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SERIES

79 EXAM QUESTIONS

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SUMMATIONS BY FORMULAS

17 BASIC QUESTIONS

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Question 1 (**)

Use standard results on summations to find the value of

$$\sum_{r=36}^{48} [(r-1)(3r-2)].$$

 , 66638

Find a simplified expression for the sum of the first n terms

$$\sum_{r=1}^n [(r-1)(3r-2)] = \sum_{r=1}^n (3r^2 - 5r + 2)$$

$$= 3 \sum_{r=1}^n r^2 - 5 \sum_{r=1}^n r + 2 \sum_{r=1}^n 1$$

$$= 3 \times \frac{1}{6} n(n+1)(2n+1) - 5 \times \frac{1}{2} n(n+1) + 2n$$

$$= \frac{1}{2} n [(2n+1)(2n+1) - 5n(n+1) + 4]$$

$$= \frac{1}{2} n [2n^2 + 2n + 1 - 5n^2 - 5n + 4]$$

$$= \frac{1}{2} n [-3n^2 - 3n + 5]$$

$$= \frac{1}{2} n^2 (-3n - 3 + \frac{5}{n})$$

You use have

$$\sum_{r=36}^{48} [(r-1)(3r-2)] = \sum_{r=1}^{48} [(r-1)(3r-2)] - \sum_{r=1}^{35} [(r-1)(3r-2)]$$

$$= \frac{1}{2} (48^2 (-3 \cdot 48 - 3 + \frac{5}{48})) - \frac{1}{2} (35^2 (-3 \cdot 35 - 3 + \frac{5}{35}))$$

$$= 108288 - 41630$$

$$= \underline{\underline{66638}}$$

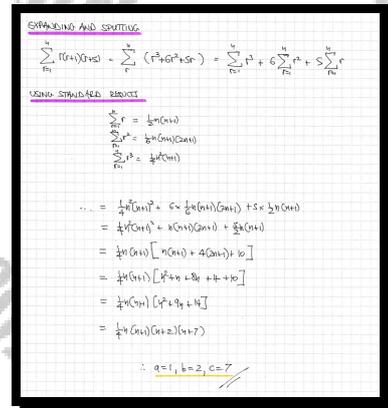
Question 2 (**)

Use standard results on summations to show that

$$\sum_{r=1}^n r(r+1)(r+5) = \frac{1}{4}n(n+a)(n+b)(n+c),$$

where a , b , and c are positive integers to be found.

, $a=1, b=2, c=7$



Question 3 (**)

Use standard results on summations to show that

$$\sum_{r=1}^n [r^2(r-1)] = \frac{1}{12}n(n-1)(n+1)(3n+2) + m,$$

where m is an integer to be found.

, $m = -22$

EXPAND THE SUMMATION & USE STANDARD RESULTS

$$\sum_{r=1}^n r^2(r-1) = \sum_{r=1}^n (r^3 - r^2) = \left[\sum_{r=1}^n (r^3) \right] - (0 + 4 + 16)$$

$$= \sum_{r=1}^n (r^3 - r^2) - 22$$

Using $\sum_{r=1}^n r^3 = \frac{1}{4}(n+1)^2$ & $\sum_{r=1}^n r^2 = \frac{1}{6}n(n+1)(2n+1)$

$$\dots = \frac{1}{4}(n+1)^2 - \frac{1}{6}n(n+1)(2n+1) - 22$$

$$\dots = \frac{1}{12}n(n+1)[3(n+1) - 2(2n+1)] - 22$$

$$\dots = \frac{1}{12}n(n+1)[3n^2 + 3n - 4n - 2] - 22$$

$$\dots = \frac{1}{12}n(n+1)(3n^2 - n - 2) - 22$$

$$\dots = \frac{1}{12}n(n+1)(3n+2) - 22$$

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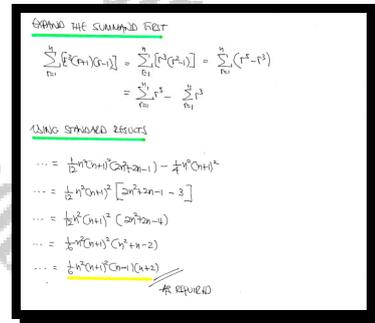
Question 4 ()**

Use standard results on summations to show that

$$\sum_{r=1}^n [r^3(r+1)(r-1)] = \frac{1}{6}n^2(n+1)^2(n-1)(n+2).$$

You may assume without proof that $\sum_{r=1}^n r^5 = \frac{1}{12}n^2(n+1)^2(2n^2+2n-1)$

, proof



Question 5 (**)

$$F(r) \equiv \sum_{n=1}^r [n(n-1)(n+2)].$$

Use standard results on summations express $F(n)$ in fully factorized form.

, $F(r) = \frac{1}{12} r(r+1)(r-1)(3r+10)$

START BY EXPANDING THE SUMMATION

$$F(r) = \sum_{n=1}^r [n(n-1)(n+2)] = \sum_{n=1}^r n(n^2+n-2) = \sum_{n=1}^r (n^3+n^2-2n)$$

$$= \sum_{n=1}^r n^3 + \sum_{n=1}^r n^2 - 2 \times \sum_{n=1}^r n$$

USING STANDARD RESULTS ON SUMMATIONS

$$\Rightarrow F(r) = \frac{1}{4} r^2(r+1)^2 + \frac{1}{6} r(r+1)(r+2) - 2 \times \frac{1}{2} r(r+1)$$

$$\Rightarrow F(r) = \frac{1}{4} r^2(r+1)^2 + \frac{1}{6} r(r+1)(r+2) - r(r+1)$$

$$\Rightarrow F(r) = \frac{1}{12} r(r+1) [2r(r+1)^2 + (r+2)(r+1) - 4(r+1)]$$

$$\Rightarrow F(r) = \frac{1}{12} r(r+1) [2r^2 + 4r + 2 + r^2 + 3r + 2 - 4r]$$

$$\Rightarrow F(r) = \frac{1}{12} r(r+1) (3r^2 + 3r - 2)$$

$3r^2 - 4r - 2 = (3r+2)(r-1)$

$$F(r) = \frac{1}{12} r(r+1)(3r+10)(r-1)$$

Question 6 (**+)

Find, in fully simplified factorized form, an expression for the sum of the first n terms of the following series.

$$(5 \times 3) + (11 \times 7) + (17 \times 11) + (23 \times 15) + \dots$$

, $n^2(8n+7)$

WRITE THE EXPRESSION IN COMPACT NOTATION

$$(5 \times 3) + (11 \times 7) + (17 \times 11) + \dots = \sum_{k=1}^n (6k-1)(6k+1)$$

USING STANDARD RESULTS

$$\sum_{k=1}^n (6k-1)(6k+1) = \sum_{k=1}^n (36k^2 - 1)$$

$$= 36 \sum_{k=1}^n k^2 - 1 \times \sum_{k=1}^n 1$$

$$= 36 \times \frac{1}{6} n(n+1)(2n+1) - 1 \times \frac{1}{2} n(n+1) + 1$$

$$= 6n(n+1)(2n+1) - \frac{1}{2} n(n+1) + 1$$

$$= n [4(2n+1)(2n+1) - (2n+1) + 2]$$

$$= n [8n^2 + 16n + 4 - 2n - 1 + 2]$$

$$= n [8n^2 + 14n + 5]$$

$$= n^2 (8n+7)$$

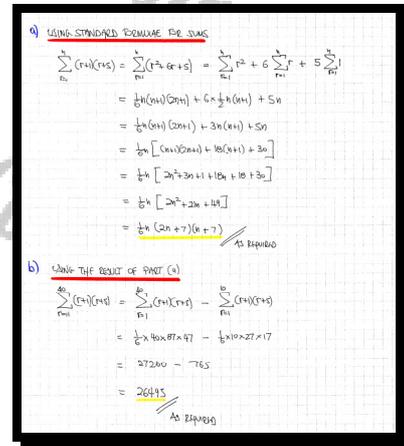
Question 7 (*)**

Show by using standard summation results that ...

a) ...
$$\sum_{r=1}^n (r+1)(r+5) = \frac{1}{6}n(n+7)(2n+7).$$

b) ...
$$\sum_{r=11}^{40} (r+1)(r+5) = 26495.$$

,



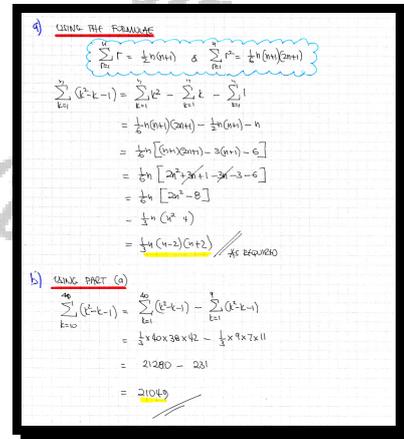
Question 8 (***)

