SIMPLE HARMONIC MOTION

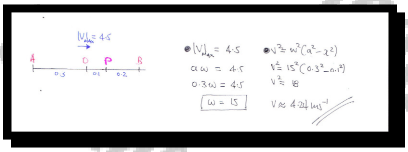
SIMPLE HARMONIC MOTION KINEMATICS

Question 1 (**)

A particle P is moving on a straight line with simple harmonic motion of amplitude 0.3 m. It passes through the centre of the oscillation O with speed 4.5 ms^{-1}.

Calculate the speed of P when $|OP| = 0.1 \text{ m}$.

$$\boxed{|v| = 3\sqrt{2} \approx 4.24 \text{ ms}^{-1}}$$

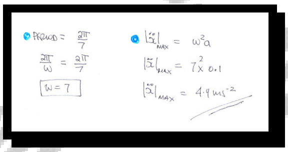


Question 2 (**)

A particle P is moving on a straight line with simple harmonic motion of amplitude 0.1 m and period $\frac{2\pi}{7} \text{ s}$.

Calculate the maximum acceleration of P.

$$\boxed{|\ddot{x}|_{\max} = 4.9 \text{ ms}^{-2}}$$



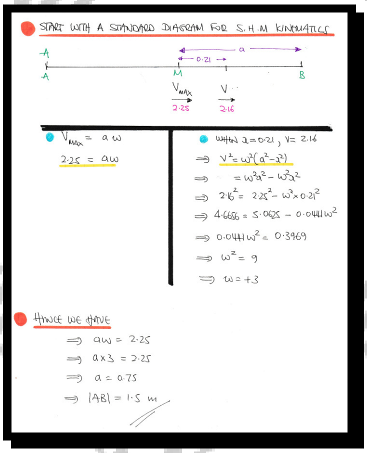
Question 3 (**)

A particle is moving in a straight line between two points A and B, with simple harmonic motion.

During this motion its greatest speed is 2.25 ms^{-1}. When the particle is at a distance of 21 cm from the midpoint of AB its speed is 2.16 ms^{-1}.

Find the distance AB.

$\boxed{}$, $\boxed{|AB| = 1.5 \text{ m}}$

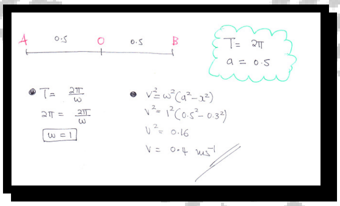


Question 4 (**)

A particle P is moving on a straight line with simple harmonic motion, centre at O, and period 2π s.

Find the speed of P when it is at a distance of 0.3 m from O, given that it comes to instantaneous rest at a distance 0.5 m from O.

$\boxed{\text{speed} = 0.4 \text{ ms}^{-1}}$

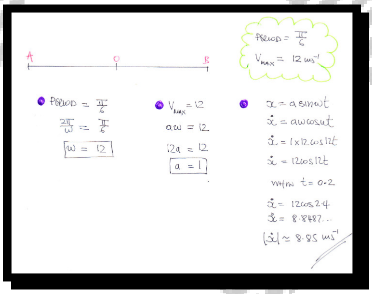


Question 5 (**+)

A particle P is moving on a straight line with simple harmonic motion of period $\frac{\pi}{6}$ s.

Given that the maximum speed of P is 12 ms^{-1}, find the speed of P 0.2 s after passing through the centre of the oscillation.

$$\boxed{\text{speed} = 8.8487... \text{ ms}^{-1}}$$

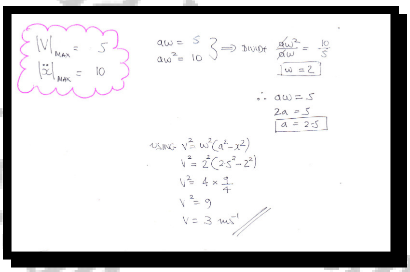


Question 6 (**+)

A particle P is moving on a straight line with simple harmonic motion of maximum speed 5 ms^{-1} and maximum acceleration 10 ms^{-2}.

Calculate the speed of P when it is 2 m from the centre of the oscillation.

$$\boxed{\text{speed} = 3 \text{ ms}^{-1}}$$



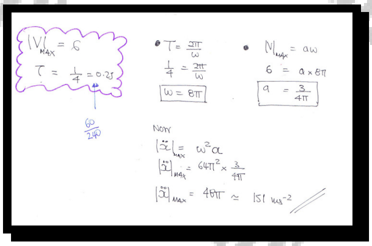
Question 7 **(**+)**

A particle P is moving on a straight line with simple harmonic motion, centre at O.

P passes through O with speed 6 ms^{-1} and performs 240 complete oscillations every minute.

Calculate the maximum acceleration of P.

$\boxed{\approx 151 \text{ ms}^{-2}}$

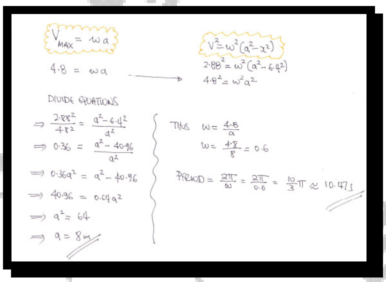


Question 8 **(**+)**

A particle P is moving in a straight line with simple harmonic motion, achieving a maximum speed of 4.8 ms^{-1}. When P is at a distance of 6.4 m from the centre of the motion, its speed is 2.88 ms^{-1}.

Determine in any order the amplitude and the period of the motion.

$\boxed{a = 8 \text{ m}}$, $\boxed{T = \frac{10\pi}{3} \approx 10.47 \text{ s}}$



Question 9 (***)

A particle is about to move in a straight line with simple harmonic motion.

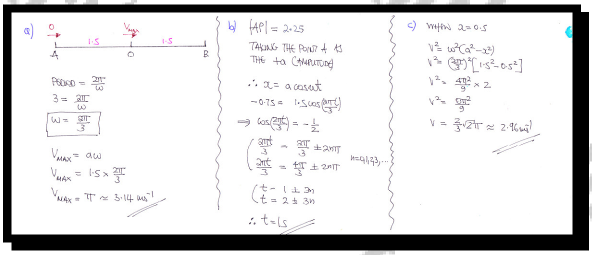
It is released from rest from a point A and travels directly to a point O, arriving there $0.75\ \text{s}$ later with maximum speed $V\ \text{ms}^{-1}$.

a) Given that $AO = 1.5\ \text{m}$, determine the value of V.

b) Find the time it takes the particle to cover the first $2.25\ \text{m}$ of the motion.

c) Calculate the speed of the particle when is at a distance of $0.5\ \text{m}$ from O.

$\boxed{V = \pi \approx 3.14}$, $\boxed{t = 1\ \text{s}}$, $\boxed{v = \frac{1}{2}\pi\sqrt{2} \approx 2.96\ \text{ms}^{-1}}$

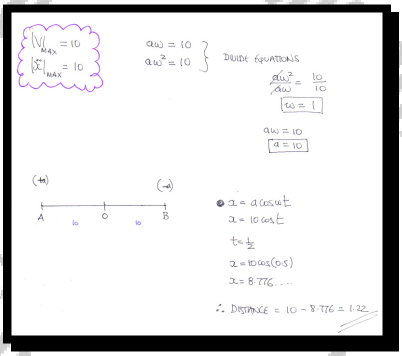


Question 10 (***)

A particle P is moving on a straight line with simple harmonic motion of maximum speed $10\ \text{ms}^{-1}$ and maximum acceleration $10\ \text{ms}^{-2}$.

Calculate the distance of P from one the endpoints of the oscillation $0.5\ \text{s}$ after passing through the centre point of the motion.

$\boxed{d \approx 1.22\ \text{m}}$



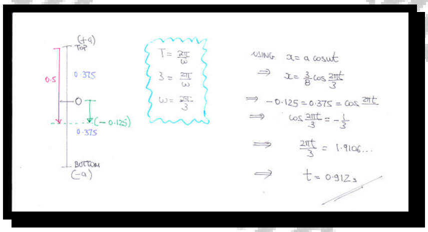
Question 11 (***)

A particle P is moving on a vertical straight line with simple harmonic motion.

It takes 3 s for a complete oscillation and the distance between the highest and the lowest level of the motion is 0.75 m.

Calculate the time P takes to travel 0.5 m from the highest point of the motion.

$\boxed{t \approx 0.912 \text{ s}}$



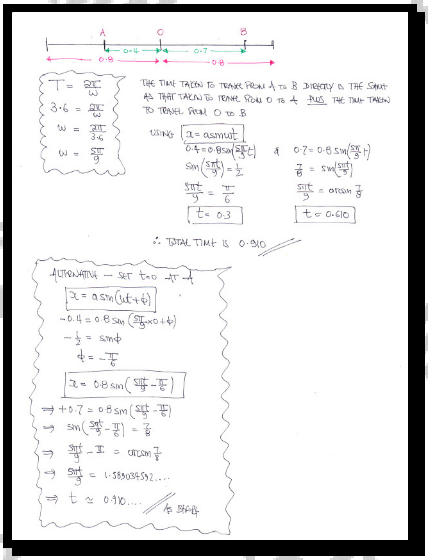
Question 12 (***)

Three points A, O and B lie in that order on a straight line.

A particle P is moving on this line with simple harmonic motion of period 3.6 s, amplitude 0.8 m and centre at O.

Given that OA is 0.4 m and OB is 0.7 m, calculate the time taken by P to travel directly from A to B.

$\boxed{t \approx 0.910 \text{ s}}$



Question 13 (***)

A boat moored at a harbour is moving up and down, taking 2.5 s to move from its highest point to its lowest point, where the vertical distance between these two points is 0.8 m. The boat is modelled as a particle moving with simple harmonic motion in a vertical direction.

The point A is 0.1 m below the highest point of the motion and the point B is 0.65 m below the highest point of the motion.

a) Determine the vertical speed of the boat as it passes through A.

b) Calculate the least time taken by the boat to move from A to B.

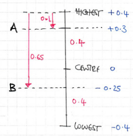
$\boxed{}$, $\boxed{|V_A| \approx 0.332 \text{ ms}^{-1}}$, $\boxed{t \approx 1.21 \text{ s}}$

a) STARTING WITH A DIAGRAM

- AMPLITUDE $a = 0.4$
- PERIOD $= 2 \times 2.5 = 5$

$\frac{2\pi}{\omega} = 5$

$\omega = \frac{2\pi}{5}$



USING $v^2 = \omega^2(a^2 - x^2)$

$v^2 = \left(\frac{2\pi}{5}\right)^2(0.4^2 - 0.3^2)$

$v^2 = 0.1105...$

$|v| \simeq 0.332 \text{ ms}^{-1}$

b) USING $x = a\cos\omega t$, WITH $t=0$ AT THE HIGHEST POINT

- UP TO "A"

$0.3 = 0.4\cos\left(\frac{2\pi t}{5}\right)$

$0.75 = \cos\left(\frac{2\pi t}{5}\right)$

$\frac{2\pi t}{5} \approx 0.7227...$

$t_1 \approx 0.57513...$

- UP TO "B"

$-0.25 = 0.4\cos\left(\frac{2\pi t}{5}\right)$

$-0.625 = \cos\left(\frac{2\pi t}{5}\right)$

$\frac{2\pi t}{5} = 2.2459...$

$t_2 \simeq 1.78725...$

$\therefore$ REQUIRED TIME $= t_2 - t_1 = 1.78725... - 0.57513...$

$\simeq 1.21$

Question 14 (***+)

A particle is moving in a straight line between two points A and B, which are 0.4 m apart, with simple harmonic motion.

The point C is 0.1 away from A.

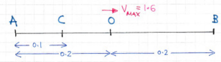
a) If the greatest speed of the particle during its motion is 1.6 ms^{-1}, determine the speed of the particle as it passes through C.

