MOMENT OF INERTIA CALCULATIONS

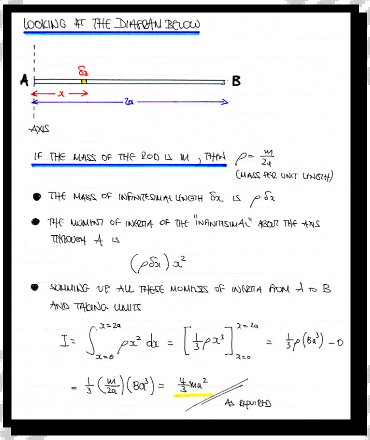
MOMENT OF INERTIA BY CALCULUS

Question 1 (**)

Use integration to show that the moment of inertia I of a thin uniform rod AB, of length $2a$ and mass m, about an axis through A and perpendicular to the length of the rod is given by

$$I = \frac{4}{3}ma^2.$$

[ ], proof



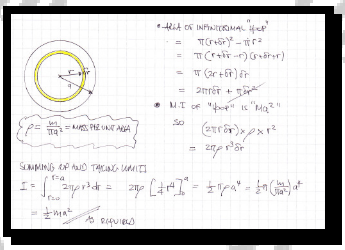
Question 2 (**)

A uniform circular disc, of mass m and radius a, is free to rotate about an axis L, through the centre of the disc and perpendicular to the plane of the disc

Prove, using integration, that the moment of inertia of the disc about L is $\frac{1}{2}ma^2$.

[*You may assume without proof that the standard result of the moment of inertia of a uniform hoop about an axis through its centre and perpendicular to its plane.*]

proof



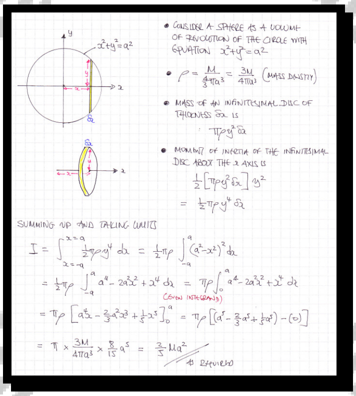
Question 3 (**+)

Show that the moment of inertia of a uniform solid sphere of mass m and radius a, about one of its diameters is

$$\frac{2}{5}ma^2 .$$

[*In this proof, you may assume standard results for the moment of inertia of uniform circular discs.*]

$\boxed{\text{proof}}$



- CONSIDER A SPHERE AS A VOLUME OF REVOLUTION OF THE CIRCLE WITH EQUATION $x^2+y^2=a^2$
- $\rho = \frac{M}{\frac{4}{3}\pi a^3} = \frac{3M}{4\pi a^3}$ (MASS DENSITY)
- MASS OF AN INFINITESIMAL DISC OF THICKNESS δx IS
  $\pi \rho y^2 \delta x$
- MOMENT OF INERTIA OF THE INFINITESIMAL DISC ABOUT THE x AXIS IS
  $\frac{1}{2}\left[\pi\rho y^2 \delta x\right] y^2$
  $= \frac{1}{2}\pi\rho y^4 \delta x$

SUMMING UP AND TAKING LIMITS

$$I = \int_{x=-a}^{x=a} \frac{1}{2}\pi\rho y^4 \, dx = \frac{1}{2}\pi\rho \int_{-a}^{a} (a^2-x^2)^2 \, dx$$

$$= \frac{1}{2}\pi\rho \int_{-a}^{a} a^4 - 2a^2x^2 + x^4 \, dx = \pi\rho \int_{0}^{a} a^4 - 2a^2x^2 + x^4 \, dx$$

(EVEN INTEGRAND)

$$= \pi\rho \left[a^4x - \frac{2}{3}a^2x^3 + \frac{1}{5}x^5\right]_0^a = \pi\rho\left[\left(a^5 - \frac{2}{3}a^5 + \frac{1}{5}a^5\right) - (0)\right]$$

$$= \pi \times \frac{3M}{4\pi a^3} \times \frac{8}{15}a^5 = \frac{2}{5}Ma^2$$

AS REQUIRED

Question 4 (***)

A uniform circular lamina L has mass m and radius a.

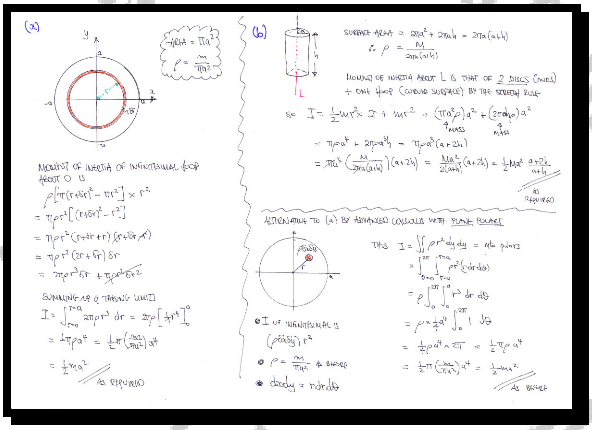
a) Show by integration that the moment of inertia of L about a perpendicular axis through the plane of the lamina and though its centre is $\frac{1}{2}ma^2$.

A closed hollow cylinder C has mass M, radius a and height h. The entire cylinder is made of the same material with uniform density.

b) Show that the moment of inertia of C about its axis of symmetry is

$$\frac{1}{2}Ma^2\left(\frac{a+2h}{a+h}\right).$$

proof



(a)

AREA $= \pi a^2$

$\rho = \frac{m}{\pi a^2}$

MOMENT OF INERTIA OF INFINITESIMAL HOOP ABOUT O IS

$\rho\left[\pi(r+\delta r)^2 - \pi r^2\right] \times r^2$

$= \pi\rho r^2\left[(r+\delta r)^2 - r^2\right]$

$= \pi\rho r^2(r+\delta r+r)(\cancel{r}+\delta r-\cancel{r})$

$= \pi\rho r^2(2r+\delta r)\delta r$

$= 2\pi\rho r^3\delta r + \cancel{\pi\rho r^2\delta r^2}$

SUMMING UP & TAKING LIMITS

$I = \int_{r=0}^{r=a} 2\pi\rho r^3\,dr = 2\pi\rho\left[\frac{1}{4}r^4\right]_0^a$

$= \frac{1}{2}\pi\rho a^4 = \frac{1}{2}\pi\left(\frac{m}{\pi a^2}\right)a^4$

$= \frac{1}{2}ma^2$

AS REQUIRED

(b)

SURFACE AREA $= 2\pi a^2 + 2\pi ah = 2\pi a(a+h)$

$\therefore \rho = \frac{M}{2\pi a(a+h)}$

MOMENT OF INERTIA ABOUT L IS THAT OF 2 DISCS (ENDS) + ONE HOOP (CURVED SURFACE) BY THE STRETCH RULE

SO $I = \frac{1}{2}mr^2 \times 2 + mr^2 = \underbrace{(\pi a^2\rho)}_{\text{MASS}}a^2 + \underbrace{(2\pi ah\rho)}_{\text{MASS}}a^2$

$= \pi\rho a^4 + 2\pi\rho a^3h = \pi\rho a^3(a+2h)$

$= \pi a^3\left(\frac{M}{2\pi a(a+h)}\right)(a+2h) = \frac{Ma^2}{2(a+h)}(a+2h) = \frac{1}{2}Ma^2\frac{a+2h}{a+h}$

AS REQUIRED

ALTERNATIVE TO (a) BY ADVANCED CALCULUS WITH PLANE POLARS

- I OF INFINITESIMAL IS $(\rho\delta x\delta y)r^2$
- $\rho = \frac{m}{\pi a^2}$ AS BEFORE
- $dxdy = r\,dr\,d\theta$

THUS $I = \iint \rho r^2\,dx\,dy$ = into polars

$= \int_{\theta=0}^{2\pi}\int_{r=0}^{r=a}\rho r^2(r\,dr\,d\theta)$

$= \rho\int_0^{2\pi}\int_0^a r^3\,dr\,d\theta$

$= \rho \times \frac{1}{4}a^4\int_0^{2\pi} 1\,d\theta$

$= \frac{1}{4}\rho a^4 \times 2\pi = \frac{1}{2}\pi\rho a^4$

$= \frac{1}{2}\pi\left(\frac{m}{\pi a^2}\right)a^4 = \frac{1}{2}ma^2$

AS BEFORE

Question 5 (***+)

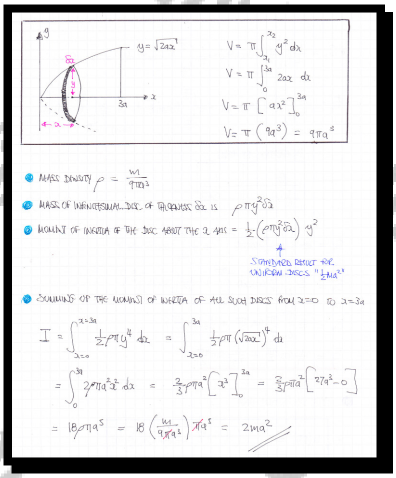
The finite region R is bounded by the x axis, the straight line with equation $x = 3a$ and the curve with equation $y = \sqrt{2ax}$, where a is a positive constant.

