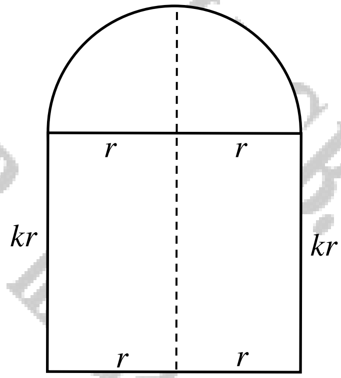
CENTRE OF MASS OF SOLIDS

Question 1 (**)



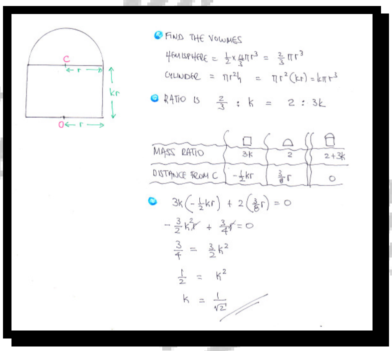
A uniform composite solid S consists of a solid hemisphere of radius r and a solid circular cylinder of radius r and height kr, where k is a positive constant

The circular face of the hemisphere is joined to one of the circular faces of the cylinder, so that the centres of the two faces coincide. The other circular face of the cylinder has centre O.

The centre of mass of S lies on the common plane of the cylinder and the hemisphere.

Determine the exact value of k.

$$\boxed{k = \frac{1}{\sqrt{2}}}$$



FIND THE VOLUMES

HEMISPHERE $= \frac{1}{2} \times \frac{4}{3}\pi r^3 = \frac{2}{3}\pi r^3$

CYLINDER $= \pi r^2 h = \pi r^2 (kr) = k\pi r^3$

RATIO IS $\frac{2}{3} : k = 2 : 3k$

	□	◠	
MASS RATIO	$3k$	2	$2+3k$
DISTANCE FROM C	$-\frac{1}{2}kr$	$\frac{3}{8}r$	0

$3k\left(-\frac{1}{2}kr\right) + 2\left(\frac{3}{8}r\right) = 0$

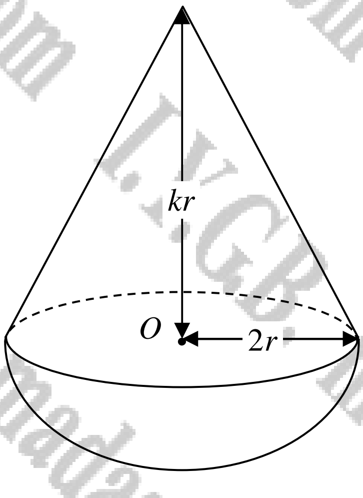
$-\frac{3}{2}k^2 r + \frac{3}{4}r = 0$

$\frac{3}{4} = \frac{3}{2}k^2$

$\frac{1}{2} = k^2$

$k = \frac{1}{\sqrt{2}}$

Question 2 (**)



A uniform solid S, consists of a hemisphere of radius $2r$ and a right circular cone of radius $2r$ and height kr, where k is a constant such that $k > 2\sqrt{3}$. The centre of the plane face of the hemisphere is at O and this plane face coincides with the plane face at the base of the cone, as shown in the figure above.

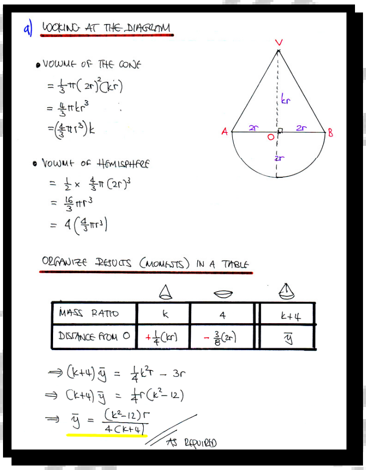
a) Show that the distance of the centre of mass of S from O is

$$\frac{k^2 - 12}{4(k+4)} r.$$

The point P lies on the circumference of the base of the cone. S is suspended by a string and hangs freely in equilibrium. The angle between OP and the vertical when S is in equilibrium is θ.

b) Given that $\tan\theta = 0.3$ determine the value of k.

$\boxed{\text{FM}}$, $\boxed{k = 6}$


a) LOOKING AT THE DIAGRAM

• VOLUME OF THE CONE

$= \frac{1}{3}\pi(2r)^2(kr)$

$= \frac{4}{3}\pi k r^3$

$= \left(\frac{4}{3}\pi r^3\right)k$

• VOLUME OF HEMISPHERE

$= \frac{1}{2} \times \frac{4}{3}\pi(2r)^3$

$= \frac{16}{3}\pi r^3$

$= 4\left(\frac{4}{3}\pi r^3\right)$

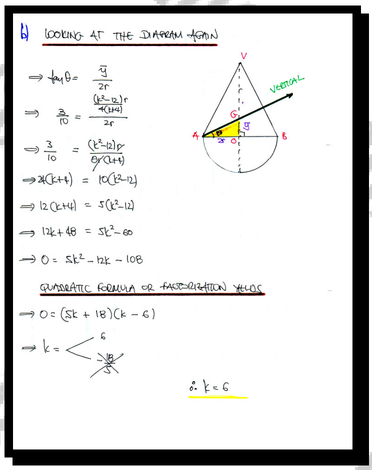
ORGANIZE RESULTS (MOMENTS) IN A TABLE

MASS RATIO	k	4	$k+4$
DISTANCE FROM O	$+\frac{1}{4}(kr)$	$-\frac{3}{8}(2r)$	$\bar{y}$

$\Rightarrow (k+4)\bar{y} = \frac{1}{4}k^2 r - 3r$

$\Rightarrow (k+4)\bar{y} = \frac{1}{4}r(k^2 - 12)$

$\Rightarrow \bar{y} = \frac{(k^2-12)r}{4(k+4)}$ // AS REQUIRED


b) LOOKING AT THE DIAGRAM AGAIN

$\Rightarrow \tan\theta = \frac{\bar{y}}{2r}$

$\Rightarrow \frac{3}{10} = \frac{\frac{(k^2-12)r}{4(k+4)}}{2r}$

$\Rightarrow \frac{3}{10} = \frac{(k^2-12)r}{8r(k+4)}$

$\Rightarrow 24(k+4) = 10(k^2-12)$

$\Rightarrow 12(k+4) = 5(k^2-12)$

$\Rightarrow 12k + 48 = 5k^2 - 60$

$\Rightarrow 0 = 5k^2 - 12k - 108$

QUADRATIC FORMULA OR FACTORIZATION YIELDS

$\Rightarrow 0 = (5k+18)(k-6)$

$\Rightarrow k = 6, \cancel{-\frac{18}{5}}$

$\therefore k = 6$

Question 3 (***)

A uniform solid spindle is made up by joining together the circular faces of two right circular cones.

The common circular face of the two cones has radius r and centre at the point O.

The smaller cone has height h and the larger cone has height kh, $k > 1$.

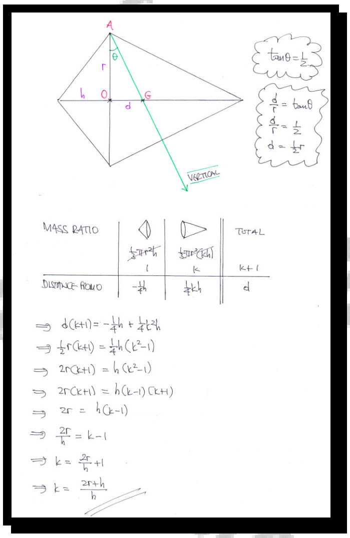
The point A lies on the circumference of the common circular face of the two cones.

The spindle is suspended from A and hangs freely in equilibrium with AO at an angle of $\arctan\frac{1}{2}$ to the vertical.

Show that

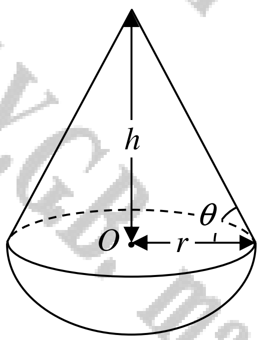
$$k = \frac{2r+h}{h}.$$

$\boxed{\text{proof}}$



Question 4 (***)

figure 1



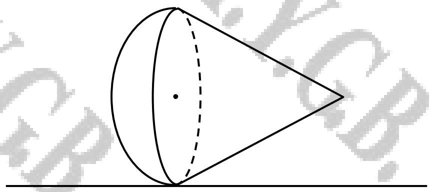


figure 2

Figure 1 above, shows a uniform solid S, consisting of a hemisphere of radius r and a right circular cone of radius r and height h. The centre of the plane face of the hemisphere is at O and this plane face coincides with the plane face at the base of the cone. The curved surface of the cone makes an angle θ with its base. The distance of the centre of mass of S above the level of O is denoted by $\bar{y}$.

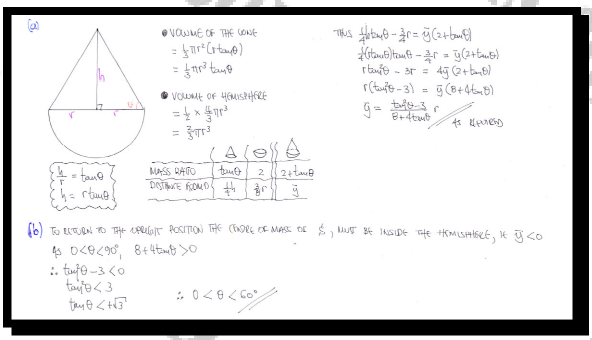
a) Show clearly that

$$\bar{y} = \frac{\tan^2\theta - 3}{8 + 4\tan\theta} r .$$

Figure 2 shows S held still on a horizontal surface so that the common plane of the hemisphere and the cone is perpendicular to the surface. When S is released it eventually returns to an upright position.

b) Determine the range of values of θ.

$\boxed{0 < \theta < 60°}$



(a)

• VOLUME OF THE CONE
$= \frac{1}{3}\pi r^2 (r\tan\theta)$
$= \frac{1}{3}\pi r^3 \tan\theta$

• VOLUME OF HEMISPHERE
$= \frac{1}{2} \times \frac{4}{3}\pi r^3$
$= \frac{2}{3}\pi r^3$

$\frac{h}{r} = \tan\theta$
$h = r\tan\theta$

	cone	hemisphere	S
MASS RATIO	$\tan\theta$	2	$2+\tan\theta$
DISTANCE FROM O	$\frac{1}{4}h$	$\frac{3}{8}r$	$\bar{y}$

THUS $\frac{1}{4}h\tan\theta - \frac{3}{4}r = \bar{y}(2+\tan\theta)$
$\frac{1}{4}(r\tan\theta)\tan\theta - \frac{3}{4}r = \bar{y}(2+\tan\theta)$
$r\tan^2\theta - 3r = 4\bar{y}(2+\tan\theta)$
$r(\tan^2\theta - 3) = \bar{y}(8+4\tan\theta)$
$\bar{y} = \frac{\tan^2\theta-3}{8+4\tan\theta} r$ AS REQUIRED

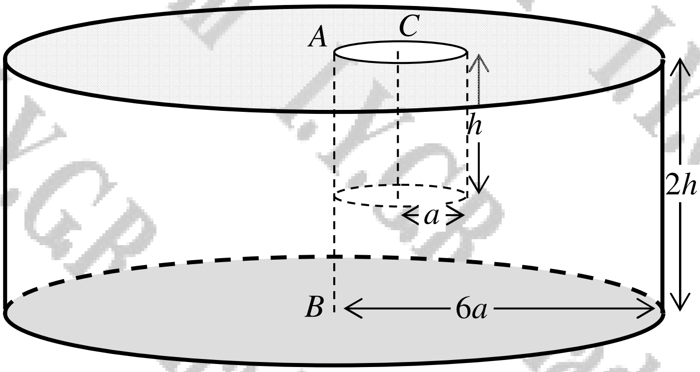
(b) TO RETURN TO THE UPRIGHT POSITION THE CENTRE OF MASS OF S, MUST BE INSIDE THE HEMISPHERE, IE $\bar{y} < 0$

AS $0<\theta<90°$, $8+4\tan\theta > 0$

$\therefore \tan^2\theta - 3 < 0$
$\tan^2\theta < 3$
$\tan\theta < +\sqrt{3}$

$\therefore 0 < \theta < 60°$

Question 5 (***)



A uniform solid right circular cylinder has height $2h$ and radius $6a$. The centre of one plane face is A and the centre of the other plane face is B.

