SERIES
79 EXAM QUESTIONS
SUMMATIONS
BY FORMULAS
17 BASIC QUESTIONS
Question 1 (***)

Use standard results on summations to find the value of

\[ \sum_{r=36}^{48} [(r-1)(3r-2)]. \]

\[ 66638 \]
Question 2 (**)

Use standard results on summations to show that

\[
\sum_{r=1}^{n} r(r+1)(r+5) = \frac{1}{4}n(n+a)(n+b)(n+c),
\]

where \(a\), \(b\), and \(c\) are positive integers to be found.

\[
\begin{align*}
&= a = 1, b = 2, c = 7
\end{align*}
\]
Question 3 (***)

Use standard results on summations to show that

\[ \sum_{r=1}^{n} \left[ r^2 (r-1) \right] = \frac{1}{12} n(n-1)(n+1)(3n+2) + m, \]

where \( m \) is an integer to be found.

\[ m = -22 \]
Question 4  (**

Use standard results on summations to show that

\[ \sum_{r=1}^{n} [r^3 (r+1)(r-1)] = \frac{1}{6} n^2 (n+1)^2 (n-1)(n+2). \]

You may assume without proof that

\[ \sum_{r=1}^{n} r^5 = \frac{1}{12} n^2 (n+1)^2 \left(2n^2 + 2n - 1\right) \]
Question 5 (**)

\[ F(r) = \sum_{n=1}^{r} [n(n-1)(n+2)]. \]

Use standard results on summations express \( F(n) \) in fully factorized form.

\[ F(r) = \frac{1}{12} r(r+1)(r-1)(3r+10) \]

Question 6 (**+)

Find, in fully simplified factorized form, an expression for the sum of the first \( n \) terms of the following series.

\[ (5 \times 3) + (11 \times 7) + (17 \times 11) + (23 \times 15) + \ldots \]

\[ n^2(8n + 7) \]
Question 7 (***+)

Show by using standard summation results that …

a) \[ \sum_{r=1}^{n} (r+1)(r+5) = \frac{1}{6} n(n+7)(2n+7). \]

b) \[ \sum_{r=1}^{40} (r+1)(r+5) = 26495. \]
Question 8 (***)

Show by using standard summation results that …

a) \[ \sum_{k=1}^{n} (k^2 - k - 1) = \frac{1}{3} n (n + 2) (n - 2). \]

b) \[ \sum_{k=10}^{40} (k^2 - k - 1) = 21049. \]
Question 9 (**+)**
Find, in fully factorized form, an expression for the sum
\[
\sum_{p=1}^{k} (p^3 + p^2) = \frac{1}{12} k(k+1)(k+2)(3k+1)
\]

Question 10 (**+)**
Find, in fully factorized form, an expression for the sum
\[
\sum_{r=1}^{2n} (3r^2 - \frac{1}{2}) = 2n^2 (4n+3)
\]
Question 11  (***)

Use standard results on summations to show that

\[
\sum_{r=1}^{n-2} (r+1)^2 = \frac{1}{12} n(n-1)(n-2)(3n-1).
\]
Question 12 (***)

It is given that

$$\sum_{r=1}^{n} [(3r+a)(r+2)] = n(n+2)(n+b).$$

Determine the values of each of the constants $a$ and $b$.

$$a = 1, \quad b = 3$$
Question 13  (***)

Show clearly that

$$(1 \times 3) + (2 \times 4) + (3 \times 5) + \ldots + (n-5)(n-3) = \frac{1}{6}(n+6)(2n+11)(n+5).$$

proof

Question 14  (***)

Use standard results on summations to show that

$$\sum_{r=1}^{n} (3r^2 + r - 1) \equiv n^2(n+2).$$

proof
Question 15  (***)
Use standard results on summations to show that
\[ \sum_{n=1}^{k} (18n^2 + 28n + 5) = k(k+2)(6k+11). \]

proof

Question 16  (***)
Use standard results on summations to find the value of the following sum.
\[ \sum_{k=2}^{16} [(k-1)(k+2)]. \]

