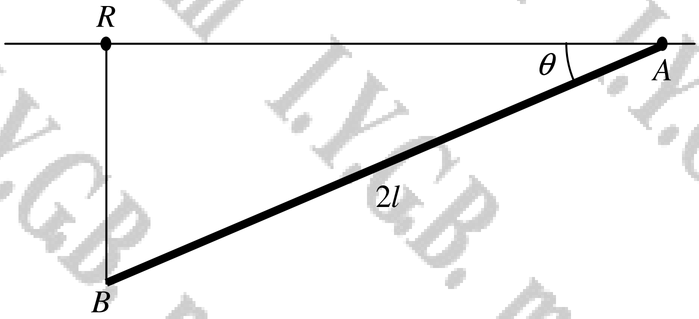
# EQUILIBRIUM
# and
# POTENTIAL ENERGY

**Question 1 (**)**

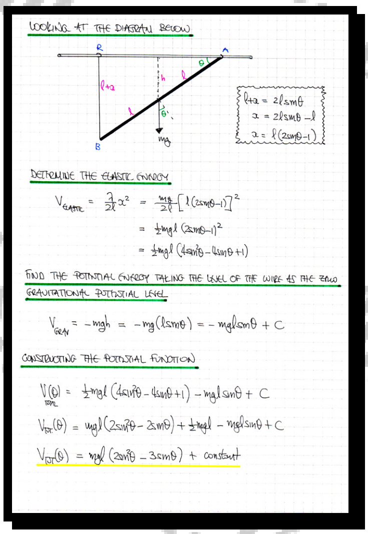


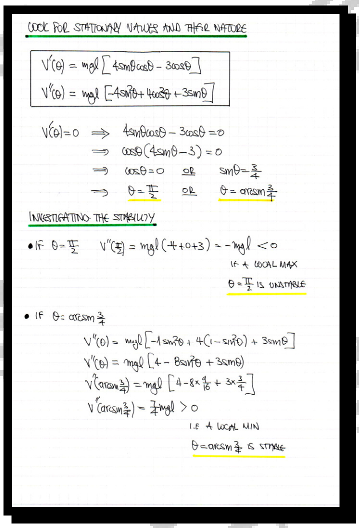

A small light smooth ring is attached to the end $A$ of a uniform rod $AB$, of mass m and length $2l$. The ring is threaded on a smooth horizontal wire. Another small light smooth ring $R$ is threaded on the same wire and a light elastic spring has one end attached to $R$ and the other end attached to $B$. The system rest is equilibrium with $B$ vertically below $R$, as shown in the figure above.

The angle $BAR$ is denoted by $\theta$, $0 \le \theta \le \dfrac{\pi}{2}$.

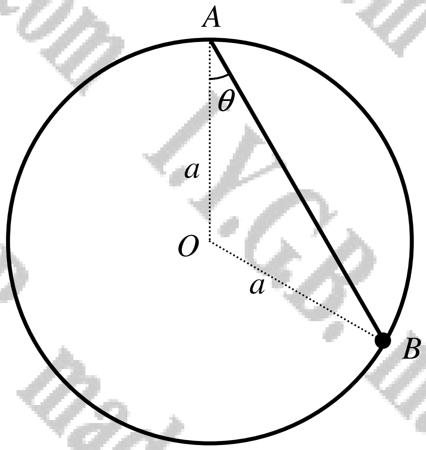
Find the two positions of equilibrium of the system and determine their stability.

$\boxed{\phantom{xxxx}}$, $\boxed{\theta = \dfrac{\pi}{2}, \text{ unstable}}$, $\boxed{\theta = \arcsin\left(\dfrac{3}{4}\right), \text{ stable}}$





**Question 2 (**+)**




The figure above shows a small bead $B$ of mass $m$, threaded on a smooth circular wire of radius $a$, which is fixed in a vertical plane. The centre of the circle is at $O$, and the highest point of the circle is at $A$.

A light elastic string of natural length $a$ and modulus of elasticity $3mg$ has one end attached to $A$ and the other end attached to $B$.

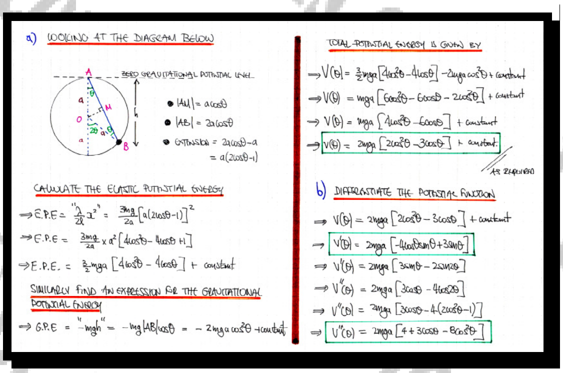
The acute angle $OAB$ is denoted by $\theta$.

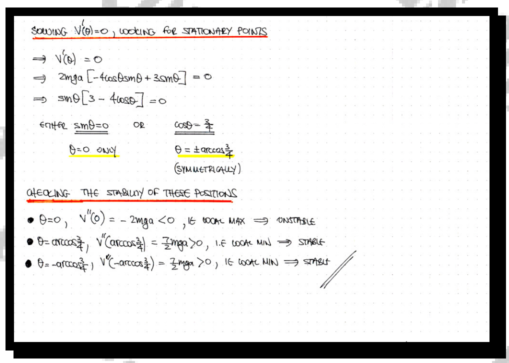
**a)** Given that the string is taut, show that the potential energy of the system is

$$2mga\left(2\cos^2\theta - 3\cos\theta\right) + \text{constant}.$$

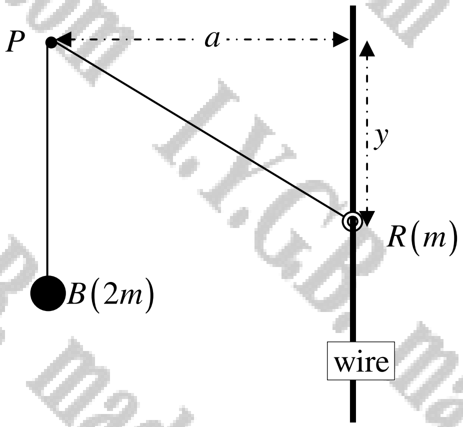
**b)** Hence find the three positions of equilibrium of the system and determine their stability.

$\boxed{\phantom{xxxx}}$, $\boxed{\theta = 0, \text{ unstable}}$, $\boxed{\theta = \arccos\left(\frac{3}{4}\right), \text{ stable}}$, $\boxed{\theta = -\arccos\left(\frac{3}{4}\right), \text{ stable}}$





**Question 3** (***)




A ring $R$, of mass $m$, is threaded on a smooth wire which is securely taut in a vertical direction. The ring is connected to a particle $B$, of mass $2m$, by a light inextensible string which passes over a smooth peg $P$. The peg is at a distance $a$ away from the wire.

