GENERAL CIRCULAR MOTION

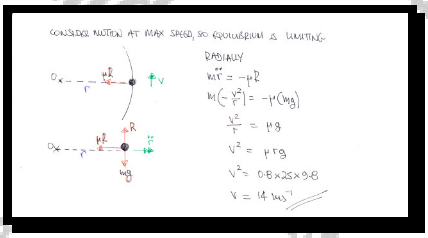
Question 1 (**)

A parcel is placed on a rough flat horizontal surface at the back of a delivery van.

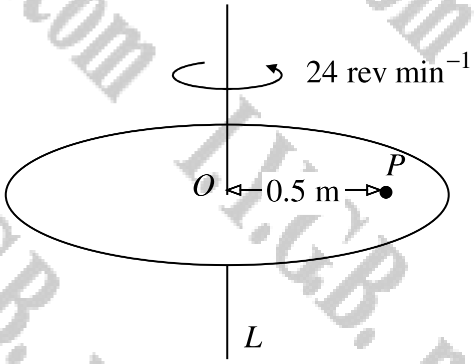
The van travels around a circular bend of radius 25 m at constant speed and the parcel does not slide. The coefficient of friction between the parcel and surface at the back of the van particle is 0.8.

Calculate the greatest speed of the van around this bend.

$$\boxed{V_{\max} = 14 \text{ ms}^{-1}}$$



Question 2 (**)

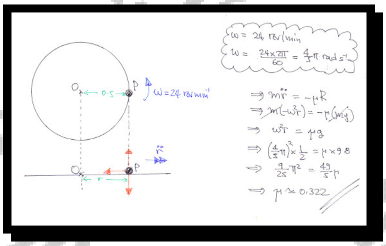


A rough turntable is rotating in a horizontal plane about a vertical axis L passing through its centre O, with constant angular speed of 24 revolutions per minute.

A particle P is located 0.5 m from O and it is at the point of slipping.

Calculate the value of the coefficient of friction between P and the turntable.

$\boxed{\mu \approx 0.322}$

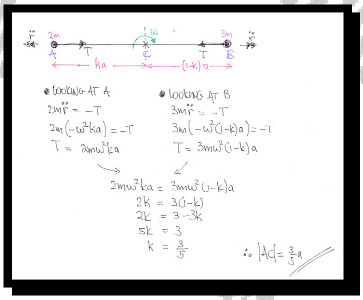


Question 3 **(***)**

A light rod AB, of length a, has particles of masses $2m$ and $3m$ attached at A and B, respectively. The rod is made to rotate with constant angular velocity about the point C on the rod, so that the tensions in the sections AC and CB are equal.

Show clearly that $AC = \frac{3}{5}a$.

proof



Question 4 (***)

A rough circular plate rotates horizontally with constant angular velocity ω rad s^{-1}, about s smooth vertical axis through its centre.

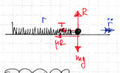
A particle of mass 0.5 kg lies at a point on the plate at a distance of 0.75 m from the centre of the plate.

The particle is connected to the axis through the centre of the plate by an elastic string of natural length 0.6 m, and modulus of elasticity 15 N.

If the coefficient of friction between the plate and the particle is 0.45, calculate the minimum and the maximum value of ω.

$\boxed{}$, $\boxed{\omega_{\min} \approx 2.03}$, $\boxed{\omega_{\max} \approx 3.98}$

DRAWING A DIAGRAM WITH PARTICLE SLIPPING OUTWARDS (ω MAX)



$m = 0.5$ kg
$r = 0.75$
$l = 0.6$
$\lambda = 15$
$\mu = 0.45$

- $R = mg$
- $m\ddot{r} = -T - \mu R$

$m(-\omega^2 r) = -\frac{\lambda}{l}x - \mu(mg)$

$m\omega^2 r = \frac{\lambda}{l}x + \mu mg$

$0.5\omega^2(0.75) = \frac{15}{0.6}(0.75-0.6) + 0.45 \times 0.5 \times 9.8$

$\frac{3}{8}\omega^2 = \frac{15}{4} + 2.205$

$\frac{3}{8}\omega^2 = 5.955$

$\omega^2 = 15.88$

$\omega_{MAX} = 3.98$ rad s^{-1}

TO FIND ω_{MIN} EVERYTHING IS THE SAME, EXCEPT FRICTION IS ACTING OUTWARDS

$\vdots$

$\frac{3}{8}\omega^2 = \frac{15}{4} - 2.205$

$\frac{3}{8}\omega^2 = 1.545$

$\omega^2 = 4.12$

$\omega_{MIN} \simeq 2.03$ rad s^{-1}

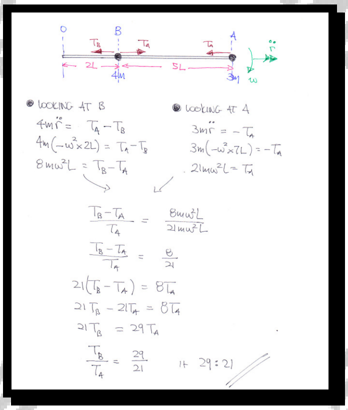
Question 45 (***+)

A particle of mass $3m$ is attached at the point A of light rigid rod OA, of length $7L$. A second particle of mass $4m$ is attached at the point B on the rod, where $OB = 2L$.

The rod is made to rotate with constant angular velocity about O.

Determine the ratio of the tensions in the sections OB and BA.

$\boxed{29:21}$



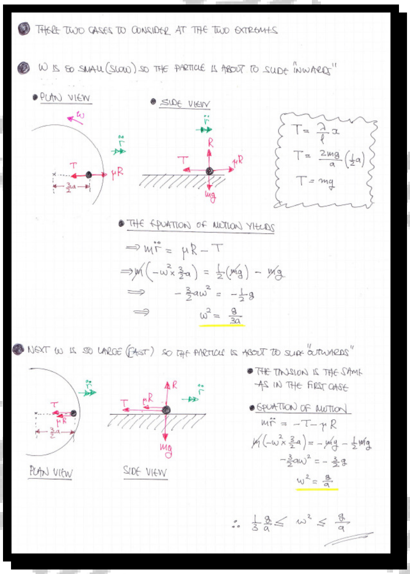
Question 6 (****)

A rough disc rotates in a horizontal plane with constant angular velocity ω about a fixed vertical axis. A particle of mass m is connected to the axis by a light elastic string of natural length a and modulus of elasticity of $2mg$.

As the disc rotates the particle lies at rest on the disc at a distance of $\frac{3}{2}a$ from the axis.

Given that the coefficient of friction between the disc and the particle is $\frac{1}{2}$, find the range of ω^2, in terms of a and g.

$$\boxed{\frac{g}{3a} \le \omega^2 \le \frac{g}{a}}$$


THERE TWO CASES TO CONSIDER AT THE TWO EXTREMES

ω IS SO SMALL (SLOW) SO THE PARTICLE IS ABOUT TO SLIDE "INWARDS"

- PLAN VIEW
- SIDE VIEW

$T = \frac{\lambda}{l}x$

$T = \frac{2mg}{a}\left(\frac{1}{2}a\right)$

$T = mg$

- THE EQUATION OF MOTION YIELDS

$\Longrightarrow m\ddot{r} = \mu R - T$

$\Longrightarrow \cancel{m}\left(-\omega^2 \times \frac{3}{2}a\right) = \frac{1}{2}(\cancel{m}g) - \cancel{m}g$

$\Longrightarrow -\frac{3}{2}a\omega^2 = -\frac{1}{2}g$

$\Longrightarrow \omega^2 = \frac{g}{3a}$

NEXT ω IS SO LARGE (FAST) SO THE PARTICLE IS ABOUT TO SLIDE "OUTWARDS"

PLAN VIEW

SIDE VIEW

- THE TENSION IS THE SAME AS IN THE FIRST CASE
- EQUATION OF MOTION

$m\ddot{r} = -T - \mu R$

$\cancel{m}\left(-\omega^2 \times \frac{3}{2}a\right) = -\cancel{m}g - \frac{1}{2}\cancel{m}g$

$-\frac{3}{2}a\omega^2 = -\frac{3}{2}g$

$\omega^2 = \frac{g}{a}$

$\therefore \frac{1}{3}\frac{g}{a} \le \omega^2 \le \frac{g}{a}$

Question 7 (*****)

A satellite is moving in a circular orbit above the Earth's equator.

The orbit is described as geostationary, which means that angular velocity of the satellite is identical to that of Earth's rotation, so it appears in a fixed position relative to an observer on the Earth.

The radius of the orbit, measured from the centre of the Earth, is r.

a) Show that

$$r = \sqrt[3]{\frac{GMT^2}{4\pi^2}},$$

where M is the mass of the earth, T is the period of the motion and G is the universal gravitation constant.

b) Determine the value of r, and hence find the minimum number of satellites needed to view all the points on the earth's equator.

You may assume

- earth's mass, $M = 5.97 \times 10^{24}$ kg
- earth's radius, $R = 6.37 \times 10^{6}$ m
- universal gravitation constant, $G = 6.67 \times 10^{-11}\ \text{m}^3\ \text{kg}^{-1}\ \text{s}^{-2}$

, $\boxed{3}$

a) LOOKING AT THE DIAGRAM

$\Rightarrow F = G\frac{mM}{r^2}$

$\Rightarrow \cancel{m}(\omega^2 r) = G\frac{\cancel{m}M}{r^2}$

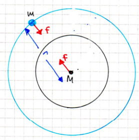
$\Rightarrow \omega^2 r^3 = GM$

$\Rightarrow \left(\frac{2\pi}{T}\right)^2 r^3 = GM$

$\Rightarrow \frac{4\pi^2 r^3}{T^2} = GM$

$\Rightarrow r^3 = \frac{GMT^2}{4\pi^2}$

$\Rightarrow r = \sqrt[3]{\frac{GMT^2}{4\pi^2}}$ //



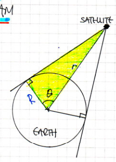
b) $M = 5.97 \times 10^{24}$ kg
$G = 6.67 \times 10^{-11}\ \text{m}^3\ \text{kg}^{-1}\ \text{s}^{-2}$
$T = 24$ HOURS $= 8.64 \times 10^4$ s
$R = 6.37 \times 10^6$ m

$r = \left[\frac{6.67 \times 10^{-11} \times 5.97 \times 10^{24} \times (8.64 \times 10^4)^2}{4\pi^2}\right]^{\frac{1}{3}}$

$r \approx \sqrt[3]{7.5295\ldots \times 10^{22}}$

$r \approx 4.22 \times 10^7$

LOOKING AT THE FOLLOWING DIAGRAM



$\Rightarrow \cos\theta = \frac{R}{r}$

$\Rightarrow \cos\theta = \frac{6.37 \times 10^6}{4.22 \times 10^7}$

$\Rightarrow \cos\theta = 0.1509\ldots$

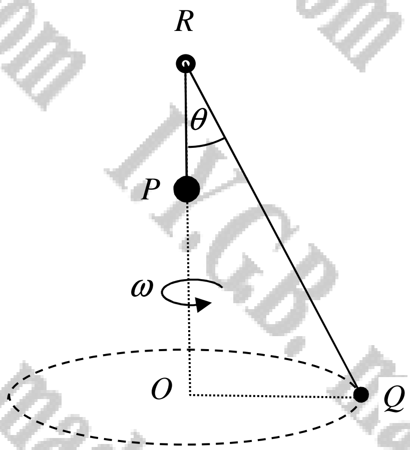
$\Rightarrow \theta \approx 81.3\ldots$

$\Rightarrow 2\theta \approx 162.63\ldots$

$\therefore \frac{360}{162.63} \approx 2.21\ldots$ IE MINIMUM OF 3 //

CONICAL PENDULUM MOTION

Question 1 (**)



The figure above shows two particles P and Q of respective masses of $3m$ and m attached to the ends of a light inextensible string of length $7l$. The string is threaded through a fixed smooth ring R.

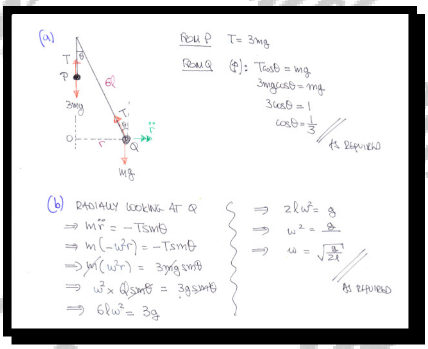
When Q is moving in a horizontal circle with constant angular velocity ω, P remains in equilibrium at a vertical distance l below R. The centre of the circle that Q describes has its centre at O, where O is vertically below P, and the angle RQ makes with the vertical is denoted by θ.

Show clearly that …

a) … $\cos\theta = \frac{1}{3}$.

b) … $\omega = \sqrt{\frac{g}{2l}}$.

$\boxed{\text{proof}}$


(a) FROM P $T = 3mg$

FROM Q $(\uparrow)$: $T\cos\theta = mg$

$3mg\cos\theta = mg$

$3\cos\theta = 1$

$\cos\theta = \frac{1}{3}$ // AS REQUIRED

(b) RADIALLY LOOKING AT Q

$\Rightarrow m\ddot{r} = -T\sin\theta$

$\Rightarrow m(-\omega^2 r) = -T\sin\theta$

$\Rightarrow \cancel{m}(\omega^2 r) = 3\cancel{m}g\sin\theta$

$\Rightarrow \omega^2 \times 6l\cancel{\sin\theta} = 3g\cancel{\sin\theta}$

$\Rightarrow 6l\omega^2 = 3g$

$\Rightarrow 2l\omega^2 = g$

$\Rightarrow \omega^2 = \frac{g}{2l}$

$\Rightarrow \omega = \sqrt{\frac{g}{2l}}$ // AS REQUIRED

Question 2 (**)

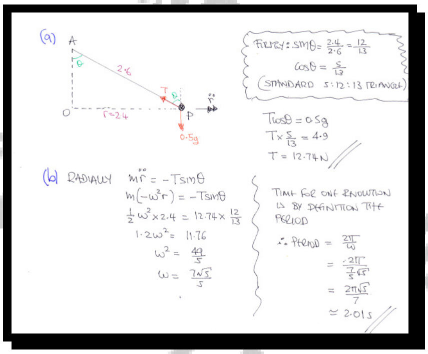
A light inextensible string of length 2.6 m has one end attached to a fixed point A and the other end attached to a particle P of mass 0.5 kg.

P is moving with constant speed in a horizontal circle with radius 2.4 m and centre at the point O, which is vertically below A.

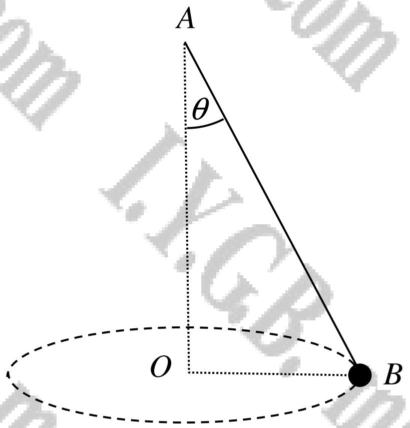
a) Determine the tension in the string.

b) Calculate the time it takes P to make one complete revolution.

$\boxed{T = 12.74 \text{ N}}$, $\boxed{t \approx 2.01 \text{ s}}$



Question 3 (***)



The figure above shows a particle B of mass 6 kg attached to one end of a light inextensible string. The other end of the string is attached to a fixed point A.

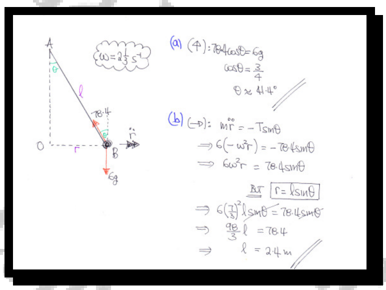
The particle moves with constant angular speed $2\frac{1}{3}$ rad s^{-1} in a horizontal circle with centre O, where O is vertically below Q. The angle AOB is denoted by θ.

