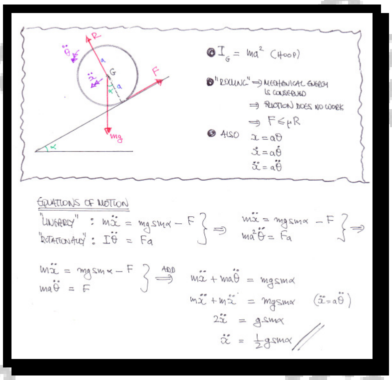
ROTATIONAL AND TRANSLATIONAL MOTION

Question 1 (**)

A uniform circular hoop rolls without slipping, with its plane vertical, down the line of greatest slope of a rough fixed plane, inclined at an angle α to the horizontal.

Find the magnitude of the acceleration of the centre of the hoop, in terms of g and θ.

$$\boxed{\ddot{x} = \frac{1}{2} g \sin\alpha}$$



- $I_G = ma^2$ (HOOP)
- "ROLLING" ⇒ MECHANICAL ENERGY IS CONSERVED ⇒ FRICTION DOES NO WORK ⇒ $F \leqslant \mu R$
- ALSO $x = a\theta$, $\dot{x} = a\dot{\theta}$, $\ddot{x} = a\ddot{\theta}$

EQUATIONS OF MOTION

"LINEARLY" : $m\ddot{x} = mg\sin\alpha - F$
"ROTATIONALLY" : $I\ddot{\theta} = Fa$
$\Rightarrow$ $m\ddot{x} = mg\sin\alpha - F$, $ma^2\ddot{\theta} = Fa$ $\Rightarrow$

$m\ddot{x} = mg\sin\alpha - F$
$ma\ddot{\theta} = F$
$\xrightarrow{\text{ADD}}$
$m\ddot{x} + ma\ddot{\theta} = mg\sin\alpha$
$m\ddot{x} + m\ddot{x} = mg\sin\alpha$ $\quad(\ddot{x} = a\ddot{\theta})$
$2\ddot{x} = g\sin\alpha$
$\ddot{x} = \frac{1}{2}g\sin\alpha$

Question 2 (**)

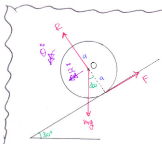
A uniform spherical shell, of radius a, is rolling without slipping, down the line of a greatest slope of rough fixed plane, inclined at an angle of $30°$ to the horizontal.

The spherical shell started from rest.

After time T, of rolling without slipping down the plane for a distance of $15a$, it has angular speed Ω.

Determine, in terms of a and g, an expression for T and an expression for Ω.

$$\boxed{T = 10\sqrt{\frac{a}{g}}}, \quad \boxed{\Omega = 3\sqrt{\frac{g}{a}}}$$



- $I_o = \frac{2}{3}ma^2$
- ROLLING WITHOUT SLIPPING IMPLIES
  - FRICTION DOES NO WORK (NO WORK IS EXCHANGED)
  - $F \leq \mu R$
  - $x = a\theta$, $\dot{x} = a\dot{\theta}$, $\ddot{x} = a\ddot{\theta}$

LINEARLY: $m\ddot{x} = mg\sin 30 - F$
ROTATIONALLY: $I\ddot{\theta} = F \times a$
ELIMINATE THE FRICTION

$$\left.\begin{array}{l} m\ddot{x} = \frac{1}{2}mg - F \\ \frac{2}{3}ma^2\ddot{\theta} = Fa \end{array}\right\} \Rightarrow \left.\begin{array}{l} m\ddot{x} = \frac{1}{2}mg - F \\ \frac{2}{3}ma\ddot{\theta} = F \end{array}\right\} \stackrel{\text{ADD}}{\Longrightarrow} m\ddot{x} + \frac{2}{3}ma\ddot{\theta} = \frac{1}{2}mg$$

$\Rightarrow \ddot{x} + \frac{2}{3}a\ddot{\theta} = \frac{1}{2}g$

$\Rightarrow \ddot{x} + \frac{2}{3}\ddot{x} = \frac{1}{2}g$

$\Rightarrow \frac{5}{3}\ddot{x} = \frac{1}{2}g$

$\Rightarrow \ddot{x} = \frac{3}{10}g$

NOW KINEMATICS

$u = 0$
$a = \frac{3}{10}g$
$s = 15a$
$t = ?$
$V = ?$

$V^2 = \cancel{u^2} + 2as$
$V^2 = 2\left(\frac{3}{10}g\right)(15a)$
$V^2 = 9ag$
$V = 3\sqrt{ag}$

$\therefore \omega = 3\sqrt{\frac{g}{a}}$

&

$v = \cancel{u} + at$
$3\sqrt{ag} = \frac{3}{10}gt$
$\frac{10}{g}\sqrt{ag} = t$
$t = 10\sqrt{\frac{a}{g}}$

Question 3 (**)

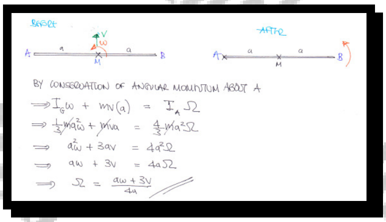
A uniform rod AB, of mass m and length $2a$ is rotating with constant angular velocity ω about M, the centre of the rod.

The centre of the rod has constant speed v.

At a certain instant A becomes fixed. The sense of direction of the rotation of the rod remains unchanged after A becomes fixed.

Determine the angular velocity with which the rod begins to rotate about A, in terms of v, a and ω.

$$\boxed{\Omega = \frac{3v + a\omega}{4a}}$$



Question 4 (**+)

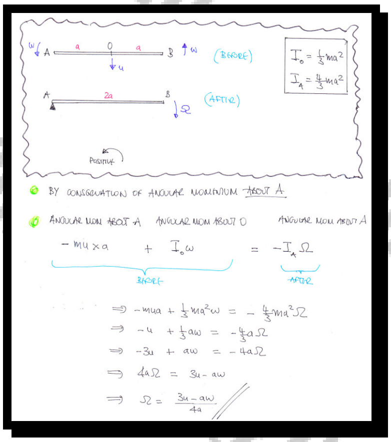
A uniform rod AB, of mass m and length $2a$ is falling freely under gravity, rotating with constant angular velocity ω about O, the centre of the rod.

When the rod is in horizontal position and O has speed u, A becomes fixed.

The sense of direction of the rotation of the rod rchanges after A becomes fixed.

Determine the angular velocity with which the rod begins to rotate about A, in terms of u, a and ω.

$$\boxed{\Omega = \frac{3u - a\omega}{4a}}$$



- BY CONSERVATION OF ANGULAR MOMENTUM ABOUT A
- ANGULAR MOM ABOUT A ANGULAR MOM ABOUT O ANGULAR MOM ABOUT A

$$\underbrace{-mu \times a + I_O\omega}_{\text{BEFORE}} = \underbrace{-I_A\Omega}_{\text{AFTER}}$$

$$\Rightarrow -mua + \tfrac{1}{3}ma^2\omega = -\tfrac{4}{3}ma^2\Omega$$
$$\Rightarrow -u + \tfrac{1}{3}a\omega = -\tfrac{4}{3}a\Omega$$
$$\Rightarrow -3u + a\omega = -4a\Omega$$
$$\Rightarrow 4a\Omega = 3u - a\omega$$
$$\Rightarrow \Omega = \frac{3u - a\omega}{4a}$$

Question 5 (**+)

A uniform solid cylinder, of radius a, is rolling without slipping with its axis horizontal, down a rough fixed plane, inclined at an angle of $30°$ to the horizontal.

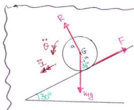
a) Find the angular acceleration of the cylinder, in terms of a and g.

The coefficient of friction between the cylinder and the plane is μ.

b) Show that

$$\mu \geq \tfrac{1}{9}\sqrt{3}.$$

$$\boxed{\ddot{\theta} = \frac{g}{3a}}$$



$\boxed{I_G = \tfrac{1}{2}ma^2}$

ROLLING WITHOUT SLIPPING IMPLIES
- FRICTION DOES NOT WORK
- MECHANICAL ENERGY IS CONSERVED
- $F \leqslant \mu R$
- $\boxed{\begin{array}{l} x = a\theta \\ \dot{x} = a\dot{\theta} \\ \ddot{x} = a\ddot{\theta} \end{array}}$

a) EQUATION OF MOTION

$\Rightarrow \left(\begin{array}{l} \text{LINEARLY: } m\ddot{x} = mg\sin 30 - F \\ \text{ROTATIONALLY: } I\ddot{\theta} = Fa \end{array} \right)$

$\Rightarrow \left(\begin{array}{l} m\ddot{x} = \tfrac{1}{2}mg - F \\ \left(\tfrac{1}{2}ma^2\right)\ddot{\theta} = Fa \end{array} \right)$

$\Rightarrow \left(\begin{array}{l} m\ddot{x} = \tfrac{1}{2}mg - F \\ \tfrac{1}{2}ma\ddot{\theta} = F \end{array} \right)$ ADD

$\Rightarrow m\ddot{x} + \tfrac{1}{2}ma\ddot{\theta} = \tfrac{1}{2}mg$

$\Rightarrow \ddot{x} + \tfrac{1}{2}a\ddot{\theta} = \tfrac{1}{2}g$

- BUT $\ddot{x} = a\ddot{\theta}$

$\Rightarrow a\ddot{\theta} + \tfrac{1}{2}a\ddot{\theta} = \tfrac{1}{2}g$

$\Rightarrow \tfrac{3}{2}a\ddot{\theta} = \tfrac{1}{2}g$

$\Rightarrow 3a\ddot{\theta} = g$

$\Rightarrow \ddot{\theta} = \frac{g}{3a}$

b) $I\ddot{\theta} = Fa$

$\Rightarrow \left(\tfrac{1}{2}ma^2\right)\left(\frac{g}{3a}\right) = Fa$

$\Rightarrow F = \tfrac{1}{6}mg$

- BUT $F \leqslant \mu R$

$\Rightarrow \tfrac{1}{6}mg \leqslant \mu(mg\cos 30)$

$\Rightarrow \tfrac{1}{6} \leqslant \mu\cos 30$

$\Rightarrow \mu \geqslant \frac{1}{6\cos 30}$

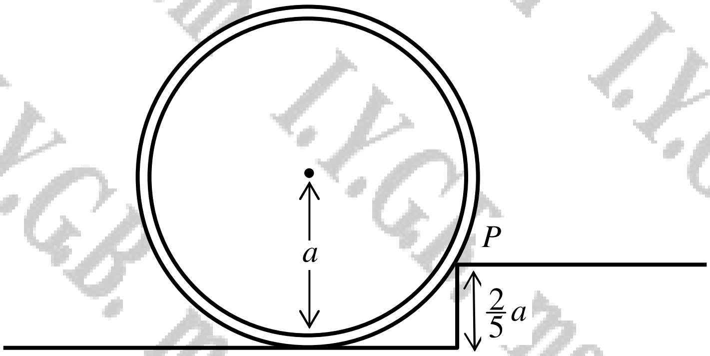
$\Rightarrow \mu \geqslant \frac{1}{6\left(\frac{\sqrt{3}}{2}\right)}$

$\Rightarrow \mu \geqslant \frac{1}{3\sqrt{3}}$

$\Rightarrow \mu \geqslant \tfrac{1}{9}\sqrt{3}$

AS REQUIRED

Question 6 (+)**



A uniform circular hoop, of radius a, is rolling without slipping on a rough horizontal plane, with constant angular speed ω.