, 1600
Question 17 (***)

Use standard results on summations to show that

\[ \sum_{r=1}^{2n} r^3 - \sum_{r=1}^{n} (6r-3)^2 = f(n), \]

where \( f(n) \) is written as a product of 4 linear factors.

\[ f(n) = n(n-1)(2n+1)(2n-3) \]
SUMMATIONS
BY FORMULAS
15 STANDARD QUESTIONS
Question 1 (***)

Find the sum of the first $n$ terms of the series

$$1\cdot 2\cdot 3 + 2\cdot 3\cdot 5 + 3\cdot 4\cdot 7 + 4\cdot 5\cdot 9 + \ldots$$

Express the answer as a product of linear factors.
Question 2  (***)

By using standard results, show that

\[
\sum_{r=n+1}^{4n} (2r-1)^2 \equiv n(84n^2-1).
\]
Question 3 \((***+)\)

Determine the value of \(a\) and the value of \(b\) given that

\[
\sum_{r=1}^{n} r(r+a)(r+b) = \frac{1}{12} n(n+1)(n+2)(3n+17).
\]

\[a = 1, \quad b = 4 \quad \text{or the other way round}\]
Question 4 (***+)

Find, in fully factorized form, an expression for the following sum.

\[ \sum_{r=n}^{2n} (r^3 - 2r). \]

\[ \sum_{r=n}^{2n} (r^3 - 2r) = \frac{3}{4} n (5n - 4)(n + 1)^2 \]
Question 5  (***)

It is thought that for some values of the constants \( p \) and \( q \) that

\[
\sum_{r=1}^{n} r^2(r+p) \equiv n(n+1)(n+2)(3n+q).
\]

Use a detailed method to show that there exist no such values of \( p \) and \( q \).
Question 6  (****)

Use standard results on summations to solve the following equation.

\[ \sum_{r=1}^{k} (r^3 - 1) = 89976. \]

\[ k = 24 \]
Question 7  (***)

It is given that

\[ \sum_{r=1}^{n} (Ar^3 + Br^2 + Cr) = n(n+1)(n+2)(4n-5). \]

Use a detailed method to find the value of each of the integer constants, \( A \), \( B \) and \( C \).

\[ \boxed{A = 16, \quad B = -3, \quad C = -19} \]
Question 8  (****)

Show by a detailed method that

$$\sum_{r=0}^{n} \left[ 2r(2r^2 - 3r - 1) + n + 1 \right] = (n^2 - 1)^2.$$
Question 9  (****)

The sum, \( S_n \), of the first \( n \) terms of a series whose general term is denoted by \( u_n \) is given by the following expression.

\[
S_n = n^2(n+1)(n+2).
\]

a) Find the first term of the series.

b) Show clearly that …

i. \( u_n = n(n+1)(4n-1) \)

ii. \[
\sum_{r=n+1}^{2n} u_r = 3n^2(n+1)(5n+2).
\]
Question 10 (****)

Use standard summation results to prove that

\[ \sum_{r=1}^{n} (n-r)^2 = \frac{1}{2} n(n-1)(2n-1). \]
Question 11 (****)

Use standard results on summations to solve the following equation

\[
\sum_{r=3}^{9} \left( \frac{r^3}{k} + (r-1)(r+1) \right) = 304.5.
\]

\[ k = 4 \]
Question 12 (****)

\[ \sum_{r=1}^{n} (ar^2 + br + c) = n^3 + 5n^2 + 6n, \]

where \( a \), \( b \) and \( c \) are integer constants.

