GRAVITATION PROBLEMS

Question 1 (**)

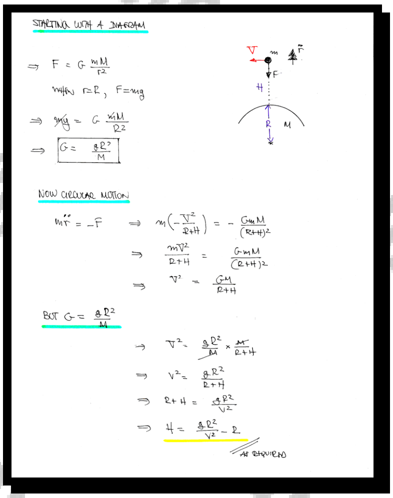
The weight of a satellite of mass M when it is on the surface of the Earth is Mg.

This satellite is moving with constant speed V in a circular orbit above the Earth's equator, at a height H above the surface of the Earth.

If the radius of the Earth is m, show that

$$H = \frac{gR^2}{V^2} - R,$$

$\boxed{}$, $\boxed{\text{proof}}$


STARTING WITH A DIAGRAM

$\Rightarrow F = G\frac{mM}{r^2}$

when $r=R$, $F=mg$

$\Rightarrow \cancel{m}g = G\frac{\cancel{m}M}{R^2}$

$\Rightarrow \boxed{G = \frac{gR^2}{M}}$

NOW CIRCULAR MOTION

$m\ddot{r} = -F \Rightarrow m\left(-\frac{V^2}{R+H}\right) = -\frac{GmM}{(R+H)^2}$

$\Rightarrow \frac{mV^2}{R+H} = \frac{GmM}{(R+H)^2}$

$\Rightarrow V^2 = \frac{GM}{R+H}$

BUT $G = \frac{gR^2}{M}$

$\Rightarrow V^2 = \frac{gR^2}{\cancel{M}} \times \frac{\cancel{M}}{R+H}$

$\Rightarrow V^2 = \frac{gR^2}{R+H}$

$\Rightarrow R+H = \frac{gR^2}{V^2}$

$\Rightarrow H = \frac{gR^2}{V^2} - R$

AS REQUIRED

Question 2 (**+)

A particle P is projected vertically upwards from a point on the surface of the Earth and moves in a straight line directly away from the centre of the Earth.

When P is at a distance x from the centre of the Earth, the gravitational force exerted by the Earth on P has magnitude $\frac{k}{x^2}$, where k is a positive constant, and is directed towards the centre of the Earth.

At the surface of the Earth the acceleration due to gravity is g.

The Earth is modelled as a fixed sphere of radius R.

a) If air resistance is ignored, show that the motion of P is governed by the differential equation

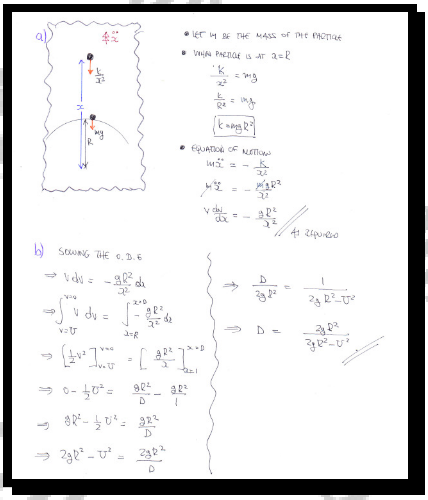
$$v\frac{dv}{dx} = -\frac{gR^2}{x^2},$$

where v is the speed of P.

When P is projected from the surface of the Earth with speed U, where $U < 2gR$, the greatest height of P above the surface of the Earth is D.

b) Find the value of D, in terms of U, R and g.

$$\boxed{D = \frac{2gR^2}{2gR - U^2}}$$



Question 3 (***)

The Earth is modelled as a fixed sphere of radius R.

