

Created by T. Madas

CENTRE OF MASS

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Question 1 ()**

Four particles A , B , C and D have masses 1kg, 2kg, 3kg and 4kg, respectively.

The respective coordinates of the particles A , B , C and D are $(2,3)$, $(4,0)$, $(-1,5)$ and $(-3,-4)$.

- a) Find the coordinates of the centre of mass of this system of four particles.

A fifth particle E of mass 10 kg is placed at the point P , so that the centre of mass of the **five** particles is now at the point with coordinates $(3,1)$.

- b) Find the coordinates of P .

$$\boxed{(-0.5, 0.2)}, \quad \boxed{P(6.5, 1.8)}$$

(a)

MASS	A	B	C	D	Total
1	2	2	3	4	10
\bar{x}	2	4	-1	-3	$2\bar{x}$
\bar{y}	3	0	5	-4	3

Hence $(1 \times 2) + (2 \times 4) + (3 \times -1) + (4 \times -3) = 10\bar{x} \Rightarrow$
 $(1 \times 2) + (2 \times 0) + (3 \times 5) + (4 \times -4) = 10\bar{y} \Rightarrow$
 $2 + 8 - 3 - 12 = 10\bar{x} \Rightarrow 10\bar{x} = -5 \Rightarrow \bar{x} = -0.5$
 $3 + 0 + 15 - 16 = 10\bar{y} \Rightarrow 10\bar{y} = 2 \Rightarrow \bar{y} = 0.2$
 $\therefore G(-0.5, 0.2)$

(b)

MASS	A-D	E	Total
\bar{x}	-0.5	\bar{x}	3
\bar{y}	0.2	1	1

$[1 \times (-0.5)] + [10 \times \bar{x}] = 2 \times 3 \Rightarrow -0.5 + 10\bar{x} = 6$
 $(1 \times 0.2) + (10\bar{y}) = 2 \times 1 \Rightarrow 0.2 + 10\bar{y} = 2$
 $\Rightarrow \bar{x} = 6.5$
 $\bar{y} = 1.8$
 $\therefore (6.5, 1.8)$

Question 2 ()**

Three particles A , B and C have masses 2 kg, 3 kg and 5 kg, respectively.

The respective coordinates of these three particles are $(2, 2)$, $(0, -5)$ and $(-3, 1)$.

- a) Find the coordinates of the centre of mass of this system of three particles.

A fourth particle D of mass 10 kg is placed at the point with coordinates $(2, 3)$.

- b) Find the coordinates of the centre of mass of the system of the **four** particles.

A fifth particle E of mass k kg is placed at the point with coordinates $(-1, \lambda)$.

The coordinates of the centre of mass of the **five** particles is now at the origin.

- c) Determine the values of k and λ .

$(-1.1, -0.6)$, $(0.45, 1.2)$, $k = 9$, $\lambda = -\frac{8}{3}$

(a)

MASS	A	B	C	TOTAL
MASS RATIO	2	3	5	10
x	2	0	-3	3
y	2	-5	1	0

$(2 \times 2) + (3 \times 0) + (5 \times -3) = 10\bar{x} \Rightarrow 10\bar{x} = -11$
 $(2 \times 2) + (3 \times -5) + (5 \times 1) = 10\bar{y} \Rightarrow 10\bar{y} = -6$
 $\bar{x} = -1.1$ $\bar{y} = -0.6$ $(-1.1, -0.6)$

(b)

MASS	A+B+C	D	TOTAL
MASS RATIO	10	10	20
x	-1.1	2	2
y	-0.6	3	3

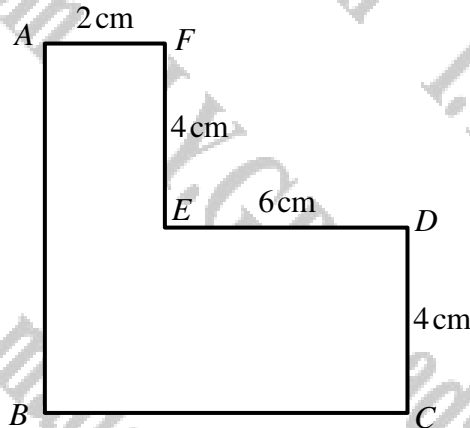
$(-1.1) + 1 \times 2 = 2\bar{x} \Rightarrow 2\bar{x} = 0.9 \Rightarrow \bar{x} = 0.45$
 $(-0.6) + 1 \times 3 = 2\bar{y} \Rightarrow 2\bar{y} = 2.4 \Rightarrow \bar{y} = 1.2$
 $\therefore (0.45, 1.2)$

(c)

MASS	A+D	E	TOTAL
MASS RATIO	20	k	20+k
x	0.45	-1	0
y	1.2	λ	0

$20(0.45) + k(-1) = 0 \Rightarrow 9 - k = 0 \Rightarrow k = 9$
 $20(1.2) + k\lambda = 0 \Rightarrow 24 + k\lambda = 0 \Rightarrow \lambda = -\frac{24}{9} = -\frac{8}{3}$

Question 3 (**)



The figure above shows a uniform lamina $ABCDEF$ where all corners are right angles and $|AF| = 2\text{ cm}$, $|FE| = 4\text{ cm}$, $|ED| = 6\text{ cm}$ and $|DC| = 4\text{ cm}$.

- a) Determine the position of the centre of mass of the lamina from AB and BC .

The lamina is suspended freely through a smooth pivot at B and hangs in equilibrium under its own weight.

- b) Find the size of the angle that AB makes with the vertical.

$3.4\text{ cm from } AB$, $2.8\text{ cm from } BC$, $\approx 50.5^\circ$

(a)

Area	Mass Ratio	Centroid (x, y)
$2 \times 4 = 8$	2	(1, 2)
$4 \times 6 = 24$	3	(4, 2)
$6 \times 4 = 24$	3	(6, 2)

$(2 \times 1) + (3 \times 4) = 5x$
 $(2 \times 2) + (3 \times 2) = 5y$
 $5x = 14 \Rightarrow x = 2.8$
 $5y = 17 \Rightarrow y = 3.4$

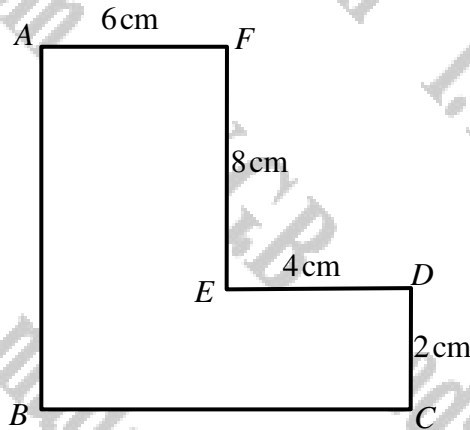
$\therefore 3.4\text{ cm from } AB$
 $2.8\text{ cm from } BC$

(b)

Weight (vertical)

$\tan \theta = \frac{3.4}{2.8}$
 $\tan \theta = \frac{17}{14}$
 $\theta \approx 50.5^\circ$

Question 4 (**)



The figure above shows a uniform lamina $ABCDEF$ where all corners are right angles and $|AF| = 6\text{ cm}$, $|FE| = 8\text{ cm}$, $|ED| = 4\text{ cm}$ and $|DC| = 2\text{ cm}$.

- a) Determine the position of the centre of mass of the lamina from AB and BC .

The lamina is suspended freely through a smooth pivot at F and hangs in equilibrium under its own weight.

- b) Find the size of the angle that AF makes with the vertical.

$\approx 3.59\text{ cm from } AB$, $\approx 4.53\text{ cm from } BC$, $\approx 66.2^\circ$

(a)

Area	\bar{x}	\bar{y}	$\bar{x}A$	$\bar{y}A$
2010	12	5	17	
2	3	5	2	
4	6	1	9	

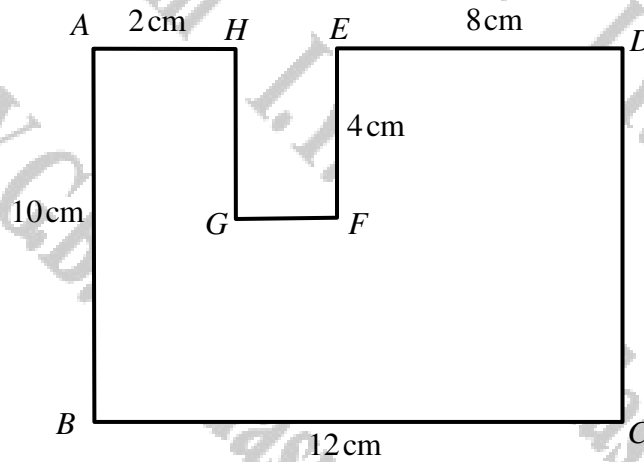
Thus $(12x) + (5y) = 17x$ $\Rightarrow 17x = 61$ $\Rightarrow x = \frac{61}{17} \approx 3.59$
 $(2x) + (5y) = 17y$ $\Rightarrow 17y = 27$ $\Rightarrow y = \frac{27}{17} \approx 1.59$

$\therefore 3.59\text{ cm from } AB$ & $4.53\text{ cm from } BC$

(b)

$\tan(\theta) = \frac{3.59}{4.53} = \frac{61}{81} = \frac{10}{13.4}$
 $\theta \approx 66.2^\circ$

Question 5 (***)



The figure above shows a uniform lamina $ABCDEFGH$ where all corners are right angles and $|AH| = 2\text{ cm}$, $|ED| = 8\text{ cm}$, $|EF| = 4\text{ cm}$, $|BC| = 12\text{ cm}$ and $|AB| = 10\text{ cm}$.

- a) Determine the position of the centre of mass of the lamina from AB and BC .

The lamina is suspended freely through a smooth pivot at D and hangs in equilibrium under its own weight.

- b) Find the size of the angle that DC makes with the vertical.

$\approx 6.21\text{ cm from } AB$, $\approx 4.79\text{ cm from } BC$, $\approx 48.0^\circ$

Area and CM of parts:

Part	Area	CM (x, y)
1 (Rectangle AHG)	4	(1, 5)
2 (Rectangle BCDEFG)	68	(8, 2)

Overall CM:

$$\bar{x} = \frac{4(1) + 68(8)}{4 + 68} = \frac{4 + 544}{72} = \frac{548}{72} \approx 7.61$$

$$\bar{y} = \frac{4(5) + 68(2)}{72} = \frac{20 + 136}{72} = \frac{156}{72} \approx 2.17$$

Angle of DC with vertical:

$$\tan \theta = \frac{\bar{x} - x_D}{\bar{y} - y_D} = \frac{7.61 - 12}{2.17 - 10} = \frac{-4.39}{-7.83} \approx 0.56$$

$$\theta \approx 29.1^\circ$$

Angle of DC with horizontal:

$$\theta = 90^\circ - 29.1^\circ = 60.9^\circ$$

Question 6 (***)

A uniform rectangular lamina $ABCD$ has mass 1.84 kg is loaded with a particle of mass 0.46 kg attached at the corner C . It is further given that $|AB| = |CD| = 16 \text{ cm}$ and $|BC| = |DA| = 12 \text{ cm}$.

