CENTRE OF MASS

Question 1 (**)

Four particles A, B, C and D have masses $1\,\text{kg}$, $2\,\text{kg}$, $3\,\text{kg}$ and $4\,\text{kg}$, respectively.

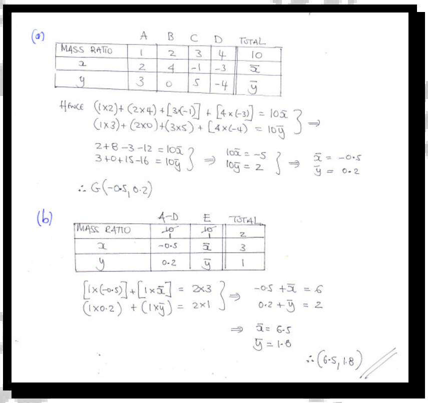
The respective coordinates of the particles A, B, C and D are $(2,3)$, $(4,0)$, $(-1,5)$ and $(-3,-4)$.

a) Find the coordinates of the centre of mass of this system of four particles.

A fifth particle E of mass 10 kg is placed at the point P, so that the centre of mass of the **five** particles is now at the point with coordinates $(3,1)$.

b) Find the coordinates of P.

$\boxed{(-0.5,0.2)}$, $\boxed{P(6.5,1.8)}$



(a)

	A	B	C	D	TOTAL
MASS RATIO	1	2	3	4	10
x	2	4	-1	-3	$\bar{x}$
y	3	0	5	-4	$\bar{y}$

HENCE $\left.\begin{array}{l}(1\times2)+(2\times4)+[3\times(-1)]+[4\times(-3)]=10\bar{x}\\(1\times3)+(2\times0)+(3\times5)+[4\times(-4)]=10\bar{y}\end{array}\right\}\Rightarrow$

$\left.\begin{array}{l}2+8-3-12=10\bar{x}\\3+0+15-16=10\bar{y}\end{array}\right\}\Rightarrow\left.\begin{array}{l}10\bar{x}=-5\\10\bar{y}=2\end{array}\right\}\Rightarrow\begin{array}{l}\bar{x}=-0.5\\\bar{y}=0.2\end{array}$

$\therefore G(-0.5,0.2)$

(b)

	A–D	E	TOTAL
MASS RATIO	$\cancel{10}$ 1	$\cancel{10}$ 1	2
x	-0.5	$\bar{x}$	3
y	0.2	$\bar{y}$	1

$\left.\begin{array}{l}[1\times(-0.5)]+[1\times\bar{x}]=2\times3\\(1\times0.2)+(1\times\bar{y})=2\times1\end{array}\right\}\Rightarrow\begin{array}{l}-0.5+\bar{x}=6\\0.2+\bar{y}=2\end{array}$

$\Rightarrow\begin{array}{l}\bar{x}=6.5\\\bar{y}=1.8\end{array}$

$\therefore(6.5,1.8)$

Question 2 (**)

Three particles A, B and C have masses 2 kg, 3 kg and 5 kg, respectively.

The respective coordinates of these three particles are $(2,2)$, $(0,-5)$ and $(-3,1)$.

a) Find the coordinates of the centre of mass of this system of three particles.

A fourth particle D of mass 10 kg is placed at the point with coordinates $(2,3)$.

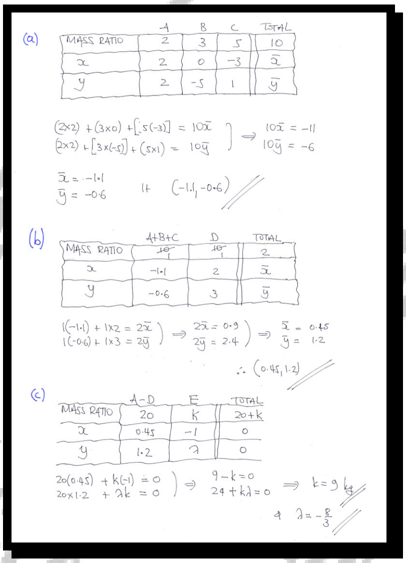
b) Find the coordinates of the centre of mass of the system of the **four** particles.

A fifth particle E of mass k kg is placed at the point with coordinates $(-1,\lambda)$.

The coordinates of the centre of mass of the **five** particles is now at the origin.

c) Determine the values of k and λ.

$\boxed{(-1.1,-0.6)}$, $\boxed{(0.45,1.2)}$, $\boxed{k=9}$, $\boxed{\lambda=-\frac{8}{3}}$



(a)

	A	B	C	TOTAL
MASS RATIO	2	3	5	10
x	2	0	−3	$\bar{x}$
y	2	−5	1	$\bar{y}$

$(2\times2)+(3\times0)+[5(-3)]=10\bar{x}$
$(2\times2)+[3\times(-5)]+(5\times1)=10\bar{y}$
$\Rightarrow 10\bar{x}=-11$, $10\bar{y}=-6$

$\bar{x}=-1.1$, $\bar{y}=-0.6$ ∴ $(-1.1,-0.6)$

(b)

	A+B+C	D	TOTAL
MASS RATIO	$\cancel{10}$ 1	$\cancel{10}$ 1	2
x	−1.1	2	$\bar{x}$
y	−0.6	3	$\bar{y}$

$1(-1.1)+1\times2=2\bar{x}$
$1(-0.6)+1\times3=2\bar{y}$
$\Rightarrow 2\bar{x}=0.9$, $2\bar{y}=2.4$ $\Rightarrow \bar{x}=0.45$, $\bar{y}=1.2$

∴ $(0.45,1.2)$

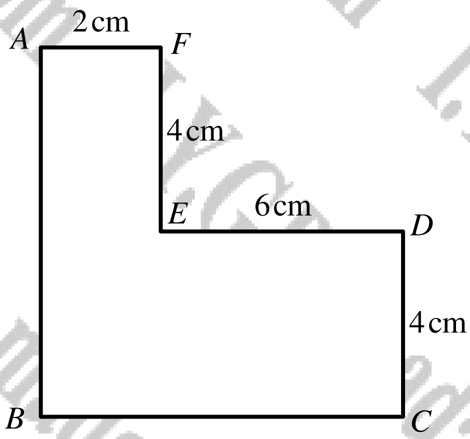
(c)

	A−D	E	TOTAL
MASS RATIO	20	k	$20+k$
x	0.45	−1	0
y	1.2	λ	0

$20(0.45)+k(-1)=0$
$20\times1.2+\lambda k=0$
$\Rightarrow 9-k=0$, $24+k\lambda=0$ $\Rightarrow k=9$ kg

& $\lambda=-\frac{8}{3}$

Question 3 (**)



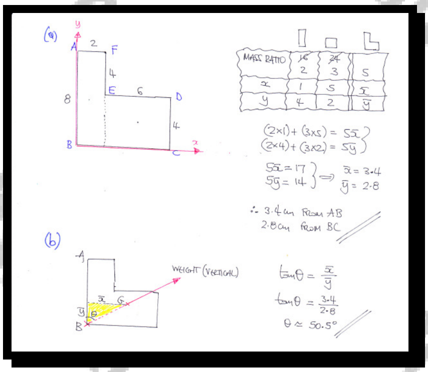
The figure above shows a uniform lamina $ABCDEF$ where all corners are right angles and $|AF| = 2\text{cm}$, $|FE| = 4\text{cm}$, $|ED| = 6\text{cm}$ and $|DC| = 4\text{cm}$.

a) Determine the position of the centre of mass of the lamina from AB and BC.

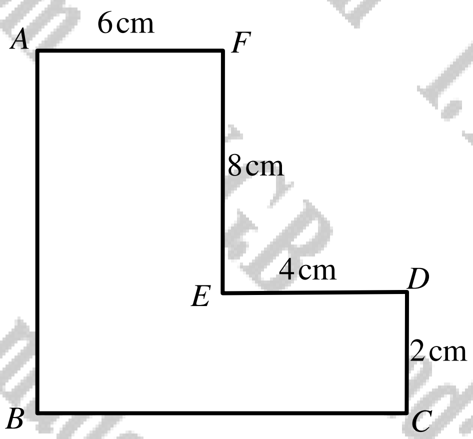
The lamina is suspended freely though a smooth pivot at B and hangs in equilibrium under its own weight.

b) Find the size of the angle that AB makes with the vertical.

$\boxed{3.4\text{cm from } AB}$, $\boxed{2.8\text{cm from } BC}$, $\boxed{\approx 50.5°}$



Question 4 (**)



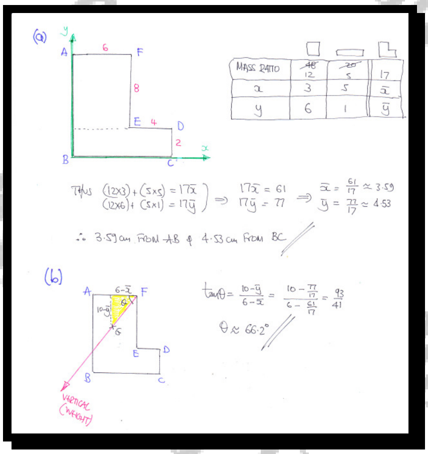
The figure above shows a uniform lamina $ABCDEF$ where all corners are right angles and $|AF| = 6\text{cm}$, $|FE| = 8\text{cm}$, $|ED| = 4\text{cm}$ and $|DC| = 2\text{cm}$.

a) Determine the position of the centre of mass of the lamina from AB and BC.

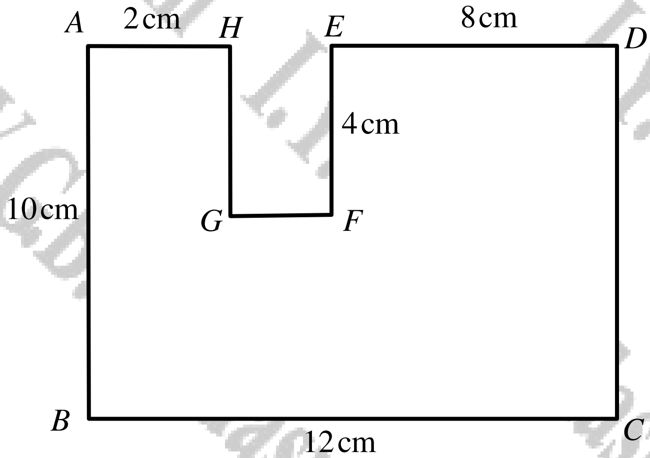
The lamina is suspended freely though a smooth pivot at F and hangs in equilibrium under its own weight.

b) Find the size of the angle that AF makes with the vertical.

$\boxed{\approx 3.59\text{cm from } AB}$, $\boxed{\approx 4.53\text{cm from } BC}$, $\boxed{\approx 66.2°}$



Question 5 (**+)



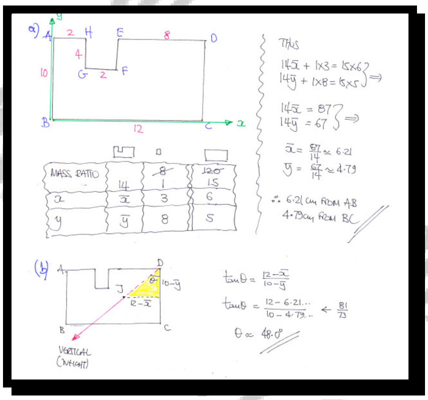
The figure above shows a uniform lamina $ABCDEFGH$ where all corners are right angles and $|AH| = 2\text{cm}$, $|ED| = 8\text{cm}$, $|EF| = 4\text{cm}$, $|BC| = 12\text{cm}$ and $|AB| = 10\text{cm}$.

a) Determine the position of the centre of mass of the lamina from AB and BC.

The lamina is suspended freely though a smooth pivot at D and hangs in equilibrium under its own weight.

b) Find the size of the angle that DC makes with the vertical.