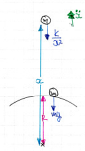
A particle P is fired vertically upwards from a point on the surface of the Earth with speed $\sqrt{\frac{3}{2}gR}$, and moves in a straight line directly away from the centre of the Earth.

When P is at a distance x from the centre of the Earth, the gravitational force exerted by the Earth on P has magnitude inversely proportional to the square of x, and is directed towards the centre of the Earth.

Assuming that air resistance is ignored, determine the speed of P, in terms of R and g, when it is a height of $2R$ above the surface of the Earth.

$\boxed{v=\sqrt{\frac{1}{6}gR}}$

START WITH A DIAGRAM & CONSIDER THE PARTICLE ON THE SURFACE OF THE EARTH, IN ORDER TO GET AN EXPRESSION FOR THE PROPORTIONALITY CONSTANT



ON EARTH'S SURFACE $x=R$

$$mg = \frac{k}{R^2}$$
$$k = mgR^2$$

NOW LOOKING AT THE EQUATION OF MOTION IN GENERAL

$$\Rightarrow m\ddot{x} = -\frac{k}{x^2}$$
$$\Rightarrow \cancel{m}v\frac{dv}{dx} = -\frac{\cancel{m}gR^2}{x^2}$$
$$\Rightarrow v\,dv = -\frac{gR^2}{x^2}\,dx$$

INTEGRATE SUBJECT TO THE CONDITION $x=R$, $v=\sqrt{\frac{3}{2}gR}$

$$\Rightarrow \int_{v=\sqrt{\frac{3}{2}gR}}^{v=V} v\,dv = \int_{x=R}^{x=3R} -\frac{gR^2}{x^2}\,dx$$
$$\Rightarrow \left[\frac{1}{2}v^2\right]_{v=\sqrt{\frac{3}{2}gR}}^{v=V} = \left[\frac{gR^2}{x}\right]_{x=R}^{x=3R}$$
$$\Rightarrow \frac{1}{2}V^2 - \frac{1}{2}\left(\frac{3}{2}gR\right) = gR^2\left[\frac{1}{3R}-\frac{1}{R}\right]$$
$$\Rightarrow \frac{1}{2}V^2 - \frac{3}{4}gR = gR^2\left(-\frac{2}{3R}\right)$$
$$\Rightarrow \frac{1}{2}V^2 - \frac{3}{4}gR = -\frac{2}{3}gR$$
$$\Rightarrow \frac{1}{2}V^2 = \frac{1}{12}gR$$
$$\Rightarrow V^2 = \frac{1}{6}gR$$
$$\Rightarrow |V| = \sqrt{\frac{1}{6}gR}$$

Question 4 (***)

A particle P, of mass m, is projected vertically upwards from a point on the surface of the Earth and travels in a straight line away from the centre of the earth. The Earth is modelled as a fixed sphere of radius R. At the surface of the Earth the acceleration due to gravity is g.

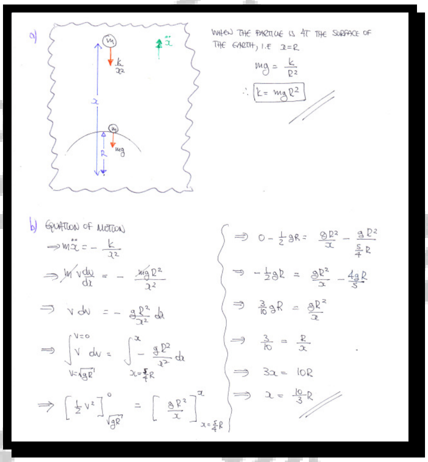
When P is at a distance x from the centre of the Earth, the force exerted by the Earth on P has magnitude $\frac{k}{x^2}$ towards the centre of the Earth, where k is a positive constant.

a) Show that $k = mgR^2$.

When P is at a height $\frac{1}{4}R$ above the surface of the Earth, the speed of P is $\sqrt{gR}$.

b) Given that air resistance can be ignored, find, in terms of R, the greatest distance from the centre of the Earth reached by P.

$$\boxed{x = \frac{10}{3}R}$$



Question 5 (***+)

A particle P, of mass m, is fired vertically upwards from a point on the surface of the Earth and moves in a straight line directly away from the centre of the Earth.

