightarrow x = \frac{1}{2}gt^2 + 8ut\ln 2 - 4u\int \ln(4-t)\,dt$

$\Rightarrow x = \frac{1}{2}gt^2 + 8ut\ln 2 - 4u\int \ln z\,(-dz)$

SUBSTITUTION
$z = 4-t$
$t = 4-z$
$dt = -dz$

$\Rightarrow x = \frac{1}{2}gt^2 + 8ut\ln 2 + 4u\int \ln z\,dz$

$\Rightarrow x = \frac{1}{2}gt^2 + 8ut\ln 2 + 4u\left[z\ln z - z\right] + B$

$\Rightarrow x = \frac{1}{2}gt^2 + 8ut\ln 2 + 4u\left[(4-t)\ln(4-t) - (4-t)\right] + B$

$\Rightarrow x = \frac{1}{2}gt^2 + 8ut\ln 2 + 4u(4-t)\ln(4-t) + 4ut + B$

- APPLY CONDITIONS AGAIN

$t=0$ $x=0$

$0 = 16u\ln 4 + B$

$B = -32u\ln 2$

$\Rightarrow x = \frac{1}{2}gt^2 + 8ut\ln 2 + 4u(4-t)\ln(4-t) + 4ut - 32u\ln 2$

- NOW WHEN THE FUEL IS EXHAUSTED $m = M$

$m = 2M - \frac{1}{2}Mt$

$M = 2M - \frac{1}{2}Mt$

$1 = 2 - \frac{1}{2}t$

$2 = 4 - t$

$t = 2$

$\Rightarrow x = 2g + 16u\ln 2 + 8u\ln 2 + 8u - 32u\ln 2$

$\Rightarrow x = 2g + 8u - 8u\ln 2$

$\Rightarrow x = 2g + 8u(1-\ln 2)$

Question 16 (**+)**

A spacecraft is travelling in a straight line in deep space where all external forces can be assumed to be negligible.

The spacecraft decelerates by ejecting fuel at a constant speed u relative to the spacecraft, and in the **direction of motion** of the spacecraft.

At time t, the spacecraft has speed v and mass m.

At time $t = 0$, the spacecraft has speed U and mass m_0.

a) Show clearly, by the momentum impulse principle, that while the spacecraft is ejecting fuel,

$$m\frac{dv}{dt} - u\frac{dm}{dt} = 0 .$$

b) Find an expression for the mass of the spacecraft, in terms of m_0, u and U, when it comes to rest.

The spacecraft comes to rest when $t = T$.

c) Given further that $m = m_0 e^{-\sqrt{kt}}$, where k is a positive constant, show that the distance covered by the spacecraft in decelerating from U to rest is $\frac{1}{3}UT$.

proof

a) AT TIME t AT TIME $t+\delta t$

$\xrightarrow{v}$ $\boxed{m}$ $\xrightarrow{v+\delta v}$ $\boxed{m+\delta m}$ $\xrightarrow{u+v}$ $\boxed{-\delta m}$

POSITIVE

BY THE IMPULSE MOMENTUM PRINCIPLE

$0 = \left[(m+\delta m)(v+\delta v) + (-\delta m)(u+v)\right] - \left[mv\right]$

NO EXTERNAL FORCES

$0 = \cancel{mv} + m\delta v + \cancel{v\delta m} + \delta m\delta v - u\delta m - \cancel{v\delta m} - \cancel{mv}$

$\Rightarrow 0 = m\delta v - u\delta m + \delta m\delta v$

$\Rightarrow 0 = \frac{m\delta v}{\delta t} - u\frac{\delta m}{\delta t} + \frac{\delta m\delta v}{\delta t}$

$\Rightarrow 0 = m\frac{dv}{dt} - u\frac{dm}{dt}$ AS REQUIRED

b) $m\frac{dv}{dt} - u\frac{dm}{dt} = 0$

$\Rightarrow m\frac{dv}{dt}\cdot\frac{dt}{dm} - u\frac{dm}{dt}\frac{dt}{dm} = 0\cdot\frac{dt}{dm}$

$\Rightarrow m\frac{dv}{dm} = u$

$\Rightarrow 1\,dv = \frac{u}{m}\,dm$

$\Rightarrow \int_{U}^{0} 1\,dv = \int_{m_0}^{m} \frac{u}{m}\,dm$

OR $\frac{\frac{dv}{dt}}{\frac{dm}{dt}} = \frac{u}{m}$

OR BACK TO THE BEGINNING, DIVIDE BY δm INSTEAD OF δt

$\Rightarrow \left[v\right]_{U}^{0} = \left[u\ln m\right]_{m_0}^{m}$

$\Rightarrow -U = u\ln m - u\ln m_0$

$\Rightarrow -U = u\ln\frac{m}{m_0}$

$\Rightarrow -\frac{U}{u} = \ln\frac{m}{m_0}$

$\Rightarrow \frac{m}{m_0} = e^{-\frac{U}{u}}$

$\Rightarrow m = m_0 e^{-\frac{U}{u}}$

c) NOW

$m\frac{dv}{dt} - u\frac{dm}{dt} = 0$

$\Rightarrow \frac{dv}{dt} = \frac{u}{m}\frac{dm}{dt}$

$\Rightarrow \frac{dv}{dt} = \frac{u}{\cancel{m}}\times\left(-\frac{1}{2}\sqrt{k}\,t^{-\frac{1}{2}}\cancel{m}\right)$

$m = m_0 e^{-\sqrt{kt}}$

$m = m_0 e^{-\sqrt{k}\times t^{\frac{1}{2}}}$

$\frac{dm}{dt} = -\frac{1}{2}\sqrt{k}\,m_0 t^{-\frac{1}{2}} e^{-\sqrt{kt}}$

$\frac{dm}{dt} = -\frac{1}{2}\sqrt{k}\,t^{-\frac{1}{2}} m$

$\Rightarrow \frac{dv}{dt} = -\frac{u\sqrt{k}}{2}t^{-\frac{1}{2}}$

$\Rightarrow \int_{v=U}^{v} 1\,dv = \int_{t=0}^{t} -\frac{u\sqrt{k}}{2}t^{-\frac{1}{2}}$

$\Rightarrow \left[v\right]_{U}^{v} = \left[-u\sqrt{k}\,t^{\frac{1}{2}}\right]_{0}^{t}$

