

Created by T. Madas

# SIMPLE HARMONIC MOTION

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# SIMPLE HARMONIC MOTION KINEMATICS

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**Question 1** (\*\*)

A particle  $P$  is moving on a straight line with simple harmonic motion of amplitude  $0.3 \text{ m}$ . It passes through the centre of the oscillation  $O$  with speed  $4.5 \text{ ms}^{-1}$ .

Calculate the speed of  $P$  when  $|OP| = 0.1 \text{ m}$ .

$$|v| = 3\sqrt{2} \approx 4.24 \text{ ms}^{-1}$$

Handwritten solution for Question 1:

- Diagram: A horizontal line with points A, O, P, B. O is the center. A and B are at distance 0.3 from O. P is at distance 0.1 from O.
- Given:  $v_0 = 4.5$
- Amplitude  $A = 0.3$
- Displacement  $x = 0.1$
- Equation:  $v^2 = \omega^2(A^2 - x^2)$
- Calculation:  $v^2 = 15^2(0.3^2 - 0.1^2)$
- Result:  $v = 3\sqrt{2}$
- Final answer:  $v \approx 4.24 \text{ ms}^{-1}$

**Question 2** (\*\*)

A particle  $P$  is moving on a straight line with simple harmonic motion of amplitude  $0.1 \text{ m}$  and period  $\frac{2\pi}{7} \text{ s}$ .

Calculate the maximum acceleration of  $P$ .

$$|\ddot{x}|_{\max} = 4.9 \text{ ms}^{-2}$$

Handwritten solution for Question 2:

- Period  $T = \frac{2\pi}{7}$
- Angular frequency  $\omega = \frac{2\pi}{T} = 7$
- Equation:  $|\ddot{x}|_{\max} = \omega^2 a$
- Calculation:  $|\ddot{x}|_{\max} = 7^2 \times 0.1$
- Result:  $|\ddot{x}|_{\max} = 4.9 \text{ ms}^{-2}$

**Question 3 (\*\*)**

A particle is moving in a straight line between two points  $A$  and  $B$ , with simple harmonic motion.

During this motion its greatest speed is  $2.25 \text{ ms}^{-1}$ . When the particle is at a distance of  $21 \text{ cm}$  from the midpoint of  $AB$  its speed is  $2.16 \text{ ms}^{-1}$ .

Find the distance  $AB$ .

,  $|AB| = 1.5 \text{ m}$

STATE WITH A SHORTER ANSWER FOR S.H.M. kinematics

Diagram: A horizontal line with points A, M, and B. M is the midpoint. A distance of 0.21 is marked from M to the right. A velocity vector  $V = 2.16$  is shown pointing right from that point. The maximum velocity  $V_{max} = 2.25$  is indicated at the ends of the line.

$V_{max} = a\omega$   
 $2.25 = a\omega$

With  $x = 0.21$ ,  $v = 2.16$   
 $\Rightarrow v = \omega\sqrt{a^2 - x^2}$   
 $\Rightarrow = \omega\sqrt{a^2 - 0.21^2}$   
 $\Rightarrow 2.16^2 = 2.25^2 - \omega^2 \cdot 0.21^2$   
 $\Rightarrow 4.6656 = 5.0625 - 0.0441\omega^2$   
 $\Rightarrow 0.0441\omega^2 = 0.3969$   
 $\Rightarrow \omega^2 = 9$   
 $\Rightarrow \omega = +3$

Hence we give  
 $\Rightarrow a\omega = 2.25$   
 $\Rightarrow 4 \times 3 = 2.25$   
 $\Rightarrow a = 0.75$   
 $\Rightarrow |AB| = 1.5 \text{ m}$

**Question 4 (\*\*)**

A particle  $P$  is moving on a straight line with simple harmonic motion, centre at  $O$ , and period  $2\pi \text{ s}$ .

Find the speed of  $P$  when it is at a distance of  $0.3 \text{ m}$  from  $O$ , given that it comes to instantaneous rest at a distance  $0.5 \text{ m}$  from  $O$ .

speed =  $0.4 \text{ ms}^{-1}$

Diagram: A horizontal line with points A, O, and B. O is the center. Distances of 0.5 are marked from O to A and O to B. A cloud contains  $T = 2\pi$  and  $a = 0.5$ .

$T = \frac{2\pi}{\omega}$   
 $2\pi = \frac{2\pi}{\omega}$   
 $\omega = 1$

$v = \omega\sqrt{a^2 - x^2}$   
 $v = 1\sqrt{0.5^2 - 0.3^2}$   
 $v = 0.4$   
 $v = 0.4 \text{ ms}^{-1}$

**Question 5** (\*\*+)

A particle  $P$  is moving on a straight line with simple harmonic motion of period  $\frac{\pi}{6}$  s.

Given that the maximum speed of  $P$  is  $12 \text{ ms}^{-1}$ , find the speed of  $P$  0.2 s after passing through the centre of the oscillation.

speed =  $8.8487\dots \text{ ms}^{-1}$

Handwritten solution for Question 5:

- Diagram: A horizontal line with a central point 'O' and two points 'A' and 'B' on either side, representing simple harmonic motion.
- Calculations:
  - $T = \frac{\pi}{6}$
  - $\omega = \frac{2\pi}{T} = \frac{2\pi}{\pi/6} = 12$
  - $V_{\text{max}} = 12 \text{ ms}^{-1}$
  - $a = a \sin \omega t$
  - $12 = 12 \cos \omega t$
  - $\omega t = 12 \cos 12t$
  - When  $t = 0.2$
  - $\omega = 12 \cos 2.4$
  - $\omega = 8.8487\dots$
  - $|\dot{x}| = 8.85 \text{ ms}^{-1}$

**Question 6** (\*\*+)

A particle  $P$  is moving on a straight line with simple harmonic motion of maximum speed  $5 \text{ ms}^{-1}$  and maximum acceleration  $10 \text{ ms}^{-2}$ .

Calculate the speed of  $P$  when it is 2 m from the centre of the oscillation.

speed =  $3 \text{ ms}^{-1}$

Handwritten solution for Question 6:

- Given:  $V_{\text{max}} = 5$ ,  $a_{\text{max}} = 10$
- Relationships:  $a\omega = 5$ ,  $a\omega^2 = 10$
- Dividing:  $\frac{a\omega}{a\omega^2} = \frac{5}{10} \Rightarrow \frac{1}{\omega} = \frac{1}{2} \Rightarrow \omega = 2$
- Substituting:  $a \cdot 2 = 5 \Rightarrow 2a = 5 \Rightarrow a = 2.5$
- Using  $v^2 = \omega^2(a^2 - x^2)$
- $v^2 = 2^2(2.5^2 - 2^2)$
- $v^2 = 4 \times \frac{1}{4}$
- $v^2 = 1$
- $v = 1 \text{ ms}^{-1}$

**Question 7 (\*\*+)**

A particle  $P$  is moving on a straight line with simple harmonic motion, centre at  $O$ .

$P$  passes through  $O$  with speed  $6 \text{ ms}^{-1}$  and performs 240 complete oscillations every minute.

Calculate the maximum acceleration of  $P$ .

$$\approx 151 \text{ ms}^{-2}$$

Handwritten solution for Question 7:

- $v_{\text{max}} = 6$
- $T = \frac{2\pi}{\omega}$
- $\frac{1}{4} = \frac{2\pi}{\omega}$
- $\omega = 8\pi$
- $N_{\text{osc}} = a\omega$
- $6 = a \times 8\pi$
- $a = \frac{3}{4\pi}$
- Now:
- $|a|_{\text{max}} = \omega^2 a$
- $|a|_{\text{max}} = (8\pi)^2 \times \frac{3}{4\pi}$
- $|a|_{\text{max}} = 4\pi \times 3 = 12\pi \approx 151 \text{ ms}^{-2}$

**Question 8 (\*\*+)**

A particle  $P$  is moving in a straight line with simple harmonic motion, achieving a maximum speed of  $4.8 \text{ ms}^{-1}$ . When  $P$  is at a distance of  $6.4 \text{ m}$  from the centre of the motion, its speed is  $2.88 \text{ ms}^{-1}$ .

Determine in any order the amplitude and the period of the motion.

$$a = 8 \text{ m}, \quad T = \frac{10\pi}{3} \approx 10.47 \text{ s}$$

Handwritten solution for Question 8:

- $v_{\text{max}} = \omega a$
- $4.8 = \omega a$
- $v = \omega \sqrt{a^2 - x^2}$
- $2.88 = \omega \sqrt{a^2 - 6.4^2}$
- $4.8^2 = \omega^2 a^2$
- Divide equations:
- $\frac{2.88^2}{4.8^2} = \frac{a^2 - 6.4^2}{a^2}$
- $0.36 = \frac{a^2 - 40.96}{a^2}$
- $0.36a^2 = a^2 - 40.96$
- $40.96 = 0.64a^2$
- $a^2 = 64$
- $a = 8 \text{ m}$
- Then:  $\omega = \frac{4.8}{8} = 0.6$
- $\omega = \frac{2\pi}{T} = 0.6$
- Period:  $T = \frac{2\pi}{\omega} = \frac{2\pi}{0.6} = \frac{10\pi}{3} \approx 10.47 \text{ s}$

**Question 9 (\*\*\*)**

A particle is about to move in a straight line with simple harmonic motion.

It is released from rest from a point  $A$  and travels directly to a point  $O$ , arriving there  $0.75$  s later with maximum speed  $V$   $\text{ms}^{-1}$ .

- a) Given that  $AO = 1.5$  m, determine the value of  $V$ .
- b) Find the time it takes the particle to cover the first  $2.25$  m of the motion.
- c) Calculate the speed of the particle when is at a distance of  $0.5$  m from  $O$ .

$$V = \pi \approx 3.14, \quad t = 1 \text{ s}, \quad v = \frac{1}{2} \pi \sqrt{2} \approx 2.96 \text{ ms}^{-1}$$

The handwritten solution for Question 9 includes the following parts:

- Diagram:** A horizontal line with points A, O, and B. A is to the left of O, and B is to the right of O. The distance AO is labeled as 1.5 m. A red arrow labeled  $v_{\text{max}}$  points from A towards O.
- Part a):**
  - Period  $T = \frac{2\pi}{\omega}$
  - $\frac{1}{2} = \frac{2\pi}{\omega}$
  - $\omega = \frac{\pi}{1}$
  - $v_{\text{max}} = a\omega$
  - $v_{\text{max}} = 1.5 \times \frac{\pi}{1}$
  - $v_{\text{max}} = 1.5\pi \approx 4.71 \text{ ms}^{-1}$
- Part b):**
  - $|AP| = 2.25$
  - Take the point A as the centre
  - $\therefore a = a \cos \omega t$
  - $-0.75 = 1.5 \cos(\frac{\pi}{1} t)$
  - $\cos(\frac{\pi}{1} t) = -\frac{1}{2}$
  - $\frac{\pi}{1} t = \frac{2\pi}{3}, \frac{4\pi}{3}, \dots$
  - $t = \frac{2}{3}, \frac{4}{3}, \dots$
  - $\therefore t = 1 \text{ s}$
- Part c):**
  - With  $a = 0.5$
  - $v^2 = \omega^2(a^2 - x^2)$
  - $v^2 = (\frac{\pi}{1})^2 [1.5^2 - 0.5^2]$
  - $v^2 = \frac{\pi^2}{1} \times 2$
  - $v = \frac{\pi}{1} \sqrt{2} \approx 2.96 \text{ ms}^{-1}$

**Question 10 (\*\*\*)**

A particle  $P$  is moving on a straight line with simple harmonic motion of maximum speed  $10$   $\text{ms}^{-1}$  and maximum acceleration  $10$   $\text{ms}^{-2}$ .

Calculate the distance of  $P$  from one the endpoints of the oscillation  $0.5$  s after passing through the centre point of the motion.

$$d \approx 1.22 \text{ m}$$

The handwritten solution for Question 10 includes the following parts:

- Parameters:**
  - $v_{\text{max}} = 10$
  - $a_{\text{max}} = 10$
  - $a\omega = 10$
  - $a\omega^2 = 10$
- Divide equations:**
  - $\frac{a\omega}{a\omega^2} = \frac{10}{10}$
  - $\frac{1}{\omega} = 1$
  - $\omega = 1$
  - $a\omega = 10$
  - $a = 10$
- Diagram:** A horizontal line with points A, O, and B. A is to the left of O, and B is to the right of O. A red arrow labeled (+) points from A towards O, and a blue arrow labeled (-) points from O towards B.
- Calculations:**
  - $a = a \cos \omega t$
  - $a = 10 \cos t$
  - $t = \frac{1}{2}$
  - $a = 10 \cos(0.5)$
  - $a = 8.776 \dots$
  - $\therefore \text{Distance} = 10 - 8.776 = 1.22$

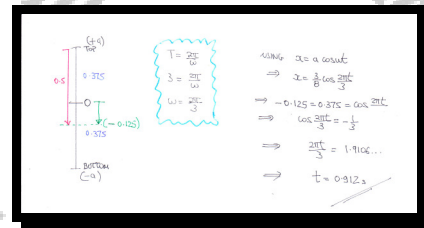
**Question 11** (\*\*\*)

A particle  $P$  is moving on a vertical straight line with simple harmonic motion.

It takes 3 s for a complete oscillation and the distance between the highest and the lowest level of the motion is 0.75 m.

Calculate the time  $P$  takes to travel 0.5 m from the highest point of the motion.

$$t \approx 0.912 \text{ s}$$





**Question 12** (\*\*\*)

Three points  $A$ ,  $O$  and  $B$  lie in that order on a straight line.

A particle  $P$  is moving on this line with simple harmonic motion of period  $3.6$  s, amplitude  $0.8$  m and centre at  $O$ .

