

Created by T. Madas

COLLISIONS

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Question 1 ()**

A smooth sphere is moving with speed $u \text{ ms}^{-1}$ on a smooth horizontal plane when it strikes at right angles a fixed smooth vertical wall. The sphere is modelled as a particle. The coefficient of restitution between the sphere and the wall is $\frac{1}{3}$.

Find the fraction of the kinetic energy is lost by the sphere, as a result of the impact with the wall.

$\frac{8}{9}$

Handwritten solution for Question 1:

- Diagram: A sphere moving with speed u towards a vertical wall. After impact, it moves with speed v .
- Equations:
 - $e = \frac{\text{Sep}}{\text{App}}$
 - $\frac{1}{3} = \frac{v}{u}$
 - $v = \frac{1}{3}u$
 - Initial KE: $KE_{\text{before}} = \frac{1}{2}mu^2$
 - Final KE: $KE_{\text{after}} = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{1}{3}u\right)^2 = \frac{1}{18}mu^2$
 - Loss of KE: $\frac{1}{2}mu^2 - \frac{1}{18}mu^2 = \frac{9}{18}mu^2 - \frac{1}{18}mu^2 = \frac{8}{18}mu^2 = \frac{4}{9}mu^2$
 - Fraction lost: $\frac{\frac{4}{9}mu^2}{\frac{1}{2}mu^2} = \frac{8}{9}$

Question 2 (+)**

A smooth sphere is moving with speed $u \text{ ms}^{-1}$ on a smooth horizontal plane when it strikes at right angles a fixed smooth vertical wall.

One quarter of the kinetic energy is lost by the sphere, as a result of the impact with the wall. The sphere is modelled as a particle.

Find the coefficient of restitution between the sphere and the wall.

$e = \frac{\sqrt{3}}{2}$

Handwritten solution for Question 2:

- Diagram: A sphere moving with speed u towards a vertical wall. After impact, it moves with speed v .
- Equations:
 - Initial KE: $KE_{\text{before}} = \frac{1}{2}mu^2$
 - Final KE: $KE_{\text{after}} = \frac{1}{2}mv^2 = \frac{1}{2}m(eu)^2 = \frac{1}{2}me^2u^2$
 - KE loss: $\frac{1}{2}mu^2 - \frac{1}{2}me^2u^2$
 - Given: $\frac{\frac{1}{2}mu^2 - \frac{1}{2}me^2u^2}{\frac{1}{2}mu^2} = \frac{1}{4}$
 - Simplify: $1 - e^2 = \frac{1}{4}$
 - $1 - \frac{1}{4} = e^2$
 - $e^2 = \frac{3}{4}$
 - $e = \frac{\sqrt{3}}{2}$

Question 3 (*)**

A smooth sphere A of mass 3 kg is moving with speed 4 ms^{-1} on a smooth horizontal plane when it collides directly with a smooth sphere B of mass 5 kg moving with speed 2 ms^{-1} , in the opposite direction as B . The two spheres are modelled as particles and the coefficient of restitution between them is $\frac{5}{6}$.

- a) Calculate the speed of A and the speed of B after the collision.

After the collision between A and B , sphere B collides directly with a smooth vertical wall. The coefficient of restitution between B and the wall is $\frac{1}{5}$.

- b) Find the magnitude of the impulse exerted by the wall onto B .

$$\boxed{|V_A| = \frac{23}{8} \text{ ms}^{-1}}, \quad \boxed{|V_B| = \frac{17}{8} \text{ ms}^{-1}}, \quad \boxed{|I| = 12.75 \text{ Ns}}$$

(a)

BY CONSERVATION OF MOMENTUM
 $(4 \times 3) - (2 \times 5) = 3x + 5y$
 $3x + 5y = 2$

BY RESTITUTION
 $e = \frac{5y - 4x}{2 - 4}$
 $\frac{5}{6} = \frac{2 - 4x}{-2}$
 $5 - 10x = 2 - 4x$
 $Y - X = 5$

$3x + 5(4 + x) = 2$
 $3x + 20 + 5x = 2$
 $8x = -18$
 $X = -\frac{18}{8}$ (MIND! \Rightarrow 'A' HAS REBOUNDED)
 $X = -2.25$

$Y = X + 5$
 $Y = -\frac{18}{8} + 5$
 $Y = \frac{22}{8} = 2.75$

\therefore SPEED OF A IS 2.25 ms^{-1} (BOTH REBOUNDED)
 SPEED OF B IS 2.75 ms^{-1}

(b)

SPEED OF B AFTER REBOUNDED IS "e x APPROACH SPEED"
 $\text{i.e. } \frac{1}{5} \times \frac{17}{8} = \frac{17}{40} = 0.425$

TAKING 'RIGHT' AS POSITIVE
 $I = \text{MOM AFTER} - \text{MOM BEFORE}$
 $I = -\frac{17}{40} \times 5 - \frac{17}{8} \times 5$
 $I = -\frac{51}{4}$
 $|I| = \frac{51}{4} = 12.75 \text{ Ns}$

Question 4 (*)**

Two smooth spheres P and Q of respective masses $3m$ and $4m$ are moving towards each other, both with speed u , when they collide directly. As a result of the collision the direction of the motion of Q is reversed and its speed is halved.

The spheres are modelled as particles moving on a smooth horizontal plane.

- a) Find the coefficient of restitution between the two spheres.

Consequently, sphere Q collides directly with a third sphere R of mass $6m$ which is initially at rest. The collision between Q and R is perfectly elastic.

Sphere R is also modelled as a particle.

- b) Show that after the collision between Q and R , they will be no more collisions between P and Q .

$$e = \frac{3}{4}$$

(a)

By conservation of momentum
 $3mu - 4mu = 3mX + 2mu$
 $-3mu = 3mX$
 $X = -u$ (it reverses)

By definition
 $e = \frac{\text{sep. app.}}{\text{app. sep.}}$
 $e = \frac{u + \frac{1}{2}u}{2u} = \frac{3}{4}$
 $e = \frac{3}{4}$

(b)

By conservation of momentum
 $2mu + 0 = 4mY + 6mW$
 $2Y + 3W = u$

By definition
 $e = 1$ (perfectly elastic)
 $\frac{W - Y}{\frac{1}{2}u} = 1$
 $-Y + W = \frac{1}{2}u$
 $W = Y + \frac{1}{2}u$

$2Y + 3(Y + \frac{1}{2}u) = u$
 $2Y + 3Y + \frac{3}{2}u = u$
 $5Y = -\frac{1}{2}u$
 $Y = -\frac{1}{10}u$ (it reverses)

Hence after two collisions

As $\frac{1}{10}u < u$, no more collisions between P & Q

Question 5 (*)**

A smooth sphere P of mass m is moving with speed u on a smooth horizontal plane when it collides directly with a smooth sphere Q of mass $4m$ which is initially at rest. The spheres are modelled as particles and the coefficient of restitution between the two spheres is e .

The magnitude of the impulse of P on Q is $\frac{22}{15}mu$.

- a) Find the value of e .

It is now given that $m = 2$ kg and $u = 30$ ms⁻¹.

- b) Show that the kinetic energy loss due to the collision is 220 J.

$$e = \frac{5}{6}$$

(a)

Diagram: Sphere P (mass m) moving right with speed u . Sphere Q (mass $4m$) at rest. After collision, P has speed X and Q has speed Y . Impulse on Q is $\frac{22}{15}mu$.

• IMPULSE ON Q IS $\frac{22}{15}mu$
 $\frac{22}{15}mu = 4mY - 4m \times 0$
 $Y = \frac{11}{30}u$

• BY CONSERVATION OF MOMENTUM
 $mu + 0 = mX + 4mY$
 $u = X + \frac{44}{30}u$
 $X = -\frac{14}{15}u$

• DEFINITION
 $e = \frac{Y - X}{u - 0}$
 $e = \frac{\frac{11}{30}u - (-\frac{14}{15}u)}{u}$
 $e = \frac{\frac{11}{30} + \frac{28}{30}}{1}$
 $e = \frac{39}{30}$
 $e = \frac{13}{10}$ (Incorrect)

(b) IF $m=2$ $u=30$

• K.E. BEFORE = $\frac{1}{2} \times 2 \times 30^2 + 0 = 900$ J

• K.E. AFTER = $\frac{1}{2} \times 2 \times 14^2 + \frac{1}{2} \times 8 \times 11^2 = 196 + 484 = 680$ J

\therefore LOSS OF $900 - 680 = 220$ J

Question 6 (*)**

A smooth sphere A of mass $3m$ is moving with speed $5u$ on a smooth horizontal plane. It collides directly with a smooth sphere B of mass m moving with speed $3u$ in the opposite direction as A . The spheres are modelled as particles and the coefficient of restitution between A and B is e .