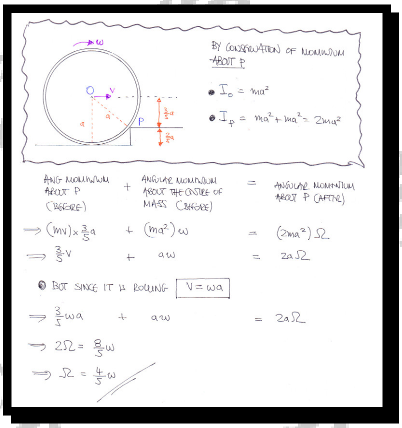
The hoop reaches a vertical step of height $\frac{2}{5}a$, which is at right angles to its direction of motion, as shown in the figure above.

When the hoop touches the step at the point P, it begins to rotate about P, without slipping or loss of contact, with angular speed Ω.

By considering angular momentum conservation, show that

$$\Omega = \frac{4}{5}\omega.$$

$\boxed{\text{proof}}$


BY CONSERVATION OF MOMENTUM ABOUT P

- $I_O = ma^2$
- $I_P = ma^2 + ma^2 = 2ma^2$

ANG MOMENTUM ABOUT P (BEFORE) + ANGULAR MOMENTUM ABOUT THE CENTRE OF MASS (BEFORE) = ANGULAR MOMENTUM ABOUT P (AFTER)

$\Rightarrow (mv) \times \frac{3}{5}a + (ma^2)\omega = (2ma^2)\Omega$

$\Rightarrow \frac{3}{5}v + a\omega = 2a\Omega$

- BUT SINCE IT IS ROLLING $\boxed{v = \omega a}$

$\Rightarrow \frac{3}{5}\omega a + a\omega = 2a\Omega$

$\Rightarrow 2\Omega = \frac{8}{5}\omega$

$\Rightarrow \Omega = \frac{4}{5}\omega$

Question 7 (***)

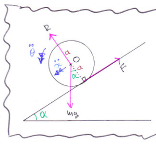
A uniform circular hoop is rolling without slipping down a rough fixed plane, inclined at an angle α to the horizontal.

The coefficient of friction between the disc and the plane is μ.

Using a detailed method, show that

$$\mu \geq \frac{1}{2}\tan\alpha .$$

proof



- ROLLING WITHOUT SLIPPING IMPLIES
  - FRICTION DOES NO WORK SO KE+PE = CONSTANT
  - $F \leq \mu R$
  - $x = a\theta$, $\dot{x} = a\dot{\theta}$, $\ddot{x} = a\ddot{\theta}$
- MOMENT OF INERTIA $= ma^2$

LINEARLY : $m\ddot{x} = mg\sin\alpha - F$
ROTATIONALLY : $I\ddot{\theta} = F \times a$

$$\Rightarrow \left\{\begin{array}{l} m\ddot{x} = mg\sin\alpha - F \\ ma^2\ddot{\theta} = Fa \end{array}\right\} \Rightarrow \left\{\begin{array}{l} m\ddot{x} = mg\sin\alpha - F \\ ma\ddot{\theta} = F \end{array}\right\} \Rightarrow \text{ADDING} \Rightarrow m\ddot{x} + ma\ddot{\theta} = mg\sin\alpha$$

$$m\ddot{x} + m\ddot{x} = mg\sin\alpha$$
$$2\ddot{x} = g\sin\alpha$$
$$\boxed{\ddot{x} = \tfrac{1}{2}g\sin\alpha}$$

HENCE $m\ddot{x} = mg\sin\alpha - F$

$$F = mg\sin\alpha - m\ddot{x}$$
$$F = mg\sin\alpha - \tfrac{1}{2}mg\sin\alpha$$
$$\boxed{F = \tfrac{1}{2}mg\sin\alpha}$$

ALSO $\boxed{R = mg\cos\alpha}$

BUT $F \leq \mu R$

$$\tfrac{1}{2}mg\sin\alpha \leq \mu mg\cos\alpha$$
$$\tfrac{1}{2}\sin\alpha \leq \mu\cos\alpha$$
$$\tfrac{1}{2}\tan\alpha \leq \mu$$
$$\mu \geq \tfrac{1}{2}\tan\alpha$$

AS REQUIRED

Question 8 (***)

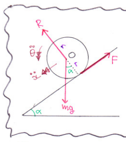
A uniform circular disc is rolling without slipping down a rough fixed plane, inclined at an angle α to the horizontal.

The coefficient of friction between the disc and the plane is μ.

Using a detailed method, show that

$$\mu \geq \frac{1}{3}\tan\alpha .$$

proof



ROLLING $\Rightarrow$ FRICTION DOES NO WORK

$\Rightarrow$ MECHANICAL ENERGY IS CONSERVED

$\Rightarrow F \leqslant \mu R$

$\Rightarrow \boxed{\begin{array}{l} x = r\theta \\ \dot{x} = r\dot{\theta} \\ \ddot{x} = r\ddot{\theta} \end{array}}$

EQUATION OF MOTION

LINEARLY : $m\ddot{x} = mg\sin\alpha - F$

ROTATIONALLY : $I\ddot{\theta} = Fr$

$$\Rightarrow \left\{ \begin{array}{l} m\ddot{x} = mg\sin\alpha - F \\ \left(\frac{1}{2}mr^2\right)\ddot{\theta} = Fr \end{array} \right\} \Rightarrow$$

$$\Rightarrow \left\{ \begin{array}{l} m\ddot{x} = mg\sin\alpha - F \\ \frac{1}{2}mr\ddot{\theta} = F \end{array} \right\} \Rightarrow \text{ADD} \Rightarrow m\ddot{x} + \frac{1}{2}mr\ddot{\theta} = mg\sin\alpha$$

BUT $\boxed{\ddot{x} = r\ddot{\theta}}$

$\Rightarrow r\ddot{\theta} + \frac{1}{2}r\ddot{\theta} = g\sin\alpha$

$\Rightarrow \frac{3}{2}r\ddot{\theta} = g\sin\alpha$

$\Rightarrow \frac{1}{2}r\ddot{\theta} = \frac{1}{3}g\sin\alpha$

$\boxed{\text{BUT } F = \frac{1}{2}mr\ddot{\theta}}$

$\Rightarrow F = \frac{1}{3}mg\sin\alpha$

$\boxed{\text{BUT } F \leq \mu R}$

$\Rightarrow \frac{1}{3}mg\sin\alpha \leq \mu(mg\cos\alpha)$

$\Rightarrow \frac{1}{3}\tan\alpha < \mu$

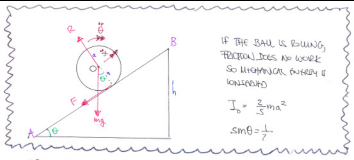
$\Rightarrow \mu > \frac{1}{3}\tan\alpha$ // AS REQUIRED

Question 9 (***)

The centre O of a uniform solid sphere has an initial speed of 5 ms^{-1} up a rough fixed plane, inclined at an angle θ to the horizontal, where $\sin\theta = \frac{1}{7}$. The centre of sphere comes to instantaneous rest after covering a distance d m up the plane.

Given that the sphere was rolling without slipping in its journey up the plane, find the value of d.

$\boxed{d = 12.5 \text{ m}}$



- BY ENERGIES TAKING THE LEVEL OF A AS THE ZERO POTENTIAL LEVEL

V_A : INITIAL SPEED $= 5$

ω_A INITIAL ANGULAR SPEED $= \frac{5}{a}$ (USING $v = \omega r$)

FINALLY $V_B = 0$, $\omega_B = 0$

$$\Rightarrow \underbrace{\tfrac{1}{2}mV_A^2 + \tfrac{1}{2}I\omega_A^2}_{KE_A} + \underbrace{0}_{PE_A} = \underbrace{0}_{KE_B} + \underbrace{mgh}_{P.E_B}$$

$$\Rightarrow \tfrac{1}{2}\cancel{m} \times 5^2 + \tfrac{1}{2} \times \left(\tfrac{2}{5}\cancel{m}a^2\right) \times \left(\tfrac{5}{a}\right)^2 = \cancel{m}g(d\sin\theta)$$

$$\Rightarrow \tfrac{25}{2} + 5 = gd \times \tfrac{1}{7}$$

$$\Rightarrow 17.5 = d \times 9.8 \times \tfrac{1}{7}$$

$$\Rightarrow d = 12.5 \text{ m}$$

Question 10 (***)

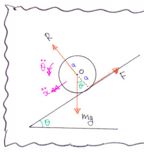
A uniform solid sphere is rolling without slipping down a rough fixed plane, inclined at an angle θ to the horizontal.

The coefficient of friction between the sphere and the plane is μ.

Using a detailed method, show that

$$\mu \geq \frac{2}{7}\tan\theta.$$

$\boxed{\text{proof}}$



- ROLLING WITHOUT SLIPPING IMPLIES
  - FRICTION DOES NO WORK, SO MECHANICAL ENERGY IS CONSERVED
  - $F \leq \mu R$
  - $x = a\theta$, $\dot{x} = a\dot{\theta}$, $\ddot{x} = a\ddot{\theta}$
- $I_O = \frac{2}{5}ma^2$

LINEARLY : $m\ddot{x} = mg\sin\theta - F$
ROTATIONALLY : $I\ddot{\theta} = Fa$

$\Rightarrow$ $m\ddot{x} = mg\sin\theta - F$, $\frac{2}{5}ma^2\ddot{\theta} = Fa$ $\Rightarrow$

$m\ddot{x} = mg\sin\theta - F$, $\frac{2}{5}ma\ddot{\theta} = F$ $\Rightarrow$ ADD THE EQUATIONS

$$m\ddot{x} + \frac{2}{5}ma\ddot{\theta} = mg\sin\theta$$
$$\ddot{x} + \frac{2}{5}a\ddot{\theta} = g\sin\theta$$
$$\ddot{x} + \frac{2}{5}\ddot{x} = g\sin\theta$$
$$\frac{7}{5}\ddot{x} = g\sin\theta$$
$$\boxed{\ddot{x} = \frac{5}{7}g\sin\theta}$$

NOW $m\ddot{x} = mg\sin\theta - F$

$\Rightarrow \frac{5}{7}mg\sin\theta = mg\sin\theta - F$

$\Rightarrow \boxed{F = \frac{2}{7}mg\sin\theta}$

ALSO $\boxed{R = mg\cos\theta}$

$\Rightarrow F \leq \mu R$

$\frac{2}{7}\cancel{mg}\sin\theta \leq \mu(\cancel{mg}\cos\theta)$

$\frac{2}{7}\tan\theta \leq \mu$

$\mu \geq \frac{2}{7}\tan\theta$

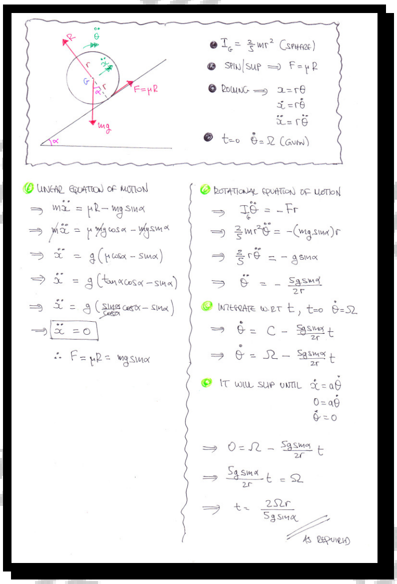
Question 11 (***)

A uniform solid sphere, of radius r, is rotating about a diameter with constant angular velocity Ω. The rotating sphere is gently placed on a rough fixed plane, inclined at an angle α to the horizontal, the rotation direction being such so that the sphere would move up the line of greatest slope of the plane.