Show by using standard summation results that ...

a) ...
$$\sum_{k=1}^n (k^2 - k - 1) = \frac{1}{3}n(n+2)(n-2).$$

b) ...
$$\sum_{k=10}^{40} (k^2 - k - 1) = 21049.$$

, proof



Question 9 (+)**

Find, in fully factorized form, an expression for the sum

$$\sum_{p=1}^k (p^3 + p^2).$$

, $\frac{1}{12}k(k+1)(k+2)(3k+1)$

USING STANDARD RESULTS

$$\begin{aligned} \sum_{p=1}^k (p^3 + p^2) &= \frac{1}{6}k(k+1)(2k+1) + \frac{1}{4}k^2(k+1) \\ &= \frac{1}{12}k(k+1)[2(2k+1) + 3k(k+1)] \\ &= \frac{1}{12}k(k+1)(4k+2+3k^2+3k) \\ &= \frac{1}{12}k(k+1)(3k^2+7k+2) \\ &= \frac{1}{12}k(k+1)(3k+1)(k+2) \end{aligned}$$

Question 10 (+)**

Find, in fully factorized form, an expression for the sum

$$\sum_{r=1}^{2n} \left(3r^2 - \frac{1}{2}\right).$$

, $2n^2(4n+3)$

USING STANDARD SUMMATION FORMULAE

$$\begin{aligned} \sum_{r=1}^{2n} \left(3r^2 - \frac{1}{2}\right) &= 3 \sum_{r=1}^{2n} r^2 - \frac{1}{2} \sum_{r=1}^{2n} 1 \\ &= 3 \times \frac{1}{6}(2n)(2n+1)(2n+1) - \frac{1}{2} \times 2n \\ &= \frac{1}{2}(2n)^2(2n+1) \\ &= n(2n+1)(2n+1) \\ &= n[(2n+1)(2n+1) - 1] \\ &= n[4n^2 + 4n + 1 - 1] \\ &= n(4n^2 + 4n) \\ &= 2n^2(4n+3) \end{aligned}$$

Question 11 (***)

Use standard results on summations to show that

$$\sum_{r=1}^{n-2} r(r+1)^2 = \frac{1}{12}n(n-1)(n-2)(3n-1).$$

, proof

GENUS: THE STANDARD SUMMATIONS TO n

$$\sum_{r=1}^n r(r+1)^2 = \sum_{r=1}^n (r^2 + 2r^2 + r) = \sum_{r=1}^n r^2 + 2\sum_{r=1}^n r^2 + \sum_{r=1}^n r$$

$$= \frac{1}{6}n(n+1)(2n+1) + 2 \times \frac{1}{2}n(n+1) + \frac{1}{2}n(n+1)$$

$$= \frac{1}{6}n(n+1)^2 + \frac{1}{2}n(n+1)(2n+1) + \frac{1}{2}n(n+1)$$

$$= \frac{1}{6}n(n+1) [3n(n+1) + 4(n+1) + 3]$$

$$= \frac{1}{6}n(n+1) (3n^2 + 3n + 4n + 4 + 3)$$

$$= \frac{1}{6}n(n+1) (3n^2 + 7n + 7)$$

$$= \frac{1}{6}n(n+1)(n+2)(3n+5)$$

FINALLY WRP $n \rightarrow n-2$

$$\sum_{r=1}^{n-2} r(r+1)^2 = \frac{1}{6}(n-2)(n-2+1)(n-2+2) [3(n-2) + 5]$$

$$= \frac{1}{6}(n-2)(n-1)n(3n-1)$$

$$= \frac{1}{12}n(n-1)(n-2)(3n-1)$$

Question 12 (***)

It is given that

$$\sum_{r=1}^n [(3r+a)(r+2)] \equiv n(n+2)(n+b).$$

Determine the values of each of the constants a and b .

, $a=1$, $b=3$

PROCEED AS FOLLOWS

$$\rightarrow \sum_{r=1}^n (3r+a)(r+2) \equiv n(n+2)(n+b)$$

$$\rightarrow \sum_{r=1}^n (3r^2 + (6+a)r + 2a) \equiv n^3 + (2+b)n^2 + 2bn$$

$$\rightarrow 3 \sum_{r=1}^n r^2 + (6+a) \sum_{r=1}^n r + 2a \sum_{r=1}^n 1 \equiv n^3 + (2+b)n^2 + 2bn$$

$$\rightarrow 3 \times \frac{1}{6}n(n+1)(n+1) + (6+a) \times \frac{1}{2}n(n+1) + 2an \equiv n^3 + (2+b)n^2 + 2bn$$

$$\rightarrow \frac{1}{2}n(n+1)(n+1) + \frac{1}{2}(6+a)n(n+1) + 2an \equiv n^3 + (2+b)n^2 + 2bn$$

COMPARE THE LHS WITH

$$\rightarrow n^3 + 2n^2 + \frac{1}{2}n + \frac{1}{2}(6+a)n^2 + \frac{1}{2}(6+a)n + 2an \equiv n^3 + (2+b)n^2 + 2bn$$

LOOK AT THE COEFFICIENTS OF n^2 & n

$\bullet \frac{1}{2} + \frac{1}{2}(6+a) = 2+b$ $3 + 6+a = 4+2b$ $a = 2b-5$	$\bullet \frac{1}{2} + \frac{1}{2}(6+a) + 2a = 2b$ $1 + 6+a + 4a = 4b$ $7 + 5a = 4b$
--	--

$$\begin{aligned} 7 + 5(2b-5) &= 4b \\ 7 + 10b - 25 &= 4b \\ 6b &= 18 \\ b &= 3 \\ \therefore a &= 1 \end{aligned}$$

Question 13 (*)**

Show clearly that

$$(1 \times 3) + (2 \times 4) + (3 \times 5) + \dots + (n-5)(n-3) = \frac{1}{6}(n+6)(2n+11)(n+5).$$

, proof

USE THE SUMMATION PROPERTY

$$(1 \times 3) + (2 \times 4) + (3 \times 5) + \dots + (n-5)(n-3) = \sum_{r=1}^{n-5} r(r+2)$$

USE THE SUMMATION PROPERTY

$$\sum_{r=1}^{n-5} r(r+2) = \sum_{r=1}^{n-5} (r^2 + 2r) = \sum_{r=1}^{n-5} r^2 + 2 \sum_{r=1}^{n-5} r$$

$$= \frac{1}{6}n(n+1)(2n+1) + 2 \times \frac{1}{2}n(n+1)$$

$$= \frac{1}{6}n(n+1)(2n+1) + n(n+1)$$

$$= \frac{1}{6}n(n+1)(2n+1 + 6)$$

$$= \frac{1}{6}n(n+1)(2n+7)$$

USE THE SUMMATION PROPERTY

$$f(n) = \sum_{r=1}^{n-5} r(r+2) = \frac{1}{6}n(n+1)(2n+7)$$

$$f(n-5) = \sum_{r=1}^{n-5} r(r+2) = \frac{1}{6}(n-5)(n-4)(2n-3) + 1$$

$$= \frac{1}{6}(n-5)(n-4)(2n-3)$$

AS REQUIRED

Question 14 (*)**

Use standard results on summations to show that

$$\sum_{r=1}^n (3r^2 + r - 1) \equiv n^2(n+2).$$

, proof

USE THE DIFFERENCE PROPERTY OF THE SUMMATION

$$\sum_{r=1}^n (3r^2 + r - 1) = \sum_{r=1}^n 3r^2 + \sum_{r=1}^n r - \sum_{r=1}^n 1$$

$$= 3 \sum_{r=1}^n r^2 + \sum_{r=1}^n r - 2 \sum_{r=1}^n 1$$

$$= 3 \times \frac{1}{6}n(n+1)(2n+1) + \frac{1}{2}n(n+1) - n$$

$$= \frac{1}{2}n(n+1)(2n+1) + \frac{1}{2}n(n+1) - n$$

$$= \frac{1}{2}n [(n+1)(2n+1) + (n+1) - 2]$$

$$= \frac{1}{2}n [2n^2 + 3n + 1 + n + 1 - 2]$$

$$= \frac{1}{2}n [2n^2 + 4n]$$

$$= \frac{1}{2}n (4n^2 + 4n)$$

$$= n^2(n+2)$$

AS REQUIRED

Question 15 (***)