- a) Determine the position of the centre of mass of the **loaded** lamina from the edge AD and from the edge AB .

The lamina is suspended in equilibrium with AB horizontal by two vertical strings one attached at A and one attached at B .

- b) Calculate the tension in each of the two strings.

$9.6 \text{ cm from } AD$, $7.2 \text{ cm from } AB$, $T_A = 9.016 \text{ N}$, $T_B = 13.524 \text{ N}$

(a)

MASS RATIO	1.84g	0.46g	5
x	8	16	5x
y	6	12	5y

$(4 \times 8) + (1 \times 16) = 5x$
 $(4 \times 6) + (1 \times 12) = 5y$
 $5x = 48 \Rightarrow x = 9.6$
 $5y = 36 \Rightarrow y = 7.2$

$\therefore 9.6 \text{ cm from } AD \text{ \& } 7.2 \text{ cm from } AB$

(b)

$T_1 + T_2 = 1.84g + 0.46g$
 $T_1 + T_2 = 2.3g$

$T_2 \times 16 = 2.3g \times 9.6$
 $16T_2 = 2.3g \times 9.6$
 $T_2 = 1.38g$
 $\& T_1 = 0.92g$

$\therefore \text{Tension at } A = 0.92g = 9.016 \text{ N}$
 $\text{Tension at } B = 1.38g = 13.524 \text{ N}$

Question 7 (***)

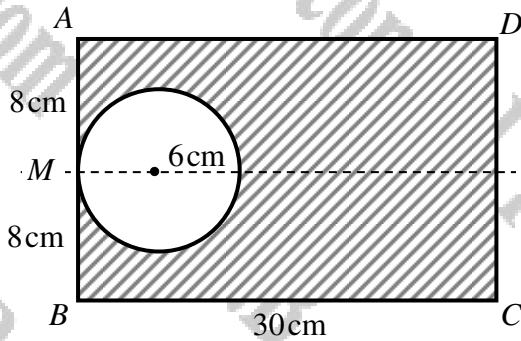


figure 1

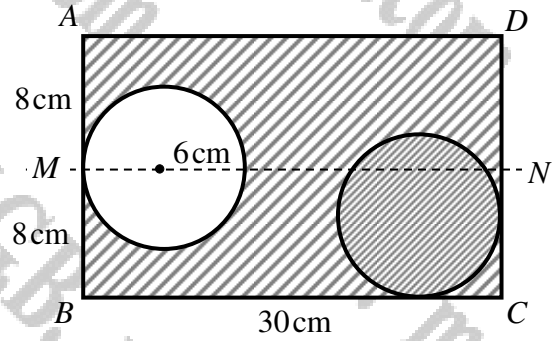


figure 2

Figure 1 shows a rectangular lamina $ABCD$ where $|AB| = 16\text{ cm}$ and $|BC| = 30\text{ cm}$.

The points M and N are the midpoints of AB and CD .

A circle of radius 6 cm whose centre lies on MN at a distance of 6 cm from AB , is removed from the lamina $ABCD$, forming a composite S .

- a) Determine the position of the centre of mass of S from AB .

The circular section removed in part (a) is now attached to a new position on S so that BC and CD are now tangents to the circular section. The new composite is shown in figure 2 and is denoted by T .

- b) Determine the distance of the centre of mass of T from AB and BC .

$\bar{x} = 17.77\text{ cm}$, $\approx 17.8\text{ cm}$ from AB , $\bar{y} = 7.53\text{ cm}$ from BC and 19.24 cm from AB

(a)

MASS RATIO	$40 - 3\pi$	3π	40
\bar{x}	$\frac{600 - 18\pi}{40 - 3\pi}$	24	\bar{x}
\bar{y}	8	6	\bar{y}

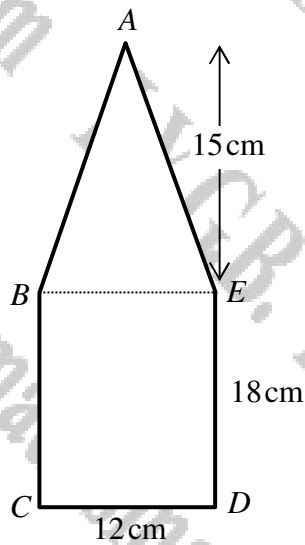
$(40 - 3\pi)\bar{x} + 6(3\pi) = 40 \times 8$
 $(40 - 3\pi)\bar{x} = 600 - 18\pi$
 $\bar{x} = \frac{600 - 18\pi}{40 - 3\pi}$
 $\bar{x} \approx 17.77\text{ cm}$
 $\therefore 17.8\text{ cm}$ from AB

(b)

MASS RATIO	$40 - 3\pi$	3π	40
\bar{x}	$\frac{600 - 18\pi}{40 - 3\pi}$	24	\bar{x}
\bar{y}	8	6	\bar{y}

$(40 - 3\pi)\bar{x} + 6(3\pi) = 40 \times 8$
 $(40 - 3\pi)\bar{x} = 600 - 18\pi$
 $\bar{x} = \frac{600 - 18\pi}{40 - 3\pi}$
 $\bar{x} \approx 19.24$
 $\bar{y} = \frac{320 - 6\pi}{40} \approx 7.53$
 $\therefore 19.24\text{ cm}$ from AB
 7.53 cm from BC

Question 8 (***)



The figure above shows a uniform lamina $ABCDE$ consisting of a rectangle $BCDE$ and an isosceles triangle ABE where $|AB| = |AE|$.

It is further given that $|CD| = 12\text{ cm}$, $|ED| = 18\text{ cm}$ and the height of the triangle measured from A is 15 cm .

The lamina is suspended freely through a smooth pivot at B and hangs in equilibrium under its own weight.

Find the size of the angle that BC makes with the vertical.

$\approx 50.9^\circ$

MASS RATIO:	$\frac{24G}{12}$	$\frac{90G}{5}$	$\frac{17G}{5}$
	-9	$\frac{1}{5}$	$\frac{1}{5}$

$12(-9) + (5 \times 5) = 17g$
 $\Rightarrow 11g = -83$
 $g = -\frac{83}{17}$

NOW SUSPENSION POINT IS
 $\tan \theta = \frac{83}{17}$
 $\theta = \frac{102}{83}$
 $\theta \approx 50.9^\circ$

Question 9 (***)

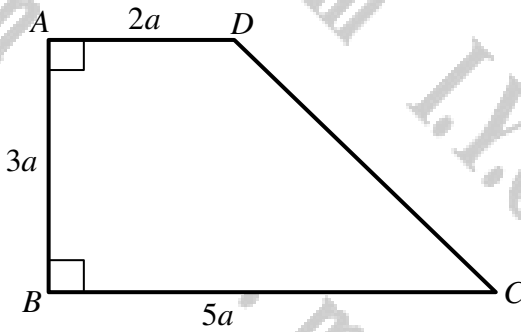


figure 1

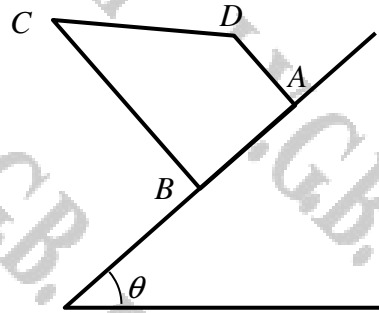


figure 2

Figure 1 above shows a lamina $ABCD$ which is in the shape of a right angled trapezium, where $\angle DAB = \angle ABC = 90^\circ$.

It is further given that $|AB| = 3a$, $|BC| = 5a$ and $|AD| = 2a$.

- a) Determine the position of the centre of mass of the lamina from AB and BC .

The lamina is next placed on plane inclined at an angle θ to the horizontal, as shown in figure 2. The plane is sufficiently rough to prevent the lamina from sliding.

- b) Given that the lamina is at the point of toppling find the value of θ .

$\frac{13a}{7}$ from AB , $\frac{9a}{7}$ from BC , $\approx 34.7^\circ$

(a)

Area	$\frac{1}{2}(2a+5a) \times 3a = \frac{21}{2}a^2$
\bar{x}	$\frac{1}{21} \left(2a \times \frac{3a}{2} + 5a \times \frac{3a}{2} \right) = \frac{13a}{7}$
\bar{y}	$\frac{1}{21} \left(2a \times \frac{3a}{2} + 5a \times \frac{3a}{2} \right) = \frac{9a}{7}$

$\therefore \frac{13a}{7}$ from AB and $\frac{9a}{7}$ from BC

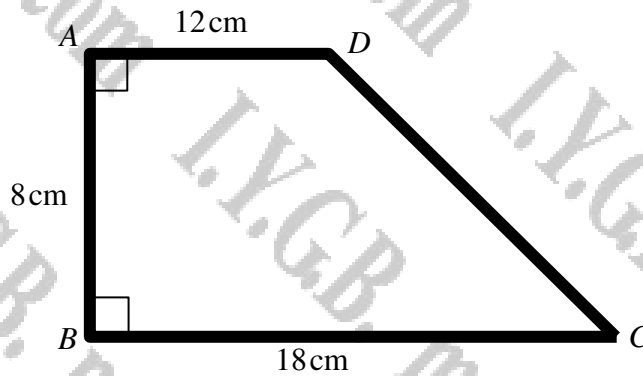
(b)

Inclined plane at θ to horizontal. At point of toppling, the weight acts vertically through the pivot B.

$\tan \theta = \frac{\bar{y}}{\bar{x}} = \frac{9a/7}{13a/7} = \frac{9}{13}$

$\theta = 34.7^\circ$

Question 10 (***)



The figure above shows a **framework** consisting of four small **uniform rods** AB , BC , CD and AD .

It is given that $|AB| = 8\text{ cm}$, $|BC| = 18\text{ cm}$, $|AD| = 12\text{ cm}$ and $\angle ABC = \angle DAB = 90^\circ$.

- a) Determine the distance of the position of the centre of mass of the framework from AB and BC .

The framework is suspended freely through a smooth pivot at C and hangs in equilibrium under its own weight.

- b) Find the size of the angle that DC makes with the vertical.