A hole is made by removing a solid right circular cylinder of radius a and height h from the end with centre A. The axis of the cylindrical hole made is parallel to AB and meets the end with centre A at the point C, where $AC = a$.

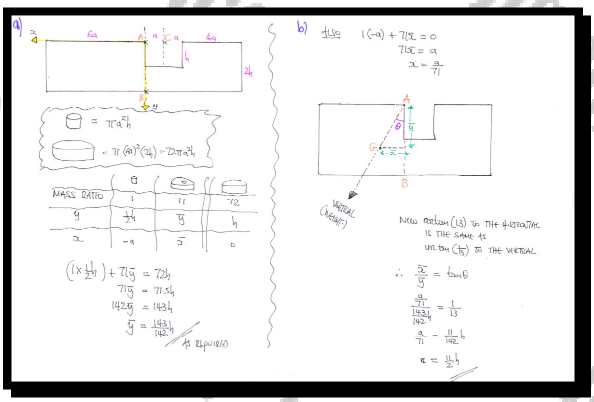
One plane face of the cylindrical hole made, coincides with the plane face through A of the cylinder. The resulting composite solid is shown in the figure above.

a) Show that the centre of mass of the composite is at a vertical distance $\frac{143}{142}h$ from the plane face containing A.

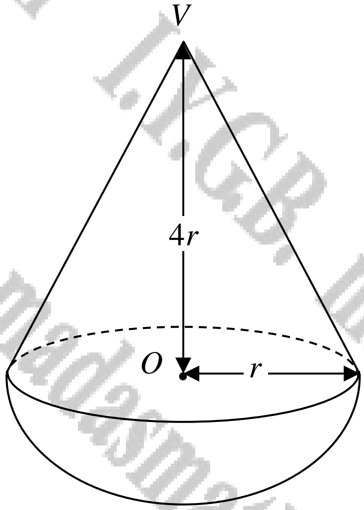
The composite is freely suspended from A and hangs in equilibrium with the axis AB inclined at an angle $\arctan(13)$ to the horizontal.

b) Express a in terms of h.

$\boxed{a = \frac{11}{2}h}$



Question 6 (*+)**



A uniform solid S, consists of a hemisphere of radius r and mass M, and a right circular cone of radius r, height $4r$ and mass m. The centre of the plane face of the hemisphere is at O and this plane face coincides with the plane face at the base of the cone, as shown in the figure above. The point P lies on the circumference of the base of the cone. S is placed on a horizontal surface, so that VP is in contact with the surface, where VP is the vertex of the cone.

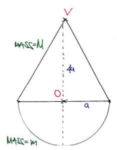
Given that S remains in equilibrium in that position, show that

$$m \le 10M .$$

$\boxed{}$, $\boxed{\text{proof}}$

- START BY FINDING THE LOCATION OF THE CENTRE OF MASS, ALONG THE AXIS OF SYMMETRY FROM A REFERENCE POINT, SAY O IN THE DIAGRAM

	CONE	HEMISPHERE	TOTAL
MASS RATIO	M	m	$M+m$
DISTANCE OF THE CENTRE OF MASS FROM O	$\frac{1}{4}\times 4a$	$-\frac{3}{8}a$	$\bar{y}$



$$\Rightarrow (M+m)\bar{y} = Ma - \frac{3}{8}ma$$

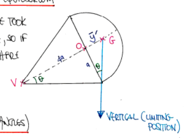
$$\Rightarrow 8(M+m)\bar{y} = (8M-3m)a$$

$$\Rightarrow \bar{y} = \frac{8M-3m}{8(M+m)}a$$

- NOW LOOKING AT THE OBJECT IN EQUILIBRIUM

NOTE IN THE ABOVE CALCULATION WE TOOK $\bar{y}$ TO BE POSITIVE IN THE CONE, SO IF WE TAKE POSITIVE IN THE HEMISPHERE

$$\bar{y}' = \frac{3m-8M}{8(M+m)}a$$



- NOW $|OG| = \frac{1}{4}a$ (SIMILAR TRIANGLES)

$$\Rightarrow \bar{y}' \le \frac{1}{4}a$$

$$\Rightarrow \frac{3m-8M}{8(M+m)}\cancel{a} \le \frac{1}{4}\cancel{a}$$

$$\Rightarrow 3m - 8M \le 2M + 2m$$

$$\Rightarrow m \le 10M$$

AS REQUIRED

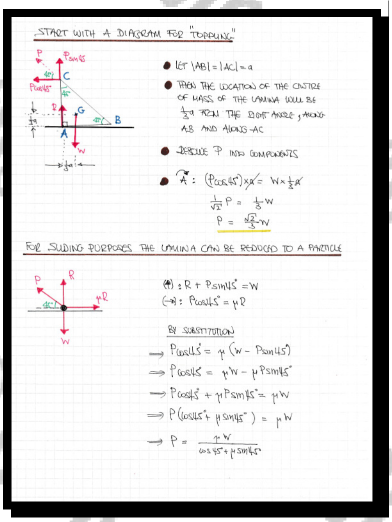
Question 7 (****)

A uniform lamina is in the shape of an isosceles right angled triangle ABC, where $\measuredangle BAC = 90°$.

The lamina is placed with AB in contact with a rough horizontal plane and C vertically above A. A gradually increasing force is applied at C, in the direction BC, until equilibrium is broken. The line of action of this force lies in the vertical plane containing the lamina.

Given that the lamina slides before it topples determine the range of possible values of the coefficient of friction between the lamina and the plane.

$\boxed{}$, $\boxed{0 < \mu < \frac{1}{2}}$



START WITH A DIAGRAM FOR "TOPPLING"

- LET $|AB| = |AC| = a$
- THEN THE LOCATION OF THE CENTRE OF MASS OF THE LAMINA WILL BE $\frac{1}{3}a$ FROM THE RIGHT ANGLE, ALONG AB AND ALONG AC
- RESOLVE P INTO COMPONENTS
- $\curvearrowright A$: $(P\cos 45°) \times a = W \times \frac{1}{3}a$

$\frac{1}{\sqrt{2}}P = \frac{1}{3}W$

$P = \frac{\sqrt{2}}{3}W$

FOR SLIDING PURPOSES THE LAMINA CAN BE REDUCED TO A PARTICLE

$(\uparrow)$: $R + P\sin 45° = W$

$(\rightarrow)$: $P\cos 45° = \mu R$

BY SUBSTITUTION

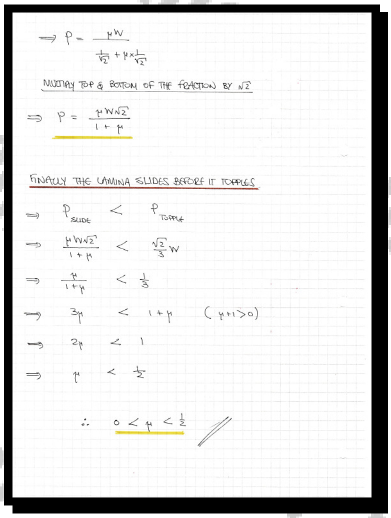
$\Rightarrow P\cos 45° = \mu(W - P\sin 45°)$

$\Rightarrow P\cos 45° = \mu W - \mu P\sin 45°$

$\Rightarrow P\cos 45° + \mu P\sin 45° = \mu W$

$\Rightarrow P(\cos 45° + \mu\sin 45°) = \mu W$

$\Rightarrow P = \frac{\mu W}{\cos 45° + \mu\sin 45°}$



$\Rightarrow P = \frac{\mu W}{\frac{1}{\sqrt{2}} + \mu \times \frac{1}{\sqrt{2}}}$

MULTIPLY TOP & BOTTOM OF THE FRACTION BY $\sqrt{2}$

$\Rightarrow P = \frac{\mu W\sqrt{2}}{1 + \mu}$

FINALLY THE LAMINA SLIDES BEFORE IT TOPPLES

$\Rightarrow P_{SLIDE} < P_{TOPPLE}$

$\Rightarrow \frac{\mu W\sqrt{2}}{1+\mu} < \frac{\sqrt{2}}{3}W$

$\Rightarrow \frac{\mu}{1+\mu} < \frac{1}{3}$

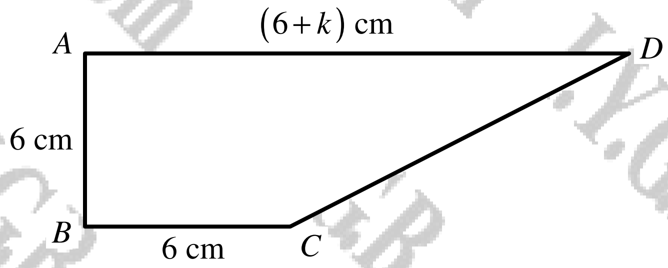
$\Rightarrow 3\mu < 1 + \mu \quad (\mu + 1 > 0)$

$\Rightarrow 2\mu < 1$

$\Rightarrow \mu < \frac{1}{2}$

$\therefore 0 < \mu < \frac{1}{2}$ //

Question 8 (****)



The figure above shows the cross section of a solid uniform right prism which in the shape of a right angled trapezium $ABCD$. It is further given that $|AB| = |BC| = 6$ cm, $|AD| = (6+k)$ cm and $\measuredangle DAB = \measuredangle ABC = 90°$.

a) By treating $ABCD$ as a uniform lamina, find in terms of the constant k the position of the centre of mass of $ABCD$, relative to the vertex A.

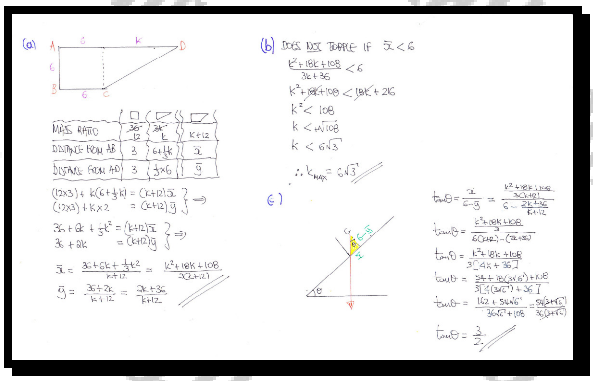
The prism is resting with $ABCD$ perpendicular to a horizontal surface and the face which contains BC, in contact with this horizontal surface.

b) Calculate the greatest value of k which allows the prism **not** to topple.