- Find, in terms of e and u , the speeds of the two spheres after the collision.
- Show that A cannot possibly reverse direction as a result of the collision.
- Given further that the total kinetic energy of the two spheres after the collision is $\frac{39}{2}mu^2$, show clearly that $e = \frac{1}{4}$

$$V_A = u(3 - 2e), \quad V_B = 3u(1 + 2e)$$

(a)

• BY CONSERVATION OF MOMENTUM
 $15mu - 3mu = 3mX + mY$
 $12u = 3X + Y$

• BY RESTITUTION
 $e = \frac{\text{SEP}}{\text{APP}}$
 $e = \frac{Y - X}{8u}$
 $-X + Y = 8eu$

SUBSTITUTE
 $3X + Y = 12u$
 $-X + Y = 8eu \Rightarrow 4X = 12u - 8eu$
 $X = 3u - 2eu$
 $X = u(3 - 2e)$

AND $Y = X + 8eu$
 $Y = (3u - 2eu) + 8eu$
 $Y = 3u + 6eu$
 $Y = 3u(1 + 2e)$

(b)

$0 < e < 1$
 $0 < 2e < 2$
 $-2 < -2e < 0$
 $1 < 3 - 2e < 3$
 $u < u(3 - 2e) < 3u$
 $u < X < 3u$

• IF $e = 0$
 $X = 3u$ (to the right)

• IF $e = 1$
 $X = u$ (to the right)

WE CHECK GET A NEGATIVE VALUE FOR X IF $e < -1$
 ∴ ALWAYS TO THE RIGHT

(c) $K.E. = \frac{39}{2}mu^2$

$$\Rightarrow \frac{1}{2}(3u)^2 + \frac{1}{2}(u)^2 = \frac{39}{2}mu^2$$

$$\Rightarrow 3X^2 + Y^2 = 39u^2$$

$$\Rightarrow 3[u(3 - 2e)]^2 + [3u(1 + 2e)]^2 = 39u^2$$

$$\Rightarrow 3u^2(3 - 2e)^2 + 9u^2(1 + 2e)^2 = 39u^2$$

$$\Rightarrow (3 - 2e)^2 + 3(1 + 2e)^2 = 13$$

$$\Rightarrow 9 - 12e + 4e^2 + 12e^2 + 12e + 3 = 13$$

$$\Rightarrow 16e^2 = 1$$

$$\Rightarrow e = \frac{1}{4}$$

(*)

Question 7 (*)**

A smooth sphere P of mass m is moving with speed $4u$ on a smooth horizontal plane. It collides directly with a smooth sphere Q of mass $2m$ moving with speed u in the opposite direction as P . The spheres are modelled as particles and the coefficient of restitution between two spheres is e .

- a) Find, in terms of e and u , the speeds of the two spheres after their collision.

The total kinetic energy of the two spheres after the collision is mu^2 .

- b) Find the value of e and hence show that P is at rest after the collision.

$$V_P = \frac{2}{3}u(1-5e), \quad V_Q = \frac{1}{3}u(5e+2), \quad e = \frac{1}{5}$$

(a)

Diagram: Sphere P (mass m) moving right with speed 4u. Sphere Q (mass 2m) moving left with speed u. After collision, P has speed X and Q has speed Y.

By conservation of momentum: $4mu - 2mu = mX + 2mY$
 $X + 2Y = 2u$

By restitution: $e = \frac{Y - X}{4u}$
 $e = \frac{Y - X}{4u}$
 $-X + Y = 4eu$

Add equations:
 $3Y = 2u + 4eu$
 $Y = \frac{2}{3}u(1 + 2e)$

Substitute Y into momentum equation:
 $X + 2(\frac{2}{3}u(1 + 2e)) = 2u$
 $X + \frac{4}{3}u(1 + 2e) = 2u$
 $X = 2u - \frac{4}{3}u(1 + 2e) = 2u - \frac{4}{3}u - \frac{8}{3}ue = \frac{2}{3}u(3 - 2 - 4e) = \frac{2}{3}u(1 - 4e)$

(b) KE after = mu^2
 $\frac{1}{2}mX^2 + \frac{1}{2}(2m)Y^2 = mu^2$
 $\frac{1}{2}X^2 + Y^2 = u^2$
 $X^2 + 2Y^2 = 2u^2$
 $\frac{4}{9}u^2(1 - 4e)^2 + 2 \times \frac{4}{9}u^2(1 + 2e)^2 = 2u^2$
 $\frac{4}{9}(1 - 4e)^2 + \frac{8}{9}(1 + 2e)^2 = 2$
 $2(1 - 4e)^2 + 4(1 + 2e)^2 = 9$
 $2(1 - 8e + 16e^2) + 4(1 + 4e + 4e^2) = 9$
 $2 - 16e + 32e^2 + 4 + 16e + 16e^2 = 9$
 $75e^2 = 3$
 $e^2 = \frac{1}{25}$
 $e = \frac{1}{5}$

Check: If $e = \frac{1}{5}$
 $X = \frac{2}{3}u(1 - \frac{4}{5}) = \frac{2}{3}u(\frac{1}{5}) = \frac{2}{15}u$
 $Y = \frac{2}{3}u(1 + \frac{4}{5}) = \frac{2}{3}u(\frac{9}{5}) = \frac{12}{15}u = \frac{4}{5}u$

Question 8 (*)**

A smooth sphere P of mass m is moving with constant speed on a smooth horizontal plane when it collides directly with a smooth sphere Q of mass km , where k is a positive constant, which is initially at rest. After the collision both spheres are moving in the same direction as the original direction of P with speeds u and $4u$.

The spheres are modelled as particles and the coefficient of restitution between the two spheres is e .

- a) Show clearly that $4k = \frac{3}{e} - 1$.
- b) Hence, find the range of the possible values of k .
- c) Given further that $k = 2$...
 - i. ... calculate the value of e
 - ii. ... find the kinetic energy loss of the system, due to the collision.

$$k \geq \frac{1}{2}, \quad e = \frac{1}{3}, \quad \text{loss} = 24mu^2$$

Handwritten solution for Question 8:

(a) Conservation of momentum: $mX + 0 = mu + 4ku$
 $X = u + 4ku$
 $X = u(1 + 4k)$

By restitution: $e = \frac{4u - X}{X - 0}$
 $e = \frac{4u - u(1 + 4k)}{u(1 + 4k)}$
 $e = \frac{3u - 4ku}{u(1 + 4k)}$
 $e = \frac{3 - 4k}{1 + 4k}$
 $e(1 + 4k) = 3 - 4k$
 $e + 4ek = 3 - 4k$
 $4ek + 4k = 3 - e$
 $4k(e + 1) = 3 - e$
 $4k = \frac{3 - e}{e + 1}$
 $4k = \frac{3}{e} - 1$ (EQUATION)

(b) $0 < e < 1$ (SMALLER OR EQUAL TO BECAUSE PARTICLES DID NOT COLLIDE)
 $\Rightarrow 1 < \frac{3}{e} - 1 < 1$
 $\Rightarrow \frac{1}{e} > 1$
 $\Rightarrow \frac{1}{e} > 2$
 $\Rightarrow \frac{1}{2} > e$
 $\Rightarrow k > \frac{1}{2}$

(c) (i) If $k = 2$
 $4e = \frac{3}{e} - 1$
 $8 = \frac{3}{e} - 1$
 $9 = \frac{3}{e}$
 $e = \frac{1}{3}$

(ii) $X = u(1 + 4k) = 9u$
 • KE before = $\frac{1}{2}mX^2 = \frac{1}{2}m(9u)^2 = \frac{81}{2}mu^2$
 • KE after = $\frac{1}{2}mX^2 + \frac{1}{2}(4m)(4u)^2 = \frac{33}{2}mu^2$
 ∴ A LOSS OF $\frac{81}{2}mu^2 - \frac{33}{2}mu^2 = 24mu^2$

Question 9 (*)**

A smooth sphere A of mass m is moving with constant speed on a smooth horizontal plane when it collides directly with a smooth sphere B of mass $3m$, which is initially at rest. After the collision A reverses direction and its speed is v .

The two spheres are modelled as particles and the coefficient of restitution between the two spheres is $\frac{3}{4}$.

- a) Determine, in terms of v , the speed A before the collision and the speed of B after the collision.

Consequently, B collides directly with a third sphere C of mass km , where k is a positive constant. It is further given that C was initially at rest and the coefficient of restitution between B and C is $\frac{6}{7}$.

