VECTOR MOMENTS

Question 1 (**)

The vectors $\mathbf{i}$, $\mathbf{j}$ *and* $\mathbf{k}$ *are unit vectors mutually perpendicular to one another.*

Relative to a fixed origin O, a light rigid rod has its ends located at the points $A(0,-7,4)$ and $B(4,9,-2)$. A force $\mathbf{F}$ acts at the midpoint of the rod.

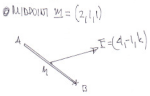
In standard vector notation, when $\mathbf{F}=(4\mathbf{i}-\mathbf{j}+k\mathbf{k})\,\text{N}$, where k is a constant, the magnitude of the moment of $\mathbf{F}$ about B has magnitude $9\sqrt{17}$ Nm.

Show clearly that $k=\pm 1$

$\boxed{\text{proof}}$

$\underline{a}=(0,-7,4)$
$\underline{b}=(4,9,-2)$

• MIDPOINT $\underline{m}=(2,1,1)$



• $\overrightarrow{MB}=\underline{m}-\underline{b}=(2,1,1)-(4,9,-2)$
$=(-2,-8,3)$

$\underline{G}_B=\overrightarrow{BM}\wedge\underline{F}$

$\underline{G}_B=\begin{vmatrix}\underline{i} & \underline{j} & k\\ -2 & -8 & 3\\ 4 & -1 & k\end{vmatrix}=(3-8k,\,2k+12,\,34)$

$|\underline{G}_B|=\sqrt{(3-8k)^2+(2k+12)^2+34^2}$

$9\sqrt{17}=\sqrt{64k^2\cancel{-48k}+9+4k^2+\cancel{48k}+144+1156}$

$1377=68k^2+1309$

$68=68k^2$

$k^2=1$

$k=\pm 1$

Question 2 (**)

The unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are oriented in the positive x direction, positive y direction and positive z direction, respectively.

Three forces $(3\mathbf{i}+\mathbf{j}+4\mathbf{k})\,\text{N}$, $\left(\frac{1}{2}\mathbf{i}-\mathbf{j}+\frac{1}{2}a\mathbf{k}\right)\text{N}$ and $(a\mathbf{i}+2\mathbf{j}-a\mathbf{k})\,\text{N}$, where a is a constant, act at the points $A(1,-2,a)$, $B(6,2,8)$ and $C(1,0,-1)$, respectively.

Distances are measured in m, relative to a fixed origin O.

a) Given that the moment of the system of the three forces about C is zero, determine the value of a.

b) Find the magnitude of the moment of the system of the three forces about O, showing clearly that its value is independent of a.

$\boxed{\text{M456}}$, $\boxed{a=-15}$, $\boxed{|\mathbf{G}_0| = 2\sqrt{2}\ \text{Nm}}$

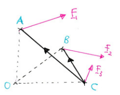
a)
$\underline{F}_1 = (3,1,4)$ | $A(1,-2,a)$
$\underline{F}_2 = \left(\frac{1}{2},-1,\frac{1}{2}a\right)$ | $B(6,2,8)$
$\underline{F}_3 = (a,2,-a)$ | $C(1,0,-1)$

RELATIVE TO C

- $\overrightarrow{CA} = \underline{a} - \underline{c} = (1,-2,a) - (1,0,-1) = (0,-2,a+1)$
- $\overrightarrow{CB} = \underline{b} - \underline{c} = (6,2,8) - (1,0,-1) = (5,2,9)$

MOMENT ABOUT C, NOTING THAT $\underline{F}_3$ HAS NO MOMENT AS IT PASSES THROUGH C

$$\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & -2 & a+1 \\ 3 & 1 & 4 \end{vmatrix} + \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 5 & 2 & 9 \\ \frac{1}{2} & -1 & \frac{1}{2}a \end{vmatrix}$$



$= [-8-a-1,\ 3a+3,\ 6] + [a+9,\ \frac{9}{2}-\frac{5}{2}a,\ -6]$
$= [-a-9,\ 3a+3,\ 6] + [a+9,\ \frac{9}{2}-\frac{5}{2}a,\ -6]$
$= [0,\ \frac{1}{2}a+\frac{15}{2},\ 0]$

AS THE MAGNITUDE IS ZERO, AND THERE IS ONLY $\underline{j}$ COMPONENT

$\Rightarrow \frac{1}{2}a + \frac{15}{2} = 0$
$\Rightarrow a + 15 = 0$
$\Rightarrow a = -15$

b) NOW MOMENT ABOUT O

$\underline{G}_0 = \sum_{i=1}^{3} (\underline{r}_i \wedge \underline{F}_i)$
$= \underline{r}_1 \wedge \underline{F}_1 + \underline{r}_2 \wedge \underline{F}_2 + \underline{r}_3 \wedge \underline{F}_3$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -2 & a \\ 3 & 1 & 4 \end{vmatrix} + \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 6 & 2 & 8 \\ \frac{1}{2} & -1 & \frac{1}{2}a \end{vmatrix} + \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 0 & -1 \\ a & 2 & -a \end{vmatrix}$$