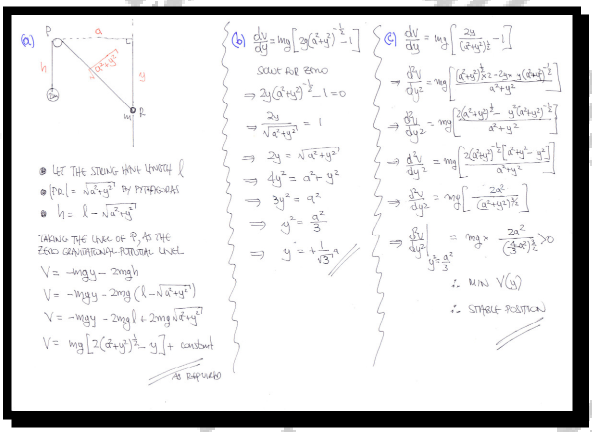
**a)** Given that $y$ is the vertical distance of the $R$ below the level of $P$, show that the total potential energy of the system $V$, is given by

$$V = mg\left[2\sqrt{a^2+y^2} - y\right] + \text{constant}.$$

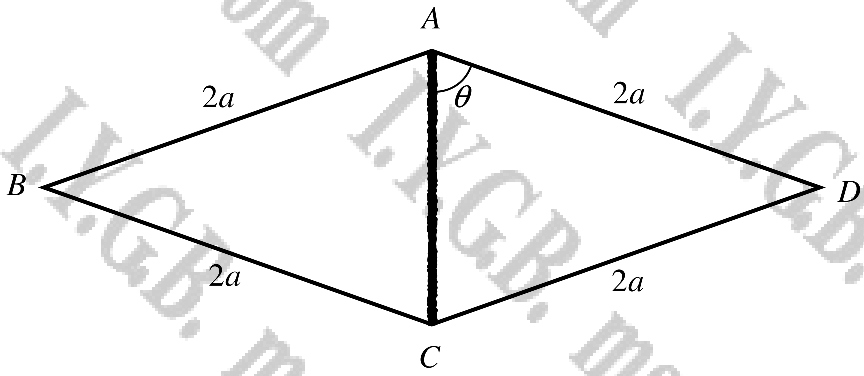
**b)** Find the value $y$, in terms of $a$ for which the system is in equilibrium.

**c)** Determine the stability of this position of equilibrium.

$y = \dfrac{1}{\sqrt{3}}a$, stable



**Question 4** (***)



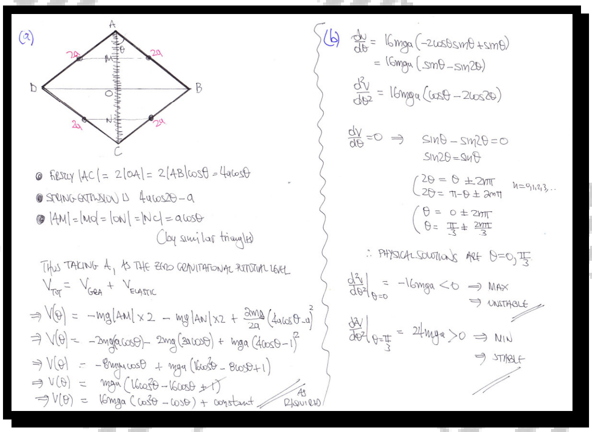

Four identical uniform rods each of mass $m$ and length $2a$ are smoothly joined together to form a rhombus $ABCD$. The vertex of the rhombus is smoothly pinned at a fixed point $A$ and a light elastic string connects $A$ and $C$ of the rhombus. The string has natural length $a$ and modulus of elasticity $2mg$. The rhombus hangs in equilibrium with $C$ vertically below $A$

**a)** Given that $\measuredangle CAD = \theta$, show that the total potential energy of the system $V$, is given by

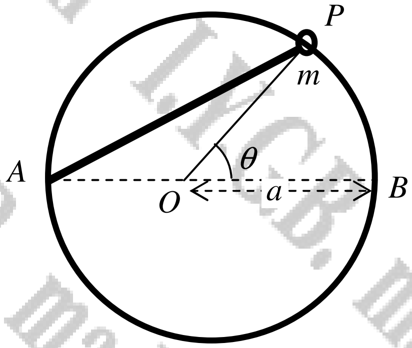
$$V = 16mga\left(\cos^2\theta - \cos\theta\right) + \text{constant}.$$

**b)** Use calculus to find the value or values of $\theta$ for which the system is in equilibrium, determining further the stability of these positions.

$\boxed{\theta = 0,\ \text{unstable}}$, $\boxed{\theta = \frac{\pi}{3},\ \text{stable}}$



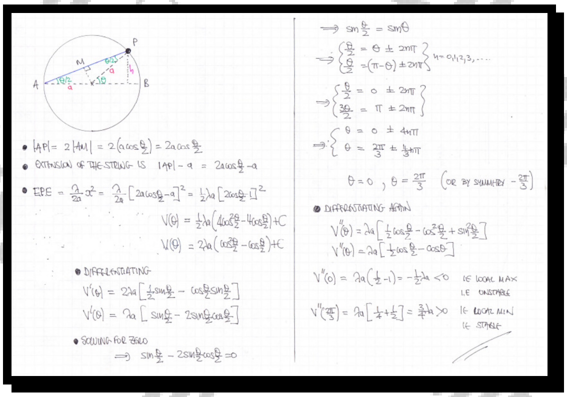
**Question 5** (***)



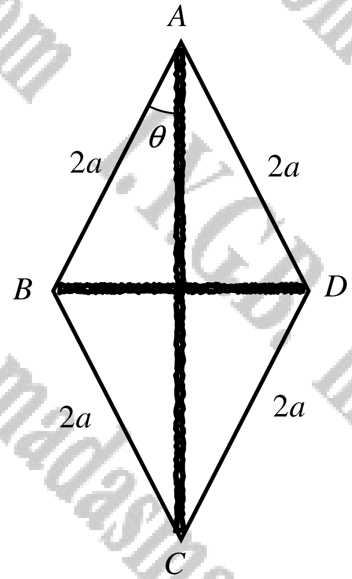

A small light ring $P$ is free to slide on a smooth wire, bent into the shape of a circular hoop of radius $a$, centred at $O$. The wire is fixed in a vertical plane. The point $B$ lies on the wire so that $AOB$ is a horizontal diameter. A light elastic string has one end attached to $A$ and the other end attached to $P$. The string has natural length $a$ and modulus of elasticity $\lambda$.

Given that the angle $POB$ is denoted by $\theta$, find each of the values of $\theta$, for which the above described system is in equilibrium and determine their stability.

$\boxed{\theta = 0°, \text{ unstable}}$, $\boxed{\theta = \pm 120°, \text{ stable}}$



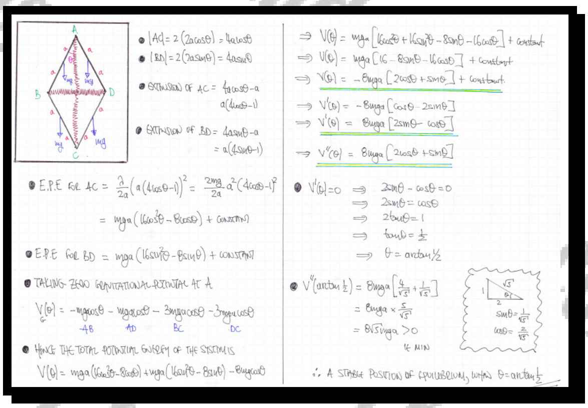
**Question 6 (***)**



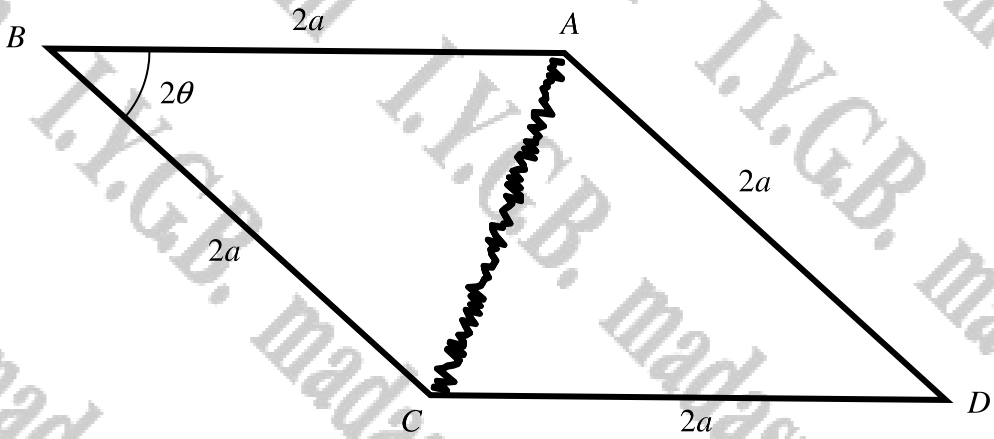

Four identical uniform rods each of mass $m$ and length $2a$ are smoothly joined together to form a rhombus $ABCD$. The vertex of the rhombus is smoothly pinned at a fixed point $A$. A light elastic string connects $A$ and $C$ with an identical string connecting $B$ to $D$. Each of the two strings has natural length $a$ and modulus of elasticity $2mg$. The rhombus hangs in equilibrium with $C$ vertically below $A$, as shown in the figure above.