The tension in the string is 78.4 N.

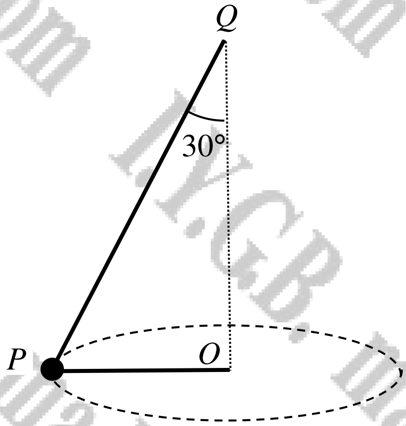
a) Show clearly that θ is approximately $41.4°$.

b) Calculate the length of the string.

$\boxed{L = 2.4 \text{ m}}$



Question 4 (***)



A particle, of mass 2 kg, is attached to one end, P, of a light inextensible string.

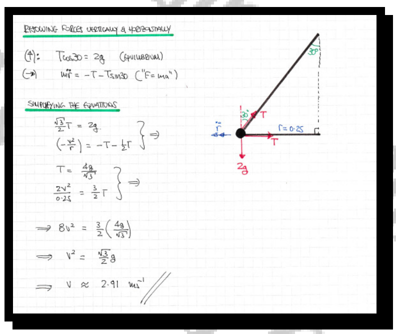
The other end of the string, Q, is attached to a fixed point.

Another light inextensible string PO, has its end P also attached to the same particle and its other end , O, attached to another fixed point, so that O is vertically below Q such that $\measuredangle OQP = 30°$.

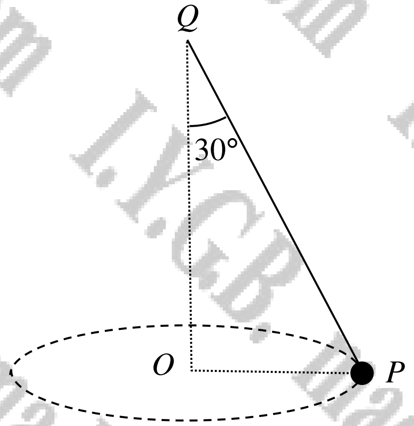
The particle moves with constant speed v in a horizontal circle of radius 0.25 m centred at O, as shown in the figure above,

Given that the tensions in both strings are equal, find the value of v.

$\boxed{\text{FM2 N}}$, $\boxed{v \approx 2.91 \text{ ms}^{-1}}$



Question 5 (***+)



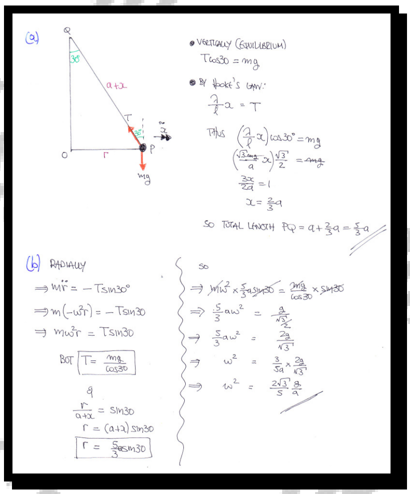
The figure above shows a particle P of mass m attached to one end of a light elastic string of natural length a and modulus of elasticity $\sqrt{3}mg$.

The other end of the string is attached to a fixed point Q. The particle moves with constant angular speed ω in a horizontal circle with centre O, where O is vertically below Q such that $\measuredangle OQP = 30°$.

a) Show that $PQ = \frac{5}{3}a$.

b) Find ω^2 in terms of a and g.

$$\boxed{\omega^2 = \frac{2\sqrt{3}g}{5a}}$$



Question 6 (***+)

A particle P of mass 2.5 kg attached to one end of a light inextensible string of length 1 m. The other end of the string is attached to a fixed point Q which is 0.5 m above a smooth horizontal surface. The particle moves with constant angular speed 1.6 s^{-1} in a horizontal circle whose centre O lies vertically below Q.

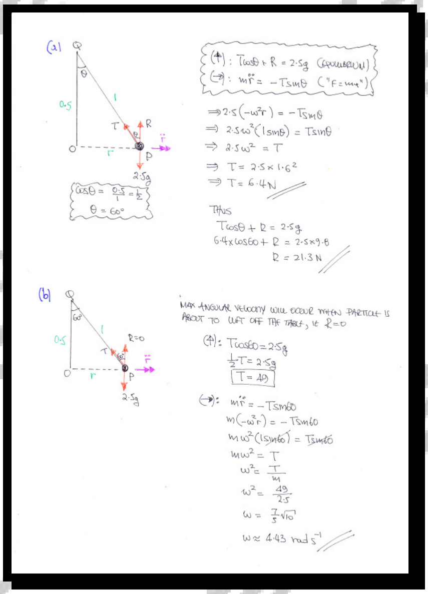
a) Determine …

  i. … the tension in the string.

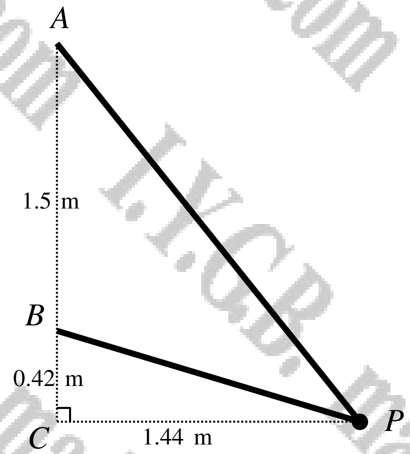
  ii. … the normal reaction between P and the table.

b) Calculate the greatest angular speed of P, so that P remains in contact with the horizontal surface.

$\boxed{T = 6.4 \text{ N}}$, $\boxed{R = 21.3 \text{ N}}$, $\boxed{\omega = \frac{7}{5}\sqrt{10} \text{ s}^{-1} \approx 4.43 \text{ s}^{-1}}$



Question 7 (***+)



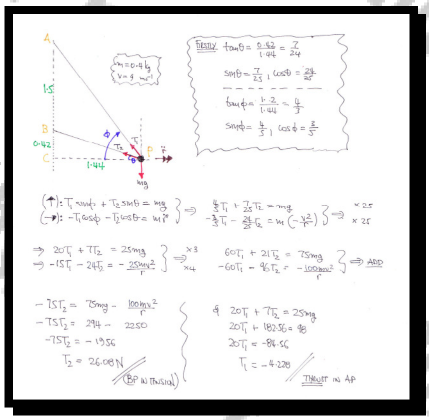
One of the two ends of each of the light rigid rods AP and BP, are attached to a particle P of mass 0.4 kg.

The particle is made to rotate with constant speed of 9 ms^{-1} in a horizontal circle with centre at C, so that A, B C and P lie in the same vertical plane, with B vertically above C and A vertically above B, as shown in the figure above.

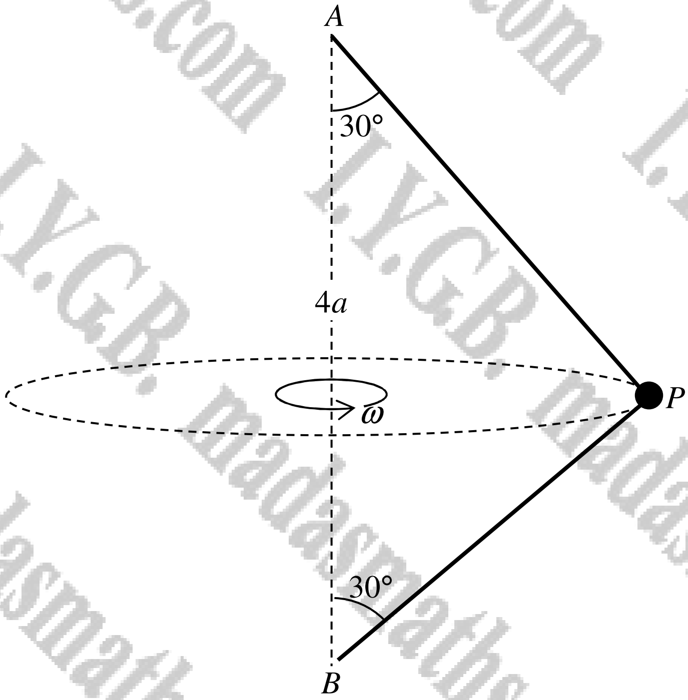
The distance of AB, BC and CP are 1.5 m, 0.42 m and 1.44 m, respectively.

Determine the magnitude and direction of the force acting along each of the two rods, AP and BP.

$\boxed{F_{BP} = 26.08 \text{ N, tension}}$, $\boxed{F_{AP} = 4.228 \text{ N, thrust}}$



Question 8 (*+)**

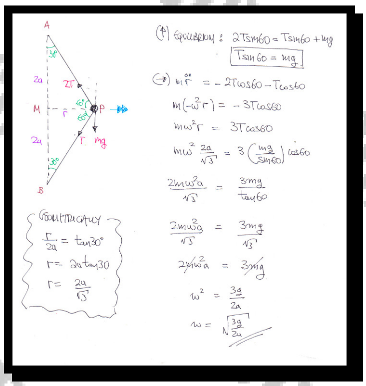


A particle P of mass m is attached at the end of two light inextensible strings AP and BP, of equal lengths. The points A, P and B lie in the same vertical plane with A located at a distance of $4a$ vertically above B, as shown in the figure above.

The particle rotates in a horizontal circle with constant angular speed ω, with both strings taut and both inclined at $30°$ to the vertical.

Given that tension in the string AP is twice as large as the tension in the string BP, show that $\omega = \sqrt{\dfrac{3g}{2a}}$.

proof



Question 9 (***+)

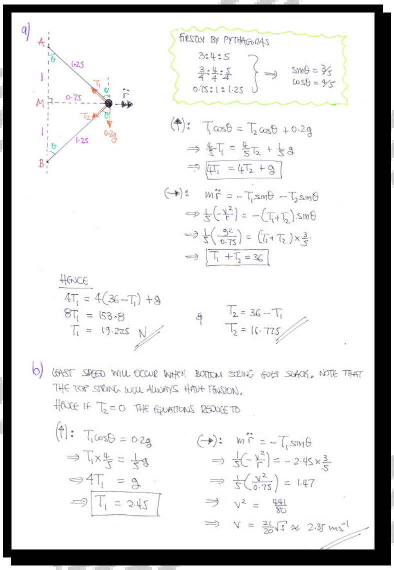
The points A and B are 2 m apart, with A vertically above B. A particle P of mass 0.2 kg is attached by two identical light inextensible strings, each of length 1.25 m to A and B.

The particle moves with constants speed V ms^{-1} in a horizontal circle whose centre is at the midpoint of AB.

a) Given that $V = 9$, determine the tension in each of the two strings.

b) Given instead that V can vary, calculate the least value of V, for which the motion described is possible.

$\boxed{T_A = 22.9 \text{ N}}$, $\boxed{T_B = 13.1 \text{ N}}$, $\boxed{V_{\min} = \frac{21}{20}\sqrt{5} \text{ ms}^{-1} \approx 2.35 \text{ ms}^{-1}}$


a)

FIRSTLY BY PYTHAGORAS

$3:4:5$

$\frac{3}{4}:\frac{4}{4}:\frac{5}{4}$

$0.75:1:1.25$

$\Rightarrow \sin\theta = \frac{3}{5}$, $\cos\theta = \frac{4}{5}$

$(\uparrow)$: $T_1\cos\theta = T_2\cos\theta + 0.2g$

$\Rightarrow \frac{4}{5}T_1 = \frac{4}{5}T_2 + \frac{1}{5}g$

$\Rightarrow \boxed{4T_1 = 4T_2 + g}$

$(\rightarrow)$: $m\ddot{r} = -T_1\sin\theta - T_2\sin\theta$

$\Rightarrow \frac{1}{5}\left(-\frac{v^2}{r}\right) = -(T_1+T_2)\sin\theta$

$\Rightarrow \frac{1}{5}\left(\frac{9^2}{0.75}\right) = (T_1+T_2)\times\frac{3}{5}$

$\Rightarrow \boxed{T_1 + T_2 = 36}$

HENCE

$4T_1 = 4(36-T_1) + g$

$8T_1 = 153.8$

$T_1 = 19.225$ N

& $T_2 = 36 - T_1$

$T_2 = 16.775$

b) LEAST SPEED WILL OCCUR WHEN BOTTOM STRING GOES SLACK. NOTE THAT THE TOP STRING WILL ALWAYS HAVE TENSION.

HENCE IF $T_2 = 0$ THE EQUATIONS REDUCE TO

$(\uparrow)$: $T_1\cos\theta = 0.2g$

$\Rightarrow T_1 \times \frac{4}{5} = \frac{1}{5}g$

$\Rightarrow 4T_1 = g$

$\Rightarrow \boxed{T_1 = 2.45}$

$(\rightarrow)$: $m\ddot{r} = -T_1\sin\theta$

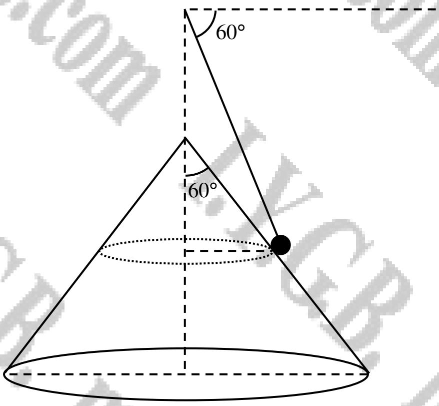
$\Rightarrow \frac{1}{5}\left(-\frac{v^2}{r}\right) = -2.45\times\frac{3}{5}$

$\Rightarrow \frac{1}{5}\left(\frac{v^2}{0.75}\right) = 1.47$

$\Rightarrow v^2 = \frac{441}{80}$

$\Rightarrow v = \frac{21}{20}\sqrt{5} \approx 2.35$ ms^{-1}

Question 10 (**+)**



The figure above shows a right circular cone of semi-vertical angle $60°$ is fixed with its axis vertical and vertex upwards.

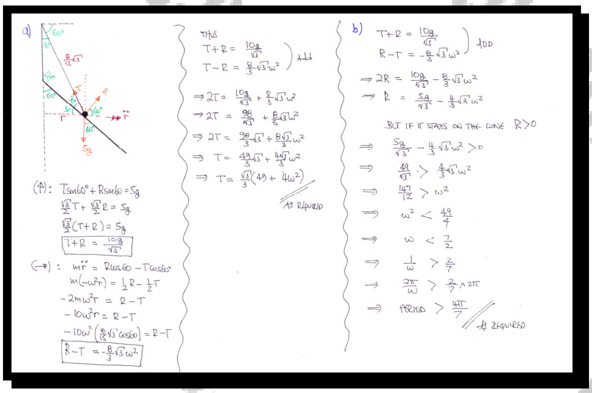
A particle of mass 5 kg is attached to one end of a light inextensible string of length $\frac{8}{15}\sqrt{3}$ m, and the other end of the string is attached to a fixed point vertically above the vertex of the cone.

The particle moves in a horizontal circle on the smooth outer surface of the cone with constant angular speed ω rad s^{-1}, with the string making a constant angle of $60°$ with the horizontal.