- b) By modelling C as a smooth particle, find the value of k , given that B is brought to rest after the collision.

$$V_A = \frac{16}{5}v, \quad V_B = \frac{7}{5}v, \quad k = \frac{7}{2}$$

(a)

By conservation of momentum:
 $mu + 0 = -mv + 3mV$
 $X = -v + 3V$

By restitution:
 $e = \frac{3V - (-v)}{u - 0} = \frac{3}{4}$
 $\frac{3}{4} = \frac{V+v}{X}$
 $3X = 4V + 4v$

$3(-v + 3V) = 4V + 4v$
 $-3v + 9V = 4V + 4v$
 $5V = 7v$
 $V = \frac{7}{5}v$ (Speed of B after)

$\therefore X = -v + 3V$
 $X = -v + \frac{21}{5}v$
 $X = \frac{16}{5}v$ (Speed of A before)

(b)

By conservation of momentum:
 $\frac{3}{4} \times 3mV + 0 = 0 + kW$
 $kW = \frac{21}{4}mV$
 $k \times \frac{3}{4} \times \frac{7}{5}v = \frac{21}{4}v$
 $6k = 21$
 $k = \frac{7}{2}$

By restitution:
 $e = \frac{0 - W}{\frac{3}{4} \times 3mV - 0} = \frac{6}{7}$
 $\frac{6}{7} = \frac{-W}{\frac{21}{4}V}$
 $W = \frac{6}{7} \times \frac{21}{4}V$
 $W = \frac{9}{2}V$

Question 10 (*)**

A smooth sphere A of mass $5m$ is moving with speed $4u$ on a smooth horizontal plane when it collides directly with a smooth sphere B of mass $3m$ moving with speed u , in the same direction as A . The two spheres are modelled as particles and the coefficient of restitution between them is e .

After the collision the speed of B is $\frac{21}{2}eu$.

- a) Show clearly that $e = \frac{1}{3}$.
- b) Find in terms of m and u the kinetic energy lost, as a result of the collision.

kinetic energy lost = $\frac{15}{2}mu^2$

(a) Conservation of momentum: $5m(4u) + 3m(u) = 5mX + 3m(\frac{21}{2}eu)$
 $25u + 3u = 5X + \frac{63}{2}eu$
 $28u = 5X + \frac{63}{2}eu$
 $28u = 5(\frac{21}{2}eu - X) + \frac{63}{2}eu$
 $28u = \frac{105}{2}eu - 5X + \frac{63}{2}eu$
 $28u = \frac{168}{2}eu - 5X$
 $28u = 84eu - 5X$
 $5X = 84eu - 28u$
 $X = \frac{84eu - 28u}{5}$

By Restitution: $e = \frac{3u - X}{4u - u}$
 $e = \frac{3u - (\frac{84eu - 28u}{5})}{3u}$
 $e = \frac{15u - 84eu + 28u}{15u}$
 $e = \frac{43u - 84eu}{15u}$
 $15ue = 43u - 84eu$
 $15ue + 84eu = 43u$
 $99eu = 43u$
 $e = \frac{43}{99}$ (Incorrect)

(b) KE before = $\frac{1}{2}(5m)(4u)^2 + \frac{1}{2}(3m)u^2 = 40mu^2 + \frac{3}{2}mu^2 = \frac{83}{2}mu^2$

SPEEDS AFTER
 (A): $X = \frac{84eu - 28u}{5} = \frac{15}{5}u = 3u$
 (B): $\frac{21}{2}eu = \frac{21}{2} \times \frac{1}{3}u = \frac{7}{2}u$

KE after = $\frac{1}{2}(5m)(3u)^2 + \frac{1}{2}(3m)(\frac{7}{2}u)^2 = \frac{45}{2}mu^2 + \frac{157}{8}mu^2 = 34mu^2$

\therefore A LOSS OF $\frac{83}{2}mu^2 - 34mu^2 = \frac{15}{2}mu^2$

Question 11 (***)

A smooth sphere A of mass 2 kg is moving with speed 12 ms^{-1} on a smooth horizontal plane when it collides directly with a smooth sphere B of mass $m \text{ kg}$, which is initially at rest. After the collision A reverses direction and its speed is v .

The two spheres are modelled as particles and the coefficient of restitution between the two spheres is e .

- a) Given that the speed of B after the collision is $2v$, show clearly that

$$m = \frac{v+12}{v}$$

After the collision, $\frac{11}{16}$ of the initial kinetic energy is conserved.

- b) Calculate ...
- i. ... the value of v .
 - ii. ... the value of e .

$v = 3$, $e = 0.75$

The image shows a handwritten solution for Question 11. It includes a diagram of two spheres, A and B, before and after collision. Sphere A has mass 2 kg and initial velocity 12 ms⁻¹ to the right. Sphere B has mass m kg and is initially at rest. After collision, sphere A has velocity v to the left and sphere B has velocity 2v to the right. The solution uses the conservation of momentum and the conservation of kinetic energy to derive the required equations.

Momentum:

$$2 \times 12 + 0 = -2v + 2mv$$

$$\Rightarrow 24 = -2v + 2mv$$

$$\Rightarrow 12 = mv - v$$

$$\Rightarrow 12 + v = mv$$

$$\Rightarrow m = \frac{v+12}{v}$$

Energy:

$$\frac{1}{2} \times 2 \times 12^2 = \frac{1}{2} \times 2 \times v^2 + \frac{1}{2} m (2v)^2$$

$$144 = v^2 + 2m(2v)^2$$

$$= v^2 + 8mv^2$$

$$= v^2 + 8 \left(\frac{v+12}{v} \right) v^2$$

$$= v^2 + 8(v+12)v$$

$$144 = v^2 + 8v^2 + 96v$$

$$144 = 9v^2 + 96v$$

$$9v^2 + 96v - 144 = 0$$

$$v^2 + 10.67v - 16 = 0$$

$$(v-3)(v+11) = 0$$

$$v = 3$$

Coefficient of Restitution:

$$e = \frac{2v - (-v)}{12 - 0} = \frac{3v}{12} = \frac{v}{4} = \frac{3}{4} = 0.75$$

Question 12 (***)

A particle A , of mass 0.2 kg , is travelling in a straight line on a smooth horizontal surface, when it collides with a particle B , of mass 1.5 kg , which is moving on the same surface and in the same direction as A .

The respective speeds of A and B just before the collision are 15 ms^{-1} and 4 ms^{-1} .

The coefficient of restitution between the two particles, e , is such so that the two particles move in the same direction after the collision.

Show that $e < \frac{6}{11}$.

, proof

The handwritten solution is as follows:

$$\begin{array}{ccc} \overset{15}{\rightarrow} & \overset{4}{\rightarrow} & \overset{x}{\rightarrow} \quad \overset{y}{\rightarrow} \\ \text{---} & \text{---} & \text{---} \\ \text{A} & \text{B} & \text{A} \quad \text{B} \\ \text{(Before)} & & \text{(After)} \end{array}$$

- By conservation of momentum**
 $(0.2 \times 15) + (1.5 \times 4) = 0.2x + 1.5y$
 $(5 \times 2) + (4 \times 15) = 2x + 15y$
 $2x + 15y = 90$
- By considering restitution**
 $\frac{y-x}{15-4} = e$
 $y-x = 11e$
 $y = x + 11e$
- It is given that both particles continue in the original direction of motion, of which "B" has to be "A" way with remainder**
- Howe we need an expression for X, then set it positive**
 $\Rightarrow 2x + 15(x + 11e) = 90$
 $\Rightarrow 2x + 15x + 165e = 90$
 $\Rightarrow 17x = 90 - 165e$
 $\Rightarrow x = \frac{15}{17}(6 - 11e)$
BUT $x > 0$
 $\Rightarrow 6 - 11e > 0$
 $\Rightarrow -11e > -6$
 $e < \frac{6}{11}$

Question 13 (****)

Three small smooth spheres A , B and C , are resting on a straight line, and in that order, on a horizontal surface.

The respective masses of A , B and C , are m , $3m$ and $7m$.

A is project towards B with speed u and a direct collision takes place.

The coefficient of restitution between A and B is 0.5 .

The coefficient of restitution between B and C is e .