A uniform solid S is generated by fully rotating R in the x axis.

If the mass of S is m, determine the moment of inertia of S about the x axis.

[*In this question, you may assume standard results for the moment of inertia of uniform circular discs.*]

$$\boxed{I = 2ma^2}$$



$y = \sqrt{2ax}$

$V = \pi \int_{x_1}^{x_2} y^2 \, dx$

$V = \pi \int_0^{3a} 2ax \, dx$

$V = \pi \left[ax^2 \right]_0^{3a}$

$V = \pi (9a^3) = 9\pi a^3$

- MASS DENSITY $\rho = \frac{m}{9\pi a^3}$
- MASS OF INFINITESIMAL DISC OF THICKNESS δx IS $\rho \pi y^2 \delta x$
- MOMENT OF INERTIA OF THE DISC ABOUT THE x AXIS $= \frac{1}{2}\left(\rho \pi y^2 \delta x\right) y^2$

STANDARD RESULT FOR UNIFORM DISCS "$\frac{1}{2}Ma^2$"

- SUMMING UP THE MOMENT OF INERTIA OF ALL SUCH DISCS FROM $x=0$ TO $x=3a$

$$I = \int_{x=0}^{x=3a} \frac{1}{2}\rho \pi y^4 \, dx = \int_{x=0}^{3a} \frac{1}{2}\rho \pi \left(\sqrt{2ax}\right)^4 dx$$

$$= \int_0^{3a} 2\rho \pi a^2 x^2 \, dx = \frac{2}{3}\rho \pi a^2 \left[x^3 \right]_0^{3a} = \frac{2}{3}\rho \pi a^2 \left[27a^3 - 0 \right]$$

$$= 18\rho \pi a^5 = 18 \left(\frac{m}{9\cancel{\pi}a^3} \right) \cancel{\pi} a^5 = 2ma^2$$

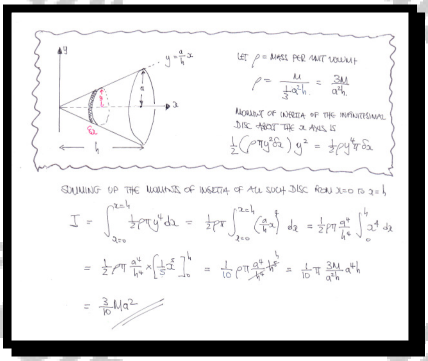
Question 6 (***+)

Show by integration that the moment of inertia of a uniform solid circular cone of mass M, height h and base radius a, about its axis of symmetry, is given by

$$\frac{3}{10}Ma^2 .$$

[*In this proof, you may assume standard results for the moment of inertia of uniform circular discs.*]

$\boxed{\text{proof}}$

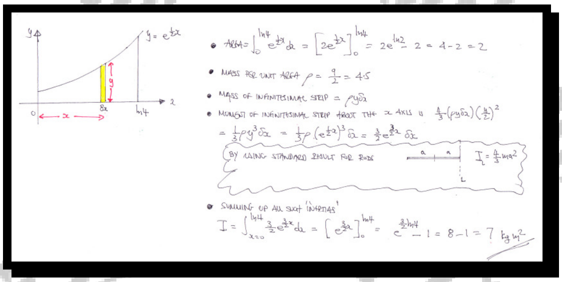


Question 7 (***+)

A uniform lamina of mass 9 kg occupies the finite region bounded by the coordinate axes, the straight line with equation $x = \ln 4$ and the curve with equation $y = e^{\frac{1}{2}x}$.

Given that distances are measured in metres, calculate the moment of inertia of the lamina about the x axis.

$\boxed{I = 7 \text{ kg m}^2}$



Question 8 (***+)

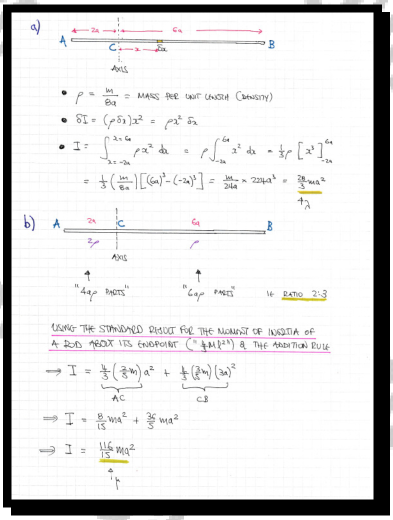
A uniform rod AB, of mass m and length $8a$, is free to rotate about an axis L which passes through the point C, where $|AC| = 2a$.

a) Given that the moment of inertia of the rod about L is λma^2, use integration to find the value of λ.

A different rod AB, also of mass m and length $8a$ is free to rotate about a smooth fixed axis L', which passes through the point C, where $|AC| = 2a$. The mass density of the section AC is twice as large as the mass density of the section CB.

b) Given that the moment of inertia of this rod about L' is μma^2, determine the value of μ.

$\boxed{}$, $\boxed{\lambda = \frac{28}{3}}$, $\boxed{\mu = \frac{115}{15}}$



a)

- $\rho = \frac{m}{8a}$ = MASS PER UNIT LENGTH (DENSITY)
- $\delta I = (\rho\,\delta x)x^2 = \rho x^2\,\delta x$
- $I = \int_{x=-2a}^{x=6a} \rho x^2\,dx = \rho\int_{-2a}^{6a} x^2\,dx = \frac{1}{3}\rho\left[x^3\right]_{-2a}^{6a}$

$= \frac{1}{3}\left(\frac{m}{8a}\right)\left[(6a)^3 - (-2a)^3\right] = \frac{m}{24a} \times 224a^3 = \underbrace{\frac{28}{3}ma^2}_{\lambda}$

b)

USING THE STANDARD RESULT FOR THE MOMENT OF INERTIA OF A ROD ABOUT ITS ENDPOINT ("$\frac{4}{3}M\ell^2$") & THE ADDITION RULE

$\Rightarrow I = \underbrace{\frac{4}{3}\left(\frac{2}{5}m\right)a^2}_{AC} + \underbrace{\frac{4}{3}\left(\frac{3}{5}m\right)(3a)^2}_{CB}$

$\Rightarrow I = \frac{8}{15}ma^2 + \frac{36}{5}ma^2$

$\Rightarrow I = \underbrace{\frac{116}{15}ma^2}_{\mu}$

Question 9 (****)

A triangular lamina OAB has $|OA| = |OB|$ and $|AB| = 2a$. The height of the lamina drawn from O to AB has length h.

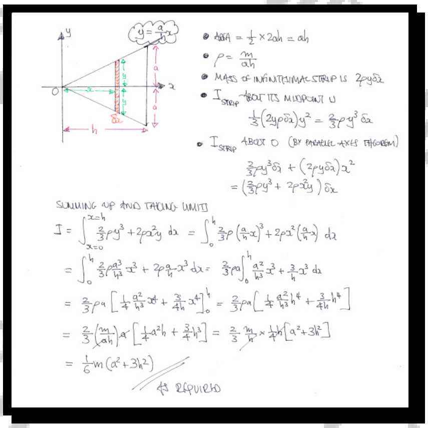
Show by integration that the moment of inertia of the lamina about an axis through its vertex through O and perpendicular to the plane of the lamina, is given by

$$\frac{1}{6}m\left(a^2 + 3h^2\right),$$

where m is the mass of the lamina.

[*In this proof, you may assume standard results for the moment of inertia of uniform rods.*]

$\boxed{\text{proof}}$



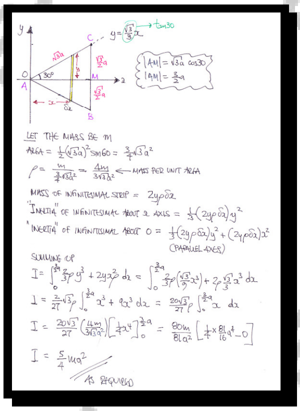
Question 10 (****)

A uniform equilateral triangular lamina ABC has mass m and side length of $\sqrt{3}a$.

Show, by integration, that the moment of inertia of the lamina about an axis through one of its vertices and perpendicular to the plane of the lamina is $\frac{5}{4}ma^2$.

[*In this proof, you may assume standard results for the moment of inertia of uniform rods.*]

$\boxed{\text{proof}}$

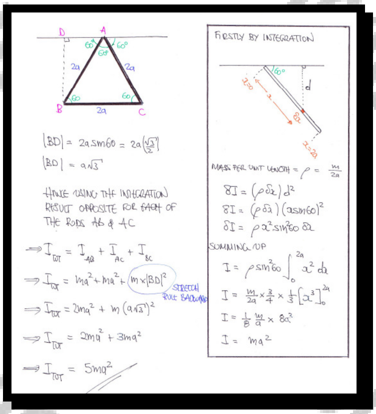


Question 11 (****)

A framework, in the shape of an equilateral triangle ABC, is formed by rigidly joining three uniform rods, each of mass m and length $2a$.