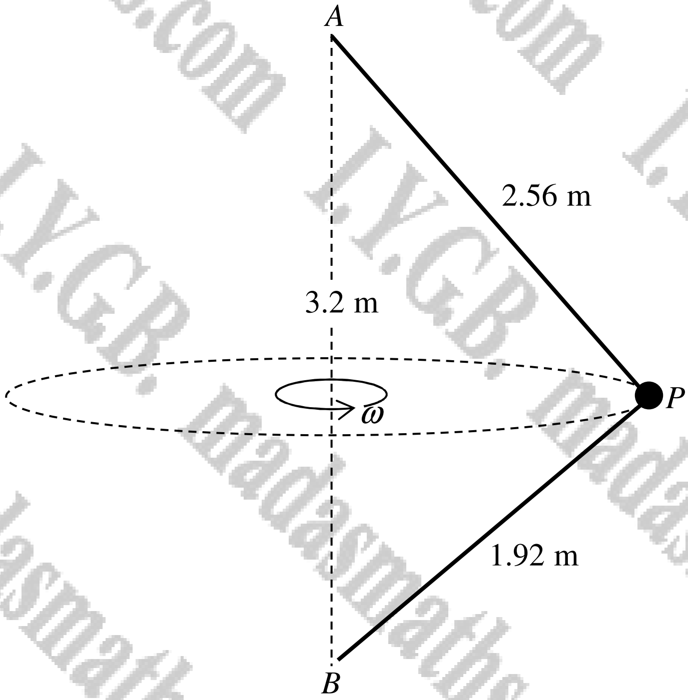
a) Show that the tension in the string is $\frac{\sqrt{3}}{3}\left(49+4\omega^2\right)$.

b) Given that the particle remains on the surface of the cone, show further that the time for the particle to make one complete revolution is at least $\frac{4\pi}{7}$.

proof



Question 11 (****+)



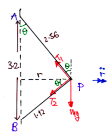
A particle P of mass m is attached at the end of two light inextensible strings AP and BP, of respective lengths 2.56 m and 1.92 m. The points A, P and B lie in the same vertical plane with A located at a distance of 3.2 m vertically above B, as shown in the figure above.

The particle rotates in a horizontal circle with constant angular speed ω, with both strings taut.

Show that the time for the particle to make a complete revolution is at most $\frac{32}{35}\pi$.

[ ], [proof]

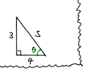
START WITH A DETAILED DIAGRAM



BY INSPECTION THE TRIANGLE IS 3:4:5 (OR USE THE COSINE RULE TO FIND $\cos\theta$)

$$1.92 : 2.56 : 3.2$$
$$192 : 256 : 320 \quad \times 10$$
$$3 : 4 : 5 \quad \div 64$$

$\therefore \sin\theta = \frac{3}{5}$, $\cos\theta = \frac{4}{5}$, $\tan\theta = \frac{3}{4}$



VERTICALLY (EQUILIBRIUM)

$$T_1\cos\theta = T_2\sin\theta + mg$$
$$\frac{4}{5}T_1 = \frac{3}{5}T_2 + mg \quad \times 5$$
$$4T_1 = 3T_2 + 5mg \quad \times 3$$
$$12T_1 = 9T_2 + 15mg$$

RADIALLY (ACCELERATION)

$$m\ddot{r} = -T_1\sin\theta - T_2\cos\theta$$
$$m(-\omega^2 r) = -\frac{3}{5}T_1 - \frac{4}{5}T_2 \quad \times 5$$
$$5m\omega^2 r = 3T_1 + 4T_2 \quad \times 4$$
$$20m\omega^2 r = 12T_1 + 16T_2$$

SOLVING BY SUBSTITUTION – WE ONLY NEED T_2 TO SET $\geqslant 0$

$$\Rightarrow 20m\omega^2 r = 9T_2 + 15mg + 16T_2$$
$$\Rightarrow 20m\omega^2 r - 15mg = 25T_2$$

NOW T_2 "MUST HAVE TENSION", I.E. $T_2 \geqslant 0$

$$\Rightarrow 20m\omega^2 r - 15mg \geqslant 0$$
$$\Rightarrow 4\omega^2 r - 3g \geqslant 0$$

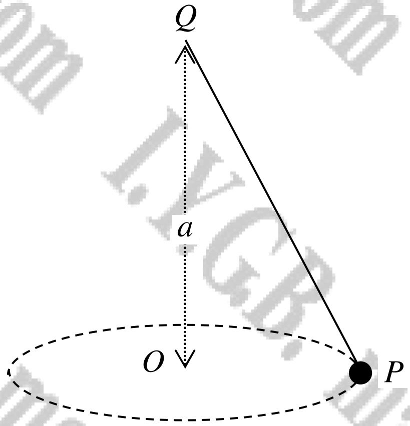
NOW LOOKING AT THE DIAGRAM

$r = 2.56\sin\theta$
$r = 2.56 \times \frac{3}{5}$
$r = 1.536$

$$\Rightarrow 4\omega^2 r \geqslant 3g$$
$$\Rightarrow 4\omega^2 \times 1.536 \geqslant 3 \times 9.8$$
$$\Rightarrow \omega^2 \geqslant \frac{1225}{256}$$
$$\Rightarrow \omega \geqslant \frac{35}{16}$$
$$\Rightarrow \frac{1}{\omega} \leqslant \frac{16}{35}$$
$$\Rightarrow \frac{2\pi}{\omega} \leqslant \frac{32\pi}{35}$$
$$\Rightarrow \text{PERIOD} \leqslant \frac{32\pi}{35}$$

// AS REQUIRED

Question 12 (****+)



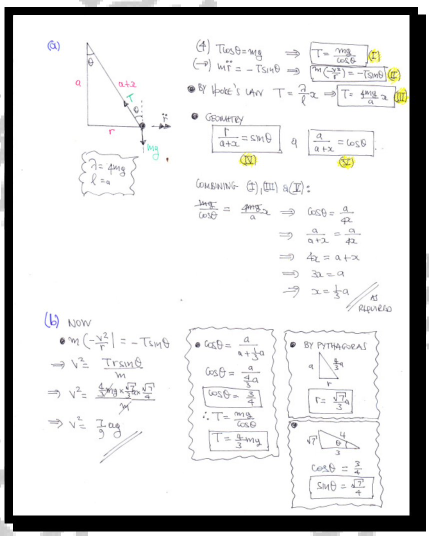
The figure above shows a particle P of mass m attached to one end of a light elastic string of natural length a and modulus of elasticity $4mg$.

The other end of the string is attached to a fixed point Q. The particle moves with constant speed v in a horizontal circle with centre O, where O is vertically below Q such that $|OQ| = a$.

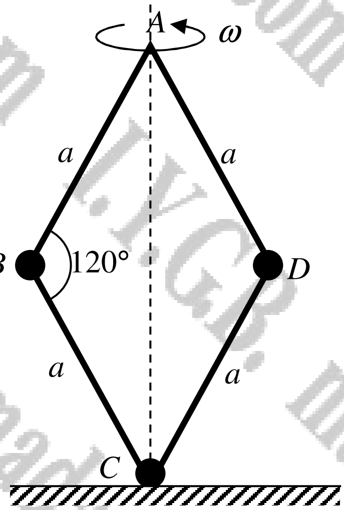
a) Show that the extension in the string is $\frac{1}{3}a$.

b) Find v^2 in terms of a and g.

$$\boxed{v^2 = \frac{7}{9}ag}$$



Question 13 (**+)**



Three small spheres B, C and D, of respective masses m, $2m$ and m, are smoothly hinged to four identical light rigid rods, each of length a. The rods form a rhombus $ABCD$. The sphere C is in contact with a smooth horizontal surface.

The system S, of the three spheres and the four rods, always lie in the same vertical plane with A vertically above C, as shown in the figure above.

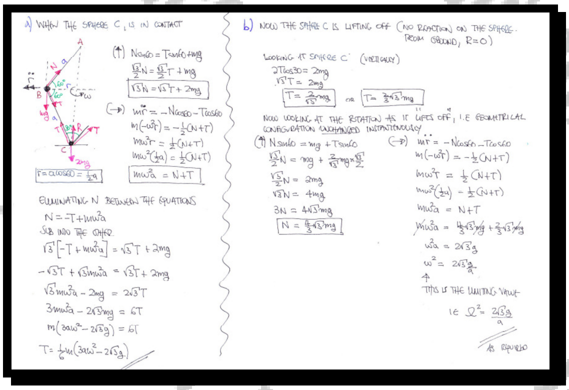
S is rotating with constant angular velocity ω, about a vertical axis AC.

a) Given $\measuredangle ABC = 120°$, find the tension in BC, in terms of m, g, a and ω.

When the angular velocity is increased to Ω, sphere C is at the point of losing contact with the surface.

b) Show clearly that $\Omega^2 = \dfrac{2\sqrt{3}g}{a}$.

$$\boxed{T = \frac{1}{6}m\left[3a\omega^2 - 2\sqrt{3}g\right]}$$



a) WHEN THE SPHERE C, IS IN CONTACT

$r = a\cos 60 = \frac{1}{2}a$

(↑) $N\sin 60 = T\sin 60 + mg$

$\frac{\sqrt{3}}{2}N = \frac{\sqrt{3}}{2}T + mg$

$\boxed{\sqrt{3}N = \sqrt{3}T + 2mg}$

(→) $m\ddot{r} = -N\cos 60 - T\cos 60$

$m(-\omega^2 r) = -\frac{1}{2}(N+T)$

$m\omega^2 r = \frac{1}{2}(N+T)$

$m\omega^2(\frac{1}{2}a) = \frac{1}{2}(N+T)$

$\boxed{m\omega^2 a = N+T}$

ELIMINATING N BETWEEN THE EQUATIONS

$N = -T + m\omega^2 a$

SUB INTO THE OTHER

$\sqrt{3}\left[-T + m\omega^2 a\right] = \sqrt{3}T + 2mg$

$-\sqrt{3}T + \sqrt{3}m\omega^2 a = \sqrt{3}T + 2mg$

$\sqrt{3}m\omega^2 a - 2mg = 2\sqrt{3}T$

$3m\omega^2 a - 2\sqrt{3}mg = 6T$

$m(3a\omega^2 - 2\sqrt{3}g) = 6T$

$T = \frac{1}{6}m(3a\omega^2 - 2\sqrt{3}g)$

b) NOW THE SPHERE C IS LIFTING OFF (NO REACTION ON THE SPHERE FROM GROUND, R=0)

LOOKING AT SPHERE C (VERTICALLY)

$2T\cos 30 = 2mg$

$\sqrt{3}T = 2mg$

$\boxed{T = \frac{2}{\sqrt{3}}mg}$ OR $\boxed{T = \frac{2}{3}\sqrt{3}mg}$

NOW LOOKING AT THE ROTATION AS IT "LIFTS OFF", I.E GEOMETRICAL CONFIGURATION UNCHANGED INSTANTANEOUSLY

(↑) $N\sin 60 = mg + T\sin 60$

$\frac{\sqrt{3}}{2}N = mg + \frac{2}{\sqrt{3}}mg \times \frac{\sqrt{3}}{2}$

$\frac{\sqrt{3}}{2}N = 2mg$

$\sqrt{3}N = 4mg$

$3N = 4\sqrt{3}mg$

$\boxed{N = \frac{4}{3}\sqrt{3}mg}$

(→) $m\ddot{r} = -N\cos 60 - T\cos 60$

$m(-\omega^2 r) = -\frac{1}{2}(N+T)$

$m\omega^2 r = \frac{1}{2}(N+T)$

$m\omega^2(\frac{1}{2}a) = \frac{1}{2}(N+T)$

$m\omega^2 a = N+T$

$\cancel{m}\omega^2 a = \frac{4}{3}\sqrt{3}\cancel{m}g + \frac{2}{3}\sqrt{3}\cancel{m}g$

$\omega^2 a = 2\sqrt{3}g$

$\omega^2 = 2\sqrt{3}\frac{g}{a}$

THIS IS THE LIMITING VALUE

IE $\Omega^2 = \frac{2\sqrt{3}g}{a}$

AS REQUIRED

BANKED CIRCULAR MOTION

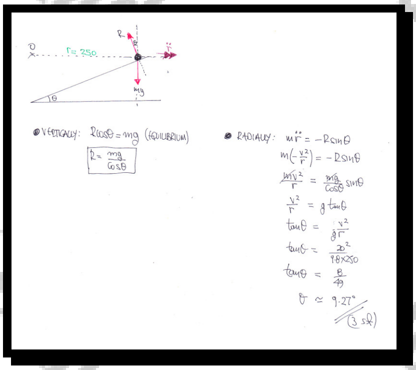
Question 1 (**)

A car is travelling round a bend which is banked at an angle θ to the horizontal. The car is modelled as a particle moving around a horizontal circle of radius 250 m.

When the car reaches a speed of 20 ms^{-1} the car experiences no sideways frictional force.

Determine the value of θ.

$\boxed{\theta \approx 9.27°}$



Question 2 (+)**

A thin hollow hemispherical bowl, of radius a, is fixed on a table with its curved surface in contact with the table and the open plane face parallel to the table.

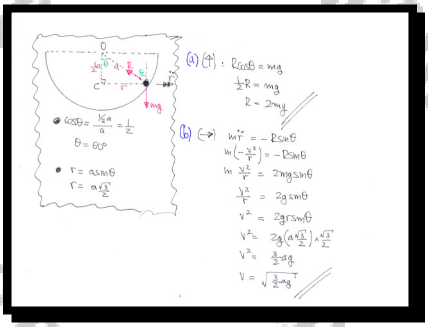
A small marble of mass m is moving in a horizontal circle round the inside smooth surface of the bowl. The centre of the circle is at a distance of $\frac{1}{2}a$ above the level of the table.

Determine …

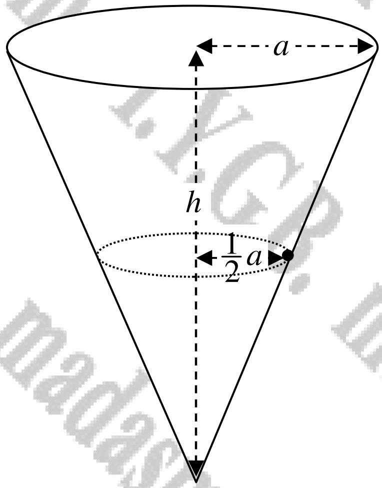
a) … the contact force between the marble and the bowl, in terms of m and g.

b) … the constant speed of the marble, in terms of a and g.

$\boxed{R = 2mg}$, $\boxed{v = \sqrt{\frac{3}{2}ag}}$



Question 3 (+)**

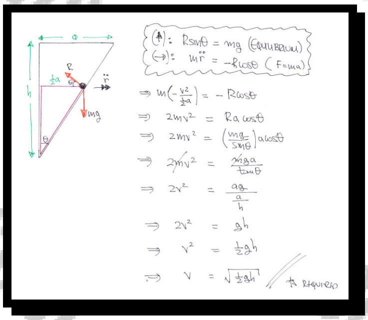


The figure above shows a particle moving on the smooth inner surface of a hollow inverted right circular cone.

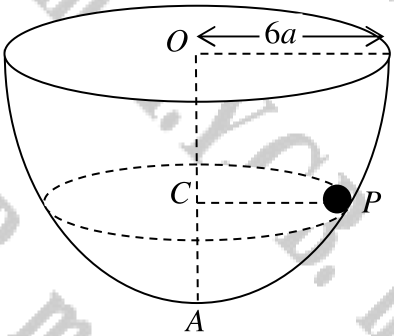
The cone has radius a and height h and the particle is moving in a horizontal circle of radius $\frac{1}{2}a$ with constant speed v.

Show clearly that $v = \sqrt{\frac{1}{2}gh}$.