$\Rightarrow v - U = -u\sqrt{k}\,t^{\frac{1}{2}}$

$\Rightarrow \boxed{v = U - u\sqrt{k}\,t^{\frac{1}{2}}}$

INTEGRATE AGAIN

$\Rightarrow \frac{dx}{dt} = U - u\sqrt{k}\,t^{\frac{1}{2}}$

$\Rightarrow \int_{x_1}^{x_2} 1\,dx = \int_{t=0}^{T} U - u\sqrt{k}\,t^{\frac{1}{2}}\,dt$

$\Rightarrow \left[x\right]_{x_1}^{x_2} = \left[Ut - \frac{2}{3}u\sqrt{k}\,t^{\frac{3}{2}}\right]_{0}^{T}$

$\Rightarrow x_2 - x_1 = UT - \frac{2}{3}u\sqrt{k}\,T^{\frac{3}{2}}$

BUT $v = U - u\sqrt{k}\,t^{\frac{1}{2}}$

when $t = T$ $v = 0$

$0 = U - u\sqrt{k}\,T^{\frac{1}{2}}$

$\boxed{u\sqrt{k} = \frac{U}{T^{1/2}}}$

THUS

$x_2 - x_1 = UT - \frac{2}{3}\left(\frac{U}{T^{1/2}}\right)T^{\frac{3}{2}}$

$x_2 - x_1 = UT - \frac{2}{3}UT$

$x_2 - x_1 = \frac{1}{3}UT$

Question 17 (**+)**

A spacecraft is moving in deep space. At time $t=0$ the mass of the spacecraft is at rest and its mass is M. At that instant the engines of the spacecraft are fired in a direction opposite to that of the motion of the spacecraft. Fuel is ejected at a constant mass rate k with speed U relative to the spacecraft.

At time t, the mass of the spacecraft is m, its speed is v and its displacement is x.

a) Show clearly that …

i. … $v = U\ln\left(\dfrac{M}{M-kt}\right)$.

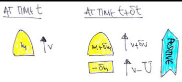
ii. … $x = \dfrac{UM}{k}\left[\dfrac{M-kt}{M}\ln\left(\dfrac{M-kt}{M}\right) - \dfrac{M-kt}{M} + 1\right]$.

The spacecraft needs to cover a **total** distance of $\dfrac{UM}{2k}$ and stops firing its engines when $m = \tfrac{1}{2}M$.

b) Determine the **total** time taken by the spacecraft to cover the distance of $\dfrac{UM}{2k}$.

$$\boxed{t = \frac{M}{k}}$$

(a)(I) AT TIME t AT TIME $t+\delta t$



BY THE IMPULSE MOMENTUM PRINCIPLE

DEEP SPACE $0\times\delta t = \left[(m+\delta m)(v+\delta v) - \delta m(v-U)\right] - mv$

$\Rightarrow 0 = \cancel{mv} + m\delta v + \cancel{v\delta m} + \delta m\delta v + U\delta m - \cancel{v\delta m} - \cancel{mv}$

$\Rightarrow 0 = m\delta v + U\delta m + \delta m\delta v$

$\Rightarrow 0 = m\dfrac{\delta v}{\delta t} + U\dfrac{\delta m}{\delta t} + \dfrac{\delta m\,\delta v}{\delta t}$

TAKING LIMITS

$\Rightarrow \boxed{0 = m\dfrac{dv}{dt} + U\dfrac{dm}{dt}}$

BUT $\boxed{\dfrac{dm}{dt} = -k} \Rightarrow \boxed{m = M-kt}$

$\Rightarrow 0 = (M-kt)\dfrac{dv}{dt} + U(-k)$

$\Rightarrow (M-kt)\dfrac{dv}{dt} = Uk$

$\Rightarrow 1\,dv = \dfrac{Uk}{M-kt}\,dt$

$\Rightarrow \displaystyle\int_{v=0}^{v} 1\,dv = \int_{t=0}^{t}\frac{Uk}{M-kt}\,dt$

$\Rightarrow [v]_0^v = \left[-U\ln(M-kt)\right]_0^t$

$\Rightarrow v = -U\ln(M-kt) + U\ln M$

$\Rightarrow v = U\ln\left(\dfrac{M}{M-kt}\right)$ // AS REQUIRED

(II) $\dfrac{dx}{dt} = -U\ln\left(\dfrac{M-kt}{M}\right)$

$\Rightarrow \displaystyle\int_{x=0}^{x} 1\,dx = \int_{t=0}^{t} -U\ln\left(\frac{M-kt}{M}\right)dt$

BY SUBSTITUTION

$T = \dfrac{M-kt}{M}$ when $t=0$, $T=1$

$dT = -\dfrac{k}{M}dt$ when $t=t$, $T = \dfrac{M-kt}{M}$

$dt = -\dfrac{M}{k}dT$

$\Rightarrow \displaystyle\int_0^x 1\,dx = \int_1^{\frac{M-kT}{M}} -U\ln T\left(-\frac{M}{k}\,dT\right)$

$\Rightarrow \displaystyle\int_0^x 1\,dx = \frac{UM}{k}\int_1^{\frac{M-kT}{M}} \ln T\,dT$

$\Rightarrow [x]_0^x = \dfrac{UM}{k}\left[T\ln T - T\right]_1^{\frac{M-kT}{M}}$

$\Rightarrow x = \dfrac{UM}{k}\left[\left[\dfrac{M-kT}{M}\ln\left(\dfrac{M-kt}{M}\right) - \dfrac{M-kT}{M}\right] - [0-1]\right]$

$\Rightarrow x = \dfrac{UM}{k}\left[\dfrac{M-kT}{M}\ln\left(\dfrac{M-kt}{M}\right) - \dfrac{M-kT}{M} + 1\right]$ // AS REQUIRED

(b) WHEN $m = \frac{1}{2}M$

- $m = M-kt$

  $\frac{1}{2}M = M-kt$

  $kt = \frac{1}{2}M$

  $\boxed{t = \dfrac{M}{2k}}$

- $v = U\ln\left(\dfrac{M}{M-kt}\right)$

  $v = U\ln\left(\dfrac{M}{M-k\times\frac{M}{2k}}\right)$

  $v = U\ln\left(\dfrac{M}{M-\frac{M}{2}}\right)$

  $\boxed{v = U\ln 2}$

- $x = \dfrac{UM}{k}\left[\dfrac{M-kt}{M}\ln\left(\dfrac{M-kt}{M}\right) - \dfrac{M-kt}{M} + 1\right]$