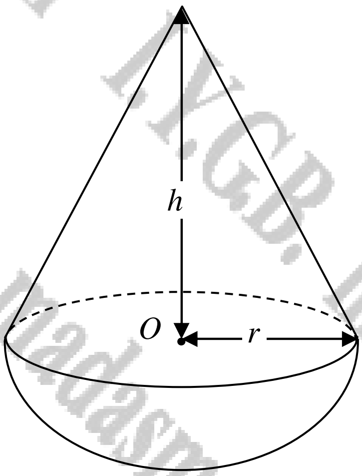
The prism is the placed on a rough plane inclined at θ to the horizontal, with BC lying on the line of greatest slope of the plane. The value of k is taken to be $3\sqrt{6}$.

c) Given the prism is about to topple, determine the exact value of $\tan\theta$.

$\boxed{\bar{x}_{AB} = \dfrac{k^2+18k+108}{3k+36}}$, $\boxed{\bar{y}_{AB} = \dfrac{2k+36}{k+12}}$, $\boxed{k_{\max} = 6\sqrt{3}}$, $\boxed{\tan\theta = \dfrac{3}{2}}$



Question 9 **(****)**



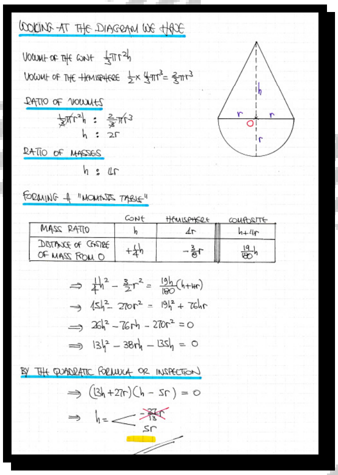
A solid S, consists of a hemisphere of radius r and a right circular cone of radius r and height h. The centre of the plane face of the hemisphere is at O and this plane face coincides with the plane face at the base of the cone.

Both the cone and the hemisphere are of uniform density, but the density of the hemisphere is twice as large as that of the cone.

The centre of mass of S lies inside the cone, at a distance of $\frac{19h}{180}$ from O.

Express h in terms of r.

$\boxed{\quad}$, $\boxed{h = 5r}$



LOOKING AT THE DIAGRAM WE HAVE

VOLUME OF THE CONE $\frac{1}{3}\pi r^2 h$

VOLUME OF THE HEMISPHERE $\frac{1}{2} \times \frac{4}{3}\pi r^3 = \frac{2}{3}\pi r^3$

RATIO OF VOLUMES

$\frac{1}{3}\pi r^2 h : \frac{2}{3}\pi r^3$

$h : 2r$

RATIO OF MASSES

$h : 4r$

FORMING A "MOMENTS TABLE"

	CONE	HEMISPHERE	COMPOSITE
MASS RATIO	h	$4r$	$h+4r$
DISTANCE OF CENTRE OF MASS FROM O	$+\frac{1}{4}h$	$-\frac{3}{8}r$	$\frac{19}{180}h$

$\Rightarrow \frac{1}{4}h^2 - \frac{3}{2}r^2 = \frac{19h}{180}(h+4r)$

$\Rightarrow 45h^2 - 270r^2 = 19h^2 + 76hr$

$\Rightarrow 26h^2 - 76rh - 270r^2 = 0$

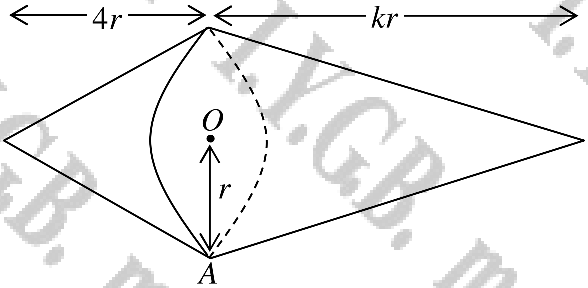
$\Rightarrow 13h^2 - 38rh - 135h = 0$

BY THE QUADRATIC FORMULA OR INSPECTION

$\Rightarrow (13h + 27r)(h - 5r) = 0$

$\Rightarrow h = \cancel{-\frac{27}{13}r}$ or $5r$

Question 10 (****)



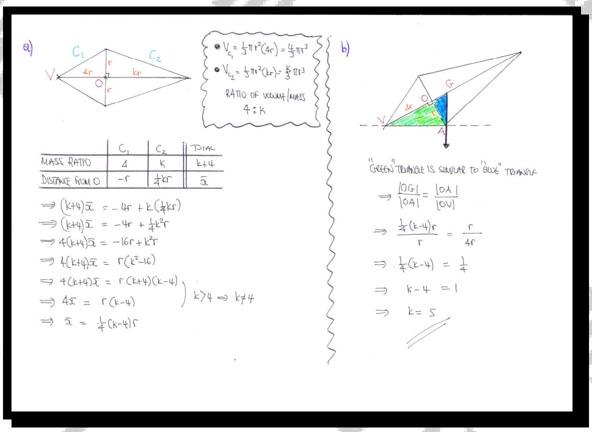
A uniform solid S, is formed by joining the plane faces of two solid right circular cones, both of radius r, so that the centres of their bases coincide at the point O, as shown in the figure above. The cone with vertex at the point V has height $4r$ and the other cone has radius kr, $k > 4$.

a) Determine the distance of the centre of mass of S from O.

The point A lies on the circumference of the common base of the two cones. When S is placed with AV in contact with a horizontal surface, it rests in equilibrium.

b) Find the greatest possible value of k.

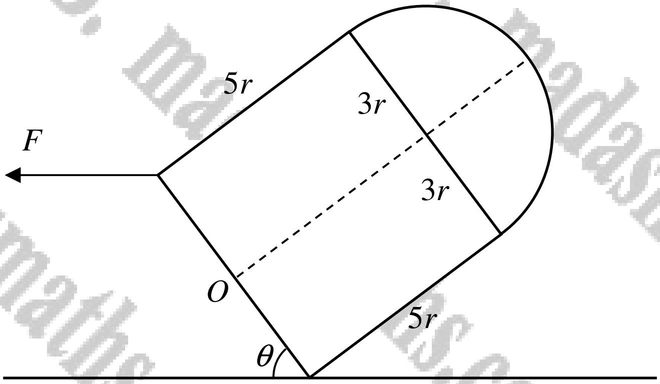
$\boxed{\bar{x} = \frac{1}{4}(k-4)r}$, $\boxed{k = 5}$



Question 11 (****+)

A composite solid C consists of a uniform solid hemisphere of radius $3r$ and a uniform solid circular cylinder of radius $3r$ and height $5r$. The circular face of the hemisphere is joined to one of the circular faces of the cylinder, so that the centres of the two faces coincide. The other circular face of the cylinder has centre O.

a) Determine, in terms of r, the distance of the centre of mass of C from O.

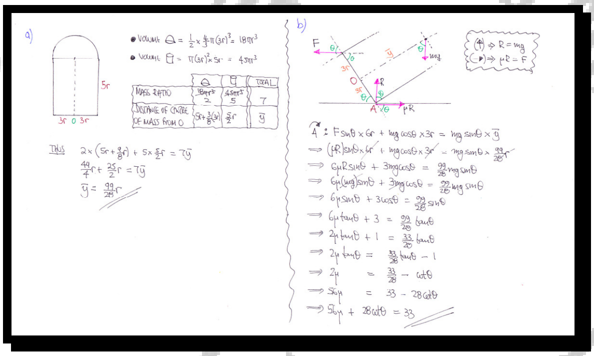


The composite is held in equilibrium by a horizontal force of magnitude F. The circular face of C has one point in contact with a fixed rough horizontal plane and is inclined at an angle θ to the horizontal. The force acts through the highest point of the circular face of C and in the vertical plane through the axis of the cylinder, as shown in the figure above. The coefficient of friction between C and the plane is μ.

b) Given that C is on the point of slipping along the plane in the same direction as F, show clearly that

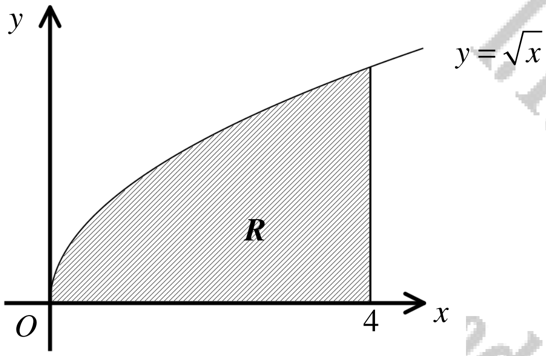
$$56\mu + 28\cot\theta = 33.$$

$$\boxed{\bar{y} = \frac{99}{28}r}$$



CENTRE OF MASS BY CALCULUS

Question 1 (**)



The figure above shows the finite region R bounded by the x axis, the curve with equation $y=\sqrt{x}$ and the straight line with equation $x=4$.

R is rotated about the x axis forming a solid of revolution S.

Use integration to determine the x coordinate of the centre of mass of S.

$\boxed{x=\frac{8}{3}}$

START BY FINDING THE VOLUME OF REVOLUTION FIRST

$$V=\pi\int_{x_1}^{x_2}(y(x))^2\,dx=\pi\int_0^4(\sqrt{x})^2\,dx=\pi\int_0^4 x\,dx$$

$$=\pi\left[\tfrac{1}{2}x^2\right]_0^4=\pi\left[8-0\right)=8\pi$$

NOW SETTING UP A "MOMENTS" EQUATION, ρ = DENSITY

$$\Rightarrow M\bar{x}=\rho\pi\int_{x_1}^{x_2}(y(x))^2x\,dx$$

$$\Rightarrow V\rho\bar{x}=\rho\pi\int_0^4(\sqrt{x})^2x\,dx$$

$$\Rightarrow 8\pi\rho\bar{x}=\rho\pi\int_0^4 x^2\,dx$$

$$\Rightarrow 8\bar{x}=\left[\tfrac{1}{3}x^3\right]_0^4$$

$$\Rightarrow 8\bar{x}=\tfrac{64}{3}-0$$

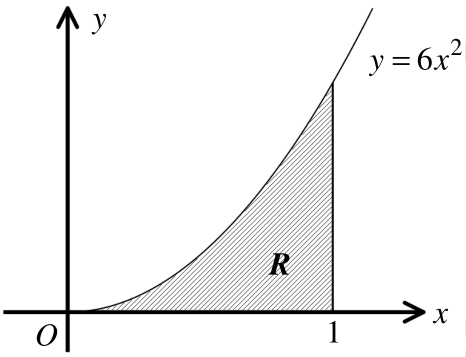
$$\therefore \bar{x}=\tfrac{8}{3}$$

FORMULA IS OF COURSE QUOTABLE

$$\bar{x}=\frac{\int_{x_1}^{x_2}y^2x\,dx}{\int_{x_1}^{x_2}y^2\,dx}=\frac{\int_0^4 x^2\,dx}{\int_0^4 x\,dx}=\frac{\left[\frac{1}{3}x^3\right]_0^4}{\left[\frac{1}{2}x^2\right]_0^4}$$

$$=\frac{64/3}{8}=\frac{8}{3}$$

Question 2 (**)



The figure above shows the finite region R bounded by the x axis, the curve with equation $y = 6x^2$ and the straight line with equation $x = 1$.