Find the moment of inertia of the framework about an axis passing through A, and parallel to BC

$\boxed{5ma^2}$



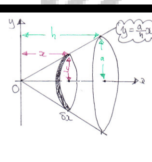
Question 12 (****)

Show by integration that the moment of inertia of a uniform solid circular cone of mass m, height h and base radius a, about an axis through its vertex and parallel to its base, is given by

$$\frac{3}{20}m\left(a^2+4h^2\right).$$

[*In this proof, you may assume standard results for the moment of inertia of uniform circular discs.*]

$\boxed{\text{proof}}$



- $\rho = \dfrac{m}{\frac{1}{3}\pi a^2 h} = \dfrac{3m}{\pi a^2 h}$ (MASS DENSITY)
- MASS OF INFINITESIMAL DISC IS
  $\pi y^2 \delta x \rho$
- I_{DISC} ABOUT A DIAMETER IS $\frac{1}{4}Mr^2$, IE A STANDARD RESULT OF DISCS BY THE PERPENDICULAR AXES THEOREM
  HERE $\frac{1}{4}(\pi y^2 \rho \delta x) y^2 = \frac{1}{4}\pi \rho y^4 \delta x$
- I_{DISC} ABOUT AN AXIS PARALLEL TO THE PLANE OF THE DISC IS
  $\frac{1}{4}\pi\rho y^4 \delta x + (\pi y^2 \rho \delta x)x^2$
  BY THE PARALLEL AXIS THEOREM
  $= \frac{1}{4}\pi\rho\left(y^4 + 4x^2y^2\right)\delta x$

SUMMING UP AND TAKING LIMITS

$$I = \int_0^h \tfrac{1}{4}\pi\rho\left(y^4+4x^2y^2\right)dx = \tfrac{1}{4}\pi\rho\int_0^h \left(\tfrac{a}{h}x\right)^4 + 4x^2\left(\tfrac{a}{h}x\right)^2 dx$$

$$= \tfrac{1}{4}\pi\rho\int_0^h \frac{a^4}{h^4}x^4 + \frac{4a^2}{h^2}x^4\,dx = \tfrac{1}{4}\pi\rho\left[\frac{a^4}{5h^4}x^5 + \frac{4a^2}{5h^2}x^5\right]_0^h$$

$$= \tfrac{1}{4}\pi\rho\left[\tfrac{1}{5}a^4h + \tfrac{4}{5}a^2h^3\right] = \tfrac{1}{4}\cancel{\pi}\times\frac{3m}{\cancel{\pi a^2 h}}\times\tfrac{1}{5}\cancel{a^2h}\times\left[a^2+4h^2\right]$$

$$= \tfrac{3}{4}m\times\tfrac{1}{5}\left(a^2+4h^2\right) = \tfrac{3}{20}m\left(a^2+4h^2\right)$$

AS REQUIRED

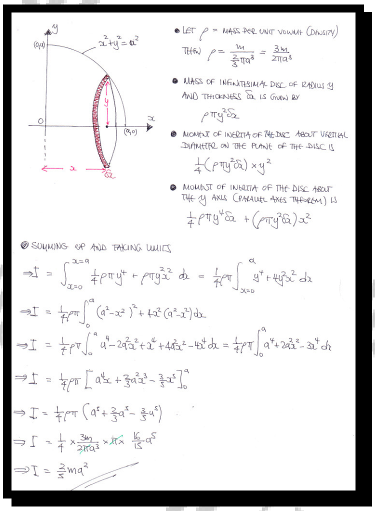
Question 13 (****)

Show by integration that the moment of inertia of a uniform solid hemisphere of mass m and radius a about a diameter of its plane face, is

$$\frac{2}{5}ma^2 .$$

[*In this proof, you may assume standard results for the moment of inertia of a uniform circular disc about one of its diameters.*]

$\boxed{\text{proof}}$



- LET ρ = MASS PER UNIT VOLUME (DENSITY)

  THEN $\rho = \dfrac{m}{\frac{2}{3}\pi a^3} = \dfrac{3m}{2\pi a^3}$
- MASS OF INFINITESIMAL DISC OF RADIUS y AND THICKNESS δx IS GIVEN BY

  $\rho \pi y^2 \delta x$
- MOMENT OF INERTIA OF THE DISC ABOUT VERTICAL DIAMETER ON THE PLANE OF THE DISC IS

  $\frac{1}{4}(\rho\pi y^2 \delta x) \times y^2$
- MOMENT OF INERTIA OF THE DISC ABOUT THE y AXIS (PARALLEL AXES THEOREM) IS

  $\frac{1}{4}\rho\pi y^4 \delta x + (\rho\pi y^2 \delta x)x^2$

- SUMMING UP AND TAKING LIMITS

$\Rightarrow I = \int_{x=0}^{x=a} \frac{1}{4}\rho\pi y^4 + \rho\pi y^2 x^2 \, dx = \frac{1}{4}\rho\pi \int_{x=0}^{a} y^4 + 4y^2x^2 \, dx$

$\Rightarrow I = \frac{1}{4}\rho\pi \int_0^a (a^2-x^2)^2 + 4x^2(a^2-x^2) \, dx$

$\Rightarrow I = \frac{1}{4}\rho\pi \int_0^a a^4 - 2a^2x^2 + x^4 + 4a^2x^2 - 4x^4 \, dx = \frac{1}{4}\rho\pi \int_0^a a^4 + 2a^2x^2 - 3x^4 \, dx$

$\Rightarrow I = \frac{1}{4}\rho\pi \left[a^4x + \frac{2}{3}a^2x^3 - \frac{3}{5}x^5 \right]_0^a$

$\Rightarrow I = \frac{1}{4}\rho\pi \left(a^5 + \frac{2}{3}a^5 - \frac{3}{5}a^5 \right)$

$\Rightarrow I = \frac{1}{4} \times \frac{3m}{2\pi a^3} \times \pi \times \frac{16}{15}a^5$

$\Rightarrow I = \frac{2}{5}ma^2$

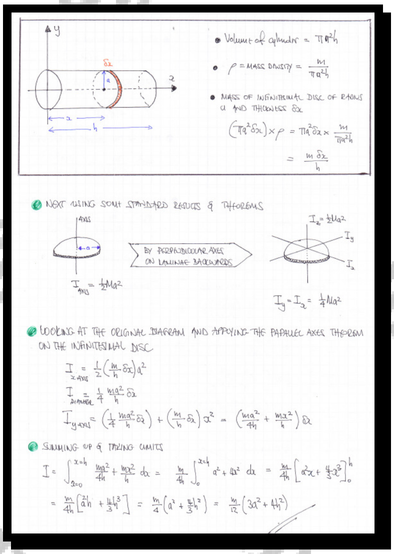
Question 14 (****)

Show by integration that the moment of inertia of a uniform solid right circular cylinder of mass m and radius a about a diameter of its plane face, is

$$\frac{m}{12}\left(3a^2+4h^2\right).$$

[*In this proof, you may assume standard results for the moment of inertia of a uniform circular disc about one of its diameters.*]

$\boxed{\text{proof}}$



Question 15 (****)

A thin uniform shell in the shape of a right circular cylinder of radius a and height h has both its circular ends removed.

The resulting open cylindrical shell has mass M.

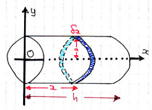
Show by integration that the moment of inertia of this shell about a diameter coplanar with one of its removed circular ends, is given by

$$\frac{1}{6}M\left(3a^2+2h^2\right).$$

[*In this proof, you may assume standard results for the moment of inertia of uniform circular hoops.*]

$\boxed{\text{M45}}$, $\boxed{\text{proof}}$

LOOKING AT THE DIAGRAM



- SURFACE AREA OF CURVED SURFACE IS $\pi a^2 h$
- ρ; (MASS PER UNIT AREA) IS
  $$\rho = \frac{M}{\pi a^2 h}$$
- MASS OF INFINITESIMAL HOOP OF THICKNESS δx IS GIVEN BY
  $$\delta m = (2\pi a\,\delta x)\rho$$
- THE MOMENT OF INERTIA OF THE INFINITESIMAL HOOP ABOUT THE x AXIS IS GIVEN BY
  $$\delta I = \delta m\, a^2$$
- MOMENT OF INERTIA OF THE HOOP ABOUT A DIAMETER (BY USING THE PERPENDICULAR AXES THEOREM) IS GIVEN BY
  $$\tfrac{1}{2}\delta I \quad \text{(USING THE THEOREM "BACKWARDS")}$$
- FINALLY, BY THE PARALLEL AXES THEOREM, THE MOMENT OF INERTIA OF THE INFINITESIMAL HOOP, ABOUT THE y AXIS IS GIVEN BY
  $$\tfrac{1}{2}\delta I + \delta m\, x^2$$