Given that the coefficient of friction between the sphere and the plane is $\tan\alpha$, show that the sphere will start rolling up the plane after a time

$$\frac{2\Omega r}{5g\sin\alpha}.$$

$\boxed{\text{proof}}$



Question 12 (***)

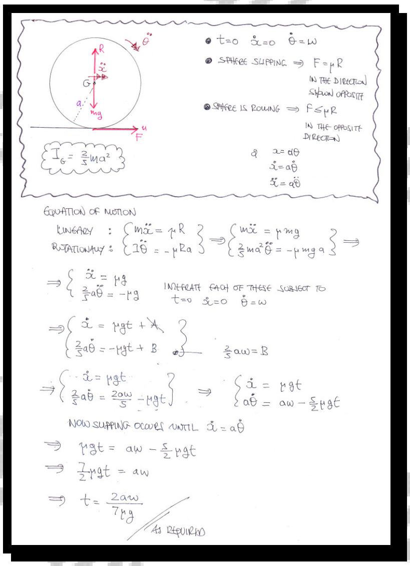
A uniform solid sphere, of radius a, is rotating about a horizontal diameter with constant angular velocity ω. The rotating sphere is gently placed on a rough horizontal surface and released. The coefficient of friction between the sphere and the surface is μ.

Show that the sphere will slip for a time

$$\frac{2a\omega}{7\mu g},$$

before it starts rolling on the horizontal surface.

$\boxed{\text{proof}}$



$I_G = \frac{2}{5}ma^2$

- $t=0 \quad \dot{x}=0 \quad \dot{\theta}=\omega$
- SPHERE SLIPPING $\Rightarrow F = \mu R$ IN THE DIRECTION SHOWN OPPOSITE
- SPHERE IS ROLLING $\Rightarrow F \leq \mu R$ IN THE OPPOSITE DIRECTION

& $x = a\theta$
$\dot{x} = a\dot{\theta}$
$\ddot{x} = a\ddot{\theta}$

EQUATION OF MOTION

LINEARY : $\left\{ m\ddot{x} = \mu R \right.$
ROTATIONALLY : $\left. I\ddot{\theta} = -\mu Ra \right\}$ $\Rightarrow \left\{ \begin{array}{l} m\ddot{x} = \mu mg \\ \frac{2}{5}ma^2\ddot{\theta} = -\mu mga \end{array} \right\} \Rightarrow$

$\Rightarrow \left\{ \begin{array}{l} \ddot{x} = \mu g \\ \frac{2}{5}a\ddot{\theta} = -\mu g \end{array} \right.$ INTEGRATE EACH OF THESE SUBJECT TO $t=0 \quad \dot{x}=0 \quad \dot{\theta}=\omega$

$\Rightarrow \left\{ \begin{array}{l} \dot{x} = \mu gt + \cancel{A} \\ \frac{2}{5}a\dot{\theta} = -\mu gt + B \end{array} \right\}$ $\quad \frac{2}{5}a\omega = B$

$\Rightarrow \left\{ \begin{array}{l} \dot{x} = \mu gt \\ \frac{2}{5}a\dot{\theta} = \frac{2a\omega}{5} - \mu gt \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} \dot{x} = \mu gt \\ a\dot{\theta} = a\omega - \frac{5}{2}\mu gt \end{array} \right.$

NOW SLIPPING OCCURS UNTIL $\dot{x} = a\dot{\theta}$

$\Rightarrow \mu gt = a\omega - \frac{5}{2}\mu gt$

$\Rightarrow \frac{7}{2}\mu gt = a\omega$

$\Rightarrow t = \frac{2a\omega}{7\mu g}$

AS REQUIRED

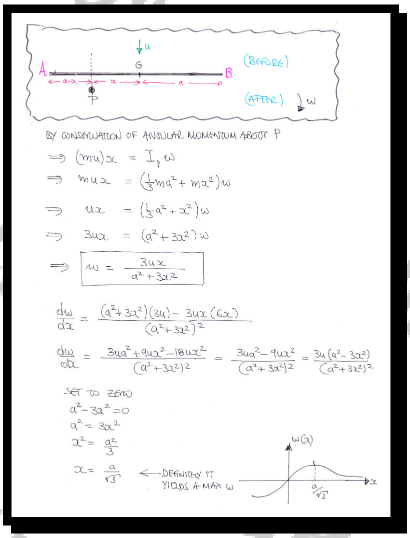
Question 13 (***)

A uniform rod AB, of mass m and length $2a$, is falling freely under gravity with speed u, in a horizontal position.

The rod hits a rough peg P at a distance x from the centre of the rod and without rebounding, begins to rotate about P with angular speed ω.

Determine, in terms of a, the value of x for which ω is greatest.

$$\boxed{x = \frac{a}{\sqrt{3}}}$$



BY CONSERVATION OF ANGULAR MOMENTUM ABOUT P

$\Rightarrow (mu)x = I_P \omega$

$\Rightarrow mux = \left(\frac{1}{3}ma^2 + mx^2\right)\omega$

$\Rightarrow ux = \left(\frac{1}{3}a^2 + x^2\right)\omega$

$\Rightarrow 3ux = (a^2 + 3x^2)\omega$

$\Rightarrow \boxed{\omega = \frac{3ux}{a^2 + 3x^2}}$

$\frac{d\omega}{dx} = \frac{(a^2+3x^2)(3u) - 3ux(6x)}{(a^2+3x^2)^2}$

$\frac{d\omega}{dx} = \frac{3ua^2 + 9ux^2 - 18ux^2}{(a^2+3x^2)^2} = \frac{3ua^2 - 9ux^2}{(a^2+3x^2)^2} = \frac{3u(a^2-3x^2)}{(a^2+3x^2)^2}$

SET TO ZERO

$a^2 - 3x^2 = 0$

$a^2 = 3x^2$

$x^2 = \frac{a^2}{3}$

$x = \frac{a}{\sqrt{3}}$ ← DEFINITELY IT YIELDS A MAX ω

Question 14 (***)

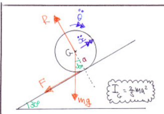
A uniform solid sphere, of radius a, is rolling without slipping up a rough fixed plane, inclined at an angle of $30°$ to the horizontal.

At $t = 0$ the angular speed of the sphere is ω.

Find the distance covered by the centre of the sphere before its angular speed is reduced to $\frac{1}{2}\omega$.

Give the distance in terms of a, ω and g.

$$\boxed{s = \frac{21\omega^2 a^2}{20g}}$$



"WITHOUT SLIPPING" $\Rightarrow x = a\theta$
$\Rightarrow \dot{x} = a\dot{\theta}$
$\Rightarrow \ddot{x} = a\ddot{\theta}$

FRICTION DOES NO WORK SO MECHANICAL ENERGY IS CONSERVED $(F < \mu R)$

METHOD A – BY THE EQUATIONS OF MOTION

- EQUATIONS OF MOTION

$$\left.\begin{aligned} (\nearrow):&\quad m\ddot{x} = -F - mg\sin 30° \\ (\curvearrowright):&\quad I\ddot{\theta} = Fa \end{aligned}\right\} \Rightarrow \left.\begin{aligned} F &= -m\ddot{x} - mg\sin 30° \\ F &= \frac{I\ddot{\theta}}{a} \end{aligned}\right\} \Rightarrow$$

- ELIMINATE F AND TIDY TO OBTAIN THE ACCELERATION(S)

$$\Rightarrow -m\ddot{x} - mg\sin 30 = \frac{I\ddot{\theta}}{a}$$
$$\Rightarrow -m\ddot{x} - \frac{1}{2}mg = \frac{\frac{2}{5}ma^2\ddot{\theta}}{a}$$
$$\Rightarrow -\ddot{x} - \frac{1}{2}g = \frac{2}{5}a\ddot{\theta}$$
$$\Rightarrow -\ddot{x} = \frac{1}{2}g + \frac{2}{5}\ddot{x}$$
$$\Rightarrow -\frac{7}{5}\ddot{x} = \frac{1}{2}g$$
$$\Rightarrow \boxed{\ddot{x} = -\frac{5}{14}g} \quad \text{or} \quad \boxed{\ddot{\theta} = -\frac{5g}{14a}}$$

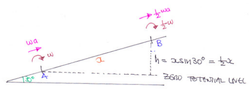
- FINALLY

"INITIAL" ANGULAR SPEED IS ω, & LINEAR SPEED IS $u = \omega a$
"FINAL" ANGULAR SPEED IS $\frac{1}{2}\omega$, & LINEAR SPEED IS $v = \frac{1}{2}\omega a$

$u = \omega a$
$a = -\frac{5}{14}g$
$s = ?$
$t =$
$v = \frac{1}{2}\omega a$

$$v^2 = u^2 + 2as \Rightarrow s = \frac{v^2 - u^2}{2a}$$
$$\Rightarrow s = \frac{\frac{1}{4}\omega^2a^2 - \omega^2a^2}{2\left(-\frac{5}{14}g\right)}$$
$$\Rightarrow s = \frac{21\omega^2a^2}{20g}$$

METHOD B – BY ENERGIES



$$\Rightarrow KE_A + \cancel{PE_A} + \cancel{W_{IN}} - \cancel{W_{OUT}} = KE_A + PE_B$$

FRICTION DOES NO WORK

$$\Rightarrow \frac{1}{2}m(\omega a)^2 + \frac{1}{2}I(\omega^2) = \frac{1}{2}m\left(\frac{1}{2}\omega a\right)^2 + \frac{1}{2}I\left(\frac{1}{2}\omega\right)^2 + mgh$$
$$\Rightarrow \frac{1}{2}\cancel{m}\omega^2a^2 + \frac{1}{2}\left(\frac{2}{5}\cancel{m}a^2\right)\omega^2 = \frac{1}{2}\cancel{m}\left(\frac{1}{4}\omega^2a^2\right) + \frac{1}{2}\left(\frac{2}{5}\cancel{m}a^2\right)\left(\frac{1}{4}\omega^2\right) + \cancel{m}g\left(\frac{1}{2}x\right)$$
$$\Rightarrow \frac{1}{2}\omega^2a^2 + \frac{1}{5}a^2\omega^2 = \frac{1}{8}\omega^2a^2 + \frac{1}{20}a^2\omega^2 + \frac{1}{2}gx$$
$$\Rightarrow \frac{21}{40}a^2\omega^2 = \frac{1}{2}gx$$
$$\Rightarrow x = \frac{21a^2\omega^2}{20g}$$

Question 15 (*+)**

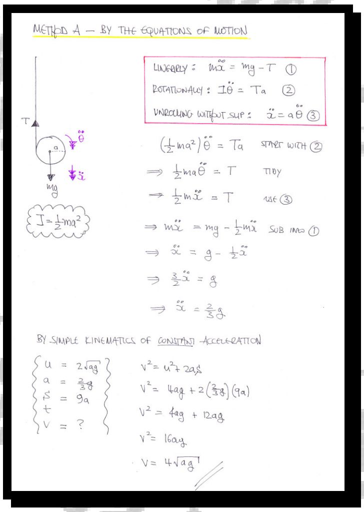
A yo-yo toy is modelled as a uniform solid disc of mass m and radius a.