Given that  $OA$  is  $0.4$  m and  $OB$  is  $0.7$  m, calculate the time taken by  $P$  to travel directly from  $A$  to  $B$ .

$$t \approx 0.910 \text{ s}$$

The diagram shows a horizontal line with points A, O, and B in order. The distance OA is 0.4 m and OB is 0.7 m. The period T is 3.6 s and the amplitude is 0.8 m.

**Method 1:**

$T = \frac{2\pi}{\omega}$   
 $3.6 = \frac{2\pi}{\omega}$   
 $\omega = \frac{2\pi}{3.6}$   
 $\omega = \frac{\pi}{1.8}$

USING  $x = a \sin(\omega t)$   
 $0.4 = 0.8 \sin\left(\frac{\pi}{1.8} t\right)$       $0.7 = 0.8 \sin\left(\frac{\pi}{1.8} t\right)$   
 $\sin\left(\frac{\pi}{1.8} t\right) = \frac{1}{2}$       $\frac{\pi}{1.8} t = \sin^{-1}\left(\frac{1}{2}\right)$   
 $\frac{\pi t}{1.8} = \frac{\pi}{6}$       $\frac{\pi t}{1.8} = \sin^{-1}\left(\frac{7}{8}\right)$   
 $t = 0.3$       $t = 0.610$

$\therefore$  TOTAL TIME IS  $0.910$

**Method 2:**

APPROACH - SET  $t=0$  AT  $A$   
 $x = a \sin(\omega t + \phi)$   
 $-0.4 = 0.8 \sin\left(\frac{\pi}{1.8} t + \phi\right)$   
 $-\frac{1}{2} = \sin\left(\frac{\pi}{1.8} t + \phi\right)$   
 $\phi = -\frac{\pi}{6}$   
 $x = 0.8 \sin\left(\frac{\pi}{1.8} t - \frac{\pi}{6}\right)$   
 $\rightarrow +0.7 = 0.8 \sin\left(\frac{\pi}{1.8} t - \frac{\pi}{6}\right)$   
 $\rightarrow \sin\left(\frac{\pi}{1.8} t - \frac{\pi}{6}\right) = \frac{7}{8}$   
 $\rightarrow \frac{\pi}{1.8} t - \frac{\pi}{6} = \sin^{-1}\left(\frac{7}{8}\right)$   
 $\rightarrow \frac{\pi t}{1.8} = 1.5809432 \dots$   
 $\rightarrow t \approx 0.910 \dots$

**Question 13** (\*\*\*)

A boat moored at a harbour is moving up and down, taking 2.5 s to move from its highest point to its lowest point, where the vertical distance between these two points is 0.8 m. The boat is modelled as a particle moving with simple harmonic motion in a vertical direction.

The point *A* is 0.1 m below the highest point of the motion and the point *B* is 0.65 m below the highest point of the motion.

- a) Determine the vertical speed of the boat as it passes through *A*.
- b) Calculate the least time taken by the boat to move from *A* to *B*.

,  $|V_A| \approx 0.332 \text{ ms}^{-1}$  ,  $t \approx 1.21 \text{ s}$

**a) SPEEDING WITH A-DIVISION**

- AMPLITUDE  $a = 0.4$
- PERIOD  $= 2 \times 2.5 = 5$
- $\frac{2\pi}{\omega} = 5$
- $\omega = \frac{2\pi}{5}$

USING  $V^2 = \omega^2 (a^2 - x^2)$

$$V^2 = \left(\frac{2\pi}{5}\right)^2 (0.4^2 - 0.3^2)$$

$$V^2 = 0.1165\dots$$

$$|V| = 0.332 \text{ ms}^{-1}$$

**b) USING  $x = a \cos t$ , WITH  $t=0$  AT THE HIGHER POINT**

- UP TO 'A'
- $0.3 = 0.4 \cos\left(\frac{2\pi t}{5}\right)$
- $0.75 = \cos\left(\frac{2\pi t}{5}\right)$
- $\frac{2\pi t}{5} \approx 0.7272\dots$
- $t_1 \approx 0.5793\dots$
- UP TO 'B'
- $-0.25 = 0.4 \cos\left(\frac{2\pi t}{5}\right)$
- $-0.625 = \cos\left(\frac{2\pi t}{5}\right)$
- $\frac{2\pi t}{5} = 2.269\dots$
- $t_2 = 1.7825\dots$

$\therefore$  LEAST TIME  $= t_2 - t_1 = 1.7825\dots - 0.5793\dots$   
 $\approx 1.21$

**Question 14** (\*\*\*)

A particle is moving in a straight line between two points  $A$  and  $B$ , which are  $0.4$  m apart, with simple harmonic motion.

The point  $C$  is  $0.1$  away from  $A$ .

- a) If the greatest speed of the particle during its motion is  $1.6 \text{ ms}^{-1}$ , determine the speed of the particle as it passes through  $C$ .

At time  $t = 0$ , the particle is at  $A$ .

- b) Determine, in terms of  $\pi$ , the time the particle takes until the time it passes through  $C$  for the eighth time.

$$|v| = \sqrt{1.92} \approx 1.39 \text{ ms}^{-1}, \quad t = \frac{23\pi}{24}$$

**a) POT THE INFORMATION GIVEN IN A DIAGRAM**

USING  $|v_{max}| = a\omega$

$$\Rightarrow 1.6 = 0.2\omega$$

$$\Rightarrow \omega = 8$$

Now using  $v = \omega^2(a^2 - x^2)$

$$\Rightarrow 1.6^2 = 8^2(0.2^2 - 0.1^2)$$

$$\Rightarrow 1.6^2 = 1.42$$

$$\Rightarrow |v| \approx 1.39 \text{ ms}^{-1} \quad // \text{ 3.s.f}$$

**b) PASSING THROUGH C FOR THE EIGHTH TIME**

SETTING  $t=0$  AT  $A$  WITH AMPLITUDE  $+0.2$  AT  $B$

$$x = a \cos \omega t$$

$$-0.1 = 0.2 \cos 8t$$

$$\cos 8t = -\frac{1}{2}$$

$$8t = \frac{2\pi}{3}$$

$$t = \frac{\pi}{12}$$

**WHAT WE FIND THE PERIOD OF THE MOTION**

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{8} = \frac{\pi}{4}$$

THE SHARON'S TIME IS

$$3 \times \frac{\pi}{4} + \frac{1}{2} \times \frac{\pi}{4} + \frac{\pi}{4} = \frac{9\pi}{8}$$

3 PERIODS      HALF PERIOD FROM B TO C AT THE VERY END

**ALTERNATIVE FOR PART (b) BY USING TRIG EQUATION**

- SET  $t=0$  AT  $A$ , SO AMPLITUDE IS NOW AT  $A$
- WE REQUIRE THE 8<sup>TH</sup> POSITIVE SOLUTION OF THE EQUATION  $+0.1 = 0.2 \cos 8t$

- SOLVING THE EQUATION

$$\Rightarrow \cos 8t = \frac{1}{2}$$

$$\Rightarrow \left( \begin{array}{l} 8t = \frac{\pi}{3} + 2n\pi \\ 8t = \frac{5\pi}{3} + 2n\pi \end{array} \right) \quad n=0,1,2,3$$

$$\Rightarrow \left( \begin{array}{l} t = \frac{\pi}{24} [1+6n] \\ t = \frac{\pi}{24} [5+6n] \end{array} \right)$$

$$\Rightarrow t = \frac{\pi}{24} \left( \frac{1}{1}, \frac{5}{1}, \frac{7}{1}, \frac{11}{1}, \frac{13}{1}, \frac{17}{1}, \frac{19}{1}, \frac{23}{1} \right)$$

1st 2nd 3rd 4th 5th 6th 7th 8th

**Question 15** (\*\*\*)

A particle is attached to one end of a light spring, whose other end is attached to a fixed point. The particle is hanging vertically in equilibrium.

The particle is then pulled downwards by a further 0.6 m and released from rest.

The motion of the particle satisfies the differential equation

$$\frac{d^2x}{dt^2} = -k^2x,$$

where  $x$  m is the additional extension of the spring from its equilibrium position, at time  $t$  s, and  $k$  is a constant. The motion has period of 2 s.

Find the first four positive values of  $t$  for which  $x = 0.3$  m.

,  $t = \frac{1}{3}, \frac{5}{3}, \frac{7}{3}, \frac{11}{3}$

$\frac{d^2x}{dt^2} = -k^2x$       PERIOD =  $\frac{2\pi}{k} = 2 \implies k = \pi$   
 AMPLITUDE = 0.6

USING THE STANDARD EQUATION FOR SIMPLE HARMONIC MOTION

$\implies a = a \cos kt$   
 $\implies 0.3 = 0.6 \cos \pi t$   
 $\implies 0.3 = 0.6 \cos \pi t$   
 $\implies 0.5 = \cos \pi t$

$\arccos(0.5) = \frac{\pi}{3}$

$\implies \pi t = \frac{\pi}{3} + 2n\pi$        $n = 0, 1, 2, \dots$   
 $\implies \pi t = \frac{5\pi}{3} + 2n\pi$

$\implies \begin{cases} t = \frac{1}{3} + 2n \\ t = \frac{5}{3} + 2n \end{cases}$

$\therefore t = \frac{1}{3}, \frac{5}{3}, \frac{7}{3}, \frac{11}{3}$

**Question 16** (\*\*\*)

A particle  $P$  is moving on a straight line with simple harmonic motion, centre at  $O$ , and period  $\pi$  s.

The point  $C$  is at a distance of 2 m from  $O$ .

It is further given that  $P$  passes through  $C$  with speed  $3 \text{ ms}^{-1}$  and returns to  $C$  after time  $T$  s, where  $T < \pi$ .

Calculate the possible values of  $T$ .

$$T \approx 0.643 \text{ s}, \quad T \approx 1.22 \text{ s}$$

The diagram shows a horizontal line representing the path of a particle in simple harmonic motion. The center is labeled  $O$ . Points  $A$ ,  $C$ , and  $B$  are marked on the line. A double-headed arrow below the line indicates the amplitude is 2 units. A note in a pink box states:  $c = \pi$ ,  $a = 2$ ,  $v = 3$ .

The handwritten solution includes the following steps:

- Given:  $c = \frac{\pi}{2}$ ,  $a = 2$ ,  $v = 3$ ,  $\omega = 2$
- Velocity equation:  $v = \omega^2(x^2 - a^2)$
- Substituting values:  $3^2 = 2^2(x^2 - 2^2)$
- Solving for  $x$ :  $9 = 4(x^2 - 4)$ ,  $\frac{9}{4} = x^2 - 4$ ,  $x^2 = \frac{25}{4}$ ,  $x = 2.5$
- Time equation:  $a \cos \omega t = x$  (since  $B$  is  $+a$ )
- Substituting values:  $2 = 2.5 \cos 2t$ ,  $2 = 2.5 \cos 2t$ ,  $\cos 2t = 0.8$
- Solving for  $t$ :  $2t = \arccos(0.8)$  (next value  $2\pi - \arccos(0.8)$ )
- Result:  $t = 0.3275 \dots$
- Final calculation:  $\therefore C \rightarrow B \rightarrow C = 2 \times 0.3275 = 0.643$
- Conclusion:  $C \rightarrow O \rightarrow A \rightarrow O \rightarrow C$
- Final result:  $\therefore T = 0.643 \dots = 2.49$

**Question 17** (\*\*\*)

A particle is moving with simple harmonic motion on a straight line with centre at  $O$ .

When the particle is passing through a point  $P$ , heading towards  $O$ , its speed is  $3 \text{ ms}^{-1}$  and its acceleration  $8 \text{ ms}^{-2}$ .

Calculate the time taken for the particle to return to  $P$  for the first time.

,  $t \approx 2.21 \text{ s}$

POTTING THE INFORMATION INTO A DIAGRAM

$v = \omega^2(a^2 - x^2)$        $|v| = \omega^2 x$   
 $3 = \omega^2(8^2 - 4^2)$        $3 = \omega^2 \cdot 2$   
 $9 = \omega^2(64 - 16)$        $\omega^2 = 4$   
 $9 = 48\omega^2$        $\omega = 2$   
 $\frac{9}{48} = \omega^2$   
 $0.1875 = \omega^2$   
 $\omega = 2.5$        $A \text{ PERIOD TO } 2\pi = \pi$

NO LONG SETTING A DISPLACEMENT EQUATION AS A FUNCTION OF TIME

Let  $t=0$  at the origin

$\Rightarrow x = a \cos \omega t$   
 $\Rightarrow x = 2.5 \cos 2t$   
 $\Rightarrow 2.5 \cos 2t = 4$       TO FIND THE TIME FROM P TO O OR FROM O TO P  
 $\Rightarrow \cos 2t = \frac{4}{2.5}$   
 $\Rightarrow 2t = \arccos\left(\frac{4}{2.5}\right)$  (BEST TIME)  
 $\Rightarrow t = \frac{1}{2} \arccos\left(\frac{4}{2.5}\right)$

THE PERIODIC TIME IS GIVEN BY

$T = \frac{2\pi}{\omega} = \frac{2\pi}{2.5} = \frac{4\pi}{2.5} = 5.024 \text{ s}$   
 $\frac{1}{2} \text{ PERIOD} = \frac{1}{2} \times 5.024 = 2.512 \text{ s}$   
 $\approx 2.21 \text{ s}$

**Question 18** (\*\*\*)

A particle  $P$  is moving in a straight line with simple harmonic motion between two points  $A$  and  $B$ , where  $|AB| = 0.8$  m.

The points  $C$  and  $D$  lie on the path of  $P$  such that  $|AC| = 0.2$  m and  $|AD| = 0.6$  m, and it takes  $\frac{2}{3}$  s for  $P$  to travel directly from  $C$  to  $D$ .

When  $t = 0$ ,  $P$  is at  $A$ .