$\boxed{\text{proof}}$



Question 4 (***)



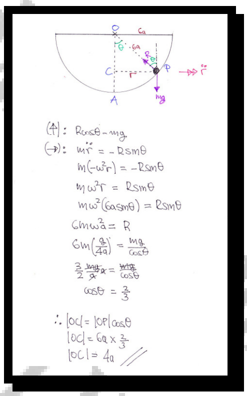
The figure above shows a thin hemispherical bowl of radius $6a$, which is fixed with its circular rim in a horizontal plane.

The centre of the circular rim is at the point O and the point A is on the inner surface of the bowl, vertically below O.

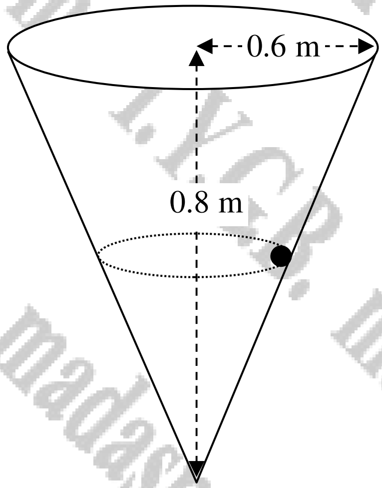
A particle P moves in a horizontal circle with centre C, where C lies on OA, on the smooth inner surface of the bowl.

Given that P moves with constant angular speed $\sqrt{\dfrac{g}{4a}}$, determine the distance OC, in terms of a.

$$\boxed{|OC| = 4a}$$



Question 5 (***)



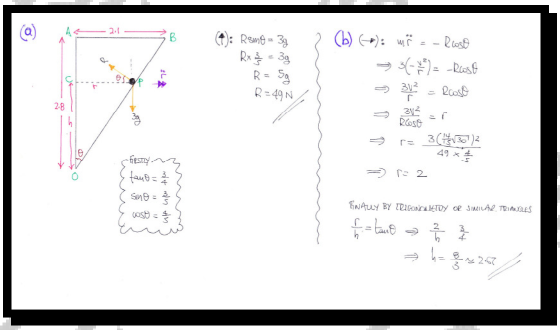
The figure above shows a particle of mass 3 kg, moving on the smooth inner surface of a hollow inverted right circular cone.

The cone has radius 2.1 m and height 2.8 m and the particle is moving in a horizontal circle with constant speed $\frac{14}{15}\sqrt{30} \text{ ms}^{-1}$.

a) Find the magnitude of the constant reaction between the particle and the cone.

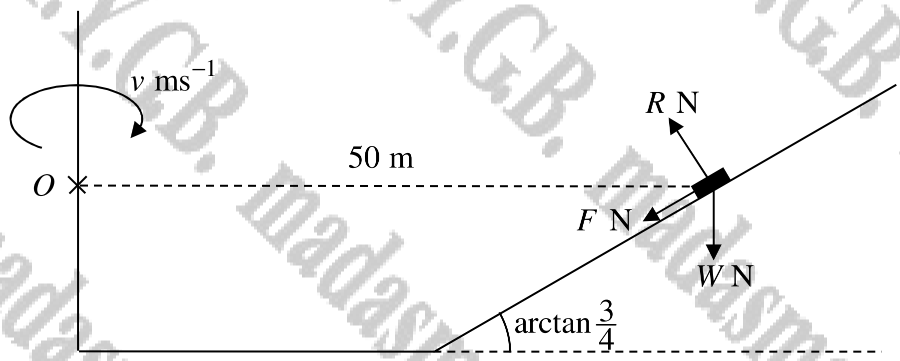
b) Calculate the height of the horizontal circle, in which the particle is moving, above the vertex of the cone.

$\boxed{R = 49 \text{ N}}$, $\boxed{h = \frac{8}{3} \approx 2.67 \text{ m}}$



Question 6 (*+)**

A sports car of mass 1000 kg is moving with constant speed v ms^{-1}, in horizontal circle with centre O and radius 50 m on a road which is banked at $\arctan\frac{3}{4}$ to the horizontal.



The figure above shows the weight of the car W N, the normal reaction R N and the frictional force between the road and the car F N.

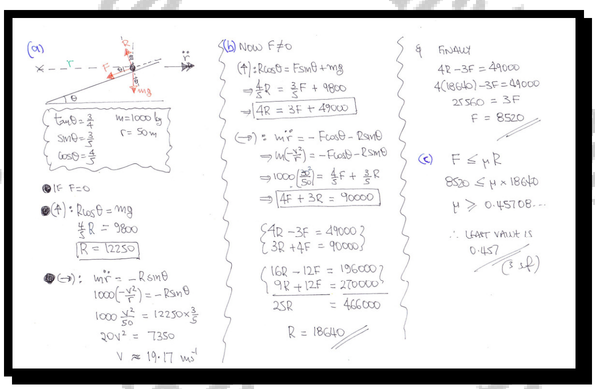
a) Given that $F=0$ determine the value of v.

Given instead that $v=30$ ms^{-1} …

b) … find value of F and the value of R.

c) … calculate the least value of the coefficient of friction between the road and the car.

$\boxed{v \approx 19.17 \text{ ms}^{-1}}$, $\boxed{R=18640}$, $\boxed{F=8520}$, $\boxed{\mu \approx 0.457}$



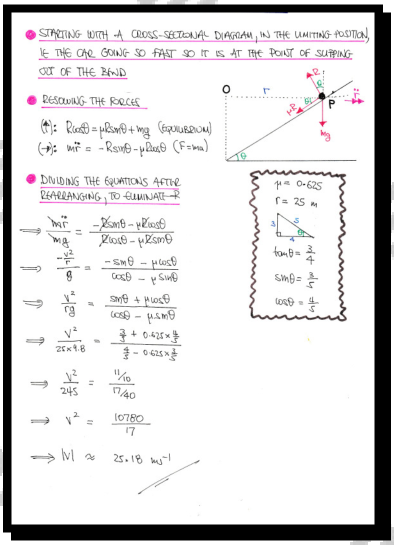
Question 7 (*+)**

A car is driven at constant speed v ms^{-1} round a bend of a race track. The track, round the bend is banked at $\arctan\frac{3}{4}$ to the horizontal and the coefficient of friction between the car tyres and the track is 0.625.

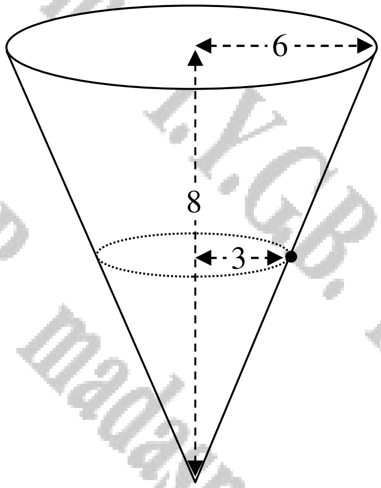
The car is modelled as a particle whose path round the bend is a horizontal circle of radius 25 m.

If the car tyres do not slip sideways as the car goes round the bend, determine the greatest value of v, correct to 2 decimal places.

$\boxed{|v| \approx 28.18 \text{ ms}^{-1}}$



Question 8 (***+)



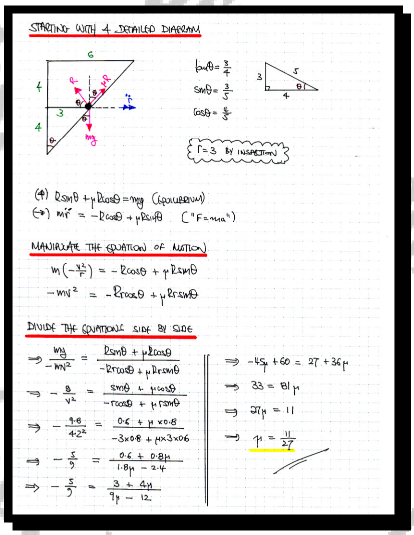
The figure above shows a particle on the rough inner surface of a hollow inverted right circular cone of radius 6 m and height 8 m, whose axis of symmetry is vertical.

When the cone rotates about its axis of symmetry with constant angular velocity the particle moves with constant speed 4.2 ms^{-1} in a horizontal circle of radius 3 m.

The cone rotates sufficiently fast for the particle to stay in contact with the cone.

Determine the smallest possible value of the coefficient of friction between the cone and the particle.

$\boxed{}$, $\boxed{\mu = \frac{11}{27}}$



STARTING WITH A DETAILED DIAGRAM

$\tan\theta = \frac{3}{4}$

$\sin\theta = \frac{3}{5}$

$\cos\theta = \frac{4}{5}$

r = 3 BY INSPECTION

$(\uparrow)$ $R\sin\theta + \mu R\cos\theta = mg$ (EQUILIBRIUM)

$(\rightarrow)$ $m\ddot{r} = -R\cos\theta + \mu R\sin\theta$ ("F = ma")

MANIPULATE THE EQUATION OF MOTION

$m\left(-\frac{v^2}{r}\right) = -R\cos\theta + \mu R\sin\theta$

$-mv^2 = -Rr\cos\theta + \mu Rr\sin\theta$

DIVIDE THE EQUATIONS SIDE BY SIDE

$\Rightarrow \frac{mg}{-mv^2} = \frac{R\sin\theta + \mu R\cos\theta}{-Rr\cos\theta + \mu Rr\sin\theta}$

$\Rightarrow -\frac{g}{v^2} = \frac{\sin\theta + \mu\cos\theta}{-r\cos\theta + \mu r\sin\theta}$

$\Rightarrow -\frac{9.8}{4.2^2} = \frac{0.6 + \mu \times 0.8}{-3\times 0.8 + \mu\times 3\times 0.6}$

$\Rightarrow -\frac{5}{9} = \frac{0.6 + 0.8\mu}{1.8\mu - 2.4}$

$\Rightarrow -\frac{5}{9} = \frac{3 + 4\mu}{9\mu - 12}$

$\Rightarrow -45\mu + 60 = 27 + 36\mu$

$\Rightarrow 33 = 81\mu$

$\Rightarrow 27\mu = 11$

$\Rightarrow \mu = \frac{11}{27}$

Question 9 (***+)

A car is moving, with constant speed v, in a circular bend banked at an angle θ to the horizontal, so that the car is at the point of slipping down the banked road.

The motion of the car takes place in horizontal circle with centre O and radius r.

The coefficient of friction between the road and the car is μ.

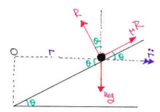
Show with detail method that

$$v^2 = \frac{rg\left(\tan\theta - \mu\right)}{1 + \mu\tan\theta}.$$

FM, proof

- VERTICALLY WE HAVE EQUILIBRIUM

$$\boxed{R\cos\theta + \mu R\sin\theta = mg}$$



- IN THE RADIAL DIRECTION, AWAY FROM O

$$\boxed{m\ddot{r} = \mu R\cos\theta - R\sin\theta}$$

- DIVIDING THE TWO EQUATIONS WE OBTAIN

$$\Rightarrow \frac{m\ddot{r}}{mg} = \frac{\mu R\cos\theta - R\sin\theta}{R\cos\theta + \mu R\sin\theta}$$

$$\Rightarrow \frac{\ddot{r}}{g} = \frac{\mu\cos\theta - \sin\theta}{\cos\theta + \mu\sin\theta}$$

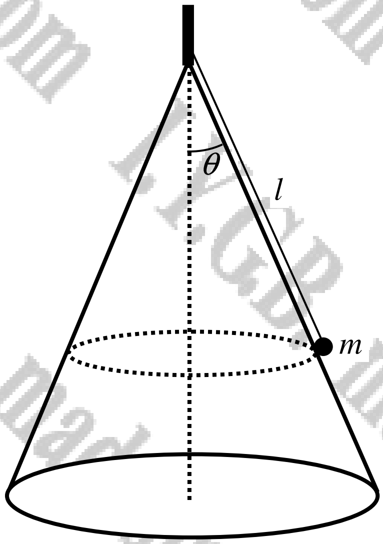
$$\Rightarrow \frac{-\frac{v^2}{r}}{g} = \frac{\frac{\mu\cos\theta}{\cos\theta} - \frac{\sin\theta}{\cos\theta}}{\frac{\cos\theta}{\cos\theta} + \mu\frac{\sin\theta}{\cos\theta}}$$

$$\Rightarrow -\frac{v^2}{rg} = \frac{\mu - \tan\theta}{1 + \mu\tan\theta}$$

$$\Rightarrow \frac{v^2}{rg} = \frac{\tan\theta - \mu}{\mu\tan\theta + 1}$$

$$\Rightarrow v^2 = \frac{rg\left(\tan\theta - \mu\right)}{1 + \mu\tan\theta}$$

Question 10 (****)



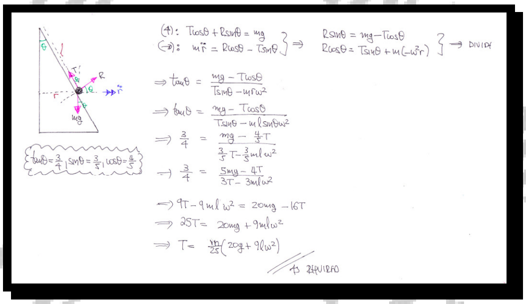
One of the two ends of a light inextensible string of length l is attached to the vertex of a right circular cone. A particle of mass m is attached to the other end of the string. The particle is made to rotate on the curved surface of the cone with constant angular velocity ω with the string taut.

A line of greatest slope on the surface of the cone forms an angle θ with the axis of symmetry of the cone, such that $\tan\theta = \frac{3}{4}$, as shown in the above figure.

Show that the tension in the string is

$$v = \frac{1}{25}m\left(20g + 9l\omega^2\right).$$

$\boxed{\text{proof}}$



$(\uparrow)$: $T\cos\theta + R\sin\theta = mg$
$(\rightarrow)$: $m\ddot{r} = R\cos\theta - T\sin\theta$ $\Rightarrow$ $R\sin\theta = mg - T\cos\theta$, $R\cos\theta = T\sin\theta + m(-\omega^2 r)$ $\rightarrow$ DIVIDE

$$\Rightarrow \tan\theta = \frac{mg - T\cos\theta}{T\sin\theta - mr\omega^2}$$

$$\Rightarrow \tan\theta = \frac{mg - T\cos\theta}{T\sin\theta - ml\sin\theta\omega^2}$$

$$\Rightarrow \frac{3}{4} = \frac{mg - \frac{4}{5}T}{\frac{3}{5}T - \frac{3}{5}ml\omega^2}$$

$$\Rightarrow \frac{3}{4} = \frac{5mg - 4T}{3T - 3ml\omega^2}$$

$$\Rightarrow 9T - 9ml\omega^2 = 20mg - 16T$$

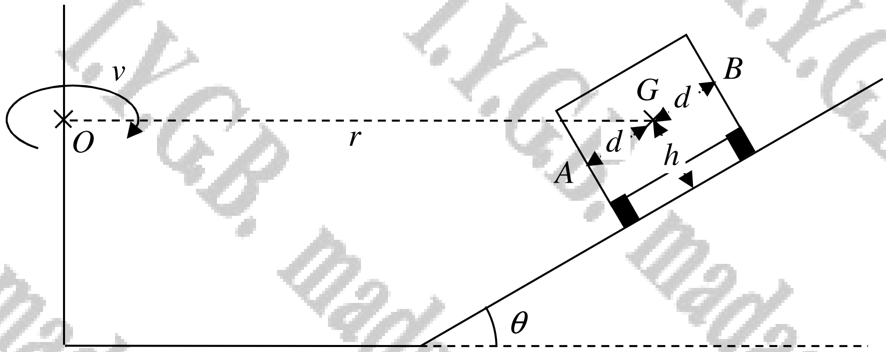
$$\Rightarrow 25T = 20mg + 9ml\omega^2$$

$$\Rightarrow T = \frac{m}{25}(20g + 9l\omega^2)$$

// AS REQUIRED

$\tan\theta = \frac{3}{4}$, $\sin\theta = \frac{3}{5}$, $\cos\theta = \frac{4}{5}$

Question 11 (*****)



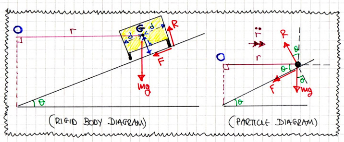
The figure above shows a car moving with constant speed v, in a horizontal circle, with centre O and radius r, on a road which is banked at an angle θ to the horizontal.