At time $t = 0$, the particle is at A.

b) Determine, in terms of π, the time the particle takes until the time it passes through C for the eighth time.

$$\boxed{|v| = \sqrt{1.92} \approx 1.39 \text{ ms}^{-1}}, \quad \boxed{t = \frac{23\pi}{24}}$$

a) PUT THE INFORMATION GIVEN IN A DIAGRAM



USING $|V_{MAX}| = a\omega$

$\Rightarrow 1.6 = 0.2\omega$

$\Rightarrow \omega = 8$

NOW USING $v^2 = \omega^2(a^2 - x^2)$

$\Rightarrow v^2 = 8^2(0.2^2 - 0.1^2)$

$\Rightarrow v^2 = 1.92$

$\Rightarrow |v| \simeq 1.39 \text{ ms}^{-1}$ // 3 s.f

b) "PASSING THROUGH C FOR THE EIGHTH TIME"

SETTING $t=0$, AT B WITH AMPLITUDE $+0.2$ AT B

$x = a\cos\omega t$

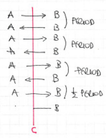
$x = 0.2\cos 8t$

$-0.1 = 0.2\cos 8t$

$\cos 8t = -\frac{1}{2}$

$8t = \frac{2\pi}{3}$

$t = \frac{\pi}{12}$



NEXT WE FIND THE PERIOD OF THE MOTION

$T = \frac{2\pi}{\omega} = \frac{2\pi}{8} = \frac{\pi}{4}$

THE REQUIRED TIME IS

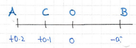
$3 \times \frac{\pi}{4} + \frac{1}{2} \times \frac{\pi}{4} + \frac{\pi}{12} = \frac{23}{24}\pi$ //

3 PERIODS — HALF PERIOD — FROM B TO C AT THE VERY END

ALTERNATIVE FOR PART (b) BY A DIRECT TRIG EQUATION

- SET $t=0$ AT A, SO + AMPLITUDE IS NOW AT A
- WE REQUIRE THE 8^{TH} POSITIVE SOLUTION OF THE EQUATION

$+0.1 = 0.2\cos 8t$



- SOLVING THE EQUATION

$\Rightarrow \cos 8t = \frac{1}{2}$

$\Rightarrow \begin{cases} 8t = \frac{\pi}{3} + 2n\pi \\ 8t = \frac{5\pi}{3} + 2n\pi \end{cases} \quad n = 0, 1, 2, 3, \ldots$

$\Rightarrow \begin{cases} t = \frac{\pi}{24}[1 + 6n] \\ t = \frac{\pi}{24}[5 + 6n] \end{cases}$

$\Rightarrow t = \frac{\pi}{24}, \frac{5\pi}{24}, \frac{7\pi}{24}, \frac{11\pi}{24}, \frac{13\pi}{24}, \frac{17\pi}{24}, \frac{19\pi}{24}, \frac{23\pi}{24}$ //

1st, 2nd, 3rd, 4th, 5th, 6th, 7th, 8th

Question 15 (***+)

A particle is attached to one end of a light spring, whose other end is attached to a fixed point. The particle is hanging vertically in equilibrium.

The particle is then pulled downwards by a further 0.6 m and released from rest.

The motion of the particle satisfies the differential equation

$$\frac{d^2x}{dt^2} = -k^2 x,$$

where x m is the additional extension of the spring from its equilibrium position, at time t s, and k is a constant. The motion has period of 2 s.

Find the first four positive values of t for which $x = 0.3$ m.

$\boxed{}$, $\boxed{t = \frac{1}{3}, \frac{5}{3}, \frac{7}{3}, \frac{11}{3}}$

$\frac{d^2x}{dt^2} = -k^2 x$ PERIOD $= \frac{2\pi}{k} = 2$ $\therefore k = \pi$

AMPLITUDE $= 0.6$

USING THE STANDARD EQUATION FOR RELEASE AT THE ENDPOINT

$\Rightarrow x = a\cos kt$

$\Rightarrow x = 0.6\cos \pi t$

$\Rightarrow 0.3 = 0.6\cos \pi t$

$\Rightarrow 0.5 = \cos \pi t$

$\arccos(0.5) = \frac{\pi}{3}$

$\Rightarrow \begin{cases} \pi t = \frac{\pi}{3} \pm 2n\pi \\ \pi t = \frac{5\pi}{3} \pm 2n\pi \end{cases}$ $n = 0, 1, 2, 3, \dots$

$\Rightarrow \begin{cases} t = \frac{1}{3} \pm 2n \\ t = \frac{5}{3} \pm 2n \end{cases}$

$\therefore t = \frac{1}{3}, 2\frac{1}{3}, \frac{5}{3}, 2 + \frac{5}{3}$

$\therefore t = \frac{1}{3}, \frac{5}{3}, \frac{7}{3}, \frac{11}{3}$

Question 16 (***+)

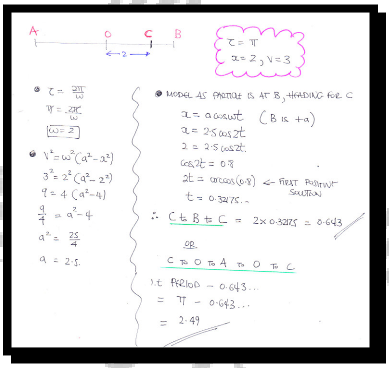
A particle P is moving on a straight line with simple harmonic motion, centre at O, and period π s.

The point C is at a distance of 2 m from O.

It is further given that P passes through C with speed 3 ms^{-1} and returns to C after time T s, where $T < \pi$.

Calculate the possible values of T.

$\boxed{T \approx 0.643 \text{ s}, \quad T \approx 1.22 \text{ s}}$



Question 17 (***+)

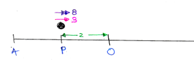
A particle is moving with simple harmonic motion on a straight line with centre at O.

When the particle is passing through a point P, heading towards O, its speed is 3 ms^{-1} and its acceleration 8 ms^{-2}.

Calculate the time taken for the particle to return to P for the first time.

$\boxed{}$, $\boxed{t \approx 2.21 \text{ s}}$

PUTTING THE INFORMATION INTO A DIAGRAM



$v = 3$
$|\ddot{x}| = 8$
$x = 2$

- $v^2 = \omega^2(a^2 - x^2)$
$9 = \omega^2(a^2 - 2^2)$
$9 = \omega^2(a^2 - 4)$
$9 = 4(a^2 - 4)$
$\frac{9}{4} = a^2 - 4$
$6.25 = a^2$
$a = 2.5$

- $|\ddot{x}| = \omega^2 x$
$8 = \omega^2 \times 2$
$\omega^2 = 4$
$\omega = 2$

& PERIOD $T = \frac{2\pi}{\omega} = \pi$

NOW SETTING A DISPLACEMENT EQUATION AS A FUNCTION OF TIME

LET $t = 0$ AT THE ORIGIN

$\Rightarrow x = a\cos\omega t$
$\Rightarrow x = 2.5\cos 2t$
$\Rightarrow 2 = 2.5\cos 2t$ TO FIND THE TIME FROM "P" TO "O" OR FROM "O" TO "P"
$\Rightarrow \cos 2t = \frac{4}{5}$
$\Rightarrow 2t = \arccos\left(\frac{4}{5}\right)$ (FIRST TIME)
$\Rightarrow t = \frac{1}{2}\arccos\left(\frac{4}{5}\right)$

THE REQUIRED TIME IS GIVEN BY

"PO" + $\underbrace{\text{"OB" + "BO"}}$ + "OP"

$\frac{1}{2}\arccos\frac{4}{5}$ + HALF PERIOD + $\frac{1}{2}\arccos\frac{4}{5} = \frac{\pi}{2} + \arccos\frac{4}{5}$

$\simeq 2.21 \text{ s}$

Question 18 (***+)

A particle P is moving in a straight line with simple harmonic motion between two points A and B, where $|AB| = 0.8$ m.

The points C and D lie on the path of P such that $|AC| = 0.2$ m and $|AD| = 0.6$ m, and it takes $\frac{2}{3}$ s for P to travel directly from C to D.

When $t = 0$, P is at A.

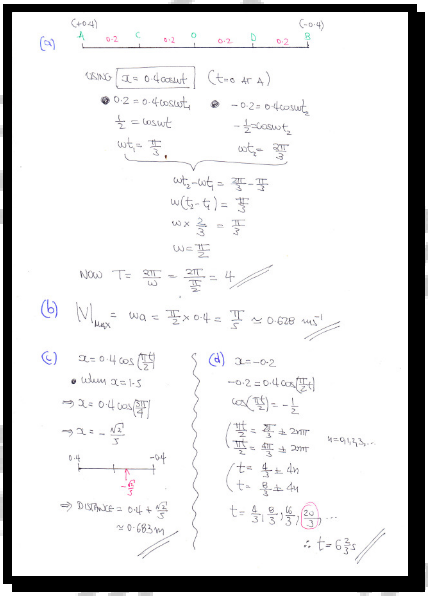
a) Show that the period of the motion is 4 s.

b) Find the maximum speed of P.

c) Find the distance of P from A when $t = 1.5$.

d) Calculate the value of t when P passes through D for the **fourth** time.

$\boxed{|v|_{\max} = \frac{\pi}{5} \approx 0.628 \text{ ms}^{-1}}$, $\boxed{d \approx 0.683 \text{ m}}$, $\boxed{t = 6\frac{2}{3} \text{ s}}$



(a) (+0.4) A 0.2 C 0.2 O 0.2 D 0.2 B (−0.4)

USING $\boxed{x = 0.4\cos\omega t}$ (t=0 AT A)

- $0.2 = 0.4\cos\omega t_1$, $\frac{1}{2} = \cos\omega t$, $\omega t_1 = \frac{\pi}{3}$
- $-0.2 = 0.4\cos\omega t_2$, $-\frac{1}{2} = \cos\omega t_2$, $\omega t_2 = \frac{2\pi}{3}$

$\omega t_2 - \omega t_1 = \frac{2\pi}{3} - \frac{\pi}{3}$

$\omega(t_2 - t_1) = \frac{\pi}{3}$

$\omega \times \frac{2}{3} = \frac{\pi}{3}$

$\omega = \frac{\pi}{2}$

NOW $T = \frac{2\pi}{\omega} = \frac{2\pi}{\frac{\pi}{2}} = 4$

(b) $|V|_{\max} = \omega a = \frac{\pi}{2} \times 0.4 = \frac{\pi}{5} \simeq 0.628 \text{ ms}^{-1}$

(c) $x = 0.4\cos\left(\frac{\pi t}{2}\right)$

- when $x = 1.5$

$\Rightarrow x = 0.4\cos\left(\frac{3\pi}{4}\right)$

$\Rightarrow x = -\frac{\sqrt{2}}{5}$

0.4 … $-\frac{\sqrt{2}}{5}$ … −0.4

$\Rightarrow$ DISTANCE $= 0.4 + \frac{\sqrt{2}}{5}$ $\simeq 0.683$ m

(d) $x = -0.2$

$-0.2 = 0.4\cos\left(\frac{\pi}{2}t\right)$

$\cos\left(\frac{\pi t}{2}\right) = -\frac{1}{2}$

$\frac{\pi t}{2} = \frac{2\pi}{3} \pm 2n\pi$, $\frac{\pi t}{2} = \frac{4\pi}{3} \pm 2n\pi$ $\quad n = 0, 1, 2, 3, \ldots$

$t = \frac{4}{3} \pm 4n$, $t = \frac{8}{3} \pm 4n$

$t = \frac{4}{3}, \frac{8}{3}, \frac{16}{3}, \frac{20}{3}, \ldots$

$\therefore t = 6\frac{2}{3}$ s

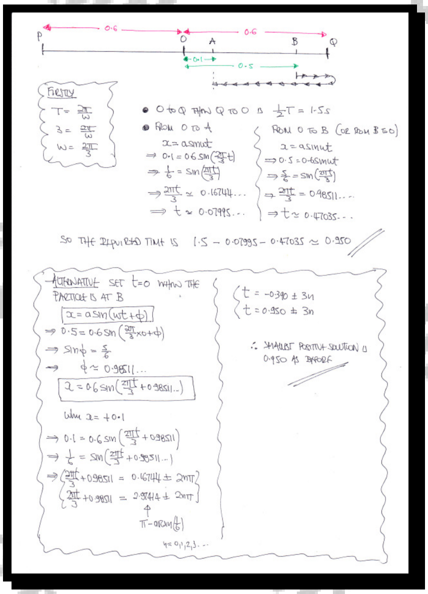
Question 19 (***+)

Three points O, A and B lie in that order on a straight line. A particle P is moving on this line with simple harmonic motion of period 3 s, amplitude 0.6 m and centre at O. It is further given that OA is 0.1 m and OB is 0.5 m.