- Show that the period of the motion is 4 s.
- Find the maximum speed of  $P$ .
- Find the distance of  $P$  from  $A$  when  $t = 1.5$ .
- Calculate the value of  $t$  when  $P$  passes through  $D$  for the **fourth** time.

$$|v|_{\max} = \frac{\pi}{5} \approx 0.628 \text{ ms}^{-1}, \quad d \approx 0.683 \text{ m}, \quad t = 6\frac{2}{3} \text{ s}$$

Handwritten solution for Question 18:

(a)  $x = 0.4 \cos \omega t$  (at  $t = 0$  at  $A$ )  
 $0.2 = 0.4 \cos \omega t$       $-0.2 = 0.4 \cos \omega t$   
 $\frac{1}{2} = \cos \omega t$       $-\frac{1}{2} = \cos \omega t$   
 $\omega t = \frac{\pi}{3}$       $\omega t = \frac{2\pi}{3}$   
 $\omega t_2 - \omega t_1 = \frac{2\pi}{3} - \frac{\pi}{3} = \frac{\pi}{3}$   
 $\omega(\frac{2}{3} - \frac{1}{3}) = \frac{\pi}{3}$   
 $\omega \times \frac{1}{3} = \frac{\pi}{3}$   
 $\omega = \frac{\pi}{2}$   
 Now  $T = \frac{2\pi}{\omega} = \frac{2\pi}{\frac{\pi}{2}} = 4$

(b)  $|v|_{\max} = \omega a = \frac{\pi}{2} \times 0.4 = \frac{\pi}{5} \approx 0.628 \text{ ms}^{-1}$

(c)  $x = 0.4 \cos(\frac{\pi}{2} t)$   
 when  $x = 1.5$   
 $\Rightarrow 1.5 = 0.4 \cos(\frac{\pi}{2} t)$   
 $\Rightarrow 1.5 = \frac{\sqrt{3}}{2}$   
 (Diagram showing a cosine wave with  $\frac{\sqrt{3}}{2}$  marked on the y-axis and  $\frac{\pi}{3}$  marked on the x-axis)  
 $\Rightarrow \text{Distance} = 0.4 + \frac{\sqrt{3}}{2} \approx 0.683 \text{ m}$

(d)  $x = -0.2$   
 $\Rightarrow -0.2 = 0.4 \cos(\frac{\pi}{2} t)$   
 $\cos(\frac{\pi}{2} t) = -\frac{1}{2}$   
 $\frac{\pi}{2} t = \frac{2\pi}{3} + 2n\pi$       $n = 0, 1, 2, \dots$   
 $t = \frac{4}{3} + 4n$   
 $t = \frac{4}{3}, \frac{16}{3}, \frac{28}{3}, \dots$   
 $\therefore t = 6\frac{2}{3} \text{ s}$

**Question 19** (\*\*\*)

Three points  $O$ ,  $A$  and  $B$  lie in that order on a straight line. A particle  $P$  is moving on this line with simple harmonic motion of period  $3$  s, amplitude  $0.6$  m and centre at  $O$ . It is further given that  $OA$  is  $0.1$  m and  $OB$  is  $0.5$  m.

At a certain instant  $P$  is observed passing through  $B$  moving in the direction  $OB$ .

Calculate the time when  $P$  reaches  $A$ .

$t \approx 0.950$  s

The diagram shows a horizontal line with points  $P$ ,  $O$ ,  $A$ , and  $B$  in order from left to right.  $O$  is the center of oscillation.  $OA = 0.1$  m and  $OB = 0.5$  m. The amplitude is  $0.6$  m. The period is  $3$  s. The particle  $P$  is shown moving from  $B$  towards  $O$ .

**Given:**  
 $T = 3$  s  
 $a = 0.6$  m  
 $\omega = \frac{2\pi}{T} = \frac{2\pi}{3}$  rad/s

**Equation of motion:**  
 $x = a \sin(\omega t + \phi)$   
 $x = 0.6 \sin\left(\frac{2\pi}{3}t + \phi\right)$

**At  $t = 0$ ,  $x = 0.5$  m (at  $B$ ):**  
 $0.5 = 0.6 \sin(\phi)$   
 $\sin \phi = \frac{5}{6}$   
 $\phi = \sin^{-1}\left(\frac{5}{6}\right) \approx 0.9273$  rad

**At  $t = t$ ,  $x = 0.1$  m (at  $A$ ):**  
 $0.1 = 0.6 \sin\left(\frac{2\pi}{3}t + \phi\right)$   
 $\sin\left(\frac{2\pi}{3}t + \phi\right) = \frac{1}{6}$   
 $\frac{2\pi}{3}t + \phi = \sin^{-1}\left(\frac{1}{6}\right) \approx 0.1674$  rad

**Solving for  $t$ :**  
 $\frac{2\pi}{3}t + 0.9273 = 0.1674$   
 $\frac{2\pi}{3}t = 0.1674 - 0.9273 = -0.7599$   
 $t = \frac{-0.7599 \times 3}{2\pi} \approx -0.290$  s

**Alternative solution:**  
 The particle is at  $B$  at  $t = 0$ . It moves towards  $O$ . The time to reach  $O$  is  $\frac{T}{4} = 0.75$  s. The time to reach  $A$  is  $\frac{T}{4} - t$ .  
 $0.1 = 0.6 \sin\left(\frac{2\pi}{3}\left(\frac{T}{4} - t\right)\right)$   
 $0.1 = 0.6 \sin\left(\frac{2\pi}{3}\left(0.75 - t\right)\right)$   
 $\sin\left(\frac{2\pi}{3}\left(0.75 - t\right)\right) = \frac{1}{6}$   
 $\frac{2\pi}{3}\left(0.75 - t\right) = \sin^{-1}\left(\frac{1}{6}\right) \approx 0.1674$   
 $0.75 - t = \frac{0.1674 \times 3}{2\pi} \approx 0.0797$   
 $t = 0.75 - 0.0797 \approx 0.6703$  s



**Question 20** (\*\*\*)

A particle  $P$  is moving with simple harmonic motion.

The motion takes place along a straight line with centre at  $O$ . The points  $O$ ,  $A$  and  $B$ , lie in that order, on this line with  $|OA| = 0.5$  m and  $|AB| = 0.7$  m.

The speed of  $P$  at  $A$  is  $6 \text{ ms}^{-1}$  and its speed at  $B$  is  $2.5 \text{ ms}^{-1}$ .

- Show that the period of the motion is  $\frac{2\pi}{5}$  s.
- Determine the acceleration of  $P$  at  $A$ .
- Calculate the time taken for  $P$  to travel directly from  $A$  to  $B$ .

$a = 12.5 \text{ ms}^{-2}$ ,  $t \approx 0.156 \text{ s}$

Handwritten solution for Question 20:

(a)  $v = \omega \sqrt{a^2 - x^2}$   
 $6 = \omega \sqrt{a^2 - 0.25}$   
 $2.5 = \omega \sqrt{a^2 - 1.44}$   
 $\Rightarrow \frac{144}{25} = \frac{a^2 - 0.25}{a^2 - 1.44}$   
 $\Rightarrow 144a^2 - 207.36 = 25a^2 - 6.25$   
 $\Rightarrow 119a^2 = 201.11$   
 $\Rightarrow a^2 = 1.69$   
 $\Rightarrow a = 1.3$

Then  $36 = \omega^2(a^2 - 0.25)$   $\Rightarrow \omega = 5$   
 $36 = \omega^2(1.3^2 - 0.25)$   
 $\omega = 5$   
 $T = \frac{2\pi}{\omega} = \frac{2\pi}{5}$

(b)  $a = -\omega^2 x$  at  $A, x = 0.5$   $\therefore |a| = 5^2 \times 0.5 = 12.5 \text{ ms}^{-2}$

(c) Using  $t=0, x=0$   
 $x = a \sin \omega t$   
 $0.5 = 1.3 \sin 5t$   
 $\frac{5}{13} = \sin 5t$   
 $5t = \arcsin\left(\frac{5}{13}\right)$   
 $t = \frac{1}{5} \arcsin\left(\frac{5}{13}\right)$

Also  $1.2 = 1.3 \sin 5t$   
 $\frac{12}{13} = \sin 5t$   
 $5t = \arcsin\left(\frac{12}{13}\right)$   
 $t = \frac{1}{5} \arcsin\left(\frac{12}{13}\right)$

Time taken from A to B is  $\frac{1}{5} \arcsin\left(\frac{12}{13}\right) - \frac{1}{5} \arcsin\left(\frac{5}{13}\right) = 0.156 \text{ s}$

**Question 21** (\*\*\*\*)

A particle is moving on a straight line with simple harmonic motion, centre at  $O$ , and period  $\frac{1}{3}\pi$  s.

When the particle is at a distance of 0.48 m from  $O$ , its speed is  $2.16 \text{ ms}^{-1}$ .

Calculate the total time within a complete oscillation, for which the particle has speed less than  $2.88 \text{ ms}^{-1}$ .

,  $t \approx 0.429 \text{ s}$

Handwritten solution for Question 21:

- $T = \frac{2\pi}{\omega}$   
 $\Rightarrow \frac{1}{3}\pi = \frac{2\pi}{\omega}$   
 $\Rightarrow \omega = 6$
- $v^2 = \omega^2(a^2 - x^2)$   
 $\Rightarrow 2.16^2 = 6^2(a^2 - 0.48^2)$   
 $\Rightarrow 0.424 = a^2 - 0.144$   
 $\Rightarrow a^2 = 0.568$   
 $\Rightarrow a = 0.754$
- NEXT FIND THE VALUE OF  $x$  WHEN  $v = 2.88$   
 $\Rightarrow v^2 = \omega^2(a^2 - x^2)$   
 $\Rightarrow 2.88^2 = 6^2(a^2 - x^2)$   
 $\Rightarrow 0.2204 = a^2 - x^2$   
 $\Rightarrow x^2 = 0.1204$   
 $\Rightarrow |x| = 0.347$
- NOW USING A DISPLACEMENT-TIME RELATIONSHIP WITH  $t=0$  AT THE END-POINT OF THE OSCILLATION  
 $x = a \cos \omega t$   
 $x = \frac{1}{2} \cos 6t$   
 $0.347 = \frac{1}{2} \cos 6t$   
 $\cos 6t = 0.694$   
 $6t = \arccos(0.694)$  (FIRST POSITIVE SOLUTION)  
 $t = \frac{1}{6} \arccos(0.694) \approx 0.15469 \dots$   
 $\therefore$  RESOURCES THAT = PERIOD - 4(0.15469...)  $\approx 0.429$

Question 22 (\*\*\*\*)

A particle  $P$  moves in a straight line with simple harmonic motion with period  $\frac{\pi}{3}$  s.

At time  $t=0$ ,  $P$  is at rest at the point  $A$  and the acceleration at that instant has magnitude  $21.6 \text{ ms}^{-2}$ .

- Find the amplitude of the motion.
- Hence state the greatest speed of  $P$  during the motion.
- Calculate the time  $P$  takes to travel a **total distance** of 2.5 m after it has first left  $A$ .

$a = 0.6 \text{ m}$ ,  $v_{\text{max}} = 3.6 \text{ ms}^{-1}$ ,  $t \approx 1.14 \text{ s}$

Handwritten solution for Question 22:

a)  $T = \frac{2\pi}{\omega}$   
 $\frac{\pi}{3} = \frac{2\pi}{\omega}$   
 $\omega = 6 \text{ rad s}^{-1}$   
 $\omega = \sqrt{a/x}$   
 $6 = \sqrt{21.6/x}$   
 $36 = 21.6/x$   
 $x = 0.6 \text{ m}$

b)  $v_{\text{max}} = a\omega$   
 $v_{\text{max}} = 0.6 \times 6$   
 $v_{\text{max}} = 3.6 \text{ ms}^{-1}$

9) Total distance of 2.5 m  
 A L 0.6 0 0.6 B  
 A L 0.6 0 0.6 B  
 A L 0.6 0 0.6 B  
 ← 0.5 →

Using  $x = a \cos \omega t$   
 $0.5 = 0.6 \cos 6t$   
 $\frac{5}{6} = \cos 6t$   
 $6t = \cos^{-1}(\frac{5}{6})$   
 $t = \frac{\cos^{-1}(\frac{5}{6})}{6}$   
 $t \approx 0.081 \dots$

This time required time is  $0.081 + 0.081 = 0.162 \approx 0.16 \text{ s}$

Question 23 (\*\*\*\*)

A particle  $P$  is at rest at some point  $B$ .

At time  $t = 0$  s,  $P$  starts moving with simple harmonic motion on a straight line, taking  $\frac{1}{3}\pi$  s to return to  $B$  for the first time. The maximum speed of  $P$  is  $3.6 \text{ ms}^{-1}$ .

- Determine the amplitude of the motion.
- Calculate the speed of the particle 1 s after leaving  $B$ .
- Find the values of  $t$ , for  $0 < t < 1$ , so that the **speed** of  $P$  is the same as that found in part (b), giving the answers correct to three decimal places.