Use standard results on summations to show that

$$\sum_{n=1}^k (18n^2 + 28n + 5) \equiv k(k+2)(6k+11).$$

, **proof**

SPLIT THE SUM INTO INDIVIDUAL COMPONENTS

$$\sum_{n=1}^k (18n^2 + 28n + 5) = \sum_{n=1}^k (18n^2) + \sum_{n=1}^k (28n) + \sum_{n=1}^k 5$$

$$= 18 \sum_{n=1}^k n^2 + 28 \sum_{n=1}^k n + 5 \sum_{n=1}^k 1$$

USE STANDARD RESULTS

$$\sum_{n=1}^k n^2 = \frac{1}{6}k(k+1)(2k+1) \quad \sum_{n=1}^k n = \frac{1}{2}k(k+1) \quad \sum_{n=1}^k 1 = k$$

SUBSTITUTE

$$= 18 \times \frac{1}{6}k(k+1)(2k+1) + 28 \times \frac{1}{2}k(k+1) + 5 \times k$$

$$= 3k(k+1)(2k+1) + 14k(k+1) + 5k$$

$$= k [3(k+1)(2k+1) + 14(k+1) + 5]$$

$$= k [6k^2 + 9k + 3 + 14k + 14 + 5]$$

$$= k [6k^2 + 23k + 22]$$

$$= k (k+2)(6k+11)$$

Question 16 (***)

Use standard results on summations to find the value of the following sum.

$$\sum_{k=2}^{16} [(k-1)(k+2)].$$

, **1600**

SPLIT BY FINDING AN EXPRESSION FOR THE SUM OF THE PARTS IN THEM
OR BY FINDING A SIMILAR EXPRESSION

$$\sum_{k=2}^{16} [(k-1)(k+2)] = \sum_{k=2}^{16} [k^2 + k - 2]$$

$$= \sum_{k=2}^{16} k^2 + \sum_{k=2}^{16} k - \sum_{k=2}^{16} 2$$

$$= \frac{1}{6}n(n+1)(2n+1) + \frac{1}{2}n(n+1) - 2 \times n$$

$$= \frac{1}{6}n [(2n+1)(n+1) + 3(n+1) - 12]$$

$$= \frac{1}{6}n [2n^2 + 3n + 1 + 3n + 3 - 12]$$

$$= \frac{1}{6}n (2n^2 + 6n - 8)$$

$$= \frac{1}{6}n (n^2 + 3n - 4)$$

$$= \frac{1}{6}n (n-1)(n+4)$$

Now let n = 16

$$\sum_{k=2}^{16} [(k-1)(k+2)] = \frac{1}{6} \times 16 \times 15 \times 20 = 1600$$

Question 17 (***)

Use standard results on summations to show that

$$\sum_{r=1}^{2n} r^3 - \sum_{r=1}^n (6r-3)^2 \equiv f(n),$$

where $f(n)$ is written as a product of 4 linear factors.

, $f(n) = n(n-1)(2n+1)(2n-3)$

USING STANDARD SUMMATION RESULTS

• $\sum_{r=1}^n r = \frac{1}{2}n(n+1)$ • $\sum_{r=1}^n r^2 = \frac{1}{6}n(n+1)(2n+1)$ • $\sum_{r=1}^n r^3 = \frac{1}{4}n^2(n+1)^2$

$$\sum_{r=1}^{2n} r^3 - \sum_{r=1}^n (6r-3)^2 = \sum_{r=1}^{2n} r^3 - \left[\sum_{r=1}^n (36r^2 - 36r + 9) \right]$$

$$= \sum_{r=1}^{2n} r^3 - 36 \sum_{r=1}^n r^2 + 36 \sum_{r=1}^n r - 9 \sum_{r=1}^n 1$$

$$= \frac{1}{4}(2n)^2(2n+1)^2 - 36 \left[\frac{1}{6}n(n+1)(2n+1) \right] + 36 \left[\frac{1}{2}n(n+1) \right] - 9n$$

$$= n^2(2n+1)^2 - 6n(n+1)(2n+1) + 18n(n+1) - 9n$$

PROCEED TO SIMPLIFY

$$= n(2n+1) [n(2n+1) - 6(n+1)] + 9n [2(n+1) - 1]$$

$$= n(2n+1) [2n^2 + n - 6n - 6] + 9n(2n+1)$$

$$= n(2n+1) [2n^2 - 5n - 6] + 18n(2n+1)$$

FACTORISE AGAIN

$$= n(2n+1) [2n^2 - 5n - 6 + 18]$$

$$= n(2n+1) [2n^2 - 5n + 12]$$

$$= n(2n+1) (2n-3)(n+4)$$

ALTERNATIVE WAY UP

$$\dots = n^2(2n+1)^2 - 6n(n+1)(2n+1) + 18n(n+1) - 9n = n(2n+1) [n(2n+1) - 6(n+1) + 18(n+1) - 9]$$

$$= n(2n+1) [2n^2 + n - 6n - 6 + 18n + 18 - 9] = n(2n+1) [2n^2 - 5n + 12]$$

• THIS LOOKS LIKE THE ANSWER, CHECK BY EXPANDING IT OUT

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SUMMATIONS

BY FORMULAS

15 STANDARD QUESTIONS

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Question 1 (***)

Find the sum of the first n terms of the series

$$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 5 + 3 \cdot 4 \cdot 7 + 4 \cdot 5 \cdot 9 + \dots$$

Express the answer as a product of linear factors.

10, proof

PROVE BY WRITING THE SERIES IN SUMMATION

$$\underbrace{(1 \cdot 2 \cdot 3) + (2 \cdot 3 \cdot 5) + (3 \cdot 4 \cdot 7) + (4 \cdot 5 \cdot 9) + \dots}_{n \text{ terms}} = \sum_{r=1}^n [r(r+1)(2r+1)]$$

EXPAND & SIMPLIFY

$$\sum_{r=1}^n [r(r+1)(2r+1)] = \sum_{r=1}^n [2r^3 + 3r^2 + r]$$
$$= 2 \sum_{r=1}^n r^3 + 3 \sum_{r=1}^n r^2 + \sum_{r=1}^n r$$

USING STANDARD RESULT

$$= 2 \cdot \frac{1}{4} n(n+1)^2 + 3 \cdot \frac{1}{2} n(n+1)(n+1) + \frac{1}{2} n(n+1)$$
$$= \frac{1}{2} n(n+1)^2 + \frac{3}{2} n(n+1)(n+1) + \frac{1}{2} n(n+1)$$
$$= \frac{1}{2} n(n+1) [n(n+1) + 3(n+1) + 1]$$
$$= \frac{1}{2} n(n+1) [n^2 + 2n + 2 + 3n + 3 + 1]$$
$$= \frac{1}{2} n(n+1)(n^2 + 5n + 6)$$
$$= \frac{1}{2} n(n+1)(n+2)(n+3)$$

Question 2 (***)

By using standard results, show that

$$\sum_{r=n+1}^{4n} (2r-1)^2 \equiv n(84n^2-1).$$

$\frac{1}{3}$, proof

PROVED-AS-FOLLOWS

$$f(n) = \sum_{r=1}^{4n} (2r-1)^2 = \sum_{r=1}^{4n} [4r^2 - 4r + 1] = 4 \sum_{r=1}^{4n} r^2 - 4 \sum_{r=1}^{4n} r + \sum_{r=1}^{4n} 1$$

USING STANDARD RESULTS

$$f(n) = 4 \times \frac{1}{6} (4n)(4n+1) - 4 \times \frac{1}{2} (4n+1) + n$$

$$f(n) = \frac{2}{3} n(4n+1) - 2n(4n+1) + n$$

$$f(n) = \frac{1}{3} n [2(4n+1) - 6(4n+1) + 3]$$

$$f(n) = \frac{1}{3} n [4n + 2 - 24n - 6 + 3]$$

$$f(n) = \frac{1}{3} n (4n^2 - 1)$$

YOU USE THIS

$$\sum_{r=n+1}^{4n} (2r-1)^2 = \sum_{r=1}^{4n} (2r-1)^2 - \sum_{r=1}^n (2r-1)^2$$

$$= f(4n) - f(n)$$

$$= \frac{1}{3} n (4(4n)^2 - 1) - \frac{1}{3} n (4n^2 - 1)$$

$$= \frac{1}{3} n (64n^2 - 1) - \frac{1}{3} n (4n^2 - 1)$$

$$= \frac{1}{3} n (20n^2 - 4n^2 + 1)$$

$$= \frac{1}{3} n (20n^2 - 3)$$

$$= n(84n^2 - 1)$$

Q.E.D.