Determine the value of \( a \), \( b \) and \( c \).

\[ a = 3 \quad b = 7 \quad c = 2 \]
Question 13  (****)

The variance $\text{Var}(n)$ of the first $n$ natural numbers is given by

$$\text{Var}(n) = \frac{1}{n} \sum_{r=1}^{n} r^2 - \left[ \frac{1}{n} \sum_{r=1}^{n} r \right]^2.$$ 

Determine a simplified expression for $\text{Var}(n)$ and hence evaluate $\text{Var}(61)$.

$$\text{Var}(n) = \frac{1}{12} (n^2 - 1) \quad \text{Var}(61) = 310$$
Question 14 \( (****) \)

\[ \sum_{r=1}^{n} \left[ r^3 - r \right], \quad n \in \mathbb{N}. \]

a) Use standard summation results to find a fully factorized expression for \( f(n) \).

b) Hence solve the equation

\[ \sum_{r=5}^{10} \left[ r^3 - r + 6k \right] - \sum_{r=1}^{12} \left[ r^2 + k^2 \right] = 70, \quad f(n) = \frac{1}{4} n(n-1)(n+1)(n+2), \quad k = -12, \quad k = 15 \]
Question 15  (****)

The function $F(n)$ is defined as

$$F(n) = \sum_{r=1}^{n} \left[ r(r-1)(n-2)(r+1) \right] \quad n \in \mathbb{N}.$$ 

Show with detailed workings that

$$F(2n) - F(n) = \frac{1}{2} n \left( n^2 - 1 \right) \left( 3n^2 - 4 \right).$$

proof
SUMMATIONS
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5 HARD QUESTIONS
Question 1  (***)

It is given that

\[ \sum_{r=1}^{20} (r-10) = 200 \quad \text{and} \quad \sum_{r=1}^{20} (r-10)^2 = 2800. \]

Find the value of

\[ \sum_{r=1}^{20} r^2. \]
Question 2  (****+)

\[ \sum_{r=1}^{n} (r+a)(r+b) \equiv \frac{1}{3} n(n-1)(n+4), \]

where \( a \) and \( b \) are integer constants.

Use a clear algebraic method to determine the value of \( a \) and the value of \( b \).

[1], 2 and –1 (in any order)
Question 3 (****+)

By using an algebraic method, find the value of

\[ 99^2 - 97^2 + 95^2 - 93^2 + \ldots + 3^2 - 1^2 \]

Answer:

\[ 5000 \]
Question 4  (***)

Show clearly that

\[ 1^3 - 2^3 + 3^3 - 4^3 + \ldots - 40^3 = -33200. \]
Question 5  (****+)

The positive integer functions \( f \) and \( g \) are defined as

\[
f(n) = \sum_{r=1}^{n} r^3 \quad \text{and} \quad g(n) = 1 + \sum_{r=1}^{n} (2r + 1).
\]

Evaluate

\[
\sum_{n=1}^{39} \left[ \frac{f(n)}{g(n)} \right].
\]
SUMMATIONS
BY FORMULAS
8 ENRICHMENT QUESTIONS
Question 1  (*****)
Use standard summation results to prove that

$$\sum_{r=n}^{2n} (n-r)^2 = \sum_{r=1}^{n} r^2.$$
Question 2

Find the sum of the first 16 terms of the following series.

\[
\frac{1^3}{1 + 3} + \frac{1^3 + 2^3}{1 + 3 + 5} + \frac{1^3 + 2^3 + 3^3}{1 + 3 + 5 + 7} + \cdots
\]

\[\boxed{\text{Answer}: 446}\]
The function \( f \) is defined for \( n \in \mathbb{N} \) as

\[
f(n) = 1 \times n^2 + 2(n-1)^2 + 3(n-2)^2 + 4(n-3)^2 + \ldots + (n-1)2^2 + n1^2.
\]

Determine a simplified expression for the sum of \( f(n) \), giving the final answer in fully factorized form.

\[
f(n) = \frac{1}{12}n(n+2)(n+1)^2
\]
Question 4 (*****)

Use an algebraic method justifying each step, to find the greatest value of $k$, $k \in \mathbb{N}$, which satisfies the following inequality.

$$\sum_{r=k+1}^{80} \left[ \frac{r-1}{\log_b(16)} \right] > 100 \,000.$$
Question 5

Use algebra to find the sum of the first 100 terms of the following sequence.