$\boxed{\approx 6.21\text{cm from } AB}$, $\boxed{\approx 4.79\text{cm from } BC}$, $\boxed{\approx 48.0°}$



Question 6 (**+)

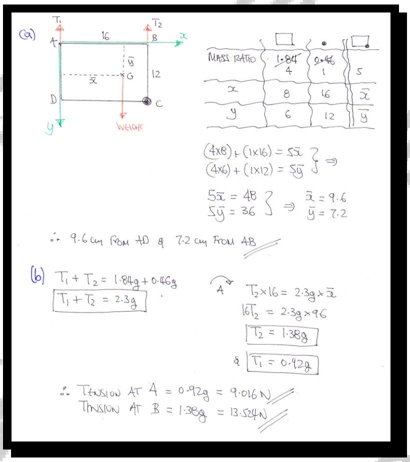
A uniform rectangular lamina $ABCD$ has mass 1.84 kg is loaded with a particle of mass 0.46 kg attached at the corner C. It is further given that $|AB| = |CD| = 16\text{cm}$ and $|BC| = |DA| = 12\text{cm}$.

a) Determine the position of the centre of mass of the **loaded** lamina from the edge AD and from the edge AB.

The lamina is suspended in equilibrium with AB horizontal by two vertical stringa one attached at A and one attached at B.

b) Calculate the tension in each of the two strings.

$\boxed{9.6 \text{ cm from } AD}$, $\boxed{7.2 \text{ cm from } AB}$, $\boxed{T_A = 9.016 \text{ N}}$, $\boxed{T_B = 13.524 \text{ N}}$



MASS RATIO	~~1.84~~ 4	~~0.46~~ 1	5
x	8	16	$\bar{x}$
y	6	12	$\bar{y}$

$(4\times8)+(1\times16)=5\bar{x}$
$(4\times6)+(1\times12)=5\bar{y}$ $\Rightarrow$

$5\bar{x}=48$
$5\bar{y}=36$ $\Rightarrow$ $\bar{x}=9.6$, $\bar{y}=7.2$

$\therefore$ 9.6 cm FROM AD & 7.2 cm FROM AB

(b) $T_1+T_2 = 1.84g+0.46g$

$\boxed{T_1+T_2=2.3g}$

A: $T_2\times16 = 2.3g\times\bar{x}$

$16T_2 = 2.3g\times9.6$

$\boxed{T_2 = 1.38g}$

& $\boxed{T_1 = 0.92g}$

$\therefore$ TENSION AT A $= 0.92g = 9.016$ N

TENSION AT B $= 1.38g = 13.524$ N

Question 7 (***)

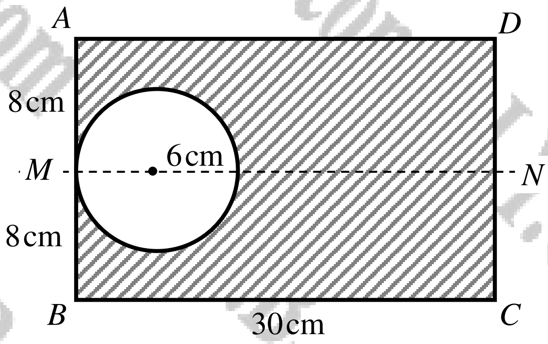


figure 1

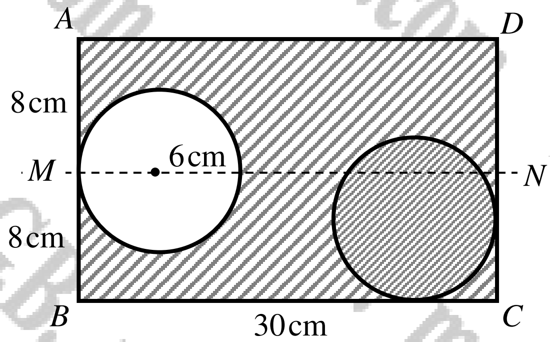


figure 2

Figure 1 shows a rectangular lamina $ABCD$ where $|AB| = 16\text{cm}$ and $|BC| = 30\text{cm}$.

The points M and N are the midpoints of AB and CD.

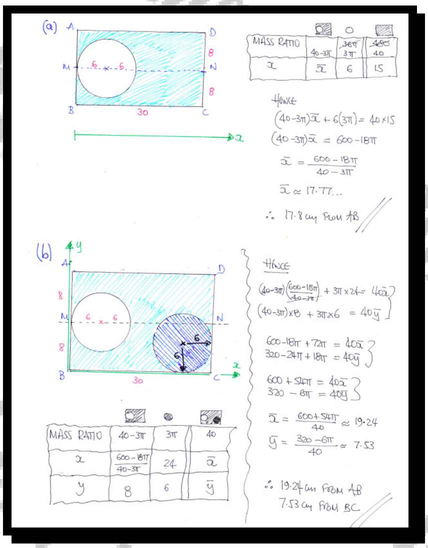
A circle of radius 6 cm whose centre lies on MN at a distance of 6 cm from AB, is removed from the lamina $ABCD$, forming a composite $\mathbf{S}$.

a) Determine the position of the centre of mass of $\mathbf{S}$ from AB.

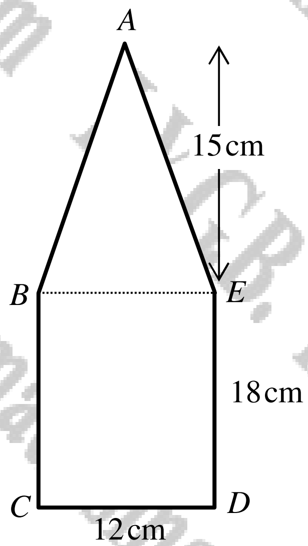
The circular section removed in part (a) is now attached to a new position on $\mathbf{S}$ so that BC and CD are now tangents to the circular section. The new composite is shown in figure 2 and is denoted by $\mathbf{T}$.

b) Determine the distance of the centre of mass of $\mathbf{T}$ from AB and BC.

$\boxed{\quad}$, $\boxed{\approx 17.8\text{cm from } AB}$, $\boxed{\approx 19.24\text{cm from } AB \text{ and } 7.53\text{cm from } BC}$



Question 8 (***)



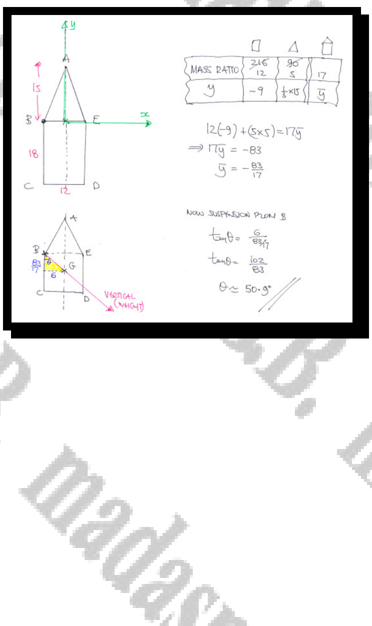
The figure above shows a uniform lamina $ABCDE$ consisting of a rectangle $BCDE$ and an isosceles triangle ABE where $|AB| = |AE|$.

It is further given that $|CD| = 12\text{cm}$, $|ED| = 18\text{cm}$ and the height of the triangle measured from A is 15cm.

The lamina is suspended freely though a smooth pivot at B and hangs in equilibrium under its own weight.

Find the size of the angle that BC makes with the vertical.

$\boxed{\approx 50.9°}$



Question 9 (***)

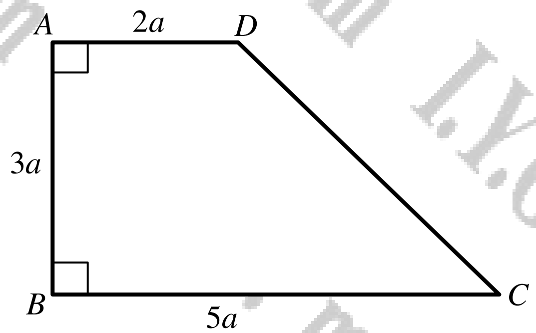


figure 1

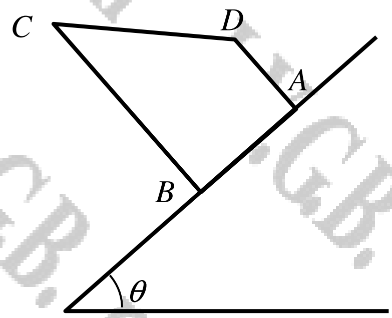


figure 2

Figure 1 above shows a lamina $ABCD$ which is in the shape of a right angled trapezium, where $\measuredangle DAB = \measuredangle ABC = 90°$.

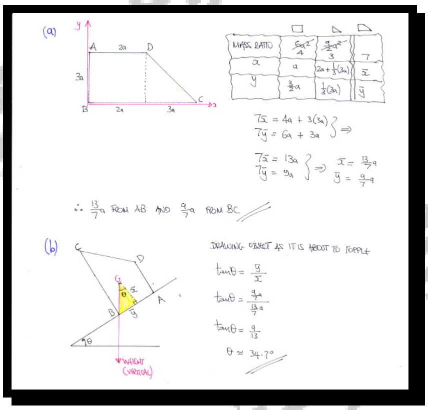
It is further given that $|AB| = 3a$, $|BC| = 5a$ and $|AD| = 2a$.

a) Determine the position of the centre of mass of the lamina from AB and BC.

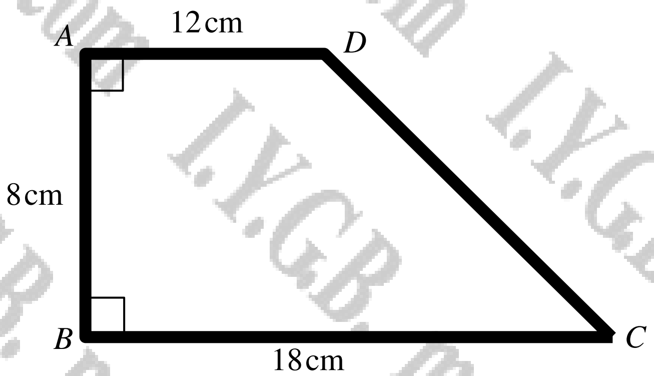
The lamina is next placed on plane inclined at an angle θ to the horizontal, as shown in figure 2. The plane is sufficiently rough to prevent the lamina from sliding.

b) Given that the lamina is at the point of toppling find the value of θ.

$\boxed{\frac{13a}{7} \text{ from } AB}$, $\boxed{\frac{9a}{7} \text{ from } BC}$, $\boxed{\approx 34.7°}$



Question 10 (***)



The figure above shows a **framework** consisting of four small **uniform rods** AB, BC, CD and AD.

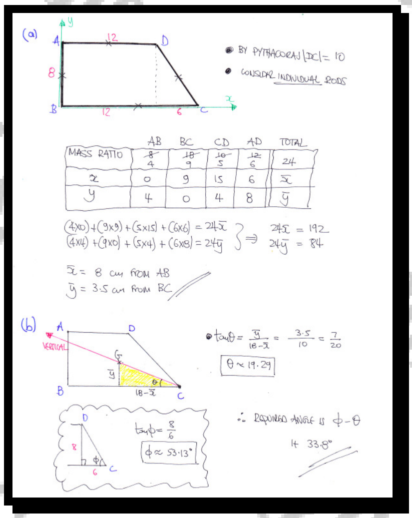
It is given that $|AB| = 8\text{cm}$, $|BC| = 18\text{cm}$, $|AD| = 12\text{cm}$ and $\measuredangle ABC = \measuredangle DAB = 90°$.

a) Determine the distance of the position of the centre of mass of the framework from AB and BC.