One end of a light inextensible string is fixed at a point on the rim of the yo-yo, and the rest of the string is wrapped several times around the rim. The disc of the yo-yo is held in a vertical plane with the other end of the string held fixed.

The yo-yo is projected vertically downwards with speed $2\sqrt{ag}$, so that the sting as it unwraps from the toy remains vertical.

Given that the string has not fully unwind, find the speed of the centre of the yo-yo, when the centre of the yo-yo has travelled a distance $9a$.

$$\boxed{v = 4\sqrt{ag}}$$



METHOD A — BY THE EQUATIONS OF MOTION

LINEARLY: $m\ddot{x} = mg - T$ ①
ROTATIONALLY: $I\ddot{\theta} = Ta$ ②
UNROLLING WITHOUT SLIP: $\ddot{x} = a\ddot{\theta}$ ③

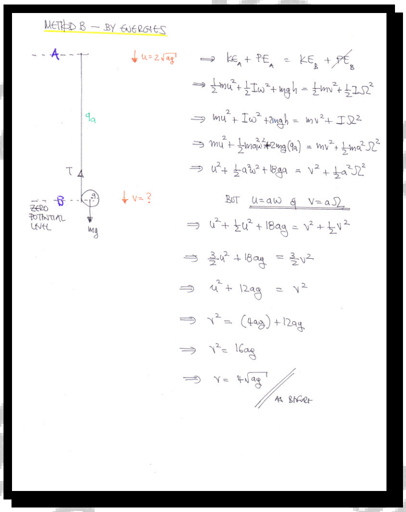
$I = \frac{1}{2}ma^2$

$\left(\frac{1}{2}ma^2\right)\ddot{\theta} = Ta$ START WITH ②
$\Rightarrow \frac{1}{2}ma\ddot{\theta} = T$ TIDY
$\Rightarrow \frac{1}{2}m\ddot{x} = T$ USE ③
$\Rightarrow m\ddot{x} = mg - \frac{1}{2}m\ddot{x}$ SUB INTO ①
$\Rightarrow \ddot{x} = g - \frac{1}{2}\ddot{x}$
$\Rightarrow \frac{3}{2}\ddot{x} = g$
$\Rightarrow \ddot{x} = \frac{2}{3}g$

BY SIMPLE KINEMATICS OF CONSTANT ACCELERATION

$u = 2\sqrt{ag}$, $a = \frac{2}{3}g$, $s = 9a$, t, $v = ?$

$v^2 = u^2 + 2as$
$v^2 = 4ag + 2\left(\frac{2}{3}g\right)(9a)$
$v^2 = 4ag + 12ag$
$v^2 = 16ag$
$v = 4\sqrt{ag}$



METHOD B — BY ENERGIES

$u = 2\sqrt{ag}$, $9a$, $v = ?$, ZERO POTENTIAL LEVEL

$\Rightarrow KE_A + PE_A = KE_B + \cancel{PE_B}$
$\Rightarrow \frac{1}{2}mu^2 + \frac{1}{2}I\omega^2 + mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\Omega^2$
$\Rightarrow mu^2 + I\omega^2 + 2mgh = mv^2 + I\Omega^2$
$\Rightarrow mu^2 + \frac{1}{2}ma^2\omega^2 + 2mg(9a) = mv^2 + \frac{1}{2}ma^2\Omega^2$
$\Rightarrow u^2 + \frac{1}{2}a^2\omega^2 + 18ga = v^2 + \frac{1}{2}a^2\Omega^2$

BUT $u = a\omega$ & $v = a\Omega$

$\Rightarrow u^2 + \frac{1}{2}u^2 + 18ag = v^2 + \frac{1}{2}v^2$
$\Rightarrow \frac{3}{2}u^2 + 18ag = \frac{3}{2}v^2$
$\Rightarrow u^2 + 12ag = v^2$
$\Rightarrow v^2 = (4ag) + 12ag$
$\Rightarrow v^2 = 16ag$
$\Rightarrow v = 4\sqrt{ag}$ AS BEFORE

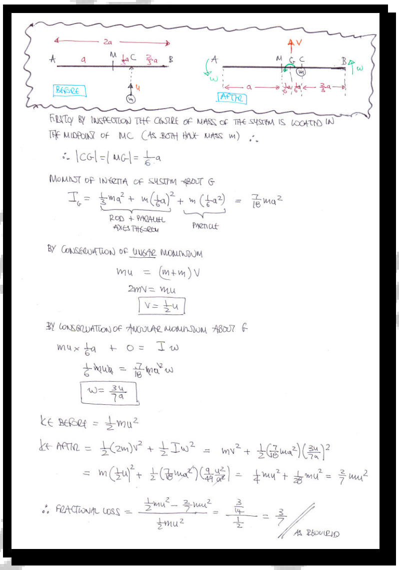
Question 16 (***+)

A thin uniform solid rod AB of mass m and length $2a$, is lying at rest on a smooth horizontal surface. A particle of mass m, moving with speed u on the same surface.

The particle, moving in a perpendicular direction to the rod, strikes the rod at the point C, where $|AC| = \frac{4}{3}a$, and immediately adheres to the rod.

Show that $\frac{3}{7}$ of the kinetic energy is lost in the collision.

proof



FIRSTLY BY INSPECTION THE CENTRE OF MASS OF THE SYSTEM IS LOCATED IN THE MIDPOINT OF MC (AS BOTH HAVE MASS m) ∴

$\therefore |CG| = |MG| = \frac{1}{6}a$

MOMENT OF INERTIA OF SYSTEM ABOUT G

$$I_G = \underbrace{\frac{1}{3}ma^2 + m\left(\frac{1}{6}a\right)^2}_{\text{ROD + PARALLEL AXES THEOREM}} + \underbrace{m\left(\frac{1}{6}a^2\right)}_{\text{PARTICLE}} = \frac{7}{18}ma^2$$

BY CONSERVATION OF LINEAR MOMENTUM

$$mu = (m+m)V$$
$$2mV = mu$$
$$\boxed{V = \frac{1}{2}u}$$

BY CONSERVATION OF ANGULAR MOMENTUM ABOUT G

$$mu \times \frac{1}{6}a + 0 = I\omega$$
$$\frac{1}{6}mua = \frac{7}{18}ma^2\omega$$
$$\boxed{\omega = \frac{3u}{7a}}$$

KE BEFORE $= \frac{1}{2}mu^2$

KE AFTER $= \frac{1}{2}(2m)V^2 + \frac{1}{2}I\omega^2 = mV^2 + \frac{1}{2}\left(\frac{7}{18}ma^2\right)\left(\frac{3u}{7a}\right)^2$

$= m\left(\frac{1}{2}u\right)^2 + \frac{1}{2}\left(\frac{7}{18}ma^2\right)\left(\frac{9}{49}\frac{u^2}{a^2}\right) = \frac{1}{4}mu^2 + \frac{1}{28}mu^2 = \frac{2}{7}mu^2$

$\therefore$ FRACTIONAL LOSS $= \dfrac{\frac{1}{2}mu^2 - \frac{2}{7}mu^2}{\frac{1}{2}mu^2} = \dfrac{\frac{3}{14}}{\frac{1}{2}} = \frac{3}{7}$ AS REQUIRED

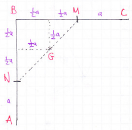
Question 17 (***+)

Two identical uniform rods AB and BC, each of mass m and length $2a$, are rigidly joined at B, so that ABC is a right angle.

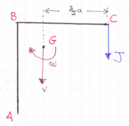
The system of the two rods lies at rest on a smooth horizontal surface, when it receives at C an impulse of magnitude J, in a direction parallel to BA.

Determine, in terms of J and m, the kinetic energy of the system after the impulse is received.

$$\boxed{\frac{37J}{40m}}$$



- THE LOCATION OF THE CENTRE OF MASS BY INSPECTION IS $\frac{1}{2}a$ FROM AB & FROM BC
- $|MG| = \sqrt{\left(\frac{1}{2}a\right)^2 + \left(\frac{1}{2}a\right)^2} = \frac{\sqrt{2}}{2}a$
- MOMENT OF INERTIA OF BC ABOUT G IS GIVEN BY
  $\frac{1}{3}ma^2 + m|MG|^2$
  $= \frac{1}{3}ma^2 + m\left(\frac{\sqrt{2}}{2}a\right)^2$
  $= \frac{5}{6}ma^2$
- SIMILARLY FOR AB, SO THE TOTAL MOMENT OF INERTIA IS
  $\frac{5}{3}ma^2$



- NEXT THE IMPULSE IS APPLIED

LINEARLY

$J = 2m(v-0)$

$J = 2mv$

$v = \frac{J}{2m}$

ROTATIONALLY

(MOMENT OF IMPULSE ABOUT G = CHANGE OF ANGULAR MOMENTUM ABOUT G)

$J \times \frac{3}{2}a = I_G\,\omega$

$J \times \frac{3}{2}a = \frac{5}{3}ma^2\omega$

$9J = 10ma\omega$

$\omega = \frac{9J}{10ma}$

TOTAL KINETIC ENERGY $= \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}(2m)\left(\frac{J}{2m}\right)^2 + \frac{1}{2}\left(\frac{5}{3}ma^2\right)\left(\frac{9J}{10ma}\right)^2$

$= \frac{J^2}{4m} + \frac{27J^2}{40m} = \frac{37J^2}{40m}$

Question 18 (***+)

A uniform solid sphere, of radius a, is projected at time $t=0$ up a line of greatest slope of a rough plane, inclined at angle θ to the horizontal.

At the instant of projection the sphere has linear speed u and no angular velocity.