, , ,

(a) **BY PYTHAGOREAN TRIC = 10**
CONSIDER INDIVIDUAL RODS

MASS RATIO	AB	BC	CD	AD	TOTAL
\bar{x}	4	9	15	6	34
\bar{y}	4	0	4	8	16

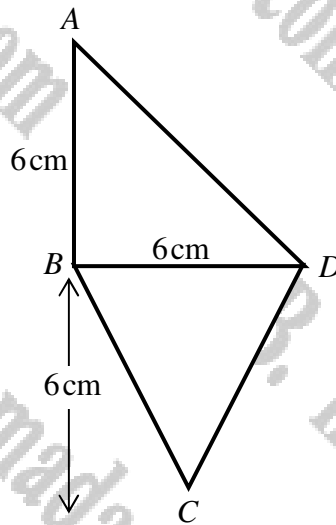
$(4 \times 4) + (9 \times 9) + (4 \times 4) + (8 \times 8) = 245$
 $(4 \times 4) + (9 \times 0) + (4 \times 4) + (8 \times 8) = 245$
 $\Rightarrow 245 = 192$
 $245 = 84$

$\bar{x} = 8\text{ cm FROM AB}$
 $\bar{y} = 3.5\text{ cm FROM BC}$

(b) $\tan \phi = \frac{3.5}{10} = \frac{3.5}{10} = \frac{7}{20}$
 $\phi \approx 19.29^\circ$

\therefore REQUIRED ANGLE $\theta = \phi - \theta$
 $\theta = 33.8^\circ$

Question 11 (***)



The figure above shows a lamina $ABCD$ consisting of a right angled isosceles triangle ABD where $\angle ABD = 90^\circ$ and an isosceles triangle BCD where $|BC| = |CD|$.

It is further given that $|AB| = 6\text{cm}$, $|BD| = 6\text{cm}$ and the height of the triangle BCD measured from C is 6cm .

- a) Explain why the centre of mass of the lamina $ABCD$ lies on BD .
- b) Find the distance of the centre of mass of the lamina $ABCD$ from B .

The lamina $ABCD$ is smoothly pivoted at A and kept in a position with BD horizontal and C below the level of BD by a horizontal force F .

F acts through D , in the direction BD .

- c) Given the mass of the lamina is m , find the size of F in terms of m and g .

2.5 cm from B, $F = \frac{5}{12}mg$

(b) BOTH TRIANGLES HAVE IDENTICAL AREAS (MASSES)
THE C.G. OF THE TRIANGLES OF MASSES IS 2cm FROM B.D.

MASS RATIO	ΔABD	ΔBCD	Σ
	2	3	5

$(2 \times 2) + (3 \times 3) = 25$
 $\bar{x} = 2.5 \text{ cm FROM B}$

(c) PIVOT
TAKING MOMENTS ABOUT A
 $mg \times 2.5 = F \times 6$
 $F = \frac{5}{12}mg$
← REQUIRED

Question 12 (***)

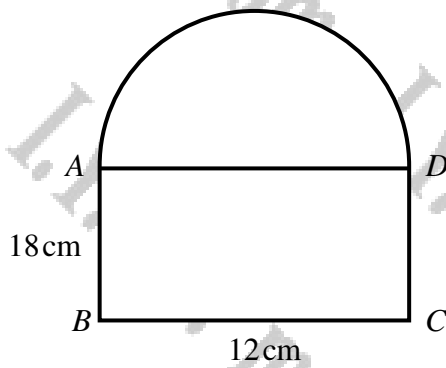


figure 1

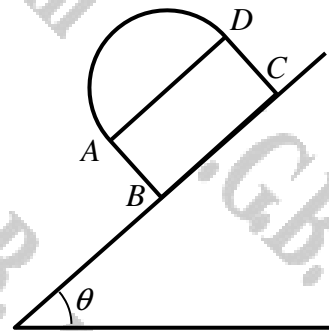


figure 2

Figure 1 above shows a uniform composite lamina consisting of a rectangle $ABCD$ and a semicircle of diameter AD .

It is further given that $|AB| = 18\text{cm}$ and $|BC| = 12\text{cm}$.

- a) Determine the position of the centre of mass of the lamina from BC .

The lamina is next placed on plane inclined at an angle θ to the horizontal, as shown in figure 2. The plane is sufficiently rough to prevent the lamina from sliding.

- b) Given that the lamina is at the point of toppling find the value of θ .

$\approx 11.4\text{cm from } BC$, $\approx 27.8^\circ$

Handwritten solution for Question 12:

(a) Diagram of the lamina with a vertical y -axis. A mass ratio table is shown:

MASS RATIO	AREA	COM DISTANCE FROM BC	COM DISTANCE FROM BC
$\frac{1}{2}\pi$	$\frac{1}{2}\pi$	$18 + \frac{6}{\pi}$	9
12	12	6	6

Calculations for the center of mass from BC :

$$\bar{y} = \frac{\pi(18 + \frac{6}{\pi}) + (12 \times 9)}{\frac{1}{2}\pi + 12} = \frac{18\pi + 116}{\pi + 12} \approx 11.4 \text{ cm from } BC$$

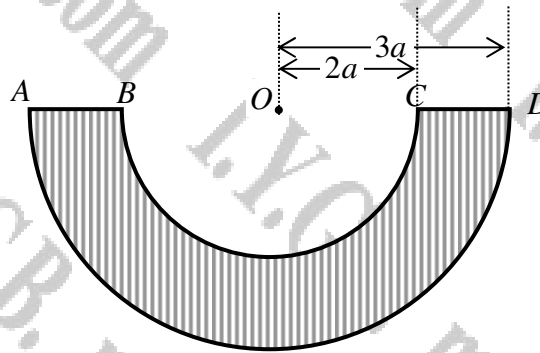
(b) Diagram of the lamina on an inclined plane. The weight acts vertically downwards from the center of mass G . The reaction force acts perpendicular to the incline at point B . The angle θ is between the incline and the horizontal.

Calculations for the angle θ :

$$\tan \theta = \frac{6}{11.4}$$

$$\theta = 27.8^\circ$$

Question 13 (***)



The figure above shows a lamina $ABDC$ consisting of a semicircle centre at O and radius $2a$ removed from a larger semicircle also with centre at O and radius $3a$.

- a) Find the distance of the centre of mass of the lamina from O .

The lamina is suspended freely through a smooth pivot at B and hangs in equilibrium under its own weight.

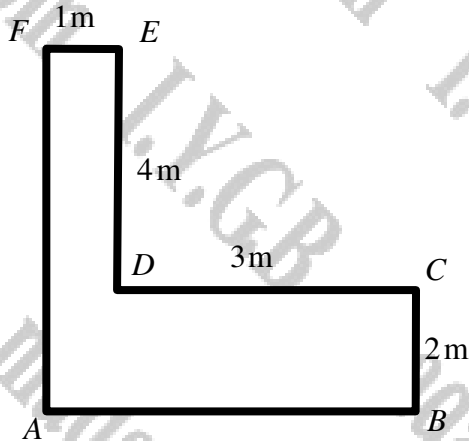
- b) Find the size of the angle that BC makes with the horizontal.

$\frac{76a}{15\pi}$ from O , 38.9°

(a) $\frac{76a}{15\pi}$ from O

(b) 38.9°

Question 14 (***)



The figure above shows a rigid framework $ABCDEF$, consisting of 6 uniform rods all of equal cross-section and of equal mass density.

It is further given that all the corners of the framework are right angles and $|BC| = 2\text{ m}$, $|CD| = 3\text{ m}$, $|DE| = 4\text{ m}$ and $|EF| = 1\text{ m}$.

- a) Find the position of the centre of mass of the framework from AB and AF .

The framework is suspended freely through a smooth pivot at F and hangs in equilibrium under its own weight.

- b) Show that the tangent of the angle which DC makes with the vertical is $\frac{18}{7}$.

, 1.4 m from AF , 2.4 m from AB , $\approx 66.2^\circ$

a) (LOOKING AT THE DIAGRAM BELOW)

ROD	MASS RATIO	x	y
AB	4	2	0
BC	2	4	1
CD	3	2.5	2
DE	4	1	4
EF	1	1	6
FA	6	0	3
TOTAL	20	2	3

HENCE WE OBTAIN

$$\begin{aligned} 20\bar{x} &= (4 \times 2) + (2 \times 4) + (3 \times 2.5) + (4 \times 1) + (1 \times 1) + (6 \times 0) \\ 20\bar{y} &= (4 \times 0) + (2 \times 1) + (3 \times 2) + (4 \times 4) + (1 \times 6) + (6 \times 3) \end{aligned}$$

$$\begin{aligned} 20\bar{x} &= 28 \\ 20\bar{y} &= 48 \end{aligned}$$

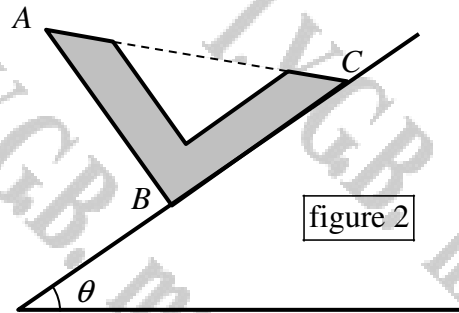
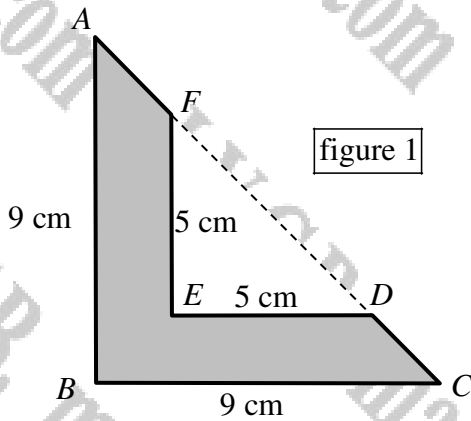
$$\begin{aligned} \bar{x} &= 1.4 \\ \bar{y} &= 2.4 \end{aligned}$$

THE CENTRE OF MASS IS 1.4m FROM AF & 2.4m FROM AB

b) (REMEMBERING THE FRAMEWORK IS IN EQUILIBRIUM)

$$\begin{aligned} \tan \theta &= \frac{6-3}{2} \\ \tan \theta &= \frac{6-2.4}{1.4} \\ \tan \theta &= \frac{3.6}{1.4} \\ \tan \theta &= \frac{36}{14} \\ \tan \theta &= \frac{18}{7} \end{aligned}$$

Question 15 (***)



A uniform lamina is in the shape of an isosceles triangle ABC with $\angle ABC = 90^\circ$ and $AB = BC = 9$ cm. An isosceles triangle DEF is removed from ABC , such that $\angle DEF = 90^\circ$ and $DE = EF = 5$ cm, forming a composite S , shown in figure 1.

- a) Find the distance of the centre of mass of S ...
- i. ... from AB .
 - ii. ... from BC .

The composite S is placed on the greatest slope of a plane inclined at an angle θ to the horizontal, as shown in figure 2. The plane is sufficiently rough to prevent S from sliding.