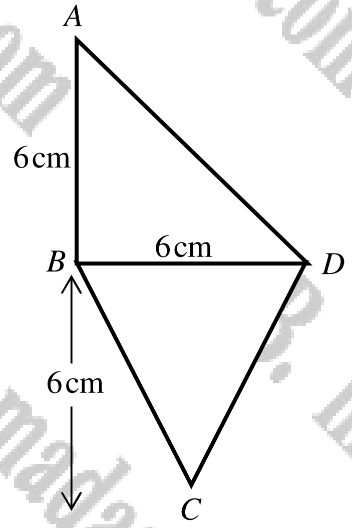
The framework is suspended freely though a smooth pivot at C and hangs in equilibrium under its own weight.

b) Find the size of the angle that DC makes with the vertical.

$\boxed{}$, $\boxed{8\text{cm from } AB}$, $\boxed{3.5\text{cm from } BC}$, $\boxed{33.8°}$



Question 11 (***+)



The figure above shows a lamina $ABCD$ consisting of a right angled isosceles triangle ABD where $\measuredangle ABD = 90°$ and an isosceles triangle BCD where $|BC| = |CD|$.

It is further given that $|AB| = 6\text{cm}$, $|BD| = 6\text{cm}$ and the height of the triangle BCD measured from C is 6cm.

a) Explain why the centre of mass of the lamina $ABCD$ lies on BD.

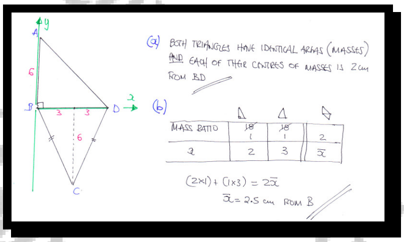
b) Find the distance of the centre of mass of the lamina $ABCD$ from B.

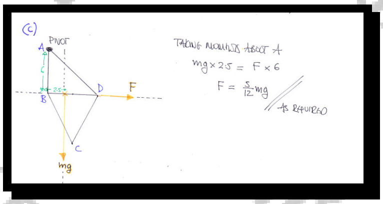
The lamina $ABCD$ is smoothly pivoted at A and kept in a position with BD horizontal and C below the level of BD by a horizontal force F.

F acts through D, in the direction BD.

c) Given the mass of the lamina is m, find the size of F in terms of m and g.

$\boxed{2.5\text{cm from } B}$, $\boxed{F = \frac{5}{12}mg}$





Question 12 (***)

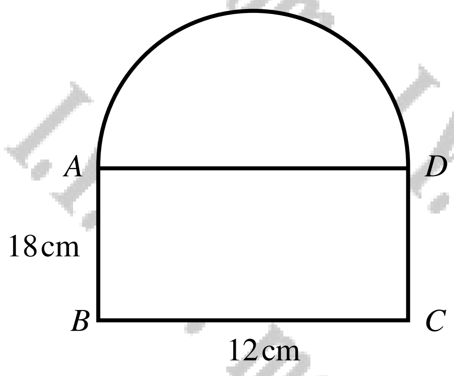


figure 1

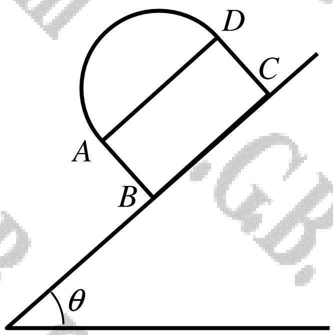


figure 2

Figure 1 above shows a uniform composite lamina consisting of a rectangle $ABCD$ and a semicircle of diameter AD.

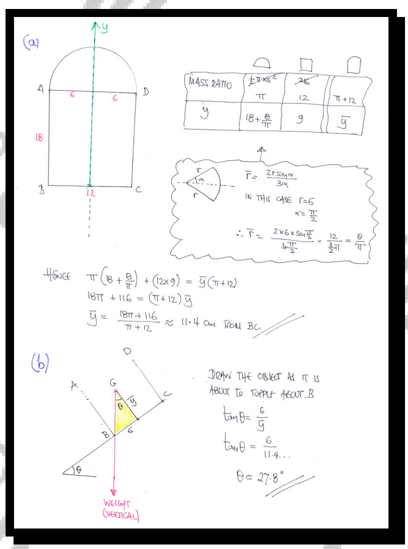
It is further given that $|AB| = 18\text{cm}$ and $|BC| = 12\text{cm}$.

a) Determine the position of the centre of mass of the lamina from BC.

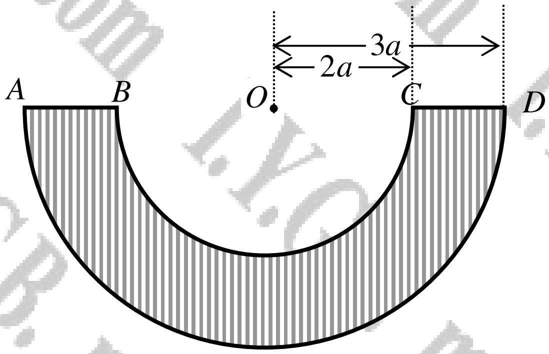
The lamina is next placed on plane inclined at an angle θ to the horizontal, as shown in figure 2. The plane is sufficiently rough to prevent the lamina from sliding.

b) Given that the lamina is at the point of toppling find the value of θ.

$\boxed{\approx 11.4\text{cm from } BC}$, $\boxed{\approx 27.8°}$



Question 13 (***+)



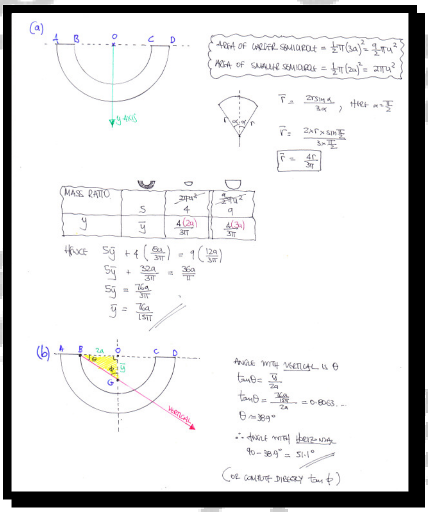
The figure above shows a lamina $ABDC$ consisting of a semicircle centre at O and radius $2a$ removed from a larger semicircle also with centre at O and radius $3a$.

a) Find the distance of the centre of mass of the lamina from O.

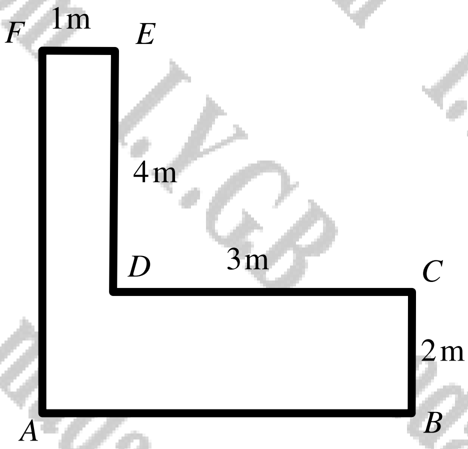
The lamina is suspended freely though a smooth pivot at B and hangs in equilibrium under its own weight.

b) Find the size of the angle that BC makes with the **horizontal**.

$\boxed{\frac{76a}{15\pi} \text{ from } O}$, $\boxed{38.9°}$



Question 14 (***+)



The figure above shows a rigid framework $ABCDEF$, consisting of 6 uniform rods all of equal cross-section and of equal mass density.

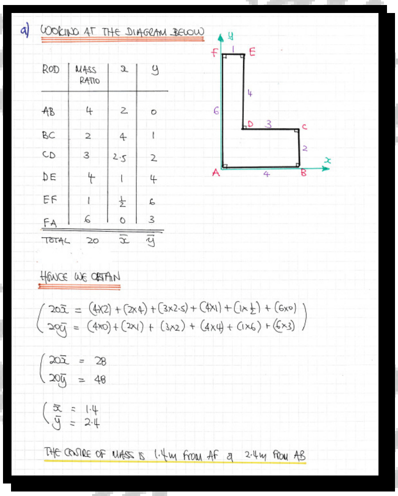
It is further given that all the corners of the framework are right angles and $|BC| = 2\text{ m}$, $|CD| = 3\text{ m}$, $|DE| = 4\text{ m}$ and $|EF| = 1\text{ m}$.

a) Find the position of the centre of mass of the framework from AB and AF.

The framework is suspended freely though a smooth pivot at F and hangs in equilibrium under its own weight.

b) Show that the tangent of the angle which DC makes with the vertical is $\frac{18}{7}$.

$\boxed{}$, $\boxed{1.4\text{ m from } AF}$, $\boxed{2.4\text{ m from } AB}$, $\boxed{\approx 66.2°}$


a) LOOKING AT THE DIAGRAM BELOW

ROD	MASS RATIO	x	y
AB	4	2	0
BC	2	4	1
CD	3	2.5	2
DE	4	1	4
EF	1	$\frac{1}{2}$	6
FA	6	0	3
TOTAL	20	$\bar{x}$	$\bar{y}$

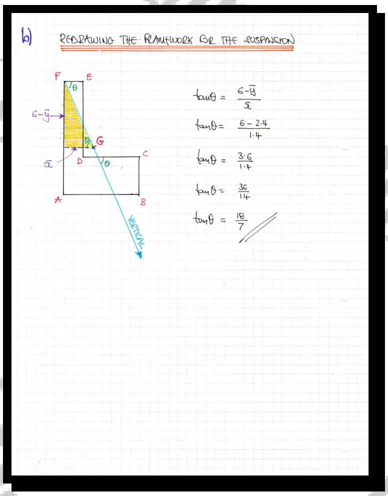
HENCE WE OBTAIN

$$\begin{cases} 20\bar{x} = (4\times2)+(2\times4)+(3\times2.5)+(4\times1)+(1\times\frac{1}{2})+(6\times0) \\ 20\bar{y} = (4\times0)+(2\times1)+(3\times2)+(4\times4)+(1\times6)+(6\times3) \end{cases}$$

$$\begin{cases} 20\bar{x} = 28 \\ 20\bar{y} = 48 \end{cases}$$

$$\begin{cases} \bar{x} = 1.4 \\ \bar{y} = 2.4 \end{cases}$$

THE CENTRE OF MASS IS 1.4m FROM AF & 2.4m FROM AB


b) REDRAWING THE FRAMEWORK FOR THE SUSPENSION

$$\tan\theta = \frac{6-\bar{y}}{\bar{x}}$$

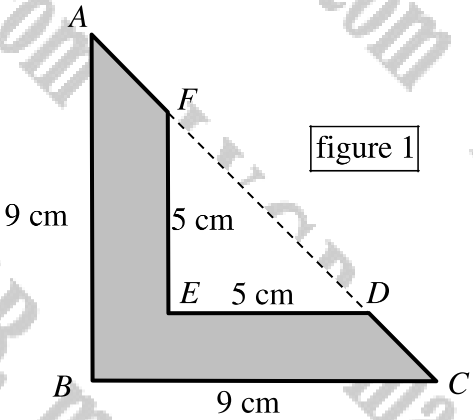
$$\tan\theta = \frac{6-2.4}{1.4}$$

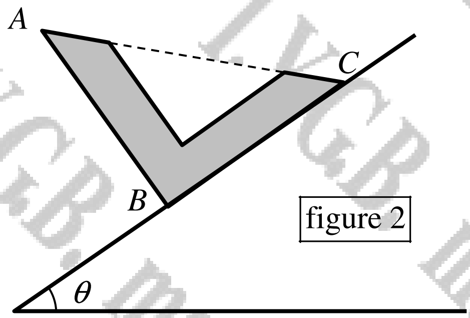
$$\tan\theta = \frac{3.6}{1.4}$$

$$\tan\theta = \frac{36}{14}$$

$$\tan\theta = \frac{18}{7}$$

Question 15 (***+)





A uniform lamina is in the shape of on isosceles triangle ABC with $\measuredangle ABC = 90°$ and $AB = BC = 9$ cm. An isosceles triangle DEF is removed from ABC, such that $\measuredangle DEF = 90°$ and $DE = EF = 5$ cm, forming a composite S, shown in figure 1.

a) Find the distance of the centre of mass of S …

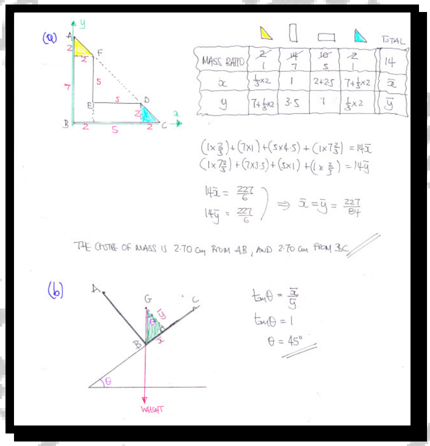
  i. … from AB .

  ii. … from BC .