At a certain instant P is observed passing through B moving in the direction OB.

Calculate the time when P reaches A.

$\boxed{t \approx 0.950 \text{ s}}$



FIRSTLY

$T = \frac{2\pi}{\omega}$

$3 = \frac{2\pi}{\omega}$

$\omega = \frac{2\pi}{3}$

- O to Q THEN Q TO O IS $\frac{1}{2}T = 1.5$ s
- FROM O TO A

$x = a\sin\omega t$

$\Rightarrow 0.1 = 0.6\sin\left(\frac{2\pi}{3}t\right)$

$\Rightarrow \frac{1}{6} = \sin\left(\frac{2\pi t}{3}\right)$

$\Rightarrow \frac{2\pi t}{3} \simeq 0.16744\ldots$

$\Rightarrow t \simeq 0.07995\ldots$

FROM O TO B (OR FROM B TO O)

$x = a\sin\omega t$

$\Rightarrow 0.5 = 0.6\sin\omega t$

$\Rightarrow \frac{5}{6} = \sin\left(\frac{2\pi t}{3}\right)$

$\Rightarrow \frac{2\pi t}{3} = 0.98511\ldots$

$\Rightarrow t \simeq 0.47035\ldots$

SO THE REQUIRED TIME IS $1.5 - 0.07995 - 0.47035 \simeq 0.950$

ALTERNATIVE SET $t=0$ WHEN THE PARTICLE IS AT B

$\boxed{x = a\sin(\omega t + \phi)}$

$\Rightarrow 0.5 = 0.6\sin\left(\frac{2\pi}{3}\times 0 + \phi\right)$

$\Rightarrow \sin\phi = \frac{5}{6}$

$\Rightarrow \phi \simeq 0.98511\ldots$

$\boxed{x = 0.6\sin\left(\frac{2\pi t}{3} + 0.98511\ldots\right)}$

When $x = +0.1$

$\Rightarrow 0.1 = 0.6\sin\left(\frac{2\pi t}{3} + 0.98511\right)$

$\Rightarrow \frac{1}{6} = \sin\left(\frac{2\pi t}{3} + 0.98511\ldots\right)$

$\Rightarrow \begin{cases} \frac{2\pi t}{3} + 0.98511 = 0.16744 \pm 2n\pi \\ \frac{2\pi t}{3} + 0.98511 = 2.97414 \pm 2n\pi \end{cases}$

$\pi - \arcsin\left(\frac{1}{6}\right)$

$n = 0, 1, 2, 3, \ldots$

$\begin{cases} t = -0.390 \pm 3n \\ t = 0.950 \pm 3n \end{cases}$

$\therefore$ SMALLEST POSITIVE SOLUTION IS 0.950 AS BEFORE

Question 20 (***+)

A particle P is moving with simple harmonic motion.

The motion takes place along a straight line with centre at O. The points O, A and B, lie in that order, on this line with $|OA| = 0.5$ m and $|AB| = 0.7$ m.

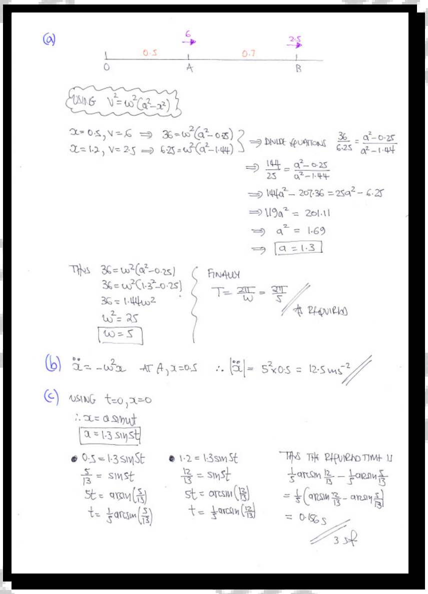
The speed of P at A is 6 ms^{-1} and its speed at B is 2.5 ms^{-1}.

a) Show that the period of the motion is $\dfrac{2\pi}{5}$ s.

b) Determine the acceleration of P at A.

c) Calculate the time taken for P to travel directly from A to B.

$\boxed{a = 12.5 \text{ ms}^{-2}}$, $\boxed{t \approx 0.156 \text{ s}}$



(a)

USING $v^2 = \omega^2(a^2 - x^2)$

$x = 0.5, v = 6 \Rightarrow 36 = \omega^2(a^2 - 0.25)$
$x = 1.2, v = 2.5 \Rightarrow 6.25 = \omega^2(a^2 - 1.44)$
$\Rightarrow$ DIVIDE EQUATIONS $\dfrac{36}{6.25} = \dfrac{a^2 - 0.25}{a^2 - 1.44}$
$\Rightarrow \dfrac{144}{25} = \dfrac{a^2 - 0.25}{a^2 - 1.44}$
$\Rightarrow 144a^2 - 207.36 = 25a^2 - 6.25$
$\Rightarrow 119a^2 = 201.11$
$\Rightarrow a^2 = 1.69$
$\Rightarrow \boxed{a = 1.3}$

THUS $36 = \omega^2(a^2 - 0.25)$
$36 = \omega^2(1.3^2 - 0.25)$
$36 = 1.44\omega^2$
$\omega^2 = 25$
$\boxed{\omega = 5}$

FINALLY $T = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{5}$ AS REQUIRED

(b) $\ddot{x} = -\omega^2 x$ AT $A, x = 0.5$ $\therefore |\ddot{x}| = 5^2 \times 0.5 = 12.5 \text{ ms}^{-2}$

(c) USING $t = 0, x = 0$
$\therefore x = a \sin \omega t$
$\boxed{x = 1.3 \sin 5t}$

• $0.5 = 1.3 \sin 5t$
$\frac{5}{13} = \sin 5t$
$5t = \arcsin\left(\frac{5}{13}\right)$
$t = \frac{1}{5}\arcsin\left(\frac{5}{13}\right)$

• $1.2 = 1.3 \sin 5t$
$\frac{12}{13} = \sin 5t$
$5t = \arcsin\left(\frac{12}{13}\right)$
$t = \frac{1}{5}\arcsin\left(\frac{12}{13}\right)$

THUS THE REQUIRED TIME IS
$\frac{1}{5}\arcsin\frac{12}{13} - \frac{1}{5}\arcsin\frac{5}{13}$
$= \frac{1}{5}\left(\arcsin\frac{12}{13} - \arcsin\frac{5}{13}\right)$
$= 0.156$ s 3 sf

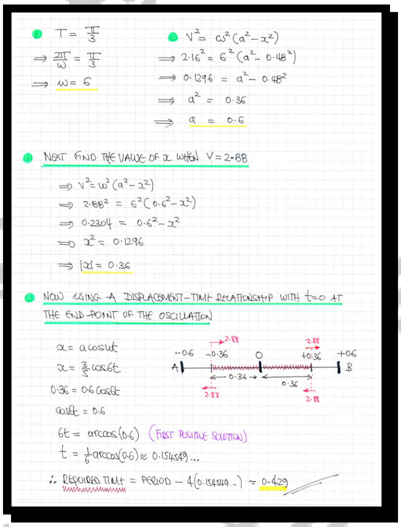
Question 21 (****)

A particle is moving on a straight line with simple harmonic motion, centre at O, and period $\frac{1}{3}\pi$ s.

When the particle is at a distance of 0.48 m from O, its speed is 2.16 ms^{-1}.

Calculate the total time within a complete oscillation, for which the particle has speed less than 2.88 ms^{-1}.

$\boxed{}$, $\boxed{t \approx 0.429 \text{ s}}$



- $T = \frac{\pi}{3}$

$\Rightarrow \frac{2\pi}{\omega} = \frac{\pi}{3}$

$\Rightarrow \omega = 6$

- $v^2 = \omega^2(a^2 - x^2)$

$\Rightarrow 2.16^2 = 6^2(a^2 - 0.48^2)$

$\Rightarrow 0.1296 = a^2 - 0.48^2$

$\Rightarrow a^2 = 0.36$

$\Rightarrow a = 0.6$

- NEXT FIND THE VALUE OF x WHEN $V = 2.88$

$\Rightarrow v^2 = \omega^2(a^2 - x^2)$

$\Rightarrow 2.88^2 = 6^2(0.6^2 - x^2)$

$\Rightarrow 0.2304 = 0.6^2 - x^2$

$\Rightarrow x^2 = 0.1296$

$\Rightarrow |x| = 0.36$

- NOW USING A DISPLACEMENT–TIME RELATIONSHIP WITH $t=0$ AT THE END-POINT OF THE OSCILLATION

$x = a\cos\omega t$

$x = \frac{3}{5}\cos 6t$

$0.36 = 0.6\cos 6t$

$\cos 6t = 0.6$

$6t = \arccos(0.6)$ (FIRST POSITIVE SOLUTION)

$t = \frac{1}{6}\arccos(0.6) \approx 0.154549\ldots$

$\therefore$ REQUIRED TIME $=$ PERIOD $- 4(0.154549\ldots) \approx 0.429$

Question 22 (****)

A particle P moves in a straight line with simple harmonic motion with period $\frac{\pi}{3}$ s.

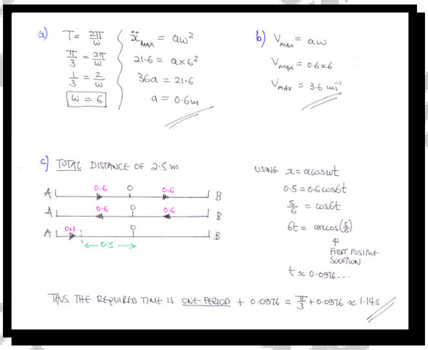
At time $t = 0$, P is at rest at the point A and the acceleration at that instant has magnitude 21.6 ms^{-2}.

a) Find the amplitude of the motion.

b) Hence state the greatest speed of P during the motion.

c) Calculate the time P takes to travel a **total distance** of 2.5 m after it has first left A.

$\boxed{a = 0.6 \text{ m}}$, $\boxed{v_{\text{max}} = 3.6 \text{ ms}^{-1}}$, $\boxed{t \approx 1.14 \text{ s}}$



a) $T = \frac{2\pi}{\omega}$

$\frac{\pi}{3} = \frac{2\pi}{\omega}$

$\frac{1}{3} = \frac{2}{\omega}$

$\boxed{\omega = 6}$

$\ddot{x}_{\text{MAX}} = a\omega^2$

$21.6 = a \times 6^2$

$36a = 21.6$

$a = 0.6 \text{ m}$

b) $V_{\text{MAX}} = a\omega$

$V_{\text{MAX}} = 0.6 \times 6$

$V_{\text{MAX}} = 3.6 \text{ ms}^{-1}$

c) TOTAL DISTANCE OF 2.5 m

USING $x = a\cos\omega t$

$0.5 = 0.6\cos 6t$

$\frac{5}{6} = \cos 6t$

$6t = \arccos\left(\frac{5}{6}\right)$

FIRST POSITIVE SOLUTION

$t \approx 0.0976\ldots$

THUS THE REQUIRED TIME IS ONE PERIOD $+ 0.0976 = \frac{\pi}{3} + 0.0976 \approx 1.14 \text{ s}$

Question 23 (****)

A particle P is at rest at some point B.