Question 3 (***)

Determine the value of a and the value of b given that

$$\sum_{r=1}^n r(r+a)(r+b) \equiv \frac{1}{12}n(n+1)(n+2)(3n+17).$$

, $a=1, b=4$ or the other way round

USE STANDARD SERIES AND EQUIVALENT EXPONENTS

$$\sum_{r=1}^n r(r+a)(r+b) = \frac{1}{2}n(n+1)(n+2)(3n+17)$$

$$\sum_{r=1}^n [r^3 + (a+b)r^2 + ab r] = \frac{1}{2}n(n+1)(n+2)(3n+17)$$

$$\sum_{r=1}^n r^3 + (a+b)\sum_{r=1}^n r^2 + ab\sum_{r=1}^n r = \frac{1}{2}n(n+1)(n+2)(3n+17)$$

$$\frac{1}{4}n^2(n+1)^2 + \frac{1}{2}(a+b)n(n+1)(n+2) + abn(n+1) = \frac{1}{2}n(n+1)(n+2)(3n+17)$$

DUCE A FACTOR OF $n(n+1)$ ALL THE WAY THROUGH

$$\frac{1}{4}n(n+1) + \frac{1}{2}(a+b)(n+1) + ab = \frac{1}{2}(n+2)(3n+17)$$

$$3n(n+1) + 2(a+b)(n+1) + 6ab = (n+2)(3n+17)$$

$$3n^2 + 3n + 4(a+b)n + 2(a+b) + 6ab = 3n^2 + 7n + 6n + 34$$

$$3n^2 + [3 + 4(a+b)]n + [2(a+b) + 6ab] = 3n^2 + 13n + 34$$

SETTING TWO EXPRESSIONS

$$\begin{array}{l} 3 + 4(a+b) = 13 \\ 2(a+b) + 6ab = 34 \end{array} \rightarrow \begin{array}{l} 4(a+b) = 10 \\ a+b = \frac{5}{2} \\ a+b = 2.5 \end{array} \rightarrow \begin{array}{l} 2(a+b) = 5 \\ 6ab = 34 - 5 = 29 \\ ab = \frac{29}{6} \end{array}$$

BY INSPECTING, SUBSTITUTION, OR REARRANGING EACH TECHNIQUE

$a=1, b=4$ (OR THE OTHER WAY ROUND)

Question 4 (***)

Find, in fully factorized form, an expression for the following sum.

$$\sum_{r=n}^{2n} (r^3 - 2r)$$

$$\boxed{}, \quad \sum_{r=n}^{2n} (r^3 - 2r) = \frac{3}{4}n(5n-4)(n+1)^2$$

USING THE STANDARD SUMMATION FORMULAE

$$\sum_{r=1}^k r = \frac{1}{2}k(k+1)$$

$$\sum_{r=1}^k r^2 = \frac{1}{6}k(k+1)(2k+1)$$

Hence we now find:

$$\sum_{r=n}^{2n} (r^3 - 2r) = \sum_{r=1}^{2n} r^3 - 2 \sum_{r=1}^{2n} r$$

$$= \left[\frac{1}{4}(2n)^2(2n+1) - \frac{1}{4}(n-1)^2n \right] - 2 \left[\frac{1}{2}(2n)(2n+1) - \frac{1}{2}(n-1)n \right]$$

$$= \frac{1}{4}(2n)^2(2n+1) - \frac{1}{4}(n-1)^2n - 2n(2n+1) + n(n-1)$$

$$= \frac{1}{4}n \left[4n(2n+1) - (n-1)^2 - 8(2n+1) + 4(n-1) \right]$$

IT WILL BE A MESS TO EXPAND TO SEE WHICH IT BET TO EXPAND THE TERMS INSIDE THE BRACKET IN PINKS

$$= \frac{1}{4}n \left[4 \left[4n^2 + 4n \right] - (n^2 - 2n + 1) - 16n - 8 + 4n - 4 \right]$$

$$= \frac{1}{4}n \left[n \left[16n^2 + 16n - n^2 + 2n - 1 - 16n - 8 + 4n - 4 \right] \right]$$

$$= \frac{1}{4}n \left[n \left[15n^2 + 12n - 13 \right] \right]$$

$$= \frac{1}{4}n \left[n \left[3n(5n+4) - 13 \right] \right]$$

$$= \frac{1}{4}n \left[3n(5n+4) - 13 \right]$$

$$= \frac{3}{4}n \left[n(15n+12) - 13 \right]$$

ALTERNATIVE BY EXPANDING & CHECK FROM THE OTHER PERSPECTIVE

$$= \frac{1}{4}n \left[4n(2n+1) - (n-1)^2 - 8(2n+1) + 4(n-1) \right]$$

$$= \frac{1}{4}n \left[8n^2 + 4n - n^2 + 2n - 1 - 16n - 8 + 4n - 4 \right]$$

$$= \frac{1}{4}n \left[7n^2 + 10n - 13 \right]$$

$$= \frac{3}{4}n \left[3n^2 + 6n - 13 \right]$$

LOOKING FOR PATTERNS

$n=1$ $3(1)^2 + 6(1) - 13 = -4 \neq 0$
 $n=1$ $3(1)^2 + 6(1) - 13 = -4 \neq 0$ $\therefore (n+1)$ is a factor

LONG DIVIDE

$3n^2 + 6n - 13$	$\div (n+1)$	$= 3n + 3$	$- 4$
$3n^2 + 3n$			
$3n - 13$			
$3n + 3$			
-16			
-16			
0			

$$= \frac{3}{4}n(3n+3)(n+1) - 4n$$

$$= \frac{3}{4}n(3n+3)(n+1) - 4n$$

$$= \frac{3}{4}n(3n+3)(n+1) - 4n$$

Question 5 (***)

It is thought that for some values of the constants p and q that

$$\sum_{r=1}^n r^2(r+p) \equiv n(n+1)(n+2)(3n+q).$$

Use a detailed method to show that there exist no such values of p and q .

, proof

EXPAND THE LHS & COMPARE COEFFICIENTS

$$\Rightarrow \sum_{r=1}^n r^2(r+p) = \sum_{r=1}^n r^3 + p \sum_{r=1}^n r^2 \equiv qn(n+1)(n+2)(3n+q)$$

$$\Rightarrow \sum_{r=1}^n (r^3 + pr^2) = \sum_{r=1}^n r^3 + p \sum_{r=1}^n r^2 \equiv qn(n+1)(n+2)(3n+q)$$

$$\Rightarrow \frac{1}{4}n^2(n+1) + \frac{1}{6}p(n+1)(2n+1) \equiv qn(n+1)(n+2)(3n+q)$$

$$\Rightarrow n(n+1) \left[\frac{1}{4}n + \frac{1}{6}p \right] \equiv qn(n+1)(n+2)(3n+q)$$

$$\Rightarrow \frac{1}{4}n + \frac{1}{6}p \equiv q(n+2)(3n+q)$$

$$\Rightarrow \frac{1}{4}n^2 + \frac{1}{4}n + \frac{1}{6}pn + \frac{1}{6}p \equiv (3q^2 + n + 6n + 2)q$$

$$\equiv \frac{1}{4}n^2 + \left(\frac{1}{4} + \frac{1}{6}p\right)n + \frac{1}{6}p \equiv 3qn^2 + 7qn + 2q$$

Now look at each power

$[n^2]: \frac{1}{4} = 3q$
 $q = \frac{1}{12}$

$[n^1]: \frac{1}{4} + \frac{1}{6}p = 7q$
 $\frac{1}{4} + \frac{1}{6}p = \frac{7}{12}$
 $3 + 4p = 7$
 $4p = 4$
 $p = 1$

BUT NOW $[n^0]$ YIELDS INCONSISTENCY SINCE $\frac{1}{6}p = 2q$
 $\frac{1}{6} \neq 2 \cdot \frac{1}{12}$

Question 6 (***)

Use standard results on summations to solve the following equation.

$$\sum_{r=1}^k (r^3 - 1) = 89976.$$

$$\boxed{}, k = 24$$

START BY GETTING A POSITIVE EXPRESSION USING STANDARD RESULTS

$$\sum_{r=1}^k (r^3 - 1) = \sum_{r=1}^k r^3 - \sum_{r=1}^k 1 = \frac{1}{4}k^2(k+1)^2 - k$$

$$= \frac{1}{4}k(k^2+2k+1)^2 - k = \frac{1}{4}k(k^3+2k^2+k-4)$$

Now k-1 is an obvious zero of the cubic, so (k-1) is a factor

$$= \frac{1}{4}k [k^2(k-1) + 3k(k-1) + 4(k-1)]$$

no one cancelled
ALGEBRAIC DIVISION

$$= \frac{1}{4}k(k-1)(k^2+3k+4)$$

IRREDUCIBLE

Now solving by trial & error as k is a positive integer

$$f(1) = \frac{1}{4}k(k-1)(k^2+3k+4)$$

$$f(2) = \frac{1}{4} \times 2 \times 1 \times (4+6+8) = 2015 < 89976$$

$$f(3) = \frac{1}{4} \times 3 \times 2 \times 9 \times (9+9+12) = 4050 < 89976$$

$$f(4) = \frac{1}{4} \times 4 \times 3 \times 21 \times (16+12+16) = 26195 > 89976$$

$$f(5) = \frac{1}{4} \times 5 \times 4 \times 24 \times (25+15+16) = 105600 > 89976$$

$$f(6) = \frac{1}{4} \times 6 \times 5 \times 28 \times (36+18+16) = 89976$$

$\therefore k = 24$

Question 7 (****)

It is given that

$$\sum_{r=1}^n (Ar^3 + Br^2 + Cr) = n(n+1)(n+2)(4n-5).$$

Use a detailed method to find the value of each of the integer constants, A , B and C .