SUMMING UP AND TAKING LIMITS

$$I = \int_{x=0}^{x=h}\left(\tfrac{1}{2}dI + x^2\,dm\right) = \int_{x=0}^{x=h}\tfrac{1}{2}a^2\,dm + x^2\,dm$$

$$= \int_{x=0}^{x=h}\left(\tfrac{1}{2}a^2 + x^2\right)dm$$

$$\Rightarrow I = \int_{x=0}^{x=h}\left(\tfrac{1}{2}a^2+x^2\right)(2\pi a\rho\,dx)$$

$$\Rightarrow I = \int_{x=0}^{x=h} 2\pi a\rho\left(\tfrac{1}{2}a^2+x^2\right)\,dx$$

$$\Rightarrow I = 2\pi a\rho\int_0^h \tfrac{1}{2}a^2 + x^2\ dx$$

$$\Rightarrow I = \cancel{2\pi a}\times\frac{M}{\cancel{2\pi a}h}\left[\tfrac{1}{2}a^2x + \tfrac{1}{3}x^3\right]_0^h$$

$$\Rightarrow I = \frac{M}{h}\left[\left(\tfrac{1}{2}a^2h + \tfrac{1}{3}h^3\right) - 0\right]$$

$$\Rightarrow I = M\left(\tfrac{1}{2}a^2 + \tfrac{1}{3}h^2\right)$$

$$\Rightarrow I = \tfrac{1}{6}M\left(3a^2 + 2h^2\right)$$

// AS REQUIRED

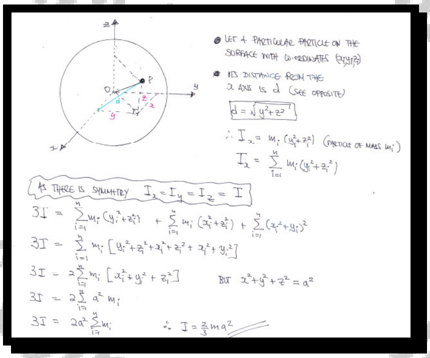
Question 16 (*****)

Show that the moment of inertia of a thin uniform spherical shell of mass m and radius a, about one of its diameters is

$$\frac{2}{3}ma^2 .$$

proof

[*In this proof, you may use valid symmetry arguments instead of calculus.*]



- LET A PARTICULAR PARTICLE ON THE SURFACE WITH CO-ORDINATES (x_i, y_i, z_i)
- ITS DISTANCE FROM THE x AXIS IS d (SEE OPPOSITE)

$\boxed{d = \sqrt{y^2+z^2}}$

$\therefore I_x = m_i(y_i^2+z_i^2)$ (PARTICLE OF MASS m_i)

$I_x = \sum_{i=1}^{n} m_i(y_i^2+z_i^2)$

$\boxed{\text{AS THERE IS SYMMETRY } I_x = I_y = I_z = I}$

$3I = \sum_{i=1}^{n} m_i(y_i^2+z_i^2) + \sum_{i=1}^{n} m_i(x_i^2+z_i^2) + \sum_{i=1}^{n} (x_i^2+y_i)^2$

$3I = \sum_{i=1}^{n} m_i\left[y_i^2+z_i^2+x_i^2+z_i^2+x_i^2+y_i^2\right]$

$3I = 2\sum_{i=1}^{n} m_i\left[x_i^2+y_i^2+z_i^2\right]$ BUT $x^2+y^2+z^2=a^2$

$3I = 2\sum_{i=1}^{n} a^2 m_i$

$3I = 2a^2\sum_{i=1}^{n} m_i$ $\therefore I = \frac{2}{3}ma^2$

MOMENT OF INERTIA
BY
STANDARD RESULTS

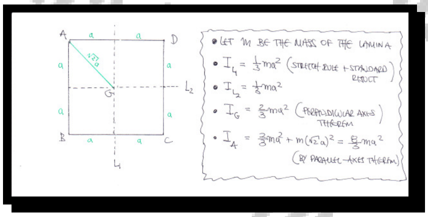
Question 1 (**)

A uniform square lamina has mass m and side length $2a$.

The lamina is free to rotate in about an axis L, which is perpendicular to the lamina and passes through one of the vertices of the lamina.

Calculate the moment of inertia of the lamina about L, in terms of m and a.

$$\boxed{I = \frac{8}{3}ma^2}$$



Question 2 (**)

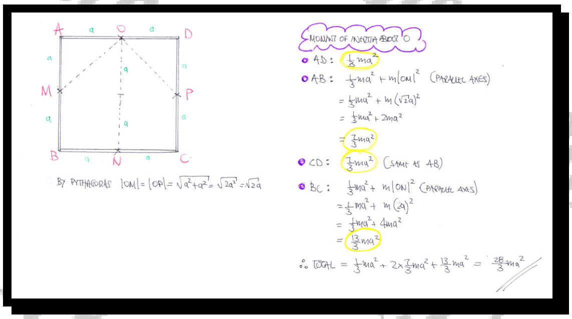
A square frame $ABCD$ consists of four uniform rods AB, BC, CD and DA.

Each of the rods has mass m and length $2a$.

The frame is free to rotate about an axis L, which is perpendicular to the plane of the frame and passes through the midpoint of AB.

Calculate the moment of inertia of the lamina about L, in terms of m and a.

$$\boxed{I = \frac{28}{3}ma^2}$$



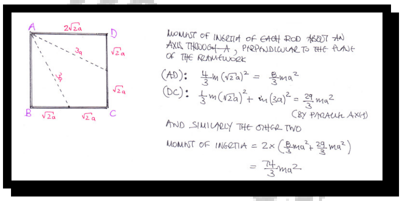
Question 3 (**)

Four uniform rods, each of mass m and length $2\sqrt{2}\,a$, are rigidly joined together to form a square framework $ABCD$.

The framework is free to rotate about an axis L, which is perpendicular to plane of the framework and passes through A.

Calculate the moment of inertia of $ABCD$ about L, in terms of m and a.

$$\boxed{I = \frac{74}{3}ma^2}$$

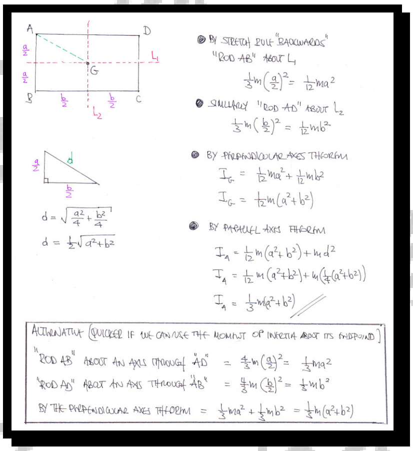


Question 4 (**)

A rectangular lamina $ABCD$, has mass m, length a and width b are rigidly joined together to form a square framework.

Calculate the moment of inertia of $ABCD$ about an axis through A perpendicular to the plane of $ABCD$.

$$\boxed{I = \frac{1}{3}m\left(a^2+b^2\right)}$$



Question 5 (**)

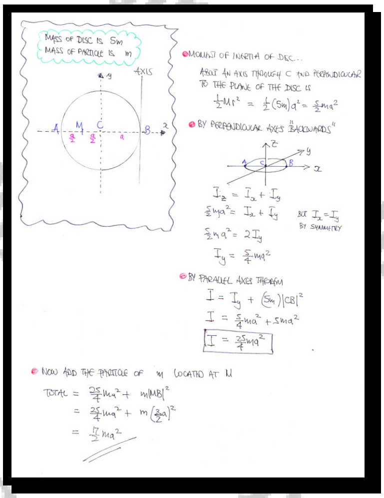
A uniform circular disc, with centre C, has mass $5m$ and radius a.

The straight line AB is a diameter of the disc.

A particle of mass m is attached to the disc at the point M, where M is the midpoint of AC. The disc is free to rotate about an axis L, which lies in the plane of the disc and is a tangent to the disc at B.

Find the moment of inertia of the loaded disc about L.

$$\boxed{I = \frac{17}{2}ma^2}$$



• MOMENT OF INERTIA OF DISC...

ABOUT AN AXIS THROUGH C AND PERPENDICULAR TO THE PLANE OF THE DISC IS

$\frac{1}{2}Mr^2 = \frac{1}{2}(5m)a^2 = \frac{5}{2}ma^2$

• BY PERPENDICULAR AXES "BACKWARDS"

$I_z = I_x + I_y$

$\frac{5}{2}ma^2 = I_x + I_y$ BUT $I_x = I_y$ BY SYMMETRY

$\frac{5}{2}ma^2 = 2I_y$

$I_y = \frac{5}{4}ma^2$

• BY PARALLEL AXES THEOREM

$I = I_y + (5m)|CB|^2$

$I = \frac{5}{4}ma^2 + 5ma^2$

$\boxed{I = \frac{25}{4}ma^2}$

• NOW ADD THE PARTICLE OF m LOCATED AT M

TOTAL $= \frac{25}{4}ma^2 + m|MB|^2$

$= \frac{25}{4}ma^2 + m\left(\frac{3}{2}a\right)^2$

$= \frac{17}{2}ma^2$

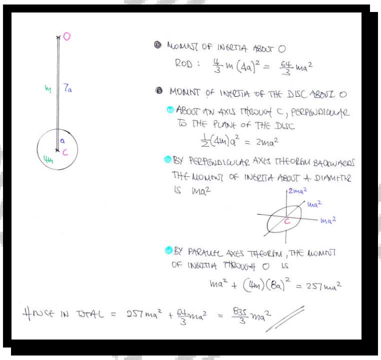
Question 6 (**+)

A compound pendulum consists of a thin uniform rod OC of length $8a$ and mass m is rigidly attached at C to the centre of a thin uniform circular disc of radius a and mass $4m$. The rod is in the same vertical plane as the disc. The pendulum is free to rotate in this vertical plane, through a smooth vertical axis through O, perpendicular to the plane of the disc.