At time $t = 0$ s, P starts moving with simple harmonic motion on a straight line, taking $\frac{1}{3}\pi$ s to return to B for the first time. The maximum speed of P is 3.6 ms^{-1}.

a) Determine the amplitude of the motion.

b) Calculate the speed of the particle 1 s after leaving B.

c) Find the values of t, for $0 < t < 1$, so that the **speed** of P is the same as that found in part (b), giving the answers correct to three decimal places.

$\boxed{a = 0.6 \text{ m}}$, $\boxed{v \approx 1.01 \text{ ms}^{-1}}$, $\boxed{t \approx 0.047,\ 0.476,\ 0.571}$

(a) A O B

(−a) (+a)

$T = \frac{\pi}{3}$

$\frac{2\pi}{\omega} = \frac{\pi}{3}$

$\frac{2}{\omega} = \frac{1}{3}$

$\omega = 6$

$V_{MAX} = \omega a$

$3.6 = 6a$

$a = 0.6 \text{ m}$

(b) USING $x = a\cos\omega t$ (START AT B)

$x = 0.6\cos 6t$

$\dot{x} = -3.6\sin 6t$

when $t=1$ $\quad v \approx 1.01 \text{ ms}^{-1}$

(c) WHEN $v \simeq 1.01$

$-3.6\sin 6t \approx 1.01$

$\sin 6t = \sin 6$

$\begin{cases} 6t = 6 \pm 2n\pi \\ 6t = (\pi - 6) \pm 2n\pi \end{cases}$ $n = 0,1,2,3,\ldots$

$\begin{cases} t = 1 \pm \frac{n\pi}{3} & \text{(I)} \\ t = \frac{\pi}{6} - 1 \pm \frac{n\pi}{3} & \text{(II)} \end{cases}$

$\therefore t = 0.5708\ldots$ FROM (II)

WHEN $v \simeq -1.01$

$-3.6\sin 6t \simeq -1.01$

$\sin 6t = \sin(-6)$

$\begin{cases} 6t = -6 \pm 2n\pi \\ 6t = \pi + 6 \pm 2n\pi \end{cases}$ $n = 0,1,2,3$

$\begin{cases} t = -1 \pm \frac{n\pi}{3} & \text{(III)} \\ t = \frac{\pi}{6} + 1 \pm \frac{n\pi}{3} & \text{(IV)} \end{cases}$

$\therefore t = 0.0472\ldots$ FROM (III)

$t = 0.4764\ldots$ FROM (IV)

Thus $t \simeq 0.047, 0.476, 0.571$

Question 24 (****)

Three points A, O and B lie in that order on a straight line.

Two particles, P_1 and P_2, are moving on this line with simple harmonic motion between A and B, where O is the centre of the motion.

At time $t = 0$ s, P_1 is observed at the midpoint of OB moving towards B.

The subsequent displacement of P_1 from O is given by

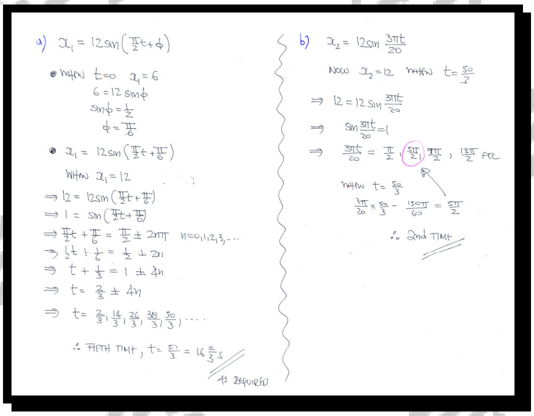
$$x_1 = 12\sin\left(\frac{\pi t}{2} + \varphi\right), \quad 0 < \varphi < \frac{\pi}{2}.$$

a) Show that P_1 arrives at B for the **fifth** time when $t = 16\frac{2}{3}$ s.

At time $t = 0$ s, P_2 is observed passing through O moving towards B. When P_1 arrives at B for the **fifth** time, P_2 also arrives at B for the k^{th} time, for $t > 0$.

b) Determine by calculation the value of k.

$\boxed{k = 2}$



a) $x_1 = 12\sin\left(\frac{\pi}{2}t + \phi\right)$

• WHEN $t = 0$ $x_1 = 6$

$6 = 12\sin\phi$

$\sin\phi = \frac{1}{2}$

$\phi = \frac{\pi}{6}$

• $x_1 = 12\sin\left(\frac{\pi}{2}t + \frac{\pi}{6}\right)$

WHEN $x_1 = 12$

$\Rightarrow 12 = 12\sin\left(\frac{\pi}{2}t + \frac{\pi}{6}\right)$

$\Rightarrow 1 = \sin\left(\frac{\pi}{2}t + \frac{\pi}{6}\right)$

$\Rightarrow \frac{\pi}{2}t + \frac{\pi}{6} = \frac{\pi}{2} \pm 2n\pi \quad n = 0, 1, 2, 3, \ldots$

$\Rightarrow \frac{1}{2}t + \frac{1}{6} = \frac{1}{2} \pm 2n$

$\Rightarrow t + \frac{1}{3} = 1 \pm 4n$

$\Rightarrow t = \frac{2}{3} \pm 4n$

$\Rightarrow t = \frac{2}{3}, \frac{14}{3}, \frac{26}{3}, \frac{38}{3}, \frac{50}{3}, \ldots$

$\therefore$ FIFTH TIME, $t = \frac{50}{3} = 16\frac{2}{3}$ s

AS REQUIRED

b) $x_2 = 12\sin\frac{3\pi t}{20}$

NOW $x_2 = 12$ WHEN $t = \frac{50}{3}$

$\Rightarrow 12 = 12\sin\frac{3\pi t}{20}$

$\Rightarrow \sin\frac{3\pi t}{20} = 1$

$\Rightarrow \frac{3\pi t}{20} = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \frac{13\pi}{2}$ ETC

WHEN $t = \frac{50}{3}$

$\frac{3\pi}{20} \times \frac{50}{3} = \frac{150\pi}{60} = \frac{5\pi}{2}$

$\therefore$ 2nd TIME

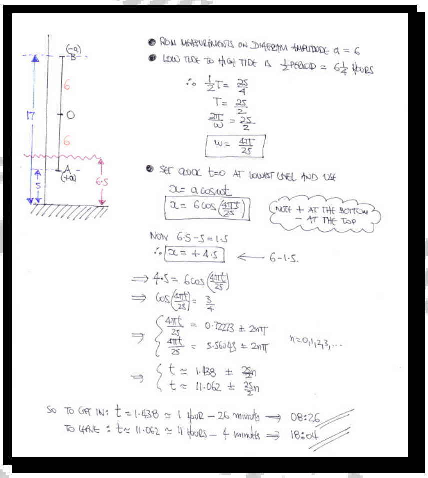
Question 25 (****)

The level of the sea in a harbour is assumed to rise and fall with simple harmonic motion. On a certain day low tide occurs at 07.00 hours when the depth of the sea will be 5 m. The next high tide will occur at 13.15 hours when the depth of the sea will be 17 m.

A ship wishes to enter the harbour that day and needs a minimum sea depth of 6.5 m.

Calculate, to the nearest minute, the earliest time it can enter the harbour on this day and the time by which it must leave.

$\boxed{08:26}$, $\boxed{18:04}$



Question 26 (****)

The level of the sea in a harbour is assumed to rise and fall with simple harmonic motion. On a certain day low tide occurs at 15.00 hours when the depth of the sea will be 8 m. The next high tide will occur at 03.30 hours when the depth of the sea will be 18 m.

A ship wishes to enter the harbour that day and needs a minimum sea depth of 12 m.

Calculate, to the nearest minute, the earliest time it can enter the harbour and the time by which it must leave.

$\boxed{20:27}$, $\boxed{10:33}$

D=18 — H (−5)
5m
D=13 — O
5m
D=8 — L (+5)

$\boxed{a=5}$, HALF PERIOD IS $12\frac{1}{2}$ HOURS
PERIOD IS 25 HOURS

$$\frac{2\pi}{\omega} = 25$$
$$\boxed{\omega = \frac{2\pi}{25}}$$

AT $t=0$ (13:00) LOW TIDE

$\Longrightarrow x = a\cos\omega t$, WITH TIME STARTING AT $\underline{\underline{+a}}$

$\Longrightarrow x = 5\cos\frac{2\pi t}{25}$

NOW DEPTH OF 12m IS AT DISPLACEMENT +1

$\Longrightarrow 1 = 5\cos\frac{2\pi t}{25}$

$\Longrightarrow \cos\frac{2\pi t}{25} = \frac{1}{5}$

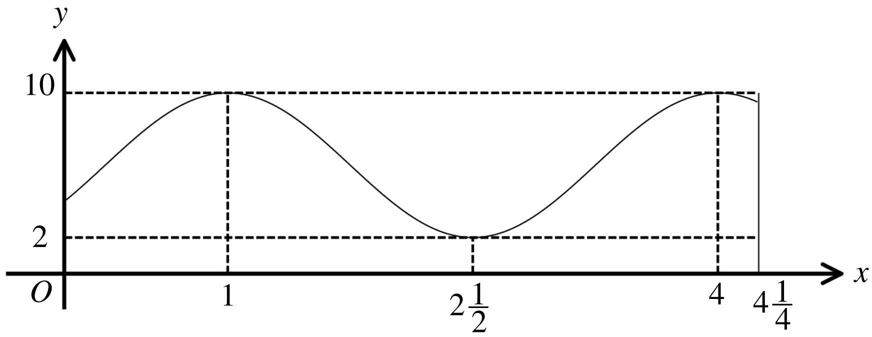
$$\Longrightarrow \frac{2\pi t}{25} = \begin{cases} \arccos\frac{1}{5} \pm 2n\pi \\ \left(2\pi - \arccos\frac{1}{5}\right) \pm 2n\pi \end{cases} \quad n = 0,1,2,3\ldots$$

$$\Longrightarrow t = \begin{cases} \frac{25}{2\pi}\arccos\frac{1}{5} \pm 25n \\ \frac{25}{2\pi}\left(2\pi - \arccos\frac{1}{5}\right) \pm 25n \end{cases}$$

$\therefore\ t_1 = \frac{25}{2\pi}\arccos\frac{1}{5} \approx 5.44882\ldots \approx 20:27$

$t_2 = 25 - \frac{25}{2\pi}\arccos\frac{1}{5} \approx 19.5511\ldots \approx 10:33$

Question 27 (****)



The graph above shows the height, y m, of a particle P at time t s, given by

$$y = A\sin(\omega t - \varphi) + B,$$

where A, B, ω and φ are positive constants.

a) Show algebraically that P is moving with simple harmonic motion.

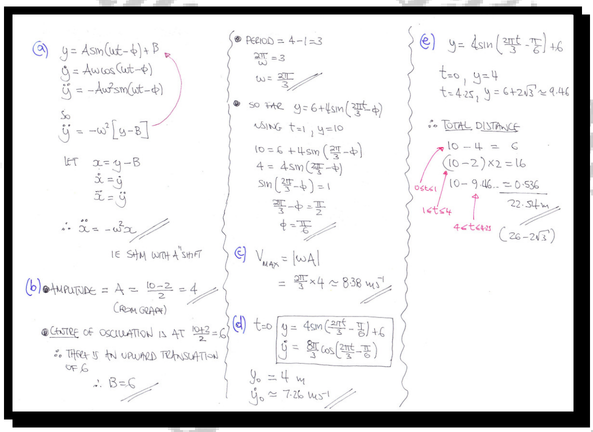
b) Determine the exact values, where appropriate, of A, B, ω and φ.

c) Calculate the maximum speed of P.

d) Find the initial height and the initial velocity of P.

e) Calculate the total distance travelled by P, for $0 \le t \le 14$.