When P is at a distance x from the centre of the Earth, the gravitational force exerted by the Earth on P has magnitude $\frac{\lambda}{x^2}$, where λ is a constant, and is directed towards the centre of the Earth.

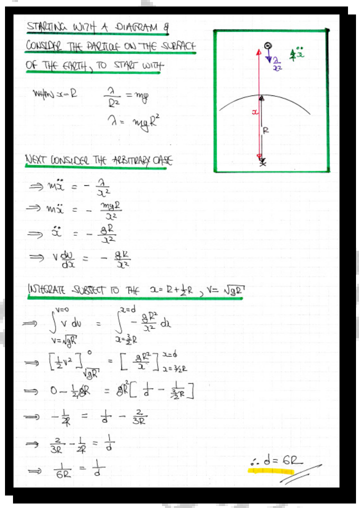
At the surface of the Earth the acceleration due to gravity is g.

The Earth is modelled as a fixed sphere of radius R.

When P is at a height $\frac{1}{2}R$ above the surface of the Earth, the speed of P is $\sqrt{gR}$.

Given that air resistance can be ignored, find, in terms of R, the greatest distance from the centre of the Earth reached by P.

$\boxed{}$, $\boxed{x = 6R}$


STARTING WITH A DIAGRAM & CONSIDER THE PARTICLE ON THE SURFACE OF THE EARTH, TO START WITH

WHEN $x = R$ $\quad \frac{\lambda}{R^2} = mg$

$\lambda = mgR^2$

NEXT CONSIDER THE ARBITRARY CASE

$\Rightarrow m\ddot{x} = -\frac{\lambda}{x^2}$

$\Rightarrow m\ddot{x} = -\frac{mgR}{x^2}$

$\Rightarrow \ddot{x} = -\frac{gR}{x^2}$

$\Rightarrow v\frac{dv}{dx} = -\frac{gR}{x^2}$

INTEGRATE SUBJECT TO THE $x = R + \frac{1}{2}R$, $v = \sqrt{gR}$

$\Rightarrow \int_{v=\sqrt{gR}}^{v=0} v \, dv = \int_{x=\frac{3}{2}R}^{x=d} -\frac{gR^2}{x^2} \, dx$

$\Rightarrow \left[\frac{1}{2}v^2\right]_{\sqrt{gR}}^{0} = \left[\frac{gR^2}{x}\right]_{x=3/2R}^{x=d}$

$\Rightarrow 0 - \frac{1}{2}gR = gR^2\left[\frac{1}{d} - \frac{1}{\frac{3}{2}R}\right]$

$\Rightarrow -\frac{1}{2R} = \frac{1}{d} - \frac{2}{3R}$

$\Rightarrow \frac{2}{3R} - \frac{1}{2R} = \frac{1}{d}$

$\Rightarrow \frac{1}{6R} = \frac{1}{d}$

$\therefore d = 6R$

Question 6 (***+)

A particle P, of mass m, is projected vertically upwards from a point on the surface of the Earth and moves in a straight line directly away from the centre of the Earth.

The Earth is modelled as a fixed sphere of radius R.

The distance of P from the centre of the Earth is denoted by x.

At the surface of the Earth the acceleration due to gravity is g.

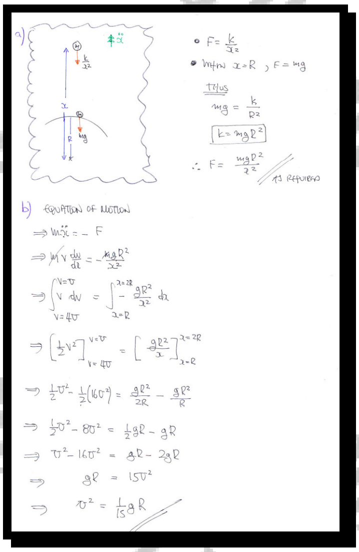
The Earth's gravitational effect is modelled as a force F, whose magnitude is inversely proportional to x^2.

a) Show that $F = \dfrac{mgR^2}{x^2}$.