Given that $\measuredangle BAC = \theta$, use calculus to find the value or values of $\theta$ for which the system is in equilibrium, determining further the stability of these positions.

$\boxed{\theta = \arctan\frac{1}{2},\ \text{stable}}$



**Question 7 (***)**




Four identical uniform rods each of mass $m$ and length $2a$ are smoothly joined together to form a rhombus $ABCD$.

The rod $AB$ is fixed in a horizontal position and a light elastic string connects the joints $A$ and $C$.

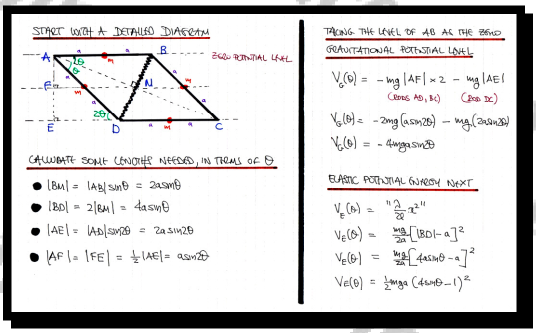
The string has natural length $a$ and modulus of elasticity $mg$.

The rhombus hangs in equilibrium with $C$ and $D$ at the same horizontal level, vertically below $AB$.

Given that $\measuredangle ABC = 2\theta$, $0° < \theta < 45°$, show that the positions of equilibrium of the system are solutions of the equation

$$2\sin 2\theta - 2\cos 2\theta - \cos\theta = 0.$$

, proof


START WITH A DETAILED DIAGRAM

ZERO POTENTIAL LEVEL

CALCULATE SOME LENGTHS NEEDED, IN TERMS OF $\theta$

- $|BM| = |AB|\sin\theta = 2a\sin\theta$
- $|BD| = 2|BM| = 4a\sin\theta$
- $|AE| = |AD|\sin 2\theta = 2a\sin 2\theta$
- $|AF| = |FE| = \frac{1}{2}|AE| = a\sin 2\theta$

TAKING THE LEVEL OF AB AS THE ZERO GRAVITATIONAL POTENTIAL LEVEL

$V_G(\theta) = -mg|AF| \times 2 - mg|AE|$ (RODS AD, BC) (ROD DC)

$V_G(\theta) = -2mg(a\sin 2\theta) - mg(2a\sin 2\theta)$

$V_G(\theta) = -4mga\sin 2\theta$

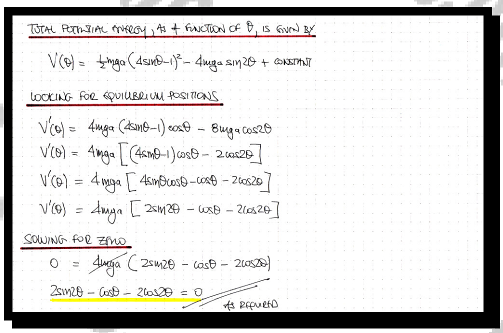
ELASTIC POTENTIAL ENERGY NEXT

$V_E(\theta) = $ "$\frac{\lambda}{2l}x^2$"

$V_E(\theta) = \frac{mg}{2a}\left[|BD| - a\right]^2$

$V_E(\theta) = \frac{mg}{2a}\left[4a\sin\theta - a\right]^2$

$V_E(\theta) = \frac{1}{2}mga(4\sin\theta - 1)^2$


TOTAL POTENTIAL ENERGY, AS A FUNCTION OF $\theta$, IS GIVEN BY

$V(\theta) = \frac{1}{2}mga(4\sin\theta - 1)^2 - 4mga\sin 2\theta + \text{CONSTANT}$

LOOKING FOR EQUILIBRIUM POSITIONS

$V'(\theta) = 4mga(4\sin\theta - 1)\cos\theta - 8mga\cos 2\theta$

$V'(\theta) = 4mga\left[(4\sin\theta - 1)\cos\theta - 2\cos 2\theta\right]$

$V'(\theta) = 4mga\left[4\sin\theta\cos\theta - \cos\theta - 2\cos 2\theta\right]$

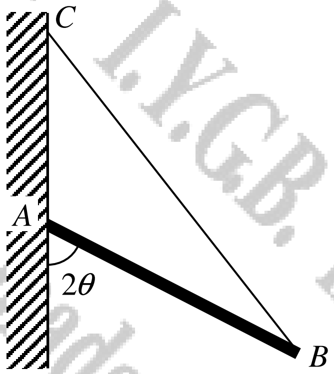
$V'(\theta) = 4mga\left[2\sin 2\theta - \cos\theta - 2\cos 2\theta\right]$

SOLVING FOR ZERO

$0 = 4mga(2\sin 2\theta - \cos\theta - 2\cos 2\theta)$

$2\sin 2\theta - \cos\theta - 2\cos 2\theta = 0$ AS REQUIRED

**Question 8 (***)**



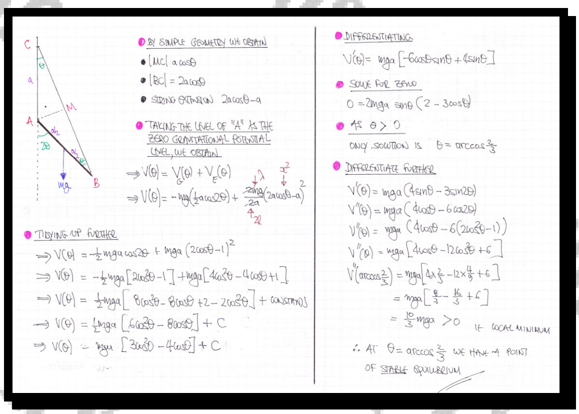

The figure above shows a uniform rod $AB$ of length $a$ metres and mass $m$, having its end $A$ smoothly hinged against a vertical wall.

A light elastic string $BC$ is attached to a point $C$ on the wall which lies vertically above $A$, such that $|AC| = |AB|$.

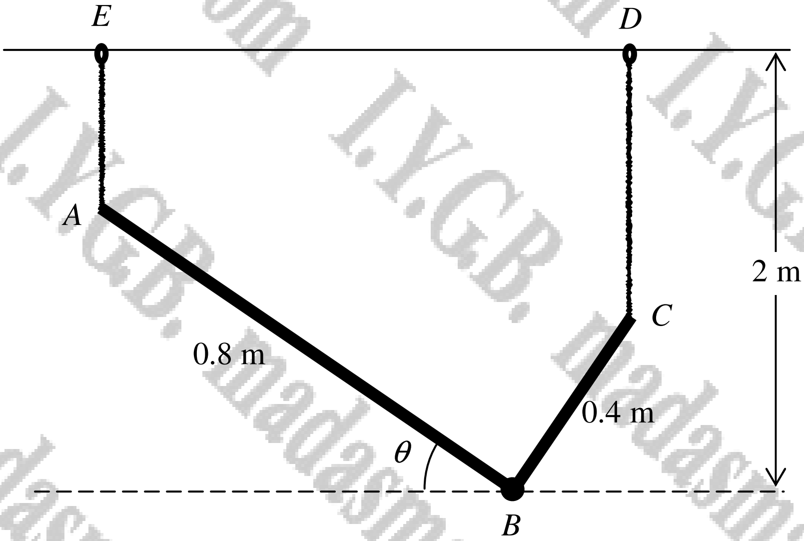
The plane $ABC$ is perpendicular to the wall and the angle $AB$ makes with the downward vertical is $2\theta$, where $\theta > 0$.