$\boxed{A=4}$, $\boxed{B=6}$, $\boxed{\omega = \frac{2\pi}{3}}$, $\boxed{\varphi = \frac{\pi}{6}}$, $\boxed{v_{\text{max}} \approx 8.38 \text{ ms}^{-1}}$, $\boxed{y_0 = 4 \text{ m}}$, $\boxed{v_0 \approx 7.26 \text{ ms}^{-1}}$, $\boxed{d \approx 22.54 \text{ m}}$



Question 28 (****+)

A particle is moving on the x axis and its speed, v ms^{-1}, is given by

$$\frac{1}{2}v = \sqrt{8 - 2x - x^2}\,,\ \ x_1 \le x \le x_2\,,$$

where x is the position of the particle on the x axis.

a) Show that the motion of the particle is simple harmonic.

b) Determine the value of x_1 and the value of x_2.

At time $t = 0$, the particle is observed to be 1 m from the centre of the oscillation and moving away from the centre of the oscillation.

At time $t = T$, the particle is observed to be 2 m from the centre of the oscillation for the **third** time.

c) Calculate the value of T.

Full justification for the answer to part* (c) *must be shown.

$\boxed{}$, $\boxed{x_1 = -4,\ x_2 = 2}$, $\boxed{t = 1.77 \text{ s}}$

a) DIFFERENTIATE WITH RESPECT TO x, AFTER SQUARING

$\Rightarrow \frac{1}{2}v = \sqrt{8-2x-x^2}$

$\Rightarrow \frac{1}{4}v^2 = 8-2x-x^2$

$\Rightarrow \frac{1}{2}v\frac{dv}{dx} = -2-2x$

$\Rightarrow v\frac{dv}{dx} = -4-4x$

$\Rightarrow \frac{d^2x}{dt^2} = -4(x+1)$

$\Rightarrow \ddot{x} = -4(x+1)$

$\ddot{x} = \frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{dv}{dx}\times\frac{dx}{dt}$

v

NOW LET $y = x+1$ & $\dot{y} = \dot{x}$ & $\ddot{y} = \ddot{x}$

$\ddot{y} = -4y$

$\therefore$ S.H.M WITH CENTRE $x=-1$, $\omega^2 = 4$

b) NOW THE CENTRE OF THE OSCILLATION IS AT $x=-1$

$\frac{1}{2}V_{MAX} = \sqrt{8-2(-1)-(-1)^2}$

$\frac{1}{2}V_{MAX} = \sqrt{8+2-1}$

$\frac{1}{2}V_{MAX} = 3$

$V_{MAX} = 6$

BUT $V_{MAX} = \omega a$

$\Rightarrow 6 = 2a$ $\quad(\omega^2=4)$

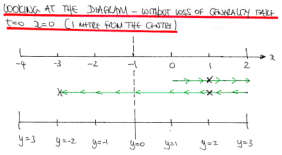
$\Rightarrow a = 3$

THUS CENTRE IS AT $x=-1$ & THE AMPLITUDE IS 3.

$\therefore\ -4 \le x \le 2$

IE $x_1 = -4$
$x_2 = 2$

c) LOOKING AT THE DIAGRAM – WITHOUT LOSS OF GENERALITY TAKE
$t=0$ $x=0$ (1 METRE FROM THE CENTRE)



USING $y = 3\sin(2t+\phi)$

ω

$t=0$ $y=1$
$1 = 3\sin\phi$
$\sin\phi = \frac{1}{3}$
$\phi = \arcsin\frac{1}{3}$

SOLVE FOR $y=-2$

$-2 = 3\sin(2t+\phi)$

$\sin(2t+\phi) = -\frac{2}{3}$

$\begin{cases} 2t+\phi = \arcsin(-\frac{2}{3}) \pm 2n\pi \\ 2t+\phi = \pi - \arcsin(-\frac{2}{3}) \pm 2n\pi \end{cases}$ $\quad n = 0,1,2,3$

$\begin{cases} 2t = \arcsin(-\frac{2}{3}) - \phi \pm 2n\pi \\ 2t = \pi - \arcsin(-\frac{2}{3}) - \phi \pm 2n\pi \end{cases}$

$\begin{cases} 2t = -\arcsin(\frac{2}{3}) - \arcsin(\frac{1}{3}) \pm 2n\pi \\ 2t = \pi + \arcsin(\frac{2}{3}) - \arcsin(\frac{1}{3}) \pm 2n\pi \end{cases}$

$\begin{cases} t = -\frac{1}{2}(\arcsin\frac{2}{3} + \arcsin\frac{1}{3}) \pm n\pi \\ t = \frac{1}{2}(\pi + \arcsin\frac{2}{3} - \arcsin\frac{1}{3}) \pm n\pi \end{cases}$

$t_1 = 1.7657417\ldots$

$t_2 = 2.606810\ldots$

$t_3 = 4.90733\ldots$

BUT NOTE THAT THE THIRD TIME IT IS 2 METRES FROM THE CENTRE OF THE OSCILLATION IS THE FIRST TIME WHEN $y=-2$

$\therefore\ t = 1.77$ s

SIMPLE HARMONIC MOTION DYNAMICS

Question 1 (**)

A particle P of mass 0.2 kg is attached to one end of a light elastic string of natural length 0.8 m and modulus of elasticity 16 N.

The other end of the string is attached to a fixed point A on a smooth horizontal surface on which P rests.

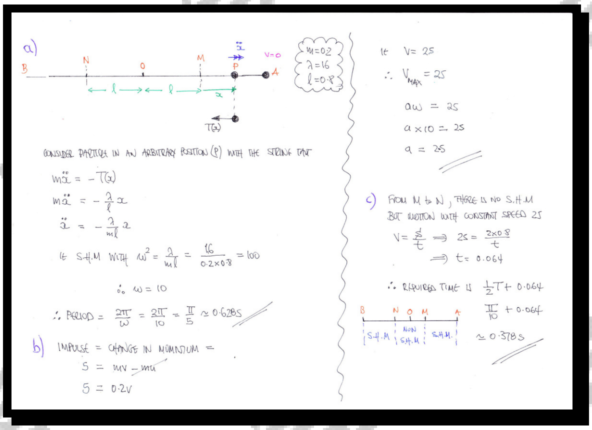
With the string at natural length, P receives an impulse of magnitude 5 Ns, in the direction AP.

a) Show that in the subsequent motion, while the string is taut, the motion of P is simple harmonic.

b) Determine its amplitude of the motion.

c) Find the time it takes P to travel between the extreme points of its motion.

$\boxed{a = 2.5 \text{ m}}$, $\boxed{t \approx 0.378 \text{ s}}$



Question 2 (**)

A particle P of mass 0.7 kg is attached to one end of a light elastic spring of natural length 0.6 m and modulus of elasticity 168 N.

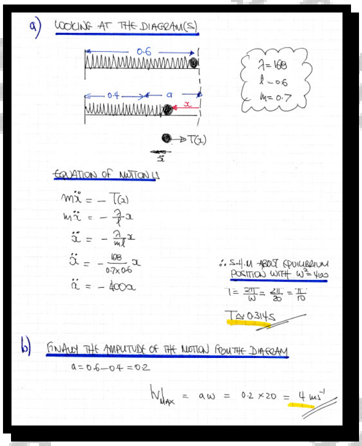
The other end of the spring is attached to a fixed point A on a smooth horizontal surface on which P rests.

P is pushed in the direction PA so that the spring has length 0.4 m, and released from rest.

a) Show that the subsequent motion of P is simple harmonic and state its period.

b) Determine the greatest speed of P.

$\boxed{\quad}$, $\boxed{\tau = \frac{\pi}{10} \approx 0.314 \text{ s}}$, $\boxed{\text{max speed} = 4 \text{ ms}^{-1}}$


a) LOOKING AT THE DIAGRAM(S)

$\lambda = 168$, $l = 0.6$, $m = 0.7$

EQUATION OF MOTION

$m\ddot{x} = -T(x)$

$m\ddot{x} = -\frac{\lambda}{l}x$

$\ddot{x} = -\frac{\lambda}{ml}x$

$\ddot{x} = -\frac{168}{0.7 \times 0.6}x$

$\ddot{x} = -400x$

$\therefore$ S.H.M ABOUT EQUILIBRIUM POSITION WITH $\omega^2 = 400$

$T = \frac{2\pi}{\omega} = \frac{2\pi}{20} = \frac{\pi}{10}$

$T \approx 0.314$ s

b) FINALLY THE AMPLITUDE OF THE MOTION FROM THE DIAGRAM

$a = 0.6 - 0.4 = 0.2$

$V_{max} = a\omega = 0.2 \times 20 = 4 \text{ ms}^{-1}$

Question 3 (**+)

A particle of mass 2 kg is attached to one end of a light elastic string of natural length 3 m and the other end is attached to a fixed point A. The particle hangs in equilibrium at some point E, where $|AE| = 3.7$ m.

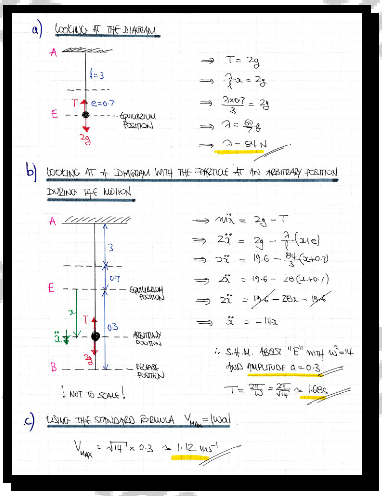
a) Find the modulus of elasticity of the string.

The particle is pulled vertically downwards from the point E to the point B, where $|AB| = 4$ m, and it is released from rest.

b) Show that in the subsequent motion, the particle moves with simple harmonic motion and determine its amplitude and its period.

c) Calculate the maximum speed of the particle during its motion.

$\boxed{}$, $\boxed{\lambda = 84 \text{ N}}$, $\boxed{T = \dfrac{2\pi}{\sqrt{14}} \approx 1.68 \text{ s}}$, $\boxed{a = 0.3 \text{ m}}$, $\boxed{v_{\max} \approx 1.12 \text{ ms}^{-1}}$


a) LOOKING AT THE DIAGRAM

A, $l = 3$, T, $e = 0.7$, E, EQUILIBRIUM POSITION, 2g

$\Rightarrow T = 2g$

$\Rightarrow \frac{\lambda}{l}x = 2g$

$\Rightarrow \frac{\lambda \times 0.7}{3} = 2g$

$\Rightarrow \lambda = \frac{60}{7}g$

$\Rightarrow \lambda = 84\text{ N}$

b) LOOKING AT A DIAGRAM WITH THE PARTICLE AT AN ARBITRARY POSITION DURING THE MOTION

A, 3, 0.7, E, EQUILIBRIUM POSITION, x, T, 0.3, $\ddot{x}$, ARBITRARY POSITION, 2g, B, RELEASE POSITION

! NOT TO SCALE !

$\Rightarrow m\ddot{x} = 2g - T$

$\Rightarrow 2\ddot{x} = 2g - \frac{\lambda}{l}(x+e)$

$\Rightarrow 2\ddot{x} = 19.6 - \frac{84}{3}(x+0.7)$

$\Rightarrow 2\ddot{x} = 19.6 - 28(x+0.7)$

$\Rightarrow 2\ddot{x} = \cancel{19.6} - 28x - \cancel{19.6}$

$\Rightarrow \ddot{x} = -14x$

$\therefore$ S.H.M. ABOUT "E" WITH $\omega^2 = 14$ AND AMPLITUDE $a = 0.3$

$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{14}} \approx 1.68\text{s}$

c) USING THE STANDARD FORMULA $V_{MAX} = |\omega a|$

$V_{MAX} = \sqrt{14} \times 0.3 \approx 1.12 \text{ ms}^{-1}$

Question 4 (**+)

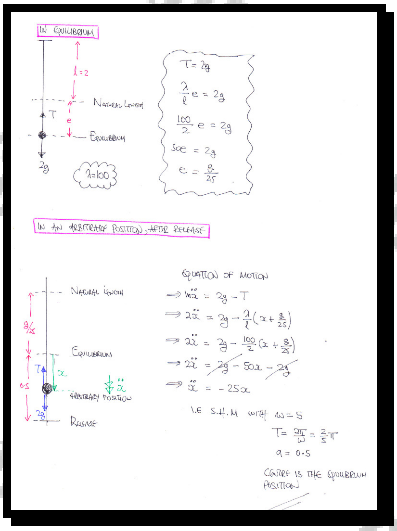
A particle P of mass 2 kg is attached to the free end of a light elastic spring of natural length 0.8 m and modulus of elasticity 100 N.