If there is a second collision between A and B , find the range of possible values of e .

$$\boxed{}, \frac{19}{21} < e \leq 1$$

LOOKING AT THE COLLISION BETWEEN A & B

BY CONSERVATION OF MOMENTUM

$$\Rightarrow mu + 0 = mX + 3mY$$

$$\Rightarrow u = X + 3Y$$

$$\Rightarrow X + 3Y = u$$

BY CONSIDERING RESTITUTION

$$\Rightarrow e = \frac{3mY - mX}{mu}$$

$$\Rightarrow \frac{1}{2} = \frac{Y - X}{u}$$

$$\Rightarrow -X + Y = \frac{1}{2}u$$

ADDING ONLY

$$4Y = \frac{3}{2}u$$

$$Y = \frac{3}{8}u$$

AND HENCE

$$X = u - 3Y$$

$$X = u - 3\left(\frac{3}{8}u\right)$$

$$X = u - \frac{9}{8}u$$

$$X = -\frac{1}{8}u$$

$\therefore A$ HAS REBOUNDED (GIVEN) WITH SPEED $\frac{1}{8}u$

NEXT THE COLLISION BETWEEN B & C

BY CONSERVATION OF MOMENTUM

$$\Rightarrow 3m\left(\frac{3}{8}u\right) + 0 = 3mW + 7mV$$

$$\Rightarrow \frac{9}{8}u = 3W + 7V$$

BY CONSIDERING RESTITUTION

$$\Rightarrow e = \frac{7mV - 3mW}{3mY}$$

$$\Rightarrow e = \frac{7V - 3W}{3Y}$$

$$\Rightarrow -3W + 7V = \frac{7}{8}ue$$

$$\Rightarrow -7V + 7W = -\frac{7}{8}ue$$

ADDING THE EQUATIONS ABOVE (WE ONLY NEED V)

$$\Rightarrow 10V = \frac{9}{8}u - \frac{7}{8}ue$$

$$\Rightarrow 10V = \frac{9}{8}u(1 - e)$$

$$\Rightarrow V = \frac{9}{80}u(1 - e) \leftarrow \text{TO THE 'RIGHT'}$$

$$\Rightarrow V = \frac{9}{80}u(1 - e) \leftarrow \text{TO THE 'LEFT'}$$

FOR A COLLISION BETWEEN B & C

$$\Rightarrow \frac{9}{80}u(1 - e) > \frac{1}{8}u$$

$$\Rightarrow 1 - e > \frac{10}{9}$$

$$\Rightarrow 1 - e > \frac{10}{9}$$

$$\Rightarrow e > \frac{19}{21}$$

OR $\frac{19}{21} < e < 1$

Question 14 (***)

A smooth sphere P of mass $2m$ is moving with speed u on a smooth horizontal plane when it collides directly with a smooth sphere Q of mass $3m$ which is initially at rest. The spheres are modelled as particles and the coefficient of restitution between the two spheres is $\frac{1}{4}$.

- a) Find, in terms of u , the speeds of P and Q after the collision.

Consequently, sphere Q strikes at right angles a fixed smooth vertical wall, and rebounds at right angles. The coefficient of restitution between Q and the wall is e .

After a second collision between the spheres, P is brought to rest.

- b) Show clearly that $e = \frac{1}{6}$.

$$V_P = \frac{1}{4}u, \quad V_Q = \frac{1}{2}u$$

(a) $\begin{matrix} u & 0 \\ \rightarrow & \rightarrow \\ \text{---} & \text{---} \\ P & Q \\ 2m & 3m \end{matrix} \quad \begin{matrix} X & Y \\ \rightarrow & \rightarrow \\ \text{---} & \text{---} \\ P & Q \\ 2m & 3m \end{matrix} \quad \text{--- POSITIVE}$

• BY CONSERVATION OF MOMENTUM $2mu = 2mX + 3mY$
 $2u = 2X + 3Y$
 \downarrow
 $2X + 3Y = 2u$
 $-2X + 2Y = \frac{1}{2}u$
 $5Y = \frac{3}{2}u$
 $Y = \frac{3}{10}u$ (SPEED OF Q)

• BY RESTITUTION $e = \frac{SEP}{APP} = \frac{Y - X}{u - 0} = \frac{1}{4}$
 $-X + Y = \frac{1}{4}u$
 $\leftarrow -2X + 2Y = \frac{1}{2}u$

AND $2u = 2X + 3Y$
 $2u = 2X + \frac{3}{2}u$
 $\frac{1}{2}u = 2X$
 $X = \frac{1}{4}u$ (SPEED OF P)

(b) $\begin{matrix} \frac{1}{2}u \\ \downarrow \\ \text{---} \\ P \\ 2m \end{matrix} \quad \begin{matrix} \frac{1}{2}u \\ \downarrow \\ \text{---} \\ Q \\ 3m \end{matrix} \quad \begin{matrix} 0 & V \\ \rightarrow & \rightarrow \\ \text{---} & \text{---} \\ P & Q \\ 2m & 3m \end{matrix} \quad \text{--- POSITIVE}$

• BY CONSERVATION OF MOMENTUM $\frac{1}{2}u(2m) - \frac{1}{2}u(3m) = 3mV$
 $u - 3eu = 3mV$
 \downarrow
 $u - 3eu = 3(\frac{1}{4}u + \frac{1}{2}u)V$
 $(1 - 3e)u = \frac{3}{2}u + \frac{3}{2}eV$
 $8 - 24e = 3 + 6e$
 $5 = 30e$
 $e = \frac{1}{6}$ AS REQUIRED

Question 15 (***)

Two smooth spheres A and B , of respective masses $4m$ and m , are moving in a straight line and in the same direction towards a smooth vertical wall. The speeds of A and B are u and $5u$ respectively. Sphere B hits the wall at right angles and rebounds so that it subsequently collides with A . Immediately after this collision the speed of A is U and the speed of B is V , with the direction of motion of each sphere reversed.

The spheres are modelled as particles and the coefficient of restitution between B and the wall is 0.8 .

- a) Show clearly that $V = 4U$.

The total kinetic energy of the two spheres immediately after their collision is $\frac{1}{4}$ of the total kinetic energy of the two spheres immediately before their collision.

- b) Calculate the coefficient of restitution between the two spheres.

$$e = \frac{1}{2}$$

(a) $\frac{5u}{\text{B}}$ $\frac{u}{\text{A}}$ $\frac{4u}{\text{A}}$ $\frac{V}{\text{B}}$ $\frac{U}{\text{A}}$

● SPEED OF B AFTER REBOUNDING ON THE WALL IS $\frac{1}{2} \times 5u = 2.5u$

BY CONSERVATION OF MOMENTUM
 $4mu - 4mU = -4mU + mV$
 $0 = V - 4U$
 $V = 4U$
 AS REQUIRED

(b) K.E. BEFORE = $\frac{1}{2}(4m)u^2 + \frac{1}{2}m(5u)^2 = 2mu^2 + 12.5mu^2 = 14.5mu^2$
 K.E. AFTER = $\frac{1}{2}(4m)U^2 + \frac{1}{2}mV^2 = 2mU^2 + \frac{1}{2}m(4U)^2 = 10mU^2$
 Now K.E. AFTER = $\frac{1}{4}$ x K.E. BEFORE!
 $10mU^2 = \frac{1}{4} \times 14.5mu^2$
 $U^2 = \frac{1}{4}u^2$
 $U = \frac{1}{2}u$
 AS GIVEN $V = 4U$, $V = 2u$
 Thus $e = \frac{\text{SEP}}{\text{APP}} = \frac{U+V}{u+4u} = \frac{\frac{1}{2}u + 2u}{5u} = \frac{\frac{5}{2}u}{5u} = \frac{1}{2}$
 $\therefore e = \frac{1}{2}$

Question 16 (***)

Two smooth spheres A and B , of respective masses $2m$ and m , are moving with constants speeds on a smooth horizontal plane, when they collide directly.

The respective speeds of A and B after the collision are $2v$ and $11v$. Before the collision the spheres were moving in opposite directions and after the collision both spheres are moving in the original direction of motion of A .

The two spheres are modelled as particles and the coefficient of restitution between them is e .

- a) Find, in terms of e and v , the speeds of the two spheres before their collision.

The total kinetic energy lost as a result of the collision is $21mv^2$.

- b) Find the value of e .

$$U_A = v\left(5 + \frac{3}{e}\right), \quad U_B = v\left(\frac{6}{e} - 5\right), \quad e = \frac{3}{4}$$

(a)

• BY (SIMULTANEOUS) OF MOMENTUM
 $2m(X - Y) = 4mv + 11mv$
 $2X - Y = 15v$

• BY RESTITUTION
 $e = \frac{\text{SEP}}{\text{APP}}$
 $e = \frac{2v - 11v}{X - Y}$
 $X + Y = \frac{9v}{e}$

↓

$$\frac{2X - Y = 15v}{X + Y = \frac{9v}{e}} \quad \text{Add}$$

$$3X = 15v + \frac{9v}{e}$$

$$X = 5v + \frac{3v}{e}$$

Speed of A

• ALSO BY ELIMINATION

$$\begin{array}{r} 2X - Y = 15v \\ 2X + 2Y = \frac{18v}{e} \end{array} \quad \text{SUBTRACT}$$

$$3Y = \frac{18v}{e} - 15v$$

$$Y = \frac{6v}{e} - 5v$$

Speed of B

(b) $KE_{\text{before}} = \frac{1}{2}(2m)X^2 + \frac{1}{2}(m)Y^2 = m\left[5v + \frac{3v}{e}\right]^2 + \frac{1}{2}m\left[\frac{6v}{e} - 5v\right]^2$

$$= mv^2\left(25 + \frac{30}{e} + \frac{9}{e^2}\right) + \frac{1}{2}mv^2\left(\frac{36}{e^2} - \frac{60}{e} + 25\right)$$

$$= mv^2\left(25 + \frac{30}{e} + \frac{9}{e^2}\right) + mv^2\left(\frac{18}{e^2} - \frac{30}{e} + \frac{25}{2}\right)$$

$$= mv^2\left[\frac{35}{2} + \frac{27}{e^2}\right]$$

$KE_{\text{after}} = \frac{1}{2}(2m)(2v)^2 + \frac{1}{2}m(11v)^2 = 4mv^2 + 12.5mv^2 = 16.5mv^2$