- b) Given that S is at the point of toppling over, calculate the value of θ .

$$\frac{227}{84} \approx 2.70 \text{ cm from } AB \text{ and from } BC, \quad \theta = 45^\circ$$

(a)

Part	Area	Centroid	Mass
$\triangle AFB$	$\frac{1}{2} \times 4 \times 4 = 8$	$(\frac{4}{3}, \frac{4}{3})$	$\frac{8}{9}$
Rectangle $FEDE$	$5 \times 5 = 25$	$(\frac{5}{2}, \frac{5}{2})$	$\frac{25}{9}$
$\triangle EDC$	$\frac{1}{2} \times 4 \times 4 = 8$	$(\frac{4}{3}, \frac{4}{3})$	$\frac{8}{9}$
Total	41	$(\frac{227}{84}, \frac{227}{84})$	$\frac{41}{9}$

$(\frac{8}{9} \times \frac{4}{3}) + (\frac{25}{9} \times \frac{5}{2}) + (\frac{8}{9} \times \frac{4}{3}) = \frac{148}{27}$
 $(\frac{8}{9} \times \frac{4}{3}) + (\frac{25}{9} \times \frac{5}{2}) + (\frac{8}{9} \times \frac{4}{3}) = \frac{148}{27}$
 $\frac{148}{27} = \frac{227}{84} \Rightarrow \bar{x} = \bar{y} = \frac{227}{84}$

THE CENTRE OF MASS IS 2.70 CM FROM AB, AND 2.70 CM FROM BC

(b)

$\tan \theta = \frac{227}{84} = 1$
 $\theta = 45^\circ$

Question 16 (****)

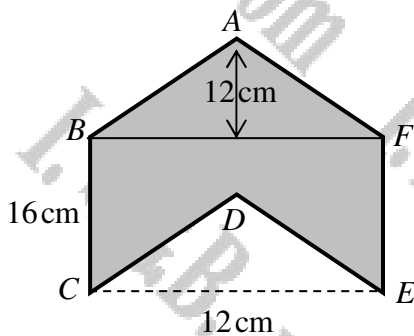


figure 1

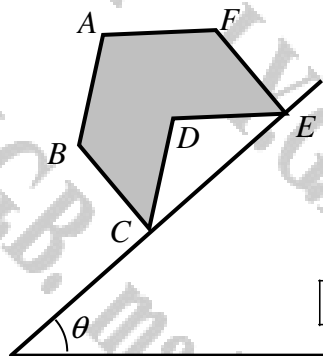


figure 2

From a rectangle $BCEF$, an isosceles triangle CDE is removed and attached to the rectangle so that the sides CE and BF coincide, and the point D is relabelled as A .

It is further given that $|CD| = |DE|$, $|BC| = 16\text{cm}$ and $|CE| = 12\text{cm}$. The height of the triangle ABF , measured from A , is 12cm .

Figure 1 above, shows the composite which is modelled as a uniform lamina.

- a) Show that the centre of mass of the lamina is located at a distance of 14cm from CE .

The lamina is next placed on plane inclined at an angle θ to the horizontal, as shown in figure 2. The plane is sufficiently rough to prevent the lamina from sliding.

- b) Given that the lamina is at the point of toppling find the value of θ .

, $\approx 23.2^\circ$

a) CONSIDER THE PROBLEM IN 2 PARTS

MASS RATIO	$\frac{2 \times 12 \times 12}{2} = 144$	12	$\frac{12 \times 16 \times 12}{2} = 1152$
	3	5	8
DISTANCE OF THE CENTRE OF MASS FROM CE	$\frac{1}{2} \times 12 = 6$	4	8

$\Rightarrow 3 \times 6 + 8 \times 8 = 69$
 $\Rightarrow 12 + 64 = 76$
 $\Rightarrow \frac{69}{76} = 0.9079$
 $\Rightarrow \frac{69}{76} \times 12 = 10.9$

NOW REPEAT WITH THE TRIANGLE ON TOP

MASS RATIO	$\frac{12 \times 12}{2} = 72$	3	192
	5	3	8
DISTANCE OF THE CENTRE OF MASS FROM CE	10.4	$\frac{1}{2} \times 12 = 6$	8

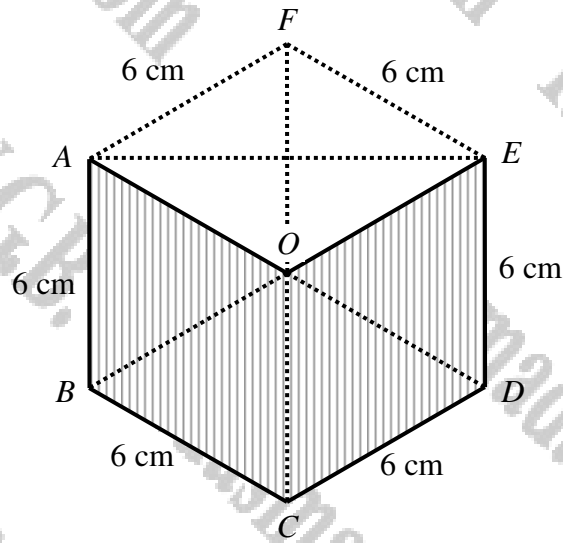
$\Rightarrow (5 \times 10.4) + (3 \times 8) = 69$
 $\Rightarrow 52 + 24 = 76$
 $\Rightarrow \frac{69}{76} = 0.9079$
 $\Rightarrow \frac{69}{76} \times 12 = 10.9$

AS EVIDENT

b) LET IT BE SUCH SO THAT THE LAMINA IS ABOUT TO TOPPLE

$\Rightarrow \tan \theta = \frac{10.9}{12}$
 $\Rightarrow \tan \theta = 0.9079$
 $\Rightarrow \theta = 42.2^\circ$

Question 17 (****)



The figure above shows a uniform lamina $ABCDEF$ in the shape of a regular hexagon of side length 6 cm, whose centre is at O . A rhombus $AEOF$ is removed from the hexagon forming a composite lamina S .

- a) Determine the distance of the centre of mass of S from O .

The composite S is suspended from the point E and hangs freely in equilibrium. The side OE makes an angle θ with the vertical.

- b) Show that $\sin \theta = \frac{1}{14}\sqrt{7}$.

1.5 cm from O

(a) **NOTE: A REGULAR HEXAGON CONSISTS OF 6 CONGRUENT EQUILATERAL TRIANGLES**

AREA RATIO	1	2	3	TOTAL
y	1	3	3	0

$(1 \times 3) + 2y = 0$
 $3 + 2y = -3$
 $2y = -6$
 $y = -3$
 $\therefore 1.5 \text{ cm BELOW } O$

(b)

By the cosine rule
 $|EG|^2 = 6^2 + 1.5^2 - 2(6)(1.5)\cos(120^\circ)$
 $|EG|^2 = 36 + 2.25 - 2(6)(1.5)(-\frac{1}{2})$
 $|EG|^2 = 36 + 2.25 + 9 = 47.25$
 $|EG| = \frac{3\sqrt{47}}{2}$

• $\sin \theta = \frac{\sin 120^\circ}{1.5}$
 $\sin \theta = \frac{\frac{\sqrt{3}}{2}}{1.5}$
 $\sin \theta = \frac{\sqrt{3}}{3}$
 $\sin \theta = \frac{\sqrt{7}}{14}$ (As required)

Question 18 (****)

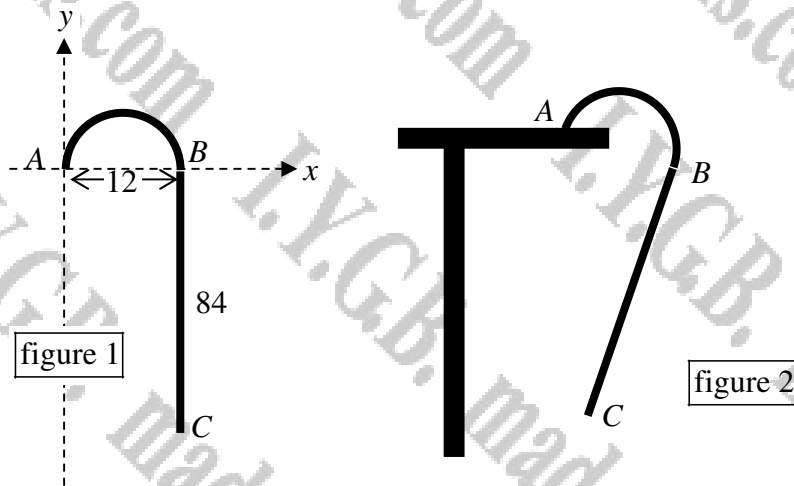


Figure 1 shows a walking stick ABC , modelled as two uniform rods AB and BC . The straight section BC has length 84 cm and the section AB is a circular arc of diameter 12 cm. A set of coordinate axes is defined with A as the origin as shown in figure 1.

- a) Find the coordinates of the centre of mass of the walking stick.

The walking stick is placed with its end A at the end of a horizontal table and rests in equilibrium under its own weight as shown in figure 2, without touching any other object.

- b) Determine the size of the angle that BC makes with the vertical.

, $(\bar{x}, \bar{y}) \approx (10.9, -33.6)$, $\approx 18.0^\circ$

(a)

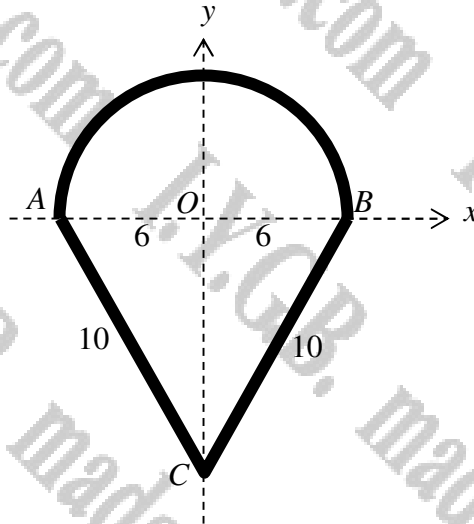
	MASS RATIO	$\frac{x(\text{cm})}{12}$	$\frac{y(\text{cm})}{84}$
\bar{x}	6	12	$\frac{14.4}{3}$
\bar{y}	$\frac{12}{2}$	-82	$\frac{1}{3}$

$\bar{x} = \frac{6 \times 12 + 12 \times \frac{14.4}{3}}{12 + 12} = \frac{72 + 57.6}{24} = 10.9$
 $\bar{y} = \frac{\frac{12}{2} \times (-82) + 12 \times \frac{1}{3}}{12 + 12} = \frac{-492 + 4}{24} = -33.6$

(b)

• ADD EXTRA PIECE AD
 • AS AD || BC THEY BOTH INCLINED TO THE VERTICAL AT THE SAME ANGLE θ
 • $\tan \theta = \frac{3}{9} = \frac{10.9}{33.6}$
 $\theta \approx 18.0^\circ$

Question 19 (****)



The figure above shows a **framework** consisting of three thin **uniform rods** AB , BC and AC . The rods BC and AC are straight lines, both of lengths 10 units. The rod AB is in the shape of a semicircular arc of radius 6 units with centre at O . A set of coordinate axes is defined with O as the origin, as shown in figure.