P is in equilibrium, hanging vertically from a fixed point A.

P is pulled vertically downwards a further 0.5 m and released from rest.

Show that in the subsequent motion, P moves with simple harmonic motion, and determine the amplitude, the centre and the period of the oscillations.

$$\boxed{a = 0.5 \text{ m}}, \quad \boxed{\tau = \frac{2\pi}{5} \text{ s}}$$



Question 5 (***)

A particle P of mass 1.5 kg is attached to the free end of a light elastic spring of natural length 2 m and modulus of elasticity λ N.

P is in equilibrium, hanging vertically from a fixed point A.

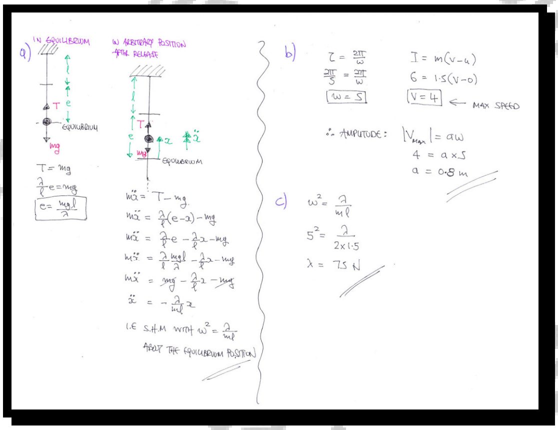
An impulse of magnitude 6 Ns is given to P, in a direction parallel to the spring towards A.

a) Show that in the subsequent motion, P moves with simple harmonic motion.

b) Given further that the period of the oscillations is $\frac{2}{5}\pi$ s, find the amplitude of the motion.

c) Determine the value of λ.

$\boxed{a = 0.8 \text{ m}}$, $\boxed{\lambda = 75}$



Question 6 (***+)

A particle P of mass 0.5 kg is attached to the free end of a light elastic string of natural length 0.8 m and modulus of elasticity 90 N.

The particle is in equilibrium, hanging vertically from a fixed point A.

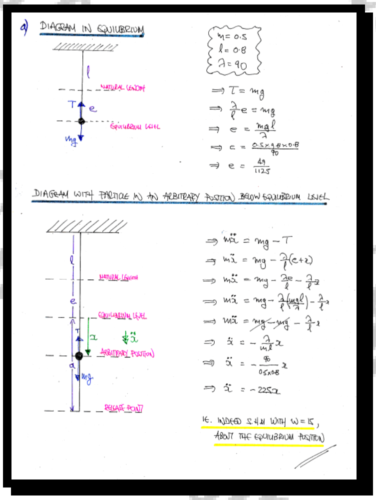
The particle is pulled vertically downwards a further 0.5 m and released from rest.

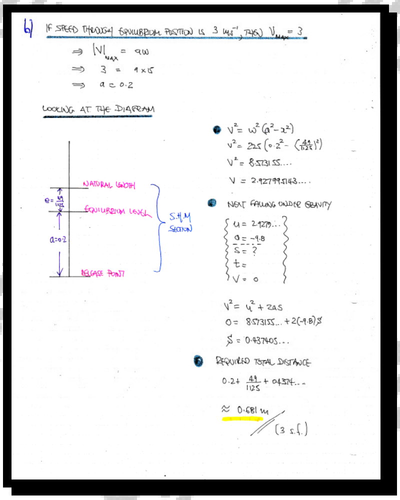
a) Show that in the subsequent motion, P moves with simple harmonic motion.

The particle is next pulled vertically downwards a further distance a m below the equilibrium position and released from rest.

b) Given that P passes through the equilibrium position with speed 3 ms^{-1}, calculate the distance P covers until it first comes to instantaneous rest.

$\boxed{}$, $\boxed{d \approx 0.681 \text{ m}}$





Question 7 (***+)

Two fixed points A and B lie on a smooth horizontal surface, so that the distance between them is $2.5\ \text{m}$.

A particle P of mass $0.5\ \text{kg}$ is attached to one end of a light elastic string S_A and the other end of S_A is attached to A.

A second light elastic string S_B is also attached to P while the other end of S_B is attached to B.

Both strings are identical in every aspect, each of natural length $0.75\ \text{m}$ and modulus of elasticity $24.5\ \text{N}$.

The point C lies on the straight line segment AB, so that $AC = 1\ \text{m}$.

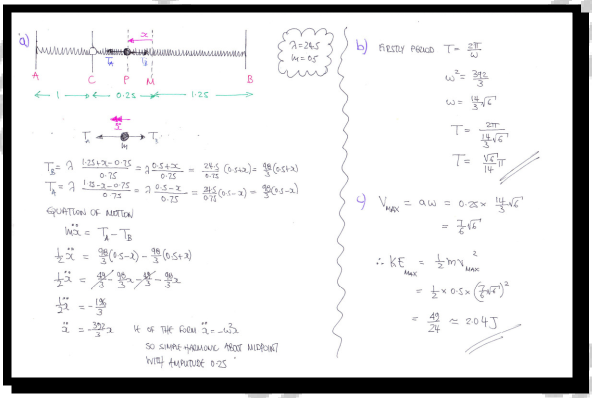
At time $t = 0\ \text{s}$, P is released from rest from C and moves without any resistance.

a) Show that in the subsequent motion, P moves with simple harmonic motion.

b) Determine the period of the motion.

c) Calculate the maximum kinetic energy of P.

$$\boxed{\tau = \frac{\sqrt{6}}{14}\pi \approx 0.550\ \text{s}}, \quad \boxed{K.E_{\text{max}} = 2.04\ \text{J}}$$



Question 8 (****)

A light elastic string, of natural length a and modulus of elasticity $4mg$, has one end attached to a fixed point A and the other end is attached to a particle P of mass m.

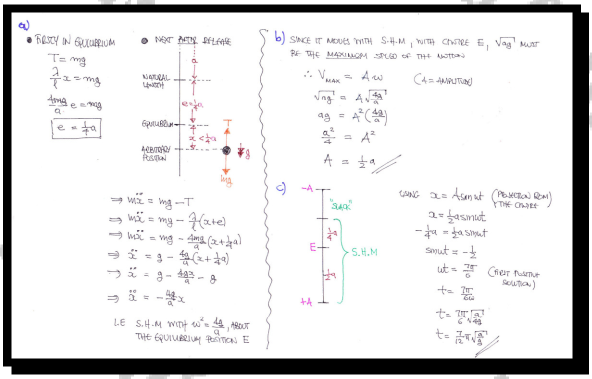
Initially P hangs freely at rest in equilibrium at the point E. At time $t = 0$, P is projected vertically downwards from E with speed $\sqrt{ag}$.

a) Prove that, while the string is taut, P moves with simple harmonic motion.

b) Find, in terms of a, the amplitude of the simple harmonic motion.

c) Determine, in terms of a and g, the time at which the string first goes slack.

$$\boxed{\text{amplitude} = \frac{1}{2}a}, \quad \boxed{t = \frac{7\pi}{12}\sqrt{\frac{a}{g}}}$$



a)

• FIRSTLY IN EQUILIBRIUM

$T = mg$

$\frac{\lambda}{l}x = mg$

$\frac{4mg}{a}e = mg$

$\boxed{e = \frac{1}{4}a}$

• NEXT AFTER RELEASE

$\Rightarrow m\ddot{x} = mg - T$

$\Rightarrow m\ddot{x} = mg - \frac{\lambda}{l}(x+e)$

$\Rightarrow m\ddot{x} = mg - \frac{4mg}{a}(x + \frac{1}{4}a)$

$\Rightarrow \ddot{x} = g - \frac{4g}{a}(x + \frac{1}{4}a)$

$\Rightarrow \ddot{x} = g - \frac{4gx}{a} - g$

$\Rightarrow \ddot{x} = -\frac{4g}{a}x$

I.E S.H.M WITH $\omega^2 = \frac{4g}{a}$, ABOUT THE EQUILIBRIUM POSITION E

b) SINCE IT MOVES WITH S.H.M, WITH CENTRE E, $\sqrt{ag}$ MUST BE THE MAXIMUM SPEED OF THE MOTION

$\therefore V_{MAX} = A\omega$ (A = AMPLITUDE)

$\sqrt{ag} = A\sqrt{\frac{4g}{a}}$

$ag = A^2\left(\frac{4g}{a}\right)$

$\frac{a^2}{4} = A^2$

$A = \frac{1}{2}a$

c)

USING $x = A\sin\omega t$ (PROJECTION FROM THE CENTRE)

$x = \frac{1}{2}a\sin\omega t$

$-\frac{1}{4}a = \frac{1}{2}a\sin\omega t$

$\sin\omega t = -\frac{1}{2}$

$\omega t = \frac{7\pi}{6}$ (FIRST POSITIVE SOLUTION)

$t = \frac{7\pi}{6\omega}$

$t = \frac{7\pi}{6}\sqrt{\frac{a}{4g}}$

$t = \frac{7}{12}\pi\sqrt{\frac{a}{g}}$

Question 9 (****)

A smooth hollow narrow tube of length 1.6 m has one open end and one closed end.

The tube is fixed in a vertical position with the closed end at the bottom.

A light elastic **spring** of natural length 1.6 m and modulus of elasticity 98 N is placed inside the tube.

The spring has one end attached to a fixed point on the closed end of the tube and the other end of the spring is attached to a particle of mass 1.25 kg.

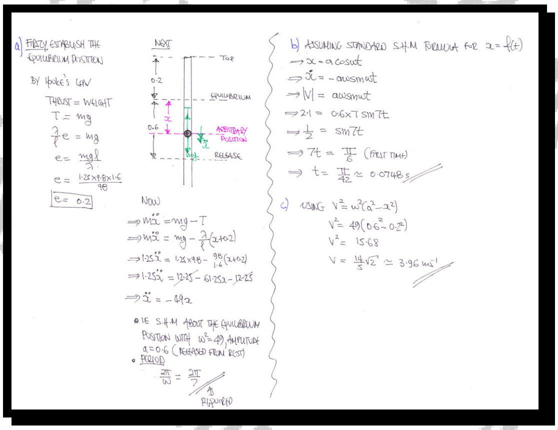
The particle is next held inside the tube at a distance 0.8 m below the open end of the tube and released from rest.

a) Show clearly that after release the motion of the particle is simple harmonic with period $\frac{2}{7}\pi \text{ s}$.

b) Calculate the time for the particle to first attain a speed of 2.1 ms^{-1}.

c) Find the speed with which the particle passes through the open end of the tube.

$\boxed{t = \frac{1}{42}\pi \approx 0.0748 \text{ s}}$, $\boxed{v = \frac{14}{5}\sqrt{2} \approx 3.96 \text{ ms}^{-1}}$



Question 10 (****)

A particle is attached to one end of a light elastic spring of natural length 3.6 m and the other end of the spring is attached to a fixed point A. The particle is hanging freely in equilibrium at the point E, where $|AE| = 5.4$ m.

The particle is then pulled vertically downwards from E to the point B, where $|AB| = 6.48$ m, and released from rest.

a) Show that the particle moves in simple harmonic motion, stating the centre of the motion.