Now Loss = $21mv^2$

$$\Rightarrow mv^2\left[\frac{35}{2} + \frac{27}{e^2}\right] - 16.5mv^2 = 21mv^2$$

$$\Rightarrow \frac{35}{2} + \frac{27}{e^2} - 16.5 = 21$$

$$\Rightarrow \frac{27}{e^2} = 4.5$$

$$\Rightarrow e^2 = \frac{27}{4.5} = \frac{6}{1} = 6$$

$$\Rightarrow e = \frac{3}{4}$$

Question 17 (***)

A smooth sphere A of mass m is moving with speed $4u$ on a smooth horizontal plane. It collides directly with a smooth sphere B of mass $6m$ moving with speed u in the same direction as A . As a result of the impact the direction of motion of A is reversed. The spheres are modelled as particles.

The coefficient of restitution between the two spheres is e .

- a) Show that after the collision the speed of B is $\frac{1}{7}u(10+3e)$, and find a similar expression for the speed of A .
- b) Deduce that $e > \frac{5}{9}$.

After the collision, B strikes at right angles a fixed smooth vertical wall, and rebounds also at right angles to the wall. The coefficient of restitution between B and the wall is $\frac{1}{2}$.

- c) Given that A and B collide again show further that $e < \frac{10}{11}$.

$$\frac{2}{7}u(9e-5)$$

$\begin{matrix} 4u & u \\ \rightarrow & \rightarrow \\ m & 6m \\ A & B \end{matrix} \quad \begin{matrix} X & Y \\ \leftarrow & \rightarrow \\ m & 6m \\ A & B \end{matrix} \quad \text{POSITIVE}$

BY CONSERVATION OF MOMENTUM
 $4mu + 6mu = -mX + 6mY$
 $10u = -X + 6Y$

BY RESTITUTION
 $e = \frac{SP}{AP}$
 $e = \frac{X+Y}{3u}$
 $X+Y = 3eu$

ADD
 $7Y = 10u + 3eu$
 $7Y = u(10+3e)$
 $Y = \frac{1}{7}u(10+3e)$

USING $-X+6Y = 10u$
 $6X+6Y = 18eu$
 $7X = 18eu - 10u$
 $7X = 2u(9e-5)$
 $X = \frac{2}{7}u(9e-5)$

(b) $X > 0$ (ie it goes to the left)
 $\frac{2}{7}u(9e-5) > 0$
 $9e-5 > 5$
 $e > \frac{5}{9}$

THIS, THE COEFFICIENT IS
 $\frac{1}{2}$

FOR ANOTHER COLLISION $\frac{1}{2} > X$
 $\frac{1}{2} \times \frac{1}{7}u(10+3e) > \frac{2}{7}u(9e-5)$
 $10+3e > 4(9e-5)$
 $10+3e > 36e-20$
 $-33e > -30$
 $e < \frac{10}{11}$

Question 18 (****)

A smooth sphere A of mass m is moving with speed u on a smooth horizontal plane when it collides directly with a smooth sphere B of mass $3m$ which is at rest.

The two spheres are modelled as particles and the coefficient of restitution between them is e .

- Find, in terms of e and u , the speeds of the two spheres after their collision.
- Given that the direction of A is unchanged after the collision find the range of the possible values of e .
- Given instead that $e = \frac{1}{2}$, show that $\frac{9}{16}$ of the kinetic energy is lost as a result of the collision.

$$V_A = \frac{1}{4}u(1-3e), \quad V_B = \frac{1}{4}u(1+e), \quad 0 \leq e < \frac{1}{3}$$

(a)
 BY CONSERVATION OF MOMENTUM $m u + 0 = m X + 3m Y$
 $u = X + 3Y$
 BY RESTITUTION $e = \frac{\text{SEPARATION}}{\text{APPROACH}}$
 $e = \frac{Y - X}{u}$
 $-X + Y = eu$
 +11 EQUATIONS
 $4Y = u + eu$
 $Y = \frac{1}{4}u(1+e)$ ← speed of B
 AND $X = u - 3Y$
 $X = u - \frac{3}{4}u(1+e) = u - \frac{3}{4}u - \frac{3}{4}eu = \frac{1}{4}u - \frac{3}{4}eu$
 $X = \frac{1}{4}u(1-3e)$ ← speed of A

(b) $X > 0 \Rightarrow \frac{1}{4}u(1-3e) > 0$
 $1-3e > 0$
 $-3e > -1$
 $e < \frac{1}{3} \therefore 0 \leq e < \frac{1}{3}$

(c) If $e = \frac{1}{2}$
 $X = -\frac{1}{4}u$ & $Y = \frac{3}{8}u$
 KE before = $\frac{1}{2}mu^2$
 KE after = $\frac{1}{2}mX^2 + \frac{1}{2}(3m)Y^2 = \frac{1}{2}m(-\frac{1}{4}u)^2 + \frac{1}{2}(3m)(\frac{3}{8}u)^2$
 $= \frac{1}{32}mu^2 + \frac{27}{128}mu^2 = \frac{31}{128}mu^2$
 $\therefore \frac{\frac{31}{128}mu^2}{\frac{1}{2}mu^2}$ IS THE FRACTION THAT REMAINS IS $\frac{31}{64}$ REMAINS
 $\therefore 1 - \frac{31}{64} = \frac{33}{64}$ IS LOST

Question 19 (***)

A smooth sphere P of mass $4m$ is moving with speed $2u$ on a smooth horizontal plane. It collides directly with a smooth sphere Q of mass m moving with speed $5u$ in the opposite direction as P . The spheres are modelled as particles and the coefficient of restitution between P and Q is e .

- a) Show that the speed of Q after the collision is $\frac{1}{5}u(3+28e)$.

As a result of the impact the direction of motion of P is reversed after the collision.

- b) Find the range of the possible values of e .

The magnitude of the impulse of Q on P is $10mu$.

- c) Determine the value of e .

$$\frac{3}{7} < e \leq 1, \quad e = \frac{11}{14}$$

(a) $\frac{2u}{4m}$ $\frac{5u}{m}$ $\frac{x}{4m}$ $\frac{y}{m}$ \rightarrow Positive \rightarrow

• BY CONSERVATION OF MOMENTUM
 $4m \cdot 2u - 5m \cdot u = 4m \cdot x + m \cdot y$
 $3u = 4x + y$

• BY RESTITUTION
 $e = \frac{5u - x}{4u - y}$
 $e = \frac{y - x}{4u - y}$
 $-x + y = 7eu$

\downarrow
 $3u = 4x + (x + 7eu)$
 $3u = 5x + 7eu$
 $3u - 7eu = 5x$
 $x = \frac{1}{5}u(3 - 7e)$

\downarrow
 $y = x + 7eu = \frac{1}{5}u(3 - 7e) + 7eu = \frac{3}{5}u - \frac{7}{5}eu + 7eu$
 $y = \frac{3}{5}u + \frac{28}{5}eu$
 $y = \frac{1}{5}u(3 + 28e)$ // As required

(b) $x < 0$
 $\frac{1}{5}u(3 - 7e) < 0$
 $3 - 7e < 0$
 $-7e < -3$
 $e > \frac{3}{7}$
 $\therefore \frac{3}{7} < e \leq 1$

(c) IMPULSE ON P = $-10mu$
 \therefore IMPULSE ON Q = $+10mu$
 $10mu = m \cdot y - (-5mu)$
 $\Rightarrow 10mu = m \cdot y + 5mu$
 $\Rightarrow 5mu = m \cdot y$
 $\Rightarrow 5u = \frac{1}{5}u(3 + 28e)$
 $\Rightarrow 25 = 3 + 28e$
 $\Rightarrow 22 = 28e$
 $\Rightarrow e = \frac{11}{14}$

Question 20 (***)

A smooth sphere A of mass m is moving with speed $2u$ on a smooth horizontal plane when it collides directly with a smooth sphere B of mass $4m$ which is at rest. As a result of the collision the direction of motion of A is reversed. The two spheres are modelled as particles and the coefficient of restitution between them is e .

- a) Find, in terms of e and u , the speeds of the two spheres after their collision.

After the collision, B strikes at right angles a fixed smooth vertical wall, and rebounds at right angles. The coefficient of restitution between B and the wall is $\frac{5}{6}$.