$= [-8-a,\ 3a-4,\ 7] + [a+8,\ 4-3a,\ -7] + [2,0,2]$
$= [-8-a+a+8+2,\ 3a-4+4-3a+0,\ 7-7+2]$
$= [2,0,2]$

SO THE MAGNITUDE OF THE MOMENT IS

$|\underline{G}_0| = \sqrt{2^2+0^2+2^2} = \sqrt{8} = 2\sqrt{2}\ \text{Nm}$

Question 3 **(**)**

The vectors $\mathbf{i}$, $\mathbf{j}$ *and* $\mathbf{k}$ *are unit vectors mutually perpendicular to one another.*

Three forces, $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$, act on a rigid body at the points with position vectors $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$, respectively. This information is summarised below.

$$\mathbf{F}_1 = (3\mathbf{i} + 4\mathbf{j} - \mathbf{k}) \text{ N acting at } \mathbf{r}_1 = (\mathbf{i} + \mathbf{j} - \mathbf{k}) \text{ m}$$

$$\mathbf{F}_2 = (\mathbf{i} - 2\mathbf{j} - \mathbf{k}) \text{ N acting at } \mathbf{r}_2 = (2\mathbf{i} + \mathbf{j} + \mathbf{k}) \text{ m}$$

$$\mathbf{F}_3 = (-4\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}) \text{ N acting at } \mathbf{r}_1 = (\mathbf{i} - \mathbf{j} + 3\mathbf{k}) \text{ m}$$

Show that the system is equivalent to a couple and find the magnitude of the vector moment of this couple.

$$\boxed{|\mathbf{G}| = \sqrt{333} \approx 18.25 \text{ Nm}}$$

- $\underline{F}_1 + \underline{F}_2 + \underline{F}_3 = (3,4,-1) + (1,-2,-1) + (-4,-2,2) = (0,0,0)$

  SO NO RESULTANT FORCE

- TAKING MOMENTS ABOUT O (COUPLE IS INDEPENDENT OF POSITION)

$$\underline{r}_1 \wedge \underline{F}_1 + \underline{r}_2 \wedge \underline{F}_2 + \underline{r}_3 \wedge \underline{F}_3 = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & -1 \\ 3 & 4 & -1 \end{vmatrix} + \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 1 & 1 \\ 1 & -2 & -1 \end{vmatrix} + \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -1 & 3 \\ -4 & -2 & 2 \end{vmatrix}$$

$$= (3,-2,1) + (1,3,-5) + (4,-14,-6)$$

$$= (8,-13,-10)$$

SO THE SYSTEM REDUCES TO A COUPLE OF MAGNITUDE

$|8,-13,-10| = \sqrt{64+169+100} \simeq 18.25 \text{ Nm}$

Question 4 (**+)

The vectors $\mathbf{i}$, $\mathbf{j}$ *and* $\mathbf{k}$ *are unit vectors mutually perpendicular to one another.*

Three forces, $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$, act on a rigid body at the points with position vectors $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$, respectively. This information is summarised below.

$$\mathbf{F}_1 = (5\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) \text{ N acting at } \mathbf{r}_1 = (\mathbf{i} + \mathbf{j} + \mathbf{k}) \text{ m}$$

$$\mathbf{F}_2 = (2\mathbf{i} - 2\mathbf{j} - 7\mathbf{k}) \text{ N acting at } \mathbf{r}_2 = (\mathbf{i} - 4\mathbf{k}) \text{ m}$$

$$\mathbf{F}_3 = (-4\mathbf{i} + 5\mathbf{j} + 8\mathbf{k}) \text{ N acting at } \mathbf{r}_1 = (2\mathbf{i} + \mathbf{j} - 5\mathbf{k}) \text{ m}$$

The system of the three forces is equivalent to a single force $\mathbf{R}$ acting at the point with position vector $(2\mathbf{i} - \mathbf{k})$ m, together with a couple of moment $\mathbf{G}$.

Determine $\mathbf{R}$ and $\mathbf{G}$ in vector form.

$\boxed{\mathbf{R} = 3\mathbf{i} + 5\mathbf{j} - 2\mathbf{k}}$, $\boxed{\mathbf{G} = 15\mathbf{i} + 10\mathbf{j} - \mathbf{k}}$

FIRSTLY $\underline{R} = \sum_{i=1}^{3} \underline{F}_i = (5,2,-3) + (2,-2,-7) + (-4,5,8) = (3,5,-2)$

NOW $\sum_{i=1}^{3} \underline{r}_i \wedge \underline{F}_i = \underline{R} \wedge (2,0,-1) + \underline{G}$

$$\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 1 \\ 5 & 2 & -3 \end{vmatrix} + \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 0 & -4 \\ 2 & -2 & -7 \end{vmatrix} + \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 1 & -5 \\ -4 & 5 & 8 \end{vmatrix} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 0 & -1 \\ 3 & 5 & -2 \end{vmatrix} + \underline{G}$$

$(-5,8,-3) + (-8,-1,-2) + (33,4,14) = (5,1,10) + \underline{G}$

$(20,11,9) = (5,1,10) + \underline{G}$

$\underline{G} = (15,10,-1)$

Question 5 (***)

The vectors $\mathbf{i}$, $\mathbf{j}$ *and* $\mathbf{k}$ *are unit vectors mutually perpendicular to one another.*

Three forces $\mathbf{F}_1 = (3\mathbf{i} - \mathbf{j} + \mathbf{k})\,\text{N}$, $\mathbf{F}_2 = (\mathbf{i} - 2\mathbf{k})\,\text{N}$ and $\mathbf{F}_3$ act on a rigid body.