, $A = 16$, $B = -3$, $C = -19$

The handwritten solution shows two methods to solve for A, B, and C.

Method 1: n(n+1) Trick

- Starts with the given equation: $\sum_{r=1}^n (Ar^3 + Br^2 + Cr) = n(n+1)(n+2)(4n-5)$
- Substitutes $n=1$ and $n=2$ to get two equations:
 - $f(1) = n(n+1)(n+2)(4n-5) = 1(2)(3)(-1) = -6$
 - $f(2) = 2(3)(4)(3) = 72$
- Substitutes these into the general form:
 - $f(1) = A(1)^3 + B(1)^2 + C(1) = A + B + C = -6$
 - $f(2) = A(2)^3 + B(2)^2 + C(2) = 8A + 4B + 2C = 72$
- Solves the system of equations to find $A = 16$, $B = -3$, and $C = -19$.

Method 2: Comparing Coefficients

- Starts with the given equation: $\sum_{r=1}^n (Ar^3 + Br^2 + Cr) = n(n+1)(n+2)(4n-5)$
- Expands the right-hand side: $n(n+1)(n+2)(4n-5) = n(4n^3 + 11n^2 + 6n - 10) = 4n^4 + 11n^3 + 6n^2 - 10n$
- Expands the left-hand side using summation formulas:
 - $\sum_{r=1}^n Ar^3 = A \frac{n^2(n+1)^2}{4}$
 - $\sum_{r=1}^n Br^2 = B \frac{n(n+1)(2n+1)}{6}$
 - $\sum_{r=1}^n Cr = C \frac{n(n+1)}{2}$
- Equates the coefficients of n^4 , n^3 , n^2 , and n on both sides to solve for A, B, and C, resulting in $A = 16$, $B = -3$, and $C = -19$.

Question 8 (***)

Show by a detailed method that

$$\sum_{r=0}^n [2r(2r^2 - 3r - 1) + n + 1] = (n^2 - 1)^2$$

, proof

EXPAND THE SUMMATION AND USE STANDARD RESULTS

$$\sum_{r=0}^n [2r(2r^2 - 3r - 1) + n + 1]$$

$$= \sum_{r=0}^n [4r^3 - 6r^2 - 2r + n + 1]$$

THE SUMMATION HAS NO DEPENDENCE ON r, SO IT MAY BE TAKEN OUT

$$= 4 \sum_{r=0}^n r^3 - 6 \sum_{r=0}^n r^2 - 2 \sum_{r=0}^n r + \sum_{r=0}^n (n + 1)$$

NOTE THE THE FIRST TERM IN THE FIRST 3 SUMMATIONS IS ZERO
SO WE MAY START THESE SUMMATIONS FROM r=1

$$= 4 \sum_{r=1}^n r^3 - 6 \sum_{r=1}^n r^2 - 2 \sum_{r=1}^n r + (n + 1) \sum_{r=0}^n 1$$

USE STANDARD RESULTS

$$= 4 \times \frac{1}{4} n(n+1)^2 - 6 \times \frac{1}{6} n(n+1) - 2 \times \frac{1}{2} n(n+1) + (n+1)(n+1)$$

$$= n^2(n+1)^2 - n(n+1)(2n+1) - n(n+1) + (n+1)^2$$

$$= n(n+1)[n(n+1) - (2n+1) - 1] + (n+1)^2$$

$$= n(n+1)[n^2 - n - 2] + (n+1)^2$$

$$= n(n+1)(n+1)(n-2) + (n+1)^2$$

$$= (n+1)^2 [n(n-2) + 1]$$

$$= (n+1)^2 (n^2 - 2n + 1)$$

$$= (n+1)^2 (n-1)^2 = [(n+1)(n-1)]^2$$

$$= (n^2 - 1)^2$$

AS REQUIRED

Question 9 (***)

The sum, S_n , of the first n terms of a series whose general term is denoted by u_n is given by the following expression.

$$S_n = n^2(n+1)(n+2).$$

a) Find the first term of the series.

b) Show clearly that ...

i. ... $u_n = n(n+1)(4n-1)$

ii. ... $\sum_{r=n+1}^{2n} u_r = 3n^2(n+1)(5n+2).$

, $u_1 = 6$

a) TELL ME WE HAVE
 $u_1 = S_1 = 1^2(1+1)(1+2) = 1 \times 2 \times 3 = 6$

b) i) USING $S_n - S_{n-1} = u_n$
 $\rightarrow u_n = n^2(n+1)(n+2) - (n-1)^2[n(n+1)]$
 $\rightarrow u_n = n^2(n+1)(n+2) - (n-1)^2 n(n+1)$
 $\rightarrow u_n = n(n+1)[n(n+2) - (n-1)^2]$
 $\rightarrow u_n = n(n+1)[n^2 + 2n - (n^2 - 2n + 1)]$
 $\rightarrow u_n = n(n+1)(4n-1)$ At 26/02/20

ii) $\sum_{r=n+1}^{2n} u_r = S_{2n} - S_n$
 $= (2n)^2(2n+1)(2n+2) - n^2(n+1)(n+2)$
 $= 4n^2(2n+1) \times 2(n+1) - n^2(n+1)(n+2)$
 $= n^2(n+1)[8(2n+1) - (n+2)]$
 $= n^2(n+1)(16n+8-n-2)$
 $= n^2(n+1)(15n+6)$
 $= 3n^2(n+1)(5n+2)$ At 26/02/20

Question 10 (***)

Use standard summation results to prove that

$$\sum_{r=1}^n (n-r)^2 = \frac{1}{2}n(n-1)(2n-1).$$

, proof

FIRST LET US NOTE THAT $\sum_{r=1}^n r^2 = \frac{1}{6}n(n+1)(2n+1)$
 NEXT PROCEED AS FOLLOWS
 $\sum_{r=1}^n (n-r)^2 = \sum_{r=1}^n (n^2 - 2nr + r^2)$
 $\rightarrow \sum_{r=1}^n (n-r)^2 = n^2 \sum_{r=1}^n 1 - 2n \sum_{r=1}^n r + \sum_{r=1}^n r^2$
 $\rightarrow \sum_{r=1}^n (n-r)^2 = n^2 \times n - 2n \times \frac{1}{2}n(n+1) + \frac{1}{6}n(n+1)(2n+1)$
 $\rightarrow \sum_{r=1}^n (n-r)^2 = n^3 - n^2(n+1) + \frac{1}{6}n(n+1)(2n+1)$
 $\rightarrow \sum_{r=1}^n (n-r)^2 = \frac{1}{6}n [(n+1)(2n+1) - 6n]$
 $\rightarrow \sum_{r=1}^n (n-r)^2 = \frac{1}{6}n [2n^2 + 3n + 1 - 6n]$
 $\rightarrow \sum_{r=1}^n (n-r)^2 = \frac{1}{6}n [2n^2 - 3n + 1]$
 $\rightarrow \sum_{r=1}^n (n-r)^2 = \frac{1}{6}n (2n-1)(n-1)$

Question 11 (***)

Use standard results on summations to solve the following equation

$$\sum_{r=3}^9 \left[\binom{r}{k}^3 + (r-1)(r+1) \right] = 304.5.$$

$$\boxed{r=4}, \quad \boxed{k=4}$$

Write in series

$$\sum_{r=3}^9 \left[\binom{r}{k}^3 + (r-1)(r+1) \right] = \sum_{r=3}^9 \left[\binom{r}{k}^3 + r^2 - 1 \right] = \sum_{r=3}^9 \binom{r}{k}^3 + \sum_{r=3}^9 (r^2 - 1)$$

Evaluate and simplify each term using $\sum_{r=1}^n r^2 = \frac{1}{6}n(n+1)(2n+1)$

$$\sum_{r=3}^9 \binom{r}{k}^3 = \frac{1}{k^3} \sum_{r=3}^9 r^3 = \frac{1}{k^3} \left[\sum_{r=1}^9 r^3 - \sum_{r=1}^2 r^3 \right]$$

$$= \frac{1}{k^3} \left[\frac{1}{4}(9+1)(9+1)(9+1) - \frac{1}{4}(2+1)(2+1)(2+1) \right]$$

$$= \frac{1}{k^3} \times 2016$$

Similarly using $\sum_{r=1}^n r^2 = \frac{1}{6}n(n+1)(2n+1)$

$$\sum_{r=3}^9 (r^2 - 1) = \sum_{r=3}^9 r^2 - \sum_{r=3}^9 1$$

$$= \frac{1}{6} \times 10 \times 11 \times 13 - (4 \times 2) - 7$$

$$= 285 - 5 - 7$$

$$= 273$$

Final answer have

$$\frac{2016}{k^3} + 273 = 304.5 \Rightarrow \frac{2016}{k^3} = 31.5$$

$$\Rightarrow k^3 = 64$$

$$\Rightarrow k = 4$$

Question 12 (****)

$$\sum_{r=1}^n (ar^2 + br + c) \equiv n^3 + 5n^2 + 6n,$$

where a , b and c are integer constants.