- b) Given that A and B collide again show that

$$\frac{1}{4} < e < \frac{11}{19}$$

$$V_A = \frac{2}{5}u(4e-1), \quad V_B = \frac{2}{5}u(1+e)$$

(a) $\frac{2u}{m}$ $\frac{0}{4m}$ $\frac{x}{m}$ $\frac{y}{4m}$ \rightarrow POSITIVE

- BY CONSERVATION OF MOMENTUM
 $2mu + 0 = -mX + 4mY$
 $-X + 4Y = 2u$
- BY RESTITUTION
 $e = \frac{\text{SEP. SPD}}{\text{APP. SPD}}$
 $e = \frac{X+Y}{2u}$
 $X+Y = 2eu$

\downarrow
 $-X + 4Y = 2u$
 $X + Y = 2eu$ Add
 $5Y = 2u(1+e)$
 $Y = \frac{2}{5}u(1+e)$ SPEED OF B

Also $-X + 4Y = 2u$ } Subtract (b) from (a)
 $4X + 4Y = 8eu$
 $5X = 8eu - 2u$
 $X = \frac{2}{5}u(4e-1)$ SPEED OF A

(b) $\frac{x}{m}$ $\frac{y}{4m}$ $\frac{0}{4m}$

AFTER BOUNCING FROM THE WALL, THERE ARE TWO SPHERES OF THE TWO SPHERES:

$\frac{x}{m}$ $\frac{y}{4m}$ $\frac{0}{4m}$

FOR ANOTHER COLLISION $\frac{y}{4m} > \frac{x}{m}$

$\frac{2}{5}u(1+e) > \frac{2}{5}u(4e-1)$
 $5 + 5e > 24e - 6$
 $-19e > -11$
 $e < \frac{11}{19}$

NOTE THAT 'A' ENTERED IN REVERSE, SO $X > 0 \Rightarrow \frac{2}{5}u(4e-1) > 0$
 $\Rightarrow 4e-1 > 0$
 $\Rightarrow e > \frac{1}{4}$

$\therefore \frac{1}{4} < e < \frac{11}{19}$ (4) (10 MARKS)

Question 21 (***)

A smooth sphere A of mass $3m$ is moving with speed $3u$ on a smooth horizontal plane. It collides directly with a smooth sphere B of mass $4m$ moving with speed u in the same direction as A . The spheres are modelled as particles and the coefficient of restitution between A and B is $\frac{1}{2}$.

- a) Find, in terms of u , the speeds of the two spheres after their collision.

After the collision between A and B , B collides directly with a third sphere C of mass $2m$ which was at rest on the same smooth horizontal plane as A and B .

The sphere C is also modelled as a particle and the coefficient of restitution between B and C is e .

- b) Given that there are no more collisions between the three spheres find the range of possible values of e .

$$V_A = \frac{9}{7}u, \quad V_B = \frac{16}{7}u, \quad 0 \leq e \leq \frac{5}{16}$$

The handwritten solution is divided into two parts, (a) and (b).

Part (a): Shows a diagram of spheres A and B moving to the right. Sphere A has mass 3m and velocity 3u. Sphere B has mass 4m and velocity u. After collision, A has velocity X and B has velocity Y. Conservation of momentum is used: $3(3u) + 4(u) = 3X + 4Y$, which simplifies to $13 = 3X + 4Y$. The coefficient of restitution is given as $e = \frac{1}{2}$, leading to $\frac{1}{2} = \frac{Y - X}{3u - u}$, which simplifies to $-X + Y = u$. Solving these two equations gives $X = \frac{9}{7}u$ and $Y = \frac{16}{7}u$.

Part (b): Shows a diagram of spheres B and C. Sphere B has mass 4m and velocity $\frac{16}{7}u$. Sphere C has mass 2m and is at rest. After collision, B has velocity V and C has velocity W. Conservation of momentum is used: $4(\frac{16}{7}u) + 2(0) = 4V + 2W$, which simplifies to $\frac{32}{7}u = 2V + W$. The coefficient of restitution is given as $e = \frac{W - V}{\frac{16}{7}u - 0}$, which simplifies to $-V + W = \frac{7}{4}eu$. Solving these two equations gives $V = \frac{32}{7}u - \frac{7}{4}eu$ and $W = \frac{32}{7}u - \frac{7}{4}eu + \frac{7}{4}eu = \frac{32}{7}u$. The condition for no further collisions is $W > V$, which leads to $\frac{32}{7}u > \frac{32}{7}u - \frac{7}{4}eu$, simplifying to $0 > -\frac{7}{4}eu$, which is always true. The condition for no collision between A and B is $V > X$, which leads to $\frac{32}{7}u - \frac{7}{4}eu > \frac{9}{7}u$, simplifying to $\frac{23}{7}u > \frac{7}{4}eu$, which gives $e < \frac{23}{7} \cdot \frac{4}{7} = \frac{92}{49}$. The condition for no collision between B and C is $V > W$, which leads to $\frac{32}{7}u - \frac{7}{4}eu > \frac{32}{7}u$, simplifying to $-\frac{7}{4}eu > 0$, which is never true. The condition for no collision between A and C is $V > W$, which leads to $\frac{32}{7}u - \frac{7}{4}eu > \frac{32}{7}u$, which is never true. The final result is $0 \leq e \leq \frac{5}{16}$.

Question 22 (***)

Three smooth spheres P , Q and R have respective masses $2m$, m and km , where k is a positive constant. The spheres lie at rest in a straight line, in that order, on a smooth horizontal plane. The spheres are modelled as particles. The coefficient of restitution between any pair of spheres is $\frac{1}{2}$.

P is then projected towards Q with speed u so that the spheres collide directly.

- a) Find, in terms of u , the speed of P and the speed of Q after the collision.

Consequently there is a collision between Q and R .

- b) Given that there is a third collision between P and Q , show that $k > \frac{1}{2}$.

$$V_P = \frac{1}{2}u, \quad V_Q = u$$

(a)

By conservation of momentum: $2u + 0 = 2x + y$
 $2u = 2x + y$

By restitution: $y - x = \frac{1}{2}u$
 $-x + y = \frac{1}{2}u$

$2u = 2x + (\frac{1}{2}u + x)$
 $2u = 3x + \frac{1}{2}u$
 $\frac{3}{2}u = 3x$
 $x = \frac{1}{4}u$

$y = x + \frac{1}{2}u$
 $y = \frac{1}{4}u + \frac{1}{2}u$
 $y = \frac{3}{4}u + \frac{1}{2}u$
 $y = u$

(b)

By conservation of momentum: $u + 0 = w + kv$
 $u = w + kv$

By restitution: $v - w = \frac{1}{2}u$
 $-w + v = \frac{1}{2}u$
 $v = w + \frac{1}{2}u$

$u = w + k(\frac{1}{2}u + w)$
 $u = w + \frac{1}{2}ku + kw$
 $u - \frac{1}{2}ku = w(1+k)$
 $\frac{1}{2}u(2-k) = w(1+k)$
 $w = \frac{1}{2}u \frac{(2-k)}{(1+k)}$

See above collision: $x > y$
 $\frac{1}{4}u > \frac{1}{2}u \left(\frac{2-k}{1+k} \right)$
 $1 > \frac{2-k}{1+k}$
 $1+k > 2-k$

$2k > 1$
 $k > \frac{1}{2}$

Question 23 (***)

A smooth sphere P of mass m is moving with speed u on a smooth horizontal plane. It collides directly with a smooth sphere Q of mass $4m$ which is initially at rest. The spheres are modelled as particles and the coefficient of restitution between P and Q is e , where $e > \frac{1}{4}$.

- a) Show that the **speed** of P after the collision is $\frac{1}{5}u(4e-1)$ and find a similar expression for the speed of Q .

Three smooth spheres A , B and C lie in a straight line in that order on the same smooth horizontal plane. The masses of A and C are $4m$ each, while the mass of B is m . The three spheres are modelled as particles and the coefficient of restitution between any of these spheres is 0.75 .

The spheres are initially at rest when B is projected towards C with speed u .

- b) Show that after B and C collide, there will be another collision between A and B , and no more collisions between the spheres thereafter.

, $v_Q = \frac{1}{5}u(1+e)$

Handwritten solution for part (a):

Diagram: Sphere P (mass m) moving right with speed u towards sphere Q (mass 4m) at rest. After collision, P has speed X and Q has speed Y.

By conservation of momentum:
 $mu + 0 = mX + 4mY$
 $\Rightarrow X + 4Y = u$

By restitution:
 $e = \frac{Y - X}{u}$
 $\Rightarrow -X + Y = eu$

Solving the equations:
 $5Y = u(1+e)$
 $Y = \frac{1}{5}u(1+e)$

Sub Y into X + 4Y = u:
 $X + \frac{4}{5}u(1+e) = u$
 $X = \frac{1}{5}u - \frac{4}{5}eu$
 $X = \frac{1}{5}u(1-4e)$

\therefore Speed of P is $|\frac{1}{5}u(4e-1)|$ backwards.