The force $\mathbf{F}_1$ acts through the point with position vector $(\mathbf{i} + \mathbf{j} + 2\mathbf{k})\,\text{m}$, the force $\mathbf{F}_2$ acts through the point with position vector $(\mathbf{i} - \mathbf{j})\,\text{m}$ and the force $\mathbf{F}_3$ acts through the point with position vector $(2\mathbf{i} + \mathbf{j} + \mathbf{k})\,\text{m}$.

The system of the three forces reduce to a couple $\mathbf{G}$.

a) Determine $\mathbf{G}$.

The line of action of $\mathbf{F}_3$ is changed so that the system of the three forces now reduces to the couple $(4\mathbf{i} - \mathbf{j} + \mathbf{k})\,\text{Nm}$.

b) Find a vector equation of the new line of action of $\mathbf{F}_3$.

$\boxed{(5\mathbf{i} + \mathbf{j} + 3\mathbf{k})\,\text{Nm}}$, $\boxed{\mathbf{r} = (8\mathbf{i} - \mathbf{j}\,) + \lambda(-4\mathbf{i} + \mathbf{j} + \mathbf{k})}$

(a) FIRSTLY $\sum_{i=1}^{3} \underline{F}_i = 0$ $\quad \underline{F}_1 + \underline{F}_2 + \underline{F}_3 = 0$

$(3,-1,1) + (1,0,-2) + \underline{F}_3 = 0$

$\underline{F}_3 = (-4,1,1)$

$$\underline{G} = \sum_{i=1}^{3} \underline{r}_i \wedge \underline{F}_i = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 2 \\ 3 & -1 & 1 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 0 \\ 1 & 0 & -2 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ -4 & 1 & 1 \end{vmatrix}$$

$$= (3,5,-4) + (2,2,1) + (0,-6,6)$$

$$= (5,1,3)$$

(b) NOW $\underline{r}_1 \wedge \underline{F}_1 + \underline{r}_2 \times \underline{F}_2 + (x,y,z) \wedge \underline{F}_3 = (4,-1,1)$

$$(3,5,-4) + (2,2,1) + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ -4 & 1 & 1 \end{vmatrix} = (4,-1,1)$$

$$(y-z,\ -4z-x,\ x+4y) = (-1,-8,4)$$

THUS
$$\left(\begin{array}{ccc|c} 1 & 4 & 0 & 4 \\ 0 & 1 & -1 & -1 \\ -1 & 0 & -4 & -8 \end{array}\right) \ r_{13}(1) \ \begin{pmatrix} 1 & 4 & 0 & 4 \\ 0 & 1 & -1 & -1 \\ 0 & 4 & -4 & -4 \end{pmatrix} \ \begin{array}{l} r_{23}(-4) \\ r_{21}(-4) \end{array}$$

$$\begin{pmatrix} 1 & 0 & 4 & 8 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{ie} \quad \begin{array}{l} x + 4z = 8 \\ y - z = -1 \end{array}$$

LET $z = \lambda$ $\quad \begin{array}{l} x = 8 - 4\lambda \\ y = -1 + \lambda \\ z = \lambda \end{array}$ $\quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ 1 \\ 1 \end{pmatrix}$

$\underline{r} = (8,-1,0) + \lambda(-4,1,1)$

Question 6 (***)

The vectors $\mathbf{i}$, $\mathbf{j}$ *and* $\mathbf{k}$ *are unit vectors mutually perpendicular to one another.*

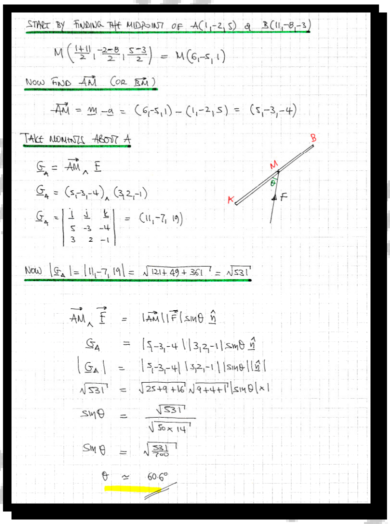
A light rod AB has its ends at $A(1,-2,5)$ and $B(11,-8,-3)$.

A single $\mathbf{F} = (3\mathbf{i} + 2\mathbf{j} - \mathbf{k})\,\text{N}$ is acting at the midpoint of the rod.