$a = 0.6 \text{ m}$ ,  $v \approx 1.01 \text{ ms}^{-1}$ ,  $t \approx 0.047, 0.476, 0.571$

Handwritten solution for Question 23:

(a)  $T = \frac{1}{3}\pi$  s,  $V_{max} = \omega a$   
 $\frac{1}{3}\pi = \frac{2\pi}{\omega}$   $3.6 = \omega a$   
 $\omega = 6$   $a = 0.6 \text{ m}$

(b) Using  $x = a \cos(\omega t)$  (START AT B)  
 $x = 0.6 \cos(6t)$   
 $\dot{x} = -3.6 \sin(6t)$   
 when  $t = 1$   $v \approx 1.01 \text{ ms}^{-1}$

(c) When  $v = 1.01$   $v = -1.01$   
 $-3.6 \sin(6t) = 1.01$   $-3.6 \sin(6t) = -1.01$   
 $\sin(6t) = -0.2806$   $\sin(6t) = 0.2806$   
 $6t = \sin^{-1}(-0.2806)$   $6t = \sin^{-1}(0.2806)$   
 $6t = -0.284$   $6t = 0.284$   
 $t = -0.047$  (7)  $t = 0.047$  (8)  
 $t = \frac{\pi}{6} - 0.047$  (9)  $t = \frac{\pi}{6} + 0.047$  (10)  
 $\therefore t = 0.5708 \dots$  FROM (9)  $\therefore t = 0.4762 \dots$  FROM (10)  
 $t = 0.4764 \dots$  FROM (10)  
 Also  $t = 0.047, 0.476, 0.571$

**Question 24** (\*\*\*\*)

Three points  $A$ ,  $O$  and  $B$  lie in that order on a straight line.

Two particles,  $P_1$  and  $P_2$ , are moving on this line with simple harmonic motion between  $A$  and  $B$ , where  $O$  is the centre of the motion.

At time  $t = 0$  s,  $P_1$  is observed at the midpoint of  $OB$  moving towards  $B$ .

The subsequent displacement of  $P_1$  from  $O$  is given by

$$x_1 = 12 \sin\left(\frac{\pi t}{2} + \phi\right), \quad 0 < \phi < \frac{\pi}{2}.$$

- a) Show that  $P_1$  arrives at  $B$  for the **fifth** time when  $t = 16\frac{2}{3}$  s.

At time  $t = 0$  s,  $P_2$  is observed passing through  $O$  moving towards  $B$ . When  $P_1$  arrives at  $B$  for the **fifth** time,  $P_2$  also arrives at  $B$  for the  $k^{\text{th}}$  time, for  $t > 0$ .

- b) Determine by calculation the value of  $k$ .

$k = 2$

Handwritten solution for Question 24:

a)  $x_1 = 12 \sin\left(\frac{\pi t}{2} + \phi\right)$   
 • When  $t = 0$ ,  $x_1 = 6$   
 $6 = 12 \sin \phi$   
 $\sin \phi = \frac{1}{2}$   
 $\phi = \frac{\pi}{6}$   
 •  $x_1 = 12 \sin\left(\frac{\pi t}{2} + \frac{\pi}{6}\right)$   
 When  $x_1 = 12$   
 $\Rightarrow 1 = \sin\left(\frac{\pi t}{2} + \frac{\pi}{6}\right)$   
 $\Rightarrow \frac{\pi t}{2} + \frac{\pi}{6} = \frac{\pi}{2} + 2n\pi \quad n = 0, 1, 2, 3, \dots$   
 $\Rightarrow \frac{1}{2}t + \frac{1}{6} = \frac{1}{2} + 2n$   
 $\Rightarrow t + \frac{1}{3} = 1 + 4n$   
 $\Rightarrow t = \frac{2}{3} + 4n$   
 $\Rightarrow t = \frac{2}{3}, \frac{14}{3}, \frac{26}{3}, \frac{38}{3}, \dots$   
 $\therefore$  Fifth time,  $t = \frac{38}{3} = 12\frac{2}{3}$  s

b)  $x_2 = 12 \sin \frac{\pi t}{2}$   
 When  $x_2 = 12$ , then  $t = \frac{2}{3}$   
 $\Rightarrow 12 = 12 \sin \frac{\pi t}{2}$   
 $\Rightarrow \sin \frac{\pi t}{2} = 1$   
 $\Rightarrow \frac{\pi t}{2} = \frac{\pi}{2} + 2n\pi$   
 $\Rightarrow t = 1 + 4n$   
 When  $t = \frac{38}{3}$   
 $\frac{38}{3} = 1 + 4n$   
 $\frac{35}{3} = 4n$   
 $n = \frac{35}{12}$   
 $n = 2$   
 $\therefore$  2nd time

**Question 25** (\*\*\*\*)

The level of the sea in a harbour is assumed to rise and fall with simple harmonic motion. On a certain day low tide occurs at 07.00 hours when the depth of the sea will be 5 m. The next high tide will occur at 13.15 hours when the depth of the sea will be 17 m.

A ship wishes to enter the harbour that day and needs a minimum sea depth of 6.5 m.

Calculate, to the nearest minute, the earliest time it can enter the harbour on this day and the time by which it must leave.

08:26, 18:04

• NEW MEASUREMENT ON DIFFERENT TIDEGAGE  $a = 6$   
 • LOW TIDE TO HIGH TIDE is  $\frac{1}{2}$  PERIOD =  $6\frac{1}{2}$  HOURS  
 $\therefore \frac{1}{2}T = 6\frac{1}{2}$   
 $T = 13$   
 $\omega = \frac{2\pi}{T} = \frac{2\pi}{13}$   
 $\omega = \frac{4\pi}{13}$

• SET CLOCK ZERO AT LOWEST LEVEL AND USE  
 $x = a \cos(\omega t)$   
 $5 = 6 \cos\left(\frac{4\pi t}{13}\right)$  (NOTE - AT THE BOTTOM - AT THE TOP)

NOW  $6 \cos = 5 \Rightarrow 1.5$   
 $\therefore \cos = \frac{5}{6} \leftarrow 0.833$

$\Rightarrow 4 - 5 = 6 \cos\left(\frac{4\pi t}{13}\right)$   
 $\Rightarrow \cos\left(\frac{4\pi t}{13}\right) = \frac{5}{6}$   
 $\Rightarrow \frac{4\pi t}{13} = 0.72273 \pm 2\pi n$      $n = 0, 1, 2, 3, \dots$   
 $\Rightarrow \frac{4\pi t}{13} = 5.56045 \pm 2\pi n$

$\Rightarrow t = 1.438 \pm \frac{13n}{4}$   
 $\Rightarrow t = 11.062 \pm \frac{13n}{4}$

SO TO GET IN:  $t = 1.438 \approx 1$  HOUR - 26 MINUTES  $\rightarrow$  08:26  
 TO LEAVE:  $t = 11.062 \approx 11$  HOURS - 4 MINUTES  $\rightarrow$  18:04

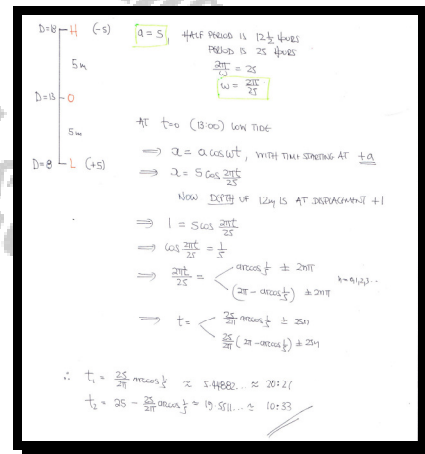
**Question 26** (\*\*\*\*)

The level of the sea in a harbour is assumed to rise and fall with simple harmonic motion. On a certain day low tide occurs at 15.00 hours when the depth of the sea will be 8 m. The next high tide will occur at 03.30 hours when the depth of the sea will be 18 m.

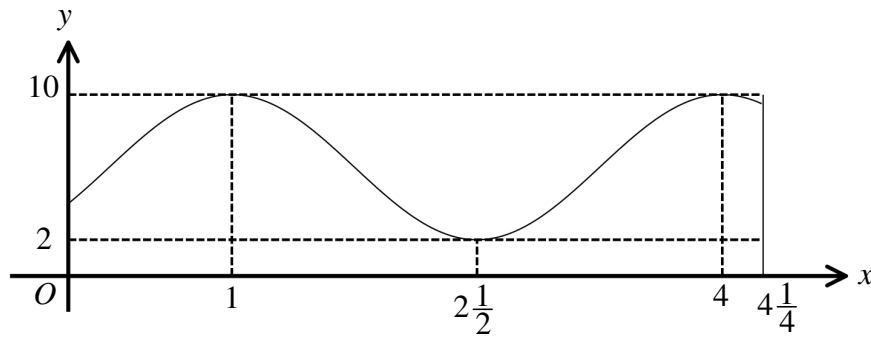
A ship wishes to enter the harbour that day and needs a minimum sea depth of 12 m.

Calculate, to the nearest minute, the earliest time it can enter the harbour and the time by which it must leave.

20:27, 10:33



Question 27 (\*\*\*\*)



The graph above shows the height,  $y$  m, of a particle  $P$  at time  $t$  s, given by

$$y = A \sin(\omega t - \phi) + B,$$

where  $A$ ,  $B$ ,  $\omega$  and  $\phi$  are positive constants.

- Show algebraically that  $P$  is moving with simple harmonic motion.
- Determine the exact values, where appropriate, of  $A$ ,  $B$ ,  $\omega$  and  $\phi$ .
- Calculate the maximum speed of  $P$ .
- Find the initial height and the initial velocity of  $P$ .
- Calculate the total distance travelled by  $P$ , for  $0 \leq t \leq 14$ .

$$A = 4, \quad B = 6, \quad \omega = \frac{2\pi}{3}, \quad \phi = \frac{\pi}{6}, \quad v_{\max} \approx 8.38 \text{ ms}^{-1}, \quad y_0 = 4 \text{ m}, \quad v_0 \approx 7.26 \text{ ms}^{-1},$$

$$d \approx 22.54 \text{ m}$$

$y = A \sin(\omega t - \phi) + B$   
 $\ddot{y} = A \omega^2 \cos(\omega t - \phi)$   
 $\ddot{y} = -A \omega^2 \sin(\omega t - \phi)$   
 $\ddot{y} = -\omega^2 [y - B]$   
 So  $\ddot{y} = -\omega^2 (y - B)$   
 Let  $x = y - B$   
 $\ddot{x} = \ddot{y}$   
 $\ddot{x} = -\omega^2 x$   
 i.e. SHM with A's SHM  
 (b) AMPLITUDE =  $A = \frac{10-2}{2} = 4$   
 (Mean value)  
 Centre of oscillation is at  $\frac{10+2}{2} = 6$   
 $\therefore$  there is an upward translation of 6  
 $\therefore B = 6$   
 Period =  $4 - 1 = 3$   
 $\frac{2\pi}{T} = \omega = \frac{2\pi}{3}$   
 So far  $y = 6 + 4 \sin(\frac{2\pi}{3}t - \phi)$   
 Using  $t=1, y=10$   
 $10 = 6 + 4 \sin(\frac{2\pi}{3} - \phi)$   
 $4 = 4 \sin(\frac{2\pi}{3} - \phi)$   
 $\sin(\frac{2\pi}{3} - \phi) = 1$   
 $\frac{2\pi}{3} - \phi = \frac{\pi}{2}$   
 $\phi = \frac{\pi}{6}$   
 $\therefore v_{\max} = |\omega A|$   
 $= \frac{2\pi}{3} \times 4 \approx 8.38 \text{ ms}^{-1}$   
 (c)  $t=0, y = 4 \sin(\frac{2\pi}{3} \cdot 0 - \frac{\pi}{6}) + 6$   
 $y = 4 \sin(-\frac{\pi}{6}) + 6$   
 $y = -2 + 6 = 4$   
 $t = 4.25, y = 6 + 4 \sin(\frac{2\pi}{3} \cdot 4.25 - \frac{\pi}{6})$   
 $\therefore$  TOTAL DISTANCE  
 $10 - 4 = 6$   
 $(10 - 2) \times 2 = 16$   
 $10 - 9.46 = 0.54$   
 $22.54 \text{ m}$   
 $(20 - 2 \times 6)$



Question 28 (\*\*\*)

A particle is moving on the  $x$  axis and its speed,  $v \text{ ms}^{-1}$ , is given by

$$\frac{1}{2}v = \sqrt{8 - 2x - x^2}, \quad x_1 \leq x \leq x_2,$$

where  $x$  is the position of the particle on the  $x$  axis.

a) Show that the motion of the particle is simple harmonic.

b) Determine the value of  $x_1$  and the value of  $x_2$ .

At time  $t = 0$ , the particle is observed to be 1 m from the centre of the oscillation and moving away from the centre of the oscillation.

At time  $t = T$ , the particle is observed to be 2 m from the centre of the oscillation for the **third** time.

c) Calculate the value of  $T$ .