P is projected with initial speed $4U$ and reaches a speed U at a height R above the surface of the Earth.

b) Ignoring air resistance, find U^2, in terms of g and R.

$$\boxed{U^2 = \frac{1}{15}gR}$$



a)

• $F = \dfrac{k}{x^2}$

• when $x = R$, $F = mg$

thus

$mg = \dfrac{k}{R^2}$

$\boxed{k = mgR^2}$

$\therefore F = \dfrac{mgR^2}{x^2}$ as required

b) EQUATION OF MOTION

$\Rightarrow m\ddot{x} = -F$

$\Rightarrow \cancel{m} v\dfrac{dv}{dx} = -\dfrac{\cancel{m}gR^2}{x^2}$

$\Rightarrow \displaystyle\int_{v=4U}^{v=U} v\,dv = \int_{x=R}^{x=2R} -\frac{gR^2}{x^2}\,dx$

$\Rightarrow \left[\frac{1}{2}v^2\right]_{v=4U}^{v=U} = \left[\frac{gR^2}{x}\right]_{x=R}^{x=2R}$

$\Rightarrow \frac{1}{2}U^2 - \frac{1}{2}(16U^2) = \frac{gR^2}{2R} - \frac{gR^2}{R}$

$\Rightarrow \frac{1}{2}U^2 - 8U^2 = \frac{1}{2}gR - gR$

$\Rightarrow U^2 - 16U^2 = gR - 2gR$

$\Rightarrow gR = 15U^2$

$\Rightarrow U^2 = \frac{1}{15}gR$

Question 7 (***+)

A piece of spaceship debris P, of mass m, is falling vertically downwards in a straight line towards the centre of the earth. The Earth is modelled as a fixed sphere of radius R. At the surface of the Earth the acceleration due to gravity is g.

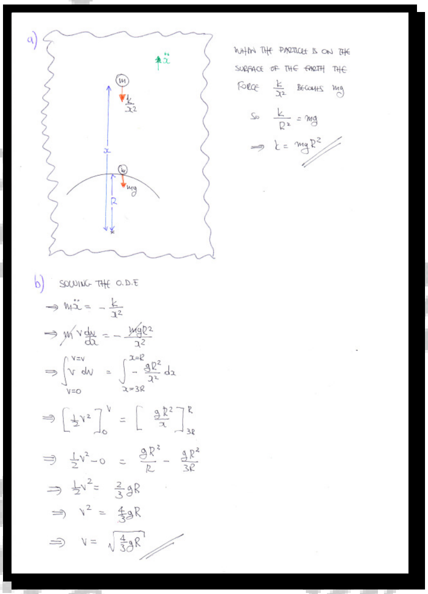
When P is at a distance x from the centre of the Earth, the force exerted by the Earth on P has magnitude $\dfrac{k}{x^2}$ towards the centre of the Earth, where k is a positive constant.

a) Show that $k = mgR^2$.

P starts falling from rest $x = 3R$

b) Ignoring air resistance, determine the speed of P as it crashes onto the surface of the earth, in terms of g and R.

$$\boxed{\text{speed} = \sqrt{\frac{4}{3}gR}}$$



Question 8 (***+)

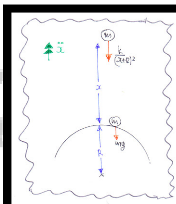
Above the Earth's surface, the magnitude of the gravitational force on a particle due to the Earth is inversely proportional to the square of the distance of the particle from the centre of the Earth. The Earth is modelled as a sphere of radius R and the acceleration due to gravity at the Earth's surface is g.

Let x denote the height of P above the surface of the Earth.

When $x = 2R$, P has speed $\sqrt{\frac{1}{2}gR}$.