By considering vector moments determine the acute angle between $\mathbf{F}$ and AB.

You may NOT use the scalar product or the cosine rule in this question

$\boxed{}$, $\boxed{\approx 60.6°}$



START BY FINDING THE MIDPOINT OF $A(1,-2,5)$ & $B(11,-8,-3)$

$$M\left(\frac{1+11}{2}, \frac{-2-8}{2}, \frac{5-3}{2}\right) = M(6,-5,1)$$

NOW FIND $\overrightarrow{AM}$ (OR $\overrightarrow{BM}$)

$$\overrightarrow{AM} = \underline{m} - \underline{a} = (6,-5,1) - (1,-2,5) = (5,-3,-4)$$

TAKE MOMENTS ABOUT A

$$\underline{G}_A = \overrightarrow{AM} \wedge \underline{F}$$

$$\underline{G}_A = (5,-3,-4) \wedge (3,2,-1)$$

$$\underline{G}_A = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 5 & -3 & -4 \\ 3 & 2 & -1 \end{vmatrix} = (11,-7,19)$$

NOW $|\underline{G}_A| = |11,-7,19| = \sqrt{121+49+361} = \sqrt{531}$

$$\overrightarrow{AM} \wedge \overrightarrow{F} = |\overrightarrow{AM}||\overrightarrow{F}|\sin\theta\, \hat{\underline{n}}$$

$$\underline{G}_A = |5,-3,-4||3,2,-1|\sin\theta\, \hat{\underline{n}}$$

$$|\underline{G}_A| = |5,-3,-4||3,2,-1||\sin\theta||\hat{\underline{n}}|$$

$$\sqrt{531} = \sqrt{25+9+16}\sqrt{9+4+1}|\sin\theta| \times 1$$

$$\sin\theta = \frac{\sqrt{531}}{\sqrt{50 \times 14}}$$

$$\sin\theta = \sqrt{\frac{531}{700}}$$

$$\theta \approx 60.6°$$

Question 7 (***)

The standard unit vectors $\mathbf{i}$ and $\mathbf{j}$ are oriented in the positive x direction and positive y direction, respectively.

The respective equations of the lines of actions of two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ are

$$\mathbf{r}_1 = \left[\mathbf{i} + 3\mathbf{j} + \lambda(3\mathbf{i} + 4\mathbf{j})\right]\text{m} \qquad \text{and} \qquad \mathbf{r}_2 = \left[2\mathbf{j} + \mu(\mathbf{i} + \mathbf{j})\right]\text{m},$$

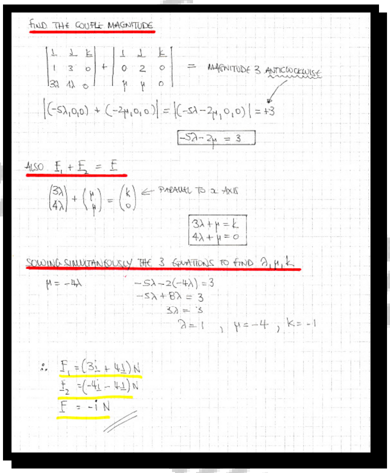
where λ and μ are scalar parameters.

It is further given that these two forces are equivalent to a single force $\mathbf{F}$ and a couple of **anticlockwise** magnitude 3 Nm.

This single force $\mathbf{F}$ is acting through the origin with a direction parallel to the x axis.

Determine in vector form expressions for $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}$.

$\boxed{\mathbf{F}_1 = (3\mathbf{i} + 4\mathbf{j})\text{N}}$, $\boxed{\mathbf{F}_2 = (-4\mathbf{i} - 4\mathbf{j})\text{N}}$, $\boxed{\mathbf{F} = -\mathbf{i}\text{ N}}$



Question 8 (***)

The vectors $\mathbf{i}$, $\mathbf{j}$ *and* $\mathbf{k}$ *are unit vectors mutually perpendicular to one another.*

Three forces $\mathbf{F}_1 = (-\mathbf{i} - 3\mathbf{j} + 4\mathbf{k})\,\text{N}$, $\mathbf{F}_2 = (-2\mathbf{i} + 4\mathbf{j} - 5\mathbf{k})\,\text{N}$ and $\mathbf{F}_3$ act on a rigid body.

The force $\mathbf{F}_1$ acts through the point with position vector $(-2\mathbf{j} + 4\mathbf{k})\,\text{m}$ and the force $\mathbf{F}_2$ acts through the point with position vector $(3\mathbf{i} - 3\mathbf{j} + 5\mathbf{k})\,\text{m}$.

The system of the three forces is in equilibrium.

a) Find a vector equation of the line of action of $\mathbf{F}_3$.

The force $\mathbf{F}_3$ is replaced by a force $\mathbf{F}_4$ acting through the point $(\mathbf{i} - \mathbf{j})\,\text{m}$.