7, 12, 19, 28, 39, 52, 67, 84, 103, ...

\[ f(n) = \frac{1}{12}n(n+2)(n+1)^2 \]
Question 6  (*****)
Evaluate the following expression

\[ \sum_{n=1}^{9} \sum_{m=n+1}^{2n} [2m + n]. \]

Detailed workings must be shown.
Question 7 (***)

Evaluate the following expression

\[ \sum_{m=1}^{19} \sum_{n=m}^{19} [2m + n]. \]

You may find reversing the order of summation useful in this question.
Question 8  (***)

The function \( f \) is defined as

\[
f(n, y) = \sum_{x=1}^{n} \frac{x^2 y^x}{k}, \quad n \in \mathbb{N}, \ y \in \mathbb{R}
\]

where \( k = \sum_{r=1}^{n} r^2 \).

Use standard results on series to show that

\[
\frac{d^2 f}{dy^2} \bigg|_{y=1} + \frac{df}{dy} \bigg|_{y=1} - \left[ \frac{df}{dy} \bigg|_{y=1} \right]^2 = \frac{3n^4 + 6n^3 - n^2 - 4n - 4}{20(2n+1)^2}
\]

You may assume without proof \( \sum_{r=1}^{n} r^4 = \frac{1}{30} n(n+1) \left( 6n^3 + 9n^2 + n - 1 \right) \).
SUMMATIONS

METHOD OF DIFFERENCES

8 BASIC QUESTIONS
Question 1 (***)

\[ f(r) = \frac{5}{(5r-1)(5r+4)}, \quad r \in \mathbb{N} \]

a) Express \( f(r) \) into partial fractions

b) Hence show that

\[
\sum_{r=1}^{n} f(r) = \frac{5n}{4(5n+4)}. \]

\[
f(r) = \frac{1}{5r-1} - \frac{1}{5r+4}\]
Question 2 \((**)\)

a) Show carefully that
\[
\frac{1}{r^2} - \frac{1}{(r+1)^2} = \frac{2r+1}{r^2(r+1)^2}.
\]

b) Hence use the method of differences to find
\[
\sum_{r=1}^{n} \frac{2r+1}{r^2(r+1)^2}.
\]
Question 3 (**) 

a) Show carefully that

\[ \frac{1}{r!} - \frac{1}{(r+1)!} = \frac{r}{(r+1)!}. \]

b) Hence find

\[ \sum_{r=1}^{n} \frac{r}{(r+1)!}. \]
Question 4 (***)

\[ f(r) = \frac{1}{r(r+2)}, \quad r \in \mathbb{N} \]

a) Express \( f(r) \) into partial fractions.

b) Hence show that

\[ \sum_{r=1}^{30} f(r) = \frac{1425}{1984}. \]

\[
\begin{align*}
\frac{1}{2} & - \frac{1}{2(r+2)} \\
\end{align*}
\]
Question 5  (***)

\[ f(r) = \frac{2}{(r+1)(r+3)}, \quad r \in \mathbb{N} \]

a) Express \( f(r) \) into partial fractions

b) Use the method of differences to find

\[ \sum_{r=1}^{n} f(r). \]

c) Hence evaluate

\[ \sum_{r=8}^{\infty} f(r). \]

\[ f(r) = \frac{1}{r+1} - \frac{1}{r+3} \]

\[ \sum_{r=1}^{n} f(r) = \frac{5}{6} - \frac{1}{n+2} - \frac{1}{n+3} \]

\[ \sum_{r=8}^{\infty} f(r) = \frac{19}{90} \]
Question 6 (***)

a) Simplify \( \frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)} \) into a single fraction.

b) Hence show that

\[
\sum_{r=1}^{20} \left[ \frac{1}{r(r+1)(r+2)} \right] = \frac{115}{462}.
\]
Question 7 (***)

\[ f(r) \equiv r^2 (r+1)^2 - (r-1)^2 r^2, \quad r \in \mathbb{N}. \]

a) Simplify \( f(r) \) as far as possible.