The centre of mass of a uniform lamina whose shape is that of R, is denoted by G.

Use integration to determine the coordinates of G.

$\boxed{G\left(\frac{3}{4}, \frac{9}{5}\right)}$

START BY FINDING THE AREA

$$\text{AREA} = \int_0^1 6x^2 \, dx = \left[2x^3\right]_0^1 = 2$$

USING "STANDARD" FORMULAS

$$\bar{x} = \frac{\int_a^b xy \, dx}{\int_a^b y \, dx}$$

$$\bar{x} = \frac{\int_0^1 x(6x^2) \, dx}{2}$$

$$\bar{x} = \frac{1}{2}\int_0^1 6x^3 \, dx$$

$$\bar{x} = \frac{1}{2}\left[\frac{3}{2}x^4\right]_0^1$$

$$\bar{x} = \frac{3}{4}$$

$$\bar{y} = \frac{\int \frac{1}{2}y^2 \, dx}{\int_a^b y \, dx}$$

$$\bar{y} = \frac{\int_0^1 \frac{1}{2}(6x^2)^2 \, dx}{2}$$

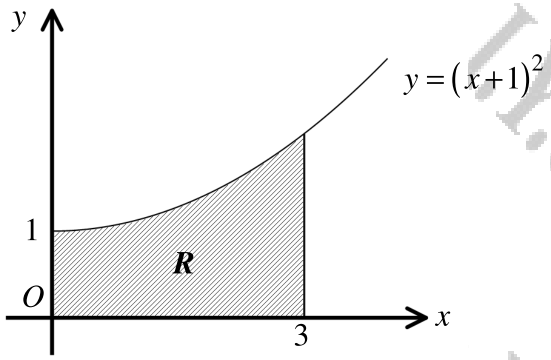
$$\bar{y} = \frac{1}{2}\int_0^1 18x^4 \, dx$$

$$\bar{y} = \frac{1}{2}\left[\frac{18}{5}x^5\right]_0^1$$

$$\bar{y} = \frac{9}{5}$$

$\therefore \left(\frac{3}{4}, \frac{9}{5}\right)$

Question 3 (**+)

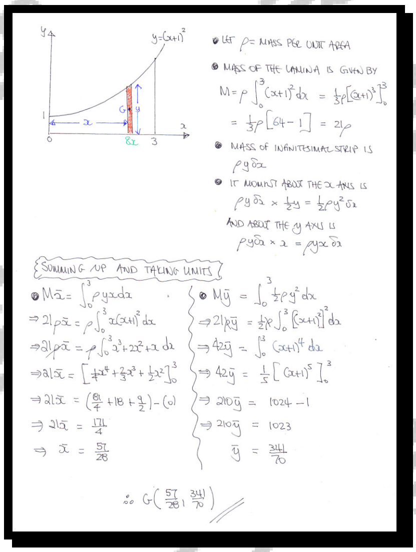


The figure above shows the finite region R bounded by the coordinate axes, the curve with equation $y=(x+1)^2$ and the straight line with equation $x=3$.

The centre of mass of a uniform lamina whose shape is that of R, is denoted by G.

Use a detailed calculus method to determine the coordinates of G.

$$\boxed{G\left(\frac{57}{28},\frac{341}{70}\right)}$$



- LET ρ = MASS PER UNIT AREA
- MASS OF THE LAMINA IS GIVEN BY

$M=\rho\int_0^3 (x+1)^2\,dx = \frac{1}{3}\rho\left[(x+1)^3\right]_0^3$

$= \frac{1}{3}\rho\left[64-1\right] = 21\rho$

- MASS OF INFINITESIMAL STRIP IS $\rho y\,\delta x$
- IT MOMENT ABOUT THE x AXIS IS $\rho y\,\delta x \times \frac{1}{2}y = \frac{1}{2}\rho y^2\,\delta x$

AND ABOUT THE y AXIS IS $\rho y\,\delta x \times x = \rho y x\,\delta x$

SUMMING UP AND TAKING LIMITS

- $M\bar{x} = \int_0^3 \rho y x\,dx$

$\Rightarrow 21\rho\bar{x} = \rho\int_0^3 x(x+1)^2\,dx$

$\Rightarrow 21\rho\bar{x} = \rho\int_0^3 x^3+2x^2+x\,dx$

$\Rightarrow 21\bar{x} = \left[\frac{1}{4}x^4+\frac{2}{3}x^3+\frac{1}{2}x^2\right]_0^3$

$\Rightarrow 21\bar{x} = \left(\frac{81}{4}+18+\frac{9}{2}\right)-(0)$

$\Rightarrow 21\bar{x} = \frac{171}{4}$

$\Rightarrow \bar{x} = \frac{57}{28}$

- $M\bar{y} = \int_0^3 \frac{1}{2}\rho y^2\,dx$

$\Rightarrow 21\rho\bar{y} = \frac{1}{2}\rho\int_0^3 \left[(x+1)^2\right]^2\,dx$

$\Rightarrow 42\bar{y} = \int_0^3 (x+1)^4\,dx$

$\Rightarrow 42\bar{y} = \frac{1}{5}\left[(x+1)^5\right]_0^3$

$\Rightarrow 210\bar{y} = 1024-1$

$\Rightarrow 210\bar{y} = 1023$

$\bar{y} = \frac{341}{70}$

$\therefore G\left(\frac{57}{28},\frac{341}{70}\right)$

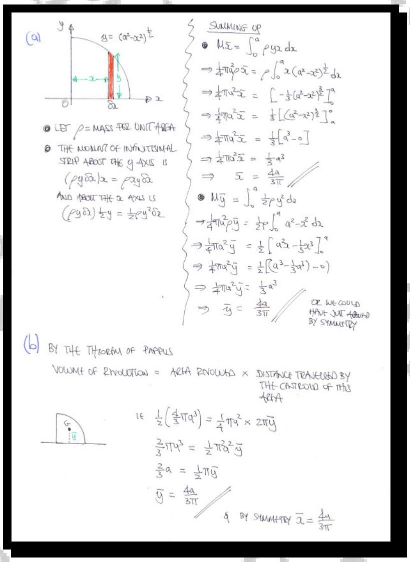
Question 4 (***)

A uniform lamina is in the shape of a quarter circle of radius a.

a) Use a calculus method to show that the centre of mass of the lamina is at a distance of $\dfrac{4a}{3\pi}$ from both the straight edges of the lamina.

b) Use one of the theorems of Pappus to verify the result of part **(a)**.

$\boxed{\text{proof}}$



(a)

- LET ρ = MASS PER UNIT AREA
- THE MOMENT OF INFINITESIMAL STRIP ABOUT THE y AXIS IS
  $(\rho y \delta x)x = \rho x y \delta x$
  AND ABOUT THE x AXIS IS
  $(\rho y \delta x)\frac{1}{2}y = \frac{1}{2}\rho y^2 \delta x$

SUMMING UP

- $M\bar{x} = \int_0^a \rho y x \, dx$

$\Rightarrow \frac{1}{4}\pi a^2 \rho \bar{x} = \rho \int_0^a x(a^2-x^2)^{\frac{1}{2}} \, dx$

$\Rightarrow \frac{1}{4}\pi a^2 \bar{x} = \left[-\frac{1}{3}(a^2-x^2)^{\frac{3}{2}}\right]_0^a$

$\Rightarrow \frac{1}{4}\pi a^2 \bar{x} = \frac{1}{3}\left[(a^2-x^2)^{\frac{3}{2}}\right]_a^0$

$\Rightarrow \frac{1}{4}\pi a^2 \bar{x} = \frac{1}{3}\left[a^3 - 0\right]$

$\Rightarrow \frac{1}{4}\pi a^2 \bar{x} = \frac{1}{3}a^3$

$\Rightarrow \bar{x} = \frac{4a}{3\pi}$

- $M\bar{y} = \int_0^a \frac{1}{2}\rho y^2 \, dx$

$\Rightarrow \frac{1}{4}\pi a^2 \rho \bar{y} = \frac{1}{2}\rho \int_0^a a^2 - x^2 \, dx$

$\Rightarrow \frac{1}{4}\pi a^2 \bar{y} = \frac{1}{2}\left[a^2 x - \frac{1}{3}x^3\right]_0^a$

$\Rightarrow \frac{1}{4}\pi a^2 \bar{y} = \frac{1}{2}\left[\left(a^3 - \frac{1}{3}a^3\right) - 0\right)$

$\Rightarrow \frac{1}{4}\pi a^2 \bar{y} = \frac{1}{3}a^3$

$\Rightarrow \bar{y} = \frac{4a}{3\pi}$

OR WE COULD HAVE JUST ARGUED BY SYMMETRY

(b) BY THE THEOREM OF PAPPUS

VOLUME OF REVOLUTION = AREA REVOLVED × DISTANCE TRAVELLED BY THE CENTROID OF THIS AREA

IE $\frac{1}{2}\left(\frac{4}{3}\pi a^3\right) = \frac{1}{4}\pi a^2 \times 2\pi \bar{y}$

$\frac{2}{3}\pi a^3 = \frac{1}{2}\pi^2 a^2 \bar{y}$

$\frac{2}{3}a = \frac{1}{2}\pi \bar{y}$

$\bar{y} = \frac{4a}{3\pi}$

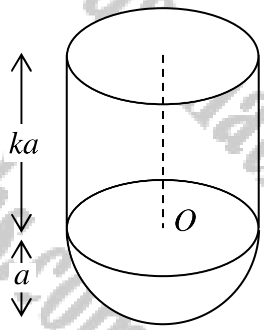
& BY SYMMETRY $\bar{x} = \frac{4a}{3\pi}$

Question 5 **(***)**

The volume of a hemisphere of radius r is $\frac{1}{2}\pi r^3$.

a) Show, by calculus, that the centre of mass of a hemisphere of radius r is at a distance of $\frac{3}{8}r$ from the centre of its plane face.

A composite C is formed by joining a uniform solid right circular cone, of base radius a and height ka, where k is a positive constant, to a uniform solid hemisphere of radius a. The plane face of the hemisphere and one of the plane faces of the cylinder coincide, both having O as a centre, as shown in the figure below.