The composite S is placed on the greatest slope of a plane inclined at an angle θ to the horizontal, as shown in figure 2. The plane is sufficiently rough to prevent S from sliding.

b) Given that S is at the point of toppling over, calculate the value of θ.

$\boxed{\frac{227}{84} \approx 2.70 \text{ cm from } AB \text{ and from } BC}$, $\boxed{\theta = 45°}$



Question 16 (****)

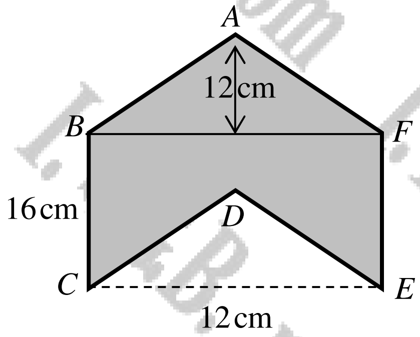


figure 1

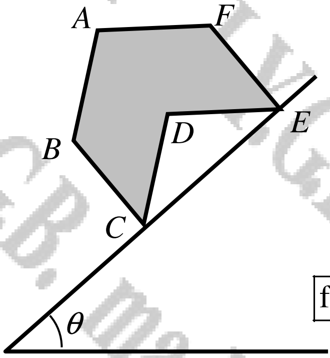


figure 2

From a rectangle $BCEF$, an isosceles triangle CDE is removed and attached to the rectangle so that the sides CE and BF coincide, and the point D is relabelled as A.

It is further given that $|CD| = |DE|$, $|BC| = 16\text{cm}$ and $|CE| = 12\text{cm}$. The height of the triangle ABF, measured from A, is 12 cm.

Figure 1 above, shows the composite which is modelled as a uniform lamina.

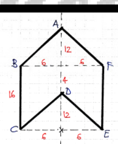
a) Show that the centre of mass of the lamina is located at a distance of 14 cm from CE.

The lamina is next placed on plane inclined at an angle θ to the horizontal, as shown in figure 2. The plane is sufficiently rough to prevent the lamina from sliding.

b) Given that the lamina is at the point of toppling find the value of θ.

$\boxed{\approx 23.2°}$

a) CONSIDER THE PROBLEM IN 2 PARTS



MASS RATIO	$\frac{1}{2}\times12\times12$ $\cancel{72}$ 3	5	12×16 $\cancel{192}$ 8
DISTANCE OF THE CENTRE OF MASS FROM CE	$\frac{1}{3}\times12$ 4	$\bar{y}$	8

$\Rightarrow 3\times4 + 5\bar{y} = 64$
$\Rightarrow 12 + 5\bar{y} = 64$
$\Rightarrow 5\bar{y} = 52$
$\Rightarrow \bar{y} = 10.4$

NOW REPEAT WITH THE TRIANGLE ON "TOP"

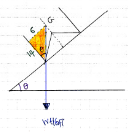
MASS RATIO	$192-72$ $\cancel{120}$ 5	$\cancel{72}$ 3	$\cancel{192}$ 8
DISTANCE OF THE CENTRE OF MASS FROM CE	10.4	$16+\frac{1}{3}\times12$ 20	$\bar{\bar{y}}$

$\Rightarrow (5\times10.4) + (3\times20) = 8\bar{\bar{y}}$
$\Rightarrow 52 + 60 = 8\bar{\bar{y}}$
$\Rightarrow 8\bar{\bar{y}} = 112$
$\Rightarrow \bar{\bar{y}} = 14$

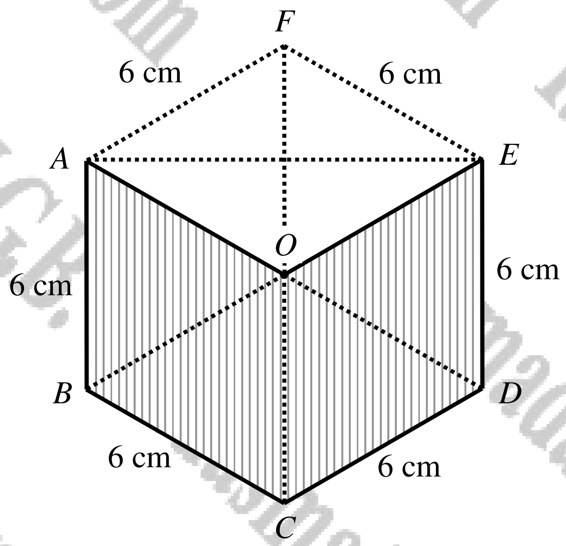
AS REQUIRED

b) LET θ BE SUCH SO THAT THE LAMINA IS ABOUT TO TOPPLE

$\Rightarrow \tan\theta = \frac{6}{14}$
$\Rightarrow \tan\theta = \frac{3}{7}$
$\Rightarrow \theta \approx 23.2°$



Question 17 (****)



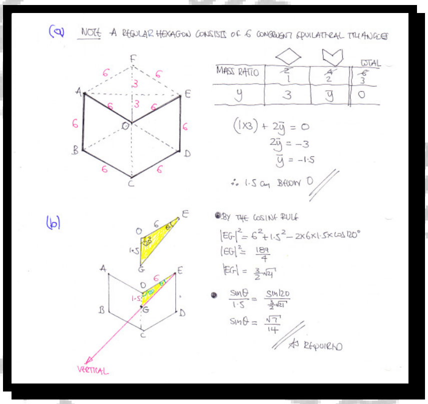
The figure above shows a uniform lamina $ABCDEF$ in the shape of a regular hexagon of side length 6 cm, whose centre is at O. A rhombus $AEOF$ is removed from the hexagon forming a composite lamina S.

a) Determine the distance of the centre of mass of S from O.

The composite S is suspended from the point E and hangs freely in equilibrium. The side OE makes an angle θ with the vertical.

b) Show that $\sin\theta = \frac{1}{14}\sqrt{7}$.

$\boxed{1.5 \text{ cm from } O}$



Question 18 (****)

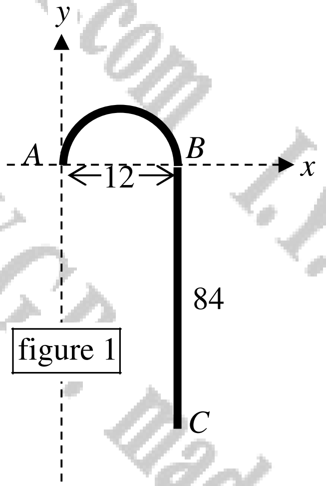


figure 1

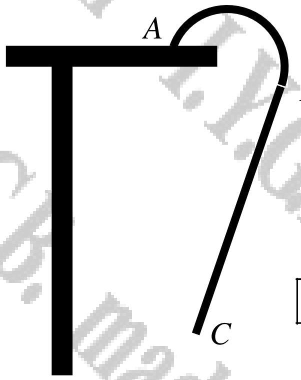


figure 2

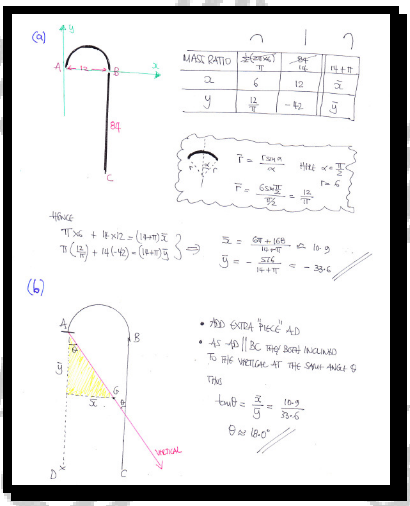
Figure 1 shows a walking stick ABC, modelled as two uniform rods AB and BC. The straight section BC has length 84 cm and the section AB is a circular arc of diameter 12 cm. A set of coordinate axes is defined with A as the origin as shown in figure 1.

a) Find the coordinates of the centre of mass of the walking stick.

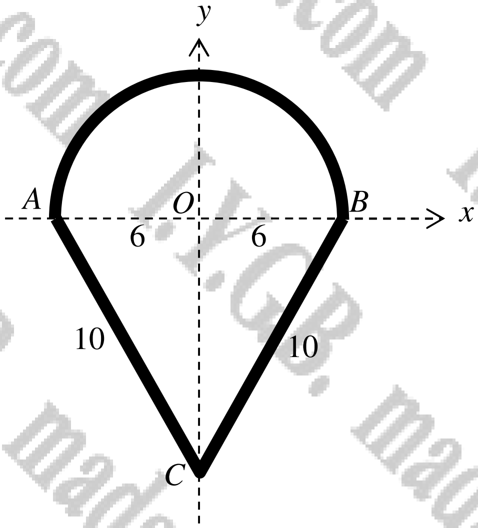
The walking stick is placed with its end A at the end of a horizontal table and rests in equilibrium under its own weight as shown in figure 2, without touching any other object.

b) Determine the size of the angle that BC makes with the vertical.

$\boxed{}$, $\boxed{(\bar{x},\bar{y}) \approx (10.9,-33.6)}$, $\boxed{\approx 18.0°}$



Question 19 (****)



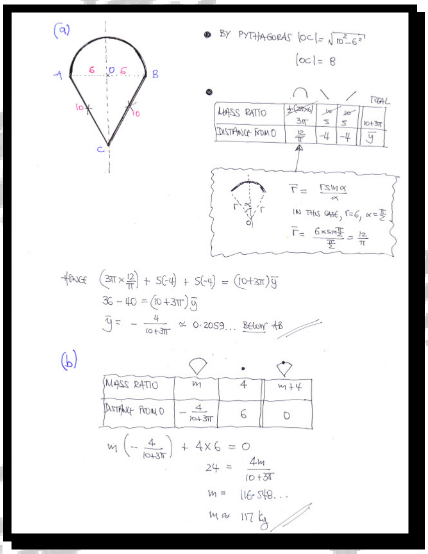
The figure above shows a **framework** consisting of three thin **uniform rods** AB, BC and AC. The rods BC and AC are straight lines, both of lengths 10 units. The rod AB is in the shape of a semicircular arc of radius 6 units with centre at O. A set of coordinate axes is defined with O as the origin, as shown in figure.

a) Determine the position of the centre of mass of the framework from O.

A particle of mass 4 kg is attached to the midpoint of AB. The centre of mass of the **loaded framework** is now at O.

b) Find the mass of the framework.