- a) Determine the position of the centre of mass of the framework from O .

A particle of mass 4 kg is attached to the midpoint of AB . The centre of mass of the **loaded framework** is now at O .

- b) Find the mass of the framework.

≈ 0.206 from O , ≈ 117 kg

(a)

Diagram: A small version of the framework diagram with O at the origin. The semicircular arc AB has radius 6. The straight rods AC and BC have length 10. The vertical distance from O to C is labeled OC .

• BY PYTHAGORAS $OC = \sqrt{6^2 + 6^2}$
 $OC = 8.5$

• MASS RATIO

	AB	AC	BC
MASS	$\frac{1}{2}\pi(6)^2$	10	10
DISTANCE FROM O	0	8.5	8.5

• $F = \frac{Wx}{L}$
 IN THIS CASE, $r = 6$, $\theta = \frac{\pi}{2}$
 $F = \frac{6 \times \pi \times 6^2}{2 \times 6} = 36\pi$

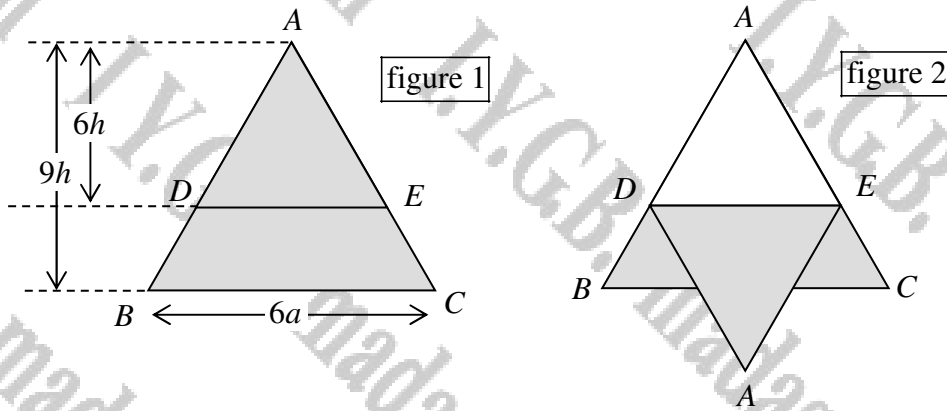
\therefore FORCE $(36\pi \times \frac{6}{2}) + 5(-4) + 5(-4) = (10+2m)y$
 $36 - 40 = (10+2m)y$
 $y = -\frac{4}{10+2m} \approx 0.2059$ Below x -axis

(b)

	AB	AC	BC
MASS RATIO	$\frac{1}{2}\pi(6)^2$	10	10
DISTANCE FROM O	0	8.5	8.5

$m \left(-\frac{4}{10+2m} \right) + 4 \times 6 = 0$
 $-4m = 4m(10+2m)$
 $-4 = 4m(10+2m)$
 $-1 = m(10+2m)$
 $-1 = 10m + 2m^2$
 $2m^2 + 10m + 1 = 0$
 $m \approx 117$ kg

Question 20 (****)



A uniform lamina ABC is the shape of an isosceles triangle where $AB = AC$, $BC = 6a$. The vertical height of ABC is $9h$, as shown in figure 1. The lamina is to be folded along DE , where DE is parallel to BC and at a perpendicular distance of $6h$ from A , as shown in figure 2.

- a) Show that the centre of mass of the **trapezium** $BDEC$ is $\frac{7}{5}h$ from BC .
- b) Determine the position of the centre of mass of the **folded lamina** from BC .

The folded lamina is suspended from the point D and hangs freely in equilibrium. The side DE is inclined at $\arctan \frac{2}{9}$ to the vertical.

- c) Express a in terms of h .

, ,

a) **2x SIMILARITY**

$\frac{DE}{BC} = \frac{6h}{9h} = \frac{2}{3}$

$DE = \frac{2}{3} \times 6a = 4a$

Area of $\triangle ABC = \frac{1}{2} \times 6a \times 9h = 27ah$

Area of $\triangle ADE = \frac{1}{2} \times 4a \times 6h = 12ah$

Area of Trapezium $BDEC = 27ah - 12ah = 15ah$

Centres of mass of similar figures

FIGURE	Area	Distance of CM from base
$\triangle ABC$	$27ah$	$\frac{3}{4} \times 9h = \frac{27}{4}h$
$\triangle ADE$	$12ah$	$\frac{3}{4} \times 6h = \frac{9}{2}h$

$\Rightarrow 4x + 5y = 27h$

$\Rightarrow 20x + 25y = 135h$

$\Rightarrow 16x = 108h$

$\Rightarrow x = \frac{27}{4}h$

$\Rightarrow y = \frac{3}{4}h$

$\Rightarrow \frac{3}{4}h$ is correct

b) **Centres of mass of the folded lamina**

Distance of CM of $\triangle ADE$ from DE is $\frac{3}{4} \times 6h = \frac{9}{2}h$

Distance of CM of Trapezium $BDEC$ from DE is $\frac{7}{5}h$

Now take moments

FIGURE	Area	Distance of CM from DE
$\triangle ADE$	$12ah$	$\frac{9}{2}h$
Trapezium $BDEC$	$15ah$	$\frac{7}{5}h$

$\Rightarrow 12ah \times \frac{9}{2}h = 15ah \times \frac{7}{5}h$

$\Rightarrow 54ah^2 = 21ah^2$

$\Rightarrow 54 = 21$

$\Rightarrow a = 4h$

c) **Finally looking at the diagram above**

$\tan \theta = \frac{2}{9}$

$\frac{2}{9} = \frac{2h}{9a}$

$4a = 27h - 9a$

$4a = 27h - 9a$

$4a + 9a = 27h$

$13a = 27h$

$a = \frac{27}{13}h$

Question 21 (****)

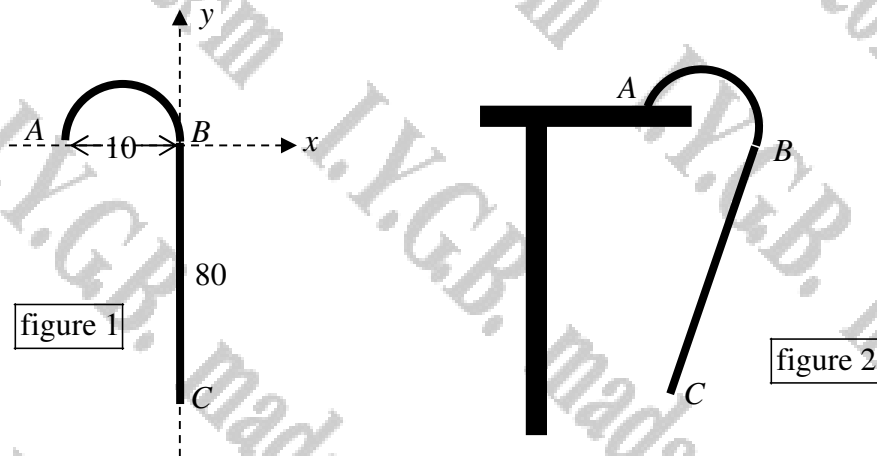


Figure 1 shows a walking stick ABC , modelled as two uniform rods AB and BC . The straight section BC has length 80 cm and the section AB is a circular arc of diameter 10 cm. The semicircular section of the walking stick is **four times** as dense as the straight section. A set of coordinate axes is defined with B as the origin as shown in figure 1.

- a) Find the coordinates of the centre of mass of the walking stick.

The walking stick is placed with its end A at the end of a horizontal table and rests in equilibrium under its own weight as shown in figure 2, without touching any other object.

- b) Determine the size of the angle that BC makes with the vertical .

, $(\bar{x}, \bar{y}) \approx (-2.2, -21.0)$, $\approx 20.4^\circ$

a) PROCEDURE AS FOLLOWS

MASS	DISTO	$\frac{1}{2}(x_1+y_1)^2$	$\frac{1}{2}(x_1-y_1)^2$	$\frac{1}{2}(x_1^2+y_1^2)$
4	5	10	0	10
16	40	320	1600	1920

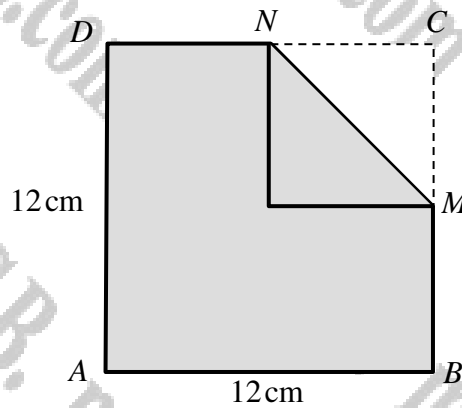
$\bar{x} = \frac{10 + 320}{20} = \frac{330}{20} = 16.5$
 $\bar{y} = \frac{0 + 1600}{20} = \frac{1600}{20} = 80$

HENCE WE HAVE
 $T(-5) + 4(0) = (16.5)R$
 $T(\frac{10}{4}) + 4(40) = (16.5)R$

b) DRAWING A DIAGRAM SHOWING THE VERTICAL THROUGH G

$\tan \theta = \frac{10 - 151}{151}$
 $\tan \theta = \frac{10 - 22}{21}$
 $\tan \theta = 0.5714...$
 $\theta \approx 20.4^\circ$

Question 22 (****)



The figure above shows a lamina $ABCD$ in the shape of a square of side length 12 cm, made of sheet metal of uniform material and uniform thickness. The points M and N are the midpoints of BC and CD , respectively.

The triangular section MCN is folded over the lamina forming a new composite lamina L , as shown in the figure.

- a) Find the position of the centre of mass of L from A .

A smooth pin is attached to L at D and L is kept in a equilibrium by a horizontal force F acting at B in the direction AB .