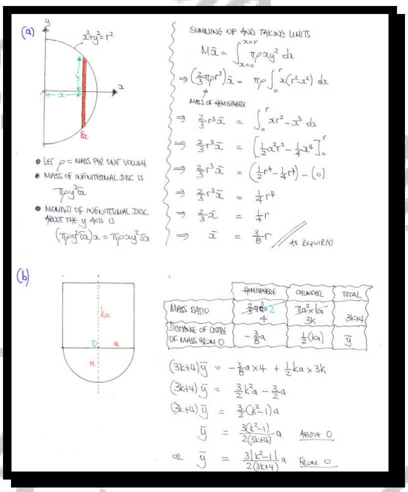


The mass density of the hemisphere is twice the mass density of the cylinder.

b) Show that that the distance of the centre of mass of C from O is

$$\frac{3\left|k^2-1\right|a}{2(3k+4)}.$$

proof



(a)

$x^2+y^2=r^2$

- LET ρ = MASS PER UNIT VOLUME
- MASS OF INFINITESIMAL DISC IS $\pi\rho y^2\delta x$
- MOMENT OF INFINITESIMAL DISC ABOUT THE y AXIS IS $(\pi\rho y^2\delta x)x = \pi\rho xy^2\delta x$

SUMMING UP AND TAKING LIMITS

$$M\bar{x} = \int_{x=0}^{x=r} \pi\rho x y^2 \, dx$$

$$\Rightarrow \left(\tfrac{2}{3}\pi\rho r^3\right)\bar{x} = \pi\rho\int_0^r x(r^2-x^2)\,dx$$

MASS OF HEMISPHERE

$$\Rightarrow \tfrac{2}{3}r^3\bar{x} = \int_0^r xr^2 - x^3\,dx$$

$$\Rightarrow \tfrac{2}{3}r^3\bar{x} = \left[\tfrac{1}{2}x^2r^2 - \tfrac{1}{4}x^4\right]_0^r$$

$$\Rightarrow \tfrac{2}{3}r^3\bar{x} = \left(\tfrac{1}{2}r^4 - \tfrac{1}{4}r^4\right) - (0)$$

$$\Rightarrow \tfrac{2}{3}r^3\bar{x} = \tfrac{1}{4}r^4$$

$$\Rightarrow \tfrac{2}{3}\bar{x} = \tfrac{1}{4}r$$

$$\Rightarrow \bar{x} = \tfrac{3}{8}r$$ AS REQUIRED

(b)

ka, O, a, a

	HEMISPHERE	CYLINDER	TOTAL
MASS RATIO	$\cancel{\frac{2}{3}\pi a^3} \times 2$ 4	$\cancel{\pi a^2 \times ka}$ $3k$	$3k+4$
DISTANCE OF CENTRE OF MASS FROM O	$-\frac{3}{8}a$	$\frac{1}{2}(ka)$	$\bar{y}$

$$(3k+4)\bar{y} = -\tfrac{3}{8}a \times 4 + \tfrac{1}{2}ka \times 3k$$

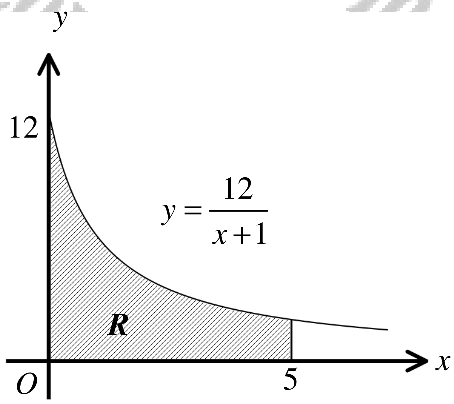
$$(3k+4)\bar{y} = \tfrac{3}{2}k^2a - \tfrac{3}{2}a$$

$$(3k+4)\bar{y} = \tfrac{3}{2}(k^2-1)a$$

$$\bar{y} = \frac{3(k^2-1)}{2(3k+4)}a$$ ABOUT O

OR $$\bar{y} = \frac{3|k^2-1|}{2(3k+4)}a$$ FROM O

Question 6 (***+)



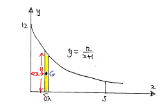
The figure above shows the finite region R bounded by the coordinate axes, the curve with equation $y = \dfrac{12}{x+1}$ and the straight line with equation $x = 5$.

The centre of mass of a uniform lamina whose shape is that of R, is denoted by G.

Use integration to determine the exact coordinates of G.

$\boxed{}$, $\boxed{G\left(\dfrac{5}{\ln 6} - 1, \dfrac{5}{\ln 6}\right)}$

LET ρ BE THE MASS PER UNIT AREA



AREA UNDER THE CURVE IS

$$A = \int_0^5 \frac{12}{x+1}\,dx = \Big[12\ln(x+1)\Big]_0^5 = 12\ln 6 - \cancel{12\ln 1} = 12\ln 6$$

THE MASS OF AN INFINITESIMAL STRIP OF HEIGHT y AND THICKNESS δx IS

$$\delta m = \rho y\,\delta x$$

THE "MOMENTS" OF THE INFINITESIMAL ABOUT THE x & THE y AXIS ARE

$$(\rho y \delta x)x \quad \& \quad (\rho y \delta x) \times \tfrac{1}{2}y$$

SUMMING UP AND TAKING LIMITS

- $M\bar{x} = \int_0^5 \rho y x\,dx$

$(12\ln 6)\rho\bar{x} = \rho\int_0^5 \frac{12x}{x+1}\,dx$

$\bar{x} = \frac{12}{12\ln 6}\int_0^5 \frac{x+1-1}{x+1}\,dx$

$\bar{x} = \frac{1}{\ln 6}\int_0^5 1 - \frac{1}{x+1}\,dx$

$\Rightarrow \bar{x} = \frac{1}{\ln 6}\Big[x - \ln(x+1)\Big]_0^5$

$\Rightarrow \bar{x} = \frac{1}{\ln 6}\Big[(5 - \ln 6) - (\cancel{\ln 1})\Big]$

$\Rightarrow \bar{x} = \frac{5}{\ln 6} - 1$

- $M\bar{y} = \int_0^5 \frac{1}{2}\rho y^2\,dx$

$(12\ln 6)\rho\bar{y} = \frac{1}{2}\rho\int_0^5 \frac{144}{(x+1)^2}$

$\bar{y} = \frac{1}{24\ln 6}\Big[-\frac{144}{x+1}\Big]_0^5$

$\bar{y} = \frac{144}{24\ln 6}\Big[\frac{1}{x+1}\Big]_5^0$

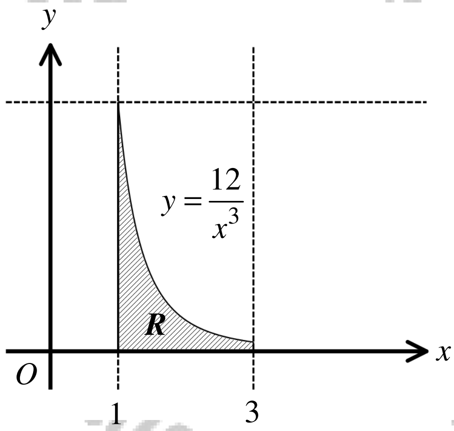
$\Rightarrow \bar{y} = \frac{6}{\ln 6}\Big[1 - \frac{1}{6}\Big]$

$\Rightarrow \bar{y} = \frac{6}{\ln 6} \times \frac{5}{6}$

$\Rightarrow \bar{y} = \frac{5}{\ln 6}$

$\therefore G\left(\frac{5}{\ln 6} - 1, \frac{5}{\ln 6}\right)$

Question 7 (***+)

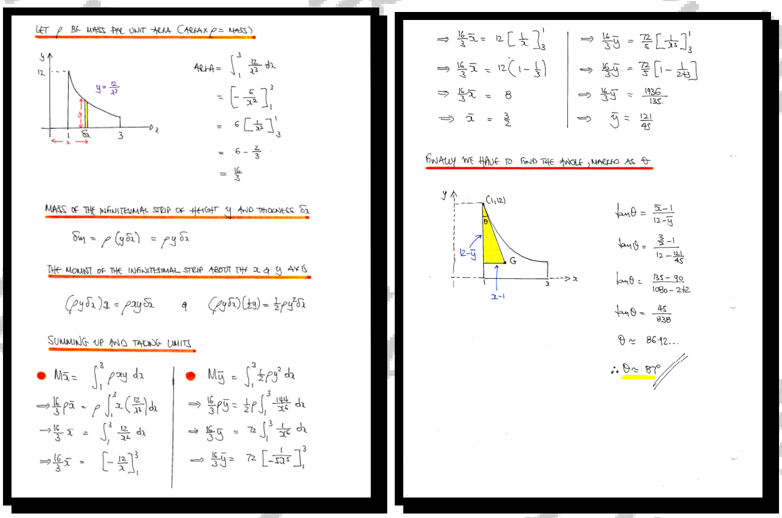


The figure above shows the finite region R bounded by the x axis, the curve with equation $y = \dfrac{60}{x^3}$ and the straight lines with equations $x = 1$ and $x = 3$.

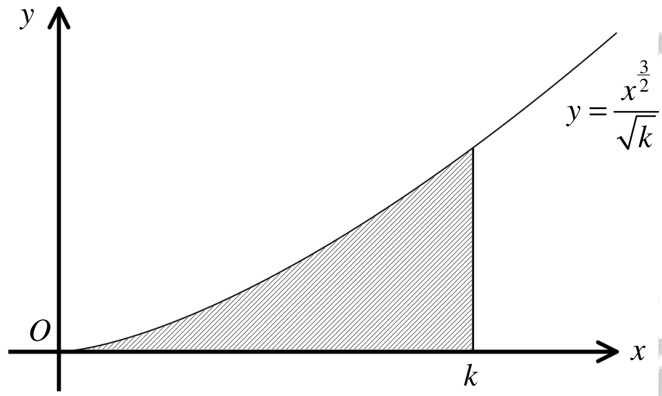
A uniform lamina whose shape is that of R, is suspended from the point $(1,12)$ and hangs freely under gravity.

Determine the angle the longer straight edge of the lamina makes with the vertical.

$\boxed{}$, $\boxed{\theta \approx 87°}$



Question 8 (***+)



The figure above shows the curve with equation

$$y = \frac{x^{\frac{3}{2}}}{\sqrt{k}} ,$$

where k is a positive constant

The finite region bounded by the curve, the coordinate axes and the straight line with equation $x = k$ region is revolved by $360°$ about the x axis, forming a solid of revolution. This solid is carefully placed with its plane face on a rough plane inclined at an angle θ to the horizontal and is at the point of toppling without any slipping.

Determine the value of $\tan\theta$.

$\boxed{\text{FM2-M}}$, $\boxed{\tan\theta = 5}$

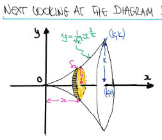
- START BY FINDING THE VOLUME OF REVOLUTION

$$V = \pi\int_{x_1}^{x_2} (y(x))^2 \, dx = \pi\int_0^k \left(\frac{1}{\sqrt{k}}x^{\frac{3}{2}}\right)^2 dx$$

$$V = \pi\int_0^k \frac{x^3}{k} \, dx = \frac{\pi}{4k}\left[x^4\right]_0^k$$

$$V = \frac{\pi}{4k}\left[k^4 - 0\right] = \frac{1}{4}\pi k^3$$

- NEXT LOOKING AT THE DIAGRAM BELOW



- THE MASS OF THE INFINITESIMAL DISC OF THICKNESS δx IS

$$\delta m = \rho\pi y^2 \delta x \qquad (\rho = \text{DENSITY})$$

- THE "MOMENT" OF THE INFINITESIMAL DISC ABOUT THE y AXIS IS GIVEN BY

$$(\rho\pi y^2 \delta x)x = \rho\pi x y^2 \delta x$$

- SUMMING UP, TAKING LIMITS AND CARRYING OUT THE RESULTING INTEGRATIONS

$$\Rightarrow M\bar{x} = \int_{x=0}^{x=k} \rho\pi x y^2 \, dx$$

$$\Rightarrow \left(\frac{1}{4}\pi k^3 \rho\right)\bar{x} = \rho\pi\int_{x=0}^{x=k} x\left(\frac{1}{\sqrt{k}}x^{\frac{3}{2}}\right)^2 dx$$