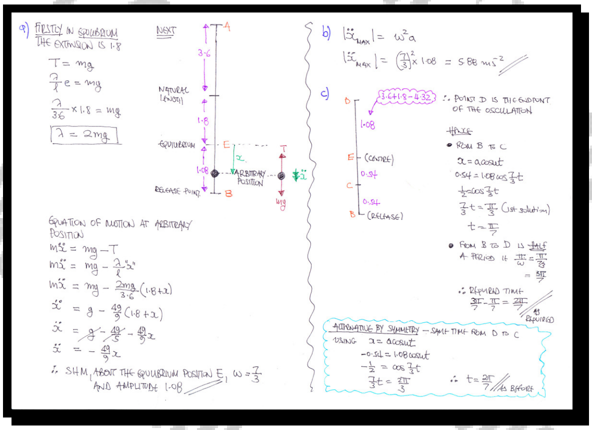
b) Find the greatest magnitude of the acceleration of the particle.

The point C is the midpoint of EB.

The point D lies vertically below A, where $|AD| = 4.32$ m.

c) Show further that the time taken by the particle to move directly from C to D is $\frac{2}{7}\pi$.

$\boxed{|\ddot{x}_{\text{max}}| = 5.88 \text{ ms}^{-2}}$



Question 11 (****)

Two fixed points A and B lie on a smooth horizontal surface, so that the distance between them is 4.2 m.

A particle P of mass 0.25 kg is attached to one end of a light elastic string S_A and the other end of S_A is attached to A.

A second light elastic string S_B is also attached to P while the other end of S_B is attached to B.

The natural length of S_A is 1.8 m and its modulus of elasticity is 20 N, while the natural length of S_B is 1.2 m and its modulus of elasticity is 40 N.

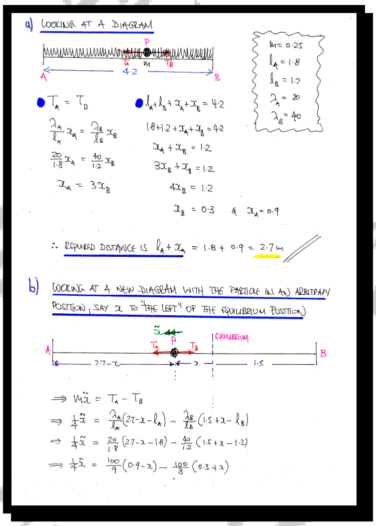
P rests in equilibrium at some point O between A and B.

a) Show by calculation that $|OA| = 2.7$.

P is then displaced from its equilibrium position O to a new position C, and released from rest.

b) Given that when P is at C, both strings are taut, show further that in the subsequent motion, P moves with simple harmonic motion, stating its period.

$\boxed{}$, $\boxed{t = \dfrac{3\pi}{20} \approx 0.471 \text{ s}}$



a) LOOKING AT A DIAGRAM

$m = 0.25$, $l_A = 1.8$, $l_B = 1.2$, $\lambda_A = 20$, $\lambda_B = 40$

- $T_A = T_B$

$\frac{\lambda_A}{l_A} x_A = \frac{\lambda_B}{l_B} x_B$

$\frac{20}{1.8} x_A = \frac{40}{1.2} x_B$

$x_A = 3x_B$

- $l_A + l_B + x_A + x_B = 4.2$

$1.8 + 1.2 + x_A + x_B = 4.2$

$x_A + x_B = 1.2$

$3x_B + x_B = 1.2$

$4x_B = 1.2$

$x_B = 0.3$ & $x_A = 0.9$

∴ REQUIRED DISTANCE IS $l_A + x_A = 1.8 + 0.9 = 2.7$ m //

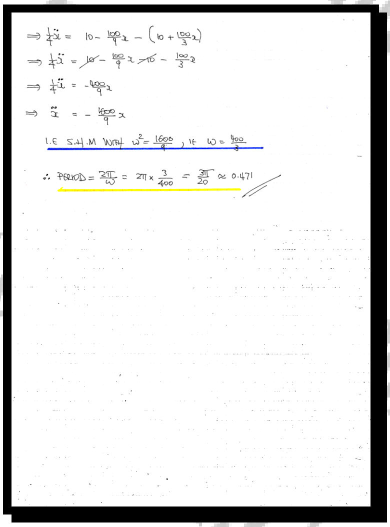
b) LOOKING AT A NEW DIAGRAM WITH THE PARTICLE IN AN ARBITRARY POSITION, SAY x TO "THE LEFT" OF THE EQUILIBRIUM POSITION)

$\Rightarrow m\ddot{x} = T_A - T_B$

$\Rightarrow \frac{1}{4}\ddot{x} = \frac{\lambda_A}{l_A}(2.7 - x - l_A) - \frac{\lambda_B}{l_B}(1.5 + x - l_B)$

$\Rightarrow \frac{1}{4}\ddot{x} = \frac{20}{1.8}(2.7 - x - 1.8) - \frac{40}{1.2}(1.5 + x - 1.2)$

$\Rightarrow \frac{1}{4}\ddot{x} = \frac{100}{9}(0.9 - x) - \frac{100}{3}(0.3 + x)$



$\Rightarrow \frac{1}{4}\ddot{x} = 10 - \frac{100}{9}x - \left(10 + \frac{100}{3}x\right)$

$\Rightarrow \frac{1}{4}\ddot{x} = \cancel{10} - \frac{100}{9}x \cancel{-10} - \frac{100}{3}x$

$\Rightarrow \frac{1}{4}\ddot{x} = -\frac{400}{9}x$

$\Rightarrow \ddot{x} = -\frac{1600}{9}x$

I.E S.H.M WITH $\omega^2 = \frac{1600}{9}$, IE $\omega = \frac{400}{3}$

∴ PERIOD $= \frac{2\pi}{\omega} = 2\pi \times \frac{3}{400} = \frac{3\pi}{20} \approx 0.471$ //

Question 12 (****)

Two fixed points A and B lie on a smooth horizontal surface, so that the distance between them is 4.13 m.

A particle P of mass m kg is attached to one end of a light elastic spring S_A and the other end of S_A is attached to A.

A second light elastic spring S_B is also attached to P while the other end of S_B is attached to B.

The natural length of S_A is 0.8 m and its modulus of elasticity is 120 N, while the natural length of S_B is 1.5 m and its modulus of elasticity is 80 N.

The particle rests in equilibrium at some point O between A and B.

a) Show by calculation that $|OA| = 1.28$.

The point C lies on the straight line segment AOB, between A and O.

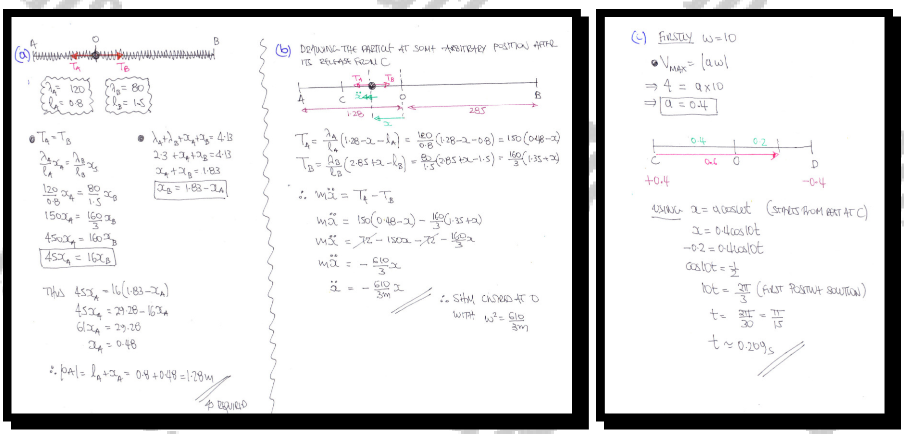
At time $t = 0$ s, P is released from rest from C and moves without any resistance.

b) Show that in the subsequent motion, P moves with simple harmonic motion.

The angular frequency of P is 10 Hz and its maximum speed is 4 ms^{-1}.

c) Determine the time taken for P to travel a distance of 0.6 m from C.

$$\boxed{t = \frac{\pi}{15} \approx 0.209 \text{ s}}$$



k

Question 13 (****)

A particle of mass 0.75 kg is attached to a fixed point A by a light elastic string of modulus of elasticity 78 N.

The particle is released from rest from A and falls vertically without any air resistance, coming to rest at a point C, 4 m below A.

a) Show by calculation that the natural length of the string is 2.6 m.

b) Show that when the extension in the string is x m

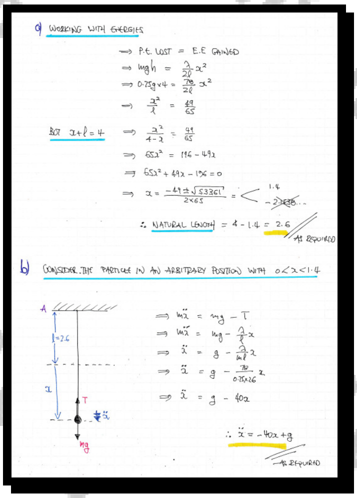
$$\frac{d^2x}{dt^2} = -40x + g \,.$$

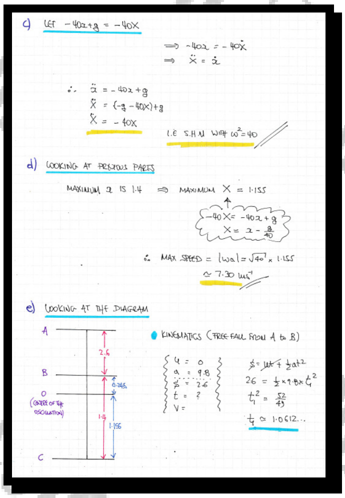
c) Use a suitable substitution to demonstrate that the above differential equation represents simple harmonic motion.

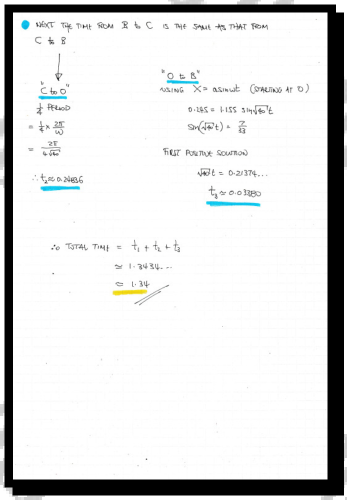
d) Determine the maximum speed of the particle during its motion.

e) Calculate, correct to 4 decimal places, the time it takes the particle to move from A to C.

$\boxed{v_{max} = 7.30 \text{ ms}^{-1}}$, $\boxed{t \approx 1.3434 \text{ s}}$







Question 14 (****)

Two fixed points A and B lie on a smooth horizontal surface, such that $|AB| = 5$ m. A particle P of mass 0.3 kg is attached to one end of a light elastic string S_A and the other end of S_A is attached to A.

A second light elastic string S_B is also attached to P while the other end of S_B is attached to B.

The natural length of S_A is 1 m and its modulus of elasticity is 90 N, while the natural length of S_B is 2 m and its modulus of elasticity is 60 N.

The particle rests in equilibrium at some point O between A and B.

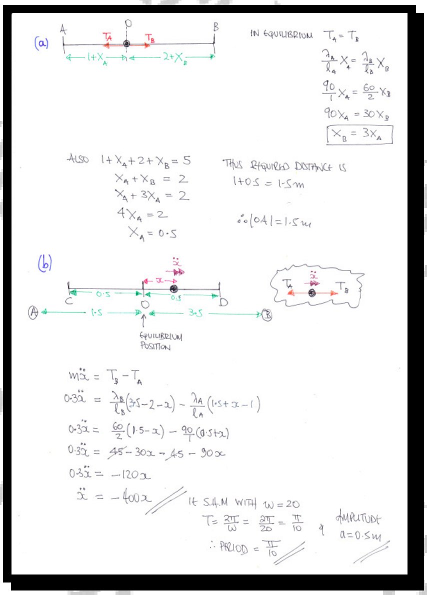
a) Determine the distance OA.