Given that the natural length of the string is $a$ and its modulus of elasticity is $2mg$, find the value of $\theta$ when the rod is in equilibrium and determine its stability.

stable at $\theta = \arccos \frac{2}{3}$



**Question 9** (***)



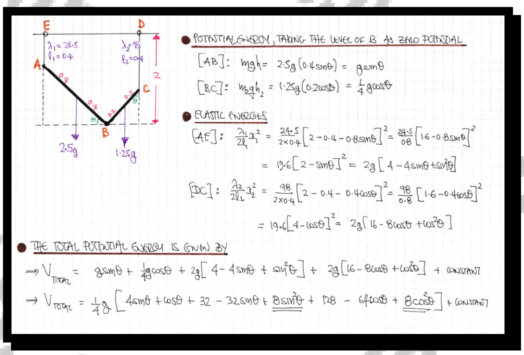

A uniform rod $AB$, of mass 2.5 kg and length 0.8 m, is rigidly joined to another uniform rod $BC$, of mass 1.25 kg and length 0.4 m, so that $\measuredangle ABC = 90°$. The rigid structure $ABC$ is freely pivoted at $B$, so it can rotate in a vertical plane.

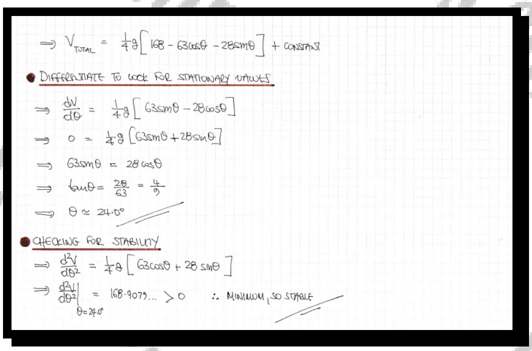
A light elastic string $AE$, of natural length 0.4 m and modulus of elasticity 24.5 N, and another light elastic string $CD$, of natural length 0.4 m and modulus of elasticity 98 N, are attached to the rigid structure at $A$ and $C$, respectively. A smooth horizontal wire is threaded through at $D$ and $E$, so that $AE$ and $CD$ remain vertical at all times, as shown in the figure above.

The vertical distance of $B$ below the wire is 2 m and the acute angle $AB$ makes with the horizontal is denoted by $\theta$, $0 \le \theta \le 90°$.

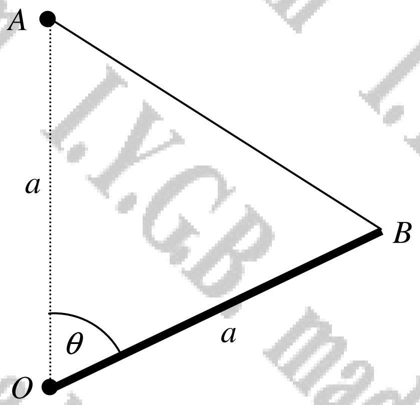
By considering potential energy, find the value of $\theta$ for which $ABC$ is in equilibrium, and determine its stability.

$\boxed{\phantom{xx}}$, $\boxed{\theta \approx 24.0°, \text{ stable}}$



**Question 10 (***+)**




The figure above shows a uniform rod $OB$ of mass $m$ and length $a$, freely hinged at $O$, so that the rod can rotate about $O$ in a vertical plane.

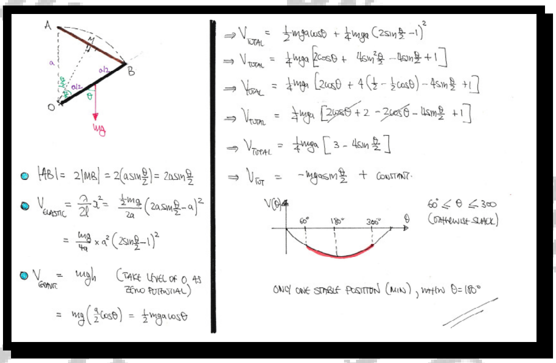
The point $A$ is fixed at a distance $a$, vertically above $O$.

A light elastic string of natural length $a$ and modulus of elasticity $\frac{1}{2}mg$ has one end attached to $A$ and the other end attached to $B$.

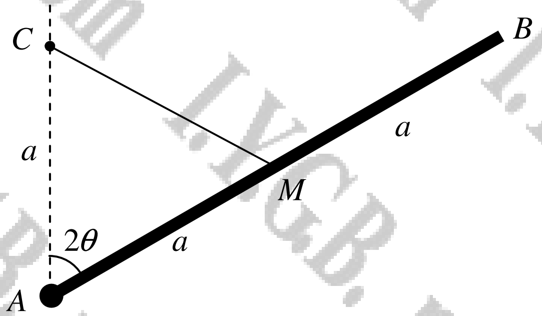
The acute angle $OAB$ is denoted by $\theta$.

By considering potential energy find the value of $\theta$ at any positions of equilibrium of the system and determine their stability.

$\theta = 180°$, stable



**Question 11 (***+)**




A uniform rod $AB$, of mass $m$ and length $2a$, is smoothly hinged to a fixed point $A$ so it can rotate in a vertical plane. A light elastic string of natural length $a$ and modulus of elasticity $\lambda$ has one of its ends attached to the midpoint of the rod, $M$. The other end of the string is attached to a fixed point $C$ which lies at a vertical distance $a$ above $A$, as shown in the figure. The potential energy of the system is $V$.

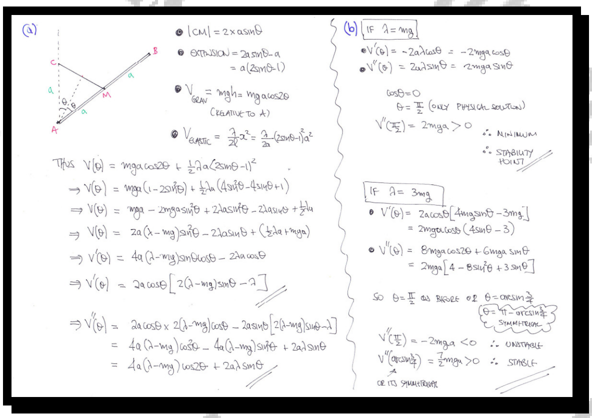
**a)** Given that $\measuredangle CAM = 2\theta$, with $0 \le \theta < \pi$, show that

- **i.** … $V'(\theta) = 2a\cos\theta\left[2(\lambda - mg)\sin\theta - \lambda\right]$
- **ii.** … $V''(\theta) = 4a(\lambda - mg)\cos 2\theta + 2a\lambda\sin\theta$.