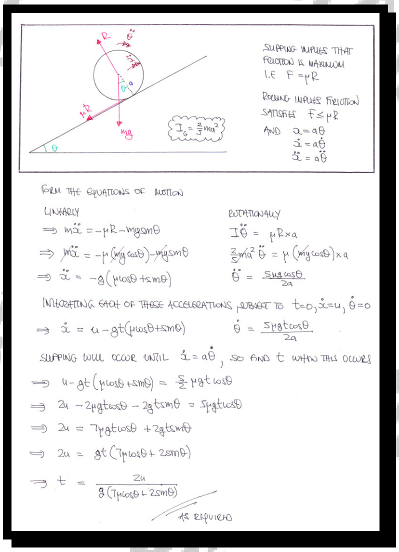
The coefficient of friction between the sphere and the plane is μ.

Show that the sphere will slip until

$$t = \frac{2u}{g\left(7\mu\cos\theta + 2\sin\theta\right)},$$

before it starts rolling up the plane.

proof



SLIPPING IMPLIES THAT FRICTION IS MAXIMUM I.E $F = \mu R$

ROLLING IMPLIES FRICTION SATISFIES $F \leq \mu R$

AND $x = a\theta$, $\dot{x} = a\dot{\theta}$, $\ddot{x} = a\ddot{\theta}$

FORM THE EQUATIONS OF MOTION

LINEARLY

$\Rightarrow m\ddot{x} = -\mu R - mg\sin\theta$

$\Rightarrow \cancel{m}\ddot{x} = -\mu(\cancel{m}g\cos\theta) - \cancel{m}g\sin\theta$

$\Rightarrow \ddot{x} = -g(\mu\cos\theta + \sin\theta)$

ROTATIONALLY

$I\ddot{\theta} = \mu R \times a$

$\frac{2}{5}\cancel{m}a^2\ddot{\theta} = \mu(\cancel{m}g\cos\theta) \times a$

$\ddot{\theta} = \frac{5\mu g\cos\theta}{2a}$

INTEGRATING EACH OF THESE ACCELERATIONS, SUBJECT TO $t=0$, $\dot{x}=u$, $\dot{\theta}=0$

$\Rightarrow \dot{x} = u - gt(\mu\cos\theta + \sin\theta)$ $\qquad \dot{\theta} = \frac{5\mu gt\cos\theta}{2a}$

SLIPPING WILL OCCUR UNTIL $\dot{x} = a\dot{\theta}$, SO FIND t WHEN THIS OCCURS

$\Rightarrow u - gt(\mu\cos\theta + \sin\theta) = \frac{5}{2}\mu gt\cos\theta$

$\Rightarrow 2u - 2\mu gt\cos\theta - 2gt\sin\theta = 5\mu gt\cos\theta$

$\Rightarrow 2u = 7\mu gt\cos\theta + 2gt\sin\theta$

$\Rightarrow 2u = gt(7\mu\cos\theta + 2\sin\theta)$

$\Rightarrow t = \frac{2u}{g(7\mu\cos\theta + 2\sin\theta)}$

AS REQUIRED

Question 19 (***+)

A uniform spherical shell, of radius a, is projected at time $t = 0$ down a line of greatest slope of a rough plane, inclined at angle α to the horizontal. At the instant of projection the spherical shell has linear speed V and no angular velocity.

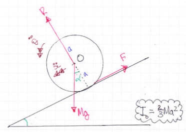
The coefficient of friction between the spherical shell and the plane is μ.

Show that the spherical shell will slip until

$$t = \frac{2V}{g(5\mu\cos\alpha - 2\sin\alpha)},$$

before it starts rolling down the plane.

$\boxed{\text{proof}}$



$$I_O = \tfrac{2}{3}Ma^2$$

• EQUATIONS OF MOTION

LINEARLY $\quad M\ddot{x} = Mg\sin\alpha - F$

ROTATIONALLY $\quad I\ddot{\theta} = Fa$

AS THE SPHERE SLIDES TO START WITH $F = \mu R$

• $\cancel{M}\ddot{x} = \cancel{M}g\sin\alpha - \mu(\cancel{M}g\cos\alpha)$

$\ddot{x} = g(\sin\alpha - \mu\cos\alpha)$

$\dot{x} = V + gt(\sin\alpha - \mu\cos\alpha)$

$(t=0,\ \dot{x}=V)$

• SIMILARLY

$\tfrac{2}{3}\cancel{M}a^{\cancel{2}}\ddot{\theta} = \mu(\cancel{M}g\cos\alpha)\cancel{a}$

$\ddot{\theta} = \dfrac{3\mu g}{2a}\cos\alpha$

$\dot{\theta} = \dfrac{3\mu g t}{2a}\cos\alpha$

$(t=0,\ \dot{\theta}=0)$

• WHEN ROLLING $\quad \dot{x} = a\dot{\theta}$

$\Rightarrow V + gt(\sin\alpha - \mu\cos\alpha) = \cancel{a}\left[\dfrac{3\mu g t\cos\alpha}{2\cancel{a}}\right]$

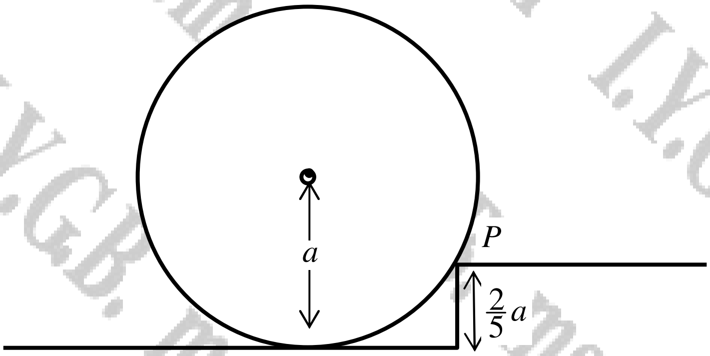
$\Rightarrow 2V + 2gt(\sin\alpha - \mu\cos\alpha) = 3\mu g t\cos\alpha$

$\Rightarrow 2V = 3\mu g t\cos\alpha - 2gt(\sin\alpha - \mu\cos\alpha)$

$\Rightarrow 2V = gt\left[3\mu\cos\alpha - 2\sin\alpha + 2\mu\cos\alpha\right]$

$\Rightarrow t = \dfrac{2V}{g}(5\mu\cos\alpha - 2\sin\alpha)$

Question 20 (***+)



A uniform solid sphere of radius a, is rolling without slipping on a rough horizontal plane, with constant speed V.

The sphere reaches a vertical step of height $\frac{2}{5}a$, which is at right angles to its direction of motion, as shown in the figure above.

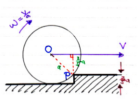
When the sphere touches the step at the point P, it begins to rotate about P, without slipping or loss of contact.

Show that

$$V < \frac{147}{125}ag .$$

proof

LOOKING AT THE DIAGRAM BELOW



MOMENT OF INERTIA OF THE SPHERE

$I_O = \frac{2}{5}ma^2$

$I_P = \frac{2}{5}ma^2 + ma^2 = \frac{7}{5}ma^2$

LET THE ANGULAR VELOCITY ABOUT P, BE Ω

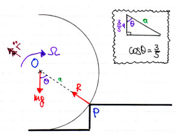
BY CONSERVATION OF MOMENTUM ABOUT P

$\Rightarrow \underbrace{(I_O\omega) + (mV)\times\frac{3}{5}a}_{\text{JUST BEFORE IMPACT}} = \underbrace{I_P\Omega}_{\text{AFTER IMPACT AT P}}$

$\Rightarrow \frac{2}{5}ma^2\omega + \frac{3}{5}mVa = \frac{7}{5}ma^2\Omega$ $\div \frac{1}{5}ma$

$\Rightarrow \boxed{2a\omega + 3V = 7a\Omega}$

NEXT WE CONSIDER THE INSTANT AFTER THE IMPACT



RADIALLY, AS IT ROTATES ABOUT P

$\Rightarrow m\ddot{r} = R - mg\cos\theta$

$\Rightarrow m(-\Omega^2 a) = R - mg\cos\theta$

$\Rightarrow R = mg\cos\theta - ma\Omega^2$

FOR ROTATION ABOUT P, $R>0$

$\Rightarrow \frac{3}{5}\cancel{m}g - \cancel{m}a\Omega^2 > 0$

$\Rightarrow a\Omega^2 < \frac{3}{5}g$ $\times 49a$

$\Rightarrow 49a^2\Omega^2 < \frac{147}{5}ag$

$\Rightarrow (2a\omega + 3V)^2 < \frac{147}{5}ag$

$\Rightarrow \left(2a\frac{V}{a} + 3V\right)^2 < \frac{147}{5}ag$

$\Rightarrow (5V)^2 < \frac{147}{5}ag$

$\Rightarrow 25V^2 < \frac{147}{5}ag$

$\Rightarrow V^2 < \frac{147}{125}ag$

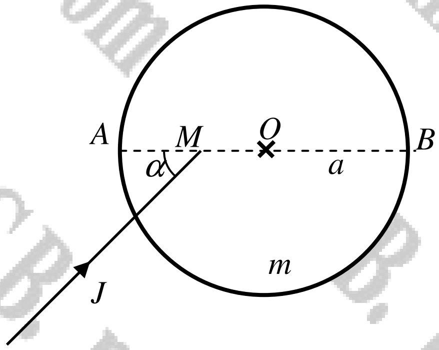
AS REQUIRED

IN ORDER TO "CREATE"

$7a\omega = 2a\omega + 3V$

OBTAINED EARLIER

Question 21 (***+)



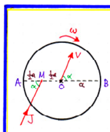
A uniform circular disc, of mass m and radius a, is lying flat on a smooth horizontal surface. The points A and B lie on the circumference of the disc, so that AB is a diameter and the point M is the midpoint of AO, where O is the centre of the disc.

The disc is initially at rest, until a horizontal impulse J is applied at M, at an angle α to AB, as shown in the figure above.

Show that the kinetic energy generated by the impulse is

$$\frac{J^2}{4m}\left(2+\sin^2\alpha\right).$$

[ ], [proof]



AS THE DISC IS UNCONSTRAINED, IT WILL AQUIRE

- LINEAR SPEED V, PARALLEL TO THE DIRECTION OF J
- ANGULAR SPEED ω, ABOUT THE CENTRE OF MASS

MOMENT OF INERTIA OF THE DISC IS

$$I_o = \frac{1}{2}ma^2$$

BY CONSERVATION OF LINEAR MOMENTUM

$$\Rightarrow J = m(v-u)$$
$$\Rightarrow J = m(v-0)$$
$$\Rightarrow J = mv$$

BY CONSERVATION OF ANGULAR MOMENTUM ABOUT O

$$\Rightarrow (J\sin\alpha)\times\frac{1}{2}a = I_o(\omega-0)$$

[MOMENT OF MOMENTUM = CHANGE IN ANGULAR MOMENTUM ABOUT O]

$$\Rightarrow \frac{1}{2}Ja\sin\alpha = \left(\frac{1}{2}ma^2\right)\omega$$
$$\Rightarrow J\sin\alpha = ma\omega$$

FINALLY ENERGIES

- K.E BEFORE $= 0$ (AT REST)
- KE AFTER $= \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$

$$= \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{1}{2}ma^2\omega^2\right)$$
$$= \frac{1}{2}mv^2 + \frac{1}{4}ma^2\omega^2$$
$$= \frac{1}{4m}\left[2m^2v^2 + m^2a^2\omega^2\right]$$
$$= \frac{1}{4m}\left[2J^2 + J^2\sin^2\alpha\right]$$
$$= \frac{J^2}{4m}\left[2+\sin^2\alpha\right]$$

AS REQUIRED

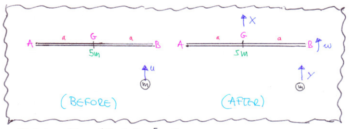
Question 22 (***+)

A thin uniform solid rod AB of mass $5m$ and length $2a$, is lying at rest on a smooth horizontal surface. A particle of mass m, moving with speed u on the same surface. The particle, moving in a perpendicular direction to the rod, strikes the rod at B. The rod begins to rotate with constant angular velocity ω.