- b) Given that the weight of L is W , determine ...
- ... the value of F .
 - ... the magnitude of the reaction force at the pin at D .

$$\boxed{\frac{23}{4}\sqrt{2} \approx 8.13 \text{ cm from } O}, \quad \boxed{F = \frac{23}{48}W}, \quad \boxed{R \approx 1.11W}$$

(a)

SHAPE	Area	Centroid	Mass
Square $ABMN$	36	$(6, 6)$	36
Triangle MNC	9	$(9, 9)$	9
Total	45	(\bar{x}, \bar{y})	45

$$45\bar{x} = 36(6) + 9(9) = 216 + 81 = 297$$

$$\bar{x} = \frac{297}{45} = 6.6$$

$$45\bar{y} = 36(6) + 9(9) = 216 + 81 = 297$$

$$\bar{y} = \frac{297}{45} = 6.6$$

$$\therefore \text{Distance from } A = \sqrt{6.6^2 + 6.6^2} = \frac{23}{4}\sqrt{2} \approx 8.13$$

(b)

$$\sum \tau_D = 0$$

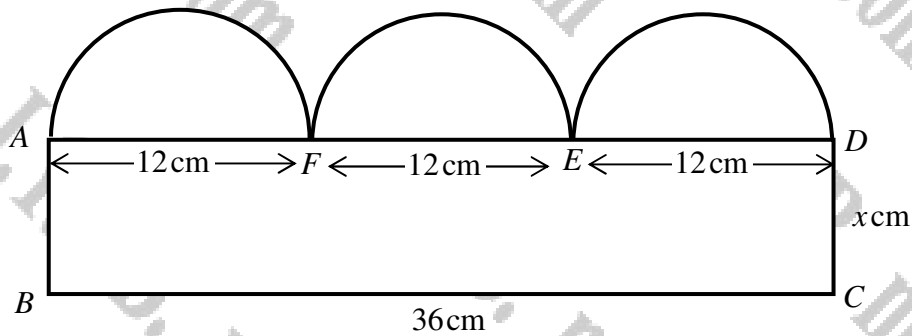
$$W \times \frac{23}{4}\sqrt{2} = F \times 12$$

$$W \times \frac{23}{4} = 12F$$

$$F = \frac{23}{48}W$$

(c) $x = F = \frac{23}{48}W$
 $y = W$
 Magnitude $= \sqrt{x^2 + y^2} \approx 1.11W$

Question 23 (****)



The figure above shows a rectangle $ABCD$ where $|BC| = 36\text{ cm}$ and $|DC| = x\text{ cm}$.

The straight edges of three identical semicircles of diameter 12 cm are attached to AD forming a composite S , modelled as a uniform lamina.

- a) Show that the distance of the centre of mass of S from AD is

$$\frac{|24 - x^2|}{2x + 3\pi}$$

The composite S is suspended from B and hangs freely in equilibrium under its own weight, with BC making an angle θ with the horizontal.

- b) Given that $x = 4$, show further that

$$\tan \theta = \frac{72 + 27\pi}{20 + 6\pi}$$

proof

(a)

MASS RATIO	$\frac{36x}{2x}$	$\frac{36\pi}{4}$	$\frac{36\pi}{4}$	$\frac{36\pi}{4}$	TOTAL
DISTANCE FROM AD	$\frac{1}{2}x$	$\frac{6}{\pi}$	$\frac{6}{\pi}$	$\frac{6}{\pi}$	$2x + 3\pi$

TRIA: $2x(\frac{1}{2}x) + \pi(\frac{6}{\pi}) \times 3 = 5(2x + 3\pi)$

$-x^2 + 24 = 5(2x + 3\pi)$

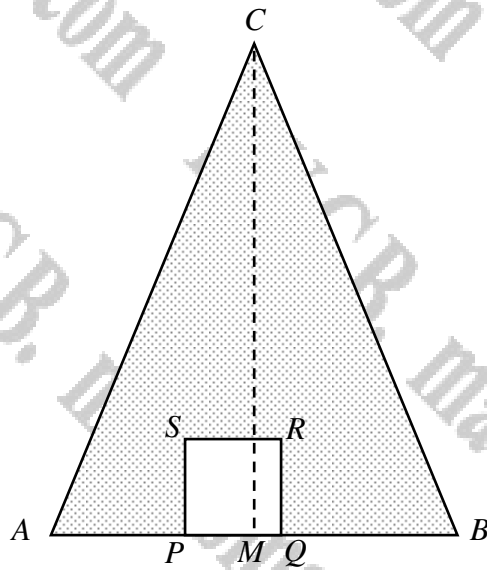
$\bar{y} = \frac{24 - x^2}{2x + 3\pi}$ ← THIS CAN BE + OR - DEPENDING ON THE SIZE OF x

∴ DISTANCE WILL BE $\frac{|24 - x^2|}{2x + 3\pi}$

(b) $x = 4$

∴ $\bar{y} = \frac{72 + 27\pi}{20 + 6\pi}$

Question 24 (****)



The figure above shows a uniform lamina, formed by removing a square $PQRS$ from a triangle ABC . The triangle ABC is isosceles with $AC = BC$ and $AB = 12$ cm.

The midpoint of AB is the point M and $MC = 18$ cm.

The vertices of the square, P and Q lie on AB and $PQ = 3$ cm. The centre of mass of the lamina is at the point G .

- a) Find the distance of G from AB .

The centre of the square is the point O . When the lamina is freely suspended from A and hangs in equilibrium, the edge AB is inclined at 46° to the vertical.

- b) Determine the distance of O from MC .

≈ 6.41 cm, ≈ 2.08 cm

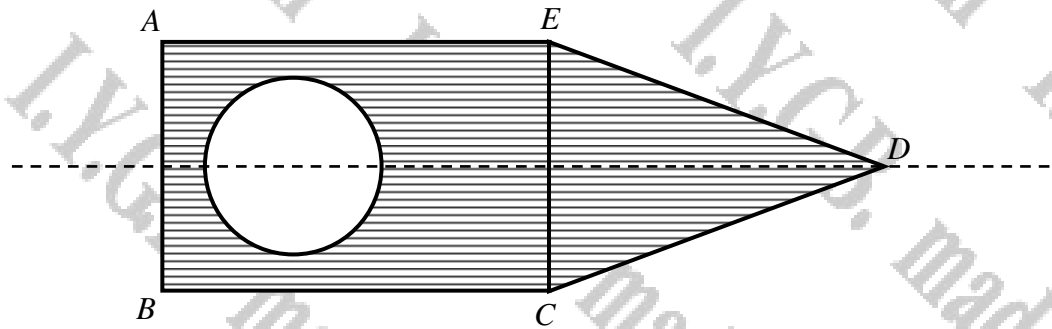
a)

SHAPE	AREA	CM FROM AB	CM FROM MC
$\triangle ABC$	$\frac{1}{2} \times 12 \times 18 = 108$	6	12
$\square PQRS$	$3 \times 3 = 9$	1.5	0
Lamina	$108 - 9 = 99$	$\frac{108 \times 6 - 9 \times 1.5}{99} = 6.41$	$\frac{108 \times 12 - 9 \times 0}{99} = 12.73$

b)

When suspended from A, the lamina hangs in equilibrium. The center of mass G is vertically below A. The center of the square O is vertically below G. The distance of O from MC is 2.08 cm.

Question 25 (****)



The figure above shows a uniform lamina $ABCDE$, formed by combining a rectangle $ABCE$ and a triangle ECD . A circular disc of radius 4 cm is removed from the rectangle, so that the resulting lamina has a single line of symmetry. The centre of the disc is 6 cm from AB . The triangle ECD is isosceles with $ED = CD$. It is also given that $BC = 17$ cm and $AB = 10$ cm.

The centre of mass of the lamina $ABCDE$, with the disc removed, lies on EC .

Determine the length of the height of the triangle ECD , which lies along the line of symmetry of the lamina.

, ≈ 23.14 cm

FINALLY CONSIDER THE SHAPES WITH THE FILE

SHAPE	MASS RATIO	\bar{x}	\bar{y}
Rectangle	$(170-16\pi)$	16π	170
Circle	16π	6	$9\frac{1}{2}$
Triangle	170	170	170

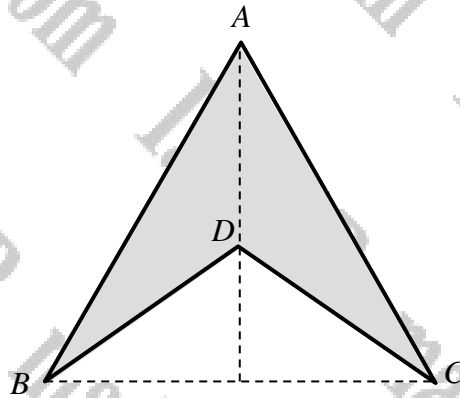
$\Rightarrow (170-16\pi)\bar{x} + (5 \times 16\pi) = 170 \times \frac{17}{2}$
 $\Rightarrow (170-16\pi)\bar{x} + 96\pi = 1444.5$
 $\Rightarrow (170-16\pi)\bar{x} = 1444.5 - 96\pi$
 $\Rightarrow \bar{x} = \frac{1444.5 - 96\pi}{170 - 16\pi}$

NOT CONSIDER THE CIRCULAR SHAPE

SHAPE	MASS RATIO	\bar{x}	\bar{y}
Rectangle	$170-16\pi$	16π	$170-8\pi+5h$
Triangle	170	170	170

$\Rightarrow (170-16\pi)\bar{x} + 5h(17+\frac{17}{2}) = 17(170-16\pi+5h)$
 $\Rightarrow 1444.5 - 96\pi + 8.5h^2 = 2890 - 272\pi + 85h$
 $\Rightarrow \frac{5}{2}h^2 = 1445 - 176\pi$
 $\Rightarrow h^2 = \frac{2}{5}(1445 - 176\pi)$
 $\Rightarrow h^2 = 535 - 247.68\pi \dots$
 $\Rightarrow h \approx 23.14$
 2 dp

Question 26 (****)



The figure above shows a logo $ABDC$.

The logo is formed by removing an isosceles triangle BDC from a uniform lamina ABC , which is in the shape of an equilateral triangle of side 6 m.