Show that the moment of inertia of the pendulum about the above described axis is

$$\frac{835}{3}ma^2.$$

$\boxed{\text{proof}}$

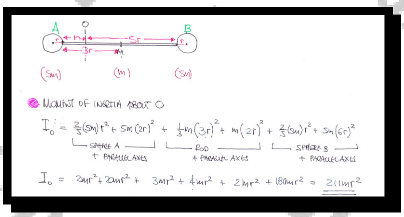


Question 7 (**+)

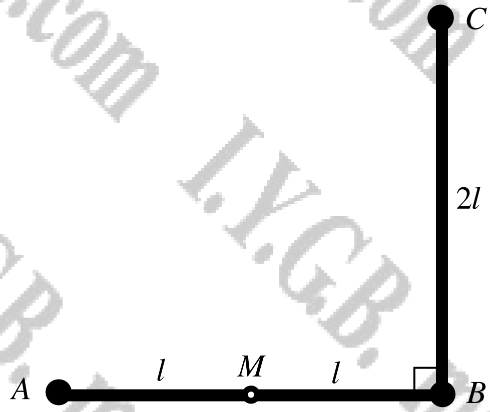
Two uniform spheres, each of mass $5m$ and radius r, are attached to each of the ends of a thin uniform rod AB, of mass m and length $6r$. The centres of the spheres are collinear with AB, and are located $8r$ apart. The system is free to rotate about an axis on a point on the rod O, where $|AO| = r$. This axis is perpendicular to AB.

Determine the moment of inertia of the system about O.

$\boxed{211mr^2}$



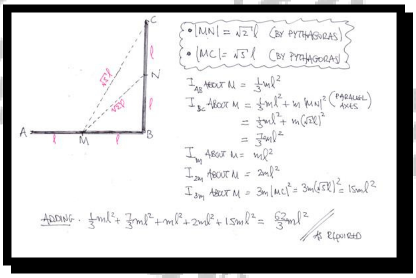
Question 8 (+)**



Two identical uniform rods AB and BC, each of mass m and length l are rigidly joined at B, so that $\measuredangle ABC = 90°$. Three particles of masses m, $2m$ and $3m$ are fixed at A, B and C, respectively. The system of the two rods and the three particles can rotate freely in a vertical plane about a horizontal axis through M, where M is the midpoint of AB.

Show clearly that the moment of inertia of the system about an axis through M and perpendicular to the plane ABC is $\dfrac{62}{3}ml^2$.

$\boxed{\text{proof}}$



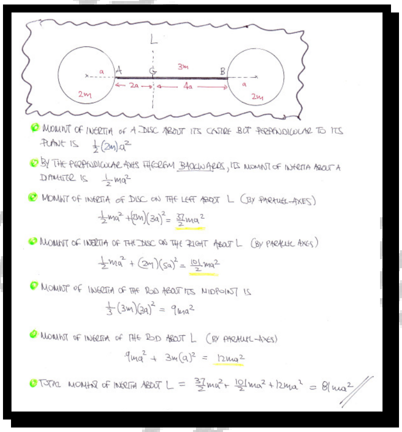
Question 9 (**+)

Two uniform discs, each of mass $2m$ and radius a, are attached to each of the ends of a thin uniform rod AB, of mass $3m$ and length $6a$. The system lies in the same plane the centres of the discs being collinear with AB, and located $8a$ apart.

The system is free to rotate about an axis on a point on the rod C, where $|AC| = 2a$. This axis is perpendicular to AB but lies in the same plane as the system.

Determine the moment of inertia of the system about C.

$\boxed{81ma^2}$



MOMENT OF INERTIA OF A DISC ABOUT ITS CENTRE BUT PERPENDICULAR TO ITS PLANE IS $\frac{1}{2}(2m)a^2$

BY THE PERPENDICULAR AXES THEOREM BACKWARDS, ITS MOMENT OF INERTIA ABOUT A DIAMETER IS $\frac{1}{2}ma^2$

MOMENT OF INERTIA OF DISC ON THE LEFT ABOUT L (BY PARALLEL-AXES)

$$\frac{1}{2}ma^2 + (2m)(3a)^2 = \frac{37}{2}ma^2$$

MOMENT OF INERTIA OF THE DISC ON THE RIGHT ABOUT L (BY PARALLEL AXES)

$$\frac{1}{2}ma^2 + (2m)(5a)^2 = \frac{101}{2}ma^2$$

MOMENT OF INERTIA OF THE ROD ABOUT ITS MIDPOINT IS

$$\frac{1}{3}(3m)(3a)^2 = 9ma^2$$

MOMENT OF INERTIA OF THE ROD ABOUT L (BY PARALLEL-AXES)

$$9ma^2 + 3m(a)^2 = 12ma^2$$

TOTAL MOMENT OF INERTIA ABOUT L $= \frac{37}{2}ma^2 + \frac{101}{2}ma^2 + 12ma^2 = 81ma^2$

Question 10 (**+)

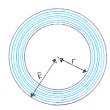
A disc of radius r and centre O is removed from a larger uniform disc of radius R and centre O, forming an annulus of mass M.

Use standard results to show that the moment of inertia of the annulus about an axis through O and perpendicular to its plane, is

$$\frac{1}{2}M\left(R^2 - r^2\right),$$

and use this result to deduce the moment of inertia of a circular hoop of mass M, about an axis through its centre and perpendicular to the plane of the hoop.

$\boxed{\text{proof}}$



- THE LAMINA (ANNULUS) HAS MASS M
- THE AREA IS
  $\pi R^2 - \pi r^2 = \pi(R^2 - r^2)$
- MASS PER UNIT AREA IS
  $\rho = \frac{M}{\pi(R^2-r^2)}$
- MASS OF DISC OF RADIUS R IS $\pi\rho R^2$
  MASS OF DISC OF RADIUS r IS $\pi\rho r^2$

- HENCE THE MOMENT OF INERTIA OF THE ANNULUS BY SUBTRACTION IS GIVEN BY

$\Rightarrow I = \frac{1}{2}(\pi\rho R^2)R^2 - \frac{1}{2}(\pi\rho r^2)r^2$

$\Rightarrow I = \frac{1}{2}\pi\rho\left[R^4 - r^4\right]$

$\Rightarrow I = \frac{1}{2}\cancel{\pi}\left(\frac{M}{\cancel{\pi}(R^2-r^2)}\right)(R^4 - r^4)$

$\Rightarrow I = \frac{1}{2}M\frac{R^4 - r^4}{R^2 - r^2}$

$\Rightarrow I = \frac{1}{2}M\frac{\cancel{(R^2 - r^2)}(R^2+r^2)}{\cancel{R^2 - r^2}}$

$\Rightarrow I = \frac{1}{2}M(R^2 + r^2)$

- FINALLY TAKING THE LIMIT AS $r \to R$

$\Rightarrow I = \frac{1}{2}M(R^2 + R^2)$

$\Rightarrow I = \frac{1}{2}M(2R^2)$

$\Rightarrow I = MR^2$

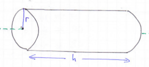
Question 11 (**+)

A thin uniform shell in the shape of a right circular cylinder of radius r and height h, with both its circular ends made of the same material and having the same thickness.

The resulting closed cylindrical shell has mass m.

Find the moment of inertia of the shell about its axis of symmetry.

$$\boxed{\frac{mr^2(r+2h)}{2(r+h)}}$$



AREA OF THE CYLINDER

$= \pi r^2 + \pi r^2 + 2\pi r h$

$= 2\pi r^2 + 2\pi r h$

$= 2\pi r(r+h)$

$\therefore$ MASS PER UNIT AREA $= \dfrac{m}{2\pi r(r+h)}$

- MASS OF EACH OF THE TWO CIRCULAR BASES $= \pi r^2 \times \dfrac{m}{2\pi r(r+h)} = \dfrac{mr}{2(r+h)}$
- MASS OF THE CURVED SURFACE $= 2\pi r h \times \dfrac{m}{2\pi r(r+h)} = \dfrac{mh}{r+h}$
- NEXT, BY THE STRETCH RULE, THE CURVED SURFACE CAN BE REDUCED TO A CIRCULAR HOOP OF RADIUS r, AND OF THE SAME MASS

  EACH OF THE BASES IS MODELLED AS A DISC
- MOMENT OF INERTIA OF THE CYLINDER ABOUT ITS AXIS IS GIVEN BY

$$\underbrace{2 \times \frac{1}{2}\left[\frac{mr}{2(r+h)}\right]r^2}_{\text{TWO BASES}} + \underbrace{\left[\frac{mh}{r+h}\right]r^2}_{\text{CURVED SURFACE}}$$

- $I_{AXIS} = \dfrac{mr^3}{2(r+h)} + \dfrac{mr^2h}{r+h}$

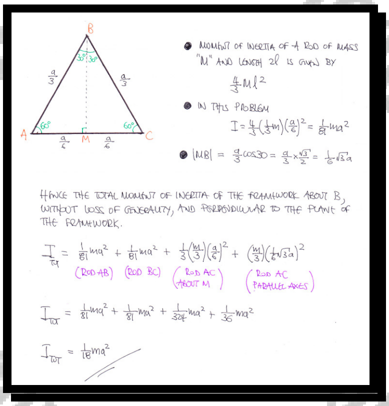
$$I_{AXIS} = \frac{mr^2(r+2h)}{2(r+h)}$$

Question 12 (***)

A thin uniform wire AB, of mass m and length $3a$, is bent into the shape of an equilateral triangle.