The coefficient of restitution between the rod and the particle is $\frac{1}{3}$.

Determine ω, in terms of u and a, and find the speed of the particle after it strikes the rod, in terms of u.

$$\boxed{\omega = \frac{4u}{9a}}, \quad \boxed{\frac{7u}{27}}$$



ROD : $I_G = \frac{1}{3}(5m)a^2 = \frac{5}{3}ma^2$

BY CONSERVATION OF LINEAR MOMENTUM : $0 + mu = 5mX + mY$

$$\boxed{5X + Y = u} \quad (\text{I})$$

BY CONSERVATION OF ANGULAR MOMENTUM ABOUT G

$$0 + (mu) \times a = I_G \omega + (mY)a$$

(ROD) MOMENT OF MOMENTUM OF PARTICLE ABOUT G ROD MOMENT OF MOMENTUM OF PARTICLE ABOUT G

$$mua = \frac{5}{3}ma^2\omega + mYa$$

$$\boxed{u = \frac{5}{3}a\omega + Y} \quad (\text{II})$$

BY RESTITUTION

$$\frac{(X + \omega a) - Y}{u} = \frac{1}{3}$$

$$\boxed{X + \omega a - Y = \frac{1}{3}u} \quad (\text{III})$$

THREE EQUATIONS & THREE UNKNOWNS X, Y & ω (IN TERMS OF a, u)

SUB (I) INTO (II) & (III)

$$\boxed{Y = u - 5X} \quad (\text{I})$$

$$\left.\begin{aligned} \cancel{u} &= \frac{5}{3}a\omega + \cancel{u} - 5X \\ X + \omega a - (u - 5X) &= \frac{1}{3}u \end{aligned}\right\} \Rightarrow \left.\begin{aligned} X &= \frac{1}{3}a\omega \\ 6X &= \frac{4}{3}u - \omega a \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \left.\begin{aligned} 6X &= 2a\omega \\ 6X &= \frac{4}{3}u - a\omega \end{aligned}\right\} \Rightarrow 2a\omega = \frac{4}{3}u - a\omega$$

$$\Rightarrow 3a\omega = \frac{4}{3}u$$

$$\Rightarrow \omega = \frac{4u}{9a}$$

WE ALSO REQUIRE Y

FIRSTLY $X = \frac{1}{3}a\omega$

$$X = \frac{1}{3}a\left(\frac{4u}{9a}\right)$$

$$\boxed{X = \frac{4}{27}u}$$

FINALLY $Y = u - 5X$

$$Y = u - 5\left(\frac{4}{27}u\right)$$

$$Y = \frac{7}{27}u$$

Question 23 (***+)

A uniform sphere of mass m and radius a lies at rest on rough horizontal ground. The coefficient of friction between the ground and the sphere is μ.

The sphere is set in motion by a horizontal impulse of magnitude J, applied at a height $\frac{1}{2}a$ above the ground. The impulse is applied in a vertical plane through the centre of the sphere. The sphere begins to move with speed U, along a straight line.

a) Calculate the magnitude of the initial angular velocity of the sphere and hence deduce that initially the sphere is slipping.

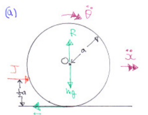
The sphere stops slipping when $t = T$.

b) Show clearly that $T = \dfrac{9U}{14\mu g}$

c) Show further that once the sphere stops slipping it moves with constant velocity, and determine its magnitude.

$$\boxed{|\omega| = \frac{5U}{4a}}, \quad \boxed{|v| = \frac{5U}{14}}$$

(a)



$\boxed{I_0 = \frac{2}{5}ma^2}$

• BY CONSERVATION OF LINEAR MOMENTUM

$J = m(U - 0)$

$\boxed{J = mU}$ $\overset{+}{\rightarrow}$

• BY CONSERVATION OF ANGULAR MOMENTUM ABOUT O

$-J \times \frac{1}{2}a = I(\omega - 0)$

$-\frac{1}{2}Ja = \frac{2}{5}ma^2\omega$

$-\frac{1}{2}\cancel{m}U\cancel{a} = \frac{2}{5}\cancel{m}a^{\cancel{2}}\omega$

$\omega = -\frac{5U}{4a}$

SPHERE CANNOT BE ROLLING AS $\dot{x} \neq a\dot{\theta}$

$U \neq a\left(-\frac{5U}{4a}\right)$ NOT EVEN THE SIGNS AGREE

(b) SLIPPING $\Rightarrow F = \mu R = \mu mg$

• "ROTATIONALLY"

$I\ddot{\theta} = F \times a$

$\Rightarrow \frac{2}{5}ma^2\ddot{\theta} = \mu mga$

$\Rightarrow \frac{2}{5}a\ddot{\theta} = \mu g$

$\Rightarrow \ddot{\theta} = \frac{5\mu g}{2a}$

$\Rightarrow \dot{\theta} = \frac{5\mu g}{2a}t + C$

when $t=0$, $\dot{\theta} = -\frac{5U}{4a}$

$\Rightarrow \boxed{\dot{\theta} = \frac{5\mu g}{2a}t - \frac{5U}{4a}}$

• "TRANSLATIONALLY"

$m\ddot{x} = -F$

$\Rightarrow m\ddot{x} = -\mu mg$

$\Rightarrow \ddot{x} = -\mu g$

$\Rightarrow \dot{x} = -\mu g t + D$

when $t=0$ $\dot{x} = U$

$\Rightarrow \boxed{\dot{x} = U - \mu g t}$

ROLLING $\Rightarrow \dot{x} = a\dot{\theta}$

$\Rightarrow U - \mu g t = \left[\frac{5\mu g}{2a}t - \frac{5U}{4a}\right]a$

$\Rightarrow U - \mu g t = \frac{5}{2}\mu g t - \frac{5}{4}U$

$\Rightarrow \frac{9}{4}U = \frac{7}{2}\mu g t$

$\Rightarrow t = \frac{9U}{14\mu g}$ AS REQUIRED

(c) ONCE ROLLING BEGINS $\ddot{x} = a\ddot{\theta}$

$\left.\begin{array}{l} I\ddot{\theta} = Fa \\ m\ddot{x} = -F \end{array}\right\} \Rightarrow \left.\begin{array}{l} \frac{2}{5}ma^2\ddot{\theta} = Fa \\ m\ddot{x} = -F \end{array}\right\} \Rightarrow \begin{array}{l} \frac{2}{5}ma\ddot{\theta} = F \\ m\ddot{x} = -F \end{array}$

$\left.\begin{array}{l} a\ddot{\theta} = \frac{5F}{2m} \\ \ddot{x} = \frac{-F}{m} \end{array}\right\} \Rightarrow \frac{5F}{2m} = -\frac{F}{m}$

$\Rightarrow m \neq 0$

$\Rightarrow \frac{5}{2}F = -F$

$\Rightarrow \frac{7}{2}F = 0$

$F = 0$

$\therefore \ddot{x} = 0$

$\therefore \dot{x} = \text{constant}$

USING $\dot{x} = U - \mu g t$

$\dot{x} = U - \mu g\left(\frac{9U}{14\mu g}\right)$

$\dot{x} = U - \frac{9}{14}U$

$\dot{x} = \frac{5}{14}U$ AS REQUIRED

Question 24 (***+)

A uniform solid circular cylinder, of radius a, is rolling without slipping with its axis horizontal, down a rough fixed plane, inclined at an angle θ to the horizontal.

The cylinder began to roll from rest.

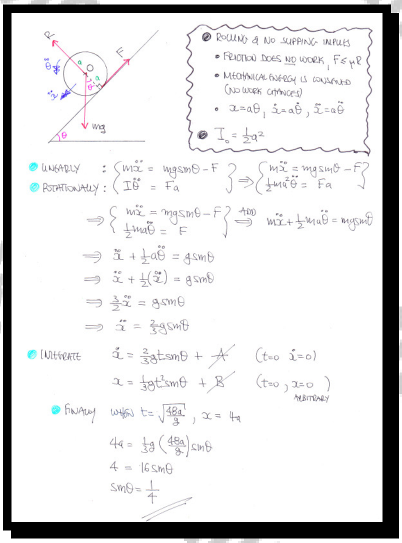
Let t be the time since the cylinder began to roll and x be the distance its axis travelled down the plane.

The cylinder began to slip when $t = \sqrt{\dfrac{48a}{g}}$ and $x = 4a$.

Show that

$$\sin\theta = \frac{1}{4}.$$

$\boxed{\text{proof}}$



- ROLLING & NO SLIPPING IMPLIES
  - FRICTION DOES NO WORK, $F \leqslant \mu R$
  - MECHANICAL ENERGY IS CONSERVED (NO WORK CHANGES)
  - $x = a\theta$, $\dot{x} = a\dot{\theta}$, $\ddot{x} = a\ddot{\theta}$
- $I_0 = \frac{1}{2}a^2$

- LINEARLY :
- ROTATIONALLY :

$$\begin{Bmatrix} m\ddot{x} = mg\sin\theta - F \\ I\ddot{\theta} = Fa \end{Bmatrix} \Rightarrow \begin{Bmatrix} m\ddot{x} = mg\sin\theta - F \\ \frac{1}{2}ma^2\ddot{\theta} = Fa \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} m\ddot{x} = mg\sin\theta - F \\ \frac{1}{2}ma\ddot{\theta} = F \end{Bmatrix} \overset{\text{ADD}}{\Longrightarrow} m\ddot{x} + \frac{1}{2}ma\ddot{\theta} = mg\sin\theta$$

$$\Rightarrow \ddot{x} + \frac{1}{2}a\ddot{\theta} = g\sin\theta$$

$$\Rightarrow \ddot{x} + \frac{1}{2}(\ddot{x}) = g\sin\theta$$

$$\Rightarrow \frac{3}{2}\ddot{x} = g\sin\theta$$

$$\Rightarrow \ddot{x} = \frac{2}{3}g\sin\theta$$

- INTEGRATE

$$\dot{x} = \frac{2}{3}gt\sin\theta + \cancel{A} \quad (t=0 \ \dot{x}=0)$$

$$x = \frac{1}{3}gt^2\sin\theta + \cancel{B} \quad (t=0, x=0 \text{ ARBITRARY})$$

- FINALLY WHEN $t = \sqrt{\frac{48a}{g}}$, $x = 4a$

$$4a = \frac{1}{3}g\left(\frac{48a}{g}\right)\sin\theta$$

$$4 = 16\sin\theta$$

$$\sin\theta = \frac{1}{4}$$

Question 25 (***+)

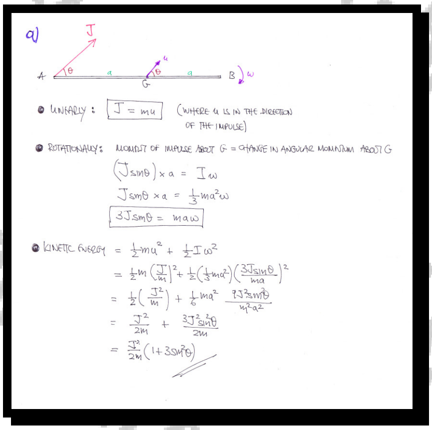
A rigid uniform rod AB of length $2a$ and mass m lies at rest on a smooth horizontal surface when it receives an impulse of magnitude J at A. The direction of the impulse is at an acute angle θ to AB, as shown in the figure above.