  $x = \dfrac{UM}{k}\left[\frac{1}{2}\ln\frac{1}{2} - \frac{1}{2} + 1\right]$

  $x = \dfrac{UM}{k}\left[\frac{1}{2} - \frac{1}{2}\ln 2\right]$

  $\boxed{x = \dfrac{UM}{2k}[1-\ln 2]}$

- EXTRA DISTANCE TO COVER $= \dfrac{UM}{2k} - \dfrac{UM}{2k}(1-\ln 2) = \dfrac{UM}{2k}\ln 2$
- ONCE ENGINES STOPPED, CRAFT IS MOVING WITH CONSTANT SPEED $U\ln 2$
- EXTRA TIME $= \dfrac{\frac{\cancel{U}M}{2k}\cancel{\ln 2}}{\cancel{U}\cancel{\ln 2}} = \dfrac{M}{2k}$
- TOTAL TIME $= \underbrace{\dfrac{M}{2k}}_{\text{TIME TO BURN FUEL FROM } M \text{ to } \frac{1}{2}M} + \underbrace{\dfrac{M}{2k}}_{\text{EXTRA TIME}} = \dfrac{M}{k}$ //

Question 18 (**+)**

A small motorboat, of mass M, is travelling in a straight line across still water with constant speed U. The boat's engine provides a constant driving force and the resistance to motion is $2v$, where v is the speed of the boat at any given time.

At time $t = 0$, a leak develops and water starts flooding the interior of the boat whose mass increases at constant rate k. The boat's engine provides the same constant driving force and the resistance to motion remains unchanged.

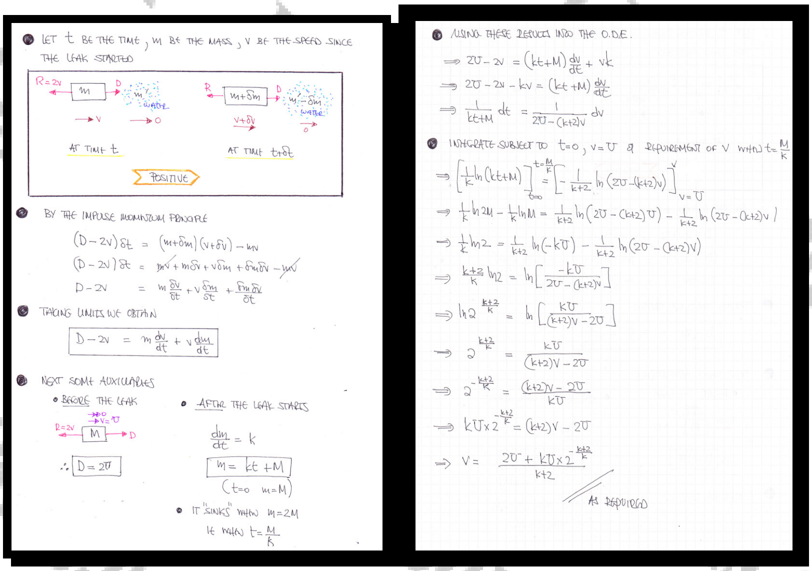
The boat sinks when its mass of the water in boat equals the mass of the boat.

Show that the speed of the boat the instant it sinks is

$$\frac{2U + kU \times 2^{-\frac{k+2}{k}}}{k+2}.$$

You may assume this speed is greater than $\dfrac{2U}{k+2}$.

proof



LET t BE THE TIME, m BE THE MASS, v BE THE SPEED SINCE THE LEAK STARTED

BY THE IMPULSE MOMENTUM PRINCIPLE

$$(D - 2v)\delta t = (m + \delta m)(v + \delta v) - mv$$
$$(D - 2v)\delta t = \cancel{mv} + m\delta v + v\delta m + \delta m \delta v - \cancel{mv}$$
$$D - 2v = m\frac{\delta v}{\delta t} + v\frac{\delta m}{\delta t} + \frac{\delta m \delta v}{\delta t}$$

TAKING LIMITS WE OBTAIN

$$\boxed{D - 2v = m\frac{dv}{dt} + v\frac{dm}{dt}}$$

NEXT SOME AUXILIARIES

- BEFORE THE LEAK

$$\therefore \boxed{D = 2U}$$

- AFTER THE LEAK STARTS

$$\frac{dm}{dt} = k$$
$$\boxed{m = kt + M}$$
$$(t = 0 \quad m = M)$$

- IT "SINKS" WHEN $m = 2M$

IE WHEN $t = \frac{M}{k}$

USING THESE RESULTS INTO THE O.D.E.

$$\Rightarrow 2U - 2v = (kt + M)\frac{dv}{dt} + vk$$
$$\Rightarrow 2U - 2v - kv = (kt + M)\frac{dv}{dt}$$
$$\Rightarrow \frac{1}{kt + M}\,dt = \frac{1}{2U - (k+2)v}\,dv$$

INTEGRATE SUBJECT TO $t = 0$, $v = U$ & REQUIREMENT OF V WHEN $t = \frac{M}{k}$

$$\Rightarrow \left[\frac{1}{k}\ln(kt + M)\right]_{t=0}^{t=\frac{M}{k}} = \left[-\frac{1}{k+2}\ln\left(2U - (k+2)v\right)\right]_{v=U}^{V}$$
$$\Rightarrow \frac{1}{k}\ln 2M - \frac{1}{k}\ln M = \frac{1}{k+2}\ln\left(2U - (k+2)U\right) - \frac{1}{k+2}\ln\left(2U - (k+2)V\right)$$
$$\Rightarrow \frac{1}{k}\ln 2 = \frac{1}{k+2}\ln(-kU) - \frac{1}{k+2}\ln\left(2U - (k+2)V\right)$$
$$\Rightarrow \frac{k+2}{k}\ln 2 = \ln\left[\frac{-kU}{2U - (k+2)V}\right]$$
$$\Rightarrow \ln 2^{\frac{k+2}{k}} = \ln\left[\frac{kU}{(k+2)V - 2U}\right]$$
$$\Rightarrow 2^{\frac{k+2}{k}} = \frac{kU}{(k+2)V - 2U}$$
$$\Rightarrow 2^{-\frac{k+2}{k}} = \frac{(k+2)V - 2U}{kU}$$
$$\Rightarrow kU \times 2^{-\frac{k+2}{k}} = (k+2)V - 2U$$
$$\Rightarrow V = \frac{2U + kU \times 2^{-\frac{k+2}{k}}}{k+2}$$

AS REQUIRED

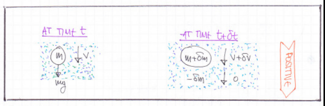
Question 19 (****+)

A spherical raindrop of radius a falls from rest. The radius of the raindrop increases at constant rate k, $k > 0$, as it picks moisture from the stationary cloud.