$\boxed{\approx 0.206 \text{ from } O}$, $\boxed{\approx 117 \text{ kg}}$



(a)

• BY PYTHAGORAS $|OC| = \sqrt{10^2 - 6^2}$

$|OC| = 8$

•

				TOTAL
MASS RATIO	$\cancel{\frac{1}{2}(2\pi \times 6)}$ 3π	$\cancel{10}$ 5	$\cancel{10}$ 5	$10+3\pi$
DISTANCE FROM O	$\frac{12}{\pi}$	-4	-4	$\bar{y}$

$\bar{r} = \frac{r \sin\alpha}{\alpha}$

IN THIS CASE, $r=6$, $\alpha = \frac{\pi}{2}$

$\bar{r} = \frac{6 \times \sin\frac{\pi}{2}}{\frac{\pi}{2}} = \frac{12}{\pi}$

HENCE $\left(3\pi \times \frac{12}{\pi}\right) + 5(-4) + 5(-4) = (10+3\pi)\bar{y}$

$36 - 40 = (10+3\pi)\bar{y}$

$\bar{y} = -\frac{4}{10+3\pi} \simeq 0.2059\ldots$ BELOW AB

(b)

MASS RATIO	m	4	$m+4$
DISTANCE FROM O	$-\frac{4}{10+3\pi}$	6	0

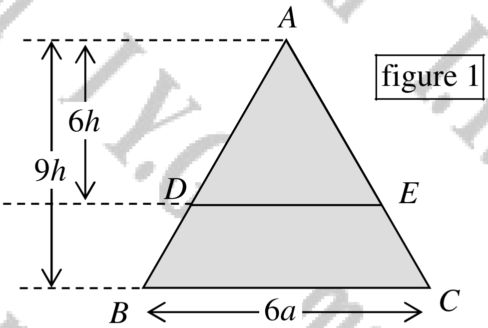
$m\left(-\frac{4}{10+3\pi}\right) + 4 \times 6 = 0$

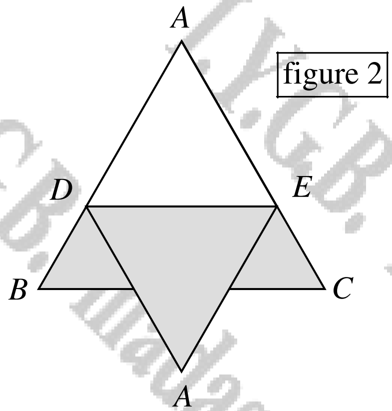
$24 = \frac{4m}{10+3\pi}$

$m = 116.548\ldots$

$m \approx 117$ kg

Question 20 (****)





A uniform lamina ABC is the shape of an isosceles triangle where $AB = AC$, $BC = 6a$. The vertical height of ABC is $9h$, as shown in figure 1. The lamina is to be folded along DE, where DE is parallel to BC and at a perpendicular distance of $6h$ from A, as shown in figure 2.

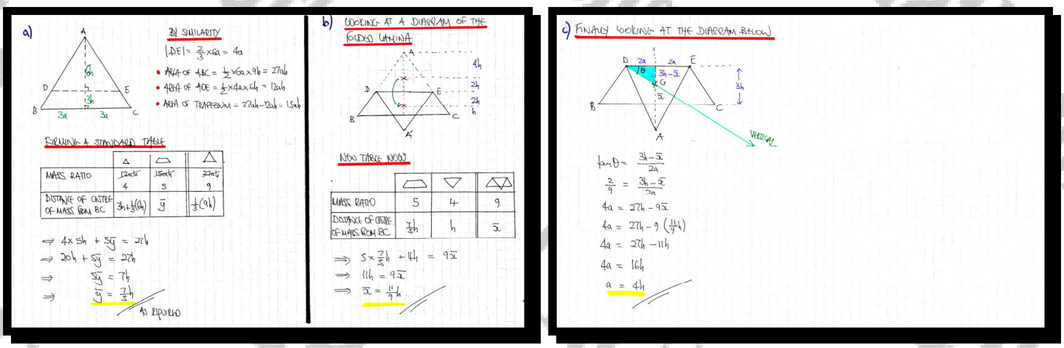
a) Show that the centre of mass of the **trapezium** $BDEC$ is $\frac{7}{5}h$ from BC.

b) Determine the position of the centre of mass of the **folded lamina** from BC.

The folded lamina is suspended from the point D and hangs freely in equilibrium. The side DE is inclined at $\arctan\frac{2}{9}$ to the vertical.

c) Express a in terms of h.

$\boxed{}$, $\boxed{\frac{11}{9}h \text{ from } BC}$, $\boxed{a = 4h}$



Question 21 (****)

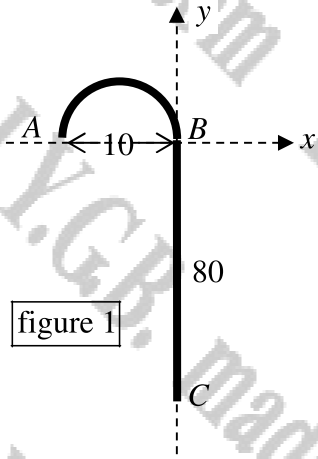


figure 1

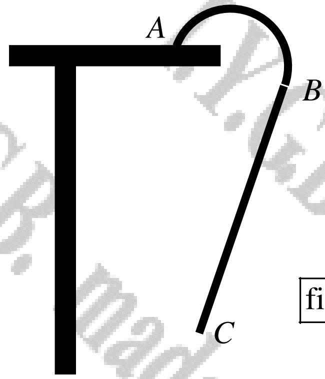


figure 2

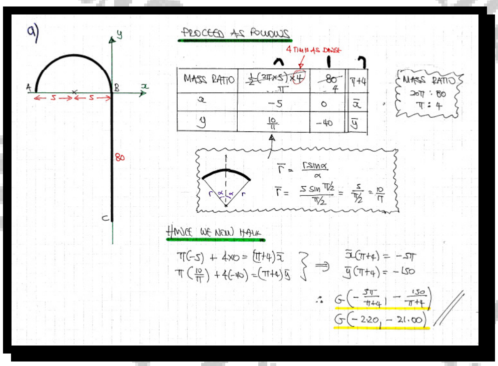
Figure 1 shows a walking stick ABC, modelled as two uniform rods AB and BC. The straight section BC has length 80 cm and the section AB is a circular arc of diameter 10 cm. The semicircular section of the walking stick is **four times** as dense as the straight section. A set of coordinate axes is defined with B as the origin as shown in figure 1.

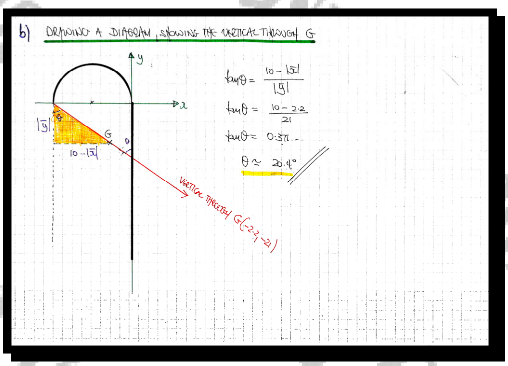
a) Find the coordinates of the centre of mass of the walking stick.

The walking stick is placed with its end A at the end of a horizontal table and rests in equilibrium under its own weight as shown in figure 2, without touching any other object.

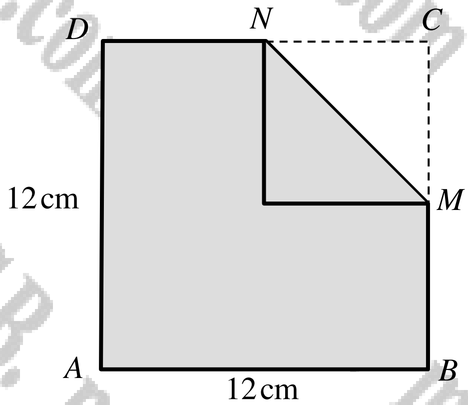
b) Determine the size of the angle that BC makes with the vertical.

$\boxed{}$, $\boxed{(\bar{x},\bar{y}) \approx (-2.2,-21.0)}$, $\boxed{\approx 20.4°}$





Question 22 (****)



The figure above shows a lamina $ABCD$ in the shape of a square of side length 12 cm, made of sheet metal of uniform material and uniform thickness. The points M and N are the midpoints of BC and CD, respectively.

The triangular section MCN is folded over the lamina forming a new composite lamina L, as shown in the figure.

a) Find the position of the centre of mass of L from A.

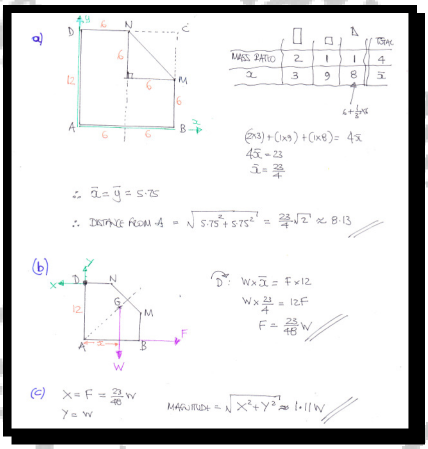
A smooth pin is attached to L at D and L is kept in a equilibrium by a horizontal force F acting at B in the direction AB.

b) Given that the weight of L is W, determine …

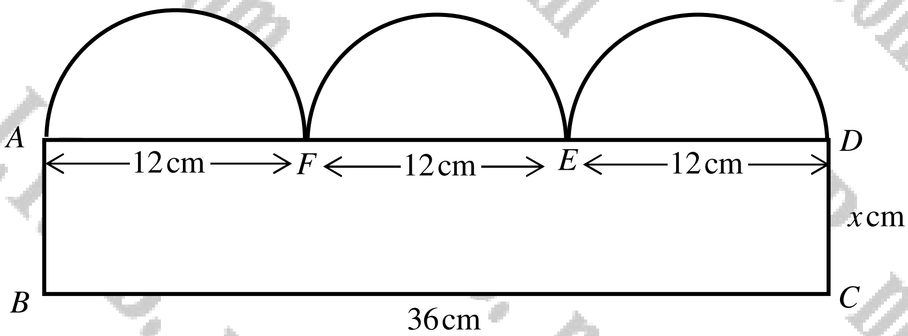
i. … the value of F.

ii. … the magnitude of the reaction force at the pin at D.

$\boxed{\frac{23}{4}\sqrt{2} \approx 8.13 \text{ cm from } O}$, $\boxed{F = \frac{23}{48}W}$, $\boxed{R \approx 1.11W}$



Question 23 (****)



The figure above shows a rectangle $ABCD$ where $|BC| = 36\text{cm}$ and $|DC| = x\text{cm}$.

The straight edges of three identical semicircles of diameter 12 cm are attached to AD forming a composite **S**, modelled as a uniform lamina.

a) Show that the distance of the centre of mass of **S** from AD is

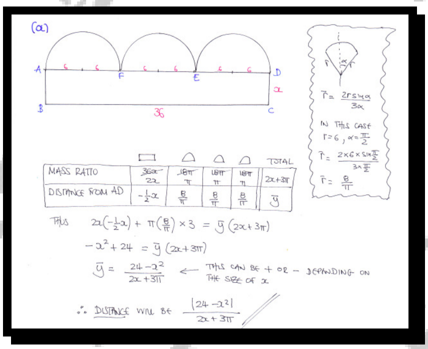
$$\frac{|24 - x^2|}{2x + 3\pi}.$$

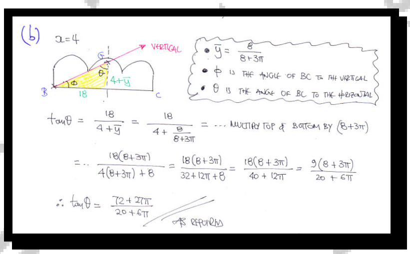
The composite **S** is suspended from B and hangs freely in equilibrium under its own weight, with BC making an angle θ with the horizontal.

b) Given that $x = 4$, show further that

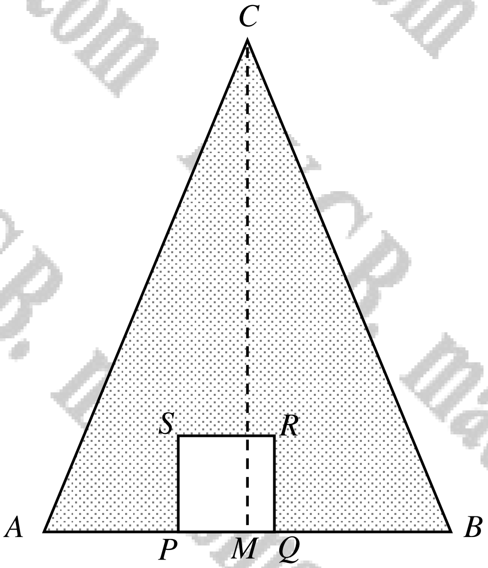
$$\tan\theta = \frac{72 + 27\pi}{20 + 6\pi}.$$

$\boxed{\text{proof}}$





Question 24 (****)



The figure above shows a uniform lamina, formed by removing a square $PQRS$ from a triangle ABC. The triangle ABC is isosceles with $AC = BC$ and $AB = 12$ cm.