The car has **width** $|AB| = 2d$, as shown in the figure and its centre of mass G is located at the midpoint of AB and at a height h above the ground.

Assuming that the car is at the point of toppling about G, in an "up the bend" direction, show that

$$v^2 = \frac{rg\left(d + h\tan\theta\right)}{h - d\tan\theta}.$$

$\boxed{}$, $\boxed{\text{proof}}$



TREATING THE CAR AS A PARTICLE FIRST

(RADIALLY): $m\ddot{r} = -R\sin\theta - F\cos\theta$

(TRANSVERSLY): $R\cos\theta = mg + F\sin\theta$

LOOKING AT THE CAR AS A RIGID BODY ABOUT TO TOPPLE, ABOUT G

G: $R \times d = f \times h$

ELIMINATE F (OR R) IE $F = \frac{Rd}{h}$

$\Rightarrow \begin{cases} m\left(-\frac{v^2}{r}\right) = -\frac{Rd}{h}\cos\theta - R\sin\theta \\ R\cos\theta = mg + \frac{Rd}{h}\sin\theta \end{cases}$

$\Rightarrow \begin{cases} \frac{mv^2}{r} = \frac{Rd}{h}\cos\theta + R\sin\theta \\ R\cos\theta = mg + \frac{Rd}{h}\sin\theta \end{cases}$

$\Rightarrow \begin{cases} mhv^2 = Rdr\cos\theta + Rrh\sin\theta \\ hR\cos\theta = mgh + Rd\sin\theta \end{cases}$

$\Rightarrow \begin{cases} mhv^2 = rR(d\cos\theta + h\sin\theta) \\ mgh = R(h\cos\theta - d\sin\theta) \end{cases}$

DIVIDING YIELDS

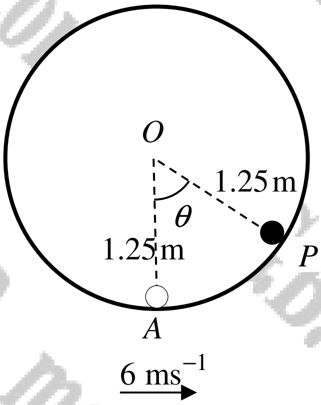
$\Rightarrow \frac{v^2}{g} = r\left(\frac{d\cos\theta + h\sin\theta}{h\cos\theta - d\sin\theta}\right)$

DIVIDE "TOP & BOTTOM" OF THE FRACTION BY $\cos\theta$

$\Rightarrow v^2 = rg\left(\frac{d + h\tan\theta}{h - d\tan\theta}\right)$

MOTION IN A VERTICAL CIRCLE

Question 1 (**)



A hollow right circular cylinder is fixed with its axis horizontal. The inner surface of the cylinder is smooth and has radius 1.25 m. A particle P of mass 0.5 kg is projected horizontally with speed 6 ms^{-1} from a point A, which is the lowest point of a vertical cross section of the cylinder, which is perpendicular to the axis of the cylinder. While P is in contact with the cylinder, its speed is $v \text{ ms}^{-1}$ and the contact force exerted by the cylinder onto P is R N. The angle AOP is θ, where O is the vertical cross section, as shown in the figure.

a) Show clearly that …

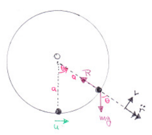
i. … $v^2 = 11.5 + 24.5\cos\theta$.

ii. … $R = 4.6 + 14.7\cos\theta$.

b) Determine the speed of P at the instant when it leaves the surface.

$\boxed{\approx 1.96 \text{ ms}^{-1}}$

(a)



$a = 1.25 \text{ m}$
$u = 6 \text{ ms}^{-1}$
$m = 0.5 \text{ kg}$

(I) BY ENERGY TAKING THE LOWEST LEVEL OF THE CYLINDER AS THE ZERO POTENTIAL LEVEL

$$KE_{\text{BOTTOM}} + \cancel{PE_{\text{BOTTOM}}} = KE_\theta + PE_\theta$$
$$\tfrac{1}{2}mu^2 = \tfrac{1}{2}mv^2 + mg(a - a\cos\theta)$$
$$u^2 = v^2 + 2ag(1-\cos\theta)$$
$$v^2 = u^2 - 2ag(1-\cos\theta)$$
$$v^2 = 36 - 24.5 + 24.5\cos\theta$$
$$v^2 = 11.5 + 24.5\cos\theta$$

AS REQUIRED

(II) NOW RADIALLY

$$m\ddot{r} = mg\cos\theta - R$$
$$m\left(-\frac{v^2}{a}\right) = mg\cos\theta - R$$
$$R = mg\cos\theta + \frac{m}{a}v^2$$
$$R = 4.9\cos\theta + \tfrac{2}{5}\left[11.5 + 24.5\cos\theta\right]$$
$$R = 4.9\cos\theta + 4.6 + 9.8\cos\theta$$
$$R = 4.6 + 14.7\cos\theta$$

AS REQUIRED

(b) WHEN IT LEAVES THE SURFACE $R=0$

$$0 = 4.6 + 14.7\cos\theta$$
$$14.7\cos\theta = -4.6$$
$$\boxed{\cos\theta = -\frac{46}{147}}$$

$$v^2 = 11.5 + 24.5\cos\theta$$
$$v^2 = 11.5 + 24.5\left(-\frac{46}{147}\right)$$
$$v^2 = \frac{23}{6}$$
$$v \approx 1.96 \text{ ms}^{-1}$$

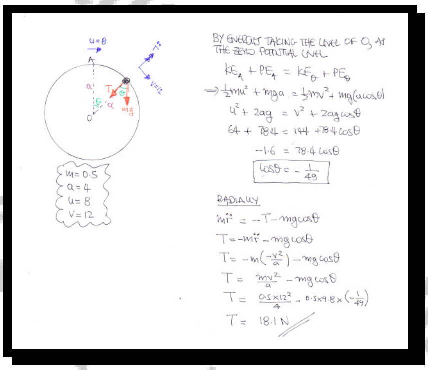
Question 2 (**)

A light inextensible string of length 4 m has one end attached to a fixed point O and the other end attached to a particle P of mass 0.5 kg.

P is moving in a vertical circle with centre O and radius 4 m. When P is at the highest point of the circle it has speed 8 ms^{-1}.

Determine the tension in the string when the speed of P is 12 ms^{-1}.

$\boxed{T = 18.1 \text{ N}}$



Question 3 (**+)

A particle is attached to one end of a light rigid rod of length a and the other end of the rod is freely hinged to a fixed point O.

The particle is originally at rest, vertically below O, when it is projected horizontally with speed u.

The particle moves in a complete vertical circle.

a) Show clearly that $u \ge \sqrt{4ag}$

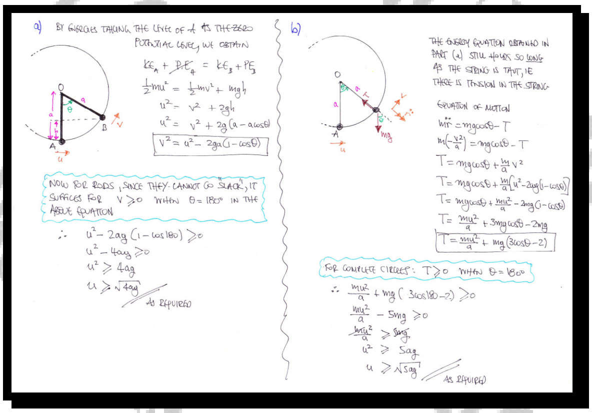
The particle is then attached to one end of a light inextensible string of length a and the other end of the string is attached to a fixed point O.

The particle is hanging freely at rest, with the string vertical, when it is projected horizontally with speed u.

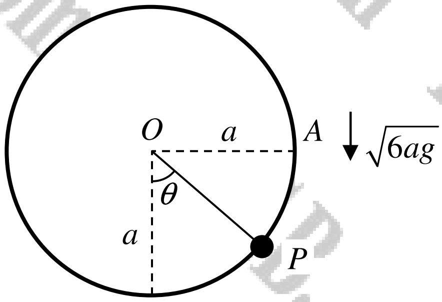
The particle moves in a complete vertical circle.

b) Show further that $u \ge \sqrt{5ag}$

$\boxed{\text{proof}}$



Question 4 (***)



A particle P of mass M attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. The particle is held at a point A with the string taut and OA is horizontal. The particle is projected downwards with speed $\sqrt{6ag}$ from A.

When the string makes an angle θ with the downward vertical through O the string it still taut, the tension denoted by T, as shown in the figure above.

a) Assuming that air resistance can be ignored, show that

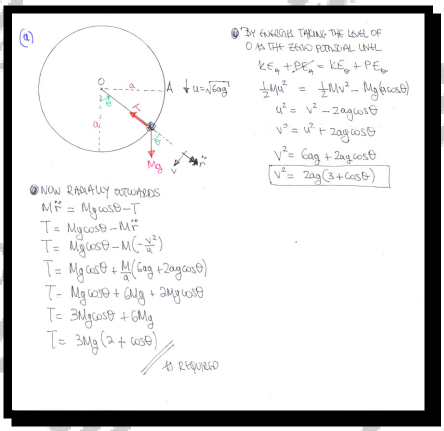
$$T = 3Mg\left(2+\cos\theta\right).$$

b) State, with justification, whether P performs a full circle in the subsequent motion.

When $T = 5Mg$ the speed of P is 5.6 ms^{-1}.

c) Determine the value of a.

$\boxed{a = 0.6 \text{ m}}$



(a)

O a A $u=\sqrt{6ag}$ θ a T Mg θ v $\ddot{r}$

• BY ENERGIES TAKING THE LEVEL OF O AS THE ZERO POTENTIAL LEVEL

$KE_A + \cancel{PE_A} = KE_\theta + PE_\theta$

$\frac{1}{2}Mu^2 = \frac{1}{2}Mv^2 - Mg(a\cos\theta)$

$u^2 = v^2 - 2ag\cos\theta$

$v^2 = u^2 + 2ag\cos\theta$

$v^2 = 6ag + 2ag\cos\theta$

$\boxed{v^2 = 2ag(3+\cos\theta)}$

• NOW RADIALLY OUTWARDS

$M\ddot{r} = Mg\cos\theta - T$

$T = Mg\cos\theta - M\ddot{r}$

$T = Mg\cos\theta - M\left(-\frac{v^2}{a}\right)$

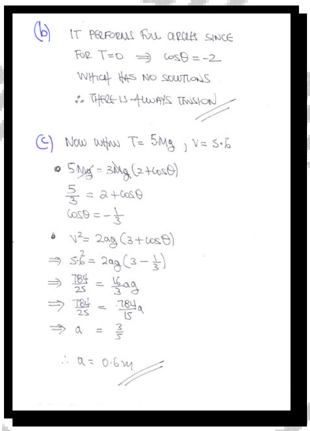
$T = Mg\cos\theta + \frac{M}{a}(6ag + 2ag\cos\theta)$

$T = Mg\cos\theta + 6Mg + 2Mg\cos\theta$

$T = 3Mg\cos\theta + 6Mg$

$T = 3Mg(2+\cos\theta)$

// AS REQUIRED



(b) IT PERFORMS FULL CIRCLES SINCE FOR $T=0 \Rightarrow \cos\theta = -2$ WHICH HAS NO SOLUTIONS

∴ THERE IS ALWAYS TENSION //

(c) NOW WHEN $T = 5Mg$, $V = 5.6$

• $5\cancel{Mg} = 3\cancel{Mg}(2+\cos\theta)$

$\frac{5}{3} = 2 + \cos\theta$

$\cos\theta = -\frac{1}{3}$

• $V^2 = 2ag(3+\cos\theta)$

$\Rightarrow 5.6^2 = 2ag\left(3 - \frac{1}{3}\right)$

$\Rightarrow \frac{784}{25} = \frac{16}{3}ag$

$\Rightarrow \frac{784}{25} = \frac{784}{15}a$

$\Rightarrow a = \frac{3}{5}$

$\therefore a = 0.6\text{ m}$ //

Question 5 (***+)

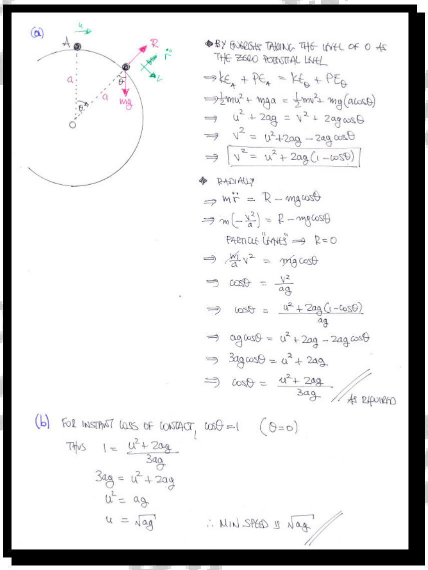
A particle P is projected horizontally with speed u from the highest point of a fixed smooth sphere of radius a and centre O. In the subsequent motion P slides down the sphere and loses contact with the sphere when OP makes an angle θ with the upward vertical.

a) Show clearly that

$$\cos\theta = \frac{u^2 + 2ag}{3ag}.$$

b) Determine the minimum value of u in terms of a and g, for which P leaves the surface of the sphere the instant it is projected.

$$\boxed{u_{\text{min}} = \sqrt{ag}}$$



- BY ENERGY TAKING THE LEVEL OF O AS THE ZERO POTENTIAL LEVEL

$\Rightarrow KE_A + PE_A = KE_\theta + PE_\theta$

$\Rightarrow \frac{1}{2}mu^2 + mga = \frac{1}{2}mv^2 + mg(a\cos\theta)$

$\Rightarrow u^2 + 2ag = v^2 + 2ag\cos\theta$

$\Rightarrow v^2 = u^2 + 2ag - 2ag\cos\theta$

$\Rightarrow \boxed{v^2 = u^2 + 2ag(1-\cos\theta)}$

- RADIALLY

$\Rightarrow m\ddot{r} = R - mg\cos\theta$

$\Rightarrow m\left(-\frac{v^2}{a}\right) = R - mg\cos\theta$

PARTICLE "LEAVES" $\Rightarrow R = 0$

$\Rightarrow \frac{\cancel{m}}{a}v^2 = \cancel{m}g\cos\theta$

$\Rightarrow \cos\theta = \frac{v^2}{ag}$

$\Rightarrow \cos\theta = \frac{u^2 + 2ag(1-\cos\theta)}{ag}$

$\Rightarrow ag\cos\theta = u^2 + 2ag - 2ag\cos\theta$

$\Rightarrow 3ag\cos\theta = u^2 + 2ag$

$\Rightarrow \cos\theta = \frac{u^2 + 2ag}{3ag}$ // AS REQUIRED

(b) FOR INSTANT LOSS OF CONTACT, $\cos\theta = 1$ $(\theta = 0)$

THUS $1 = \frac{u^2 + 2ag}{3ag}$

$3ag = u^2 + 2ag$

$u^2 = ag$

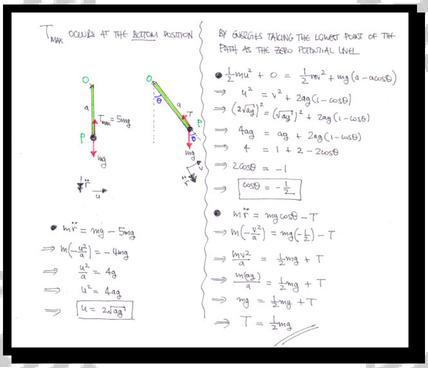
$u = \sqrt{ag}$ $\therefore$ MIN SPEED IS $\sqrt{ag}$ //

Question 6 (***+)

A light rigid rod OP of length a has a particle of mass m attached at P. The rod is rotating in a vertical plane about a fixed smooth horizontal axis through O.