Given that the centre of mass of the logo is located at D , determine the perpendicular height of the triangle ABC , measured from the vertex D to the side BC .

$$\frac{3\sqrt{3}}{2}$$

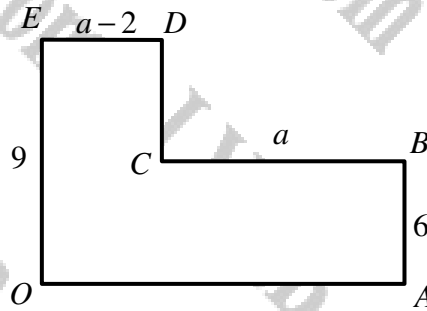
\bullet First $|AM| = |AB|\cos 30 = 6 \times \frac{\sqrt{3}}{2} = 3\sqrt{3}$

- \bullet Area of $\triangle ABC = \frac{1}{2}|AB||AC|\sin 60^\circ = \frac{1}{2} \times 6 \times 6 \times \frac{\sqrt{3}}{2} = 9\sqrt{3}$
- \bullet Area of $\triangle BDC = \frac{1}{2}|BC||AD| = \frac{1}{2} \times 6 \times d = 3d$

MASS RATIO	$\frac{3\sqrt{3}}{3\sqrt{3}}$	$\frac{3\sqrt{3}-3d}{3\sqrt{3}}$	$\frac{3d}{3\sqrt{3}}$
DISTANCE OF CENTRE OF MASS FROM M	$\frac{1}{3}d$	d	$\frac{1}{3}(3\sqrt{3})$

This $\frac{1}{3}d^2 + d(3\sqrt{3}-d) = 3\sqrt{3} \times \sqrt{3}$
 $\frac{1}{3}d^2 + 3\sqrt{3}d - d^2 = 9$
 $0 = \frac{2}{3}d^2 - 3\sqrt{3}d + 9$
 $0 = 2d^2 - 9\sqrt{3}d + 27$
 By quadratic formula
 $d = \frac{9\sqrt{3} \pm \sqrt{(9\sqrt{3})^2 - 4 \times 2 \times 27}}{2 \times 2} = \frac{9\sqrt{3} \pm \sqrt{243 - 216}}{4} = \frac{9\sqrt{3} \pm \sqrt{27}}{4}$
 $d = \frac{9\sqrt{3} \pm 3\sqrt{3}}{4} \rightarrow d = 3 \text{ or } 1.5\sqrt{3}$

Question 27 (****+)



The figure above shows a uniform lamina $OABCDE$ where all corners are right angles. The following lengths are marked in the figure in terms of suitable units

$$|AB|=6, \quad |BC|=a, \quad |ED|=a-2 \quad \text{and} \quad |OE|=9,$$

where a is a positive constant.

- a) Show that the position of the centre of mass of the lamina from OE is

$$\frac{9a^2 - 20a + 12}{10a - 12},$$

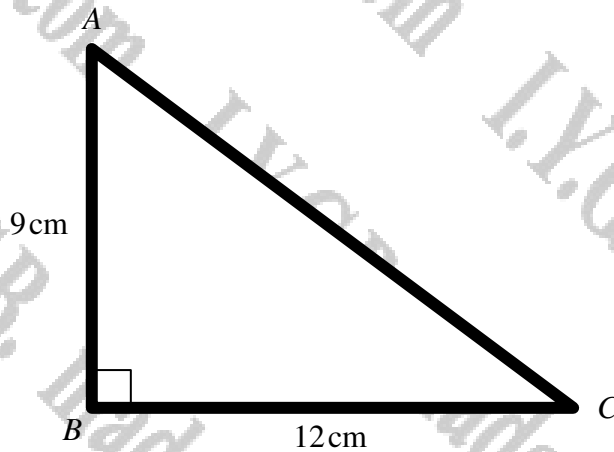
and find a similar expression for the position of the centre of mass of the lamina from OA .

The lamina is suspended freely through a smooth pivot at O and hangs in equilibrium under its own weight. The side OA lies at an angle of $\arctan \frac{5}{6}$ to the vertical.

- b) Show clearly that $a = 6$.

$$\frac{39a - 54}{10a - 12}$$

Question 28 (***)



The figure above shows a **framework** consisting of three small **uniform rods** AB , BC and AC .

It is further given that $|AB| = 9\text{ cm}$, $|BC| = 12\text{ cm}$ and $\angle ABC = 90^\circ$.

- b) Find the position of the centre of mass of the framework from AB and BC .

The framework is suspended freely through a smooth pivot at A and hangs in equilibrium under its own weight.

- c) Show clearly that the tangent of the angle that AC makes with the vertical is exactly $\frac{7}{24}$.

4.5 cm from AB , 3 cm from BC

(b)

• BY PYTHAGORAS
 $|AC| = 15$

ROD	AB	BC	AC	TOTAL
MASS	3	4	5	12
CM FROM B	4.5	6	7.5	
Σ	13.5	24	37.5	75

$(3 \times 4.5) + (4 \times 6) + (5 \times 7.5) = 75$
 $(3 \times 4.5) + (4 \times 6) + (5 \times 7.5) = 75 \Rightarrow$
 $13.5 + 24 + 37.5 = 75$
 $75 = 75$
 $\bar{x} = \frac{24}{12} = 2$
 $\bar{y} = \frac{37.5}{12} = 3.125$

$\therefore 4.5\text{ cm from } AB, 3\text{ cm from } BC$

(c)

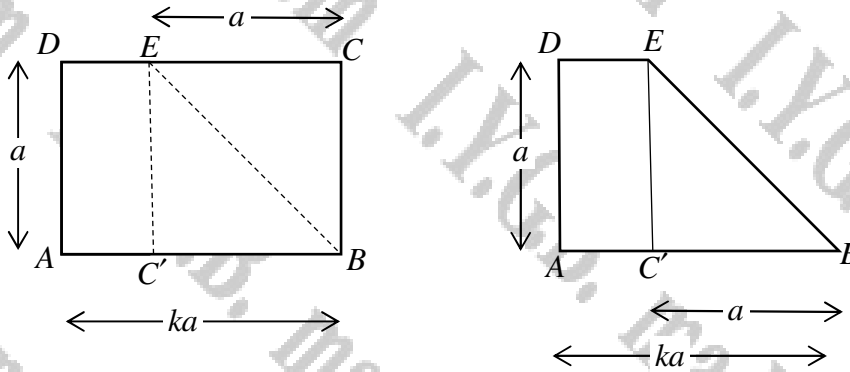
• $\tan \theta = \frac{y}{x} = \frac{3.125}{4.5} = \frac{25}{36} = \frac{5}{7.2}$
 • $\tan \phi = \frac{BC}{AB} = \frac{12}{9} = \frac{4}{3}$

Required angle is $\phi - \theta$

$\tan(\phi - \theta) = \frac{\tan \phi - \tan \theta}{1 + \tan \phi \tan \theta}$
 $\tan(\phi - \theta) = \frac{\frac{4}{3} - \frac{5}{7.2}}{1 + \frac{4}{3} \times \frac{5}{7.2}}$
 $\tan(\phi - \theta) = \frac{\frac{32}{72} - \frac{50}{72}}{1 + \frac{20}{27}}$
 $\tan(\phi - \theta) = \frac{\frac{-18}{72}}{\frac{47}{27}} = \frac{-18}{72} \times \frac{27}{47} = \frac{-18 \times 27}{72 \times 47} = \frac{-18 \times 3}{8 \times 47} = \frac{-54}{376} = \frac{-27}{188}$

Required

Question 29 (***)



A rectangular lamina $ABCD$ has $|AD|=a$ and $|AB|=ka$, where a and k are positive constants with $k > 1$. The point E lies on CD so that $|CE|=a$.

The lamina is folded over along EB so that the vertex C is now touching the point C' on AB , as shown in the figures above.

A set of cartesian coordinate axes is defined with origin at A , AB the direction of x increasing and AD the direction of y increasing.

Determine the coordinates of the centre of mass of the folded lamina, giving the answer in terms of a and k .

, $\bar{x} = \frac{(3k^2 - 1)a}{6k} \cap \bar{y} = \frac{(3k - 1)a}{6k}$

START WITH A DIAGRAM - TREAT THE FIGURES AS 'TABLE DENSITY'

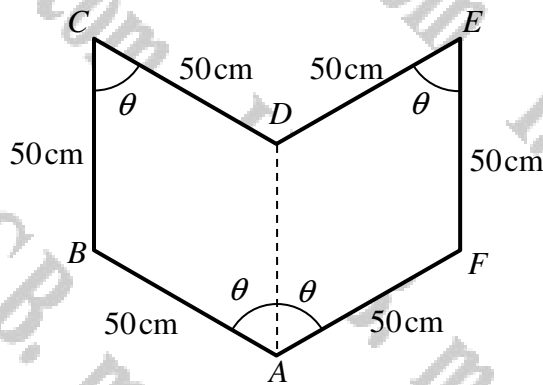
	RECTANGLE	TRIANGLE	COMPOSITE
MASS RATIO	$\frac{(k-1)a^2}{(k-1)a^2}$ $k-1$	$\frac{\frac{1}{2}a^2 \times 2}{a^2}$ 1	k
\bar{x}	$\frac{1}{2}(k-1)a$	$(k-1)a + \frac{1}{3}a$	\bar{x}
\bar{y}	$\frac{1}{2}a$	$\frac{2}{3}a$	\bar{y}

FINDING & SOLVING EQUATIONS

- $k\bar{x} = \frac{1}{2}(k-1)a + 1 \times [(k-1)a + \frac{1}{3}a]$
 $\Rightarrow k\bar{x} = \frac{1}{2}(k-1)a + (k-1)a + \frac{1}{3}a$
 $\Rightarrow k\bar{x} = \frac{3}{2}(k-1)a + \frac{1}{3}a$
 $\Rightarrow k\bar{x} = a \left[\frac{3(k-1)}{2} + \frac{1}{3} \right]$
 $\Rightarrow k\bar{x} = \frac{(3k^2 - 1)a}{6}$
 $\Rightarrow \bar{x} = \frac{(3k^2 - 1)a}{6k}$
- $k\bar{y} = \frac{1}{2}(k-1)a + 1 \times \frac{2}{3}a$
 $\Rightarrow k\bar{y} = \frac{1}{2}(k-1)a + \frac{2}{3}a$
 $\Rightarrow k\bar{y} = \frac{(3k - 3 + 4)a}{6}$
 $\Rightarrow k\bar{y} = \frac{(3k - 1)a}{6}$
 $\Rightarrow \bar{y} = \frac{(3k - 1)a}{6k}$

$\therefore \left[\frac{(3k^2 - 1)a}{6k}, \frac{(3k - 1)a}{6k} \right]$

Question 30 (****+)



The figure above shows a uniform plane lamina $ABCDEF$, made of two congruent rhombuses, each of side length 50 cm.

It is given that $\angle BAD = \angle DAF = \angle BCD = \angle DFE = \theta$.

- a) Given further that the centre of mass of the lamina is 50 cm from A , show that $\cos \theta = \frac{4}{5}$.

The weight of the lamina is W . A particle of weight kW , where k is a positive constant is fixed to the lamina at A . Another particle of weight $\frac{1}{5}W$ is fixed to the lamina at C .

The lamina is freely suspended from F and hangs in equilibrium with AD horizontal.

- b) Find the value of k .