Determine the value of a , b and c .

$$a=3, \quad b=7, \quad c=2$$

$$\begin{aligned} \sum_{r=1}^n ar^2 + br + c &= a \sum_{r=1}^n r^2 + b \sum_{r=1}^n r + c \sum_{r=1}^n 1 \\ &= \frac{a}{6} n(n+1)(2n+1) + \frac{b}{2} n(n+1) + cn \\ &= \frac{a}{6} n [2n^2 + 3n + 1] + \frac{b}{2} n [2n + 1] + cn \\ &= \frac{a}{6} n [2n^2 + 3n + 1] + \frac{b}{2} n [2n + 1] + cn \\ &= \frac{a}{6} n^3 + \frac{a}{2} n^2 + \frac{a}{6} n + \frac{b}{2} n^2 + \frac{b}{2} n + cn \\ \text{Now } \frac{a}{6} n^3 + \frac{a}{2} n^2 + \frac{a}{6} n + \frac{b}{2} n^2 + \frac{b}{2} n + cn &= n^3 + 5n^2 + 6n \\ \frac{a}{6} n^3 &= n^3 \quad \frac{a}{2} n^2 = 5n^2 \quad \frac{a}{6} n + \frac{b}{2} n + cn = 6n \\ \therefore a &= 6 \quad a + b = 10 \quad a + 3b + 6c = 36 \\ & \quad \quad \quad b = 7 \quad 3 + 21 + 6c = 36 \\ & \quad \quad \quad \quad \quad \quad c = 2 \end{aligned}$$

Question 13 (****)

The variance $\text{Var}(n)$ of the first n natural numbers is given by

$$\text{Var}(n) = \frac{1}{n} \sum_{r=1}^n r^2 - \left[\frac{1}{n} \sum_{r=1}^n r \right]^2.$$

Determine a simplified expression for $\text{Var}(n)$ and hence evaluate $\text{Var}(61)$.

$$\boxed{\text{Var}(n) = \frac{1}{12}(n^2 - 1)}, \quad \boxed{\text{Var}(61) = 310}$$

Handwritten derivation of the variance formula for the first n natural numbers:

$$\begin{aligned} \text{Variance} &= \frac{\sum_{r=1}^n r^2}{n} - \left(\frac{\sum_{r=1}^n r}{n} \right)^2 \\ &= \frac{\frac{1}{6}n(n+1)(2n+1)}{n} - \left(\frac{\frac{1}{2}n(n+1)}{n} \right)^2 \\ &= \frac{1}{6}(n+1)(2n+1) - \frac{1}{4}(n+1)^2 \\ &= \frac{1}{6}(n+1)[2(2n+1) - 3(n+1)] \\ &= \frac{1}{6}(n+1)(4n+2-3n-3) \\ &= \frac{1}{6}(n+1)(n-1) \end{aligned}$$

If $n=61$
 Variance = $\frac{1}{12} \times 62 \times 60 = 62 \times 5 = 310$

Question 14 (****)

$$f(n) = \sum_{r=1}^n [r^3 - r], \quad n \in \mathbb{N}.$$

- a) Use standard summation results to find a fully factorized expression for $f(n)$.
- b) Hence solve the equation

$$\sum_{r=5}^{10} [r^3 - r + 6k] - \sum_{r=1}^{12} [r^2 + k^2] = 70$$

$$\boxed{}, \quad \boxed{f(n) = \frac{1}{4}n(n-1)(n+1)(n+2)}, \quad \boxed{k = -12, k = 15}$$

a) $\sum_{r=1}^n (r^3 - r) = \sum_{r=1}^n r^3 - \sum_{r=1}^n r = \frac{1}{4}n^2(n+1)^2 - \frac{1}{2}n(n+1)$
 $= \frac{1}{4}n(n+1)[n(n+1) - 2] = \frac{1}{4}n(n+1)(n^2 + n - 2)$
 $= \frac{1}{4}n(n+1)(n-1)(n+2)$

b) EVALUATE IN SECTIONS
 $\Rightarrow \sum_{r=5}^{10} [r^3 - r + 6k] - \sum_{r=1}^{12} [r^2 + k^2] = 70$
 $\Rightarrow \sum_{r=5}^{10} (r^3 - r) + 6k \sum_{r=5}^{10} 1 - \sum_{r=1}^{12} r^2 - \sum_{r=1}^{12} k^2 = 70$
 $\Rightarrow \left[\sum_{r=1}^{10} (r^3 - r) - \sum_{r=1}^4 (r^3 - r) \right] + 6k(1+1+1+1+1) - \sum_{r=1}^{12} r^2 - k^2 \left(\frac{12 \times 13}{2} \right) = 70$
 $\Rightarrow \frac{1}{4} \times 10 \times 11 \times 12 - \frac{1}{4} \times 4 \times 5 \times 6 + 6k \times 5 - \frac{1}{2} \times 12 \times 13 \times 12 - k^2 \times 78 = 70$
 $\Rightarrow 2770 - 90 + 36k - 650 - 12k^2 = 70$
 $\Rightarrow 0 = 12k^2 - 36k - 2460$
 $\Rightarrow k^2 - 3k - 190 = 0$
 $\Rightarrow (k-15)(k+12) = 0$
 $\Rightarrow k = \begin{matrix} 15 \\ -12 \end{matrix}$

Question 15 (****)

The function $F(n)$ is defined as

$$F(n) = \sum_{r=1}^n [r(r-1)(n-2)(r+1)] \quad n \in \mathbb{N}.$$

Show with detailed workings that

$$F(2n) - F(n) = \frac{1}{2}n(n^2 - 1)(31n^2 - 4).$$

 , proof

SUM THE SERIES USING STRICKED RESULTS

$$F(n) = \sum_{r=1}^n [(r-1)(r-2)(r+1)] = (n-2) \sum_{r=1}^n [r(r-1)]$$

$$= (n-2) \sum_{r=1}^n (r^2 - r) = (n-2) \left[\sum_{r=1}^n r^2 - \sum_{r=1}^n r \right]$$

USE: $\sum_{r=1}^n r = \frac{1}{2}n(n+1)$ and $\sum_{r=1}^n r^2 = \frac{1}{6}n(n+1)(2n+1)$

$$\Rightarrow F(n) = (n-2) \left[\frac{1}{6}n(n+1)(2n+1) - \frac{1}{2}n(n+1) \right]$$

$$\Rightarrow F(n) = (n-2) \times \frac{1}{6}n(n+1) [2(n+1) - 3]$$

$$\Rightarrow F(n) = \frac{1}{6}n(n+1)(n-2)(n-1)(n+2)$$

$$\Rightarrow F(n) = \frac{1}{6}n(n^2-1)(n^2-4)$$

EVALUATE THE RESULT

$$F(2n) - F(n) = \frac{1}{6}(2n) [(2n)^2-1] [(2n)^2-4] - \frac{1}{6}n(n^2-1)(n^2-4)$$

$$= \frac{1}{6} [2n(4n^2-1)(4n^2-4) - n(n^2-1)(n^2-4)]$$

$$= \frac{1}{6} [8n(4n^2-1)(n^2-1) - n(n^2-1)(n^2-4)]$$

$$= \frac{1}{6}n(n^2-1) [8(4n^2-1) - (n^2-4)]$$

$$= \frac{1}{6}n(n^2-1) [32n^2 - 8 - n^2 + 4]$$

$$= \frac{1}{6}n(n^2-1)(31n^2-4)$$

PROVED

Created by T. Madas

SUMMATIONS

BY FORMULAS

5 HARD QUESTIONS

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Question 1 (****+)

It is given that

$$\sum_{r=1}^{20} (r-10) = 200 \quad \text{and} \quad \sum_{r=1}^{20} (r-10)^2 = 2800.$$