The shape of the raindrop remains spherical at all times as is falling under gravity and it is assumed that air resistance to the motion of the raindrop is negligible.

Determine a simplified expression for the distance fallen by the raindrop in time t, in terms of k, a, g and t.

$$\boxed{x = \frac{g}{4k}\left[\frac{1}{2}kt^2 + at + \frac{a^4}{2k(kt+a)^2} - \frac{a^2}{2k}\right]}$$



- IMPULSE = CHANGE IN MOMENTUM

$$\Rightarrow mg\,\delta t = \left[(m+\delta m)(v+\delta v) - \delta m \times 0\right] - mv$$
$$\Rightarrow mg\,\delta t = \cancel{mv} + m\delta v + v\delta m + \delta m\delta v - \cancel{mv}$$
$$\Rightarrow mg = m\frac{\delta v}{\delta t} + v\frac{\delta m}{\delta t} + \frac{\delta m\delta v}{\delta t}$$

- TAKING LIMITS WE OBTAIN

$$\Rightarrow m\frac{dv}{dt} + v\frac{dm}{dt} = mg$$
$$\Rightarrow \frac{dv}{dt} + \frac{v}{m}\frac{dm}{dt} = g$$
$$\Rightarrow \boxed{\frac{dv}{dt} = g - \frac{v}{m}\frac{dm}{dt}}$$

- NOW THE RADIUS INCREASES AT CONSTANT RATE & SHAPE IS SPHERICAL

$$\therefore \frac{dr}{dt} = k \qquad \& \qquad m = \frac{4}{3}\pi\rho r^3$$

(WHERE ρ = MASS DENSITY)

$$r = kt + a$$

WHERE a IS THE INITIAL RADIUS

$$m = \frac{4}{3}\pi\rho(kt+a)^3$$

$$\& \quad \frac{dm}{dt} = 4\pi\rho(kt+a)^2 \times k$$

- HENCE RETURNING TO THE O.D.E WE OBTAIN

$$\Rightarrow \frac{dv}{dt} = g - \frac{v}{\frac{4}{3}\cancel{\pi\rho}(kt+a)^3} \times \cancel{4\pi\rho}k(kt+a)^2$$
$$\Rightarrow \frac{dv}{dt} = g - \frac{3vk}{kt+a}$$
$$\Rightarrow \frac{dv}{dt} + \left(\frac{3k}{kt+a}\right)v = g$$

- SOLVING THE O.D.E — INTEGRATING FACTOR

$$e^{\int \frac{3k}{kt+a}\,dt} = e^{3\ln|kt+a|} = (kt+a)^3$$

THUS WE HAVE

$$\Rightarrow \frac{d}{dt}\left[v(kt+a)^3\right] = g(kt+a)^3$$
$$\Rightarrow v(kt+a)^3 = \int g(kt+a)^3\,dt$$

INTEGRATE SUBJECT TO THE CONDITION $t=0$, $v=0$

$$\Rightarrow \left[v(kt+a)^3\right]_{(0,0)}^{(v,t)} = \left[\frac{g}{4k}(kt+a)^4\right]_0^t$$
$$\Rightarrow v(kt+a)^3 - 0 = \frac{g}{4k}(kt+a)^4 - \frac{g}{4k}a^4$$
$$\Rightarrow v(kt+a)^3 = \frac{g}{4k}\left[(kt+a)^4 - a^4\right]$$
$$\Rightarrow v = \frac{g}{4k}\left[kt+a - \frac{a^4}{(kt+a)^3}\right]$$
$$\Rightarrow \frac{dx}{dt} = \frac{g}{4k}\left[kt+a - \frac{a^4}{(kt+a)^3}\right]$$

- INTEGRATE AGAIN SUBJECT TO $t=0$, $x=0$

$$\Rightarrow \int_0^x 1\,dx = \frac{g}{4k}\int_0^t kt + a - a^4(kt+a)^{-3}\,dt$$
$$\Rightarrow [x]_0^x = \frac{g}{4k}\left[\frac{1}{2}kt^2 + at + \frac{a^4}{2k}(kt+a)^{-2}\right]_0^t$$
$$\Rightarrow x = \frac{g}{4k}\left[\frac{1}{2}kt^2 + at + \frac{a^4}{2k}(kt+a)^{-2} - \frac{a^2}{2k}\right]$$

Question 20 (****+)

A raindrop, whose shape remains spherical at all times, absorbs water as it falls vertically under gravity through a stationary cloud.

The raindrop is initially at rest and its radius is a.

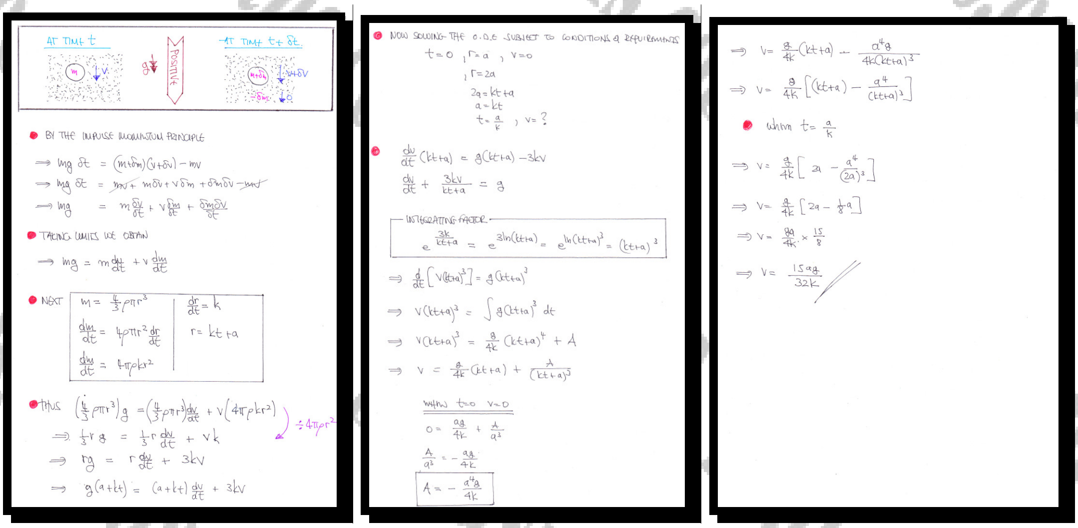
The radius of the raindrop increases at a constant rate k.