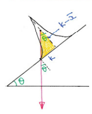
$$\Rightarrow \frac{1}{4}k^3\bar{x} = \int_{x=0}^{x=k} \frac{1}{k}x^4 \, dx$$

$$\Rightarrow \frac{1}{4}k^3\bar{x} = \frac{1}{5k}\left[x^5\right]_0^k$$

$$\Rightarrow \frac{1}{4}k^3\bar{x} = \frac{1}{5}k^4$$

$$\Rightarrow \bar{x} = \frac{4}{5}k$$

- FINALLY DRAWING THE SOLID ON THE INCLINED PLANE



$$\tan\theta = \frac{k}{k - \bar{x}}$$

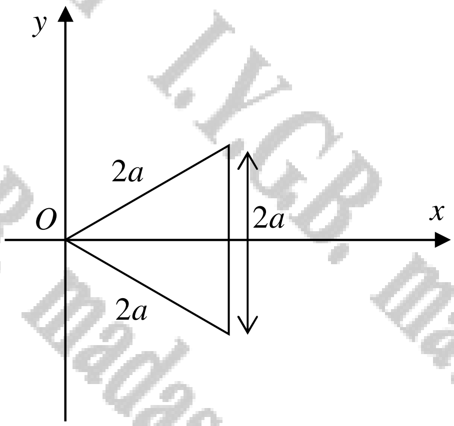
$$\tan\theta = \frac{k}{k - \frac{4}{5}k}$$

$$\tan\theta = \frac{1}{1 - \frac{4}{5}}$$

$$\tan\theta = \frac{1}{\frac{1}{5}}$$

$$\tan\theta = 5$$

Question 9 (****)

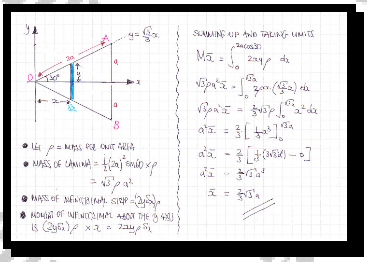


The figure above shows a uniform lamina in the shape of an equilateral triangle of side length $2a$.

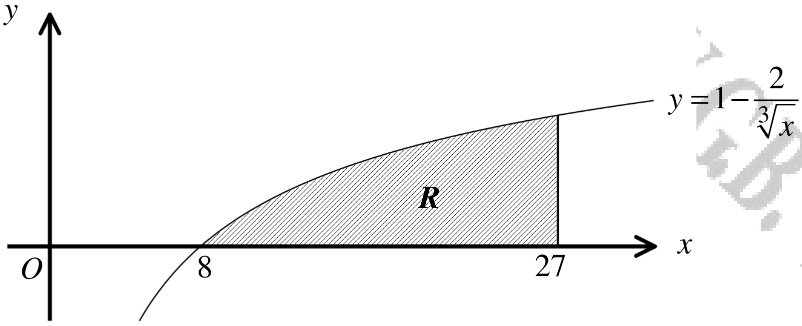
The lamina is referred in the Cartesian plane with one vertex at the origin O and an axis of symmetry along the x axis.

Use integration, with a detailed method, to find, in terms of a, the x coordinate of the centre of mass of the lamina.

$$\boxed{\bar{x} = \frac{2}{3}\sqrt{3}a}$$



Question 10 (***+)



The figure above shows the finite region R, bounded by the x axis, the curve with equation $y=1-\frac{2}{\sqrt[3]{x}}$ and the line with equation $x=27$.

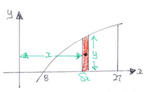
a) Use integration to determine, in exact form the coordinates, of the centre of mass of a lamina whose shape is that of R.

A shape whose area is 0.5 square units is removed from R so that the coordinates of the **resulting** shape are now located at $(20, 0.1)$.

b) Determine the coordinates of the centre of mass of the shape that was removed.

$$\boxed{\left(\frac{793}{40},\frac{1}{8}\right)=(19.825,0.125)}, \quad \boxed{(18.6,0.3)}$$

(a)



AREA UNDER CURVE IS
$\int_8^{27} 1-2x^{-\frac{1}{3}}\,dx$
$=\left[x-3x^{\frac{2}{3}}\right]_8^{27}$
$=(27-27)-(8-12)$
$=4$

- LET ρ = MASS PER UNIT AREA
- MOMENT OF INFINITESIMAL STRIP ABOUT THE ...

... y AXIS IS $(\rho y\,\delta x)x = \rho x y\,\delta x$
... x AXIS IS $(\rho y\,\delta x)\frac{1}{2}y = \frac{1}{2}\rho y^2\,\delta x$

- SUMMING UP AND TAKING LIMITS

$\Rightarrow M\bar{x} = \int_8^{27} \rho x y\,dx$
$\Rightarrow 4\rho\bar{x} = \int_8^{27} \rho x(1-2x^{-\frac{1}{3}})\,dx$
$\Rightarrow 4\bar{x} = \int_8^{27} x-2x^{\frac{2}{3}}\,dx$
$\Rightarrow 4\bar{x} = \left[\frac{1}{2}x^2-\frac{6}{5}x^{\frac{5}{3}}\right]_8^{27}$
$\Rightarrow 4\bar{x} = \left(\frac{729}{2}-\frac{1458}{5}\right)-\left(32-\frac{192}{5}\right)$
$\Rightarrow 4\bar{x} = \frac{793}{10}$
$\Rightarrow \bar{x} = 19.825$

$\Rightarrow M\bar{y} = \int_8^{27} \frac{1}{2}\rho y^2\,dx$
$\Rightarrow 4\rho\bar{y} = \frac{1}{2}\rho\int_8^{27} (1-2x^{-\frac{1}{3}})^2\,dx$
$\Rightarrow 8\bar{y} = \int_8^{27} 1-4x^{-\frac{1}{3}}+4x^{-\frac{2}{3}}\,dx$
$\Rightarrow 8\bar{y} = \left[x-6x^{\frac{2}{3}}+12x^{\frac{1}{3}}\right]_8^{27}$
$\Rightarrow 8\bar{y} = (27-54+36)-(8-24+24)$
$\Rightarrow 8\bar{y} = 1$
$\Rightarrow \bar{y} = \frac{1}{8}$

(b)

MASS RATIO	~~3.5~~ 7	0. 1	~~4~~ 8
x	20	$\bar{x}$	19.825
y	0.1	$\bar{y}$	0.125

THUS

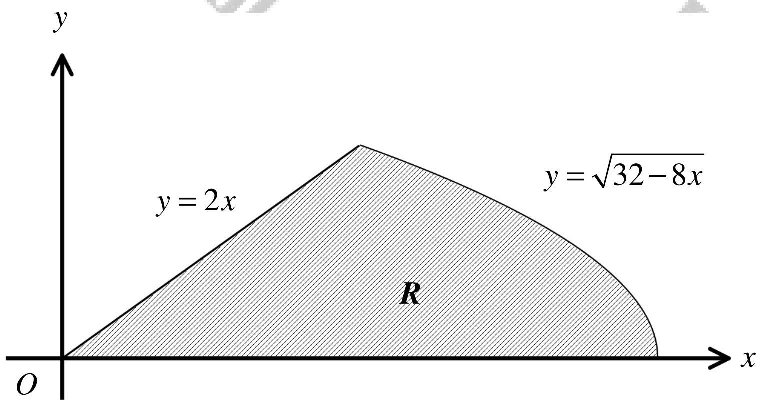
$7\times 20 + 1\bar{x} = 8\times 19.825$
$7\times 0.1 + 1\bar{y} = 8\times 0.125$ $\Rightarrow$

$140+\bar{x} = 158.6$
$0.7+\bar{y} = 1$

$\bar{x} = 18.6$
$\bar{y} = 0.3$

$\therefore (18.6, 0.3)$

Question 11 (****)

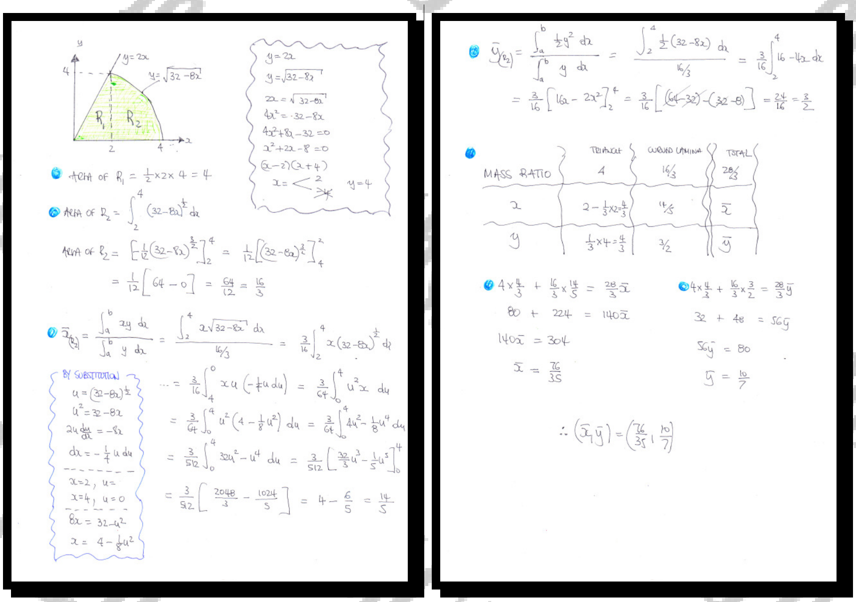


The figure above shows the finite region R bounded by the coordinate axes, the straight line with equation $y = 2x$ and the curve with equation $y = \sqrt{32 - 8x}$.

The centre of mass of a uniform lamina whose shape is that of R, is denoted by G.

Use integration to determine the exact coordinates of G.

$$\boxed{G\left(\frac{75}{35}, \frac{10}{7}\right)}$$



Question 12 (****)

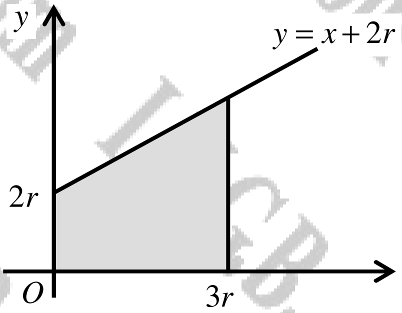


figure 1

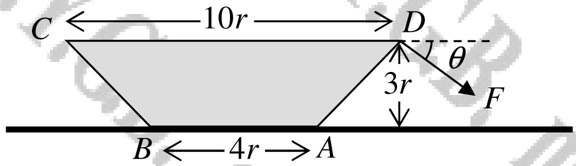


figure 2

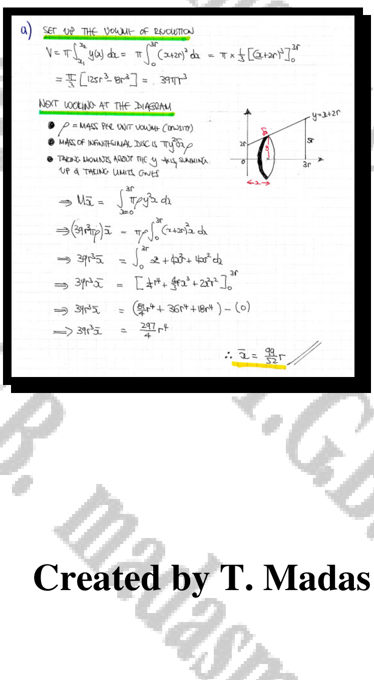
The finite region bounded by the coordinate axes and the lines with equations $y = x + 2r$ and $x = 3r$ is shown shaded in figure 1. This region is rotated about the x axis to form a frustum of a uniform right circular cone.