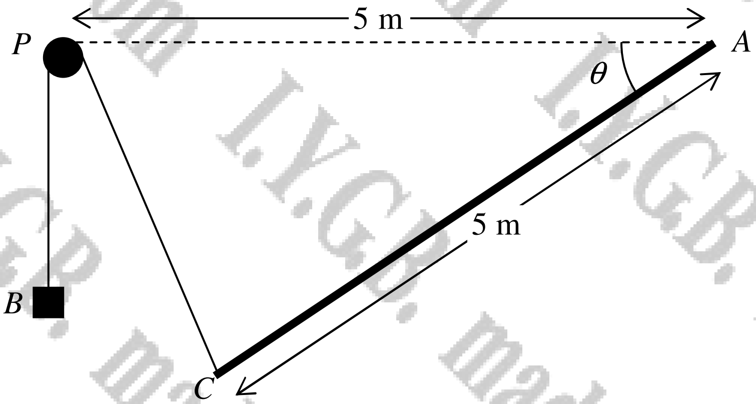
**b)** Find the values of $\theta$ for which the system is in equilibrium and investigate their stability in each of the cases

- **i.** $\lambda = mg$.
- **ii.** $\lambda = 3mg$.

$\boxed{\lambda = mg,\ \text{stable at } \theta = \frac{\pi}{2}}$ , $\boxed{\lambda = 3mg,\ \text{unstable at } \theta = \frac{\pi}{2},\ \text{stable at } \theta = \arcsin\frac{3}{4}}$



**Question 12** (***+)




The figure above shows a uniform rod $AC$, of length 5 m and mass 18 kg, with its end $A$ smoothly hinged to a fixed point.

One end of a light inextensible string is attached to the other end of the rod $B$. The string passes over a small smooth pulley which is fixed at the point $P$, where $AP$ is horizontal where $|AP| = 5$ m.

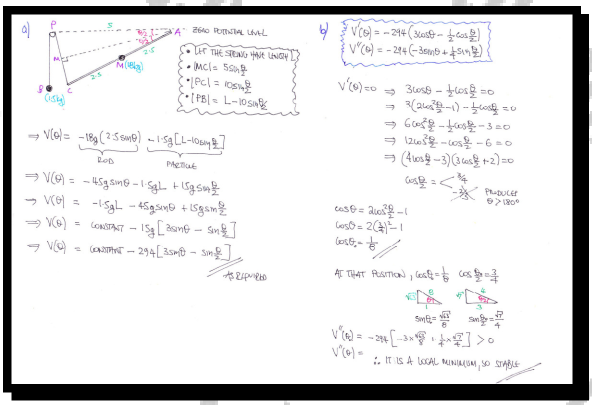
A small box $B$, of mass 1.5 kg is attached to the other end of the string and the particle hangs vertically below $P$. The acute angle $PAC$ is denoted by $\theta$.

**a)** Show that the potential energy of the system is

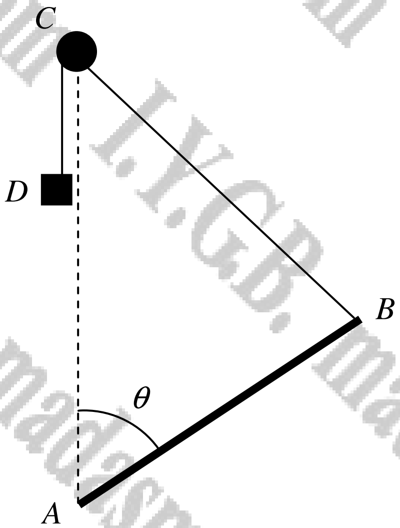
$$\text{constant} - 294\left[3\sin\theta - \sin\left(\tfrac{1}{2}\theta\right)\right].$$

**b)** Find the exact value of $\cos\theta$ when the system is in equilibrium and determine the stability of this equilibrium position.

$\boxed{\cos\theta = \frac{1}{8}, \text{ stable}}$



**Question 13** (***+)



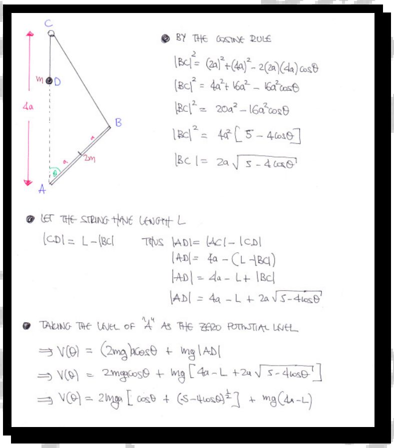

The figure above shows a uniform rod $AB$, of length $2a$ and mass $m$, with its end $A$ smoothly hinged to a fixed point.

One end of a light inextensible string is attached to the other end of the rod $B$. The string passes over a small smooth pulley which is fixed at the point $C$, where $AC$ is vertical with $|AC| = 4a$.

A particle $D$, of mass $2m$ is attached to the other end of the string and the particle hangs vertically below $C$. The acute angle $CAB$ is denoted by $\theta$, where $0 < \theta° < 180$.

Find the exact value of $\cos\theta$ when the system is in equilibrium.

$$\boxed{\cos\theta = \frac{1}{4}}$$



BY THE COSINE RULE

$|BC|^2 = (2a)^2 + (4a)^2 - 2(2a)(4a)\cos\theta$

$|BC|^2 = 4a^2 + 16a^2 - 16a^2\cos\theta$

$|BC|^2 = 20a^2 - 16a^2\cos\theta$

$|BC|^2 = 4a^2[5 - 4\cos\theta]$

$|BC| = 2a\sqrt{5 - 4\cos\theta}$

LET THE STRING HAVE LENGTH L

$|CD| = L - |BC|$ THUS $|AD| = |AC| - |CD|$

$|AD| = 4a - (L - |BC|)$

$|AD| = 4a - L + |BC|$

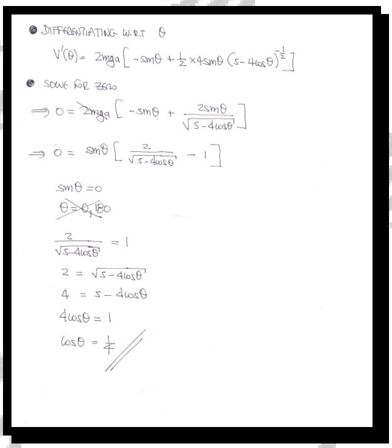
$|AD| = 4a - L + 2a\sqrt{5 - 4\cos\theta}$

TAKING THE LEVEL OF "A" AS THE ZERO POTENTIAL LEVEL

$\Rightarrow V(\theta) = (2mg)a\cos\theta + mg|AD|$

$\Rightarrow V(\theta) = 2mga\cos\theta + mg[4a - L + 2a\sqrt{5 - 4\cos\theta}]$

$\Rightarrow V(\theta) = 2mga[\cos\theta + (5 - 4\cos\theta)^{\frac{1}{2}}] + mg(4a - L)$



DIFFERENTIATING W.R.T $\theta$

$V'(\theta) = 2mga\left[-\sin\theta + \frac{1}{2} \times 4\sin\theta(5 - 4\cos\theta)^{-\frac{1}{2}}\right]$

SOLVE FOR ZERO

$\Rightarrow 0 = 2mga\left[-\sin\theta + \frac{2\sin\theta}{\sqrt{5 - 4\cos\theta}}\right]$