At time t the speed of the raindrop is v.

Find, in terms of a, g an k, the speed of the raindrop when its radius is $2a$.

You may assume that the only force acting on the raindrop is its weight.

$$\boxed{v = \frac{15ag}{3k}}$$



Question 21 (****+)

A raindrop absorbs water as it falls vertically under gravity through a cloud. In this model the cloud is assumed to consist of stationary water particles. You may assume that the only force acting on the raindrop is its weight.

The mass of the raindrop increases at the constant rate of 0.01 g s^{-1}.

At time t, the mass of the raindrop is m and its speed is v.

The raindrop starts from rest at $t = 0$, and its mass at that instant is 0.05 g.

Determine the speed of the raindrop when its mass reaches twice its initial mass.

$$\boxed{v = \frac{15}{4}g = 36.75 \text{ ms}^{-1}}$$

AT TIME t — AT TIME $t + \delta t$

(m) ↓v — (m+δm) ↓v+δv

M ↓0 — M−δm ↓0

POSITIVE

BY THE IMPULSE-MOMENTUM PRINCIPLE

$\Rightarrow mg\,\delta t = (m+\delta m)(v+\delta v) - mv$

$\Rightarrow mg\,\delta t = \cancel{mv} + m\delta v + v\delta m + \delta m\delta v - \cancel{mv}$

$\Rightarrow mg = m\frac{\delta v}{\delta t} + v\frac{\delta m}{\delta t} + \frac{\delta m \delta v}{\delta t}$

TAKING LIMITS

$\Rightarrow mg = m\frac{dv}{dt} + v\frac{dm}{dt}$

NOW WE ARE GIVEN

$\frac{dm}{dt} = 0.01 \text{ g/s} = 0.00001 \text{ kg s}^{-1}$

$m = 0.00001t + 0.00005$

USING $t=0$, $m = 0.05\text{g}$

COMBINING INTO THE EQUATION OF MOTION

$\Rightarrow g = \frac{dv}{dt} + \frac{v}{m}\frac{dm}{dt}$

$\Rightarrow \frac{dv}{dt} = g - \frac{v}{m}\frac{dm}{dt}$

$\Rightarrow \frac{dv}{dt} = g - \frac{v}{\frac{1}{100000}t + \frac{5}{100000}} \times \frac{1}{100{,}000}$

$\Rightarrow \frac{dv}{dt} = g - \frac{v}{t+5}$

$\Rightarrow \boxed{\frac{dv}{dt} + \frac{v}{t+5} = g}$

SOLVING THE O.D.E SUBJECT TO THE CONDITION $t=0$, $v=0$;
WE REQUIRE THE TIME WHEN THE MASS DOUBLE

$m = 0.00001t + 0.00005$

$0.00010 = 0.00001t + 0.00005$

$10 = t + 5$

$\boxed{t = 5}$

WE REQUIRE THE SPEED WHEN THE MASS DOUBLES

INTEGRATING FACTOR

$e^{\int \frac{1}{t+5}\,dt} = e^{\ln(t+5)} = t+5$

HENCE WE OBTAIN

$\Rightarrow \frac{d}{dt}\left(v(t+5)\right) = g(t+5)$

$\Rightarrow v(t+5) = \int g(t+5)\,dt$

$\Rightarrow v(t+5) = \frac{1}{2}g(t+5)^2 + A$

$\Rightarrow v = \frac{A}{t+5} + \frac{1}{2}g(t+5)$

WHEN $t=0$, $v=0$ $\Rightarrow 0 = \frac{A}{5} + \frac{5}{2}g$

$\Rightarrow 0 = 2A + 25g$

$\Rightarrow A = -\frac{25}{2}g$

$\Rightarrow \boxed{v = \frac{1}{2}g(t+5) - \frac{25}{2}g\frac{1}{t+5}}$

when $t=5$

$\Rightarrow v = 5g - \frac{25}{2}g \times \frac{1}{10}$

$\Rightarrow v = 5g - \frac{5}{4}g$

$\Rightarrow v = \frac{15}{4}g$ // 36.75 ms^{-1}

Question 22 (****+)

A scientist is about to conduct an experiment with a rocket. His rocket will have an initial mass 784 kg, of which 90% is the fuel for its flight. It will be initially at rest on the surface of the earth pointing vertically upwards.

The rocket will begin to propel itself upwards by ejecting mass backwards at constant rate 17.64 kg s^{-1}, with speed 175 ms^{-1} relative to the rocket.

The rocket will be modelled as a particle moving without air resistance. The motion is assumed to take place close to the surface of the earth so that g, the gravitational acceleration, will be constant throughout the motion.

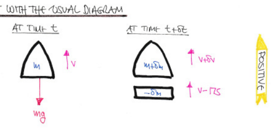
a) Calculate, correct to 2 decimal places, the speed of the rocket at the instant the fuel runs out.

b) Show that the displacement of the rocket at the instant the fuel runs out is negative.

c) Explain the flaw in the scientist's experiment.