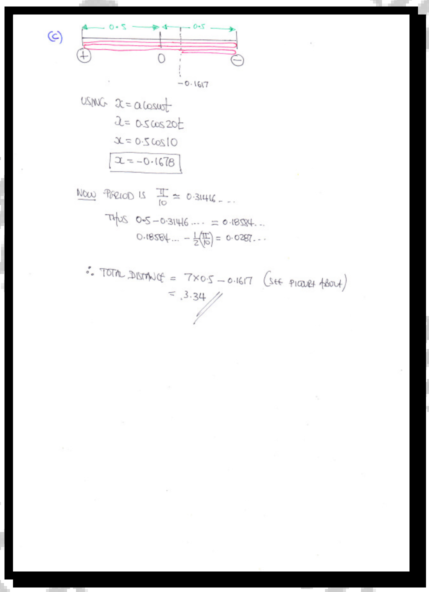
At time $t = 0$ s, P is released from rest from a point on the line segment AB such that AO is 1 m, and moves without any resistance.

b) Show that in the subsequent motion, P moves with simple harmonic motion and determine its amplitude and its period.

c) Calculate the total distance P covers in the first 0.5 s of its motion.

$\boxed{|OA| = 1.5 \text{ m}}$, $\boxed{a = 0.5 \text{ m}}$, $\boxed{T = \frac{\pi}{10} \text{ s}}$, $\boxed{d \approx 3.34 \text{ m}}$





Question 15 (****+)

A particle of mass m is attached to one end of a light elastic string of stiffness k, and the other end is attached to a fixed point A. The particle is hanging in equilibrium with the string in a vertical position. The particle is next pulled a vertical distance a below its equilibrium position and released from rest.

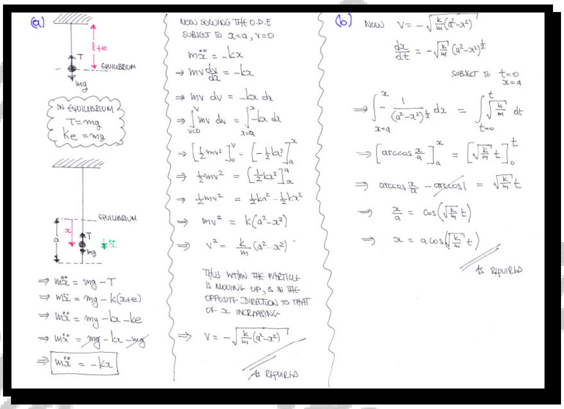
At time t, the displacement of the particle below its equilibrium position is x and the velocity of the particle is v.

Show, by forming and solving suitable differential equations, that while the particle is moving upwards with the string taut …

a) … $v = -\sqrt{\frac{k}{m}\left(a^2 - x^2\right)}$.

b) … $x = a\cos\left(\sqrt{\frac{k}{m}}\,t\right)$.

proof



(a)

$l+e$
EQUILIBRIUM
T
mg

IN EQUILIBRIUM
$T = mg$
$ke = mg$

EQUILIBRIUM
a
x
T
mg
$\ddot{x}$

$\Rightarrow m\ddot{x} = mg - T$
$\Rightarrow m\ddot{x} = mg - k(x+e)$
$\Rightarrow m\ddot{x} = mg - kx - ke$
$\Rightarrow m\ddot{x} = \cancel{mg} - kx - \cancel{mg}$
$\Rightarrow \boxed{m\ddot{x} = -kx}$

NOW SOLVING THE O.D.E SUBJECT TO $x=a$, $v=0$

$m\ddot{x} = -kx$
$\Rightarrow mv\frac{dv}{dx} = -kx$
$\Rightarrow mv\,dv = -kx\,dx$
$\Rightarrow \int_{v=0}^{v} mv\,dv = \int_{x=a}^{x} -kx\,dx$
$\Rightarrow \left[\frac{1}{2}mv^2\right]_0^v = \left[-\frac{1}{2}kx^2\right]_a^x$
$\Rightarrow \frac{1}{2}mv^2 = \left[\frac{1}{2}kx^2\right]_x^a$
$\Rightarrow \frac{1}{2}mv^2 = \frac{1}{2}ka^2 - \frac{1}{2}kx^2$
$\Rightarrow mv^2 = k(a^2 - x^2)$
$\Rightarrow v^2 = \frac{k}{m}(a^2 - x^2)$

THUS WHEN THE PARTICLE IS MOVING UP, IS IN THE OPPOSITE DIRECTION TO THAT OF x INCREASING

$\Rightarrow v = -\sqrt{\frac{k}{m}(a^2 - x^2)}$

AS REQUIRED

(b) NOW $v = -\sqrt{\frac{k}{m}(a^2 - x^2)}$

$\frac{dx}{dt} = -\sqrt{\frac{k}{m}}(a^2 - x^2)^{\frac{1}{2}}$

SUBJECT TO $t=0$, $x=a$

$\Rightarrow \int_{x=a}^{x} -\frac{1}{(a^2 - x^2)^{\frac{1}{2}}}\,dx = \int_{t=0}^{t} \sqrt{\frac{k}{m}}\,dt$
$\Rightarrow \left[\arccos\frac{x}{a}\right]_a^x = \left[\sqrt{\frac{k}{m}}\,t\right]_0^t$
$\Rightarrow \arccos\frac{x}{a} - \cancel{\arccos 1} = \sqrt{\frac{k}{m}}\,t$
$\Rightarrow \frac{x}{a} = \cos\left(\sqrt{\frac{k}{m}}\,t\right)$
$\Rightarrow x = a\cos\left(\sqrt{\frac{k}{m}}\,t\right)$

AS REQUIRED

Question 16 (****+)

A light elastic string with natural length 0.8 m, has one of its ends attached to a fixed point O on a smooth plane inclined at angle θ to the horizontal, where $\sin\theta = 0.75$. A particle of mass 2.5 kg is attached to the other end of the string.

The particle rests in equilibrium at the point A on the plane, where OA lies along a line of greatest slope with $|OA| = 1.2$ m.

The particle is then pulled down to a point B, where OAB is a straight line with $|OB| = 1.7$ m, and released from rest.

a) Show that, while the string remains taut, the particle is moving with simple harmonic motion.

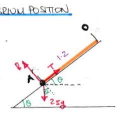
Give all the relevant details of this motion.

The point M is the midpoint of AB.

b) Calculate, correct to 4 decimal places, the time taken by the particle to move directly from M to the point where the string becomes slack for the first time.

$\boxed{}$, $\boxed{t \approx 0.3385}$

a) FIRST WE NEED TO FIND λ IN EQUILIBRIUM POSITION



$\Rightarrow T = 2.5g\sin\theta$

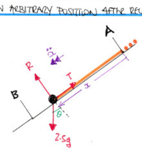
$\Rightarrow \frac{\lambda}{l}x = 2.5g \times \frac{3}{4}$

$\Rightarrow \frac{\lambda}{0.8}(1.2-0.8) = 18.375$

$\Rightarrow \lambda = 36.75\text{ N}$

$l = 0.8$ m
$m = 2.5$ kg
$\sin\alpha = \frac{3}{4}$

NEXT WE CONSIDER THE PARTICLE IN AN ARBITRARY POSITION AFTER RELEASE



$\Rightarrow m\ddot{x} = mg\sin\theta - T$

$\Rightarrow m\ddot{x} = mg\sin\theta - \frac{\lambda}{l}(x+0.4)$

$\Rightarrow \ddot{x} = g\sin\theta - \frac{\lambda}{ml}(x+0.4)$

$\Rightarrow \ddot{x} = \frac{3}{4}g - \frac{36.75}{2.5\times0.8}(x+0.4)$

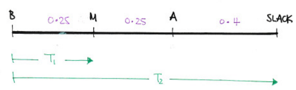
$\Rightarrow \ddot{x} = 7.35 - 18.375(x+0.4)$

$\Rightarrow \ddot{x} = \cancel{7.35} - 18.375x - \cancel{7.35}$

$\Rightarrow \ddot{x} = -18.375x$

S.H.M ABOUT A, WITH $\omega^2 = 18.375$, AND AMPLITUDE 0.5

b) LOOKING AT THE S.H.M KINEMATICS NEXT



USING THE START S.H.M EQUATION, MEASURING TIME FROM THE END-POINT, STARTING WITH PLUS AMPLITUDE

$\Rightarrow x = a\cos\omega t$

$\Rightarrow x = 0.5\cos\omega t$, $(\omega^2 = 18.375)$

- $0.25 = 0.5\cos\omega T_1$
  $0.5 = \cos\omega T_1$
  $\omega T_1 = \frac{\pi}{3}$
  $T_1 = \frac{\pi}{3\omega}$
  $T_1 \simeq 0.244295$

- $-0.4 = 0.5\cos\omega T_2$
  $-0.8 = \cos\omega T_2$
  $\omega T_2 = \arccos(-0.8)$
  $T_2 = \frac{\arccos(-0.8)}{\omega}$
  $T_2 = 0.582767\ldots$

$\therefore$ THE REQUIRED TIME IS $T_2 - T_1 = 0.3385$

Question 17 (****+)

Two fixed points A and B lie on a smooth horizontal surface, such that $|AB| = 7$ m. A particle P of mass 0.3 kg is attached to one end of a light elastic string S_A and the other end of S_A is attached to A.

A second light elastic string S_B is also attached to P while the other end of S_B is attached to B.

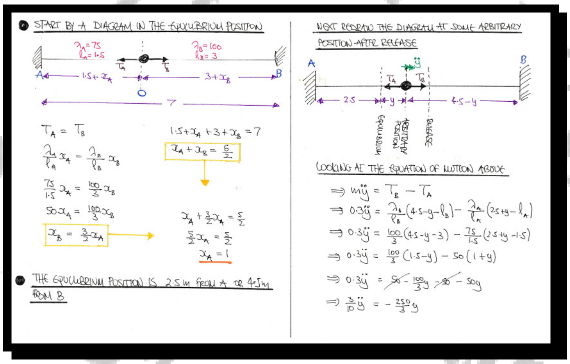
The natural length of S_A is 1.5 m and its modulus of elasticity is 75 N, while the natural length of S_B is 3 m and its modulus of elasticity is 100 N.

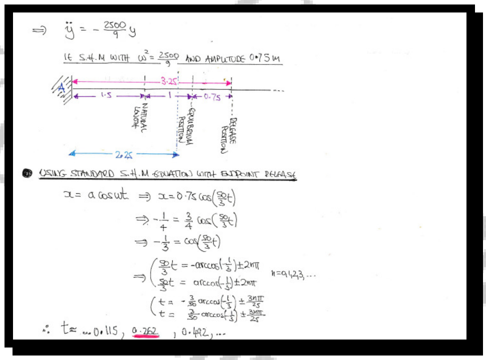
At time $t = 0$ s, P is released from rest from so that $|AP| = 3.25$ m.

At time $t = T$ s, P is moving towards B for the first time and $|AP| = 2.25$ m.

Determine the value of T.

$\boxed{}$, $\boxed{T \approx 0.262 \text{ s}}$





Question 18 (****+)

Two fixed points A and B lie on a smooth horizontal surface, such that $|AB| = 5$ m. A particle P of mass 0.5 kg is attached to one end of a light elastic string S_A and the other end of S_A is attached to A.

A second light elastic string S_B is also attached to P while the other end of S_B is attached to B.

The natural length of S_A is 1.5 m and its modulus of elasticity is 30 N, while the natural length of S_B is 0.8 m and its modulus of elasticity is 20 N.

At time $t = 0$ s, P is released from rest from so that S_A is at natural length, and moves without any resistance.

Calculate the length PB, when the particle next gets to instantaneous rest.

$\boxed{d \approx 0.481 \text{ m}}$