Given that the greatest force acting along the rod is $5mg$, find, in terms of mg, the magnitude of the force acting along the rod when the speed of the particle is $\sqrt{ag}$.

$$\boxed{F = \frac{1}{2}mg}$$



T_{MAX} OCCURS AT THE BOTTOM POSITION

- $m\ddot{r} = mg - 5mg$

$\Rightarrow m\left(-\frac{u^2}{a}\right) = -4mg$

$\Rightarrow \frac{u^2}{a} = 4g$

$\Rightarrow u^2 = 4ag$

$\Rightarrow \boxed{u = 2\sqrt{ag}}$

BY ENERGIES TAKING THE LOWEST POINT OF THE PATH AS THE ZERO POTENTIAL LEVEL

- $\frac{1}{2}mu^2 + 0 = \frac{1}{2}mv^2 + mg(a - a\cos\theta)$

$\Rightarrow u^2 = v^2 + 2ag(1-\cos\theta)$

$\Rightarrow (2\sqrt{ag})^2 = (\sqrt{ag})^2 + 2ag(1-\cos\theta)$

$\Rightarrow 4ag = ag + 2ag(1-\cos\theta)$

$\Rightarrow 4 = 1 + 2 - 2\cos\theta$

$\Rightarrow 2\cos\theta = -1$

$\Rightarrow \boxed{\cos\theta = -\frac{1}{2}}$

- $m\ddot{r} = mg\cos\theta - T$

$\Rightarrow m\left(-\frac{v^2}{a}\right) = mg\left(-\frac{1}{2}\right) - T$

$\Rightarrow \frac{mv^2}{a} = \frac{1}{2}mg + T$

$\Rightarrow \frac{m(ag)}{a} = \frac{1}{2}mg + T$

$\Rightarrow mg = \frac{1}{2}mg + T$

$\Rightarrow T = \frac{1}{2}mg$

Question 7 (***+)

A machine component consists of a particle P, of mass 2 kg, attached to one end of a light rigid rod OP, of length 0.1 m.

The particle is made to rotate at 750 revolutions per minute in a vertical circle with centre at O.

Determine the least and the greatest magnitude of tension experienced by the rod.

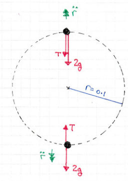
$\boxed{}$, $\boxed{|T_{\text{min}}| \approx 1214 \text{ N}}$, $\boxed{|T_{\text{max}}| \approx 1253 \text{ N}}$

START BY CONVERTING THE ANGULAR SPEED INTO SI UNITS

$\omega = 750$ REVS / MINUTE

$\omega = \frac{750 \times 2\pi}{60} = 25\pi \text{ rad s}^{-1}$

NOW LOOKING AT THE EQUATION OF RADIAL MOTION AT THE HIGHEST AND LOWEST POINT OF THE PATH



AT THE TOP

$\Rightarrow m\ddot{r} = -T - 2g$

$\Rightarrow 2(-\omega^2 r) = -T - 2g$

$\Rightarrow T = 2\omega^2 r - 2g$

$\Rightarrow T = 2(25\pi)^2(0.1) - 2g$

$\Rightarrow T \approx 1214 \text{ N}$ // (4 sf)

AT THE BOTTOM

$\Rightarrow m\ddot{r} = 2g - T$

$\Rightarrow 2(-\omega^2 r) = 2g - T$

$\Rightarrow T = 2g + 2\omega^2 r$

$\Rightarrow T = 2g + 2(25\pi)^2(0.1)$

$\Rightarrow T \approx 1253 \text{ N}$ // 4 sf

Question 8 (***+)

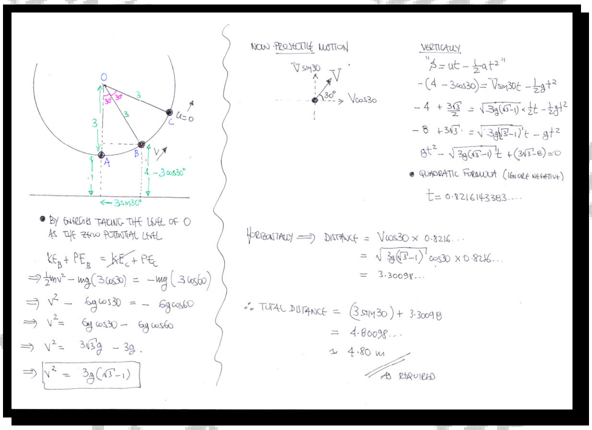
A heavy body is swinging on the end of a light inextensible string of length 3 m, whose other end is attached to a fixed point O, 4 m above level horizontal ground.

The body moves in a vertical plane through O, so that in the extreme positions of its motion the string makes an angle of $60°$ through the **downward** vertical through O.

At an instant when the string makes an angle of $30°$ with the **downward** vertical through O, and the body is moving upwards the body breaks free from the string.

Calculate the horizontal displacement of the body from O, at the point where it hits the ground.

$\boxed{d = 4.80\text{ m}}$



• BY ENERGY TAKING THE LEVEL OF O AS THE ZERO POTENTIAL LEVEL

$KE_B + PE_B = \cancel{KE_C} + PE_C$

$\Rightarrow \frac{1}{2}mv^2 - mg(3\cos 30) = -mg(3\cos 60)$

$\Rightarrow v^2 - 6g\cos 30 = -6g\cos 60$

$\Rightarrow v^2 = 6g\cos 30 - 6g\cos 60$

$\Rightarrow v^2 = 3\sqrt{3}g - 3g$.

$\Rightarrow \boxed{v^2 = 3g(\sqrt{3}-1)}$

VERTICALLY

"$s = ut - \frac{1}{2}at^2$"

$-(4 - 3\cos 30) = V\sin 30 t - \frac{1}{2}gt^2$

$-4 + \frac{3\sqrt{3}}{2} = \sqrt{3g(\sqrt{3}-1)} \times \frac{1}{2}t - \frac{1}{2}gt^2$

$-8 + 3\sqrt{3} = \sqrt{3g(\sqrt{3}-1)}\,t - gt^2$

$gt^2 - \sqrt{3g(\sqrt{3}-1)}\,t + (3\sqrt{3}-8) = 0$

• QUADRATIC FORMULA (IGNORE NEGATIVE)

$t = 0.8216143383\ldots$

HORIZONTALLY $\Rightarrow$ DISTANCE $= V\cos 30 \times 0.8216\ldots$

$= \sqrt{3g(\sqrt{3}-1)}\cos 30 \times 0.8216\ldots$

$= 3.30098\ldots$

$\therefore$ TOTAL DISTANCE $= (3\sin 30) + 3.30098$

$= 4.80098\ldots$

$\approx 4.80\text{ m}$

AS REQUIRED

Question 9 (***+)

A particle P is attached to one end of a light inextensible string of length $5a$.

The other end of the string is attached to a fixed point O and P is hanging in equilibrium vertically below O.

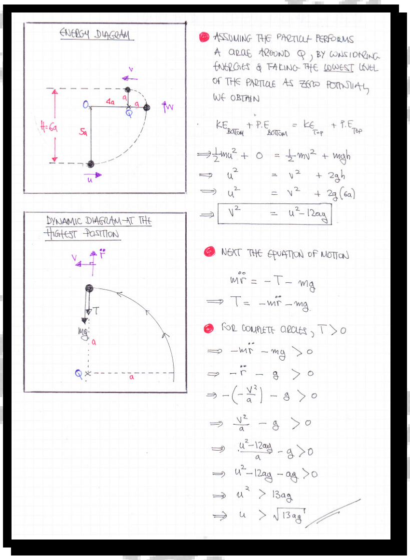
P is then projected horizontally with speed u.

When OP is horizontal, the string meets a small smooth peg at Q, where $|OQ| = 4a$.

Given that P describes a complete circle around Q, show that

$$u > \sqrt{13ag}.$$

$\boxed{\text{proof}}$


ENERGY DIAGRAM

DYNAMIC DIAGRAM AT THE HIGHEST POSITION

- ASSUMING THE PARTICLE PERFORMS A CIRCLE AROUND Q, BY CONSIDERING ENERGIES & TAKING THE LOWEST LEVEL OF THE PARTICLE AS ZERO POTENTIAL WE OBTAIN

$KE_{BOTTOM} + P.E_{BOTTOM} = KE_{TOP} + P.E_{TOP}$

$\Rightarrow \frac{1}{2}mu^2 + 0 = \frac{1}{2}mv^2 + mgh$

$\Rightarrow u^2 = v^2 + 2gh$

$\Rightarrow u^2 = v^2 + 2g(6a)$

$\Rightarrow \boxed{v^2 = u^2 - 12ag}$

- NEXT THE EQUATION OF MOTION

$m\ddot{r} = -T - mg$

$\Rightarrow T = -m\ddot{r} - mg$

- FOR COMPLETE CIRCLES, $T > 0$

$\Rightarrow -m\ddot{r} - mg > 0$

$\Rightarrow -\ddot{r} - g > 0$

$\Rightarrow -\left(-\frac{v^2}{a}\right) - g > 0$

$\Rightarrow \frac{v^2}{a} - g > 0$

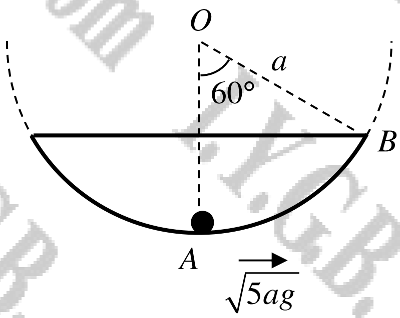
$\Rightarrow \frac{u^2 - 12ag}{a} - g > 0$

$\Rightarrow u^2 - 12ag - ag > 0$

$\Rightarrow u^2 > 13ag$

$\Rightarrow u > \sqrt{13ag}$

Question 10 (****)



A bowl is formed by removing part of a hollow hemisphere with centre at O and radius a so that so that B is a point on the rim of the bowl and A is the lowest point on the bowl as shown in the figure above. The angle AOB is $60°$.

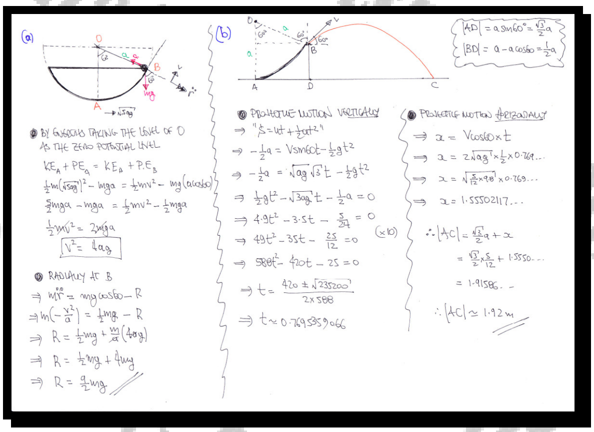
The bowl is then secured on a horizontal table, with point A in contact with the table. A particle P of mass m is placed at A and projected horizontally with speed $\sqrt{5ag}$. The surface of the bowl is assumed to be smooth and air resistance is negligible.

a) Find, in terms of m and g, the force exerted by the bowl on the particle, as it reaches B.

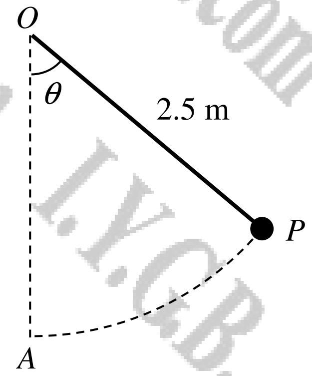
After leaving the bowl at B the particle travels freely under gravity first striking the horizontal table at C.

b) Given that $a = \frac{5}{12}$, determine the distance AC.

$\boxed{R = \frac{9}{2}mg}$, $\boxed{|AC| \approx 1.92 \text{ m}}$



Question 11 (****)



A particle of mass 5 kg is attached to one end of a light inextensible string of length 2.5 m and the other end of the string is attached to a fixed point O. The particle is at rest at a point A which lies vertically below O. The particle is then projected horizontally with speed u ms^{-1}.

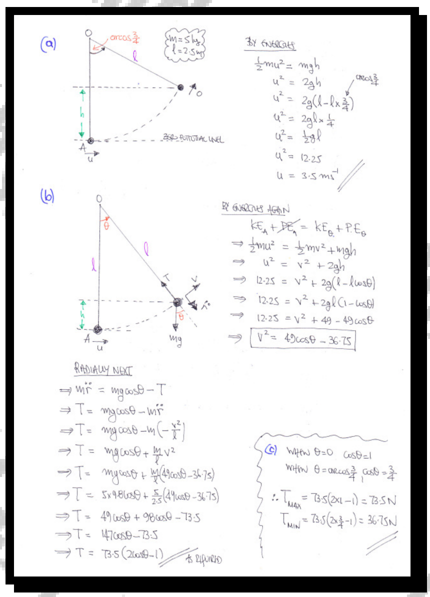
In the subsequent motion when the particle is at a general point P, its speed is v ms^{-1}, the tension in the string is T N and the angle AOP is θ. The particle comes to instantaneous rest when $\theta = \arccos\frac{3}{4}$.

a) Find the value of u.

b) Show that $T = 73.5(2\cos\theta - 1)$.

c) Find the minimum and the maximum value of T, during the described motion.

$\boxed{u = 3.5 \text{ ms}^{-1}}$, $\boxed{T_{\text{min}} = 36.75 \text{ N}}$, $\boxed{T_{\text{max}} = 73.5 \text{ N}}$



(a)

$m = 5$ kg, $l = 2.5$ m

BY ENERGIES

$\frac{1}{2}mu^2 = mgh$

$u^2 = 2gh$

$u^2 = 2g(l - l \times \frac{3}{4})$

$u^2 = 2gl \times \frac{1}{4}$

$u^2 = \frac{1}{2}gl$

$u^2 = 12.25$

$u = 3.5$ ms^{-1}

(b)

BY ENERGIES AGAIN

$KE_A + \cancel{PE_A} = KE_\theta + PE_\theta$

$\Rightarrow \frac{1}{2}mu^2 = \frac{1}{2}mv^2 + mgh$

$\Rightarrow u^2 = v^2 + 2gh$

$\Rightarrow 12.25 = v^2 + 2g(l - l\cos\theta)$

$\Rightarrow 12.25 = v^2 + 2gl(1 - \cos\theta)$

$\Rightarrow 12.25 = v^2 + 49 - 49\cos\theta$

$\Rightarrow \boxed{v^2 = 49\cos\theta - 36.75}$

RADIALLY NEXT

$\Rightarrow m\ddot{r} = mg\cos\theta - T$

$\Rightarrow T = mg\cos\theta - m\ddot{r}$

$\Rightarrow T = mg\cos\theta - m\left(-\frac{v^2}{l}\right)$

$\Rightarrow T = mg\cos\theta + \frac{m}{l}v^2$

$\Rightarrow T = mg\cos\theta + \frac{m}{l}(49\cos\theta - 36.75)$

$\Rightarrow T = 5 \times 9.8\cos\theta + \frac{5}{2.5}(49\cos\theta - 36.75)$

$\Rightarrow T = 49\cos\theta + 98\cos\theta - 73.5$

$\Rightarrow T = 147\cos\theta - 73.5$

$\Rightarrow T = 73.5(2\cos\theta - 1)$ AS REQUIRED

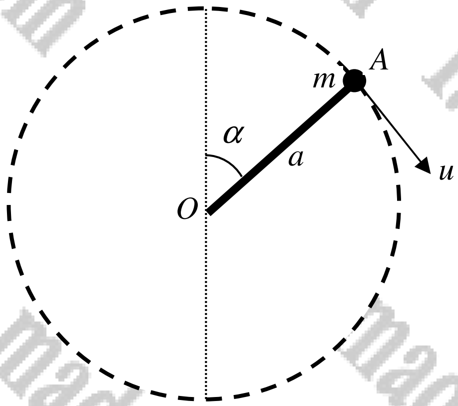
(c) WHEN $\theta = 0$, $\cos\theta = 1$

WHEN $\theta = \arccos\frac{3}{4}$, $\cos\theta = \frac{3}{4}$

$\therefore T_{MAX} = 73.5(2 \times 1 - 1) = 73.5$ N

$T_{MIN} = 73.5(2 \times \frac{3}{4} - 1) = 36.75$ N

Question 12 (****)



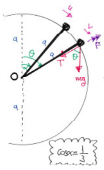
One end of a **light** rigid rod, of length a, is freely jointed to a fixed point O and the other end is attached to a particle of mass m.