$\Rightarrow 0 = \sin\theta\left[\frac{2}{\sqrt{5 - 4\cos\theta}} - 1\right]$

$\sin\theta = 0$

~~$\theta = 0, 180$~~

$\frac{2}{\sqrt{5 - 4\cos\theta}} = 1$

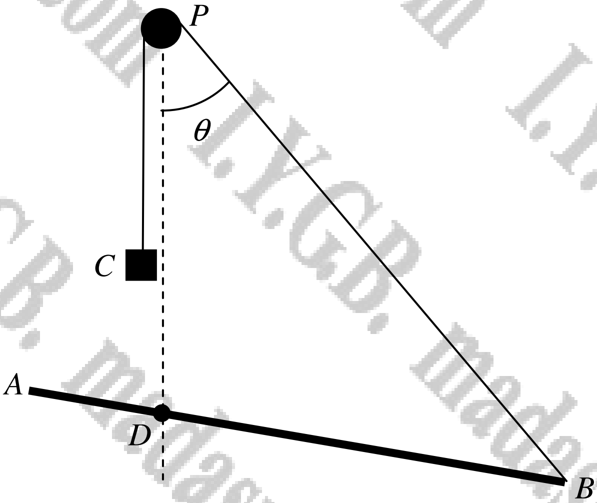
$2 = \sqrt{5 - 4\cos\theta}$

$4 = 5 - 4\cos\theta$

$4\cos\theta = 1$

$\cos\theta = \frac{1}{4}$

**Question 14** (***+)

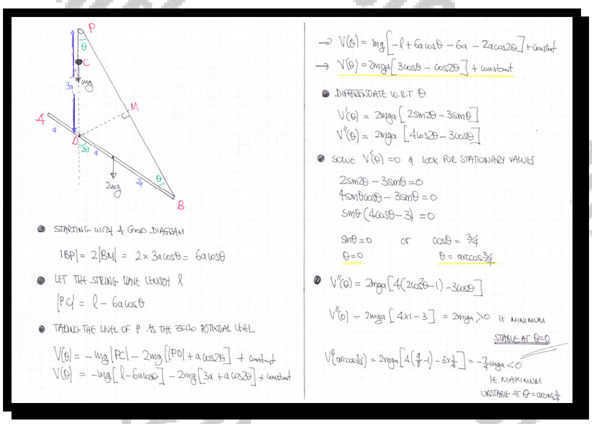



The figure above shows a uniform rod $AB$, of length $4a$ and mass $2m$, smoothly hinged to a fixed point $D$, so that $|AD| = a$. One end of a light inextensible string is attached to the end of the rod $B$. The string passes over a small smooth pulley which is fixed at the point $P$, where $DP$ is vertical with $|DP| = 3a$. A particle $C$, of mass $m$ is attached to the other end of the string and the particle hangs vertically below $P$.

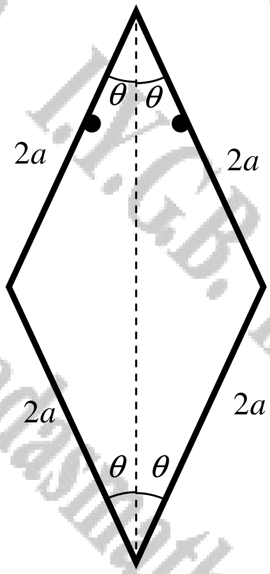
The acute angle $CPB$ is denoted by $\theta$, where $0 < \theta < \frac{1}{2}\pi$.

Find the two values of $\theta$ for which the system is in equilibrium and determine the stability of these equilibrium positions.

unstable at $\theta = \arccos\frac{3}{4}$, stable at $\theta = 0$



**Question 15 (***+)**

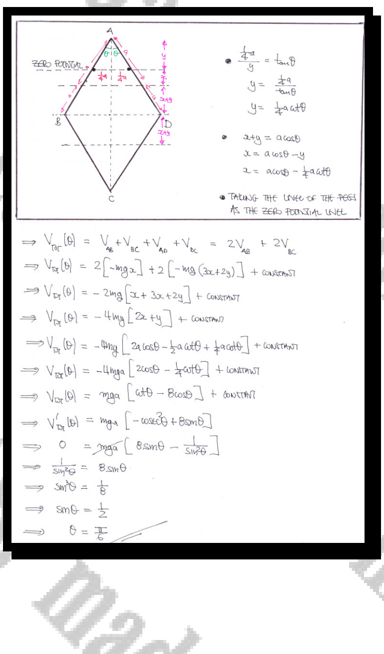



Four identical uniform rods, each of mass $m$ and length $2a$, are smoothly joined to form a rhombus. The system of the four rods is placed over two smooth pegs and rests in equilibrium as shown in the figure above.

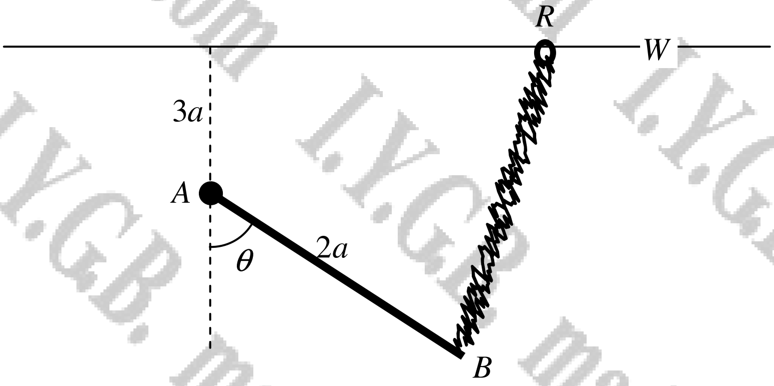
The pegs are in the same horizontal level at a distance $\frac{1}{2}a$ apart. Each of the rods makes an angle $\theta$ with the upward vertical.

Find the value of $\theta$ for which the system is in equilibrium.

$$\boxed{\theta = \frac{\pi}{6}}$$



**Question 16 (***+)**




A uniform rod $AB$ has mass $m$ and length $2a$.

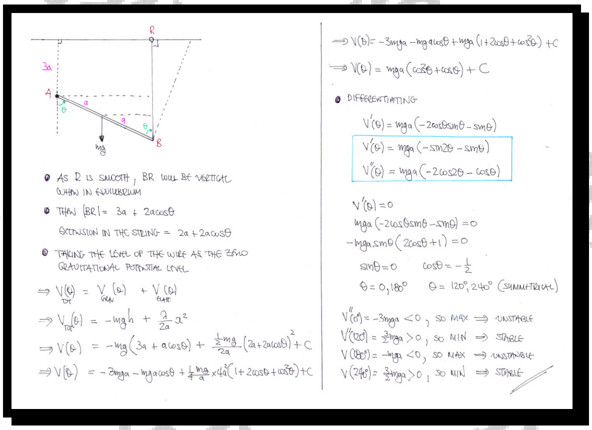
The end $A$ is freely hinged at a fixed point, which is at a distance $3a$ below a fixed smooth rigid horizontal wire $W$.

The end $B$ is attached to one end of an elastic string of natural length $a$ and modulus of elasticity $\frac{1}{2}mg$. The other end of the string is connected to a smooth ring $R$ which is threaded to $W$ as shown in the figure above.

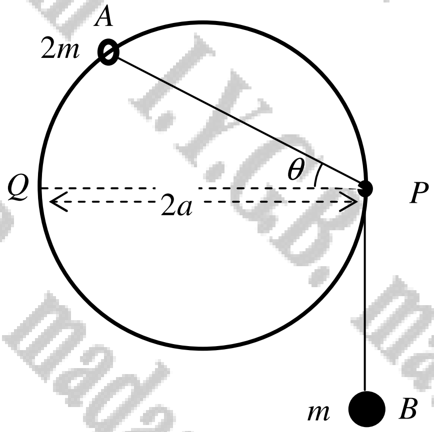
The angle the rod makes with the downward vertical through $A$ is denoted by $\theta°$.

Find each of the values of $\theta$, $0 \le \theta° < 360$, for which the above described system is in equilibrium and determine their stability.