$\boxed{}$, $\boxed{v \approx 10.95 \text{ ms}^{-1}}$

START WITH THE USUAL DIAGRAM



BY THE IMPULSE/MOMENTUM PRINCIPLE

$$-mg \times \delta t = \left[(m+\delta m)(v+\delta v) - \delta m(v-175)\right] - mv$$

$$-mg\delta t = \cancel{mv} + m\delta v + \cancel{v\delta m} + \delta m\delta v - \cancel{v\delta m} + 175\delta m - \cancel{mv}$$

$$-mg = m\frac{\delta v}{\delta t} + \frac{\delta m\delta v}{\delta t} + 175\frac{\delta m}{\delta t}$$

TAKING LIMITS YIELDS THE EQUATION OF MOTION

$$-mg = m\frac{dv}{dt} + 175\frac{dm}{dt}$$

$$\frac{dv}{dt} = -g - \frac{175}{m}\frac{dm}{dt}$$

NOW FUEL IS EJECTED AT CONSTANT RATE OF 17.64 kg/s, WITH AN INITIAL MASS OF 784 kg

I.E $\frac{dm}{dt} = -17.64$ OR $m = 784 - 17.64t$

COMBINING RESULTS

$$\Rightarrow \frac{dv}{dt} = -g - \frac{175}{784-17.64t}(-17.64)$$

$$\Rightarrow \frac{dv}{dt} = -g + \frac{3087}{784-17.64t}$$

INTEGRATING SUBJECT TO THE CONDITION $t=0, v=0$

$$\Rightarrow \int_{v=0}^{v} 1 \, dv = \int_{t=0}^{t} -g + \frac{3087}{784-17.64t} \, dt$$

$$\Rightarrow \left[v\right]_0^v = \left[-gt - \frac{3087}{17.64}\ln|784-17.64t|\right]_0^t$$

$$\Rightarrow v = \left[gt + 175\ln|784-17.64t|\right]_t^0$$

$$\Rightarrow v = (0 + 175\ln 784) - (gt + 175\ln|784-17.64t|)$$

$$\Rightarrow v = 175\ln 784 - gt - 175\ln|784-17.64t|$$

$$\Rightarrow v = 175\ln\left|\frac{784}{784-17.64t}\right| - gt$$

NOW ROCKET FUEL IS 90% OF THE INITIAL MASS – SO FUEL RUNS OUT WHEN $m = 78.4$ ($\frac{1}{10}$ OF 784)

$$m = 784 - 17.64t$$
$$78.4 = 784 - 17.64t$$
$$17.64t = 705.6$$
$$t = 40$$

HENCE WE HAVE

$$V = 175\ln\left|\frac{784}{784-17.64\times 40}\right| - (9.8\times 40)$$

$$V = 175\ln 10 - 392$$

$$V \approx 10.95 \text{ ms}^{-1}$$

b) REWRITING THE VELOCITY EXPRESSION AS FOLLOWS

$$\Rightarrow \frac{dx}{dt} = 175\ln 784 - gt - 175\ln|784-17.64t|$$

$$\Rightarrow \int_0^x 1 \, dx = \int_0^{40} \ln 784 - gt \, dt - 175\int_0^{40} \ln|784-17.64t| \, dt$$

BY SUBSTITUTION

$u = 784 - 17.64t$

$\frac{du}{dt} = -17.64$

$dt = \frac{du}{-17.64}$

$t=0 \mapsto u = 784$

$t=40 \mapsto u = 78.4$

$$\Rightarrow \left[x\right]_0^x = \left[(175\ln 784)t - \tfrac{1}{2}gt^2\right]_0^{40} - 175\int_{784}^{78.4} \ln u \, \frac{du}{-17.64}$$

$$\Rightarrow x = 7000\ln 784 - 7840 - 175 \times \frac{1}{17.64}\int_{78.4}^{784} \ln u \, du$$

$$\Rightarrow x = 7000\ln 784 - 7840 - \frac{625}{63}\left[u\ln u - u\right]_{78.4}^{784}$$

$$x = 7000\ln 784 - 7840 - \frac{625}{63}\left[(784\ln 784 - 784) - (78.4\ln 78.4 - 78.4)\right]$$

$$x = 46650.863\ldots - 7840 - 41441.763\ldots$$

$$x = -2630.899\ldots$$

c) LOOKING AT THE EQUATION OF MOTION

$$\frac{dv}{dt} = -g + \frac{3087}{784-17.64t}$$

FOR LIFT OFF THIS TERM MUST BE GREATER THAN g WHEN $t=0$

HERE IT IS ONLY 3.9375

SO THE ROCKET NEVER LEAVES THE GROUND

Question 23 (*****)

A light container C is connected to small block B of mass 5 kg by a light inextensible string. The string passes over a light smooth pulley P, which is located at the end of a rough horizontal house roof.

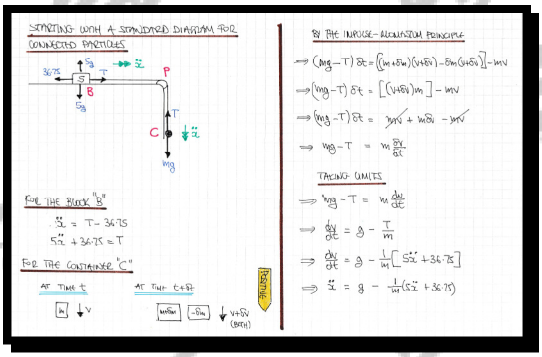
The container is initially empty hanging vertically at the end of the roof, as shown in the figure above. The string, B, P and C lie in a vertical plane at right angles to the end of the straight roof. With the block held at rest and the string taut, the container is then filled with 5 kg of water and the system is released from rest.

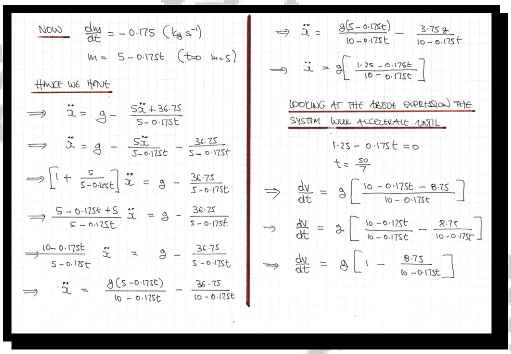
The system begins to move with water is leaking from several small holes just above the base of the container at the constant rate of 0.175 kg s^{-1}.