Handwritten solution for part (b):

Diagram 1: Sphere B (mass m) moving right with speed u towards sphere C (mass 4m) at rest.

Using momentum (e = 0.75):
 $X = \frac{1}{5}u(1-4(0.75)) = -\frac{3}{5}u$
 $Y = \frac{4}{5}u(1+0.75) = \frac{7}{5}u$

AS B REBOUNDS ANOTHER COLLISION BETWEEN A & B.

Diagram 2: Sphere A (mass 4m) at rest, sphere B (mass m) moving left with speed $\frac{3}{5}u$ towards sphere C (mass 4m) moving right with speed $\frac{7}{5}u$.

Using momentum:
 $0 - \frac{3}{5}mu = -4mV + 4mW$
 $W - V = -\frac{3}{20}u$

By restitution:
 $e = \frac{W - V}{\frac{3}{5}u}$
 $0.75 = \frac{W - V}{\frac{3}{5}u}$
 $4V + 4W = \frac{6}{5}u$

Solving the equations:
 $5V = \frac{3}{5}u$
 $W = \frac{4}{5}u$ (to the right)

AS $\frac{3}{5}u < \frac{7}{5}u$ NO MORE COLLISIONS BETWEEN B & C.

Question 24 (***)

A smooth sphere P of mass m is moving with speed u on a smooth horizontal plane when it collides directly with a smooth sphere Q of mass $3m$, which is initially at rest. The spheres are modelled as particles and the coefficient of restitution between the two spheres is e .

- a) Find, in terms of e and u , the speeds of the two spheres after their collision.

It is now given that P reverses direction as a result of the collision.

- b) State the range of the possible values of e .

After the collision, Q strikes at right angles a fixed smooth vertical wall, and rebounds at right angles.

The coefficient of restitution between the Q and the wall is $2e$.

- c) Show that there will always be another collision between P and Q .

$$\boxed{V_P = \frac{1}{4}u(1-3e)}, \quad \boxed{V_Q = \frac{1}{4}u(1+e)}, \quad \boxed{\frac{1}{3} < e \leq 1}$$

a) DRAWING A BEFORE AND AFTER DIAGRAM

BY CONSERVATION OF MOMENTUM
 $mu + 0 = mv + 3w$
 $X + 3Y = u$

BY RESTITUTION
 $e = \frac{3w - v}{u}$
 $a = \frac{3w - v}{u}$
 $-X + Y = eu$

ADD THESE
 $4Y = u + eu$
 $Y = \frac{1}{4}u(1+e)$

AND USE IN $X = Y - eu$
 $X = \frac{1}{4}u(1+e) - eu = \frac{1}{4}u(1+e-4e) = \frac{1}{4}u(1-3e)$
 i.e. $X = \frac{1}{4}u(1-3e)$

b) AS X REVERSES DIRECTION $X < 0$, OPPOSITE TO THE DIRECTION IN THE DIAGRAM

$\rightarrow \frac{1}{4}u(1-3e) < 0$
 $\rightarrow 1-3e < 0$
 $\rightarrow -3e < -1$
 $\rightarrow e > \frac{1}{3}$

$\therefore \frac{1}{3} < e \leq 1$

c) FINISH THE QUESTION WITH THE WALL

FINISH THE QUESTION IS NOW AS FOLLOWS

$\frac{1}{4}u(1+e)$
 $\frac{1}{4}u(1-3e)$

WE REQUIRE THAT

$\frac{1}{4}u(1+e) > \frac{1}{4}u(1-3e)$
 $2e(1+e) > 1-3e$
 $2e + 2e^2 > 1-3e$
 $2e^2 - e + 1 > 0$
 $e^2 - \frac{1}{2}e + \frac{1}{2} > 0$
 $(e - \frac{1}{4})^2 + \frac{7}{16} > 0$

ALWAYS IS ALWAYS TRUE

\therefore ALWAYS ANOTHER COLLISION

DISCRIMINANT
 $b^2 - 4ac = (-\frac{1}{2})^2 - 4(\frac{1}{2})(\frac{1}{2}) = -\frac{1}{4} - 2 = -\frac{9}{4} < 0$

Question 25 (***)

A smooth sphere A of mass m is moving with speed u on a smooth horizontal plane when it collides directly with a smooth sphere B of mass $3m$ which is initially at rest. The direction of motion of A is reversed as a result of the collision.

The spheres are modelled as particles and the coefficient of restitution between the two spheres is e .

- a) Find, in terms of e and u , the speeds of the two spheres after their collision.
- b) Find the range of the possible values of the speed of B .

Consequently sphere B strikes at right angles a fixed smooth vertical wall, and rebounds at right angles. The coefficient of restitution between B and the wall is $\frac{1}{4}$.

- c) Given there is another collision between the spheres show clearly that

$$\frac{1}{3} < e < \frac{5}{11}$$

$$\boxed{}, \quad V_A = \frac{1}{4}u(3e-1), \quad V_B = \frac{1}{4}u(1+e), \quad \frac{1}{3}u < V_B < \frac{1}{2}u$$

a) STATEMENT: USE A HERE AND DEFINE JARGON

BY CONSERVATION OF MOMENTUM
 $mu + 0 = -mX + 3mY$
 $u = -X + 3Y$

BY RESTITUTION (CONSERVATIONS)
 $e = \frac{\text{SEPARATION}}{\text{APPROACH}}$
 $e = \frac{3Y - X}{u}$
 $X + Y = eu$

SOLVE THE EQUATIONS
 $\Rightarrow 4Y = u + eu$
 $\Rightarrow 4Y = u(1+e)$
 $\Rightarrow Y = \frac{1}{4}u(1+e)$ ← SPEED OF B

AND FINALLY USE: X = 3Y - u
 $X = \frac{3}{4}u(1+e) - u = \frac{1}{4}u[3(1+e) - 4] = \frac{1}{4}u(3e-1)$
 $X = \frac{1}{4}u(3e-1)$ ← SPEED OF A

b) AS X IS NEGATIVE 'REVERSE' IN THE STATEMENT X > 0
 $\Rightarrow \frac{1}{4}u(3e-1) > 0$
 $\Rightarrow 3e-1 > 0$
 $\Rightarrow 3e > 1$
 $\Rightarrow e > \frac{1}{3}$

c) COLLISION WITH THE WALL

THE COEFFICIENT OF RESTITUTION IS

ANOTHER COLLISION: IMPULSES $I > X$
 $\Rightarrow \frac{1}{4}u(1+e) > \frac{1}{4}u(3e-1)$
 $\Rightarrow \frac{1}{4}(1+e) > \frac{1}{4}(3e-1)$
 $\Rightarrow 1+e > 3e-1$
 $\Rightarrow 2 > 2e$
 $\Rightarrow e < 1$

SO CHECK IT WAS FOUND (MUST) THAT $\frac{1}{3} < e < 1$
 $\therefore \frac{1}{3} < e < \frac{5}{11}$

Question 26 (****)

A particle, of mass m , lies on a smooth horizontal surface.

Initially the particle is at rest at a point O , which lies midway between a pair of fixed, smooth, parallel vertical walls, which are $2L$ apart. At time $t=0$ the particle is projected from O with speed u in a direction perpendicular to the walls.

The coefficient of restitution between the particle and each wall is e .

- a) Find, in terms of m , u and e , the magnitude of the impulse on the particle due to the **first** impact with the wall.

The particle returns to O , having bounced off each wall once, at time $t = T$.

- b) Show clearly that

$$T = \frac{L}{u} \left(1 + \frac{1}{e} \right)^2$$

$$|I| = mu(e+1)$$

Handwritten solution for part (a):

Initial velocity = u
 Final velocity = $-eu$
 Impulse = $m(-eu) - mu$
 $I = -mu(e+1)$
 \therefore Magnitude $I = mu(e+1)$

Handwritten solution for part (b):

Speed = $\frac{\text{Distance}}{\text{Time}} \Rightarrow \text{Time} = \frac{\text{Distance}}{\text{Speed}}$

Time to first wall = $\frac{L}{u}$
 Time to return to O = $\frac{L}{eu}$
 $T = \frac{L}{u} + \frac{L}{eu}$
 $T = \frac{L}{u} \left[1 + \frac{1}{e} \right]$
 $T = \frac{L}{u} \left[\frac{e+1}{e} \right]$
 $T = \frac{L}{u} \left(\frac{e+1}{e} \right)^2 = \frac{L}{u} \left(1 + \frac{1}{e} \right)^2$

Question 27 (****)

Two particles, A and B , of respective masses m and km , where k is a positive constant, lie on a smooth horizontal surface. Initially the particles are at rest at some point on the surface between a pair of fixed, smooth, parallel vertical walls.