Find the value of

$$\sum_{r=1}^{20} r^2.$$

$$\boxed{}, \quad \sum_{r=1}^{20} r^2 = 8800$$

USING THE LINEARITY OF THE SIGMA OPERATOR

$$\sum_{r=1}^n [af(r) + bg(r)] = a \sum_{r=1}^n f(r) + b \sum_{r=1}^n g(r)$$

MANIPULATING AS FOLLOWS

$$\sum_{r=1}^{20} (r-10) = 200$$

$$\sum_{r=1}^{20} r - 10 \sum_{r=1}^{20} 1 = 200$$

$$- 10 \times 20 = 200$$

$$r = 400$$

FINALLY USING THE SECOND FACT

$$\sum_{r=1}^{20} (r-10)^2 = 2800$$

$$\sum_{r=1}^{20} (r^2 - 20r + 100) = 2800$$

$$- 20 \sum_{r=1}^{20} r + 100 \sum_{r=1}^{20} 1 = 2800$$

$$- 20 \times 400 + 100 \times 20 = 2800$$

$$- 8000 + 2000 = 2800$$

$$r = 8800$$

Question 2 (****+)

$$\sum_{r=1}^n (r+a)(r+b) \equiv \frac{1}{3}n(n-1)(n+4),$$

where a and b are integer constants.

Use a clear algebraic method to determine the value of a and the value of b .

, and (in any order)

Using standard results of the summation operator

$$\sum_{r=1}^n (r+a)(r+b) \equiv \frac{1}{3}n(n-1)(n+4)$$

$$\Rightarrow \sum_{r=1}^n [r^2 + (a+b)r + ab] \equiv \frac{1}{3}n(n-1)(n+4)$$

$$\Rightarrow \sum_{r=1}^n r^2 + (a+b) \sum_{r=1}^n r + ab \sum_{r=1}^n 1 \equiv \frac{1}{3}n(n-1)(n+4)$$

$$\Rightarrow \frac{1}{6}n(n+1)(2n+1) + (a+b) \frac{1}{2}n(n+1) + abn \equiv \frac{1}{3}n(n-1)(n+4)$$

$$\Rightarrow n(n+1)(2n+1) + 3(a+b)n(n+1) + 6abn \equiv 2n(n-1)(n+4)$$

Dividing by n , $n \neq 0$, and expanding both sides

$$\Rightarrow (n+1)(2n+1) + 3(a+b)(n+1) + 6ab \equiv 2(n-1)(n+4)$$

$$\Rightarrow 2n^2 + 3n + 1 + 3(a+b)(n+1) + 6ab \equiv 2n^2 + 6n - 8$$

$$\Rightarrow 3(a+b)(n+1) + 6ab \equiv 3n - 9$$

$$\Rightarrow 3(a+b)n + 3(a+b) + 6ab \equiv 3n - 9$$

$$\therefore \begin{cases} 3(a+b) = 3 \\ a+b = 1 \end{cases} \quad \begin{cases} 3(a+b) + 6ab = -9 \\ a+b + 2ab = -3 \\ 1 + 2ab = -3 \\ 2ab = -4 \\ ab = -2 \end{cases}$$

If by inspection or solving we obtain $a=2$ & $b=-1$ (or any order), as equations are symmetric

Question 3 (***)

By using an algebraic method, find the value of

$$99^2 - 97^2 + 95^2 - 93^2 + \dots + 3^2 - 1^2$$

 , 5000

Method A
REGROUP THE TERMS

$$\begin{aligned}
 &= (99^2 - 97^2 + 95^2 - 93^2 + \dots + 3^2 - 1^2) \\
 &= [(99^2 + 95^2 + 91^2 + \dots + 3^2) - (97^2 + 93^2 + 89^2 + \dots + 1^2)] \\
 &= \sum_{k=1}^{25} [(4k-1)^2 - (4k-3)^2] \quad \text{(COMMON DIFFERENCE)} \\
 &= \sum_{k=1}^{25} [(4k-1) - (4k-3)](4k-1 + 4k-3) \quad \text{(DIFFERENCE OF SQUARES)} \\
 &= \sum_{k=1}^{25} (4k-4) \times 2 \\
 &= \sum_{k=1}^{25} (16k - 8) \\
 &= 16 \sum_{k=1}^{25} k - 8 \sum_{k=1}^{25} 1 \quad \text{(SUMMATION OF THE SQUARES)} \\
 &= 16 \times \frac{1}{2} \times 25 \times 26 - 8 \times 25 \quad \sum_{k=1}^n k = \frac{1}{2}n(n+1) \\
 &= 5000 - 200 \\
 &= 5000
 \end{aligned}$$

Method B
REGROUP THE TERMS AS PAIRS

$$\begin{aligned}
 &= 99^2 - 97^2 + 95^2 - 93^2 + \dots + 3^2 - 1^2 \\
 &= (99^2 - 97^2) + (95^2 - 93^2) + (91^2 - 89^2) + \dots + (3^2 - 1^2) \\
 &= (98-96)(98+96) + (94-92)(94+92) + (90-88)(90+88) + \dots + (3-1)(3+1) \\
 &= 2(98) + 2(96) + 2(94) + \dots + 2(4) \\
 &= 2[4 + 6 + 8 + \dots + 100 + 108 + 116] \\
 &= 2 \times 4 [1 + 3 + 5 + \dots + 45 + 47 + 49] \\
 &= 8 \times \text{Arithmetic Progression sum } n=1, d=2, n=25 \\
 &= 8 \times \frac{25}{2} [1+49] \\
 &= 8 \times \frac{25 \times 50}{2} \\
 &= 5000
 \end{aligned}$$

$$\begin{aligned}
 49 &= 1 + 2 + \dots + 25 \\
 49 &= 1 + 2 + \dots + 25 \\
 49 &= 1 + 2 + \dots + 25 \\
 49 &= 1 + 2 + \dots + 25
 \end{aligned}$$

Question 4 (****+)

Show clearly that

$$1^3 - 2^3 + 3^3 - 4^3 + \dots - 40^3 = -33200.$$

5¹, proof

Method 1

$$1^3 - 2^3 + 3^3 - 4^3 + \dots - 40^3 = (1^3 + 3^3 + \dots + 39^3) - (2^3 + 4^3 + \dots + 40^3)$$

$$= \sum_{r=1}^{39} (2r-1)^3 - \sum_{r=1}^{20} (2r)^3$$

$$= \sum_{r=1}^{39} [(2r-1)^3 - (2r)^3]$$

$$= \sum_{r=1}^{39} [8r^3 - 12r^2 + 6r - 1 - 8r^3]$$

$$= -12 \sum_{r=1}^{39} r^2 + 6 \sum_{r=1}^{39} r - \sum_{r=1}^{39} 1$$

USING SUMMATION FORMULA RESULT

$$\sum_{r=1}^n r^2 = \frac{1}{6}n(n+1)(2n+1) \quad \sum_{r=1}^n r = \frac{1}{2}n(n+1)$$

$$\dots = -12 \times \frac{1}{6} \times 39 \times 40 \times 78 + 6 \times \frac{1}{2} \times 39 \times 40 - 39$$

$$= -2 \times 20 \times 21 \times 41 + 3 \times 20 \times 21 - 1$$

$$= 20 [-21 \times 21 \times 41 + 3 \times 21 - 1]$$

$$= 20 [21(-81 + 3) - 1]$$

$$= 20 [21 \times (-78) - 1]$$

$$= 20 [-1660]$$

$$= -33200$$

Method 2

$$1^3 - 2^3 + 3^3 - 4^3 + \dots - 40^3 = (1^3 + 3^3 + \dots + 40^3) - 2(2^3 + 4^3 + \dots + 40^3)$$

$$= \sum_{r=1}^{40} (2r-1)^3 - 2 \sum_{r=1}^{20} (2r)^3$$

$$= \sum_{r=1}^{40} (2r-1)^3 - 16 \sum_{r=1}^{20} r^3$$

USING THE SUMMATION FORMULA

$$\sum_{r=1}^n r^3 = \frac{1}{4}n^2(n+1)^2$$

$$\dots = \frac{1}{4} \times 40^2 \times 41^2 - 16 \times \frac{1}{4} \times 20^2 \times 21^2$$

$$= \frac{1}{4} (1600 \times 1681) - 4 \times 20^2 \times 21^2$$

$$= 20^2 \times 41^2 - 4 \times 20^2 \times 21^2$$

$$= 20^2 [41^2 - 4 \times 21^2]$$

QUICK CALCULATIONS

41	21	441	1764	83
$\times 41$	$\times 21$	$\times 41$	$\times 21$	$\times 4(100)$
1681	441	1764	83	33200

$$\dots = 400 (1681 - 4 \times 441)$$

$$= 400 (1681 - 1764)$$

$$= 400 \times (-83)$$

$$= -33200$$

Question 5 (***)