b) Use the method of differences to show that

\[
\sum_{r=1}^{20} r^3 = 44100.
\]

\[ f(r) = 4r^3 \]
Question 8  (***)

\[ f(r) = \frac{1}{r(r+2)}, \quad r \in \mathbb{N}. \]

a) Express \( f(r) \) in partial fractions.

b) Hence prove, by the method of differences, that

\[
\sum_{r=1}^{n} f(r) = \frac{n(An+B)}{4(n+1)(n+2)},
\]

where \( A \) and \( B \) are constants to be found.

\[ A = 3, \quad B = 5 \]
SUMMATIONS

METHOD OF DIFFERENCES

8 STANDARD QUESTIONS
Question 1 (***)

Use the method of differences to show that

\[
\frac{1}{1 \times 2 \times 3} + \frac{1}{2 \times 3 \times 4} + \frac{1}{3 \times 4 \times 5} + \ldots + \frac{1}{n(n+1)(n+2)} = \frac{n(n+3)}{4(n+1)(n+2)}.
\]
Question 2 \((***)\)

\[ u_r = \frac{1}{6} r(r+1)(4r+11), \quad r \in \mathbb{N}. \]

a) Simplify \(u_r - u_{r-1}\) as far as possible.

b) By using the method of differences, or otherwise, find the sum of the first 100 terms of the following series.

\[ (1 \times 5) + (2 \times 7) + (3 \times 9) + (4 \times 11) + \ldots \]

\[ \boxed{r(2r+3)} \rightarrow 691850 \]
Question 3 (***+)

\[ f(r) = \frac{1}{(r+1)(r-1)}, \quad r \in \mathbb{N}. \]

a) Express \( f(r) \) into partial fractions.

b) Hence show that

\[ \sum_{r=2}^{n} \frac{1}{r^2 - 1} = \frac{3}{4} - \frac{1}{2n} - \frac{1}{2(n+1)}. \]

c) State the value of

\[ \sum_{r=2}^{\infty} \frac{1}{r^2 - 1}. \]
Question 4  (***)

\[ f(r) = \frac{2}{r(r+1)(r+2)}, \quad r \in \mathbb{N}. \]

a) Express \( f(r) \) into partial fractions.

b) Hence show that

\[ \sum_{r=1}^{n} f(r) = \frac{1}{2} - \frac{1}{(n+1)(n+2)}. \]

c) Find the value of the convergent infinite sum

\[ \frac{1}{5 \times 6 \times 7} + \frac{1}{6 \times 7 \times 8} + \frac{1}{7 \times 8 \times 9} + \ldots \]

\[
\left[ f(r) = \frac{1}{r} - \frac{2}{r+1} + \frac{1}{r+2} \right] \quad \frac{1}{60}
\]
Question 5 (***)

Use the method of differences to show that

$$
\frac{1}{1\times2\times3} + \frac{4}{2\times3\times4} + \frac{7}{3\times4\times5} + \ldots + \frac{3n-2}{n(n+1)(n+2)} = \frac{n^2}{(n+1)(n+2)}
$$
Question 6 (***)+

It is given that

\[
\frac{2k + 7}{(2k + 1)(2k + 3)(2k + 5)} \equiv \frac{3}{4(2k + 1)} - \frac{1}{4(2k + 3)} + \frac{1}{4(2k + 5)}.
\]

Use the method of differences to find a simplified expression for

\[
\frac{7}{1 \times 3 \times 5} \quad + \quad \frac{9}{3 \times 5 \times 7} \quad + \quad \frac{11}{5 \times 7 \times 9} \quad + \quad \ldots \quad + \quad \frac{2n + 7}{(2n + 1)(2n + 3)(2n + 5)}.
\]

Give your answer in the form \( \frac{2}{3} - f(n) \), where \( f(n) \) is a single simplified fraction.