*Full justification for the answer to part (c) must be shown.*

 ,  $x_1 = -4$ ,  $x_2 = 2$ ,  $t = 1.77 \text{ s}$

a) DIFFERENTIATE WITH RESPECT TO  $x$ , AFTER SIMPLIFYING

$$\frac{1}{2}v = \sqrt{8 - 2x - x^2}$$

$$\Rightarrow \frac{1}{2}v^2 = 8 - 2x - x^2$$

$$\Rightarrow v \frac{dv}{dx} = -2 - 2x$$

$$\Rightarrow v \frac{dv}{dx} = -2(1+x)$$

$$\Rightarrow \int v \, dv = \int -2(1+x) \, dx$$

$$\Rightarrow \frac{1}{2}v^2 = -2(x + \frac{1}{2}x^2) + C$$

Now set  $y = 2x + 1$     $\frac{dy}{dx} = 2$     $dx = \frac{1}{2}dy$

$$\frac{1}{2}v^2 = -2(\frac{1}{2}y) + C$$

$$\Rightarrow v^2 = -2y + C$$

$\therefore$  SH M WITH CENTRE  $x = -1$ ,  $\omega^2 = 4$

b) NOW THE CENTRE OF THE OSCILLATION IS AT  $x = -1$

$$\frac{1}{2}v_{\max} = \sqrt{8 - 2(-1) - (-1)^2}$$

$$\frac{1}{2}v_{\max} = \sqrt{8 - 2 + 1}$$

$$\frac{1}{2}v_{\max} = 3$$

$$v_{\max} = 6$$

But  $v_{\max} = \omega a$

$$\Rightarrow 6 = 2a \quad (\omega^2 = 4)$$

$$\Rightarrow a = 3$$

THIS CENTRE IS AT  $x = -1$  & THE AMPLITUDE IS 3

$$\therefore -4 \leq x \leq 2$$

IF  $x_1 = -4$   
 $x_2 = 2$

c) LOOKING AT THE DIAGRAM - WITHOUT LOSS OF GENERALITY TAKE  $t = 0$  (1 METRE FROM THE CENTRE)

USING  $y = 2 \sin(\omega t + \phi)$

SOLVE FOR  $\phi = -2$

$$-2 = 3 \sin(2t + \phi)$$

$$\sin(2t + \phi) = -\frac{2}{3}$$

$$2t + \phi = \arcsin(-\frac{2}{3}) \pm 2\pi n$$

$$2t + \phi = \pi - \arcsin(\frac{2}{3}) \pm 2\pi n$$

FOR  $y = 1$   
 $1 = 3 \sin(2t + \phi)$   
 $\sin(2t + \phi) = \frac{1}{3}$   
 $2t + \phi = \arcsin(\frac{1}{3})$

$$2t = \arcsin(\frac{1}{3}) - \phi \pm 2\pi n$$

$$2t = \pi - \arcsin(\frac{2}{3}) - \phi \pm 2\pi n$$

$$2t = -\arcsin(\frac{1}{3}) - \arcsin(\frac{2}{3}) \pm 2\pi n$$

$$2t = \pi + \arcsin(\frac{2}{3}) - \arcsin(\frac{1}{3}) \pm 2\pi n$$

$$t = \frac{1}{2}(-\arcsin(\frac{1}{3}) - \arcsin(\frac{2}{3})) \pm \pi n$$

$$t = \frac{1}{2}(\pi + \arcsin(\frac{2}{3}) - \arcsin(\frac{1}{3})) \pm \pi n$$

$t = 1.7657477 \dots$   
 $t_2 = 2.668210 \dots$   
 $t_3 = 4.9735 \dots$

BUT NOTE THAT THE THIRD TIME IT IS 2 METRES FROM THE CENTRE OF THE OSCILLATION IS THE FIRST TIME WHEN  $y = -2$

$\therefore t = 1.77 \text{ s}$

Created by T. Madas

# SIMPLE HARMONIC MOTION DYNAMICS

Created by T. Madas

**Question 1** (\*\*)

A particle  $P$  of mass  $0.2 \text{ kg}$  is attached to one end of a light elastic string of natural length  $0.8 \text{ m}$  and modulus of elasticity  $16 \text{ N}$ .

The other end of the string is attached to a fixed point  $A$  on a smooth horizontal surface on which  $P$  rests.

With the string at natural length,  $P$  receives an impulse of magnitude  $5 \text{ N s}$ , in the direction  $AP$ .

- Show that in the subsequent motion, while the string is taut, the motion of  $P$  is simple harmonic.
- Determine its amplitude of the motion.
- Find the time it takes  $P$  to travel between the extreme points of its motion.

$a = 2.5 \text{ m}$ ,  $t \approx 0.378 \text{ s}$

The handwritten solution includes the following steps:

- Diagram:** A horizontal line representing the string with points B, N, O, M, A marked. A particle P is shown at point A with an impulse arrow pointing right.
- Part a) SHM:**
  - Force:  $W = -T(x)$
  - Equation:  $m\ddot{x} = -\frac{\lambda}{l}x$
  - SHM condition:  $\ddot{x} = -\frac{\lambda}{ml}x$
  - Angular frequency:  $\omega^2 = \frac{\lambda}{ml} = \frac{16}{0.2 \times 0.8} = 100$
  - Period:  $T = \frac{2\pi}{\omega} = \frac{2\pi}{10} \approx 0.628 \text{ s}$
- Part b) Amplitude:**
  - Impulse = change in momentum:  $5 = mv - 0$
  - Velocity:  $v = 0.25 \text{ m/s}$
  - Energy conservation:  $\frac{1}{2}mv^2 = \frac{1}{2}\frac{\lambda}{l}a^2$
  - Amplitude:  $a = 2.5 \text{ m}$
- Part c) Time:**
  - Time to travel from A to B:  $t = \frac{a}{v} = \frac{2.5}{0.25} = 10 \text{ s}$
  - Time to travel from B to A:  $t = \frac{a}{v} = \frac{2.5}{0.25} = 10 \text{ s}$
  - Total time:  $t = 20 \text{ s}$

**Question 2 (\*\*)**

A particle  $P$  of mass  $0.7 \text{ kg}$  is attached to one end of a light elastic spring of natural length  $0.6 \text{ m}$  and modulus of elasticity  $168 \text{ N}$ .

The other end of the spring is attached to a fixed point  $A$  on a smooth horizontal surface on which  $P$  rests.

$P$  is pushed in the direction  $PA$  so that the spring has length  $0.4 \text{ m}$ , and released from rest.

- a) Show that the subsequent motion of  $P$  is simple harmonic and state its period.
- b) Determine the greatest speed of  $P$ .

,  $\tau = \frac{\pi}{10} \approx 0.314 \text{ s}$  ,  $\text{max speed} = 4 \text{ ms}^{-1}$

a) LOOKING AT THE DIAGRAM(S)

$l = 0.6$   
 $\lambda = 168$   
 $m = 0.7$

EQUATION OF MOTION

$$m\ddot{x} = -\frac{T}{a}$$

$$m\ddot{x} = -\frac{\lambda x}{l}$$

$$\ddot{x} = -\frac{168x}{0.7 \times 0.6}$$

$$\ddot{x} = -400x$$

$\therefore$  S.H. (H. ABOUT EQUATION OF MOTION WITH  $\omega^2 = 400$ )

$$1 = \frac{2\pi}{T} = \frac{2\pi}{10}$$

PERIOD

b) FINDING THE AMPLITUDE OF THE MOTION FROM THE DIAGRAM

$$a = 0.6 - 0.4 = 0.2$$

$$v_{\max} = a\omega = 0.2 \times 20 = 4 \text{ ms}^{-1}$$

**Question 3** (\*\*\*)

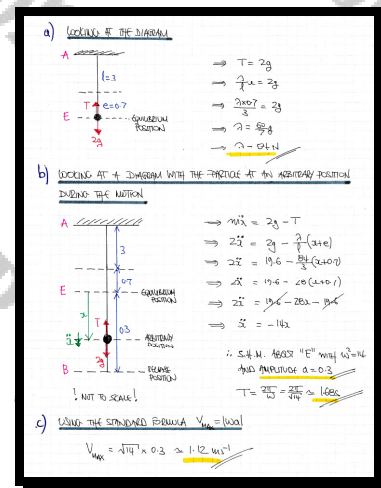
A particle of mass 2 kg is attached to one end of a light elastic string of natural length 3 m and the other end is attached to a fixed point A. The particle hangs in equilibrium at some point E, where  $|AE| = 3.7$  m.

- a) Find the modulus of elasticity of the string.

The particle is pulled vertically downwards from the point E to the point B, where  $|AB| = 4$  m, and it is released from rest.

- b) Show that in the subsequent motion, the particle moves with simple harmonic motion and determine its amplitude and its period.  
 c) Calculate the maximum speed of the particle during its motion.

$\lambda = 84 \text{ N}$ ,  $T = \frac{2\pi}{\sqrt{14}} \approx 1.68 \text{ s}$ ,  $a = 0.3 \text{ m}$ ,  $v_{\max} \approx 1.12 \text{ ms}^{-1}$



**Question 4** (\*\*\*)

A particle  $P$  of mass  $2 \text{ kg}$  is attached to the free end of a light elastic spring of natural length  $0.8 \text{ m}$  and modulus of elasticity  $100 \text{ N}$ .

$P$  is in equilibrium, hanging vertically from a fixed point  $A$ .

$P$  is pulled vertically downwards a further  $0.5 \text{ m}$  and released from rest.

Show that in the subsequent motion,  $P$  moves with simple harmonic motion, and determine the amplitude, the centre and the period of the oscillations.

$$a = 0.5 \text{ m}, \quad \tau = \frac{2\pi}{5} \text{ s}$$

The handwritten solution is divided into two parts:

**Part 1: Equilibrium**

- Diagram: A vertical line represents the spring. Point A is at the top. The natural length is  $0.8 \text{ m}$ . The equilibrium position is at a distance  $e$  from A. The particle P is at the equilibrium position.
- Equations:
 
$$T = 2g$$

$$\frac{\lambda}{l} e = 2g$$

$$\frac{100}{0.8} e = 2g$$

$$125e = 2g$$

$$e = \frac{8}{25}$$

**Part 2: SHM (Simple Harmonic Motion)**

- Diagram: Shows the particle P at a displacement  $x$  downwards from the equilibrium position. The natural length is  $0.8 \text{ m}$ . The equilibrium position is at a distance  $e$  from A. The particle P is at a distance  $e + x$  from A.
- Equations of Motion:
 
$$\Rightarrow W = 2g - T$$

$$\Rightarrow 2\ddot{x} = 2g - \frac{\lambda}{l} \left(x + \frac{8}{25}\right)$$

$$\Rightarrow 2\ddot{x} = 2g - \frac{100}{0.8} \left(x + \frac{8}{25}\right)$$

$$\Rightarrow 2\ddot{x} = 2g - 125x - 2g$$

$$\Rightarrow \ddot{x} = -25x$$
- Conclusion:
 
$$\text{is S.H.M. with } a = 5$$

$$T = \frac{2\pi}{5} = \frac{2\pi}{5} \text{ s}$$

$$a = 0.5$$
 Centre is the equilibrium position.

**Question 5 (\*\*\*)**

A particle  $P$  of mass  $1.5 \text{ kg}$  is attached to the free end of a light elastic spring of natural length  $2 \text{ m}$  and modulus of elasticity  $\lambda \text{ N}$ .

$P$  is in equilibrium, hanging vertically from a fixed point  $A$ .

An impulse of magnitude  $6 \text{ N s}$  is given to  $P$ , in a direction parallel to the spring towards  $A$ .

- Show that in the subsequent motion,  $P$  moves with simple harmonic motion.
- Given further that the period of the oscillations is  $\frac{2}{5}\pi \text{ s}$ , find the amplitude of the motion.
- Determine the value of  $\lambda$ .

$a = 0.8 \text{ m}$  ,  $\lambda = 75$

The image shows a handwritten solution for Question 5. It is divided into two parts: (a) and (b).

**Part (a):** Shows two diagrams. The first diagram, labeled 'IN EQUILIBRIUM', shows a particle of mass  $m$  hanging from a fixed point  $A$ . The forces acting on it are tension  $T$  upwards and weight  $mg$  downwards. The equilibrium position is marked. The second diagram, labeled 'IN SHM', shows the particle at a displacement  $x$  downwards from the equilibrium position. The forces are tension  $T$  upwards, weight  $mg$  downwards, and a spring force  $\frac{\lambda x}{2}$  upwards. The equations derived are:
 
$$T = mg$$

$$\frac{\lambda}{2}e = mg$$

$$e = \frac{mg}{\lambda}$$
 For SHM, the net force is:
 
$$m\ddot{x} = T - mg$$

$$m\ddot{x} = \frac{\lambda}{2}(e-x) - mg$$

$$m\ddot{x} = \frac{\lambda}{2}e - \frac{\lambda}{2}x - mg$$

$$m\ddot{x} = \frac{\lambda}{2} \cdot \frac{mg}{\lambda} - \frac{\lambda}{2}x - mg$$

$$m\ddot{x} = \frac{mg}{2} - \frac{\lambda}{2}x - mg$$

$$\ddot{x} = -\frac{\lambda}{2m}x$$
 This is SHM with  $\omega^2 = \frac{\lambda}{2m}$ . A note says 'APART FOR EQUILIBRIUM POSITION'.

**Part (b):** Shows calculations for the period and amplitude.
 
$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{\lambda}{2m}}}$$

$$6 = 1.5\sqrt{\frac{2}{\lambda}}$$

$$\omega = 5$$
 The amplitude is found using  $v_{max} = a\omega$ :
 
$$4 = a \times 5$$

$$a = 0.8 \text{ m}$$
 The period is:
 
$$T = \frac{2\pi}{5}$$

$$5^2 = \frac{\lambda}{2 \times 1.5}$$

$$\lambda = 75 \text{ N}$$

**Question 6** (\*\*\*)

A particle  $P$  of mass  $0.5 \text{ kg}$  is attached to the free end of a light elastic string of natural length  $0.8 \text{ m}$  and modulus of elasticity  $90 \text{ N}$ .

The particle is in equilibrium, hanging vertically from a fixed point  $A$ .

The particle is pulled vertically downwards a further  $0.5 \text{ m}$  and released from rest.

- a) Show that in the subsequent motion,  $P$  moves with simple harmonic motion.

The particle is next pulled vertically downwards a further distance  $a \text{ m}$  below the equilibrium position and released from rest.

- b) Given that  $P$  passes through the equilibrium position with speed  $3 \text{ ms}^{-1}$ , calculate the distance  $P$  covers until it first comes to instantaneous rest.

,  $d \approx 0.681 \text{ m}$

**Diagram in Equilibrium**

$l = 0.8$   
 $l_0 = 0.8$   
 $l = l_0 + e$   
 $\Rightarrow T = mg$   
 $\Rightarrow \frac{\lambda}{l} e = mg$   
 $\Rightarrow e = \frac{mg l}{\lambda}$   
 $\Rightarrow e = \frac{0.5 \times 9.8 \times 0.8}{90}$   
 $\Rightarrow e = \frac{3.92}{90}$

**Diagram with particle in an arbitrary position below equilibrium level**

$\Rightarrow m\ddot{x} = mg - T$   
 $\Rightarrow m\ddot{x} = mg - \frac{\lambda}{l} (e + x)$   
 $\Rightarrow m\ddot{x} = mg - \frac{\lambda}{l} e - \frac{\lambda}{l} x$   
 $\Rightarrow m\ddot{x} = mg - \frac{\lambda}{l} \left( \frac{mg l}{\lambda} \right) - \frac{\lambda}{l} x$   
 $\Rightarrow m\ddot{x} = mg - mg - \frac{\lambda}{l} x$   
 $\Rightarrow \ddot{x} = - \frac{\lambda}{ml} x$   
 $\Rightarrow \ddot{x} = - \frac{90}{0.5 \times 0.8} x$   
 $\Rightarrow \ddot{x} = -225x$

$\therefore$  Motion is SH with  $\omega = 15$ , about the equilibrium position.