The particle is projected with speed u from a point A, where OA makes an angle α with the upward vertical, as shown in the figure above. The particle moves in a complete full vertical circle with centre O, so that the greatest tension in the rod is 10 times as large as the minimum tension.

Given that $\alpha = \arccos\frac{1}{3}$, show that $u^2 = 3ag$.

[ ], [proof]

- BY ENERGIES TAKING THE LEVEL OF "O" AS THE ZERO POTENTIAL



$$\Rightarrow KE_\alpha + P.E_\alpha = KE_\theta + P.E_\theta$$
$$\Rightarrow \tfrac{1}{2}\cancel{m}u^2 + \cancel{m}ga\cos\alpha = \tfrac{1}{2}\cancel{m}v^2 + \cancel{m}ga\cos\theta$$
$$\Rightarrow u^2 + 2ag\cos\alpha = v^2 + 2ag\cos\theta$$
$$\Rightarrow u^2 + \tfrac{2}{3}ag = v^2 + 2ag\cos\theta$$
$$\Rightarrow v^2 = u^2 + \tfrac{2}{3}ag - 2ag\cos\theta$$
$$\Rightarrow v^2 = u^2 + \tfrac{2}{3}ag(1 - 3\cos\theta)$$

$\cos\alpha = \frac{1}{3}$

- USING THIS EXPRESSION WE FIND THE SPEED AT THE HIGHEST AND LOWEST POINTS OF THE PATH

  - $V^2_{TOP} = u^2 - \frac{4}{3}ag \quad (\theta = 0°)$
  - $V^2_{BOTTOM} = u^2 + \frac{8}{3}ag \quad (\theta = 180°)$

- NEXT THE EQUATION OF MOTION (RADIALLY), IN THE GENERAL POSITION OF THE PATH

$$m\ddot{r} = -T - mg\cos\theta$$
$$T = -m\ddot{r} - mg\cos\theta$$
$$T = -m\left(-\frac{v^2}{a}\right) - mg\cos\theta$$
$$T = \frac{m}{a}v^2 - mg\cos\theta$$

- WE CAN NOW SUB THE TWO VELOCITIES, FOUND EARLIER

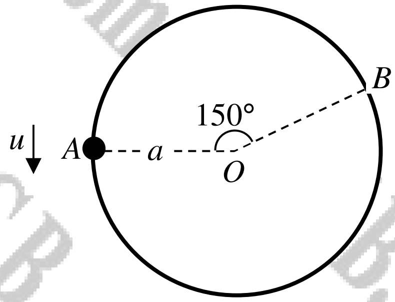
  - $T_{TOP} = \frac{m}{a}\left[u^2 - \frac{4}{3}ag\right] - mg \times \underset{\theta=0°}{1}$
  - $T_{BOTTOM} = \frac{m}{a}\left[u^2 + \frac{8}{3}ag\right] - mg\underset{\theta=180°}{(-1)}$

$$\Rightarrow \begin{cases} T_{TOP} = \frac{mu^2}{a} - \frac{4}{3}mg - mg = \frac{mu^2}{a} - \frac{7}{3}mg \\ T_{BOTTOM} = \frac{mu^2}{a} + \frac{8}{3}mg + mg = \frac{mu^2}{a} + \frac{11}{3}mg \end{cases}$$

- FINALLY WE ARE GIVEN $T_{MAX} = 10T_{MIN}$

$$\Rightarrow \frac{mu^2}{a} + \frac{11}{3}mg = 10\left[\frac{mu^2}{a} - \frac{7}{3}mg\right]$$
$$\Rightarrow \frac{\cancel{m}u^2}{a} + \frac{11}{3}\cancel{m}g = \frac{10\cancel{m}u^2}{a} - \frac{70}{3}\cancel{m}g$$
$$\Rightarrow 27g = \frac{9u^2}{a}$$
$$\Rightarrow 27ag = 9u^2$$
$$\Rightarrow u^2 = 3ag$$

Question 13 (****)



The figure above shows a particle of mass m attached to one end of a light inextensible string of length a.

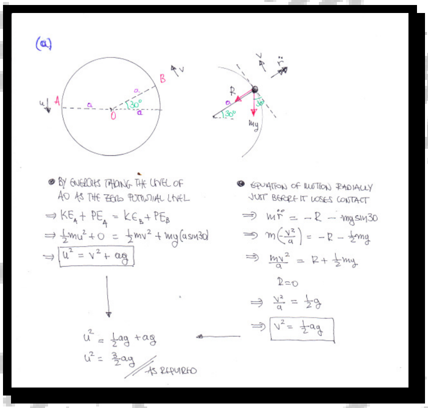
The other end of the string is attached to a fixed point O. The particle is held at a point A with the string taut and OA is horizontal. The particle is projected downwards with speed u from A. The string becomes slack for the first time at a point B such that $\measuredangle AOB = 150°$.

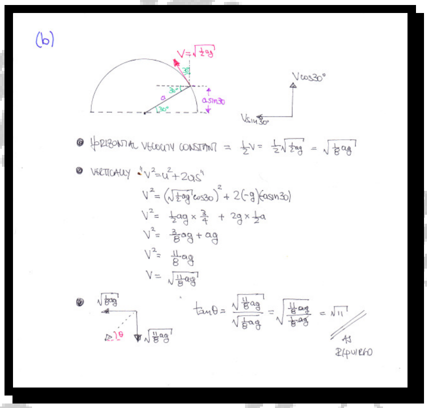
a) Assuming that air resistance can be ignored, show that $u^2 = \frac{3}{2}ag$.

After the string becomes slack the particle in its consequent motion crosses the diameter through A at the point C. The direction of the motion of the particle as it passes through C, makes an angle θ with the horizontal.

b) Show further that $\tan\theta = \sqrt{11}$.

proof





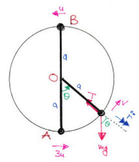
Question 14 (****)

One end of a **light** rigid rod is freely jointed to a fixed point O and the other end is attached to a point mass. The loaded rod is describing full vertical circles so that the greatest speed of the point mass is three times its least speed.

Determine the cosine of the angle which the rod makes to the downward vertical through O, when the tension in the rod is zero.

$$\boxed{\theta = \arccos\left(-\frac{5}{6}\right) = \pi - \arccos\left(\frac{5}{6}\right)}$$

LET THE MASS OF THE PARTICLE BE m AND THE LENGTH OF THE ROD BE a



BY ENERGIES, TAKING THE LEVEL OF A AS THE ZERO POTENTIAL LEVEL WE HAVE

$\Rightarrow KE_A + \cancel{PE_A} = KE_B + PE_B$

$\Rightarrow \frac{1}{2}\cancel{m}(3u)^2 + 0 = \frac{1}{2}\cancel{m}u^2 + \cancel{m}g(2a)$

$\Rightarrow \frac{9}{2}u^2 = \frac{1}{2}u^2 + 2ag$

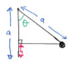
$\Rightarrow 4u^2 = 2ag$

$\Rightarrow \boxed{u^2 = \frac{1}{2}ag}$

BY ENERGIES AGAIN BETWEEN A & SOME ARBITRARY POSITION "θ"

$\Rightarrow KE_A + \cancel{P.E_A} = KE_\theta + P.E_\theta$

$\Rightarrow \frac{1}{2}\cancel{m}(3u)^2 + 0 = \frac{1}{2}\cancel{m}v^2 + \cancel{m}g\underbrace{(a - a\cos\theta)}_{h}$



$\Rightarrow \frac{9}{2}u^2 = \frac{1}{2}v^2 + ag(1-\cos\theta)$

$\Rightarrow 9u^2 = v^2 + 2ag(1-\cos\theta)$

$\Rightarrow v^2 = 9u^2 - 2ag(1-\cos\theta)$

$\Rightarrow v^2 = 9\left(\frac{1}{2}ag\right) - 2ag(1-\cos\theta)$

$\Rightarrow v^2 = \frac{9}{2}ag - 2ag + 2ag\cos\theta$

$\Rightarrow \boxed{v^2 = \frac{5}{2}ag + 2ag\cos\theta}$

FINALLY LOOKING AT THE EQUATION OF MOTION (RADIALLY)

$\Rightarrow m\ddot{r} = mg\cos\theta - T$

$\Rightarrow T = mg\cos\theta - m\ddot{r}$

$\Rightarrow T = mg\cos\theta - m\left(-\frac{v^2}{a}\right)$

$\Rightarrow T = mg\cos\theta + \frac{m}{a}v^2$

$\Rightarrow T = mg\cos\theta + \frac{m}{a}\left[\frac{5}{2}ag + 2ag\cos\theta\right]$

$\Rightarrow T = mg\cos\theta + \frac{5}{2}mg + 2mg\cos\theta$

$\Rightarrow T = 3mg\cos\theta + \frac{5}{2}mg$

$\Rightarrow T = \frac{1}{2}mg\left[6\cos\theta + 5\right]$

WHEN THE TENSION IS ZERO

$\Rightarrow 6\cos\theta + 5 = 0$

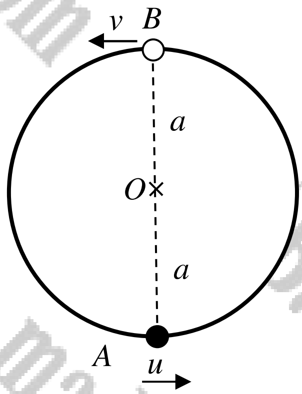
$\Rightarrow \cos\theta = -\frac{5}{6}$

$\Rightarrow \theta = \arccos\left(-\frac{\pi}{6}\right)$

OR

$\left[\theta = \pi - \arccos\frac{\pi}{6}\right]$

Question 15 (****)



A particle of mass m attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. The point A is at a distance a vertically below O, and the point B is at a distance a vertically above O.

The particle is held at a point A with the string taut. The particle then is projected horizontally with speed u and passes through B with speed v.

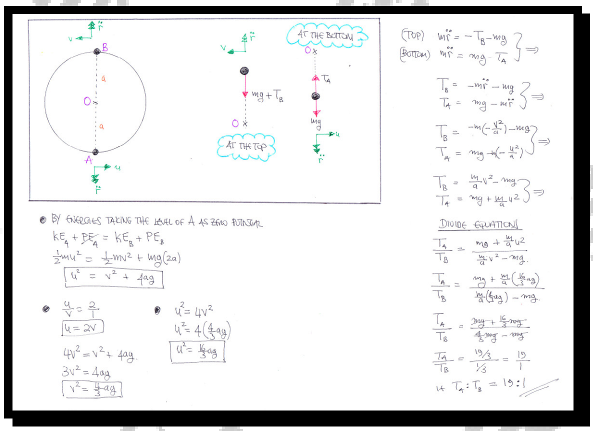
Air resistance is ignored, throughout the motion.

The tension in the string at A and B is denoted by T_A and T_B, respectively.

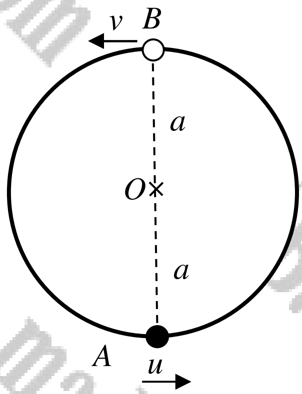
Given that the ratio $u:v = 2:1$, show that

$$T_A : T_B = 19:1.$$

proof



Question 16 (****)



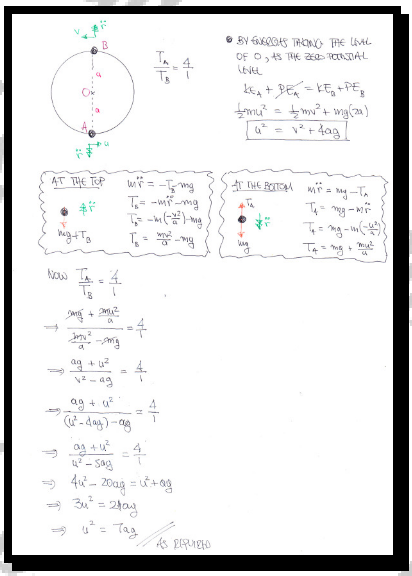
A particle of mass m attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. The point A is at a distance of a vertically below O, and the point B is at a distance of a vertically above O. The particle is held at a point A with the string taut. The particle then is projected horizontally with speed u and passes through B. Air resistance is ignored, throughout the motion.

The tension in the string at A and B is denoted by T_A and T_B, respectively.

Given that the ratio $T_A : T_B = 4:1$, show clearly that

$$u^2 = 7ag\ .$$

proof



Question 17 (****)

One end of a light inextensible string is attached to a fixed point O, and the other end is attached to a particle. Initially the particle is hanging in equilibrium vertically below O, with the string taut.

An impulse sets the particle in motion with a horizontal speed of 8.4 ms^{-1}, which consequently traces part of a vertical circle with centre at O.

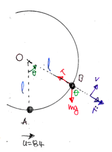
The string becomes slack when the speed of the particle is 3.5 ms^{-1}.

a) Determine the length of the string.

b) Calculate the vertical displacement of the particle from its initial position when the string becomes slack.