\[
f(n) = -\frac{n + 3}{(2n + 3)(2n + 5)}
\]
Question 7  (***)

Use the method of differences to find a simplified expression for the first \( n \) terms of the following series.

\[
\frac{1}{1 \times 3} + \frac{2}{3 \times 5} + \frac{3}{5 \times 7} + \frac{4}{7 \times 9} + \ldots
\]

Give your answer in the form \( \frac{1}{4} - f(n) \), where \( f(n) \) is a single simplified fraction.

\[
f(n) = \frac{(-1)^n}{4(2n+1)}
\]
Question 8  (***)

\[ f(r) = \frac{1}{\sqrt{r+2} + \sqrt{r}}, \quad r \geq 0. \]

a) Rationalize the denominator of \( f(r) \).

b) Find an expression for

\[ \sum_{r=1}^{n} f(r). \]

c) Show clearly that

\[ \sum_{r=1}^{48} f(r) = 3 + 2\sqrt{2} \]

\[ f(r) = \frac{\sqrt{r+2} - \sqrt{r}}{2}, \quad \sum_{r=1}^{n} f(r) = \frac{1}{2} \left( \sqrt{n+2} + \sqrt{n+1} - \sqrt{2} - 1 \right) \]
SUMMATIONS

METHOD OF DIFFERENCES

3 HARD QUESTIONS
Question 1  (***)

Consider the following infinite convergent series.

\[ \frac{3}{1 \times 2} - \frac{5}{2 \times 3} + \frac{7}{3 \times 4} - \frac{9}{4 \times 5} + \frac{11}{5 \times 6} - \ldots \]

a) Use the method of differences, to find the sum of this series.

b) Verify the answer of part (a) by using a method based on the Maclaurin expansion of \( \ln(1 + x) \).

\[ \frac{3}{1 \times 2} - \frac{5}{2 \times 3} + \frac{7}{3 \times 4} - \frac{9}{4 \times 5} + \frac{11}{5 \times 6} - \ldots \]
Question 2 \( (**+**) \)

Use partial fractions to sum the following series.

\[
\sum_{n=1}^{\infty} \frac{2n+1}{n^4 + 2n^3 + n^2}.
\]

You may assume that the series converges.
Question 3  (***/***+)

It is given that

\[ f(r) = \frac{6r^3 + 6r^2 - ar^2 - ar + 1}{r(r+1)}, \quad r \in \mathbb{N}, \]

where \( a \) is a non zero constant.

It is further given that

\[ \sum_{r=1}^{n} f(r) = \frac{n^2(n+2)(2n+1)}{n+1}. \]

Determine the value of \( a \).

\[ \boxed{a = 2} \]
SUMMATIONS

METHOD OF DIFFERENCES

15 ENRICHMENT QUESTIONS
Question 1  (*****)

Determine the exact value of the following sum.

\[
\sum_{n=2}^{20} \left[ \frac{n^3 - n^2 + 1}{n^2 - n} \right].
\]
Question 2  

\[ f(x,n) = \sum_{r=1}^{n} \frac{1}{(x-1)^r}, \quad x \in \mathbb{R}, n \in \mathbb{N}. \]

By observing the simplification of

\[ \frac{1}{(x-2)(x-1)^r} - \frac{1}{(x-2)(x-1)^{r+1}} \]

find a simplified expression for \( f(x,n) \).

\[ f(x,n) = \frac{1}{x-2} \frac{1}{(x-2)(x-1)^n} \]
Question 3

Determine, in terms of $k$ and $n$, a simplified expression for

$$
\sum_{r=2}^{n} \left[ \frac{r(1-k)^{-1}}{r(r-1)k^r} \right].
$$

$$
\frac{1}{n} \left( \frac{1}{k} \right)^n - \frac{1}{k}
$$
Question 4  (*****)

Determine the value of the following infinite convergent sum.