Find the moment of inertia of the triangle about an axis through one of its vertices and perpendicular to the plane of the triangle.

$\boxed{I = 16ma^2}$

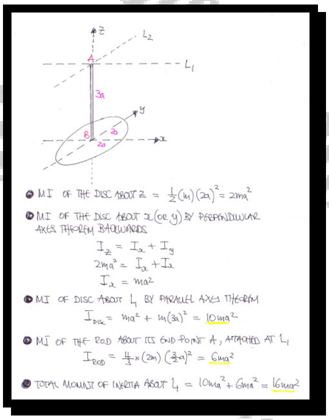


Question 13 (***)

A composite body consists of a thin uniform rod AB, of mass m and length $3a$, with the end B rigidly attached to the centre O of a uniform circular lamina, of radius $2a$ and mass m. The rod is perpendicular to the plane of the lamina. The body is free to rotate in a vertical plane about a horizontal axis through A, and perpendicular to AB.

Find the moment of inertia of the body about the above described axis.

$\boxed{I = 16ma^2}$



Question 14 (***)

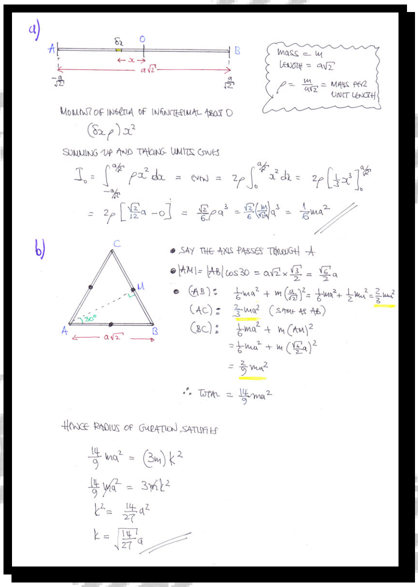
A uniform rod AB, has mass m and length $\sqrt{2}a$.

a) Use integration to find the moment of inertia of the rod about an axis through its midpoint O.

Three rods, identical to AB, are joined together to form an equilateral triangle ABC. The triangle is free to rotate about a fixed smooth axis L, which is perpendicular to the plane of ABC and passes through one of the vertices of ABC.

b) Determine the radius of gyration of ABC about L.

$\boxed{I_O = \frac{1}{6}ma^2}$, $\boxed{k = \sqrt{\frac{14}{27}}\,a}$


a)

MASS $= m$
LENGTH $= a\sqrt{2}$
$\rho = \frac{m}{a\sqrt{2}}$ = MASS PER UNIT LENGTH

MOMENT OF INERTIA OF INFINITESIMAL ABOUT O
$(\delta x\, \rho)x^2$

SUMMING UP AND TAKING LIMITS GIVES

$I_O = \int_{-a/\sqrt{2}}^{a/\sqrt{2}} \rho x^2\,dx = \text{EVEN} = 2\rho\int_0^{a/\sqrt{2}} x^2\,dx = 2\rho\left[\frac{1}{3}x^3\right]_0^{a/\sqrt{2}}$

$= 2\rho\left[\frac{\sqrt{2}}{12}a^3 - 0\right] = \frac{\sqrt{2}}{6}\rho a^3 = \frac{\sqrt{2}}{6}\left(\frac{m}{\sqrt{2}a}\right)a^3 = \frac{1}{6}ma^2$

b)

- SAY THE AXIS PASSES THROUGH A
- $|AM| = |AB|\cos 30 = a\sqrt{2}\times\frac{\sqrt{3}}{2} = \frac{\sqrt{6}}{2}a$
- (AB): $\frac{1}{6}ma^2 + m\left(\frac{a}{\sqrt{2}}\right)^2 = \frac{1}{6}ma^2 + \frac{1}{2}ma^2 = \frac{2}{3}ma^2$

  (AC): $\frac{2}{3}ma^2$ (SAME AS AB)

  (BC): $\frac{1}{6}ma^2 + m(AM)^2$

  $= \frac{1}{6}ma^2 + m\left(\frac{\sqrt{6}}{2}a\right)^2$

  $= \frac{2}{9}ma^2$

$\therefore$ TOTAL $= \frac{14}{9}ma^2$

HENCE RADIUS OF GYRATION SATISFIES

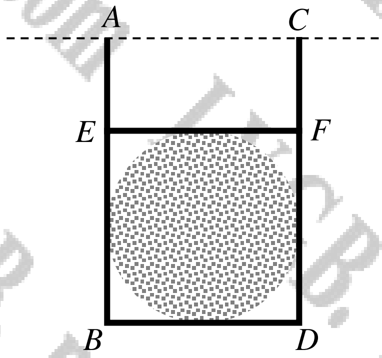
$\frac{14}{9}ma^2 = (3m)k^2$

$\frac{14}{9}\cancel{m}a^2 = 3\cancel{m}k^2$

$k^2 = \frac{14}{27}a^2$

$k = \sqrt{\frac{14}{27}}\,a$

Question 15 (***)

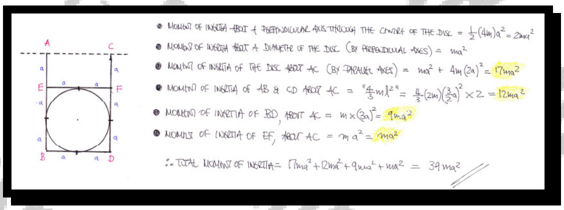


A shop sign is in the shape of a uniform circular disc of mass $4m$ and radius a.

It is suspended vertically by two uniform rods AB and CD, each of length $3a$ and mass $2m$. Two more rods EF and BD, each of length $2a$ and mass m are placed around the sign. All the rods are tangents to the disc so that $BFED$ is a square as shown in the figure.

Use standard results to determine the moment of inertia of the shop sign and the 4 rods, about a horizontal axis through A and C.

$\boxed{39ma^2}$



- MOMENT OF INERTIA ABOUT A PERPENDICULAR AXIS THROUGH THE CENTRE OF THE DISC $= \frac{1}{2}(4m)a^2 = 2ma^2$
- MOMENT OF INERTIA ABOUT A DIAMETER OF THE DISC (BY PERPENDICULAR AXES) $= ma^2$
- MOMENT OF INERTIA OF THE DISC ABOUT AC (BY PARALLEL AXES) $= ma^2 + 4m(2a)^2 = 17ma^2$
- MOMENT OF INERTIA OF AB & CD ABOUT AC $= \text{"}\frac{4}{3}ml^2\text{"} = \frac{4}{3}(2m)\left(\frac{3}{2}a\right)^2 \times 2 = 12ma^2$
- MOMENT OF INERTIA OF BD, ABOUT AC $= m \times (3a)^2 = 9ma^2$
- MOMENT OF INERTIA OF EF, ABOUT AC $= ma^2 = ma^2$

$\therefore$ TOTAL MOMENT OF INERTIA $= 17ma^2 + 12ma^2 + 9ma^2 + ma^2 = 39ma^2$

Question 16 (***)

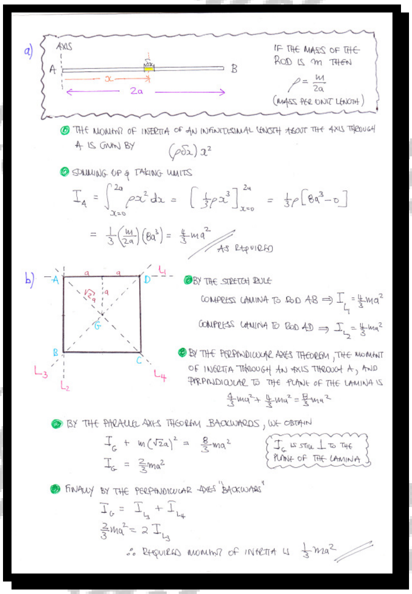
A thin uniform rod AB, of length $2a$ and mass m, is free to rotate about an axis L, which passes through A and is perpendicular to the length of the rod.

a) Use integration to show that the moment of inertia I of this rod about L is

$$I = \frac{4}{3}ma^2 .$$

b) Use this result and moment of inertia theorems, to determine the moment of inertia of a uniform square lamina, of side length $2a$ and mass m, about one of its diagonals.

$$\boxed{\frac{1}{3}ma^2}$$



a) AXIS

IF THE MASS OF THE ROD IS m THEN $\rho = \frac{m}{2a}$ (MASS PER UNIT LENGTH)