Ignoring air resistance, find the speed of P in terms of g and R, when $x = R$.

$$\boxed{v = \sqrt{\frac{5}{6}gR}}$$



WHEN THE PARTICLE IS AT THE SURFACE OF THE EARTH

$\frac{k}{(x+R)^2}$ BECOMES mg

HENCE $\frac{k}{R^2} = mg$

$$\boxed{k = mgR^2}$$

THUS

$$m\ddot{x} = -\frac{k}{(x+R)^2}$$

$$\Rightarrow \cancel{m} v\frac{dv}{dx} = -\frac{\cancel{m}gR^2}{(x+R)^2}$$

$$\Rightarrow \int_{v=\sqrt{\frac{1}{2}gR}}^{v} v\,dv = \int_{x=2R}^{x=R} -\frac{gR^2}{(x+R)^2}\,dx$$

$$\Rightarrow \left[\frac{1}{2}v^2\right]_{\sqrt{\frac{1}{2}gR}}^{v} = \left[\frac{gR^2}{x+R}\right]_{x=2R}^{x=R}$$

$$\Rightarrow \frac{1}{2}v^2 - \frac{1}{2}\left(\frac{1}{2}gR\right) = \frac{gR^2}{2R} - \frac{gR^2}{3R}$$

$$\Rightarrow \frac{1}{2}v^2 - \frac{1}{4}gR = \frac{1}{2}gR - \frac{1}{3}gR$$

$$\Rightarrow \frac{1}{2}v^2 = \frac{5}{12}gR$$

$$\Rightarrow v^2 = \frac{5}{6}gR$$

$$\Rightarrow |v| = \sqrt{\frac{5}{6}gR}$$

Question 9 (***+)

A particle P, of mass m, is falling vertically downwards in a straight line towards the centre of the earth. The Earth is modelled as a fixed sphere of radius R.

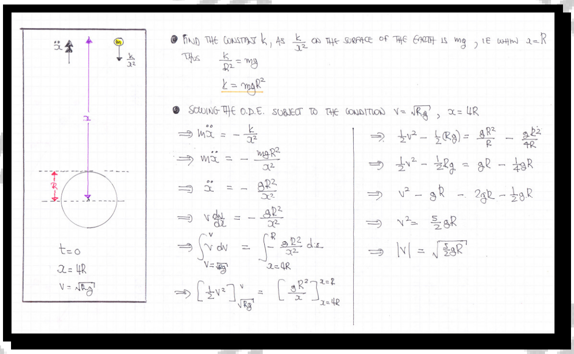
At the surface of the Earth the acceleration due to gravity is g.

When P is at a distance x from the centre of the Earth, the force exerted by the Earth on P has magnitude $\frac{k}{x^2}$ towards the centre of the Earth, where k is a positive constant.

When $x = 4R$ the speed of P is $\sqrt{gR}$.

Ignoring air resistance, determine the speed of P as it reaches the surface of the earth, in terms of g and R.

$$\boxed{\text{speed} = \sqrt{\frac{5}{2}gR}}$$



- FIND THE CONSTANT k, AS $\frac{k}{x^2}$ ON THE SURFACE OF THE EARTH IS mg, IE WHEN $x = R$

THUS $\frac{k}{R^2} = mg$

$k = mgR^2$

- SOLVING THE O.D.E. SUBJECT TO THE CONDITION $v = \sqrt{Rg}$, $x = 4R$

$\Rightarrow m\ddot{x} = -\frac{k}{x^2}$

$\Rightarrow m\ddot{x} = -\frac{mgR^2}{x^2}$

$\Rightarrow \ddot{x} = -\frac{gR^2}{x^2}$

$\Rightarrow v\frac{dv}{dx} = -\frac{gR^2}{x^2}$

$\Rightarrow \int_{v=\sqrt{Rg}}^{v} v\,dv = \int_{x=4R}^{R} -\frac{gR^2}{x^2}\,dx$