$\boxed{\theta = 0°,\ \text{unstable}}$, $\boxed{\theta = 120°,\ \text{stable}}$, $\boxed{\theta = 180°,\ \text{unstable}}$, $\boxed{\theta = 240°,\ \text{stable}}$



**Question 17 (***+)**

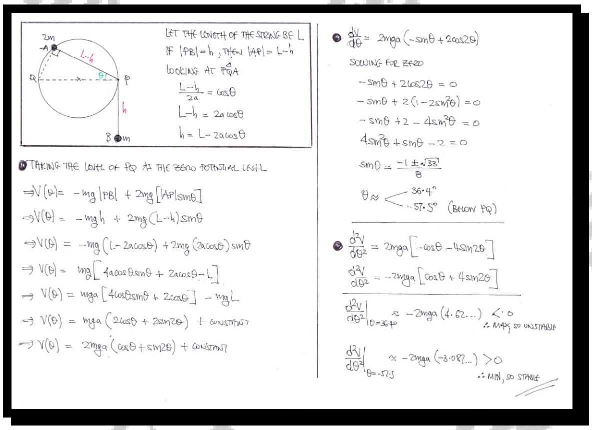



A small ring $A$, of mass $m$, is free to slide on a smooth wire, bent into the shape of a circular hoop of radius $a$. The wire is fixed in a vertical plane. A light inextensible string has one end attached to $A$ and passes over a smooth pulley $P$, located at the end of a horizontal diameter of the wire, $QP$. The other end of the string is attached to a particle of mass $2m$, which hangs freely as shown in the figure above.

The angle $QPA$ is denoted by $\theta°$.

Find each of the values of $\theta$, $0 \le \theta° < 360$, for which the above described system is in equilibrium and determine their stability.

$\boxed{\theta = 36.4°, \text{ unstable}}$, $\boxed{\theta = -57.5°, \text{ stable}}$



LET THE LENGTH OF THE STRING BE $L$
IF $|PB| = h$, THEN $|AP| = L - h$
LOOKING AT $\hat{PQA}$

$\frac{L-h}{2a} = \cos\theta$

$L - h = 2a\cos\theta$

$h = L - 2a\cos\theta$

• TAKING THE LEVEL OF PQ AS THE ZERO POTENTIAL LEVEL

$\Rightarrow V(\theta) = -mg|PB| + 2mg[|AP|\sin\theta]$

$\Rightarrow V(\theta) = -mgh + 2mg(L-h)\sin\theta$

$\Rightarrow V(\theta) = -mg(L - 2a\cos\theta) + 2mg(2a\cos\theta)\sin\theta$

$\Rightarrow V(\theta) = mg[4a\cos\theta\sin\theta + 2a\cos\theta - L]$

$\Rightarrow V(\theta) = mga[4\cos\theta\sin\theta + 2\cos\theta] - mgL$

$\Rightarrow V(\theta) = mga(2\cos\theta + 2\sin2\theta) + \text{CONSTANT}$

$\Rightarrow V(\theta) = 2mga(\cos\theta + \sin2\theta) + \text{CONSTANT}$

• $\frac{dV}{d\theta} = 2mga(-\sin\theta + 2\cos2\theta)$

SOLVING FOR ZERO

$-\sin\theta + 2\cos2\theta = 0$

$-\sin\theta + 2(1 - 2\sin^2\theta) = 0$

$-\sin\theta + 2 - 4\sin^2\theta = 0$

$4\sin^2\theta + \sin\theta - 2 = 0$

$\sin\theta = \frac{-1 \pm \sqrt{33}}{8}$

$\theta \approx 36.4°$, $-57.5°$ (BELOW PQ)

• $\frac{d^2V}{d\theta^2} = 2mga[-\cos\theta - 4\sin2\theta]$

$\frac{d^2V}{d\theta^2} = -2mga[\cos\theta + 4\sin2\theta]$

$\left.\frac{d^2V}{d\theta^2}\right|_{\theta=36.4°} \approx -2mga(4.62...) < 0$
$\therefore$ MAX, SO UNSTABLE

$\left.\frac{d^2V}{d\theta^2}\right|_{\theta=-57.5} \approx -2mga(-3.087...) > 0$
$\therefore$ MIN, SO STABLE

**Question 18** (****)

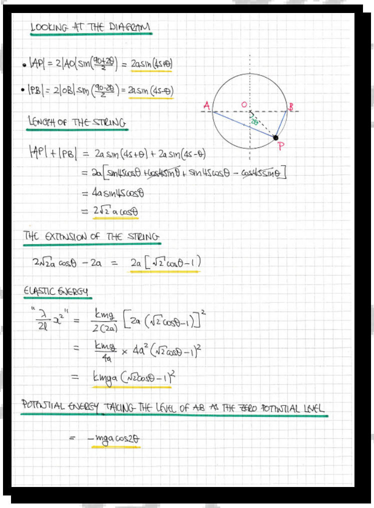
A smooth wire is bent into the shape of a circle of radius $a$, centre at $O$. The wire is fixed in a vertical plane and the straight line $AOB$ is a horizontal diameter of the circle. A small ring $P$, of mass $m$, is threaded on the wire and a light elastic string is also threaded through the ring. The two ends of the string are attached at $A$ and $B$.

The natural length of the string is $2a$ and its modulus of elasticity is $kmg$, $k > 0$.

The angle between the radius $OP$ and the downward vertical through $O$ is denoted by $2\theta$, where $-\frac{1}{2}\pi \le \theta \le \frac{1}{2}\pi$.

If $\theta = \frac{1}{6}\pi$ is a position of equilibrium for the above described system determine the exact value of $k$.

$\boxed{\phantom{k=3}}$, $\boxed{k = 3 + \sqrt{6}}$



LOOKING AT THE DIAGRAM

- $|AP| = 2|AO|\sin\left(\frac{90+2\theta}{2}\right) = 2a\sin(45+\theta)$
- $|PB| = 2|OB|\sin\left(\frac{90-2\theta}{2}\right) = 2a\sin(45-\theta)$



LENGTH OF THE STRING

$|AP| + |PB| = 2a\sin(45+\theta) + 2a\sin(45-\theta)$
$= 2a[\sin45\cos\theta + \cos45\sin\theta + \sin45\cos\theta - \cos45\sin\theta]$
$= 4a\sin45\cos\theta$
$= 2\sqrt{2}a\cos\theta$

THE EXTENSION OF THE STRING

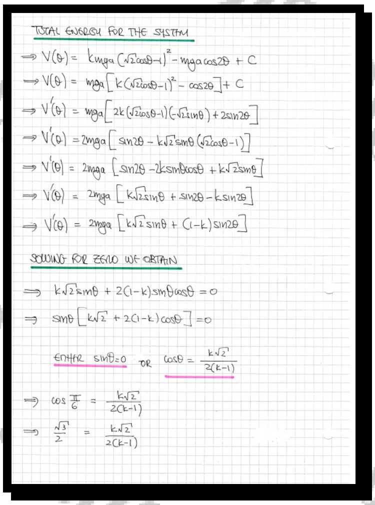
$2\sqrt{2}a\cos\theta - 2a = 2a[\sqrt{2}\cos\theta - 1]$

ELASTIC ENERGY

"$\frac{\lambda}{2l}x^2$" $= \frac{kmg}{2(2a)}\left[2a(\sqrt{2}\cos\theta - 1)\right]^2$
$= \frac{kmg}{4a} \times 4a^2(\sqrt{2}\cos\theta - 1)^2$
$= kmga(\sqrt{2}\cos\theta - 1)^2$