\[
\sum_{r=2}^{\infty} \frac{4r - 1}{r(r-1)} \left( -\frac{1}{3} \right)^r.
\]
Question 5  (*****)

Determine a simplified expression, in the form \( \ln (f(n)) \), for the following sum.

\[
\sum_{r=2}^{N} \left[ \int_{2}^{r} \frac{2}{x^2-1} \, dx \right].
\]

\[
\ln \frac{2 \times 3^{N-1}}{N(N+1)}.
\]
Question 6 (*****)

Show, by a detailed method, that

\[
\frac{48}{2 \times 3} + \frac{47}{3 \times 4} + \frac{46}{4 \times 5} + \ldots + \frac{2}{48 \times 49} + \frac{1}{49 \times 50} = A + B \sum_{r=1}^{50} \frac{1}{r},
\]

where \( A \) and \( B \) are constants to be found.

\[ A = \frac{51}{2}, \quad B = -1 \]
Question 7  (*****)

\[
\frac{3}{1^2} + \frac{5}{1^2 + 2^2} + \frac{7}{1^2 + 2^2 + 3^2} + \frac{9}{1^2 + 2^2 + 3^2 + 4^2} + \frac{11}{1^2 + 2^2 + 3^2 + 4^2 + 5^2} + \ldots
\]

Show, by a detailed method, that the sum of the first 40 terms of this series shown above is \(\frac{240}{41}\).

\[\text{proof}\]
Question 8  (*****)

By considering the simplification of

$$\arctan(2n+1) - \arctan(2n-1),$$

determine the exact value of

$$\sum_{n=1}^{\infty} \left[ \arctan \left( \frac{1}{2n^2} \right) \right].$$

$$\frac{\pi}{4}.$$
Question 9  

\[ S_n = (2 \times 1!) + (5 \times 2!) + (10 \times 3!) + (17 \times 4!) + \ldots + (n^2 + 1)n! \]

Use an appropriate method to show that

\[ S_n = n(n+1)! \]
Question 10  (*****)

By considering the trigonometric identity for $\tan(A - B)$, with $A = \arctan(n + 1)$ and $B = \arctan(n)$, sum the following series

$$\sum_{n=1}^{\infty} \arctan\left(\frac{1}{n^2 + n + 1}\right).$$

You may assume the series converges.

\[
\begin{array}{c}
\text{__}, \quad \text{__}, \quad \frac{\pi}{4} \\
\end{array}
\]
Question 11 (*****)

Determine, in terms of \( n \), a simplified expression

\[
\sum_{r=1}^{n} \left[ \frac{r^2 + 9r + 19}{(r+5)!} \right],
\]

and hence, or otherwise, deduce the value of

\[
\sum_{r=1}^{\infty} \left[ \frac{r^2 + 7r + 11}{(r+4)!} \right].
\]
Question 12  (*****)

A sequence is defined as

$$u_{r+1} = u_r + \frac{2r}{r^4 + r^2 + 1}, \quad u_1 = 0, \quad r \in \mathbb{N}.$$  

Determine the exact value of $u_{61}$.

$$u_{61} = \frac{3660}{3661}$$
Question 13 (*****)

Find the value of

\[
\sum_{r=0}^{\infty} \left[ \frac{\sin^4 \left( \pi \times 2^r \right)}{4^r} \right].
\]

Hint: Express \( \sin^4 \theta \) in terms of \( \sin^2 \theta \) and \( \sin^2 2\theta \) only.
Find the sum to infinity of the following convergent series.

\[
\frac{1}{4 \times 2!} + \frac{1}{5 \times 3!} + \frac{1}{6 \times 4!} + \frac{1}{7 \times 5!} + \frac{1}{8 \times 6!} + \ldots
\]

\[
\sum_{n=1}^{\infty} \frac{n!}{(n+1)!} = \frac{1}{6}
\]
Question 15

Evaluate the following expression

\[
\sum_{k=1}^{\infty} \left( \sum_{r=1}^{k} \right)^{-1}.
\]