**If speed through equilibrium position is  $3 \text{ ms}^{-1}$ , then  $\dot{x}_{max} = 3$**

$\Rightarrow V_{max} = \omega x$   
 $\Rightarrow 3 = 15x$   
 $\Rightarrow x = 0.2$

**Looking at the diagram**

$\bullet V^2 = \omega^2 (a^2 - x^2)$   
 $V^2 = 225 (0.2^2 - x^2)$  (at  $t=0$ )  
 $V = 2.25 \dots$   
 $V = 2.25 \dots$   
 $\bullet$  Next finding particle energy  
 $\left\{ \begin{array}{l} E = 2.25 \dots \\ E = 1.8 \\ E = 2 \\ V = 0 \end{array} \right.$   
 $V^2 = \omega^2 (a^2 - x^2)$   
 $0 = 225 (0.2^2 - x^2)$   
 $0 = 8.73 \dots - 225x^2$   
 $8.73 \dots = 225x^2$   
 $x = \sqrt{\frac{8.73 \dots}{225}}$   
 $x = 0.198 \dots$   
 $\approx 0.198 \text{ m}$  (3 s.f.)



**Question 7** (\*\*\*)

Two fixed points  $A$  and  $B$  lie on a smooth horizontal surface, so that the distance between them is 2.5 m.

A particle  $P$  of mass 0.5 kg is attached to one end of a light elastic string  $S_A$  and the other end of  $S_A$  is attached to  $A$ .

A second light elastic string  $S_B$  is also attached to  $P$  while the other end of  $S_B$  is attached to  $B$ .

Both strings are identical in every aspect, each of natural length 0.75 m and modulus of elasticity 24.5 N.

The point  $C$  lies on the straight line segment  $AB$ , so that  $AC = 1$  m.

At time  $t = 0$  s,  $P$  is released from rest from  $C$  and moves without any resistance.

- Show that in the subsequent motion,  $P$  moves with simple harmonic motion.
- Determine the period of the motion.
- Calculate the maximum kinetic energy of  $P$ .

$$\tau = \frac{\sqrt{6}}{14} \pi \approx 0.550 \text{ s}, \quad K.E_{\max} = 2.04 \text{ J}$$

The handwritten solution is divided into two parts, a) and b).

**Part a) Force Diagrams and Tension Calculations:**

- Diagram 1: Particle  $P$  at point  $C$ . Tension  $T_A$  acts to the left, and tension  $T_B$  acts to the right. The distance from  $A$  to  $C$  is 1 m, and from  $C$  to  $B$  is 1.5 m.
- Diagram 2: Particle  $P$  at point  $P$  (equilibrium position). Tension  $T_A$  acts to the left, and tension  $T_B$  acts to the right. The distance from  $A$  to  $P$  is 0.75 m, and from  $P$  to  $B$  is 1.75 m.

Calculations for tensions:

$$T_A = \lambda \frac{1.25x - 0.75}{0.75} = \lambda \frac{0.5x}{0.75} = \frac{2\lambda}{3} (0.5x) = \frac{\lambda}{3} (0.5x)$$

$$T_B = \lambda \frac{1.75 - 0.75}{1.75} = \lambda \frac{1.0}{1.75} = \frac{4\lambda}{7} (1.0 - x) = \frac{4\lambda}{7} (1 - x)$$

Equation of Motion:

$$Net = T_B - T_A$$

$$\frac{1}{2} \ddot{x} = \frac{4\lambda}{7} (1 - x) - \frac{\lambda}{3} (0.5x)$$

$$\frac{1}{2} \ddot{x} = \frac{4\lambda}{7} - \frac{4\lambda}{7}x - \frac{\lambda}{6}x$$

$$\frac{1}{2} \ddot{x} = -\frac{13\lambda}{14}x + \frac{4\lambda}{7}$$

So  $\ddot{x} = -\frac{13\lambda}{7}x + \frac{8\lambda}{7}$

It is of the form  $\ddot{x} = -\omega^2 x$

So simple harmonic about  $x = \frac{8}{13}$  with amplitude 0.25

**Part b) SHM Analysis:**

REST POSITION  $T = \frac{2\pi}{\omega}$

$$\omega^2 = \frac{24.5}{0.5}$$

$$\omega = \frac{\sqrt{49}}{1} = 7$$

$$T = \frac{2\pi}{7}$$

$$T = \frac{\sqrt{2}\pi}{4}$$

4)  $V_{\max} = a\omega = 0.25 \times \frac{13\sqrt{2}}{7}$

$$= \frac{13\sqrt{2}}{28}$$

$\therefore KE_{\max} = \frac{1}{2} m v_{\max}^2$

$$= \frac{1}{2} \times 0.5 \times \left(\frac{13\sqrt{2}}{28}\right)^2$$

$$= \frac{49}{24} \approx 2.04 \text{ J}$$

**Question 8** (\*\*\*)

A light elastic string, of natural length  $a$  and modulus of elasticity  $4mg$ , has one end attached to a fixed point  $A$  and the other end is attached to a particle  $P$  of mass  $m$ .

Initially  $P$  hangs freely at rest in equilibrium at the point  $E$ . At time  $t = 0$ ,  $P$  is projected vertically downwards from  $E$  with speed  $\sqrt{ag}$ .

- a) Prove that, while the string is taut,  $P$  moves with simple harmonic motion.
- b) Find, in terms of  $a$ , the amplitude of the simple harmonic motion.
- c) Determine, in terms of  $a$  and  $g$ , the time at which the string first goes slack.

$$\text{amplitude} = \frac{1}{2}a, \quad t = \frac{7\pi}{12} \sqrt{\frac{a}{g}}$$

**a) PROVE IN EQUILIBRIUM**

• NOT BEING RELEASED

NATURAL LENGTH  $a$

EQUILIBRIUM  $E$

• NOT BEING RELEASED

• SINCE IT MOVES WITH S.H.M., WITH CHANGE  $E$ ,  $\sqrt{ag}$  STARTS AT THE EQUILIBRIUM POSITION OF THE PARTICLE

$\therefore V_{\max} = A\omega$  (A = AMPLITUDE)

$\sqrt{ag} = A\sqrt{\frac{4g}{a}}$

$ag = A^2 \left(\frac{4g}{a}\right)$

$\frac{a^2}{4} = A^2$

$A = \frac{1}{2}a$

**b)**

$\Rightarrow \ddot{x} = \frac{4mg}{a}x - T$

$\Rightarrow \ddot{x} = \frac{4mg}{a}x - \frac{4mg}{a}\left(x + \frac{1}{2}a\right)$

$\Rightarrow \ddot{x} = \frac{4mg}{a}x - \frac{4mg}{a}x - \frac{4mg}{a}\left(\frac{1}{2}a\right)$

$\Rightarrow \ddot{x} = -\frac{4mg}{a}\left(\frac{1}{2}a\right) - g$

$\Rightarrow \ddot{x} = -\frac{2g}{a}x$

• S.H.M. WITH  $\omega = \frac{2g}{a}$ , ABOUT THE EQUILIBRIUM POSITION  $E$

**c)**

$x = A \sin \omega t$  (POSITIVE UP)

$x = \frac{1}{2}a \sin \omega t$

$-\frac{1}{2}a = \frac{1}{2}a \sin \omega t$

$\sin \omega t = -\frac{1}{2}$

$\omega t = \frac{7\pi}{6}$  (OTHER NAME: RADIANS)

$t = \frac{7\pi}{6\omega}$

$t = \frac{7\pi}{6} \sqrt{\frac{a}{g}}$

**Question 9** (\*\*\*)

A smooth hollow narrow tube of length 1.6 m has one open end and one closed end.

The tube is fixed in a vertical position with the closed end at the bottom.

A light elastic **spring** of natural length 1.6 m and modulus of elasticity 98 N is placed inside the tube.

The spring has one end attached to a fixed point on the closed end of the tube and the other end of the spring is attached to a particle of mass 1.25 kg .

The particle is next held inside the tube at a distance 0.8 m below the open end of the tube and released from rest.

- Show clearly that after release the motion of the particle is simple harmonic with period  $\frac{2}{7}\pi$  s .
- Calculate the time for the particle to first attain a speed of  $2.1 \text{ ms}^{-1}$  .
- Find the speed with which the particle passes through the open end of the tube.

$$t = \frac{1}{42}\pi \approx 0.0748 \text{ s} , \quad v = \frac{14}{5}\sqrt{2} \approx 3.96 \text{ ms}^{-1}$$

**1) FIND EQUILIBRIUM POSITION**  
 By Hooke's Law  
 Tension = weight  
 $T = mg$   
 $\frac{\lambda}{l} e = mg$   
 $e = \frac{mg l}{\lambda}$   
 $e = \frac{1.25 \times 9.8 \times 1.6}{98}$   
 $e = 0.2$

**Now**  
 $\Rightarrow 14x = mg - T$   
 $\Rightarrow 14x = mg - \frac{\lambda}{l}(l - x)$   
 $\Rightarrow 14x = 12.5 \times 9.8 - \frac{98}{1.6}(1.6 - x)$   
 $\Rightarrow 14x = 122.5 - 61.25(1.6 - x)$   
 $\Rightarrow 14x = 122.5 - 98 + 61.25x$   
 $\Rightarrow 47.25x = 24.5$   
 $\Rightarrow x = 0.52$

**2) ASSUME SHM ABOUT EQUILIBRIUM POSITION**  
 $\Rightarrow x = a \cos \omega t$   
 $\Rightarrow \dot{x} = -a\omega \sin \omega t$   
 $\Rightarrow v = a\omega \sin \omega t$   
 $\Rightarrow \frac{1}{2} = 0.52 \times \omega \sin \omega t$   
 $\Rightarrow \frac{1}{2} = 0.52 \times \omega$  (first time)  
 $\Rightarrow \omega = \frac{1}{1.04}$   
 $\Rightarrow t = \frac{1}{1.04} \approx 0.96 \text{ s}$

**3) USE ENERGY**  
 $\frac{1}{2}mv^2 = \frac{1}{2}kx^2 - \frac{1}{2}kx_0^2$   
 $\frac{1}{2} \times 1.25 v^2 = \frac{1}{2} \times 98 (0.52^2 - 0.2^2)$   
 $v^2 = \frac{98}{1.25} (0.16 - 0.04)$   
 $v = \frac{14}{5}\sqrt{2} \approx 3.96 \text{ ms}^{-1}$

**Question 10** (\*\*\*)

A particle is attached to one end of a light elastic spring of natural length 3.6 m and the other end of the spring is attached to a fixed point A. The particle is hanging freely in equilibrium at the point E, where  $|AE| = 5.4$  m.

The particle is then pulled vertically downwards from E to the point B, where  $|AB| = 6.48$  m, and released from rest.

- a) Show that the particle moves in simple harmonic motion, stating the centre of the motion.
- b) Find the greatest magnitude of the acceleration of the particle.

The point C is the midpoint of EB.

The point D lies vertically below A, where  $|AD| = 4.32$  m.

- c) Show further that the time taken by the particle to move directly from C to D is  $\frac{2}{7}\pi$ .

$|\ddot{x}_{\max}| = 5.88 \text{ ms}^{-2}$

**a) SHOWING SHM**  
 The extension is 1.8  
 $T = mg$   
 $\frac{2}{3}e = mg$   
 $\frac{2}{3} \times 1.8 = mg$   
 $\lambda = 2mg$

**EQUATION OF MOTION AT ARBITRARY EXTENSION**  
 $mg - T = m\ddot{x}$   
 $mg - \frac{3}{2}e = m\ddot{x}$   
 $mg - \frac{3}{2}(1.8 + x) = m\ddot{x}$   
 $\ddot{x} = g - \frac{3}{2}(1.8 + x)$   
 $\ddot{x} = g - \frac{9}{2} - \frac{3}{2}x$   
 $\ddot{x} = -\frac{3}{2}x$   
 $\therefore$  SHM ABOUT THE EQUILIBRIUM POSITION E,  $\omega = \frac{3}{2}$  AND AMPLITUDE 1.8

**b)  $|\ddot{x}_{\max}| = \omega^2 a$**   
 $|\ddot{x}_{\max}| = \left(\frac{3}{2}\right)^2 \times 1.8 = 5.88 \text{ ms}^{-2}$

**c)  $\frac{1}{2}(6.48 - 5.4) = 1.08$**   
 • POINT D IS THE CENTRE OF THE OSCILLATION  
 • PEAK B TO C  
 $x = a \cos \omega t$   
 $0.54 = 1.8 \cos \frac{3}{2}t$   
 $\frac{1}{3} = \cos \frac{3}{2}t$  (1st solution)  
 $\frac{3}{2}t = \frac{\pi}{3}$   
 $t = \frac{2\pi}{9}$   
 • PEAK B TO D IS  $\frac{1}{2}$  PERIOD  
 A PERIOD IS  $\frac{2\pi}{\omega} = \frac{2\pi}{3/2} = \frac{4\pi}{3}$   
 $\therefore$  TIME FROM B TO D IS  $\frac{2\pi}{3}$

**ADDITIONAL BY SYMMETRY** - SAME TIME FROM D TO C  
 $\therefore$  TIME FROM C TO D IS  $\frac{2\pi}{3}$   
 $\therefore$  TIME FROM C TO D IS  $\frac{2\pi}{7}$

**Question 11** (\*\*\*)

Two fixed points  $A$  and  $B$  lie on a smooth horizontal surface, so that the distance between them is 4.2 m.