$\boxed{l = 1.725 \text{ m}}$, $\boxed{h = 2.975 \text{ m}}$

a) CONSIDERING ENERGIES TAKING THE LEVEL OF O AS THE ZERO GRAVITATIONAL POTENTIAL LEVEL



$\Rightarrow KE_A + PE_A = KE_B + PE_B$

$\Rightarrow \frac{1}{2}mu^2 - mgl = \frac{1}{2}mv^2 - mg(l\cos\theta)$

$\Rightarrow u^2 - 2gl = v^2 - 2gl\cos\theta$

$\Rightarrow v^2 = u^2 - 2gl + 2gl\cos\theta$

$\Rightarrow \boxed{v^2 = u^2 + 2gl(\cos\theta - 1)}$

EQUATION OF MOTION RADIALLY YIELDS

$\Rightarrow m\ddot{r} = mg\cos\theta - T$

$\Rightarrow m\left(-\frac{v^2}{l}\right) = mg\cos\theta - T$

WHEN $v = 3.5$, $T = 0$

$\Rightarrow -\frac{\cancel{m}(3.5)^2}{l} = \cancel{m}g\cos\theta$

$\Rightarrow -gl\cos\theta = 12.25$

$\Rightarrow \boxed{l\cos\theta = -\frac{5}{4}}$

FINALLY FROM THE ENERGY EQUATION $u = 8.4$ $v = 3.5$

$\Rightarrow (3.5)^2 = (8.4)^2 + 2gl(\cos\theta - 1)$

$\Rightarrow 2gl(\cos\theta - 1) = -58.31$

$l(\cos\theta - 1) = -2.975$

$l\cos\theta - l = -2.975$

$-1.25 - l = -2.975$

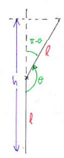
$\therefore l = \frac{69}{40} = 1.725 \text{ m}$

b) LOOKING AT THE DIAGRAM

$l\cos\theta = -\frac{5}{4}$

$\frac{69}{40}\cos\theta = -\frac{5}{4}$

$\cos\theta = -\frac{50}{69}$



VERTICAL DISPLACEMENT IS h

$h = l + l\cos\theta$

$h = l(1 + \cos\theta)$

$h = 1.725\left(1 + \frac{50}{69}\right)$

$h = 2.975 \text{ m}$

OR

$h = l + l\cos(\pi - \theta)$

$h = l + l[\cos\pi\cos\theta + \cancel{\sin\pi\sin\theta}]$

$h = l + l(-\cos\theta)$

$h = l - l\cos\theta$

$h = l(1 - \cos\theta)$

$h = 1.725\left(1 + \frac{50}{69}\right)$

$h = 2.975$

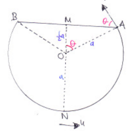
Question 18 (****)

A bowl is made from a hollow smooth sphere of radius a by cutting away the part of the sphere which is more than $\frac{1}{2}a$ above a horizontal plane through the centre of the sphere.

A particle is projected with speed u from the lowest point inside the bowl.

Show that in the subsequent motion, the particle will leave the bowl and not fall back in, if $u^2 > 5ag$.

proof



FIRSTLY $\cos\theta = \frac{\frac{1}{2}a}{a} = \frac{1}{2}$

$\theta = 60°$

$|AB| = 2|AM| = 2 \times a\sin\theta = \sqrt{3}a$

BY ENERGIES TAKING THE LEVEL OF N AS THE ZERO POTENTIAL LEVEL

$KE_N + \cancel{PE_N} = KE_A + PE_A$

$\frac{1}{2}mu^2 = \frac{1}{2}mv^2 + mg\left(a + \frac{1}{2}a\right)$

$u^2 = v^2 + 3ag$

$\boxed{v^2 = u^2 - 3ag}$

NOW PROJECTILE MOTION IF THE PARTICLE IS LAND BACK AT B

$$\left.\begin{array}{l} x = V\cos\theta \times T \\ y = V\sin\theta \times T - \frac{1}{2}gT^2 \end{array}\right\} \Rightarrow \left.\begin{array}{l} \sqrt{3}a = \frac{1}{2}VT \\ 0 = \frac{\sqrt{3}}{2}VT - \frac{1}{2}gT^2 \end{array}\right\} \Rightarrow$$

FLIGHT TIME T CAN BE FOUND

$\Rightarrow \sqrt{3}VT - gT^2 = 0$

$\Rightarrow T(\sqrt{3}V - gT) = 0$

$\Rightarrow \boxed{T = \frac{\sqrt{3}V}{g}}$

BUT PARTICLE DOESN'T FALL BACK INTO THE BOWL $\Rightarrow \frac{1}{2}VT > \sqrt{3}a$

$\Rightarrow \frac{1}{2}V \frac{\cancel{\sqrt{3}}V}{g} > \cancel{\sqrt{3}}a$

$\Rightarrow V^2 > 2ag$

$\Rightarrow u^2 - 3ag > 2ag$

$\Rightarrow u^2 > 5ag$

Question 19 (****+)

A small bead, of mass m, is threaded on a smooth circular wire, with centre O and radius a, which is fixed in a vertical plane. You may further assume that the bead can freely access all parts of the vertically fixed wire.

A light inextensible string has one of its ends attached to the bead, passes through a smooth ring at O, and has its other end attached to a particle of mass M, which is hanging freely vertically below O.

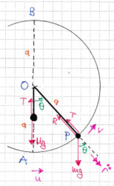
The bead is projected from the lowest part of the wire with speed u and makes complete revolutions passing through the highest part of the wire with speed $\sqrt{12ag}$.

Determine an expression for u^2, in terms of a and g, and show that

$$11m \le M \le 17m.$$

$\boxed{}$, $\boxed{u^2 = 4\sqrt{ag}}$

STARTING WITH A DETAILED DIAGRAM



BY ENERGIES TAKING THE LEVEL OF "O" AS THE ZERO POTENTIAL LEVEL

$\Rightarrow KE_A + PE_A = KE_\theta + PE_\theta$

$\Rightarrow \frac{1}{2}mu^2 - mga = \frac{1}{2}mv^2 - mg(a\cos\theta)$

$\Rightarrow u^2 - 2ag = v^2 - 2ag\cos\theta$

$\Rightarrow v^2 = u^2 - 2ag + 2ag\cos\theta$

AT THE TOP, $V = 12ag$ WHEN $\theta = \pi$

$\Rightarrow 12ag = u^2 - 2ag + 2ag\cos\pi$

$\Rightarrow 16ag = u^2$

$\Rightarrow u = 4\sqrt{ag}$

NEXT WE OBTAIN THE RADIAL EQUATION OF MOTION

$\Rightarrow m\ddot{r} = mg\cos\theta - T - R$

$\Rightarrow m\left(-\frac{v^2}{a}\right) = mg\cos\theta - Mg - R$

$\Rightarrow R = \frac{m}{a}v^2 + mg\cos\theta - Mg$

$\Rightarrow R = \frac{m}{a}\left[u^2 - 2ag + 2ag\cos\theta\right] + mg\cos\theta - Mg$

$\Rightarrow R = \frac{m}{a}(16ag - 2ag + 2ag\cos\theta) + mg\cos\theta - Mg$

$\Rightarrow R = 14mg + 3mg\cos\theta - Mg$

NOW WE HAVE $R = 0$

$\Rightarrow 0 = 14mg + 3mg\cos\theta - Mg$

$\Rightarrow 0 = 14m + 3m\cos\theta - M$

$\Rightarrow M - 14m = 3m\cos\theta$

$\Rightarrow \cos\theta = \frac{M - 14m}{3m}$

BUT $-1 \le \cos\theta \le 1$

$\Rightarrow -1 \le \frac{M-14m}{3m} \le 1$

$\Rightarrow -3m \le M - 14m \le 3m$

$\Rightarrow 11m \le M \le 17m$ //

Question 20 (****+)

A particle P of mass $3m$ is attached to one end of a light inextensible string of length a and the other end of the string is attached to a fixed point O.

P is initially is at the point A, where $OA = a$ and OA makes an angle of $60°$ with the downward vertical.

P is projected downwards from A with speed u, in a direction perpendicular to the string.

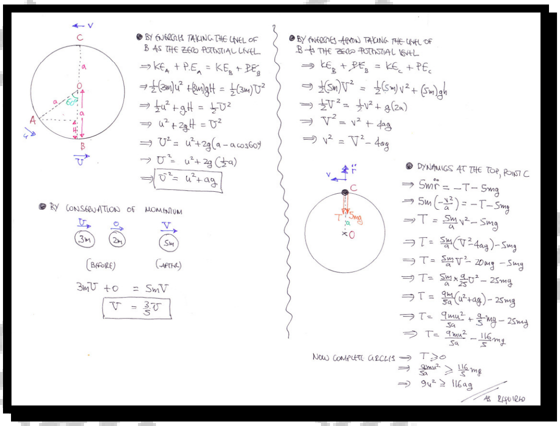
The point B is vertically below O and $OB = a$. As P passes through B, it strikes and adheres to another particle Q, of mass $2m$ which is at rest at B.

In the subsequent motion the combined particle moves in a complete circle.

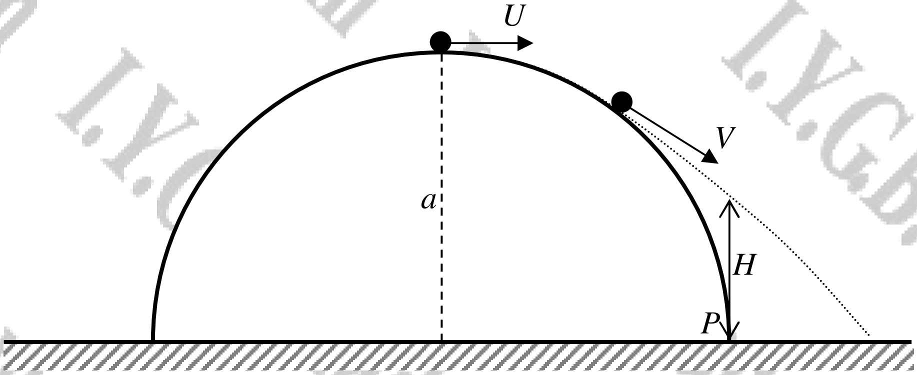
Show clearly that

$$9u^2 \geq 116ag .$$

proof



Question 21 (****+)



The figure above shows a hemisphere of radius a, with its plane face fixed on a horizontal surface.

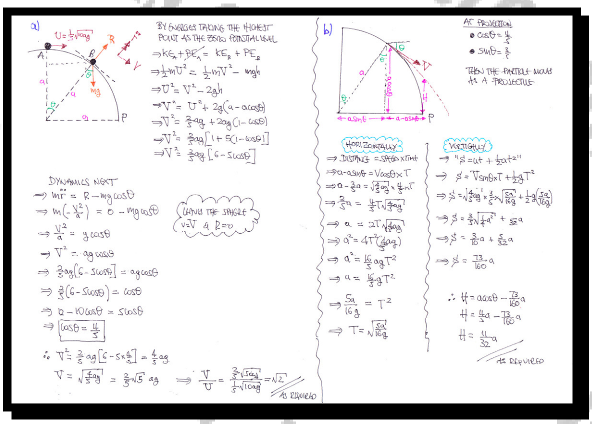
A particle is projected from the highest point of the hemisphere with horizontal velocity U and begins to move on the outer smooth surface of the hemisphere. The particle leaves the surface of the hemisphere with speed V.

a) Given that $U = \frac{1}{5}\sqrt{10ag}$, show that $\frac{V}{U} = \sqrt{2}$.

The point P lies on the rim of the plane face of the hemisphere. The path of the particle is at a vertical height H above P, before the particle lands on the horizontal surface.

b) Show further that $H = \frac{11}{32}a$.

proof



Question 22 (****+)

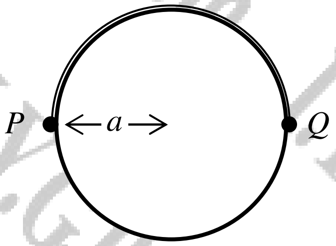


Figure 1

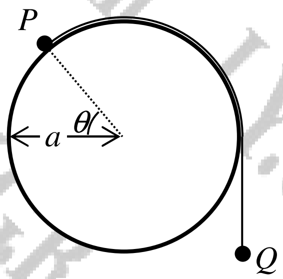


Figure 2

A rough cylinder of radius a is fixed with its axis horizontal. Two particles P and Q, of respective masses $2m$ and $3m$, are attached to the two end of a light inextensible string. The length of the string is such so that when the string is taut it allows P and Q to be held at rest diametrically opposite each other and the same horizontal level and in the same vertical plane, as shown in Figure 1

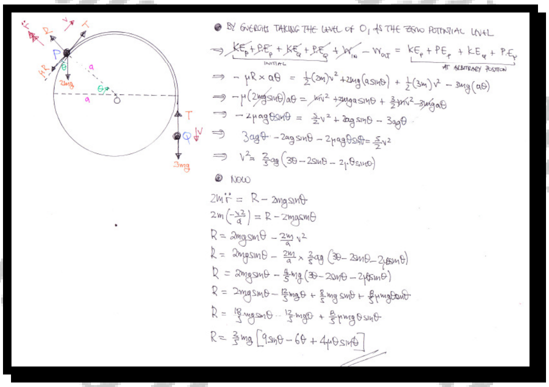
The particles are then released from rest. The coefficient of friction between P and the cylinder is sufficiently small for the particles to move. When P has rotated on the surface of the cylinder by an angle θ, P is still in contact with the cylinder, as shown in Figure 2.

Show that the contact force between P and the cylinder is

$$\frac{2}{5}mg\left(9\sin\theta - 6\theta + 4\mu\theta\sin\theta\right),$$

where μ is the coefficient of friction between P and the cylinder.

proof



- BY ENERGY TAKING THE LEVEL OF O, AS THE ZERO POTENTIAL LEVEL

$\Rightarrow \underbrace{\cancel{KE_P} + \cancel{PE_P} + \cancel{KE_Q} + \cancel{PE_Q}}_{\text{INITIAL}} + \cancel{W_{IN}} - W_{OUT} = \underbrace{KE_P + PE_P + KE_Q + PE_Q}_{\text{AT ARBITRARY POSITION}}$

$\Rightarrow -\mu R \times a\theta = \frac{1}{2}(2m)v^2 + 2mg(a\sin\theta) + \frac{1}{2}(3m)v^2 - 3mg(a\theta)$

$\Rightarrow -\mu(2mg\sin\theta)a\theta = mv^2 + 2mga\sin\theta + \frac{3}{2}mv^2 - 3mga\theta$

$\Rightarrow -2\mu ag\theta\sin\theta = \frac{5}{2}v^2 + 2ag\sin\theta - 3ag\theta$

$\Rightarrow 3ag\theta - 2ag\sin\theta - 2\mu ag\theta\sin\theta = \frac{5}{2}v^2$

$\Rightarrow v^2 = \frac{2}{5}ag(3\theta - 2\sin\theta - 2\mu\theta\sin\theta)$

- NOW

$2m\ddot{r} = R - 2mg\sin\theta$

$2m\left(-\frac{v^2}{a}\right) = R - 2mg\sin\theta$

$R = 2mg\sin\theta - \frac{2m}{a}v^2$

$R = 2mg\sin\theta - \frac{2m}{a} \times \frac{2}{5}ag(3\theta - 2\sin\theta - 2\mu\theta\sin\theta)$

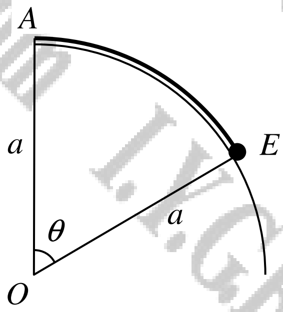
$R = 2mg\sin\theta - \frac{4}{5}mg(3\theta - 2\sin\theta - 2\mu\theta\sin\theta)$

$R = 2mg\sin\theta - \frac{12}{5}mg\theta + \frac{8}{5}mg\sin\theta + \frac{8}{5}\mu mg\theta\sin\theta$

$R = \frac{18}{5}mg\sin\theta - \frac{12}{5}mg\theta + \frac{8}{5}\mu mg\theta\sin\theta$

$R = \frac{2}{5}mg\left[9\sin\theta - 6\theta + 4\mu\theta\sin\theta\right]$

Question 23 (****+)



The figure above shows part of a fixed smooth sphere with centre O and radius a. One end of light elastic string is attached at the highest point of the sphere A and a particle P of mass m is attached to the other end. The particle is resting in equilibrium at the point E so that $\measuredangle AOE = \theta$.

The natural length of the string is $\frac{1}{2}a$ and its modulus of elasticity is $\frac{1}{2}mg$.

a) Show clearly that

$$2(\theta - \sin\theta) = 1.$$

b) Verify that $\theta \approx 1.4973$.

The particle with the string still attached to A is placed at A and is slightly disturbed.

c) Determine whether P is still in contact with the sphere when $\measuredangle AOP = \theta$.

$\boxed{\text{proof}}$