The midpoint of AB is the point M and $MC = 18$ cm.

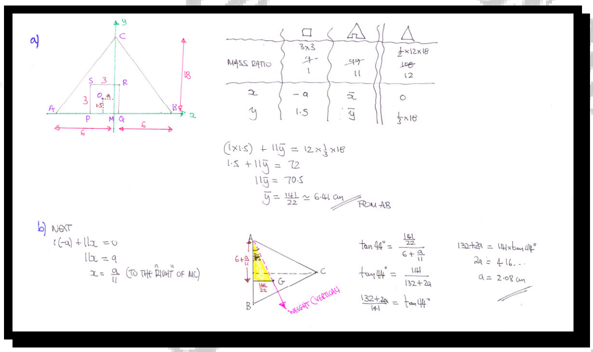
The vertices of the square, P and Q lie on AB and $PQ = 3$ cm. The centre of mass of the lamina is at the point G.

a) Find the distance of G from AB.

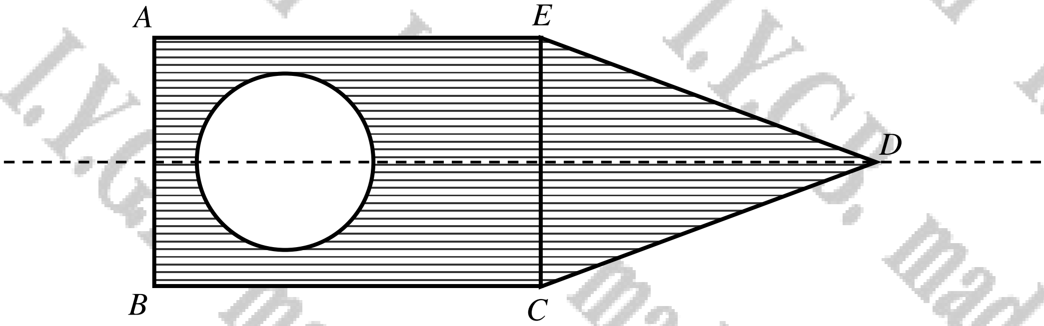
The centre of the square is the point O. When the lamina is freely suspended from A and hangs in equilibrium, the edge AB is inclined at $46°$ to the vertical.

b) Determine the distance of O from MC.

$\boxed{\approx 6.41 \text{ cm}}$, $\boxed{\approx 2.08 \text{ cm}}$



Question 25 (****)

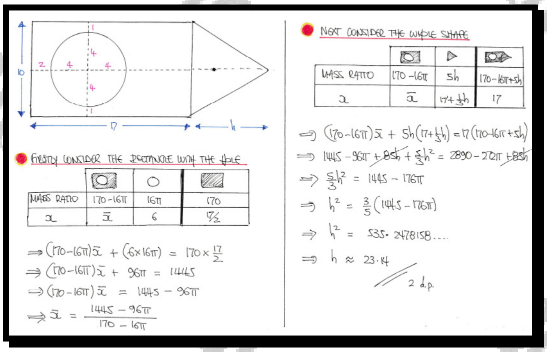


The figure above shows a uniform lamina $ABCDE$, formed by combining a rectangle $ABCE$ and a triangle ECD. A circular disc of radius 4 cm is removed from the rectangle, so that the resulting lamina has a single line of symmetry. The centre of the disc is 6 cm from AB. The triangle ECD is isosceles with $ED = CD$. It is also given that $BC = 17$ cm and $AB = 10$ cm.

The centre of mass of the lamina $ABCDE$, with the disc removed, lies on EC.

Determine the length of the height of the triangle ECD, which lies along the line of symmetry of the lamina.

$\boxed{}$, $\boxed{\approx 23.14 \text{ cm}}$



FIRSTLY CONSIDER THE RECTANGLE WITH THE HOLE

MASS RATIO	$170-16\pi$	16π	170
x	$\bar{x}$	6	$17/2$

$\Rightarrow (170-16\pi)\bar{x} + (6\times16\pi) = 170\times\frac{17}{2}$

$\Rightarrow (170-16\pi)\bar{x} + 96\pi = 1445$

$\Rightarrow (170-16\pi)\bar{x} = 1445 - 96\pi$

$\Rightarrow \bar{x} = \frac{1445-96\pi}{170-16\pi}$

NEXT CONSIDER THE WHOLE SHAPE

MASS RATIO	$170-16\pi$	$5h$	$170-16\pi+5h$
x	$\bar{x}$	$17+\frac{1}{3}h$	17

$\Rightarrow (170-16\pi)\bar{x} + 5h(17+\frac{1}{3}h) = 17(170-16\pi+5h)$

$\Rightarrow 1445 - 96\pi + \cancel{85h} + \frac{5}{3}h^2 = 2890 - 272\pi + \cancel{85h}$

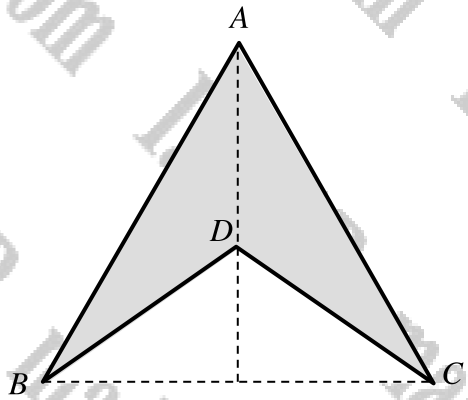
$\Rightarrow \frac{5}{3}h^2 = 1445 - 176\pi$

$\Rightarrow h^2 = \frac{3}{5}(1445 - 176\pi)$

$\Rightarrow h^2 = 535.2478158\ldots$

$\Rightarrow h \approx 23.14$ 2 d.p.

Question 26 (****)

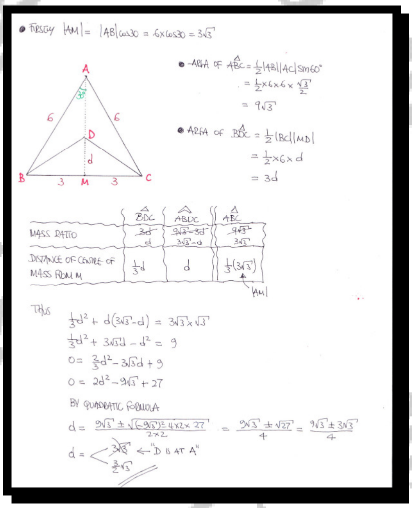


The figure above shows a logo $ABDC$.

The logo is formed by removing an isosceles triangle BDC from a uniform lamina ABC, which is in the shape of an equilateral triangle of side 6 m.

Given that the centre of mass of the logo is located at D, determine the perpendicular height of the triangle ABC, measured from the vertex D to the side BC.

$$\boxed{\frac{3}{2}\sqrt{3}}$$



• FIRSTLY $|AM| = |AB|\cos 30 = 6\times\cos 30 = 3\sqrt{3}$

• AREA OF $\triangle ABC = \frac{1}{2}|AB||AC|\sin 60°$
$= \frac{1}{2}\times 6\times 6\times \frac{\sqrt{3}}{2}$
$= 9\sqrt{3}$

• AREA OF $\triangle BDC = \frac{1}{2}|BC||MD|$
$= \frac{1}{2}\times 6\times d$
$= 3d$

	$\triangle BDC$	$ABDC$	$\triangle ABC$
MASS RATIO	$\cancel{3d}$ d	$\cancel{9\sqrt{3}-3d}$ $3\sqrt{3}-d$	$\cancel{9\sqrt{3}}$ $3\sqrt{3}$
DISTANCE OF CENTRE OF MASS FROM M	$\frac{1}{3}d$	d	$\frac{1}{3}(3\sqrt{3})$

$|AM|$

THUS

$\frac{1}{3}d^2 + d(3\sqrt{3}-d) = 3\sqrt{3}\times\sqrt{3}$

$\frac{1}{3}d^2 + 3\sqrt{3}d - d^2 = 9$

$0 = \frac{2}{3}d^2 - 3\sqrt{3}d + 9$

$0 = 2d^2 - 9\sqrt{3} + 27$

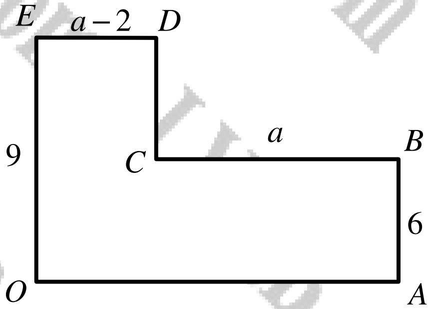
BY QUADRATIC FORMULA

$d = \frac{9\sqrt{3} \pm \sqrt{(-9\sqrt{3})^2 - 4\times 2\times 27}}{2\times 2} = \frac{9\sqrt{3}\pm\sqrt{27}}{4} = \frac{9\sqrt{3}\pm 3\sqrt{3}}{4}$

$d = \cancel{3\sqrt{3}}$ ← "D IS AT A"

$d = \frac{3}{2}\sqrt{3}$

Question 27 (****+)



The figure above shows a uniform lamina $OABCDE$ where all corners are right angles. The following lengths are marked in the figure in terms of suitable units

$$|AB| = 6, \quad |BC| = a, \quad |ED| = a-2 \quad \text{and} \quad |OE| = 9,$$

where a is a positive constant.

a) Show that the position of the centre of mass of the lamina from OE is

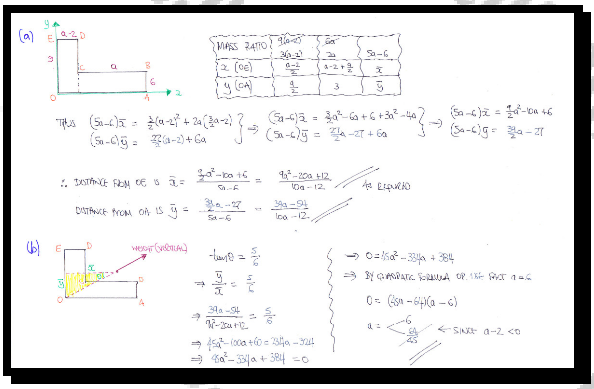
$$\frac{9a^2 - 20a + 12}{10a - 12},$$

and find a similar expression for the position of the centre of mass of the lamina from OA.