THE MOMENT OF INERTIA OF AN INFINITESIMAL LENGTH ABOUT THE AXIS THROUGH A IS GIVEN BY $(\rho\delta x)x^2$

SUMMING UP & TAKING LIMITS

$$I_A = \int_{x=0}^{2a} \rho x^2 \, dx = \left[\tfrac{1}{3}\rho x^3\right]_{x=0}^{2a} = \tfrac{1}{3}\rho\left[8a^3 - 0\right]$$

$$= \tfrac{1}{3}\left(\tfrac{m}{2a}\right)\left(8a^3\right) = \tfrac{4}{3}ma^2$$

AS REQUIRED

b) BY THE STRETCH RULE

COMPRESS LAMINA TO ROD AB $\Rightarrow I_{L_1} = \frac{4}{3}ma^2$

COMPRESS LAMINA TO ROD AD $\Rightarrow I_{L_2} = \frac{4}{3}ma^2$

BY THE PERPENDICULAR AXES THEOREM, THE MOMENT OF INERTIA THROUGH AN AXIS THROUGH A, AND PERPENDICULAR TO THE PLANE OF THE LAMINA IS

$$\tfrac{4}{3}ma^2 + \tfrac{4}{3}ma^2 = \tfrac{8}{3}ma^2$$

BY THE PARALLEL AXES THEOREM BACKWARDS, WE OBTAIN

$$I_G + m(\sqrt{2}a)^2 = \tfrac{8}{3}ma^2$$

$$I_G = \tfrac{2}{3}ma^2$$

I_G IS STILL $\perp$ TO THE PLANE OF THE LAMINA

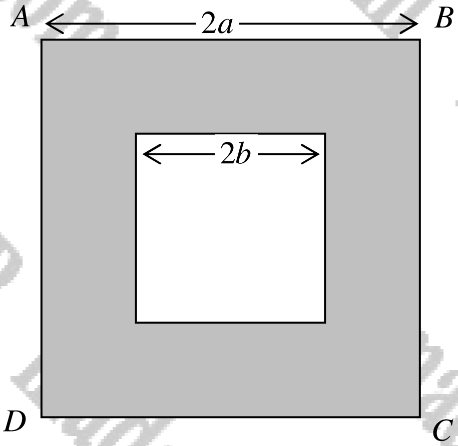
FINALLY BY THE PERPENDICULAR AXES "BACKWARDS"

$$I_G = I_{L_3} + I_{L_4}$$

$$\tfrac{2}{3}ma^2 = 2I_{L_3}$$

$\therefore$ REQUIRED MOMENT OF INERTIA IS $\frac{1}{3}ma^2$

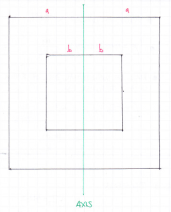
Question 17 (***)



A uniform lamina, of mass m, is formed from a square lamina $ABCD$ of side $2a$, by removing a square of side $2b$ so both squares have parallel sides and share the same centre, as shown in the figure above.

Find the moment of inertia of this lamina about an axis passing through the midpoint of AB and the midpoint of DC.

$$\boxed{\frac{1}{3}m\left(a^2+b^2\right)}$$



- AREA OF THE LAMINA IS
  $4a^2-4b^2=4(a^2-b^2)$
- MASS PER UNIT AREA
  $\rho=\frac{m}{4(a^2-b^2)}$
- MASS OF THE SQUARE OF RADIUS 2a
  $4a^2\times\frac{m}{4(a^2-b^2)}=\frac{ma^2}{a^2-b^2}$
- MASS OF THE SQUARE OF RADIUS 2b
  $4b^2\times\frac{m}{4(a^2-b^2)}=\frac{mb^2}{a^2-b^2}$

USING THE STRETCH RULE "BACKWARDS" & THE ADDITION RULE

$\Rightarrow I_{\text{BIG SQUARE}}=I_{\text{SMALL SQUARE}}+I_{\text{REQUIRED SHAPE}}$

$\Rightarrow \frac{1}{3}\left(\frac{ma^2}{a^2-b^2}\right)a^2=\frac{1}{3}\left(\frac{mb^2}{a^2-b^2}\right)b^2+I$

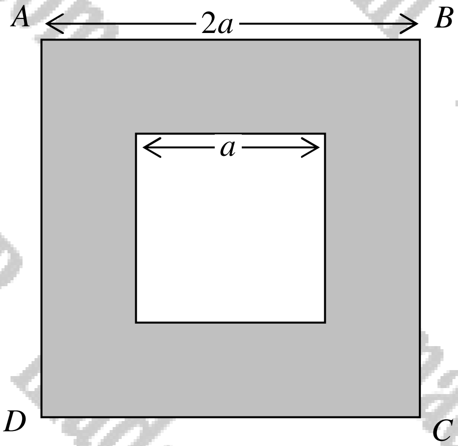
$\Rightarrow I=\frac{1}{3}\frac{ma^4}{a^2-b^4}-\frac{1}{3}\frac{mb^4}{a^2-b^2}$

$\Rightarrow I=\frac{1}{3}m\left[\frac{a^4-b^4}{a^2-b^2}\right]$

$\Rightarrow I=\frac{1}{3}m\frac{\cancel{(a^2-b^2)}(a^2+b^2)}{\cancel{a^2-b^2}}$

$\Rightarrow I=\frac{1}{3}m(a^2+b^2)$

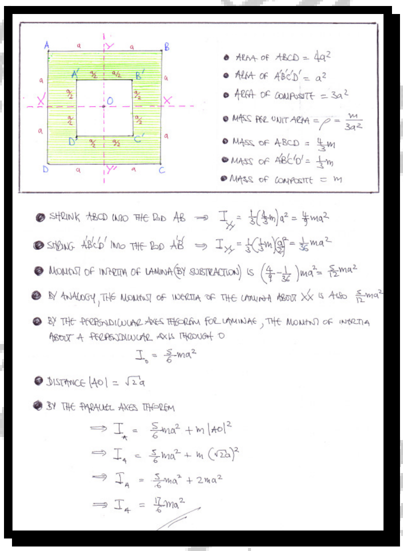
Question 18 (***)



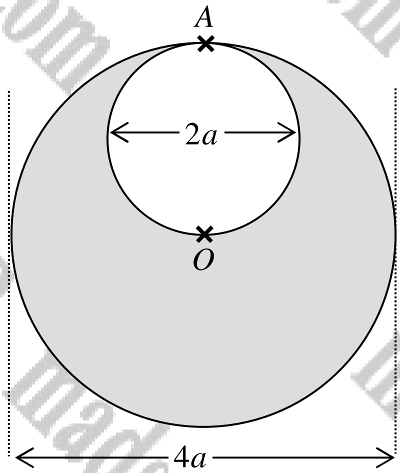
A uniform lamina, of mass m, is formed from a square lamina $ABCD$ of side $2a$, by removing a square of side a so both squares have parallel sides and share the same centre, as shown in the figure above.

Find the moment of inertia of this lamina about an axis through A and perpendicular to the plane of the lamina.

$\boxed{\frac{17}{6}ma^2}$



Question 19 (***)



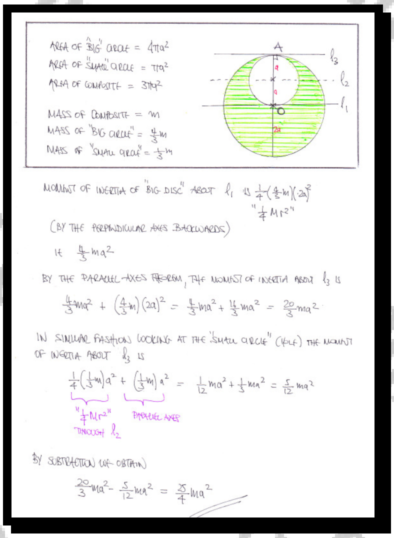
The point A lies on the circumference of a uniform circular disc of diameter $4a$.

A smaller circular disc with diameter OA is removed from the larger disc, where O is the centre of the larger disc, as shown in the figure above.

The remaining composite lamina L has mass m.

Determine its moment of inertia of L about an axis lying on the plane on L, the axis passing through A and being perpendicular to AO.

$\boxed{\frac{25}{4}ma^2}$



AREA OF "BIG" CIRCLE $= 4\pi a^2$
AREA OF "SMALL" CIRCLE $= \pi a^2$
AREA OF COMPOSITE $= 3\pi a^2$

MASS OF COMPOSITE $= m$
MASS OF "BIG CIRCLE" $= \frac{4}{3}m$
MASS OF "SMALL CIRCLE" $= \frac{1}{3}m$

MOMENT OF INERTIA OF "BIG DISC" ABOUT l_1 IS $\underbrace{\frac{1}{4}\left(\frac{4}{3}m\right)(2a)^2}_{\text{"}\frac{1}{4}Mr^2\text{"}}$

(BY THE PERPENDICULAR AXES BACKWARDS)

IE $\frac{4}{3}ma^2$

BY THE PARALLEL-AXES THEOREM, THE MOMENT OF INERTIA ABOUT l_3 IS

$$\frac{4}{3}ma^2 + \left(\frac{4}{3}m\right)(2a)^2 = \frac{4}{3}ma^2 + \frac{16}{3}ma^2 = \frac{20}{3}ma^2$$

IN SIMILAR FASHION LOOKING AT THE "SMALL CIRCLE" (HOLE) THE MOMENT OF INERTIA ABOUT l_3 IS

$$\underbrace{\frac{1}{4}\left(\frac{1}{3}m\right)a^2}_{\text{"}\frac{1}{4}Mr^2\text{" THROUGH } l_2} + \underbrace{\left(\frac{1}{3}m\right)a^2}_{\text{PARALLEL AXES}} = \frac{1}{12}ma^2 + \frac{1}{3}ma^2 = \frac{5}{12}ma^2$$

BY SUBTRACTION WE OBTAIN

$$\frac{20}{3}ma^2 - \frac{5}{12}ma^2 = \frac{25}{4}ma^2$$

Question 20 (***+)

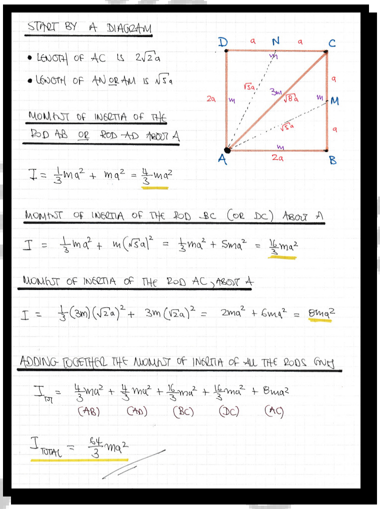
Four identical rods, each of mass m and length $2a$ are joined together to form a square rigid framework $ABCD$.