A particle  $P$  of mass 0.25 kg is attached to one end of a light elastic string  $S_A$  and the other end of  $S_A$  is attached to  $A$ .

A second light elastic string  $S_B$  is also attached to  $P$  while the other end of  $S_B$  is attached to  $B$ .

The natural length of  $S_A$  is 1.8 m and its modulus of elasticity is 20 N, while the natural length of  $S_B$  is 1.2 m and its modulus of elasticity is 40 N.

$P$  rests in equilibrium at some point  $O$  between  $A$  and  $B$ .

- a) Show by calculation that  $|OA| = 2.7$ .

$P$  is then displaced from its equilibrium position  $O$  to a new position  $C$ , and released from rest.

- b) Given that when  $P$  is at  $C$ , both strings are taut, show further that in the subsequent motion,  $P$  moves with simple harmonic motion, stating its period.

,  $t = \frac{3\pi}{20} \approx 0.471$  s

a) LOOKING AT A DIAGRAM

$k = 0.25$   
 $\lambda_A = 1.8$   
 $\lambda_B = 1.2$   
 $20 = 20$   
 $40 = 40$

$T_A = T_B$   
 $\frac{20}{1.8} x_A = \frac{40}{1.2} x_B$   
 $\frac{10}{9} x_A = \frac{10}{3} x_B$   
 $x_A = 3x_B$   
 $x_A + x_B = 4.2$   
 $3x_B + x_B = 4.2$   
 $4x_B = 4.2$   
 $x_B = 1.05$   
 $x_A = 3.15$

$\therefore$  FORWARD DISTANCE IS  $\lambda_A + x_A = 1.8 + 0.9 = 2.7$  m

b) LOOKING AT A NEW DIAGRAM WITH THE PARTICLE AT AN ARBITRARY POSITION, SAY 2 TO THE LEFT OF THE EQUILIBRIUM POSITION

$\Rightarrow \frac{1}{2} \dot{x}^2 = T_A - T_B$   
 $\frac{1}{2} \dot{x}^2 = \frac{20}{1.8} (2.7 - x) - \frac{40}{1.2} (1.5 + x - 1.2)$   
 $\frac{1}{2} \dot{x}^2 = \frac{20}{1.8} (2.7 - x) - \frac{40}{3} (1.5 + x - 1.2)$   
 $\frac{1}{2} \dot{x}^2 = \frac{100}{9} (0.9 - x) - \frac{400}{9} (0.3 + x)$

$\Rightarrow \frac{1}{2} \dot{x}^2 = 10 - \frac{100}{9} x - (10 + \frac{160}{9} x)$   
 $\Rightarrow \frac{1}{2} \dot{x}^2 = 10 - \frac{100}{9} x - 10 - \frac{160}{9} x$   
 $\Rightarrow \frac{1}{2} \dot{x}^2 = -\frac{260}{9} x$   
 $\Rightarrow \ddot{x} = -\frac{130}{9} x$

IS SHM WITH  $\omega^2 = \frac{1300}{9}$  &  $\omega = \frac{10\sqrt{13}}{3}$

$\therefore$  PERIOD =  $\frac{2\pi}{\omega} = 2\pi \times \frac{3}{10\sqrt{13}} = \frac{3\pi}{5\sqrt{13}} \approx 0.471$

**Question 12** (\*\*\*\*)

Two fixed points  $A$  and  $B$  lie on a smooth horizontal surface, so that the distance between them is 4.13 m.

A particle  $P$  of mass  $m$  kg is attached to one end of a light elastic spring  $S_A$  and the other end of  $S_A$  is attached to  $A$ .

A second light elastic spring  $S_B$  is also attached to  $P$  while the other end of  $S_B$  is attached to  $B$ .

The natural length of  $S_A$  is 0.8 m and its modulus of elasticity is 120 N, while the natural length of  $S_B$  is 1.5 m and its modulus of elasticity is 80 N.

The particle rests in equilibrium at some point  $O$  between  $A$  and  $B$ .

- a) Show by calculation that  $|OA| = 1.28$ .

The point  $C$  lies on the straight line segment  $AOB$ , between  $A$  and  $O$ .

At time  $t = 0$  s,  $P$  is released from rest from  $C$  and moves without any resistance.

- b) Show that in the subsequent motion,  $P$  moves with simple harmonic motion.

The angular frequency of  $P$  is 10 Hz and its maximum speed is  $4 \text{ ms}^{-1}$ .

- c) Determine the time taken for  $P$  to travel a distance of 0.6 m from  $C$ .

$$t = \frac{\pi}{15} \approx 0.209 \text{ s}$$

**(a)**  $T_A = T_B$   
 $\frac{120}{0.8} x_A = \frac{80}{1.5} x_B$   
 $150x_A = 160x_B$   
 $45x_A = 16x_B$   
 $T_A = T_B$   
 $2.3x_A + 2.3x_B = 4.13$   
 $x_A + x_B = 1.83$   
 $x_B = 1.83 - x_A$   
 $45x_A = 16(1.83 - x_A)$   
 $45x_A = 29.28 - 16x_A$   
 $61x_A = 29.28$   
 $x_A = 0.48$   
 $\therefore |OA| = x_A + x_B = 0.48 + 0.48 = 0.96$  (Note: student has 1.28 in final line)

**(b)** DEDUCE THE MOTION AT POINT C (ARBITRARY POSITION) FROM THE REST POSITION C  
 $T_A = \frac{120}{0.8} (1.28 - x) = \frac{160}{0.8} (0.78 - x) = 160(0.98 - x)$   
 $T_B = \frac{80}{1.5} (2.85 + x - 1.5) = \frac{80}{1.5} (1.35 + x)$   
 $\therefore m\ddot{x} = T_B - T_A$   
 $m\ddot{x} = 160(0.98 - x) - \frac{80}{1.5}(1.35 + x)$   
 $m\ddot{x} = -72x - 150x - 72x - \frac{80}{1.5}x$   
 $m\ddot{x} = -\frac{610}{3}x$   
 $\ddot{x} = -\frac{610}{3m}x$   
 $\therefore \text{SHM, COSINUS AT } t=0$   
 WITH  $\omega^2 = \frac{610}{3m}$

**(c)** **RESTING**  $\omega = 10$   
 $v_{\text{MAX}} = a\omega$   
 $4 = a \times 10$   
 $\Rightarrow a = 0.4$   
 $\text{SINCE } a = a \cos(\omega t) \text{ (SPRINGS FROM REST AT } C)$   
 $0.2 = 0.4 \cos(10t)$   
 $-0.2 = 0.4 \cos(10t)$   
 $\cos(10t) = \frac{1}{2}$   
 $10t = \frac{\pi}{3}$  (FIRST POSITIVE SOLUTION)  
 $t = \frac{\pi}{30} = \frac{\pi}{15}$   
 $t \approx 0.209 \text{ s}$

**Question 13** (\*\*\*\*)

A particle of mass 0.75 kg is attached to a fixed point A by a light elastic string of modulus of elasticity 78 N.

The particle is released from rest from A and falls vertically without any air resistance, coming to rest at a point C, 4 m below A.

- a) Show by calculation that the natural length of the string is 2.6 m.
- b) Show that when the extension in the string is  $x$  m

$$\frac{d^2x}{dt^2} = -40x + g.$$

- c) Use a suitable substitution to demonstrate that the above differential equation represents simple harmonic motion.
- d) Determine the maximum speed of the particle during its motion.
- e) Calculate, correct to 4 decimal places, the time it takes the particle to move from A to C.

  ,  $v_{\max} = 7.30 \text{ ms}^{-1}$  ,   $t \approx 1.3434 \text{ s}$

The handwritten solutions are as follows:

**Page 1:** Part (a) uses energy conservation:  $\frac{1}{2}mv^2 = \frac{1}{2}kx^2 - mgx$ . At point C,  $v=0$  and  $x=4$ . Solving  $0 = \frac{1}{2}(78)(4)^2 - (0.75)(9.8)(4)$  gives  $k = 45$ . The natural length  $l$  is found from  $k = \frac{78}{l}$ , so  $l = \frac{78}{45} = 2.6$  m.

**Page 2:** Part (b) uses Newton's second law:  $mg - kx = m \frac{d^2x}{dt^2}$ . Substituting  $k = 45$  and  $m = 0.75$  gives  $\frac{d^2x}{dt^2} = -40x + g$ . Part (c) uses the substitution  $x = \frac{g}{40} + y$  to show SHM. Part (d) finds maximum speed at  $x = \frac{g}{40} = 1.15$  m, giving  $v_{\max} = 7.30 \text{ ms}^{-1}$ .

**Page 3:** Part (e) calculates the time from A to C. It splits the motion into free fall from A to B (2.6 m) and SHM from B to C (1.4 m). Time for free fall is  $t_1 = 0.73$  s. Time for SHM is  $t_2 = 0.6134$  s. Total time is  $t = 1.3434$  s.

**Question 14** (\*\*\*\*)

Two fixed points  $A$  and  $B$  lie on a smooth horizontal surface, such that  $|AB| = 5$  m. A particle  $P$  of mass  $0.3$  kg is attached to one end of a light elastic string  $S_A$  and the other end of  $S_A$  is attached to  $A$ .

A second light elastic string  $S_B$  is also attached to  $P$  while the other end of  $S_B$  is attached to  $B$ .

The natural length of  $S_A$  is  $1$  m and its modulus of elasticity is  $90$  N, while the natural length of  $S_B$  is  $2$  m and its modulus of elasticity is  $60$  N.

The particle rests in equilibrium at some point  $O$  between  $A$  and  $B$ .

- a) Determine the distance  $OA$ .

At time  $t = 0$  s,  $P$  is released from rest from a point on the line segment  $AB$  such that  $AO$  is  $1$  m, and moves without any resistance.

- b) Show that in the subsequent motion,  $P$  moves with simple harmonic motion and determine its amplitude and its period.  
c) Calculate the total distance  $P$  covers in the first  $0.5$  s of its motion.

$$|OA| = 1.5 \text{ m}, \quad a = 0.5 \text{ m}, \quad T = \frac{\pi}{10} \text{ s}, \quad d \approx 3.34 \text{ m}$$

(a) In equilibrium  $T_A = T_B$   
 $\frac{90}{x_A} x_A = \frac{60}{x_B} x_B$   
 $\frac{90}{1} x_A = \frac{60}{2} x_B$   
 $90 x_A = 30 x_B$   
 $x_B = 3 x_A$

Also  $1 + x_A + 2 + x_B = 5$   
 $x_A + x_B = 2$   
 $x_A + 3x_A = 2$   
 $4x_A = 2$   
 $x_A = 0.5$

THIS REQUIRED DISTANCE IS  
 $1 + 0.5 = 1.5 \text{ m}$   
 $\therefore |OA| = 1.5 \text{ m}$

(b)  $m \ddot{x} = T_B - T_A$   
 $0.3 \ddot{x} = \frac{2k}{4}(1.5-2) - \frac{k}{1}(1.5-1)$   
 $0.3 \ddot{x} = \frac{60}{2}(1.5-2) - 90(0.5-1)$   
 $0.3 \ddot{x} = 90 - 30x - 45 - 90x$   
 $0.3 \ddot{x} = -120x$   
 $\ddot{x} = -400x$

It's SHM with  $\omega = 20$   
 $T = \frac{2\pi}{\omega} = \frac{2\pi}{20} = \frac{\pi}{10}$   
 $\therefore \text{PERIOD} = \frac{\pi}{10}$   
 AMPLITUDE  
 $a = 0.5 \text{ m}$

Using  $x = 0.167 \cos(20t)$   
 $x = 0.167 \cos(20t)$   
 $x = 0.167 \cos(20t)$   
 $x = 0.167 \cos(20t)$

Now period is  $T = \frac{2\pi}{\omega} = 0.314 \text{ s}$   
 This  $0.5 - 0.314 = 0.186 \text{ s}$   
 $0.186 \text{ s} - \frac{1}{2}T = 0.072 \text{ s}$

$\therefore$  Total distance =  $7 \times 0.5 - 0.167$  (See picture above)  
 $= 3.34$



**Question 15** (\*\*\*\*+)

A particle of mass  $m$  is attached to one end of a light elastic string of stiffness  $k$ , and the other end is attached to a fixed point A. The particle is hanging in equilibrium with the string in a vertical position. The particle is next pulled a vertical distance  $a$  below its equilibrium position and released from rest.

At time  $t$ , the displacement of the particle below its equilibrium position is  $x$  and the velocity of the particle is  $v$ .