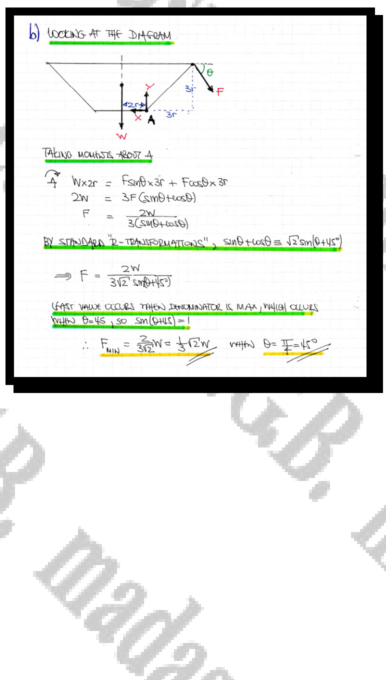
a) Use integration to find the distance of the centre of mass of the frustum of the cone from O.

The resulting frustum has weight W. The frustum is then placed on a rough horizontal surface with the plane surface of the frustum of radius $3r$ in contact with the surface. A force of magnitude , inclined at angle θ below the horizontal, is acting at a point on the circumference of the plane surface of the frustum of radius $5r$, as shown in figure 2. The frustum is at the point of toppling without sliding.

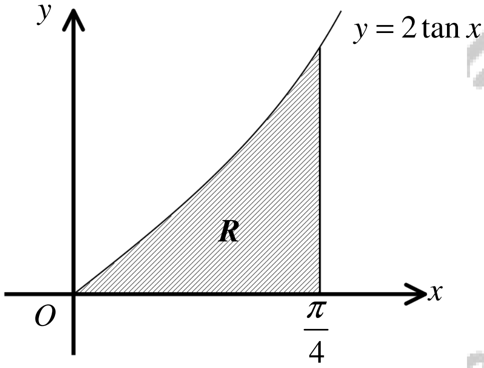
b) Given that θ can vary, determine the least value for F and the value of θ for which F takes this least value.

$\boxed{}$, $\boxed{\bar{x} = \frac{99r}{52}}$, $\boxed{F_{\text{min}} = \frac{\sqrt{2}}{3}W}$, $\boxed{\theta = 45°}$





Question 13 (****)



The figure above shows the finite region R bounded by the coordinate axes, the curve with equation $y = 2\tan x$ and the line with equation $x = \frac{\pi}{4}$.

a) Use integration to determine in exact form…

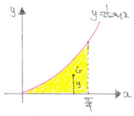
i. … the area of R.

ii. … the volume of the solid generated when R is revolved by a full turn about the x axis.

b) Hence, or otherwise, show that the y coordinate of the centre of mass of a lamina whose shape is that of R is $\frac{4-\pi}{\ln 4}$.

$\boxed{\text{area} = \ln 2}$, $\boxed{\text{volume} = \pi(4-\pi)}$

(a)



(I) AREA $= \int_0^{\frac{\pi}{4}} 2\tan x \, dx = \left[2\ln|\sec x|\right]_0^{\frac{\pi}{4}}$

$= 2\left[\ln\left|\sec\frac{\pi}{4}\right| - \ln|\sec 0|\right]$

$= 2\ln\sqrt{2} - 0$

$= \ln 2$

(II) VOLUME $= \pi\int_0^{\frac{\pi}{4}} (2\tan x)^2 \, dx = \int_0^{\frac{\pi}{4}} \tan^2 x \, dx = 4\pi\int_0^{\frac{\pi}{4}} \sec^2 x - 1 \, dx$

$= 4\pi\left[\tan x - x\right]_0^{\frac{\pi}{4}} = 4\pi\left[1 - \frac{\pi}{4}\right] = \pi(4-\pi)$

(b) BY ONE OF THE THEOREMS OF PAPPUS …

"THE VOLUME OF REVOLUTION" = "AREA OF REGION TO BE REVOLVED" × "DISTANCE TRAVELLED BY THE CENTROID OF THAT AREA"

SO $\pi(4-\pi) = \ln 2 \times 2\pi\bar{y}$

$4 - \pi = 2\ln 2 \times \bar{y}$

$\bar{y} = \frac{4-\pi}{2\ln 2}$

$\bar{y} = \frac{4-\pi}{\ln 4}$ AS REQUIRED

Question 14 (****)

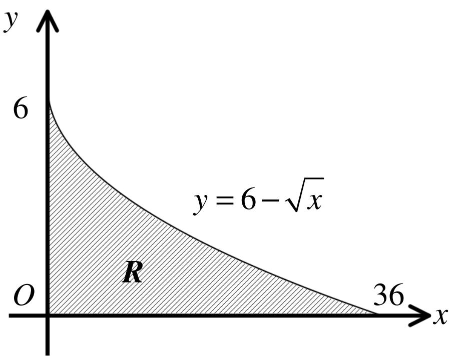


Figure 1

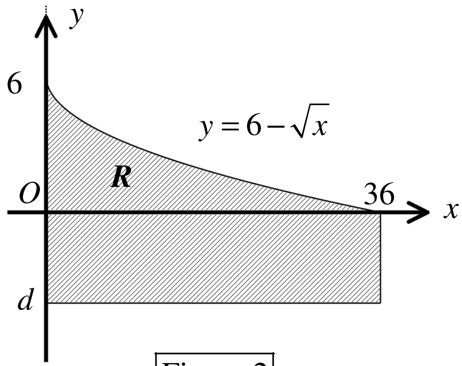


Figure 2

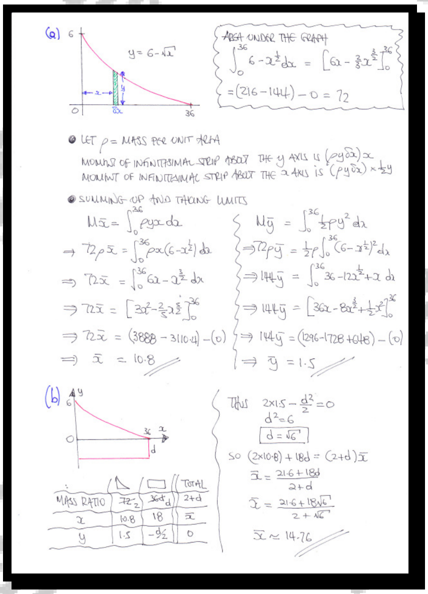
Figure 1 above shows the finite region R bounded by the coordinate axes and the curve with equation $y=6-\sqrt{x}$.

a) Use integration to determine the coordinates of the centre of mass of a uniform lamina of identical shape and measurements as that of R.

A rectangular lamina of identical thickness and density as R, measuring 36 units by d units is attached to R, as shown in figure 2. The centre of mass of the resulting composite now lies on the x axis.

b) Determine the x coordinate of the centre of mass of the composite

$\boxed{(10.8,1.5)}$, $\boxed{\bar{x}\approx 14.76}$



(a)

AREA UNDER THE GRAPH

$\int_0^{36} 6-x^{\frac{1}{2}}\,dx = \left[6x-\frac{2}{3}x^{\frac{3}{2}}\right]_0^{36}$

$=(216-144)-0=72$

- LET ρ = MASS PER UNIT AREA

MOMENT OF INFINITESIMAL STRIP ABOUT THE y AXIS IS $(\rho y\delta x)x$

MOMENT OF INFINITESIMAL STRIP ABOUT THE x AXIS IS $(\rho y\delta x)\times\frac{1}{2}y$

- SUMMING UP AND TAKING LIMITS

$M\bar{x}=\int_0^{36}\rho yx\,dx$

$\Rightarrow 72\rho\bar{x}=\int_0^{36}\rho x(6-x^{\frac{1}{2}})\,dx$

$\Rightarrow 72\bar{x}=\int_0^{36}6x-x^{\frac{3}{2}}\,dx$

$\Rightarrow 72\bar{x}=\left[3x^2-\frac{2}{5}x^{\frac{5}{2}}\right]_0^{36}$

$\Rightarrow 72\bar{x}=(3888-3110.4)-(0)$

$\Rightarrow \bar{x}=10.8$

$M\bar{y}=\int_0^{36}\frac{1}{2}\rho y^2\,dx$

$\Rightarrow 72\rho\bar{y}=\frac{1}{2}\rho\int_0^{36}(6-x^{\frac{1}{2}})^2\,dx$

$\Rightarrow 144\bar{y}=\int_0^{36}36-12x^{\frac{1}{2}}+x\,dx$

$\Rightarrow 144\bar{y}=\left[36x-8x^{\frac{3}{2}}+\frac{1}{2}x^2\right]_0^{36}$

$\Rightarrow 144\bar{y}=(1296-1728+648)-(0)$

$\Rightarrow \bar{y}=1.5$

(b)

	△	▭	TOTAL
MASS RATIO	$\cancel{72}\ 2$	$\cancel{36d}\ d$	$2+d$
x	10.8	18	$\bar{x}$
y	1.5	$-d/2$	0

THUS $2\times1.5-\frac{d^2}{2}=0$

$d^2=6$

$\boxed{d=\sqrt{6}}$

SO $(2\times10.8)+18d=(2+d)\bar{x}$

$\bar{x}=\frac{21.6+18d}{2+d}$

$\bar{x}=\frac{21.6+18\sqrt{6}}{2+\sqrt{6}}$

$\bar{x}\simeq 14.76$

Question 15 (****)

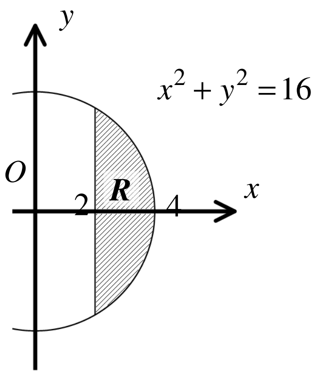


Figure 1

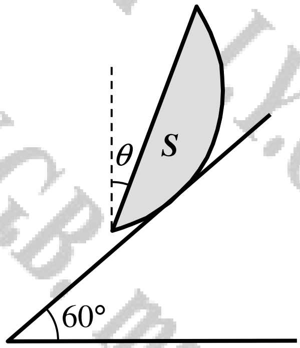


Figure 2

Figure 1 above shows the finite region R bounded by the circle and the line with respective equations $x^2 + y^2 = 16$ and $x = 2$. This region is fully revolved about the x axis to form a solid of revolution S

a) Use integration to determine the x coordinate of the centre of mass of S.

The solid S is carefully placed on a rough plane inclined at $60°$ to the horizontal and remains in equilibrium without slipping or toppling, as shown in figure 2.

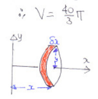
b) Determine the angle the plane face of S makes with the upward vertical, marked as θ in figure 2.