A fifth rod AC, of mass $3m$, is added to the framework for extra support.

The 5 rod framework is free to rotate about an axis L, which passes through A, and is perpendicular to the plane of $ABCD$.

Determine the moment of inertia of the framework about L.

$\boxed{I = \frac{64}{3}ma^2}$


START BY A DIAGRAM

- LENGTH OF AC IS $2\sqrt{2}a$
- LENGTH OF AN OR AM IS $\sqrt{5}a$

MOMENT OF INERTIA OF THE ROD AB OR ROD AD ABOUT A

$I = \frac{1}{3}ma^2 + ma^2 = \frac{4}{3}ma^2$

MOMENT OF INERTIA OF THE ROD BC (OR DC) ABOUT A

$I = \frac{1}{3}ma^2 + m(\sqrt{5}a)^2 = \frac{1}{3}ma^2 + 5ma^2 = \frac{16}{3}ma^2$

MOMENT OF INERTIA OF THE ROD AC, ABOUT A

$I = \frac{1}{3}(3m)(\sqrt{2}a)^2 + 3m(\sqrt{2}a)^2 = 2ma^2 + 6ma^2 = 8ma^2$

ADDING TOGETHER THE MOMENT OF INERTIA OF ALL THE RODS GIVES

$I_{TOT} = \underbrace{\frac{4}{3}ma^2}_{(AB)} + \underbrace{\frac{4}{3}ma^2}_{(AD)} + \underbrace{\frac{16}{3}ma^2}_{(BC)} + \underbrace{\frac{16}{3}ma^2}_{(DC)} + \underbrace{8ma^2}_{(AC)}$

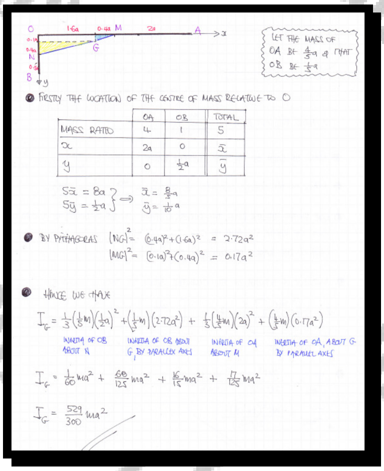
$I_{TOTAL} = \frac{64}{3}ma^2$

Question 21 (***+)

A uniform rod AB is bent at the point O, so that in the resulting $L-$shaped rigid object $\measuredangle AOB = \frac{1}{2}\pi$, $|AO| = 4a$ and $|OB| = a$.

Find the moment of inertia of the resulting object, about an axis through its centre of mass and perpendicular to the plane AOB.

$$\boxed{\frac{529}{300}ma^2}$$



LET THE MASS OF OA BE $\frac{4}{5}a$ & THAT OB BE $\frac{1}{5}a$

- FIRSTLY THE LOCATION OF THE CENTRE OF MASS RELATIVE TO O

	OA	OB	TOTAL
MASS RATIO	4	1	5
x	$2a$	0	$\bar{x}$
y	0	$\frac{1}{2}a$	$\bar{y}$

$$\left.\begin{array}{l} 5\bar{x} = 8a \\ 5\bar{y} = \frac{1}{2}a \end{array}\right\} \Rightarrow \begin{array}{l} \bar{x} = \frac{8}{5}a \\ \bar{y} = \frac{1}{10}a \end{array}$$

- BY PYTHAGORAS $|NG|^2 = (0.4a)^2 + (1.6a)^2 = 2.72a^2$

$|MG|^2 = (0.1a)^2 + (0.4a)^2 = 0.17a^2$

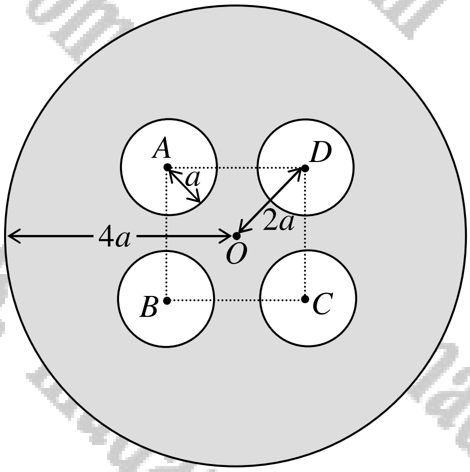
- HENCE WE HAVE

$$I_G = \underbrace{\tfrac{1}{3}\left(\tfrac{1}{5}m\right)\left(\tfrac{1}{2}a\right)^2}_{\text{INERTIA OF OB ABOUT N}} + \underbrace{\left(\tfrac{1}{5}m\right)\left(2.72a^2\right)}_{\text{INERTIA OF OB ABOUT G, BY PARALLEL AXES}} + \underbrace{\tfrac{1}{3}\left(\tfrac{4}{5}m\right)(2a)^2}_{\text{INERTIA OF OA ABOUT M}} + \underbrace{\left(\tfrac{4}{5}m\right)\left(0.17a^2\right)}_{\text{INERTIA OF OA, ABOUT G BY PARALLEL AXES}}$$

$$I_G = \frac{1}{60}ma^2 + \frac{68}{125}ma^2 + \frac{16}{15}ma^2 + \frac{17}{125}ma^2$$

$$I_G = \frac{529}{300}ma^2$$

Question 22 (*+)**



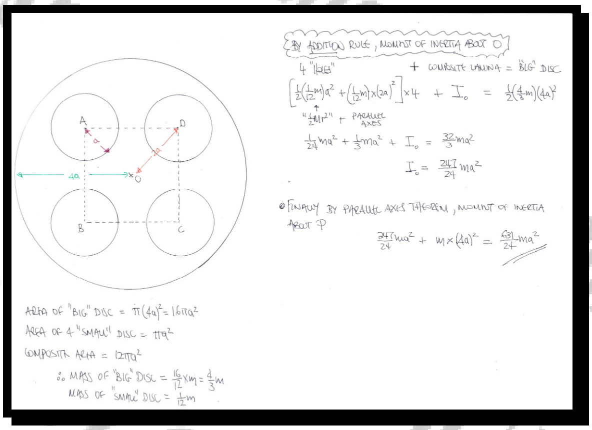
A uniform circular lamina has radius $4a$ and centre O. The points A, B, C and D lie on the lamina and are vertices of a square whose centre is at O so that $|OD| = 2a$.

Four circular discs, each of radius a, with centres A, B, C and D are removed from the lamina. The remaining lamina forms a new composite lamina of mass m.

The new lamina is free to rotate in a vertical plane about a fixed smooth horizontal axis L, which is perpendicular to the lamina and passes through a point P at the circumference of the lamina.

Calculate the moment of inertia of the lamina about L, in terms of m and a.

$$\boxed{I = \frac{631}{24} ma^2}$$



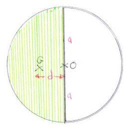
Question 23 (***+)

A uniform lamina has mass m and is in the shape of a semicircle of radius a, centred at the point O. The centre of mass of the lamina is at the point G.

The lamina is free to rotate about a fixed smooth horizontal axis L, which is perpendicular the plane of the lamina and passes through G.

Calculate the moment of inertia of the lamina about L, in terms of m and a.

$$\boxed{I = \frac{ma^2}{18\pi^2}\left(9\pi^2 - 32\right)}$$



- MOMENT OF A CIRCULAR LAMINA OF MASS 2M, ABOUT A PERPENDICULAR AXIS THROUGH ITS CENTRE O IS $\frac{1}{2}(2m)a^2 = ma^2$
- BY ADDITION RULE FOR A SEMICIRCULAR LAMINA OF MASS m, ABOUT THE SAME AXIS WOULD BE $\frac{1}{2}ma^2$
- THE POSITION OF THE CENTRE OF MASS OF A SEMICIRCULAR LAMINA IS $\frac{4r}{3\pi}$ FROM THE CENTRE, SO HERE $\frac{4a}{3\pi}$
- BY PARALLEL AXES THEOREM

$$I_O = I_G + md^2$$
$$\frac{1}{2}ma^2 = I_G + m\left(\frac{4a}{3\pi}\right)^2$$
$$\frac{1}{2}ma^2 = I_G + \frac{16ma^2}{9\pi^2}$$
$$I_G = \frac{1}{2}ma^2 - \frac{16ma^2}{9\pi^2}$$
$$I_G = \frac{1}{18}ma^2\left[9 - \frac{32}{\pi^2}\right]$$

OR $I_G = \frac{ma^2}{18\pi^2}\left[9\pi^2 - 32\right]$