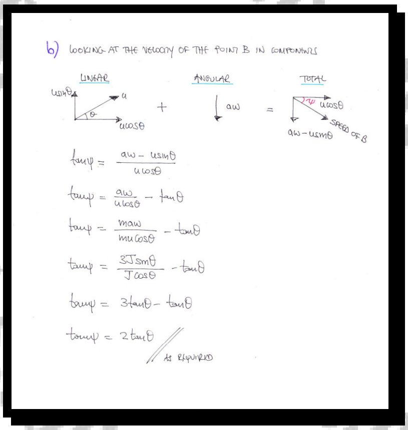
a) Find the gain in the kinetic energy of the system, as a result of this impulse, in terms of m, J and θ.

Immediately after receiving the impulse the end B, begins to move in a direction which makes an angle ψ with the AB produced.

b) Show that $\tan\psi = 2\tan\theta$

$$\boxed{\frac{J^2}{2m}\left(1+3\sin^2\theta\right)}$$





Question 26 (***+)

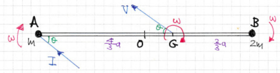
Two particles, A and B, of respective masses m and $2m$ are connected by a light rigid rod of length $2a$. The system is lies at rest on a smooth horizontal surface when it receives an impulse of magnitude I at A. The direction of the impulse is at an acute angle θ to AB, as shown in the figure above.

a) Determine the speed of each of the particles immediately after the impulse is received, in terms of m, I and θ.

b) Find the gain in the kinetic energy of the system, as a result of this impulse, in terms of m, I and θ.

$\boxed{}$, $\boxed{|V_A| = \frac{I}{3m}\sqrt{1+8\sin^2\theta}}$, $\boxed{|V_A| = \frac{I}{3m}\cos\theta}$, $\boxed{\frac{I^2}{6m}\left(1+2\sin^2\theta\right)}$

a) STARTING WITH A DIAGRAM



AS THE ROD IS LIGHT, BY INSPECTION, THE CENTRE OF MASS OF THE SYSTEM G, IS SUCH SO THAT $|AG| = \frac{4}{3}a$, $|BG| = \frac{2}{3}a$

SINCE THE SYSTEM IS NOT CONSTRAINED, ITS CENTRE OF MASS G, WILL MOVE WITH SPEED V IN THE SAME DIRECTION AS I

$\Rightarrow I = 3m(v-u)$
$\Rightarrow I = 3m(V-0)$
$\Rightarrow I = 3mV$
$\boxed{V = \frac{I}{3m}}$

THE SYSTEM WILL ALSO ACQUIRE ANGULAR VELOCITY ω, ABOUT ITS CENTRE OF MASS G

$\Rightarrow$ MOMENT OF IMPULSE ABOUT G = CHANGE IN ANG MOMENTUM ABOUT G

$\Rightarrow I\sin\theta \times |AG| = \underbrace{\left[m \times |AG|^2\right]}_{\text{MOMENT OF INERTIA OF A}} \times \omega + \underbrace{\left[2m \times |BG|^2\right]}_{\text{MOMENT OF INERTIA OF B}} \times \omega$

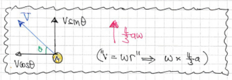
$\Rightarrow 3\cancel{m}V\sin\theta \times \frac{4}{3}\cancel{a} = \cancel{m}\left(\frac{16}{9}a^{\cancel{2}}\right)\omega + 2\cancel{m}\left(\frac{4}{9}a^{\cancel{2}}\right)\omega$

$\Rightarrow 4V\sin\theta = \frac{8}{3}a\omega$

$\Rightarrow V\sin\theta = \frac{2}{3}a\omega$

$\Rightarrow \boxed{\omega = \frac{3V\sin\theta}{2a}}$

NOW WE CAN OBTAIN THE SPEED OF EACH PARTICLE BY REFERRING TO THE DIAGRAM BELOW



$\Rightarrow (\text{SPEED})^2 = \left(V\sin\theta + \frac{4}{3}a\omega\right)^2 + (V\cos\theta)^2$

$\Rightarrow (\text{SPEED})^2 = \left(V\sin\theta + \frac{4}{3}a\,\frac{3V\sin\theta}{2a}\right)^2 + V^2\cos^2\theta$

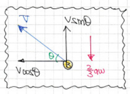
$\Rightarrow (\text{SPEED})^2 = \left(V\sin\theta + 2V\sin\theta\right)^2 + V^2\cos^2\theta$

$\Rightarrow (\text{SPEED})^2 = 9V^2\sin^2\theta + V^2\cos^2\theta$

$\Rightarrow (\text{SPEED})^2 = V^2\left(9\sin^2\theta + \cos^2\theta\right)$

$\Rightarrow (\text{SPEED})^2 = \left(\frac{I}{3m}\right)^2\left(8\sin^2\theta + 1\right)$

$\Rightarrow$ SPEED OF A $= \frac{I}{3m}\sqrt{1+8\sin^2\theta}$



$\Rightarrow (\text{SPEED})^2 = \left(V\sin\theta - \frac{2}{3}a\omega\right)^2 + (V\cos\theta)^2$

$\Rightarrow (\text{SPEED})^2 = \left(V\sin\theta - \frac{2}{3}a\,\frac{3V\sin\theta}{2a}\right)^2 + V^2\cos^2\theta$

$\Rightarrow (\text{SPEED})^2 = \cancel{\left(V\sin\theta - V\sin\theta\right)^2} + V^2\cos^2\theta$

$\Rightarrow$ SPEED OF B $= \frac{I}{3m}\cos\theta$

b) THE GAIN IN KINETIC ENERGY IS GIVEN BY

$\frac{1}{2}m\,\frac{I^2}{9m^2}\left(1+8\sin^2\theta\right) + \frac{1}{2}(2m)\,\frac{I^2}{9m^2}\cos^2\theta$

$= \frac{I^2}{18m}\left(1+8\sin^2\theta\right) + \frac{I^2}{9m}\cos^2\theta$

$= \frac{I^2}{18m}\left[1+8\sin^2\theta + 2\cos^2\theta\right]$

$= \frac{I^2}{18m}\left[3+6\sin^2\theta\right]$

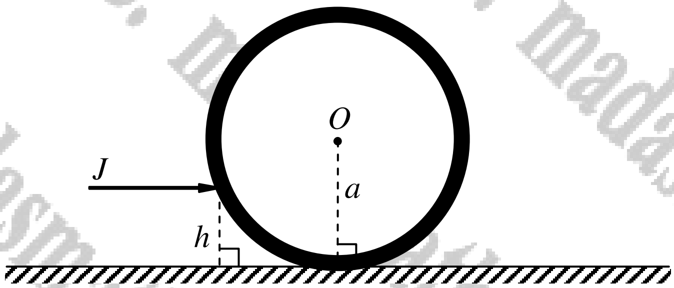
$= \frac{I^2}{6m}\left(1+2\sin^2\theta\right)$

NOTE THAT $\frac{1}{2}(3m)V^2 + \frac{1}{2}I_A\omega^2 + \frac{1}{2}I_B\omega^2$ YIELDS THE SAME ANSWER WHERE I_A & I_B ARE THE RESPECTIVE MOMENTS OF INERTIA OF A & B ABOUT G

Question 27 (****)

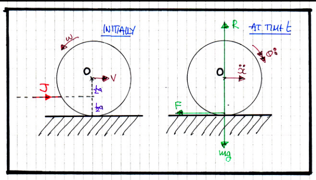
A uniform solid sphere of mass m and radius a lies at rest on a rough horizontal surface when it is set in motion by a horizontal impulse of magnitude J.

The impulse is applied at a height $\frac{1}{2}a$ above the surface, in a vertical plane through the centre of the sphere O, as shown in the figure below.



Determine the speed of O as a fraction of its original speed, when the sphere first begins to roll along the surface.

$\boxed{\frac{5}{14}}$



LOOKING AT THE INITIAL CONFIGURATION

- $I_O = \frac{2}{5}ma^2$ (STANDARD RESULT)
- BY THE CONSERVATION OF LINEAR MOMENTUM: $J = mV$, $V = \frac{J}{m}$ (INITIAL SPEED)
- BY CONSERVATION OF ANGULAR MOMENTUM ABOUT O

$$J \times \frac{1}{2}a = I\omega$$
$$J \times \frac{1}{2}a = \frac{2}{5}ma^2\omega$$
$$\omega = \frac{5J}{4ma}$$ (INITIAL ANGULAR SPEED, "BACKWARDS")

EQUATIONS OF MOTION, AT TIME t

$$\left\{\begin{matrix} m\ddot{x} = -F \\ I\ddot{\theta} = Fa \end{matrix}\right\} \Rightarrow \left\{\begin{matrix} m\ddot{x} = -F \\ \frac{2}{5}ma^2\ddot{\theta} = Fa \end{matrix}\right\} \Rightarrow \left\{\begin{matrix} m\ddot{x} = -F \\ \frac{2}{5}ma\ddot{\theta} = F \end{matrix}\right\}$$
[ADDING EQUATIONS]

$$\Rightarrow m\ddot{x} + \frac{2}{5}ma\ddot{\theta} = 0$$
$$\Rightarrow \ddot{x} + \frac{2}{5}a\ddot{\theta} = 0$$

INTEGRATE WITH RESPECT TO t

$$\Rightarrow \boxed{\dot{x} + \frac{2}{5}a\dot{\theta} = C}$$

- $t = 0$
- $\theta = 0$, $x = 0$
- $\dot{x} = V$, $\dot{\theta} = \omega$ "BACKWARDS"

$$V + \frac{2}{5}a(-\omega) = C$$
$$C = \frac{J}{m} - \frac{2}{5}a\left(\frac{5J}{4ma}\right)$$
$$C = \frac{J}{m} - \frac{J}{2m} = \frac{J}{2m}$$

$$\Rightarrow \boxed{\dot{x} + \frac{2}{5}a\dot{\theta} = \frac{J}{2m}} \text{ OR } \boxed{\dot{x} + \frac{2}{5}a\dot{\theta} = \frac{1}{2}V}$$