POTENTIAL ENERGY TAKING THE LEVEL OF AB AS THE ZERO POTENTIAL LEVEL

$= -mga\cos2\theta$



TOTAL ENERGY FOR THE SYSTEM

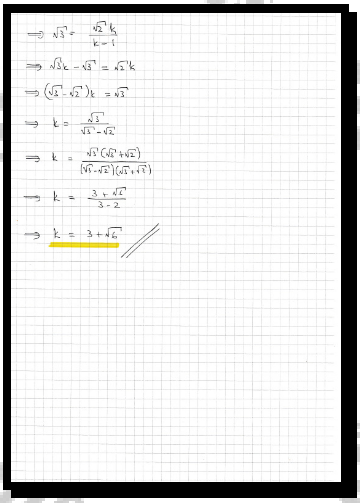
$\Rightarrow V(\theta) = kmga(\sqrt{2}\cos\theta - 1)^2 - mga\cos2\theta + C$
$\Rightarrow V(\theta) = mga\left[k(\sqrt{2}\cos\theta - 1)^2 - \cos2\theta\right] + C$
$\Rightarrow V'(\theta) = mga\left[2k(\sqrt{2}\cos\theta - 1)(-\sqrt{2}\sin\theta) + 2\sin2\theta\right]$
$\Rightarrow V'(\theta) = 2mga\left[\sin2\theta - k\sqrt{2}\sin\theta(\sqrt{2}\cos\theta - 1)\right]$
$\Rightarrow V'(\theta) = 2mga\left[\sin2\theta - 2k\sin\theta\cos\theta + k\sqrt{2}\sin\theta\right]$
$\Rightarrow V'(\theta) = 2mga\left[k\sqrt{2}\sin\theta + \sin2\theta - k\sin2\theta\right]$
$\Rightarrow V'(\theta) = 2mga\left[k\sqrt{2}\sin\theta + (1-k)\sin2\theta\right]$

SOLVING FOR ZERO WE OBTAIN

$\Rightarrow k\sqrt{2}\sin\theta + 2(1-k)\sin\theta\cos\theta = 0$
$\Rightarrow \sin\theta\left[k\sqrt{2} + 2(1-k)\cos\theta\right] = 0$

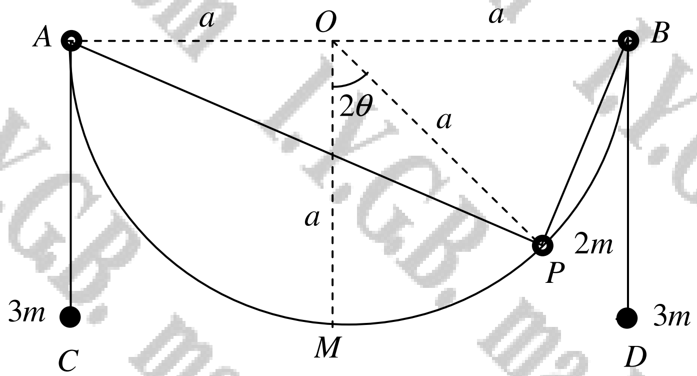
EITHER $\sin\theta = 0$ OR $\cos\theta = \frac{k\sqrt{2}}{2(k-1)}$

$\Rightarrow \cos\frac{\pi}{6} = \frac{k\sqrt{2}}{2(k-1)}$
$\Rightarrow \frac{\sqrt{3}}{2} = \frac{k\sqrt{2}}{2(k-1)}$



$\Rightarrow \sqrt{3} = \frac{\sqrt{2}k}{k-1}$
$\Rightarrow \sqrt{3}k - \sqrt{3} = \sqrt{2}k$
$\Rightarrow (\sqrt{3} - \sqrt{2})k = \sqrt{3}$
$\Rightarrow k = \frac{\sqrt{3}}{\sqrt{3} - \sqrt{2}}$
$\Rightarrow k = \frac{\sqrt{3}(\sqrt{3} + \sqrt{2})}{(\sqrt{3} - \sqrt{2})(\sqrt{3} + \sqrt{2})}$
$\Rightarrow k = \frac{3 + \sqrt{6}}{3 - 2}$
$\Rightarrow k = 3 + \sqrt{6}$

**Question 19** (****)



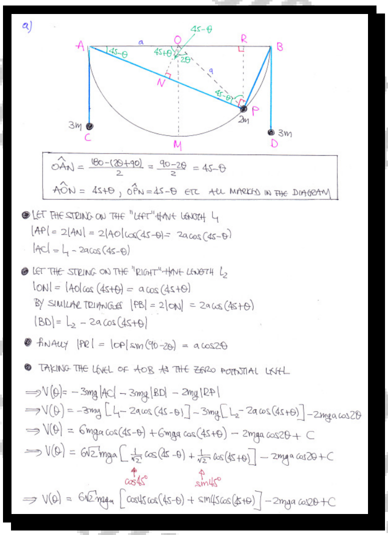

A smooth wire, with ends $A$ and $B$, is in the shape of a semicircle of radius $a$. The line $AB$ is horizontal and the midpoint of $AB$ is $O$. The wire is fixed in a vertical plane. A small ring $P$ of mass $2m$ is threaded on the wire and is attached to two light inextensible strings. One string passes through a small smooth ring fixed at $A$ and is attached to a particle of mass $3m$. The other string passes through a small smooth ring fixed at $B$ and is attached to another particle of mass $3m$. The particles hang freely under gravity, as shown in the figure above. The angle between the radius $OP$ and the downward vertical $OM$ is $2\theta$, where $-45° \le \theta \le 45°$.

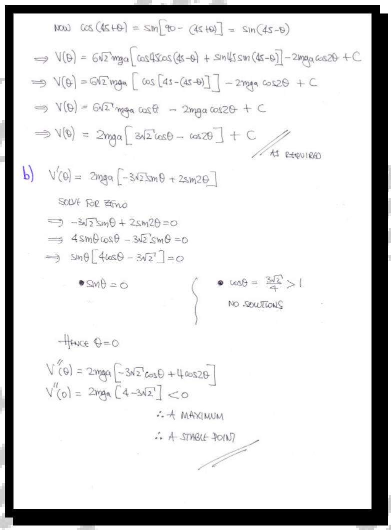
**a)** Show that the potential energy of the system is

$$2mga\left[3\sqrt{2}\cos\theta - \cos 2\theta\right] + \text{constant}.$$

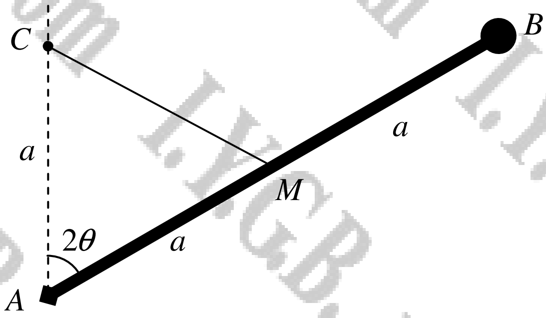
**b)** Find the value of $\theta$ for which the above described system is in equilibrium and determine its stability.

$\boxed{\theta = 0, \quad \text{stable}}$



**Question 20** (****+)




A uniform rod $AB$, of mass $m$ and length $2a$, is smoothly hinged to a fixed point $A$ so it can rotate in a vertical plane. A particle of mass $km$, where $k$ is a positive constant, is attached at $B$. A light elastic string of natural length $a$ and modulus of elasticity $2mg$ has one of its ends attached to the midpoint of the rod, $M$. The other end of the string is attached to a fixed point $C$ which lies at a vertical distance $a$ above $A$, as shown in the figure. The potential energy of the system is $V$.

Given that $\measuredangle CAM = 2\theta$ show clearly that …

**a)** … $V = 2mga\left[(1-2k)\sin^2\theta - 2\sin\theta\right] + \text{constant}$.

**b)** … the system is in equilibrium when $\cos\theta = 0$ or $\sin\theta = \dfrac{1}{1-2k}$.

**c)** … there no physical positions of equilibrium if $\sin\theta = \dfrac{1}{1-2k}$.

**d)** … there is a stable position of equilibrium for all positive values of $k$.

proof