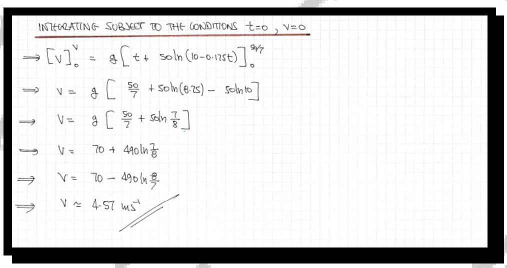
It is assumed that water is leaking in a **horizontal direction only** and the motion of the container is **vertical** at all times.

Given further that B is subject to a constant ground friction of 36.75 N, calculate the greatest speed achieved by the system.

M4 , $\boxed{v \approx 4.57 \text{ ms}^{-1}}$







Question 24 (*****)

A raindrop absorbs water as it falls vertically under gravity, through a stationary cloud. The mass m of the raindrop, increases at a rate which is directly proportional to its speed, v. The raindrop starts from rest and its mass at that instant is M.

At time t, the raindrop has fallen though a vertical distance x and its speed at that instant is v.

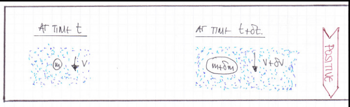
Show that

$$v^2 = \frac{2g}{3k}\left[M + kx - \frac{M}{(M+kx)^2}\right],$$

where k is a positive constant.

You may assume that the only force acting on the raindrop is its weight.

proof



BY THE IMPULSE MOMENTUM PRINCIPLE

$$\Rightarrow mg\,\delta t = (m+\delta m)(v+\delta v) - mv$$
$$\Rightarrow mg\,\delta t = \cancel{mv} + m\delta v + v\delta m + \delta m\delta v \cancel{- mv}$$
$$\Rightarrow \frac{mg\,\delta t}{\delta t} = m\frac{\delta v}{\delta t} + v\frac{\delta m}{\delta t} + \frac{\delta m\,\delta v}{\delta t}$$

TAKING LIMITS

$$\Rightarrow \boxed{mg = m\frac{dv}{dt} + v\frac{dm}{dt}}$$

NOW $\frac{dm}{dt} = kv$ (GIVEN)

$$\frac{dm}{dx}\frac{dx}{dt} = kv$$

$$\Rightarrow mg = m\frac{dv}{dt} + v\left(\frac{dm}{dx}\frac{dx}{dt}\right)$$
$$\Rightarrow mg = m\frac{dv}{dt} + v\frac{dm}{dx}v$$
$$\Rightarrow mg = m\frac{dv}{dt} + v^2\frac{dm}{dx}$$
$$\Rightarrow g = \frac{dv}{dt} + \frac{v^2}{m}\frac{dm}{dx}$$
$$\Rightarrow \frac{dv}{dt} = g - \frac{v^2}{m}\frac{dm}{dx}$$
$$\Rightarrow \boxed{v\frac{dv}{dx} = g - \frac{v^2}{m}\frac{dm}{dx}}$$

AS THE O.D.E HAS 3 VARIABLES — ELIMINATE m AS FOLLOWS

$$\Rightarrow \frac{dm}{dx}\frac{dx}{dt} = kv$$
$$\Rightarrow \frac{dm}{dx}v = kv$$
$$\Rightarrow \frac{dm}{dx} = k$$
$$\Rightarrow dm = k\,dx$$
$$\Rightarrow \left[m\right]_M^m = \left[kx\right]_{x=0}^x$$
$$\Rightarrow m - M = kx$$
$$\Rightarrow \underline{\underline{m = M + kx}}$$

RETURNING TO THE O.D.E

$$\Rightarrow v\frac{dv}{dx} = g - \frac{v^2}{m}\frac{dm}{dx}$$
$$\Rightarrow v\frac{dv}{dx} = g - \frac{v^2}{M+kx}(k)$$
$$\Rightarrow v\frac{dv}{dx} = g - \frac{kv^2}{M+kx}$$
$$\Rightarrow 2v\frac{dv}{dx} = 2g - \frac{2kv^2}{M+kx}$$
$$\Rightarrow \frac{d}{dx}\left[v^2\right] = 2g - \frac{2kv^2}{M+kx}$$
$$\Rightarrow \boxed{\frac{d(v^2)}{dx} + \frac{2k}{M+kx}v^2 = 2g}$$

REWRITE v^2 AS y FOR CONVENIENCE

$$\frac{dy}{dx} + \frac{2k}{M+kx}y = 2g$$

INTEGRATING FACTOR $e^{\int \frac{2k}{M+kx}\,dx} = e^{2\ln(M+kx)} = e^{\ln(M+kx)^2}$

$$= (M+kx)^2$$

$$\Rightarrow \frac{d}{dx}\left(y(M+kx)^2\right) = 2g(M+kx)^2$$
$$\Rightarrow y(M+kx)^2 = \int 2g(M+kx)^2\,dx$$
$$\Rightarrow y(M+kx)^2 = \frac{2}{3}g \times \frac{1}{k}(M+kx)^3 + A$$
$$\Rightarrow y = \frac{A}{(M+kx)^2} + \frac{2g}{3k}(M+kx)$$
$$\Rightarrow v^2 = \frac{A}{(M+kx)^2} + \frac{2g}{3k}(M+kx)$$

APPLY CONDITION $x=0$, $v=0$

$$0 = A + \frac{2gM}{3k}$$
$$A = -\frac{2gM}{3k}$$

$$\Rightarrow v^2 = \frac{2g}{3k}(M+kx) - \frac{2gM}{3k} \times \frac{1}{(M+kx)^2}$$
$$\Rightarrow v^2 = \frac{2g}{3k}\left[M + kx - \frac{M}{(M+kx)^2}\right]$$

AS REQUIRED

Question 25 (*****)

A particle of mass initial mass M is projected vertically upwards with speed $\sqrt{gk}$, where k is a positive constant and g is the constant gravitational acceleration.

During the upward motion the particle picks up mass from rest, so that its mass at a distance x above the level of projection is given by

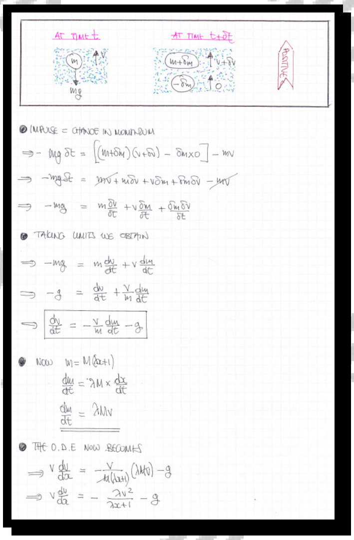
$$M(\lambda x+1),$$

where λ is a positive constant.