$\Rightarrow \left[\frac{1}{2}v^2\right]_{\sqrt{Rg}}^{v} = \left[\frac{gR^2}{x}\right]_{x=4R}^{x=R}$

$\Rightarrow \frac{1}{2}v^2 - \frac{1}{2}(Rg) = \frac{gR^2}{R} - \frac{gR^2}{4R}$

$\Rightarrow \frac{1}{2}v^2 - \frac{1}{2}Rg = gR - \frac{1}{4}gR$

$\Rightarrow v^2 - gR = 2gR - \frac{1}{2}gR$

$\Rightarrow v^2 = \frac{5}{2}gR$

$\Rightarrow |v| = \sqrt{\frac{5}{2}gR}$

Question 10 (****)

A particle P, of mass m, is projected vertically upwards with speed U, from a point on the surface of the Earth, and moves in a straight line directly away from the centre of the Earth.

When P is at a distance x from the centre of the Earth, the gravitational force exerted by the Earth on P has magnitude $\frac{mk}{x^2}$, where k is a positive constant, and is directed towards the centre of the Earth.

At the surface of the Earth the acceleration due to gravity is g.

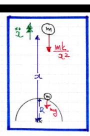
The Earth is modelled as a fixed sphere of radius R.

The kinetic energy of P at $x = 2R$ is half the kinetic energy when at $x = R$.

Ignoring air resistance, express k in terms of U and R.

$\boxed{}$, $\boxed{k = \frac{1}{2}RU^2}$

STARTING WITH A DIAGRAM



ON EARTH'S SURFACE, $x = R$

$$\frac{mk}{R^2} = mg$$
$$k = gR^2$$

NEXT THE EQUATION OF MOTION

$$\Rightarrow \cancel{m}\ddot{x} = -\frac{mk}{x^2}$$
$$\Rightarrow \ddot{x} = -\frac{gR^2}{x^2}$$
$$\Rightarrow v\frac{dv}{dx} = -\frac{gR^2}{x^2}$$
$$\Rightarrow v\,dv = -\frac{gR^2}{x^2}$$

INTEGRATE SUBJECT TO CONDITION $x = R$, $v = U$

$$\Rightarrow \int_{v=U}^{v=V} v\,dv = \int_{x=R}^{x=2R} -\frac{gR^2}{x^2}\,dx$$
$$\Rightarrow \left[\frac{1}{2}v^2\right]_{v=U}^{v=V} = \left[\frac{gR^2}{x}\right]_{x=R}^{x=2R}$$
$$\Rightarrow \frac{1}{2}V^2 - \frac{1}{2}U^2 = \frac{gR^{\cancel{2}}}{2\cancel{R}} - \frac{gR^{\cancel{2}}}{\cancel{R}}$$
$$\Rightarrow \frac{1}{2}V^2 - \frac{1}{2}U^2 = -\frac{1}{2}gR$$
$$\Rightarrow V^2 - U^2 = -gR$$
$$\Rightarrow V^2 = U^2 - gR$$

NOW CONSIDER THE KINETIC ENERGIES

INITIAL KINETIC ENERGY $= \frac{1}{2}mU^2$

"FINAL" KINETIC ENERGY $= \frac{1}{2}mV^2 = \frac{1}{2}m(U^2 - gR)$

$$\Rightarrow \frac{1}{2} \times \frac{1}{\cancel{2}}\cancel{m}U^2 = \frac{1}{\cancel{2}}\cancel{m}(U^2 - gR)$$
$$\Rightarrow U^2 = 2(U^2 - gR)$$
$$\Rightarrow U^2 = 2U^2 - 2gR$$
$$\Rightarrow U^2 = 2gR$$
$$\Rightarrow gR = \frac{1}{2}U^2$$
$$\Rightarrow gR^2 = \frac{1}{2}RU^2$$
$$\Rightarrow k = \frac{1}{2}RU^2$$

Question 11 (****)

A particle P is fired vertically upwards with speed U, from a point on the surface of the Earth and moves in a straight line directly away from the centre of the Earth.