$k = \frac{3}{8}$

CONSIDER ONE OF THE RHOMBUSES (ABC), AND FURTHER SPLIT INTO 2 TRIANGLES WITH RESPECTIVE POSITIONS OF CENTRE OF MASS G_1, G_2 (SECOND TRIANGLE)

$|AG_1| = 50 \cos \theta$
 $|AG_2| = \frac{1}{2} (50 \cos \theta) = \frac{1}{2} \times 50 \cos \theta$
 $(AG) = \frac{1}{3} (|AG_1| + 2|AG_2|) = \frac{1}{3} (50 \cos \theta + 50 \cos \theta)$
 $= \frac{2}{3} \times 50 \cos \theta$

WE NOW KNOW BOTH REQUIREMENTS

HEIGHT RATIO	VERTICAL DISTANCE OF THE CENTRE OF MASS FROM A (OR LINE BEHIND AD)
1	$ AG_1 \cos \theta$
1	$ AG_2 \cos \theta$
2	$ AG $

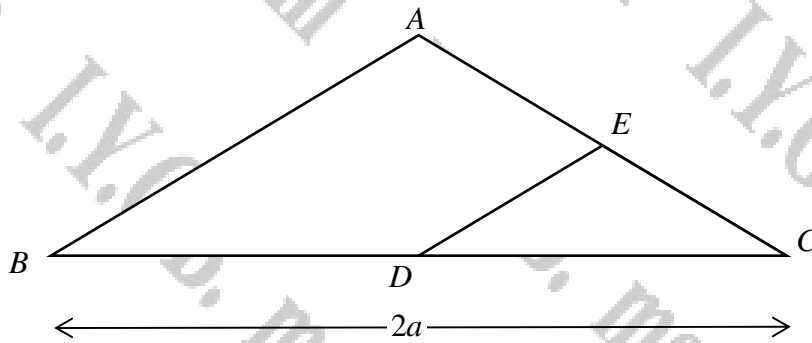
THIS $|AG| \cos \theta + |AG_2| \cos \theta = 2|AG|$
 $\Rightarrow \frac{1}{3} \times 50 \cos \theta \times \cos \theta + \frac{1}{3} \times 50 \cos \theta \times \cos \theta = 2 \times \frac{2}{3} \times 50 \cos \theta$
 $\Rightarrow 100 \cos^2 \theta = 200 \cos \theta$
 $\Rightarrow \cos^2 \theta = 2 \cos \theta$
 $\Rightarrow \frac{1}{2} + \frac{1}{2} \cos \theta = 2$
 $\Rightarrow \frac{1}{2} \cos \theta = \frac{3}{2}$
 $\Rightarrow \cos \theta = \frac{4}{5}$

\Rightarrow DOUBLE CHECK TRIANGLE ARE CORRECT

TAKING MOMENTS ABOUT F

$kW \times 50 \cos \theta = W \times (45 - 50 \cos \theta) + \frac{1}{5} W \times 50$
 $k \times 50 \times \frac{4}{5} = 45W - 50W \times \frac{4}{5} + 10W$
 $40k = 5 + 10$
 $40k = 15$
 $k = \frac{3}{8}$

Question 31 (****+)



The figure above shows a lamina ABC is in the shape of an isosceles triangle, with $|AB| = |AC|$ and $|BC| = 2a$, where a is a positive constant.

The point D is the midpoint of BC . The point E lies on AC so that $|EC| = |ED|$.

The section of the lamina defined by the triangle CDE is made of a material which is **twice as dense** as the material the rest of the lamina is made of.

A set of cartesian coordinate axes is defined with origin at D , DC the direction of x increasing and DA the direction of y increasing.

- a) If $|AD| = a$ determine the coordinates of the centre of mass of the lamina ABC , giving the answer in terms of a .

The lamina ABC is freely suspended from A and hangs in equilibrium.

- b) Show that AC is inclined at $\arctan \frac{3}{4}$ to the downward vertical.

$$\boxed{}, \quad \boxed{G\left(\frac{1}{10}a, \frac{3}{10}a\right)}$$

[solution overleaf]

a) STARTING WITH A DIAPHRAM - NOTE THAT $\angle B \hat{=} \angle C$ AND $\angle B \hat{=} \angle C$ ARE 40° EACH

FORMING A STINECOX TABLE - TREAT THE JOINTS INSTEAD AS ANOTHER JOINT

	TOWARDS ABC	TOWARDS EDC	COMPOSITE
MASS RATIO	$\frac{1}{2} \times 2$ a^2 4	$\frac{1}{2} \times 10 \times 1$ $\frac{1}{2} a^2$ 1	
α	0	$\frac{1}{2} a$	$\frac{1}{2}$
y	$\frac{1}{2} a$	$\frac{1}{2} \times 10$	$\frac{1}{2}$

FORMING 2 SCALING EQUATIONS

- $5S = (200) + 10 \times \frac{1}{2}$
- $5\alpha = \frac{1}{2} a$
- $5\gamma = -a$

- $5S = 4x \times a + 10 \times \frac{1}{2} a$
- $\frac{5\alpha}{5} = \frac{4}{5} a + \frac{1}{5} a$
- $\frac{5\gamma}{5} = \frac{3}{5} a$
- $5\gamma = \frac{3}{5} a$
- $\therefore \begin{pmatrix} -a & \frac{3}{5} a \end{pmatrix}$

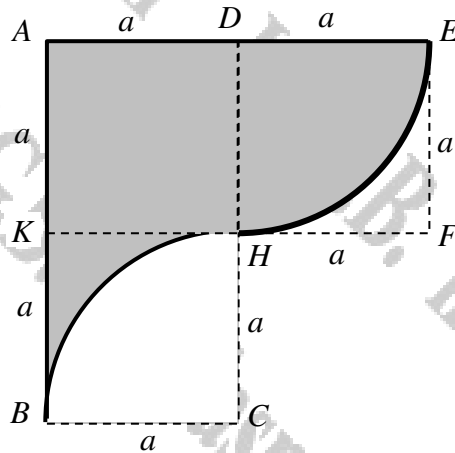
b) EXPLODING THE DIAPHRAM

$\tan \theta = \frac{10}{10}$
 $\tan \theta = \frac{10}{10}$
 $\tan \theta = \frac{10}{10}$
 $\tan \theta = \frac{10}{10}$
 $\tan \theta = \frac{10}{10}$

THE REQUIRED ANGLE IS θ

- $\Rightarrow \theta = 45^\circ$
- $\Rightarrow \tan \theta = \tan(45^\circ)$
- $\Rightarrow \tan \theta = \frac{\tan \theta + \tan \theta}{1 + \tan \theta \tan \theta}$
- $\Rightarrow \tan \theta = \frac{1 + \frac{1}{\tan \theta}}{1 + 1 \times \frac{1}{\tan \theta}}$
- $\Rightarrow \tan \theta = \frac{1 + \frac{1}{\tan \theta}}{1 + \frac{1}{\tan \theta}}$
- $\Rightarrow \tan \theta = \frac{2}{1 + \frac{1}{\tan \theta}}$
- $\Rightarrow \tan \theta = \frac{2}{\frac{\tan \theta + 1}{\tan \theta}}$
- $\Rightarrow \tan \theta = 2$
- $\Rightarrow \theta = \arctan(2)$

Question 32 (****+)



The figure above shows a uniform rectangular lamina $ABCD$, where $|AD| = |BC| = a$ and $|AB| = |DC| = 2a$. The midpoints of AB and DC are K and H , respectively.

A quarter circle of radius a and centred at C is removed from the rectangle $ABCD$.

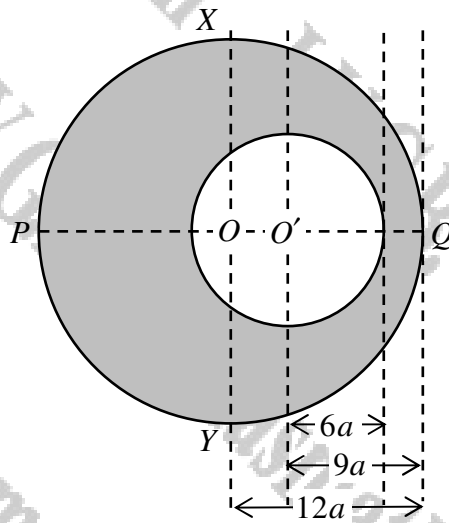
The quarter circle then attached with its centre at D and one of its straight edges along DH as shown in the figure.

Determine the position of the centre of mass of the resulting shape from AB .

, $\bar{x} = \frac{5}{6}a$

Handwritten solution showing the calculation of the center of mass for the composite shape. The student uses the method of composite shapes, calculating the area and center of mass for the rectangle and the quarter circle, and then combining them to find the overall center of mass. The final result is $\bar{x} = \frac{5}{6}a$.

Question 33 (*****)



A composite uniform lamina is modelled by the finite region bounded by two circular discs, shown shaded in the figure above. The details, of the sizes and relative positions of these discs, are as follows.

The straight line POQ is a diameter of the larger circular disc, of radius $12a$, whose centre is at the point O . The smaller circular disc, of radius $6a$, has its centre at O' , so that O' lies on OQ with $|O'Q| = 9a$.

A heavy particle is attached to the lamina at Q .

The straight line XOY is perpendicular to POQ .

When the lamina is freely suspended from X and hangs in equilibrium, with P higher than Q , POQ is inclined at $\arctan \frac{5}{12}$ to the horizontal.

Determine the ratio of the mass of the particle to the mass of the lamina.

,

[solution overleaf]

SENT BY OBTAINING THE PERIOD OF THE CIRCLE OF ARCS OF THE CANTINA WITHIN THE GREAT CIRCLE

OBJECT	COMPOSITE	HEIGHT	WID. DIST.
MASS RATIO	3	$2a\sqrt{1-x^2/a^2}$	$2a\sqrt{1-x^2/a^2}$
2. COORDINATE OF THE CENTER OF MASS	x	$2a$	0

$\Rightarrow 3x + 3a = 0$
 $\Rightarrow x = -a$ ← TO THE LEFT OF O

HOW TO OBTAIN AT MAXIMUM IN THE FOLLOWING DIAGRAM

- LET THE MASS OF THE CANTINA BE IN A UNIT OF THE PARTICLE BE
- MADE HEIGHTS
- EQUATION AT X WILL BE $(x+1)mg$
- COS IS SINCE ANGLE IS VERTICAL SO BECOME 12.624 (COS ANGLE)
- $\cos 12.624 = \frac{a}{2a}$ (SINCE COS OF MASS)
- $\cos 12.624 = \frac{1}{2}$ (SINCE COS OF MASS)
- $\cos 12.624 = \frac{1}{2}$ (SINCE COS OF MASS)
- $\cos 12.624 = \frac{1}{2}$ (SINCE COS OF MASS)

TAKING MOMENTS ABOUT O
 $\Rightarrow (x+1)mg \times a = (1+1)mg \times a \cos 12.624$
 $\Rightarrow (x+1) = 2 \cos 12.624$
 $\Rightarrow (x+1) = 2 \times \frac{1}{2}$
 $\Rightarrow x+1 = 1$
 $\Rightarrow x = 0$

PARTICLE : CANTINA
 $\frac{1}{2}m$: m
 $6m$: $7m$
 6 : 7