The lamina is suspended freely though a smooth pivot at O and hangs in equilibrium under its own weight. The side OA lies at an angle of $\arctan \frac{5}{6}$ to the vertical.

b) Show clearly that $a = 6$.

$$\boxed{\frac{39a - 54}{10a - 12}}$$



(a)

MASS RATIO	$9(a-2)$, $3(a-2)$	$6a$, $2a$	$5a-6$
x (OE)	$\frac{a-2}{2}$	$a-2+\frac{a}{2}$	$\bar{x}$
y (OA)	$\frac{9}{2}$	3	$\bar{y}$

THUS $(5a-6)\bar{x} = \frac{3}{2}(a-2)^2 + 2a\left(\frac{3}{2}a-2\right)$, $(5a-6)\bar{y} = \frac{27}{2}(a-2) + 6a$ $\Rightarrow$ $(5a-6)\bar{x} = \frac{3}{2}a^2 - 6a + 6 + 3a^2 - 4a$, $(5a-6)\bar{y} = \frac{27}{2}a - 27 + 6a$ $\Rightarrow$ $(5a-6)\bar{x} = \frac{9}{2}a^2 - 10a + 6$, $(5a-6)\bar{y} = \frac{39}{2}a - 27$

$\therefore$ DISTANCE FROM OE IS $\bar{x} = \frac{\frac{9}{2}a^2 - 10a + 6}{5a-6} = \frac{9a^2 - 20a + 12}{10a - 12}$ AS REQUIRED

DISTANCE FROM OA IS $\bar{y} = \frac{\frac{39}{2}a - 27}{5a-6} = \frac{39a - 54}{10a - 12}$

(b)

WEIGHT (VERTICAL)

$\tan\theta = \frac{5}{6}$

$\Rightarrow \frac{\bar{y}}{\bar{x}} = \frac{5}{6}$

$\Rightarrow \frac{39a - 54}{9a^2 - 20a + 12} = \frac{5}{6}$

$\Rightarrow 45a^2 - 100a + 60 = 234a - 324$

$\Rightarrow 45a^2 - 334a + 384 = 0$

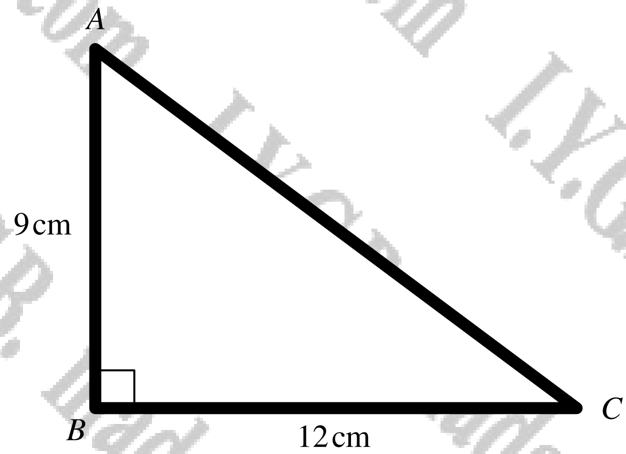
$\Rightarrow 0 = 45a^2 - 334a + 384$

$\Rightarrow$ BY QUADRATIC FORMULA OR USE FACT $a = 6$

$0 = (45a - 64)(a - 6)$

$a = 6$ or $\frac{64}{45}$ ← SINCE $a - 2 < 0$

Question 28 (****+)



The figure above shows a **framework** consisting of three small **uniform rods** AB, BC and AC.

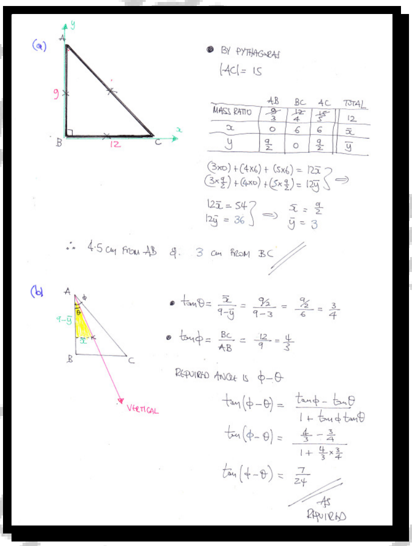
It is further given that $|AB| = 9\text{cm}$, $|BC| = 12\text{cm}$ and $\measuredangle ABC = 90°$.

b) Find the position of the centre of mass of the framework from AB and BC.

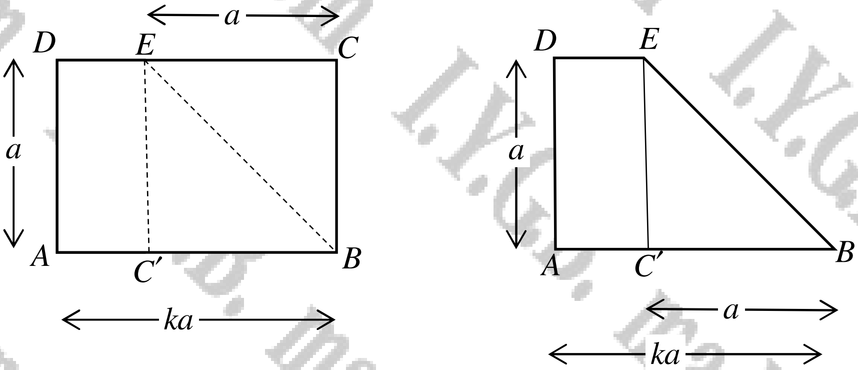
The framework is suspended freely though a smooth pivot at A and hangs in equilibrium under its own weight.

c) Show clearly that the tangent of the angle that AC makes with the vertical is exactly $\frac{7}{24}$.

$\boxed{4.5\text{cm from } AB}$, $\boxed{3\text{cm from } BC}$



Question 29 (****+)



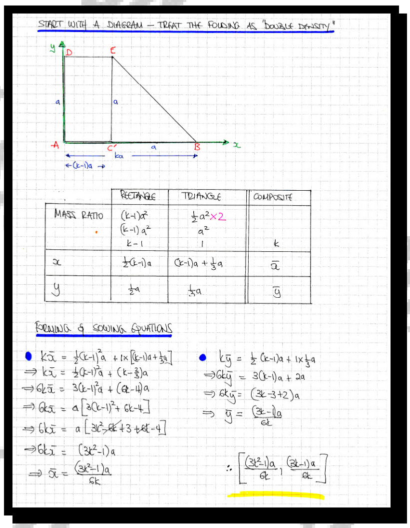
A rectangular lamina $ABCD$ has $|AD| = a$ and $|AB| = ka$, where a and k are positive constants with $k > 1$. The point E lies on CD so that $|CE| = a$.

The lamina is folded over along EB so that the vertex C is now touching the point C' on AB, as shown in the figures above.

A set of cartesian coordinate axes is defined with origin at A, AB the direction of x increasing and AD the direction of y increasing.

Determine the coordinates of the centre of mass of the folded lamina, giving the answer in terms of a and k.

$$\boxed{}\ , \ \boxed{\bar{x} = \frac{(3k^2-1)a}{6k} \ \cap \ \bar{y} = \frac{(3k-1)a}{6k}}$$



START WITH A DIAGRAM – TREAT THE FOLDING AS "DOUBLE DENSITY"

	RECTANGLE	TRIANGLE	COMPOSITE
MASS RATIO	$(k-1)a^2$ $(k-1)a^2$ $k-1$	$\frac{1}{2}a^2 \times 2$ a^2 1	 k
x	$\frac{1}{2}(k-1)a$	$(k-1)a + \frac{1}{3}a$	$\bar{x}$
y	$\frac{1}{2}a$	$\frac{1}{3}a$	$\bar{y}$

FORMING & SOLVING EQUATIONS

- $k\bar{x} = \frac{1}{2}(k-1)^2a + 1 \times \left[(k-1)a + \frac{1}{3}a\right]$

$\Rightarrow k\bar{x} = \frac{1}{2}(k-1)^2a + (k-\frac{2}{3})a$

$\Rightarrow 6k\bar{x} = 3(k-1)^2a + (6k-4)a$

$\Rightarrow 6k\bar{x} = a\left[3(k-1)^2 + 6k-4\right]$

$\Rightarrow 6k\bar{x} = a\left[3k^2 \cancel{-6k} + 3 \cancel{+6k} - 4\right]$

$\Rightarrow 6k\bar{x} = (3k^2-1)a$

$\Rightarrow \bar{x} = \frac{(3k^2-1)a}{6k}$

- $k\bar{y} = \frac{1}{2}(k-1)a + 1 \times \frac{1}{3}a$

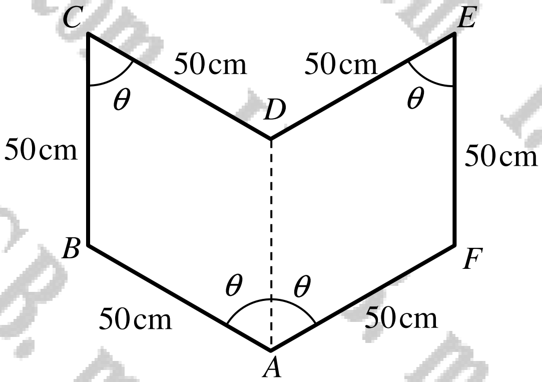
$\Rightarrow 6k\bar{y} = 3(k-1)a + 2a$

$\Rightarrow 6k\bar{y} = (3k-3+2)a$

$\Rightarrow \bar{y} = \frac{(3k-1)a}{6k}$

$\therefore \left[\frac{(3k^2-1)a}{6k}, \frac{(3k-1)a}{6k}\right]$

Question 30 (****+)



The figure above shows a uniform plane lamina $ABCDEF$, made of two congruent rhombuses, each of side length 50 cm.

It is given that $\measuredangle BAD = \measuredangle DAF = \measuredangle BCD = \measuredangle DFE = \theta$.

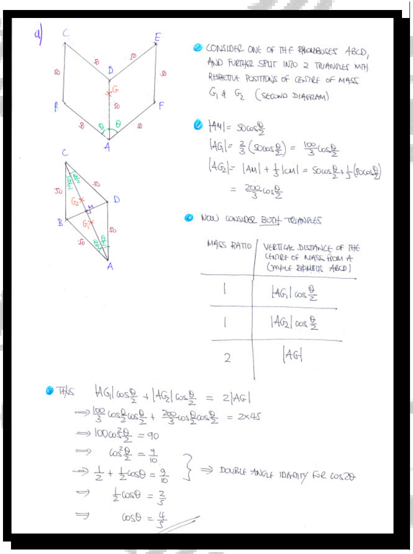
a) Given further that the centre of mass of the lamina is 50 cm from A, show that $\cos\theta = \frac{4}{5}$.

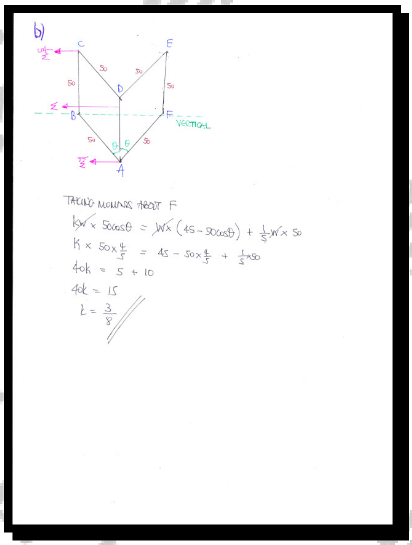
The weight of the lamina is W. A particle of weight kW, where k is a positive constant is fixed to the lamina at A. Another particle of weight $\frac{1}{5}W$ is fixed to the lamina at C.

The lamina is freely suspended from F and hangs in equilibrium with AD horizontal.

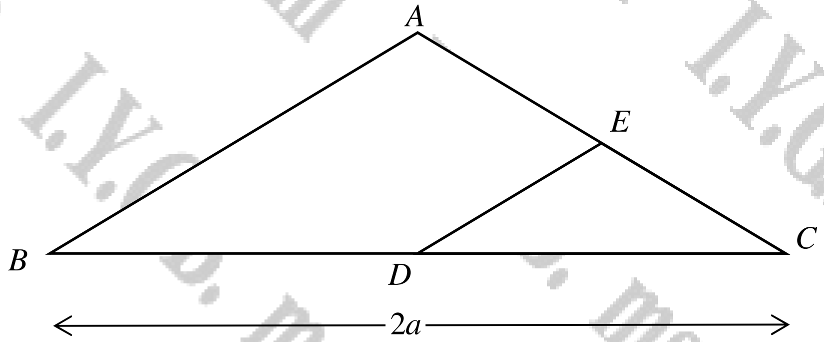
b) Find the value of k.