The system of $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_4$ is now equivalent to a single force $(\mathbf{i} - \mathbf{j} - \mathbf{k})\,\text{N}$ acting through the point $(2\mathbf{i} + \mathbf{j} + \mathbf{k})\,\text{m}$, together with a couple $\mathbf{G}$.

b) Determine the magnitude of $\mathbf{G}$.

$\boxed{\mathbf{r} = (\mathbf{i} + \mathbf{j}) + \lambda(3\mathbf{i} - \mathbf{j} + \mathbf{k})}$, $\boxed{|\mathbf{G}| = \sqrt{30} \approx 5.48 \text{ Nm}}$

(a) FIRSTLY $\sum_{i=1}^{3} F_i = 0 \Rightarrow \underline{F}_1 + \underline{F}_2 + \underline{F}_3 = \underline{0}$

$(-1,-3,4) + (-2,4,-5) + \underline{F}_3 = \underline{0}$

$\underline{F}_3 = (3,-1,1)$

$\sum_{i=1}^{3} \underline{r}_i \wedge \underline{F}_i = 0 \Rightarrow \begin{vmatrix} i & j & k \\ 0 & -2 & 4 \\ -1 & -3 & 4 \end{vmatrix} + \begin{vmatrix} i & j & k \\ 3 & -3 & 5 \\ -2 & 4 & -5 \end{vmatrix} + \begin{vmatrix} i & j & k \\ x & y & z \\ 3 & -1 & 1 \end{vmatrix} = 0$

$\Rightarrow (4,-4,-2) + (-5,5,6) + (y+z, 3z-x, -x-3y) = 0$

$\Rightarrow (y+z, 3z-x, -x-3y) = (1,-1,-4)$

THUS $\begin{pmatrix} 1 & 3 & 0 & 4 \\ -1 & 0 & 3 & -1 \\ 0 & 1 & 1 & 1 \end{pmatrix} r_{12}(1) \begin{pmatrix} 1 & 3 & 0 & 4 \\ 0 & 3 & 3 & 3 \\ 0 & 1 & 1 & 1 \end{pmatrix} r_2(\tfrac{1}{3}) \begin{pmatrix} 1 & 3 & 0 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$

$r_{23}(-1) \begin{pmatrix} 1 & 3 & 0 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} r_{21}(-3) \begin{pmatrix} 1 & 0 & -3 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

SO $\begin{matrix} x - 3y = 1 \\ y + z = 1 \end{matrix}$ $\left.\begin{matrix} x = 1 + 3z \\ y = 1 - z \end{matrix}\right\} \Rightarrow \begin{matrix} x = 1 + 3\lambda \\ y = 1 - \lambda \\ z = \lambda \end{matrix}$

LET $z = \lambda$

$\therefore \underline{r} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$

$\underline{r} = (1,1,0) + \lambda(3,-1,1)$

(b) NOW $\underline{F}_1 + \underline{F}_2 + \underline{F}_4 = (1,-1,-1)$

$(-1,-3,4) + (-2,4,-5) + \underline{F}_4 = (1,-1,-1)$

$(-3,1,-1) + \underline{F}_4 = (1,-1,-1)$

$\underline{F}_4 = (4,-2,0)$

$\underline{r}_1 \wedge \underline{F}_1 + \underline{r}_2 \wedge \underline{F}_2 + \underline{r}_4 \wedge \underline{F}_4 = \underline{r} \wedge \underline{F} + \underline{G}$

$(4,-4,-2) + (-5,5,6) + \begin{vmatrix} i & j & k \\ 4 & -2 & 0 \\ 1 & -1 & 0 \end{vmatrix} = \begin{vmatrix} i & j & k \\ 2 & 1 & 1 \\ 1 & -1 & -1 \end{vmatrix} + \underline{G}$

$(-1,1,4) + (0,0,-2) = (0,3,-3) + \underline{G}$

$\underline{G} = (-1,-2,5)$

$|G| = \sqrt{1+4+25} = \sqrt{30} \simeq 5.48 \text{ Nm}$

Question 9 (***)

The unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are oriented in the positive x direction, positive y direction and positive z direction, respectively.

Three forces

$$\mathbf{F}_1 = (3\mathbf{i} - 2\mathbf{j})\,\text{N}, \quad \mathbf{F}_2 = (4\mathbf{i} - \mathbf{j} + 2\mathbf{k})\,\text{N} \quad \text{and} \quad \mathbf{F}_2 = (3\mathbf{j} - 4\mathbf{k})\,\text{N},$$

are acting at the points $A_1(-1,1,0)$, $A_2(2,0,5)$ and $A_3(-6,2,1)$, respectively.

a) Show that the system reduces to a single force $\mathbf{F}$.

b) Find an equation of the line of action of $\mathbf{F}$.