Show, by forming and solving suitable differential equations, that while the particle is moving upwards with the string taut ...

a) ...  $v = -\sqrt{\frac{k}{m}(a^2 - x^2)}$ .

b) ...  $x = a \cos\left(\sqrt{\frac{k}{m}}t\right)$ .

proof

(a) **EQUILIBRIUM**  
 $T = mg$   
 $kx = mg$

**DISPLACED**  
 $T = mg + kx$   
 $mg - kx = mg$   
 $mg - kx - mg = -kx$   
 $\frac{d^2x}{dt^2} = -\frac{kx}{m}$

Now solving the D.E  
 subject to  $x=0, v=0$   
 $m \frac{d^2x}{dt^2} = -kx$   
 $\Rightarrow mv \frac{dv}{dx} = -kx$   
 $\Rightarrow \int mv \, dv = \int -kx \, dx$   
 $\Rightarrow \frac{1}{2}mv^2 = \left[-\frac{1}{2}kx^2\right]_0^x$   
 $\Rightarrow \frac{1}{2}mv^2 = -\frac{1}{2}kx^2 + \frac{1}{2}kx^2$   
 $\Rightarrow mv^2 = k(a^2 - x^2)$   
 $\Rightarrow v = \pm \sqrt{\frac{k}{m}(a^2 - x^2)}$

THIS WITHIN THE PARTICLE  
 IS MOVING UP, & IS THE  
 EXPANSION OF THE STRING, & THAT  
 IS AN INCREASING

$\Rightarrow v = -\sqrt{\frac{k}{m}(a^2 - x^2)}$   
 AS REQUESTED

(b) Now  $v = -\sqrt{\frac{k}{m}(a^2 - x^2)}$   
 $\frac{dx}{dt} = -\sqrt{\frac{k}{m}(a^2 - x^2)}$   
 subject to  $\frac{t=0}{x=a}$   
 $\Rightarrow \int \frac{1}{\sqrt{a^2 - x^2}} dx = \int \sqrt{\frac{k}{m}} dt$   
 $\Rightarrow \left[\arcsin \frac{x}{a}\right]_a^x = \left[\sqrt{\frac{k}{m}}t\right]_0^t$   
 $\Rightarrow \arcsin \frac{x}{a} - \arcsin 1 = \sqrt{\frac{k}{m}}t$   
 $\Rightarrow \frac{x}{a} = \cos\left(\sqrt{\frac{k}{m}}t\right)$   
 $\Rightarrow x = a \cos\left(\sqrt{\frac{k}{m}}t\right)$   
 AS REQUESTED

**Question 16** (\*\*\*\*+)

A light elastic string with natural length 0.8 m, has one of its ends attached to a fixed point  $O$  on a smooth plane inclined at angle  $\theta$  to the horizontal, where  $\sin \theta = 0.75$ . A particle of mass 2.5 kg is attached to the other end of the string.

The particle rests in equilibrium at the point  $A$  on the plane, where  $OA$  lies along a line of greatest slope with  $|OA| = 1.2$  m.

The particle is then pulled down to a point  $B$ , where  $OAB$  is a straight line with  $|OB| = 1.7$  m, and released from rest.

- a) Show that, while the string remains taut, the particle is moving with simple harmonic motion.  
Give all the relevant details of this motion.

The point  $M$  is the midpoint of  $AB$ .

- b) Calculate, correct to 4 decimal places, the time taken by the particle to move directly from  $M$  to the point where the string becomes slack for the first time.

,   $\approx 0.3385$

a) WHAT WE NEED TO FIND: A IN EQUILIBRIUM POSITION

$$\rightarrow T = 2.5g \sin \theta$$

$$\Rightarrow \frac{\lambda}{2} \cdot 2 = 2.5g \cdot \frac{3}{4}$$

$$\Rightarrow \frac{\lambda}{0.8} (2 \cdot 0.8) = 18.375$$

$$\Rightarrow \lambda = 36.75 \text{ N}$$

NEXT WE CONSIDER THE PARTICLE IN AN ARBITRARY POSITION: SETTING UP EQUATION

$$\rightarrow W \ddot{x} = mg \sin \theta - T$$

$$\Rightarrow W \ddot{x} = mg \sin \theta - \frac{\lambda}{2} (x + 0.4)$$

$$\rightarrow \ddot{x} = g \sin \theta - \frac{\lambda}{2W} (x + 0.4)$$

$$\rightarrow \ddot{x} = \frac{3}{4}g - \frac{36.75}{2 \cdot 2.5} (x + 0.4)$$

$$\Rightarrow \ddot{x} = 7.35 - 18.375(x + 0.4)$$

$$\rightarrow \ddot{x} = 7.35 - 18.375x - 7.35$$

$$\rightarrow \ddot{x} = -18.375x$$

S.H.M ABOUT A, WITH  $a^2 = 18.375$ , AND AMPLITUDE 0.5

b) LOOKING AT THE SHM KINEMATICS NEXT

USING THE SINUSOIDAL EQUATION, MEASURING TIME FROM THE END POINT, CORRELATING WITH SHM AMPLITUDE

$$\Rightarrow x = a \cos \omega t$$

$$\Rightarrow 0.25 = 0.5 \cos \omega t$$

$$0.5 = \cos \omega t$$

$$\omega t = \frac{\pi}{3}$$

$$T_1 = \frac{\pi}{3\omega}$$

$$T_1 = 0.2424 \text{ s}$$

$$-0.4 = 0.5 \cos \omega T_2$$

$$-0.8 = \cos \omega T_2$$

$$\omega T_2 = \arccos(-0.8)$$

$$T_2 = \frac{\arccos(-0.8)}{\omega}$$

$$T_2 = 0.5827 \text{ s}$$

$\therefore$  THE REQUIRED TIME IS  $T_2 - T_1 = 0.3385$

**Question 17** (\*\*\*\*+)

Two fixed points  $A$  and  $B$  lie on a smooth horizontal surface, such that  $|AB| = 7$  m. A particle  $P$  of mass  $0.3$  kg is attached to one end of a light elastic string  $S_A$  and the other end of  $S_A$  is attached to  $A$ .

A second light elastic string  $S_B$  is also attached to  $P$  while the other end of  $S_B$  is attached to  $B$ .

The natural length of  $S_A$  is  $1.5$  m and its modulus of elasticity is  $75$  N, while the natural length of  $S_B$  is  $3$  m and its modulus of elasticity is  $100$  N.

At time  $t = 0$  s,  $P$  is released from rest from so that  $|AP| = 3.25$  m.

At time  $t = T$  s,  $P$  is moving towards  $B$  for the first time and  $|AP| = 2.25$  m.

Determine the value of  $T$ .

,  $T \approx 0.262$  s

**START BY A DIAGRAM IN THE EQUILIBRIUM POSITION**

Diagram 1: Equilibrium position. Point A is on the left, B is on the right, distance AB = 7 m. Particle P is at a distance  $x_1$  from A and  $x_2$  from B. Natural length of  $S_A$  is 1.5 m, modulus is 75 N. Natural length of  $S_B$  is 3 m, modulus is 100 N. Equilibrium condition:  $1.5 + x_1 + 3 + x_2 = 7 \Rightarrow x_1 + x_2 = 2.5$ . Tension in  $S_A$  is  $T_A = \frac{75}{1.5} x_1 = 50x_1$ . Tension in  $S_B$  is  $T_B = \frac{100}{3} x_2 = \frac{100}{3}(2.5 - x_1)$ . At equilibrium,  $T_A = T_B \Rightarrow 50x_1 = \frac{100}{3}(2.5 - x_1) \Rightarrow 1.5x_1 = 2.5 - x_1 \Rightarrow 2.5x_1 = 2.5 \Rightarrow x_1 = 1$  m. Therefore, P is 1 m from A and 6 m from B.

**THE EQUILIBRIUM POSITION IS 2.5 m FROM A OR 4.5 m FROM B.**

**NEXT REDRAW THE DIAGRAM AT SOME ARBITRARY POSITION AFTER RELEASE**

Diagram 2: Particle P is displaced to the right by a distance  $y$  from its equilibrium position. The distance from A is  $1 + y$  and from B is  $6 - y$ . Tension in  $S_A$  is  $T_A = 50(1 + y)$ . Tension in  $S_B$  is  $T_B = \frac{100}{3}(6 - y - 3) = \frac{100}{3}(3 - y)$ . Net force towards A:  $F = T_A - T_B = 50(1 + y) - \frac{100}{3}(3 - y) = 50 + 50y - 100 + \frac{100}{3}y = -50 + \frac{150y + 100y}{3} = -50 + \frac{250y}{3}$ . Newton's second law:  $F = ma \Rightarrow -50 + \frac{250y}{3} = 0.3a \Rightarrow \frac{250y}{3} = 0.3a + 50 \Rightarrow y = \frac{0.9a + 150}{250} = \frac{3a}{250} + \frac{3}{5}$ . This is the equation of motion.

**LOOKING AT THE EQUATION OF MOTION ABOVE**

$\Rightarrow \ddot{y} = \frac{250}{0.3} y - 500$   
 $\Rightarrow 0.3\ddot{y} = 250y - 150$   
 $\Rightarrow 0.3\ddot{y} = \frac{250}{3}(3 - y) - \frac{150}{3}(3 - y) = \frac{100}{3}(3 - y) - \frac{150}{3}(3 - y) = -\frac{50}{3}(3 - y)$   
 $\Rightarrow 0.3\ddot{y} = \frac{50}{3}(y - 3) > 0 \Rightarrow \ddot{y} = \frac{500}{3}(y - 3)$   
 $\Rightarrow \frac{d^2y}{dt^2} = -\frac{500}{3}y$

**USING STANDARD SHM EQUATION SOME FURTHER DETAILS**

$\Rightarrow \ddot{y} = -\frac{500}{3}y$   
 IS SHM WITH  $\omega = \sqrt{\frac{500}{3}}$  AND AMPLITUDE  $0.75$  m.  
 Particle starts at  $y = 2.5$  m at  $t = 0$ .  
 $y = 2.5 \cos(\omega t)$   
 $2.5 = 0.75 \cos(\omega t)$   
 $\Rightarrow \frac{1}{3} = \frac{1}{3} \cos(\omega t)$   
 $\Rightarrow \frac{1}{3} = \cos(\omega t)$   
 $\Rightarrow \omega t = \cos^{-1}\left(\frac{1}{3}\right) = 1.1071487$   
 $\Rightarrow \frac{250}{3} t = \cos^{-1}\left(\frac{1}{3}\right) + 2n\pi$   
 $t = \frac{3}{250} \cos^{-1}\left(\frac{1}{3}\right) + \frac{6n\pi}{250}$   
 $\therefore t = 0.1115, 0.262, 0.412, \dots$

**Question 18** (\*\*\*\*+)

Two fixed points  $A$  and  $B$  lie on a smooth horizontal surface, such that  $|AB| = 5$  m. A particle  $P$  of mass  $0.5$  kg is attached to one end of a light elastic string  $S_A$  and the other end of  $S_A$  is attached to  $A$ .

A second light elastic string  $S_B$  is also attached to  $P$  while the other end of  $S_B$  is attached to  $B$ .

The natural length of  $S_A$  is  $1.5$  m and its modulus of elasticity is  $30$  N, while the natural length of  $S_B$  is  $0.8$  m and its modulus of elasticity is  $20$  N.

At time  $t = 0$  s,  $P$  is released from rest from so that  $S_A$  is at natural length, and moves without any resistance.

Calculate the length  $PB$ , when the particle next gets to instantaneous rest.

$d \approx 0.481$  m

**Diagram 1:** Particle P is at a point where string  $S_A$  is at its natural length of  $1.5$  m. The distance from A to P is  $1.5$  m, and from P to B is  $3.5$  m.

**Equilibrium Position:**

$$T_A = T_B$$

$$\frac{30}{1.5}x_1 = \frac{20}{0.8}x_2$$

$$20x_1 = 25x_2$$

$$4x_1 = 5x_2$$

Also,  $1.5 + x_1 + 0.8 + x_2 = 5$

$$2.3 + x_1 + x_2 = 5$$

$$x_1 + x_2 = 2.7$$

Substituting  $x_1 = \frac{5}{4}x_2$  into  $x_1 + x_2 = 2.7$ :

$$\frac{5}{4}x_2 + x_2 = 2.7$$

$$\frac{9}{4}x_2 = 2.7$$

$$x_2 = 1.2$$

$$x_1 = 1.5$$

**Equilibrium Position:** The particle is at a point where  $S_A$  is stretched by  $1.5$  m and  $S_B$  is stretched by  $1.2$  m.

**Initial Position:** The particle is released from rest at a point where  $S_A$  is at its natural length of  $1.5$  m. The distance from A to P is  $1.5$  m, and from P to B is  $3.5$  m.

**Newton's Second Law:**

$$m\ddot{x} = T_A - T_B$$

$$m\ddot{x} = \frac{30}{1.5}(1.5 - x) - \frac{20}{0.8}(x - 0.8)$$

$$\frac{1}{2}\ddot{x} = \frac{30}{1.5}(1.5 - x) - \frac{20}{0.8}(x - 0.8)$$

$$\frac{1}{2}\ddot{x} = 20(1.5 - x) - 25(x - 0.8)$$

$$\frac{1}{2}\ddot{x} = 30 - 20x - 25x + 20$$

$$\ddot{x} = -90x$$

$\Rightarrow$  S.H.M with  $\omega^2 = 90$  ABOUT O AND AMPLITUDE  $1.5$

**Diagram 2:** Particle P is at a point where string  $S_A$  is at its natural length of  $1.5$  m. The distance from A to P is  $1.5$  m, and from P to B is  $3.5$  m.

**Equilibrium Position:** The particle is at a point where  $S_A$  is stretched by  $1.5$  m and  $S_B$  is stretched by  $1.2$  m.

**Initial Position:** The particle is released from rest at a point where  $S_A$  is at its natural length of  $1.5$  m. The distance from A to P is  $1.5$  m, and from P to B is  $3.5$  m.

**Energy Conservation:**

At equilibrium position:

$$EPE + KE = EPE + KE$$

$$\frac{1}{2} \times \frac{30}{1.5} (1.5)^2 + \frac{1}{2} m v^2 = \frac{1}{2} \times \frac{30}{1.5} (1.5)^2 + 0$$

$$\frac{30}{2 \times 1.5} (1.5)^2 + \frac{1}{2} \times 0.5 \times v^2 = \frac{30}{2 \times 1.5} (1.5)^2$$

$$72.9 + 18.75 v^2 = 101.25$$

$$v^2 = \frac{28.35}{18.75}$$

$$v = 1.21669 \dots$$

$\therefore$  DISTANCE FROM C IS  $3 \times 0.859$

Hence distance from B is  $5 - (1.5 + 3 \times 0.859 \dots) = 0.481$