WHEN THE SPHERE BEGINS TO ROLL $\dot{x} = a\dot{\theta}$

$$\Rightarrow \dot{x} + \frac{2}{5}\dot{x} = \frac{1}{2}V$$
$$\Rightarrow \frac{7}{5}\dot{x} = \frac{1}{2}V$$
$$\Rightarrow \dot{x} = \frac{5}{14}V$$

$\therefore$ WHEN IT BEGINS TO ROLL WHEN IT REACHES $\frac{5}{14}$ OF ITS INITIAL SPEED V

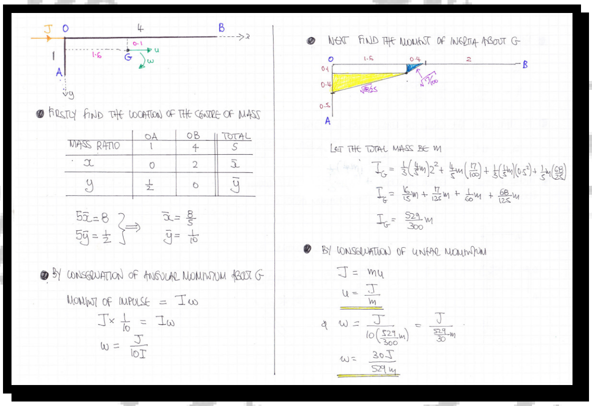
Question 28 (****+)

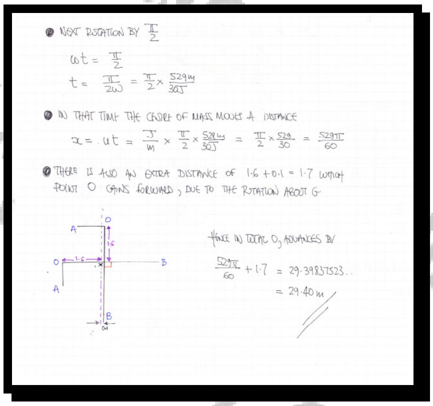
A uniform rod AB is bent at the point O, so that in the resulting $L-$shaped rigid object $\measuredangle AOB = \frac{1}{2}\pi$, $|AO| = 1$ m and $|OB| = 4$ m.

The object is placed flat on a smooth surface and an impulse is received at O in the direction OB.

In the resulting motion, determine the distance covered by O in a direction parallel to OB, until the instant the object has rotated by $\frac{1}{2}\pi$ about its centre of mass.

$\boxed{\approx 29.40 \text{ m}}$





Question 29 (****+)

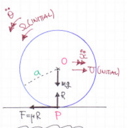
A uniform solid sphere of mass m and radius a lies at rest on a rough horizontal surface when it is set in motion by a horizontal impulse applied at a height below the centre of the sphere O.

The sphere initially begins to slide and at the same time spinning backwards. The initial speed of its centre is U and its initial angular speed about its centre is Ω.

When the sphere stops sliding, it immediately begins to roll backwards.

Show that $\Omega > \dfrac{5U}{2a}$.

proof



$I_0 = \frac{2}{5}ma^2$

- LOOKING AT THE SLIDING MOTION

$$\left.\begin{aligned} m\ddot{x} &= -\mu R \\ I\ddot{\theta} &= -(\mu R)a \end{aligned}\right\} \Rightarrow \left.\begin{aligned} \cancel{m}\ddot{x} &= -\mu \cancel{m}g \\ \tfrac{2}{5}\cancel{m}a^2\ddot{\theta} &= -\mu(\cancel{m}g)a \end{aligned}\right\} \Rightarrow$$

$$\left.\begin{aligned} \ddot{x} &= -\mu g \\ \tfrac{2}{5}a\ddot{\theta} &= -\mu g \end{aligned}\right\} \Rightarrow \boxed{\begin{aligned} \ddot{x} &= -\mu g \\ \ddot{\theta} &= -\frac{5\mu g}{2a} \end{aligned}}$$

- NEXT OBTAIN EXPRESSIONS FOR THE SPEED OF O & THE ANGULAR SPEED ABOUT O, GIVEN WHEN $t=0$, $\dot{\theta} = \Omega$, $\dot{x} = U$

- INTEGRATING EACH OF THE ACCELERATION EQUATIONS W.R.T t

$$\boxed{\dot{x} = -\mu g t + U} \quad \& \quad \boxed{\dot{\theta} = \frac{-5\mu g t}{2a} + \Omega}$$

- NOW THE SPEED OF POINT P (RELATIVE TO THE GROUND) IS $\dot{x} + a\dot{\theta}$, IN A "FORWARD" DIRECTION.

$$V = (-\mu g t + U) + a\left(-\frac{5\mu g t}{2a} + \Omega\right)$$

$$V = U - \mu g t - \tfrac{5}{2}\mu g t + a\Omega$$

$$\boxed{V = U + a\Omega - \tfrac{7}{2}\mu g t}$$

- NOW WHEN ROLLING BEGINS THE POINT IN CONTACT IS AT REST

$$0 = U + a\Omega - \tfrac{7}{2}\mu g t$$

$$\boxed{t = \frac{2(U + a\Omega)}{7\mu g}} \leftarrow \text{TIME UNTIL ROLLING STARTS}$$

- SUBSTITUTE INTO $\dot{x}$

$$\dot{x} = U - \cancel{\mu g}\left(\frac{2(U + a\Omega)}{7\cancel{\mu g}}\right)$$

$$\dot{x} = U - \tfrac{2}{7}(U + a\Omega)$$

$$\dot{x} = \tfrac{5}{7}U - \tfrac{2}{7}a\Omega$$

$$\boxed{\dot{x} = \tfrac{1}{7}(5U - 2a\Omega)}$$

- FINALLY THE SPHERE ROLLS BACK IF $\dot{x} < 0$

$$\therefore \; 5U - 2a\Omega < 0$$

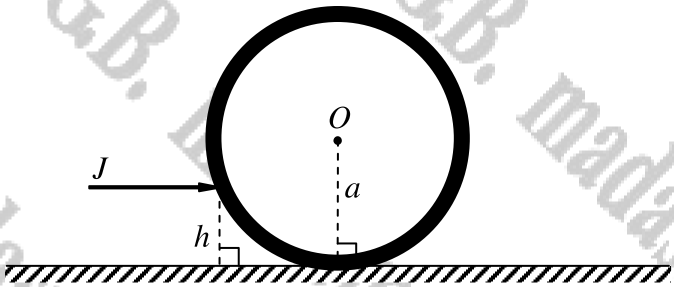
$$-2a\Omega < -5U$$

$$\Omega > \frac{5U}{2a}$$

AS REQUIRED

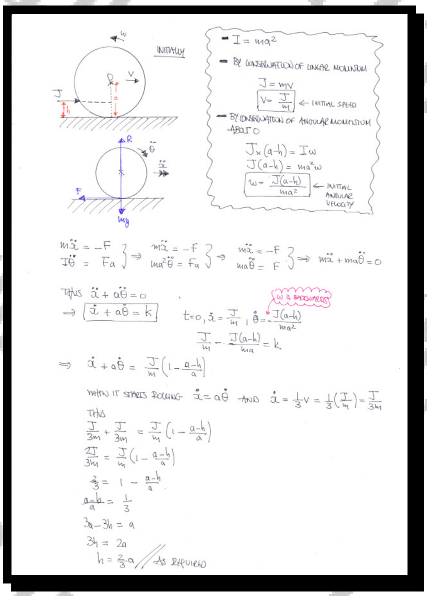
Question 30 (****+)

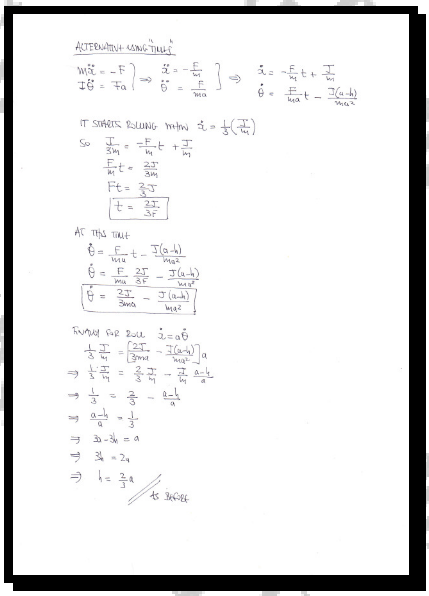
A uniform circular hoop of mass m and radius a lies at rest on a rough horizontal surface when it is set in motion by a horizontal impulse of magnitude J, applied at a height h above the surface, where $h < a$. The impulse is applied in a vertical plane through the centre of the hoop O, as shown in the figure below.



Given that the hoop first starts to roll along the surface when the speed of O is $\frac{1}{3}$ of its initial speed, show that $h = \frac{2}{3}a$.

proof





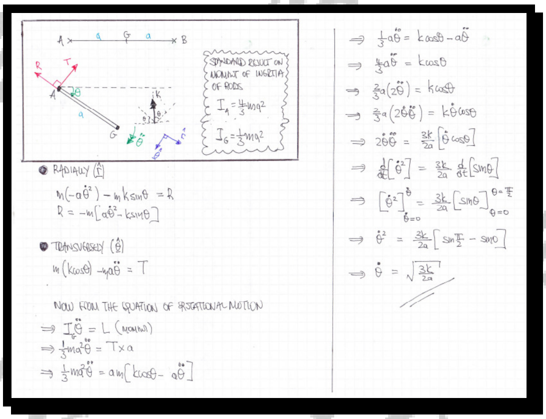
Question 31 (*****)

At time $t = 0$, the door of a train is open and at rest at right angles to the side of the train. The door is modelled as a uniform rectangular lamina, of mass m, smoothly hinged along a vertical edge. The horizontal line AB, through the centre of mass of the lamina is $2a$.

The train begins to move forward in a straight line, with constant acceleration k.

Show that the angular velocity of the door at the instant when it slams shut is $\sqrt{\dfrac{3k}{2a}}$.

$\boxed{\text{proof}}$



Question 32 (*****)

A rod AB is resting on a smooth horizontal surface, with A smoothly pivoted in a fixed position. An **identical** rod AB is also resting on a smooth horizontal surface, totally unconstrained.

Each of the two rods receives at B a horizontal impulse J, at right angles to AB.

a) Show that the kinetic energy of the pivoted rod is $\frac{3}{4}$ of the kinetic energy of the unconstrained rod.

Next consider the two rods starting from rest again.

Each of the two rods receives a horizontal impulse so the respective ends B of the rods both begin to move with speed U, at right angles to AB.

b) Show that the kinetic energy of the unconstrained rod is $\frac{3}{4}$ of the kinetic energy of the pivoted rod.

proof