$\boxed{}$, $\boxed{\mathbf{r} = -4\mathbf{i} + 3\mathbf{j} + \lambda(7\mathbf{i} - 2\mathbf{k})}$

a)

$\underline{r}_1 = (-1,1,0)$ $\quad \underline{F}_1 = (3,-2,0)$

$\underline{r}_2 = (2,0,5)$ $\quad \underline{F}_2 = (4,-1,2)$

$\underline{r}_3 = (-6,2,1)$ $\quad \underline{F}_3 = (0,3,-4)$

FIRSTLY COLLECT THE REQUIRED INFORMATION

- $\sum_{i=1}^{3} \underline{F}_i = (3,-2,0) + (4,-1,2) + (0,3,-4) = (7,0,-2) \neq (0,0,0)$

- $\underline{G}_0 = \sum_{i=1}^{3} (\underline{r}_i \wedge \underline{F}_i)$

$$\underline{G}_0 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 0 \\ 3 & -2 & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & 5 \\ 4 & -1 & 2 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -1 & 2 \\ 0 & 3 & -4 \end{vmatrix}$$

$\underline{G}_0 = (0,0,-1) + (5,16,-2) + (-11,-24,-18)$

$\underline{G}_0 = (-6,-8,-21)$

- THUS WE HAVE

$$\left(\sum_{i=1}^{3} \underline{F}_i\right) \cdot \left(\sum_{i=1}^{3} (\underline{r}_i \wedge \underline{F}_i)\right) = (7,0,-2) \cdot (-6,-8,-21) = -42 + 0 + 42 = 0$$

AS $\sum_{i=1}^{3} \underline{F}_i \neq 0$ AND $\left(\sum_{i=1}^{3} \underline{F}_i\right) \cdot \left(\sum_{i=1}^{3} (\underline{r}_i \wedge \underline{F}_i)\right) = 0$ WITHOUT $\sum_{i=1}^{3} \underline{r}_i \wedge \underline{F}_i = 0$, THEN THE SYSTEM REDUCES TO A FORCE

b)

- THE SYSTEM REDUCES TO A COUPLE — LET THE RESULTANT FORCE $\sum_{i=1}^{3} F_i$ ACT THROUGH $P(x,y,z)$

$$\Rightarrow \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ 7 & 0 & -2 \end{vmatrix} = \underline{G}_0$$

$\Rightarrow (-2y,\ 7z + 2x,\ -7y) = (-6,-8,-21)$

- THUS WE HAVE BY EQUATING COMPONENTS

$-2y = -6$ $\quad$ & $\quad 7z + 2x = -8$

$y = 3$ $\quad$ LET $z = 0$

$2x = -8$

$x = -4$

- I.E THE EQUATIONS CAN BE SATISFIED BY

$x = -4$

$y = 3$

$z = 0$

$\therefore\ \underline{r} = (-4,3,0) + \lambda(7,0,-2)$

Question 10 (***)

The standard unit vectors $\mathbf{i}$ and $\mathbf{j}$ are oriented in the positive x direction and positive y direction, respectively.

Three forces $\mathbf{F}_1 = 4\mathbf{i} + b\mathbf{j}$, $\mathbf{F}_2 = 3a\mathbf{i} + 2b\mathbf{j}$ and $\mathbf{F}_3 = 10b\mathbf{i} + 3\mathbf{j}$, where a and b are scalar constants, are acting at the points $A_1(1,2)$, $A_2(4,-2)$ and $A_3(-3,-5)$, respectively.

a) Determine the magnitude and direction of the total moment of these three forces about O.

b) Find, by direct calculation, the magnitude and direction of the total moment of these three forces about C.

$\boxed{}$, $\boxed{|\mathbf{G}_O| = 64 \text{ Nm, clockwise}}$, $\boxed{|\mathbf{G}_C| = 64 \text{ Nm, clockwise}}$

a)

FORCE	$4\mathbf{i} + b\mathbf{j}$	$3a\mathbf{i} + 2b\mathbf{j}$	$10b\mathbf{i} + 3\mathbf{j}$
POINT	$(1,2)$	$(4,-2)$	$(-3,-5)$

FIRSTLY TOTAL FORCE IS ZERO

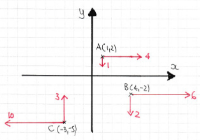
$(4\mathbf{i} + b\mathbf{j}) + (3a\mathbf{i} + 2b\mathbf{j}) + (10b\mathbf{i} + 3\mathbf{j}) = \mathbf{0}$

$(4 + 3a + 10b)\mathbf{i} + (3b + 3)\mathbf{j} = \mathbf{0}$

$3b + 3 = 0$, $3b = -3$, $b = -1$

$4 + 3a + 10b = 0$, $4 + 3a - 10 = 0$, $3a = 6$, $a = 2$

NEXT DRAW A DIAGRAM – TAKE MOMENTS ABOUT O



$-(4\times2)$, $-(1\times1)$ } $\mathbf{F}_1$ AT A

$+(6\times2)$, $-(2\times4)$ } $\mathbf{F}_2$ AT B

$-(3\times3)$, $-(10\times5)$ } $\mathbf{F}_3$ AT C

∴ TOTAL MOMENT IS

$= -8 - 1 + 12 - 8 - 9 - 50$

$= -64$

$=$ 64 Nm CLOCKWISE

b) MOMENT ABOUT C NOW

$\underbrace{-(1\times4) - (4\times7)}_{\mathbf{F}_1} \underbrace{- (6\times3) - (2\times7)}_{\mathbf{F}_2} = -4 - 28 - 18 - 14$