Given that when $x=h$, the particle come at instantaneous rest, show that

$$2(\lambda h+1)^3 = 3\lambda k+2.$$

proof



AT TIME t AT TIME $t+\delta t$

m v mg $m+\delta m$ $v+\delta v$ $-\delta m$ 0 POSITIVE

- IMPULSE = CHANGE IN MOMENTUM

$\Rightarrow -mg\,\delta t = \left[(m+\delta m)(v+\delta v) - \delta m\times 0\right] - mv$

$\Rightarrow -mg\,\delta t = \cancel{mv} + m\delta v + v\delta m + \delta m\delta v - \cancel{mv}$

$\Rightarrow -mg = m\frac{\delta v}{\delta t} + v\frac{\delta m}{\delta t} + \frac{\delta m\,\delta v}{\delta t}$

- TAKING LIMITS WE OBTAIN

$\Rightarrow -mg = m\frac{dv}{dt} + v\frac{dm}{dt}$

$\Rightarrow -g = \frac{dv}{dt} + \frac{v}{m}\frac{dm}{dt}$

$\Rightarrow \boxed{\frac{dv}{dt} = -\frac{v}{m}\frac{dm}{dt} - g}$

- NOW $m = M(\lambda x+1)$

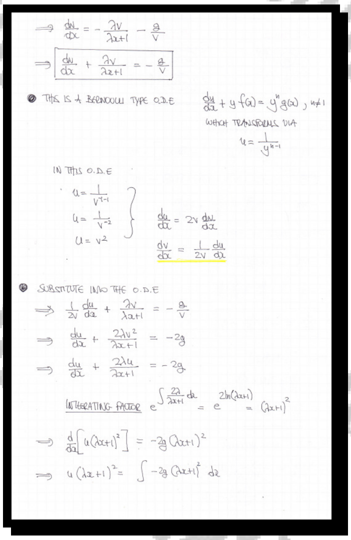
$\frac{dm}{dt} = \lambda M \times \frac{dx}{dt}$

$\frac{dm}{dt} = \lambda M v$

- THE O.D.E NOW BECOMES

$\Rightarrow v\frac{dv}{dx} = -\frac{v}{M(\lambda x+1)}(\lambda M v) - g$

$\Rightarrow v\frac{dv}{dx} = -\frac{\lambda v^2}{\lambda x+1} - g$



$\Rightarrow \frac{dv}{dx} = -\frac{\lambda v}{\lambda x+1} - \frac{g}{v}$

$\Rightarrow \boxed{\frac{dv}{dx} + \frac{\lambda v}{\lambda x+1} = -\frac{g}{v}}$

- THIS IS A BERNOULLI TYPE O.D.E $\frac{dy}{dx} + y f(x) = y^n g(x)$, $n\neq 1$ WHICH TRANSFORMS VIA $u = \frac{1}{y^{n-1}}$

IN THIS O.D.E

$u = \frac{1}{v^{-1-1}}$

$u = \frac{1}{v^{-2}}$

$u = v^2$

$\frac{du}{dx} = 2v\frac{dv}{dx}$

$\frac{dv}{dx} = \frac{1}{2v}\frac{du}{dx}$

- SUBSTITUTE INTO THE O.D.E

$\Rightarrow \frac{1}{2v}\frac{du}{dx} + \frac{\lambda v}{\lambda x+1} = -\frac{g}{v}$

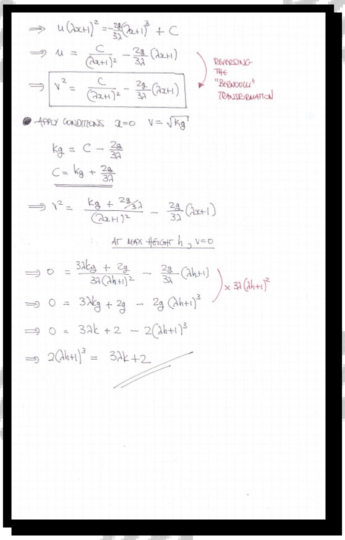
$\Rightarrow \frac{du}{dx} + \frac{2\lambda v^2}{\lambda x+1} = -2g$

$\Rightarrow \frac{du}{dx} + \frac{2\lambda u}{\lambda x+1} = -2g$

INTEGRATING FACTOR $e^{\int \frac{2\lambda}{\lambda x+1}dx} = e^{2\ln(\lambda x+1)} = (\lambda x+1)^2$

$\Rightarrow \frac{d}{dx}\left[u(\lambda x+1)^2\right] = -2g(\lambda x+1)^2$

$\Rightarrow u(\lambda x+1)^2 = \int -2g(\lambda x+1)^2\,dx$



$\Rightarrow u(\lambda x+1)^2 = -\frac{2g}{3\lambda}(\lambda x+1)^3 + C$

$\Rightarrow u = \frac{C}{(\lambda x+1)^2} - \frac{2g}{3\lambda}(\lambda x+1)$

$\Rightarrow \boxed{v^2 = \frac{C}{(\lambda x+1)^2} - \frac{2g}{3\lambda}(\lambda x+1)}$

REVERSING THE "BERNOULLI" TRANSFORMATION

- APPLY CONDITIONS $x=0$ $v=\sqrt{kg}$

$kg = C - \frac{2g}{3\lambda}$

$C = kg + \frac{2g}{3\lambda}$

$\Rightarrow v^2 = \frac{kg + 2g/3\lambda}{(\lambda x+1)^2} - \frac{2g}{3\lambda}(\lambda x+1)$

AT MAX HEIGHT h, $v=0$

$\Rightarrow 0 = \frac{3\lambda kg + 2g}{3\lambda(\lambda h+1)^2} - \frac{2g}{3\lambda}(\lambda h+1)$ $\times 3\lambda(\lambda h+1)^2$

$\Rightarrow 0 = 3\lambda kg + 2g - 2g(\lambda h+1)^3$

$\Rightarrow 0 = 3\lambda k + 2 - 2(\lambda h+1)^3$

$\Rightarrow 2(\lambda h+1)^3 = 3\lambda k+2$