$\boxed{\bar{x} = 2.7}$, $\boxed{\theta \approx 42.2°}$

(a) FIRSTLY THE VOLUME OF REVOLUTION IS $V = \pi \int_{x_1}^{x_2} y^2 \, dx$

THUS $V = \pi \int_2^4 (16 - x^2) \, dx = \pi \left[16x - \frac{1}{3}x^3\right]_2^4 = \pi \left[\left(64 - \frac{64}{3}\right) - \left(32 - \frac{8}{3}\right)\right]$

$\therefore V = \frac{40}{3}\pi$



- LET ρ = MASS PER UNIT VOLUME
- MOMENT OF INFINITESIMAL DISC ABOUT THE y AXIS IS

$(\pi y^2 \delta x \rho) \times x = \pi \rho x y^2$

- TAKING LIMITS

$M\bar{x} = \int_2^4 \pi \rho x y^2 \, dx$

$\frac{40}{3}\pi\rho\bar{x} = \pi\rho \int_2^4 x(16 - x^2) \, dx$

$\frac{40}{3}\bar{x} = \int_2^4 16x - x^3 \, dx$

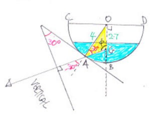
$\frac{40}{3}\bar{x} = \left[8x^2 - \frac{1}{4}x^4\right]_2^4$

$\frac{40}{3}\bar{x} = (128 - 64) - (32 - 4)$

$\frac{40}{3}\bar{x} = 36$

$\bar{x} = 2.7$

(b)



BY THE SINE RULE

$\frac{\sin\phi}{4} = \frac{\sin 30}{2.7} \Rightarrow \sin\phi = \frac{20}{27}$

$\phi = 132.2°$ (OBTUSE)

$\therefore$ OB IS AT AN ANGLE $180 - 132.2 = 47.8°$ TO THE VERTICAL

$\therefore$ OC IS AT AN ANGLE OF $90 - 47.8 = 42.2°$ TO VERTICAL

$\therefore \theta = 42.2°$

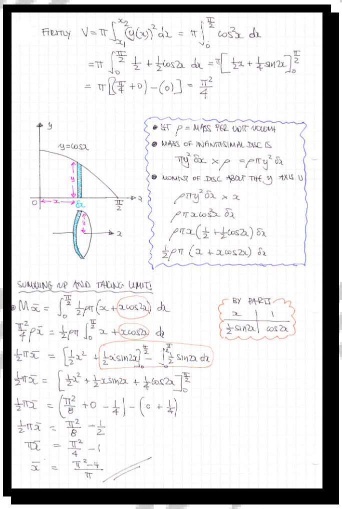
Question 16 (****+)

A finite region is bounded by the part of the curve with equation $y = \cos x$, the positive x axis and the positive y axis.

This region is rotated by 2π radians in the x axis forming a uniform solid S.

Use integration to find the x coordinate of the centre of mass of S.

$$\boxed{\frac{\pi^2 - 4}{4\pi}}$$



FIRSTLY $V = \pi \int_{x_1}^{x_2} (y(x))^2 \, dx = \pi \int_0^{\frac{\pi}{2}} \cos^2 x \, dx$

$= \pi \int_0^{\frac{\pi}{2}} \frac{1}{2} + \frac{1}{2}\cos 2x \, dx = \pi \left[\frac{1}{2}x + \frac{1}{4}\sin 2x \right]_0^{\frac{\pi}{2}}$

$= \pi \left[\left(\frac{\pi}{4} + 0 \right) - (0) \right] = \frac{\pi^2}{4}$

- LET ρ = MASS PER UNIT VOLUME
- MASS OF INFINITESIMAL DISC IS
  $\pi y^2 \delta x \times \rho = \rho \pi y^2 \delta x$
- MOMENT OF DISC ABOUT THE y AXIS IS
  $\rho \pi y^2 \delta x \times x$
  $\rho \pi x \cos^2 x \, \delta x$
  $\rho \pi x \left(\frac{1}{2} + \frac{1}{2}\cos 2x \right) \delta x$
  $\frac{1}{2}\rho \pi (x + x\cos 2x) \, \delta x$

SUMMING UP AND TAKING LIMITS

- $M\bar{x} = \int_0^{\frac{\pi}{2}} \frac{1}{2}\rho\pi (x + x\cos 2x) \, dx$

$\frac{\pi^2}{4}\rho\bar{x} = \frac{1}{2}\rho\pi \int_0^{\frac{\pi}{2}} x + x\cos 2x \, dx$

$\frac{1}{2}\pi\bar{x} = \left[\frac{1}{2}x^2 + \frac{1}{2}x\sin 2x \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \frac{1}{2}\sin 2x \, dx$

$\frac{1}{2}\pi\bar{x} = \left[\frac{1}{2}x^2 + \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x \right]_0^{\frac{\pi}{2}}$

$\frac{1}{2}\pi\bar{x} = \left(\frac{\pi^2}{8} + 0 - \frac{1}{4} \right) - \left(0 + \frac{1}{4} \right)$

$\frac{1}{2}\pi\bar{x} = \frac{\pi^2}{8} - \frac{1}{2}$

$\pi\bar{x} = \frac{\pi^2}{4} - 1$

$\bar{x} = \frac{\pi^2 - 4}{\pi}$

BY PARTS

x	1
$\frac{1}{2}\sin 2x$	$\cos 2x$

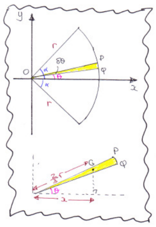
Question 17 (****+)

A circular sector of radius r subtends an angle of 2α at its centre O. The position of the centre of mass of this sector lies at the point G, along its axis of symmetry.

Use calculus to show that

$$|OG| = \frac{2r\sin\alpha}{3\alpha}.$$

proof



- CONSIDER AN INFINITESIMAL SECTOR $O\widehat{PQ}$. AS $\delta\theta \ll 1$ THE SECTOR IS AN APPROXIMATE TRIANGLE WHOSE CENTRE OF MASS LIES $\frac{2}{3}$ ALONG ITS MEDIAN FROM O, I.E $\frac{2}{3}r$
- LET ρ = MASS PER UNIT AREA
- THE MASS OF THE INFINITESIMAL IS $m_i = \rho\left(\frac{1}{2}r^2\delta\theta\right)$
- THE MASS OF THE ENTIRE LAMINA IS $\frac{1}{2}r^2(2\alpha)\rho = \rho\alpha r^2$
- $\bar{x} = \frac{\sum m_i x_i}{\sum m_i}$ (OR "TAKE MOMENTS ABOUT y")

$$\bar{x} = \frac{\sum \frac{1}{2}\rho r^2\delta\theta \times x}{\rho\alpha r^2} = \frac{\frac{1}{2}\cancel{\rho}\cancel{r^2}\sum\left(\frac{2}{3}r\cos\theta\right)}{\cancel{\rho}\alpha\cancel{r^2}}$$

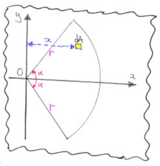
TAKING LIMITS

$$\bar{x} = \frac{1}{2\alpha}\int_{\theta=-\alpha}^{\theta=\alpha}\frac{2}{3}r\cos\theta\,d\theta = \frac{r}{3\alpha}\int_{-\alpha}^{\alpha}\cos\theta\,d\theta = \dots \text{EVEN INTEGRAND}$$

$$= \frac{2r}{3\alpha}\int_0^{\alpha}\cos\theta\,d\theta = \frac{2r}{3\alpha}\Big[\sin\theta\Big]_0^{\alpha} = \frac{2r\sin\alpha}{3\alpha}$$

AS THE CENTRE OF MASS LIES ON THE CENTRE OF SYMMETRY OF THE SECTOR THEN $\bar{x}$ IS ALSO THE CENTRE OF MASS OF THE SECTOR

I.E $\bar{r} = \frac{2r\sin\alpha}{3\alpha}$



- LET ρ = MASS PER UNIT AREA
- MASS OF INFINITESIMAL AREA IS $\rho\,dA$
- MOMENT OF INFINITESIMAL ABOUT THE x-AXIS IS $(\rho\,dA)x$
- TOTAL MASS OF THE SECTOR IS $\frac{1}{2}\rho r^2(2\alpha) = \alpha r^2\rho$
- SUMMING MOMENTS ABOUT THE y-AXIS

$$M\bar{x} = \int_{\text{SECTOR}}\rho x\,dA$$

- $\bar{x}$ WILL ALSO BE THE POSITION OF THE CENTRE OF MASS BY SYMMETRY

- TAKING LIMITS

$$\alpha r^2\rho\bar{x} = \int_{\text{SECTOR}}\rho x\,dA$$

- SWITCH INTO PLANE POLARS

$$\Rightarrow \underset{\text{CONSTANT}}{\alpha r^2\rho}\,\bar{x} = \int_{\theta=-\alpha}^{\alpha}\int_{r=0}^{r \leftarrow \text{CONSTANT}}\rho(r\cos\theta)\,r\,dr\,d\theta$$

$$\Rightarrow \underset{\text{CONSTANT}}{\alpha r^2\rho}\,\bar{x} = \rho\int_{-\alpha}^{\alpha}\int_0^{r}r^2\cos\theta\,dr\,d\theta$$

$$\Rightarrow \alpha r^2\bar{x} = \int_{-\alpha}^{\alpha}\left[\frac{1}{3}r^3\right]_0^{r}\cos\theta\,d\theta$$

$$\Rightarrow \alpha r^2\bar{x} = \frac{1}{3}r^3\int_{-\alpha}^{\alpha}\cos\theta\,d\theta$$

$$\Rightarrow \frac{3\alpha}{r}\bar{x} = \int_{-\alpha}^{\alpha}\cos\theta\,d\theta$$

$$\Rightarrow \frac{3\alpha}{r}\bar{x} = \Big[\sin\theta\Big]_{-\alpha}^{\alpha}$$

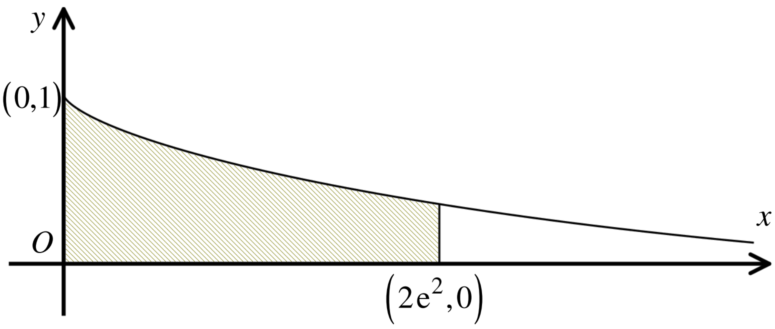
$$\Rightarrow \frac{3\alpha}{r}\bar{x} = \left[\sin\alpha - \sin(-\alpha)\right]$$

$$\Rightarrow \frac{3\alpha}{r}\bar{x} = \sin\alpha + \sin\alpha$$

$$\Rightarrow \frac{3\alpha\bar{x}}{r} = 2\sin\alpha$$

$$\Rightarrow \bar{x} = \frac{2r\sin\alpha}{3\alpha}$$

Question 18 (*****)



The figure above shows part of the curve with parametric equations

$$x = t\mathrm{e}^{t}, \quad y = \mathrm{e}^{-t}, \quad t \in \mathbb{R}.$$

A uniform lamina occupies the finite region, shown shaded in the figure, bounded by the curve, the coordinate axes and the straight line with equation $x = 2\mathrm{e}^2$.

Determine the exact coordinates of the centre of mass of this lamina.

$\boxed{}$, $\boxed{\bar{x} = \frac{1}{4}\left(3\mathrm{e}^2 - 1\right)}$, $\boxed{\bar{y} = \frac{1}{4}\left(1 - 2\mathrm{e}^{-2}\right)}$