$\boxed{k = \frac{3}{8}}$





Question 31 (****+)



The figure above shows a lamina ABC is in the shape of an isosceles triangle, with $|AB| = |AC|$ and $|BC| = 2a$, where a is a positive constant.

The point D is the midpoint of BC. The point E lies on AC so that $|EC| = |ED|$.

The section of the lamina defined by the triangle CDE is made of a material which is **twice as dense** as the material the rest of the lamina is made of.

A set of cartesian coordinate axes is defined with origin at D, DC the direction of x increasing and DA the direction of y increasing.

a) If $|AD| = a$ determine the coordinates of the centre of mass of the lamina ABC, giving the answer in terms of a.

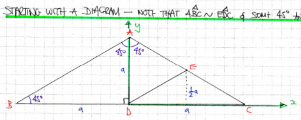
The lamina ABC is freely suspended from A and hangs in equilibrium.

b) Show that AC is inclined at $\arctan\frac{3}{4}$ to the downward vertical.

$\boxed{G\left(\frac{1}{10}a, \frac{3}{10}a\right)}$

[solution overleaf]

a) STARTING WITH A DIAGRAM — NOTE THAT $\hat{ABC} \sim \hat{EDC}$ & SOME 45° ANGLES



FORMING A "STANDARD TABLE" – TREAT THE DOUBLE DENSITY AS ANOTHER LAMINA

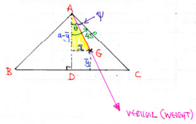
	TRIANGLE ABC	TRIANGLE EDC	COMPOSITE
MASS RATIO	$\frac{1}{2}a^2 \times 2$ a^2 4	$\frac{1}{2} \times \frac{1}{2}a \times a$ $\frac{1}{4}a^2$ 1	5
x	0	$\frac{1}{2}a$	$\bar{x}$
y	$\frac{1}{3}a$	$\frac{1}{3} \times \frac{1}{2}a$	$\bar{y}$

FORMING & SOLVING EQUATIONS

- $5\bar{x} = (4 \times 0) + 1 \times \frac{1}{2}a$
  $5\bar{x} = \frac{1}{2}a$
  $\bar{x} = \frac{1}{10}a$
- $5\bar{y} = 4 \times \frac{1}{3}a + 1 \times \frac{1}{6}a$
  $5\bar{y} = \frac{4}{3}a + \frac{1}{6}a$
  $5\bar{y} = \frac{3}{2}a$
  $\bar{y} = \frac{3}{10}a$

$\therefore \left(\frac{1}{10}a, \frac{3}{10}a\right)$

b) REDRAWING THE DIAGRAM



$\tan\theta = \dfrac{\bar{x}}{a-\bar{y}}$

$\tan\theta = \dfrac{\frac{1}{10}a}{a - \frac{3}{10}a}$

$\tan\theta = \dfrac{\frac{1}{10}}{1 - \frac{3}{10}}$

$\tan\theta = \dfrac{1}{10-3}$

$\tan\theta = \dfrac{1}{7}$

THE REQUIRED ANGLE IS ψ

$\Rightarrow \psi = 45 - \theta$

$\Rightarrow \tan\psi = \tan(45-\theta)$

$\Rightarrow \tan\psi = \dfrac{\tan 45 - \tan\theta}{1 + \tan 45 \tan\theta}$

$\Rightarrow \tan\psi = \dfrac{1 - \frac{1}{7}}{1 + 1 \times \frac{1}{7}}$

$\Rightarrow \tan\psi = \dfrac{1 - \frac{1}{7}}{1 + \frac{1}{7}}$

$\Rightarrow \tan\psi = \dfrac{7-1}{7+1}$

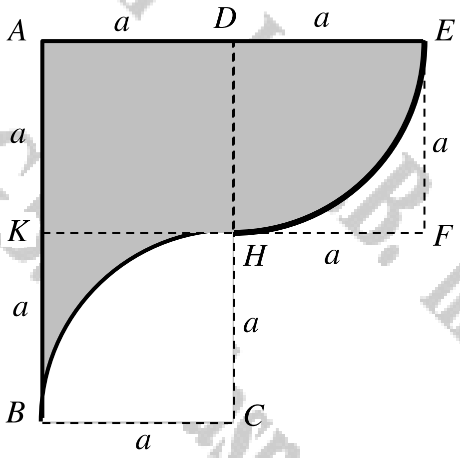
$\Rightarrow \tan\psi = \dfrac{6}{8}$

$\Rightarrow \tan\psi = \dfrac{3}{4}$

$\Rightarrow \psi = \arctan\frac{3}{4}$

AS REQUIRED

Question 32 (****+)



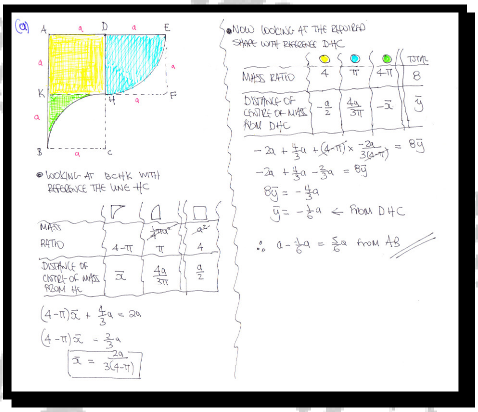
The figure above shows a uniform rectangular lamina $ABCD$, where $|AD| = |BC| = a$ and $|AB| = |DC| = 2a$. The midpoints of AB and DC are K and H, respectively.

A quarter circle of radius a and centred at C is removed from the rectangle $ABCD$.

The quarter circle then attached with its centre at D and one of its straight edges along DH as shown in the figure.

Determine the position of the centre of mass of the resulting shape from AB.

$\boxed{}$, $\boxed{\bar{x} = \frac{5}{6}a}$



(a)

• LOOKING AT BCHK WITH REFERENCE THE LINE HC

	◸	◠	□
MASS RATIO	$4-\pi$	$\cancel{\frac{1}{4}\pi a^2}$ π	$\cancel{a^2}$ 4
DISTANCE OF CENTRE OF MASS FROM HC	$\bar{x}$	$\frac{4a}{3\pi}$	$\frac{a}{2}$

$(4-\pi)\bar{x} + \frac{4}{3}a = 2a$

$(4-\pi)\bar{x} = \frac{2}{3}a$

$\boxed{\bar{x} = \frac{2a}{3(4-\pi)}}$

• NOW LOOKING AT THE REQUIRED SHAPE WITH REFERENCE DHC

	yellow	blue	green	TOTAL
MASS RATIO	4	π	$4-\pi$	8
DISTANCE OF CENTRE OF MASS FROM DHC	$-\frac{a}{2}$	$\frac{4a}{3\pi}$	$-\bar{x}$	$\bar{y}$

$-2a + \frac{4}{3}a + \cancel{(4-\pi)} \times \frac{-2a}{3\cancel{(4-\pi)}} = 8\bar{y}$

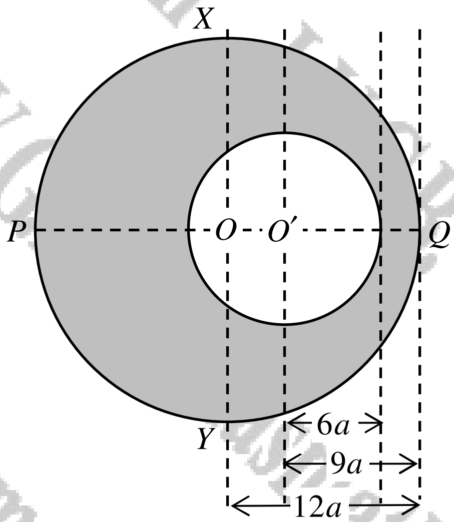
$-2a + \frac{4}{3}a - \frac{2}{3}a = 8\bar{y}$

$8\bar{y} = -\frac{4}{3}a$

$\bar{y} = -\frac{1}{6}a$ ← FROM DHC

$\therefore\ a - \frac{1}{6}a = \frac{5}{6}a$ from AB

Question 33 (*****)



A composite uniform lamina is modelled by the finite region bounded by two circular discs, shown shaded in the figure above. The details, of the sizes and relative positions of these discs, are as follows.

The straight line POQ is a diameter of the larger circular disc, of radius $12a$, whose centre is at the point O. The smaller circular disc, of radius $6a$, has its centre at O', so that O' lies on OQ with $|O'Q| = 9a$.

A heavy particle is attached to the lamina at Q.

The straight line XOY is perpendicular to POQ.

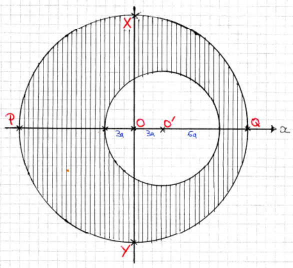
When the lamina is freely suspended from X and hangs in equilibrium, with P higher than Q, POQ is inclined at $\arctan\frac{5}{12}$ to the horizontal.

Determine the ratio of the mass of the particle to the mass of the lamina.

$\boxed{}$, $\boxed{6 : 7}$

[solution overleaf]

START BY OBTAINING THE POSITION OF THE CENTRE OF MASS OF THE LAMINA WITHOUT THE EXTRA PARTICLE

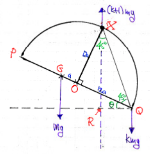


OBJECT	COMPOSITE	"HOLE"	BIG DISC
MASS RATIO	3	~~$36\pi a^2$~~ 1	~~$144\pi a^2$~~ 4
x CO-ORDINATE OF THE CENTRE OF MASS	$\bar{x}$	$3a$	0

$\Rightarrow 3\bar{x} + 3a = 0$

$\Rightarrow x = -a$ ← TO THE "LEFT" OF O

NOW LOOKING AT MOMENTS IN THE FOLLOWING DIAGRAM



- LET THE MASS OF THE LAMINA BE m & THAT OF THE PARTICLE km
- MARK WEIGHTS
- REACTION AT X WILL BE $(k+1)mg$
- $O\hat{P}X$ IS RIGHT ANGLED & ISOSCELES SO $|XQ| = 12\sqrt{2}a$ (PYTHAGORAS)
- $|QG| = 13a$ (LAMINA CENTRE OF MASS)
- $\tan\theta = \frac{5}{12}$, $\sin\theta = \frac{5}{13}$, $\cos\theta = \frac{12}{13}$
- $|RQ| = |XQ|\cos(\theta+45)$
  $|RQ| = 12\sqrt{2}a\cos(\theta+45)$

TAKING MOMENTS ABOUT Q

$\Rightarrow \cancel{mg} \times |GQ|\cos\theta = (k+1)\cancel{mg} \times |RQ|$

$\Rightarrow |GQ|\cos\theta = (k+1)\,|RQ|$

$\Rightarrow 13\cancel{a}\cos\theta = (k+1) \times 12\sqrt{2}\cancel{a}\cos(\theta+45)$

$\Rightarrow 13\cos\theta = 12(k+1)\sqrt{2}\left(\cos\theta\cos45 - \sin\theta\sin45\right)$

$\Rightarrow 13 \times \frac{12}{13} = 12(k+1)\sqrt{2}\left[\frac{12}{13} \times \frac{1}{\sqrt{2}} - \frac{5}{13} \times \frac{1}{\sqrt{2}}\right]$

$\Rightarrow \cancel{12} = \cancel{12}(k+1)\cancel{\sqrt{2}} \times \frac{1}{\cancel{\sqrt{2}}}\left(\frac{12}{13} - \frac{5}{13}\right)$

$\Rightarrow 1 = (k+1) \times \frac{7}{13}$

$\Rightarrow \frac{13}{7} = k+1$

$\Rightarrow k = \frac{6}{7}$

$\therefore$

PARTICLE	:	LAMINA
$\frac{6}{7}m$	:	m
$6m$	:	$7m$
6	:	7