$= -64$

$=$ 64 Nm CLOCKWISE

ALTERNATIVE BY CROSS PRODUCTS

a) $\mathbf{G}_O = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 0 \\ 4 & -1 & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -2 & 0 \\ 6 & -2 & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & -5 & 0 \\ -10 & 3 & 0 \end{vmatrix}$

$\mathbf{G}_O = (0,0,-9) + (0,0,4) + (0,0,-59)$

$G_O = (0,0,-64)$

I.E $|G_O| = 64$ CLOCKWISE

b) $\overrightarrow{CA} = \mathbf{a} - \mathbf{c} = (1,2) - (-3,-5) = (4,7)$

$\overrightarrow{CB} = \mathbf{b} - \mathbf{c} = (4,-2) - (-3,-5) = (7,3)$

$\mathbf{G}_C = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 7 & 0 \\ 4 & -1 & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 7 & 3 & 0 \\ 6 & -2 & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 0 \\ -10 & 3 & 0 \end{vmatrix}$

$\mathbf{G}_C = (0,0,-32) + (0,0,-32) + (0,0,0)$

$\mathbf{G}_C = (0,0,-64)$

I.E $\mathbf{G}_C = 64$ CLOCKWISE

Question 11 (***+)

The standard unit vectors $\mathbf{i}$ and $\mathbf{j}$ are oriented in the positive x direction and positive y direction, respectively.

Three forces

$$\left[(3a+1)\mathbf{i}+3\mathbf{j}\right]\text{N}, \quad \left[(a-10)\mathbf{i}-2\mathbf{j}\right]\text{N} \quad \text{and} \quad \left[\mathbf{i}+(1-a)\mathbf{j}\right]\text{N},$$

where a is a constant, act at the points $A(1,2)$, $B(2,0)$ and $C(4,-1)$, respectively.

Distances are measured in m, relative to a fixed origin O.

a) Given that the system of the three forces reduces to a couple about O, find the magnitude and direction of this couple.

b) Given instead that the system of the three forces reduces to single force $\mathbf{F}$, determine the equation of the line of action of $\mathbf{F}$.

$\boxed{\quad\quad}$, $\boxed{|\mathbf{G}_0| = 18 \text{ Nm, anticlockwise}}$, $\boxed{y = -\frac{1}{4}x}$

$\underline{F}_1 = (3a+1)\underline{i} + 3\underline{j}$ AT $A(1,2)$
$\underline{F}_2 = (a-10)\underline{i} - 2\underline{j}$ AT $B(2,0)$
$\underline{F}_3 = \underline{i} + (1-a)\underline{j}$ AT $C(4,-1)$

a) IF THE SYSTEM IS TO REDUCE TO A COUPLE ABOUT O

$$\underline{F}_1 + \underline{F}_2 + \underline{F}_3 = (0,0)$$
$$(3a+1,3) + (a-10,-2) + (1,1-a) = (0,0)$$
$$(4a-8, 2-a) = (0,0)$$

THIS IN INDEED POSSIBLE IF $a=2$

FINDING THE MOMENT ABOUT O, WITH $a=2$

$$\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 2 & 0 \\ 7 & 3 & 0 \end{vmatrix} + \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 0 & 0 \\ -8 & -2 & 0 \end{vmatrix} + \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 4 & -1 & 0 \\ 1 & -1 & 0 \end{vmatrix}$$

$$= [0,0,3-14] + [0,0,-4] + [0,0,-4+1]$$
$$= [0,0,-11] + [0,0,-4] + [0,0,-3]$$
$$= [0,0,-18]$$

HENCE, A MAGNITUDE OF 18 Nm, ANTICLOCKWISE

b) NOW FIND THE MOMENT OF THE FORCES ABOUT O, IN TERMS OF a

$$\underline{G}_0 = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 2 & 0 \\ 3a+1 & 3 & 0 \end{vmatrix} + \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 0 & 0 \\ a-10 & -2 & 0 \end{vmatrix} + \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 4 & -1 & 0 \\ 1 & 1-a & 0 \end{vmatrix}$$

$$\underline{G}_0 = (0,0,3-6a-2) + (0,0,-4) + (0,0,4-4a+1)$$
$$\underline{G}_0 = (0,0,1-6a) + (0,0,-4) + (0,0,5-4a)$$
$$\underline{G}_0 = (0,0,2-10a)$$

THIS YIELDS ZERO MOMENT IF $a = \frac{1}{5}$

THEN $\underline{F}_1 + \underline{F}_2 + \underline{F}_3 = (4a-8, 2-a) \leftarrow$ FROM EARLIER
$$= \left(-\tfrac{36}{5}, \tfrac{9}{5}\right)$$