A and B are simultaneously projected, with respective speeds u and $2u$, away from each other in directions perpendicular to the walls. After rebounding from the walls, A and B collide directly with each other.

The coefficient of restitution between **all** collisions in this question is taken to be e .

Given further that the direction of motion of A is **not** reversed after colliding with B , show that

$$e < \frac{1-2k}{3k}$$

, proof

The handwritten solution is as follows:

- STARTING WITH THE COLLISIONS WITH THE WALLS**
 AFTER REBOUNDING:
 $V_A = eu$ and $V_B = 2eu$
- NEXT THE COLLISION BETWEEN THEM**
 Diagram shows particle A moving right with velocity eu and particle B moving left with velocity $2eu$. After collision, A has velocity X and B has velocity Y . A green arrow labeled "POSITIVE" points to the right, indicating $X > 0$.
- BY CONSERVATION OF MOMENTUM**
 $m \cdot eu - 2kmeu = mX + kmY$
 $\Rightarrow eu - 2keu = X + kY$
- BY THE RESTITUTION COEFFICIENT**
 $\Rightarrow \frac{Y - X}{eu + 2eu} = e$
- ADDING THE EQUATIONS**
 $kX - kY = -3keu$
 $X + kY = eu - 2keu$
 $(k+1)X = eu - 2keu - 3keu$
 $X = \frac{eu(1-2k-3ke)}{(k+1)}$
- A DOES NOT REVERSE ITS DIRECTION**
 $X > 0$
 $\Rightarrow \frac{eu(1-2k-3ke)}{k+1} > 0$
 $\Rightarrow 1-2k-3ke > 0$ [$eu > 0, k+1 > 0$]
 $\Rightarrow -3ke > 2k-1$
 $\Rightarrow e < \frac{1-2k}{3k}$ (As required)

Question 28 (***)

A smooth sphere A of mass $3m$ is moving with speed $4u$ on a smooth horizontal plane when it collides directly with a smooth sphere B of mass $2m$ which is moving with speed u in the same direction as A . The direction of motion of A is **not** reversed as a result of the collision.

The spheres are modelled as particles and the coefficient of restitution between the two spheres is e .

- a) Show that the speed of B after the collision is $\frac{1}{5}u(14+9e)$.
- b) Given that the speed of A after the collision is $2u$ show that $e = \frac{2}{3}$.

Consequently sphere B strikes at right angles a fixed smooth vertical wall, and rebounds at right angles. The coefficient of restitution between B and the wall is $\frac{1}{4}$.

The initial collision between A and B takes place at the point P , which is at a distance d from the wall. A second collision between A and B takes place at the point Q .

- c) Find, in terms of d , the distance of Q from the wall.

$\frac{1}{6}d$

(a) $\frac{4u}{3m} + \frac{u}{2m} = \frac{X}{3m} + \frac{Y}{2m}$ $\rightarrow 4u + u = 3X + 2Y$
 • BY CONSERVATION OF MOMENTUM
 $12mu + 2mu = 3mX + 2mY$
 $14u = 3X + 2Y$
 $14u = 3(Y - 3eu) + 2Y$
 $14u = 3Y - 9eu + 2Y$
 $14u + 9eu = 5Y$
 $Y = \frac{1}{5}u(14+9e)$ as required

• BY RESTITUTION
 $e = \frac{SEP}{APP}$
 $e = \frac{Y-2u}{3u}$
 $-X+Y = 3eu$

(b) $X = 2u$ so $X = Y - 3eu$
 $2u = \frac{1}{5}u(14+9e) - 3eu$
 $2 = \frac{1}{5}(14+9e) - 3e$
 $10 = 14+9e - 15e$
 $6e = 4$
 $e = \frac{2}{3}$ as required

(c) $e = \frac{2}{3}$ $Y = \frac{1}{5}u(14+9 \cdot \frac{2}{3})$
 $Y = 4u$
 $4X = 2u$

SPHERE B REBOUNDS OFF THE WALL WITH SPEED $\frac{1}{4}(4u) = u$

SPHERE A IS MOVING TOWARDS THE WALL AS B THEY WILL COLLIDE $(\frac{1}{5}u + \frac{1}{5}u)$ FROM THE WALL
 \therefore COLLISION WITH THE WALL $\frac{1}{5}u$ FROM THE WALL

ALL S.D. TOWARDS WALL IS FIRST AFTER THE COLLISION WITH B REBOUNDS THE WALL - A MUST BE HALF WAY IF $\frac{1}{5}u$ FROM THE WALL

Question 29 (****+)

A small bouncy ball is held at a height of 1.225 m above a smooth horizontal surface. The ball is released from rest and impacts the horizontal surface with speed $V \text{ ms}^{-1}$.

- a) Determine the value of V .

The coefficient of restitution the ball and surface is e .

- b) Show that the time between the second impact and the third impact of the ball and the surface is e^2 .

The bouncy ball takes 7.5 s from the instant it was first released until the instant it comes to rest.

- c) Find the value of e .

$$V = 4.9 \text{ ms}^{-1}, \quad e = \frac{7}{8}$$

(a) $u = 0 \text{ ms}^{-1}$, $V = 2x + 20s$
 $s = 1.225$, $V^2 = 2x(1.225)$
 $t = ?$, $V^2 = 24.01$
 $V = 4.9 \text{ ms}^{-1}$

(b) LOOKING AT THE ENTIRE JOURNEY BETWEEN THE FIRST AND THE SECOND BOUNCE
 $u = 4.9$, $s = 4.9t + \frac{1}{2}at^2$
 $a = -9.8$, $0 = 4.9t - 4.9t^2$
 $s = 0$, $0 = 4.9t(e - t)$
 $t = ?$, $t = e$

THE BALL REBOUNDS THE GROUND WITH SPEED $4.9e$ (BY CONSERVATION OF MOMENTUM) - IT THEN REBOUNDS BY $e(4.9e)$
 $t = 4.9e^2$

LOOKING AT ENTIRE JOURNEY BETWEEN THE SECOND AND THIRD BOUNCE
 $u = 4.9e^2$, $s = 4.9e^2t + \frac{1}{2}at^2$
 $a = -9.8$, $0 = 4.9e^2t - 4.9e^2t^2$
 $s = 0$, $0 = 4.9e^2(e^2 - t)$
 $t = ?$, $t = e^2$

(c) FINELY LOOKING AT THE JOURNEY UP TO THE FIRST IMPACT (SEE PART (a))
 $V = 4.9at$
 $4.9 = 9.8t$
 $t = 0.5$

$\therefore 0.5 + e + e^2 + e^3 + e^4 + \dots = 7.5$
 $e + e^2 + e^3 + e^4 + \dots = 7$ ← (G.P. with a/e)
 $\frac{e}{1-e} = 7$
 $e = 7 - 7e$
 $8e = 7$ $\therefore e = \frac{7}{8}$

Question 30 (***)

A smooth sphere A of mass m is moving with speed u on a smooth horizontal surface when it collides directly with a smooth sphere B of mass $2m$ which is at rest. The spheres are modelled as particles and the coefficient of restitution between the two spheres is e .

The total amount of kinetic energy after the collision is $\frac{11}{64}mu^2$

Determine the value of e .

$$e = \frac{1}{8}$$

Handwritten solution for Question 30:

Diagram: Sphere A (mass m) moving right with speed u . Sphere B (mass $2m$) at rest. After collision, A has speed X and B has speed Y . An arrow labeled 'RESTIT' points to the right.

By conservation of momentum:
 $mu + 0 = mX + 2mY$
 $X + 2Y = u$

By coefficient of restitution (collision):
 $\frac{1}{2}mX^2 + \frac{1}{2}(2m)Y^2 = \frac{1}{64}mu^2$
 $\Rightarrow \frac{1}{2}X^2 + Y^2 = \frac{1}{64}u^2$
 $\Rightarrow X^2 + 2Y^2 = \frac{1}{32}u^2$
 BUT $X = u - 2Y$
 $\Rightarrow (u - 2Y)^2 + 2Y^2 = \frac{1}{32}u^2$
 $\Rightarrow 4Y^2 - 4uY + u^2 + 2Y^2 = \frac{1}{32}u^2$
 $\Rightarrow 6Y^2 - 4uY + \frac{31}{32}u^2 = 0$
 $\Rightarrow 192Y^2 - 128uY + 31u^2 = 0$
 BY QUADRATIC FORMULA

By coefficient of restitution:
 $e = \frac{Y - X}{u - 0}$
 $e = \frac{Y - (u - 2Y)}{u}$
 $e = \frac{3Y - u}{u}$
 $e = \frac{1}{8}$