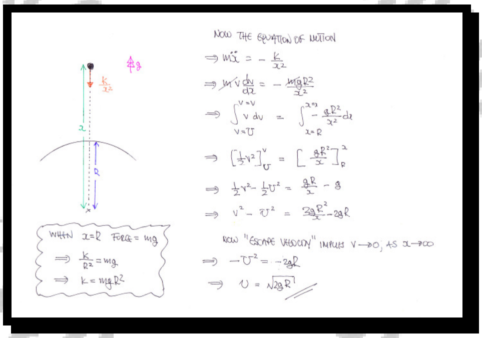
When P is at a distance x from the centre of the Earth, the gravitational force exerted by the Earth on P has magnitude inversely proportional to the distance of P from the centre of the Earth, and is directed towards the centre of the Earth.

At the surface of the Earth the acceleration due to gravity is g.

The Earth is modelled as a fixed sphere of radius R.

Given that the air resistance can be ignored, find the least value of U in terms of g and R, so that P does not fall back on to the surface of the Earth.

$$\boxed{U = \sqrt{2gR}}$$



Question 12 (*****)

A satellite is moving in a circular orbit above the Earth's equator.

The orbit is described as geostationary, which means that angular velocity of the satellite is identical to that of Earth's rotation, so it appears in a fixed position relative to an observer on the Earth.

The radius of the orbit, measured from the centre of the Earth, is r.

a) Show that

$$r = \sqrt[3]{\frac{GMT^2}{4\pi^2}},$$

where M is the mass of the earth, T is the period of the motion and G is the universal gravitation constant.

b) Determine the value of r, and hence find the minimum number of satellites needed to view all the points on the earth's equator.

You may assume

- earth's mass, $M = 5.97 \times 10^{24}$ kg
- earth's radius, $R = 6.37 \times 10^{6}$ m
- universal gravitation constant, $G = 6.67 \times 10^{-11}$ m^3 kg^{-1} s^{-2}

$\boxed{\quad}$, $\boxed{3}$

a) LOOKING AT THE DIAGRAM

$\Rightarrow F = G\frac{mM}{r^2}$

$\Rightarrow \cancel{m}(\omega^2 r) = G\frac{\cancel{m}M}{r^2}$

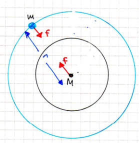
$\Rightarrow \omega^2 r^3 = GM$

$\Rightarrow \left(\frac{2\pi}{T}\right)^2 r^3 = GM$

$\Rightarrow \frac{4\pi^2 r^3}{T^2} = GM$

$\Rightarrow r^3 = \frac{GMT^2}{4\pi^2}$

$\Rightarrow r = \sqrt[3]{\frac{GMT^2}{4\pi^2}}$ //



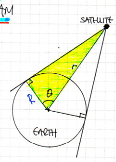
b)
$M = 5.97 \times 10^{24}$ kg
$G = 6.67 \times 10^{-11}$ m^3 kg^{-1} s^{-2}
$T = 24$ hours $= 8.64 \times 10^4$ s
$R = 6.37 \times 10^6$ m

$r = \left[\frac{6.67 \times 10^{-11} \times 5.97 \times 10^{24} \times (8.64 \times 10^4)^2}{4\pi^2}\right]^{\frac{1}{3}}$

$r \simeq \sqrt[3]{7.5295... \times 10^{22}}$

$r \approx 4.22 \times 10^7$

LOOKING AT THE FOLLOWING DIAGRAM



$\Rightarrow \cos\theta = \frac{R}{r}$

$\Rightarrow \cos\theta = \frac{6.37 \times 10^6}{4.22 \times 10^7}$

$\Rightarrow \cos\theta = 0.1509...$

$\Rightarrow \theta \simeq 81.3...$

$\Rightarrow 2\theta \simeq 162.63...$

$\therefore \frac{360}{162.63} \simeq 2.21...$ i.e. MINIMUM OF 3 //