THE LINE PASSES THROUGH THE ORIGIN (ZERO MOMENT ABOUT O)

$$y = mx + \cancel{c}$$

$$m = \frac{\frac{9}{5} - 0}{-\frac{36}{5} - 0} = -\frac{1}{4}$$

$\therefore$ REDUCES TO $\underline{F} = -\frac{36}{5}\underline{i} + \frac{9}{5}\underline{j}$, WHICH ACTS ALONG THE LINE $y = -\frac{1}{4}x$

Question 12 (***+)

The vectors $\mathbf{i}$, $\mathbf{j}$ *and* $\mathbf{k}$ *are unit vectors mutually perpendicular to one another.*

Three forces $\mathbf{F}_1 = (\mathbf{i}+\mathbf{j}+\mathbf{k})\,\text{N}$, $\mathbf{F}_2 = (\mathbf{i}+2\mathbf{k})\,\text{N}$ and $\mathbf{F}_3$ act on a rigid body.

The force $\mathbf{F}_1$ acts through the point with position vector $(\mathbf{i}+2\mathbf{j}+2\mathbf{k})\,\text{m}$ and the force $\mathbf{F}_2$ acts through the point with position vector $(-\mathbf{i}-\mathbf{j}+\mathbf{k})\,\text{m}$.

The system of the three forces is in equilibrium.

Show that the line of action of $\mathbf{F}_3$ passes through the point with position vector $-2\mathbf{k}$.

proof

$$\boxed{\begin{aligned}&\underline{F}_1 = (1,1,1) \qquad \underline{r}_1 = (1,2,2)\\ &\underline{F}_2 = (1,0,2) \qquad \underline{r}_2 = (-1,-1,1)\end{aligned}}$$

- IF IN EQUILIBRIUM $\sum_{i=1}^{3} \underline{F}_i = \underline{0}$

$\therefore \underline{F}_3 = (-2,-1,-3)$ BY INSPECTION

- ALSO THE MOMENTS MUST BE IN EQUILIBRIUM, I.E. $\sum_{i=1}^{3} (\underline{r}_i \wedge \underline{F}_i) = \underline{0}$

$$\Rightarrow \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 2 & 2 \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & -1 & 1 \\ 1 & 0 & 2 \end{vmatrix} + \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ x & y & z \\ -2 & -1 & -3 \end{vmatrix} = (0,0,0)$$

$$\Rightarrow (0,1,-1) + (-2,3,1) + (z-3y,\ 3x-2z,\ 2y-x) = (0,0,0)$$

$$\Rightarrow (-2,4,0) + (z-3y,\ 3x-2z,\ 2y-x) = (0,0,0)$$

$$\Rightarrow \left(\begin{array}{cccc} -x & +2y & & = 0 \\ 3x & & -2z & = -4 \\ & -3y & +z & = 2 \end{array}\right)$$

$$\Rightarrow \begin{pmatrix} -1 & 2 & 0 & 0 \\ 3 & 0 & -2 & -4 \\ 0 & -3 & 1 & 2 \end{pmatrix} \ r_1(-1) \begin{pmatrix} 1 & -2 & 0 & 0 \\ 3 & 0 & -2 & -4 \\ 0 & -3 & 1 & 2 \end{pmatrix} \ r_{12}(-3) \begin{pmatrix} 1 & -2 & 0 & 0 \\ 0 & 6 & -2 & -4 \\ 0 & -3 & 1 & 2 \end{pmatrix}$$

$$r_2(\tfrac{1}{2}) \begin{pmatrix} 1 & -2 & 0 & 0 \\ 0 & 3 & -1 & -2 \\ 0 & -3 & 1 & 2 \end{pmatrix} \ r_{23}(1) \begin{pmatrix} 1 & -2 & 0 & 0 \\ 0 & 3 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \ r_2(\tfrac{1}{3}) \begin{pmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & -\frac{1}{3} & -\frac{2}{3} \end{pmatrix}$$

$$r_{21}(2) \begin{pmatrix} 1 & 0 & -\frac{2}{3} & -\frac{4}{3} \\ 0 & 1 & -\frac{1}{3} & -\frac{2}{3} \end{pmatrix}$$

- IN OTHER WORDS

$$\left(\begin{array}{l} x \quad -\frac{2}{3}z = -\frac{4}{3} \\ y - \frac{1}{3}z = -\frac{2}{3} \\ z = z \end{array}\right) \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{4}{3} \\ \frac{2}{3} \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{2}{3}z \\ \frac{1}{3}z \\ z \end{pmatrix}$$

- TIDY

$$\rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{4}{3} \\ \frac{2}{3} \\ 0 \end{pmatrix} + \begin{pmatrix} 2/3 \\ 1/3 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{4}{3} \\ -\frac{2}{3} \\ 0 \end{pmatrix} - 2\begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2/3 \\ 1/3 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 2/3 \\ 1/3 \\ 1 \end{pmatrix}$$

$$\Rightarrow \underline{r} = (0,0,-2) + \mu(2,1,3)$$

I.E IT PASSES THROUGH $-2\underline{k}$