The positive integer functions f and g are defined as

$$f(n) = \sum_{r=1}^n r^3 \quad \text{and} \quad g(n) = 1 + \sum_{r=1}^n (2r+1).$$

Evaluate

$$\sum_{n=1}^{39} \left[\frac{f(n)}{g(n)} \right].$$

, 5135

$f(n) = \sum_{r=1}^n r^3$; $g(n) = 1 + \sum_{r=1}^n (2r+1)$

- Define the "individual components" in simplified form
 - $f(n) = \sum_{r=1}^n r^3 = \frac{1}{4}n^2(n+1)^2$
 - $g(n) = 1 + \sum_{r=1}^n (2r+1) = 1 + 2 \sum_{r=1}^n r + \sum_{r=1}^n 1$
 - $= 1 + 2 \times \frac{1}{2}n(n+1) + n$
 - $= 1 + n(n+1) + n = 1 + n^2 + n + n$
 - $= n^2 + 2n + 1 = (n+1)^2$
- Hence use above
 - $\sum_{n=1}^{39} \frac{f(n)}{g(n)} = \sum_{n=1}^{39} \frac{\frac{1}{4}n^2(n+1)^2}{(n+1)^2} = \sum_{n=1}^{39} \frac{1}{4}n^2$
 - $= \frac{1}{4} \times \frac{1}{2} \times (39)(40)(80)$
 - $= \frac{1}{2} \times 39 \times 40 \times 17$
 - $= 5135$

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SUMMATIONS

BY FORMULAS

8 ENRICHMENT QUESTIONS

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Question 1 (****)

Use standard summation results to prove that

$$\sum_{r=n}^{2n} (n-r)^2 = \sum_{r=1}^n r^2.$$

, proof

EXPAND AND FIND

$$\begin{aligned} \sum_{r=n}^{2n} (n-r)^2 &= \sum_{r=n}^{2n} (r^2 - 2nr + n^2) \\ &= \sum_{r=n}^{2n} r^2 - 2n \sum_{r=n}^{2n} r + \sum_{r=n}^{2n} n^2 \\ &= n^2 [2n - (n+1)] - 2n \left[\frac{1}{2}(2n)(2n+1) - \frac{1}{2}(n)(n+1) \right] \\ &\quad + \frac{1}{2}(2n)(2n+1)(2n+1) - \frac{1}{2}(n)(n+1)(n+1) \\ &= n^2(2n) - 2n \left[n(2n+1) - \frac{1}{2}n(n+1) \right] + \frac{1}{2}n(2n+1)(2n+1) - \frac{1}{2}n(n+1)(n+1) \\ &= n^2(2n) - 2n^2(2n+1) + n^2(n+1) + \frac{1}{2}n(2n+1)(2n+1) - \frac{1}{2}n(n+1)(n+1) \end{aligned}$$

FACTORISE $\frac{1}{2}n$ AT FIRST

$$\begin{aligned} &= \frac{1}{2}n \left[2n(2n) - 2n(2n+1) + n(n+1) + 2(2n+1)(2n+1) - (n+1)(n+1) \right] \\ &= \frac{1}{2}n \left[4n^2 - 4n^2 - 2n + n^2 + n + 2(4n^2 + 4n + 1) - (n^2 + 2n + 1) \right] \\ &= \frac{1}{2}n \left[2n^2 + 2n + 1 \right] \\ &= \frac{1}{2}n(2n+1)(n+1) \\ &= \sum_{r=1}^n r^2 \end{aligned}$$

Q.E.D.

Question 2 (****)

Find the sum of the first 16 terms of the following series.

$$\frac{1^3}{1} + \frac{1^3 + 2^3}{1+3} + \frac{1^3 + 2^3 + 3^3}{1+3+5} + \frac{1^3 + 2^3 + 3^3 + 4^3}{1+3+5+7} + \dots$$

SP 4, 446

Handwritten solution for Question 2:

START BY WRITING THE SERIES EXPRESSION COMPACTLY

$$\sum_{n=1}^{16} \left[\frac{1^3 + 2^3 + \dots + n^3}{1 + 3 + 5 + \dots + (2n-1)} \right]$$

USING STANDARD SUMMATION FORMULAS

$$= \sum_{n=1}^{16} \left[\frac{\frac{1}{4}n^2(n+1)^2}{\frac{1}{2}n(2n+1)} \right] = \sum_{n=1}^{16} \left[\frac{\frac{1}{4}n^2(n+1)^2}{\frac{1}{2}n(2n+1)} \right]$$

$$= \sum_{n=1}^{16} \left[\frac{\frac{1}{4}n^2(n+1)^2}{n^2} \right] = \frac{1}{4} \sum_{n=1}^{16} (n+1)^2$$

GET THE EXPAND AND USE FORMULAS APART OR TRANSLATE BY 1 STEP

$$\dots = \frac{1}{4} \sum_{k=2}^{17} k^2 = \frac{1}{4} \times \frac{1}{6} \times k(k+1)(2k+1) \Big|_{k=2}^{17} - \frac{1}{4}$$

$$= \frac{1}{4} \times \frac{1}{6} \times 17 \times 18 \times 35 - \frac{1}{4}$$

$$= \frac{1}{4} [211725 - 1]$$

$$= 446$$

Question 3 (****)

The function f is defined for $n \in \mathbb{N}$ as

$$f(n) \equiv 1 \times n^2 + 2(n-1)^2 + 3(n-2)^2 + 4(n-3)^2 + \dots + (n-1) \times 2^2 + n \times 1^2.$$

Determine a simplified expression for the sum of $f(n)$, giving the final answer in fully factorized form.

, $f(n) = \frac{1}{12}n(n+2)(n+1)^2$

The handwritten solution shows two methods for finding the sum of $f(n)$.

Method 1: Writing the series in sigma notation

$$\sum_{r=1}^n [r(n+1-r)^2] = \sum_{r=1}^n [r \{ (n+1)^2 - 2(n+1)r + r^2 \}]$$

$$= \sum_{r=1}^n [(n+1)^2 r - 2(n+1)r^2 + r^3]$$

$$= (n+1)^2 \sum_{r=1}^n r - 2(n+1) \sum_{r=1}^n r^2 + \sum_{r=1}^n r^3$$

Method 2: Using standard summation formulae

$$\sum_{r=1}^n [r(n+1-r)^2] = (n+1) \times \frac{1}{2}n(n+1) - 2(n+1) \times \frac{1}{6}n(n+1)(2n+1) + \frac{1}{24}n(n+1)^2$$

$$= \frac{1}{2}n(n+1)^2 [6(n+1) - 4(2n+1) + 3n]$$

$$= \frac{1}{2}n(n+1)^2 (6n+6 - 8n-4 + 3n)$$

$$= \frac{1}{2}n(n+1)^2 (n+2)$$

Question 4 (****)

Use an algebraic method justifying each step, to find the greatest value of k , $k \in \mathbb{N}$, which satisfies the following inequality.

$$\sum_{r=k+1}^{80} \left[\frac{r-1}{\log_8 r (16)} \right] > 100\,000.$$

, $k = 48$

STEP 1: BY MANIPULATING THE EXPONENTS

$$\frac{1}{\log_8 16} = \log_8 8^2 = r \log_8 8 = r \times \frac{2}{4}$$

SINCE $16 = 8^2$

STEP 2: SUMMING FROM $r=1$ TO n , TO GET A GENERAL EXPRESSION

$$\sum_{r=1}^n \left[\frac{r-1}{\log_8 16} \right] = \sum_{r=1}^n \left[\frac{2}{4} (r-1) \right]$$

$$= \frac{2}{4} \sum_{r=1}^n (r-1)$$

$$= \frac{2}{4} \left[\frac{1}{2} n(n+1)(2n+1) - \frac{1}{2} n(n+1) \right]$$

$$= \frac{2}{4} \left[\frac{1}{2} n(n+1)(2n+1-1) \right]$$

$$= \frac{1}{2} n(n+1)(2n)$$

$$= \frac{1}{2} n(n+1)(2n-1)$$

STEP 3: RETURNING TO THE INEQUALITY

$$\Rightarrow \sum_{r=k+1}^{80} \left[\frac{r-1}{\log_8 r (16)} \right] > 100\,000$$

$$\Rightarrow \frac{1}{2} [80(80^2-1)] - \frac{1}{2} k(k^2-1) > 100\,000$$

$\Rightarrow 80(80^2-1) - k(k^2-1) > 400\,000$

$\Rightarrow k(k^2-1) - 80(80^2-1) < -400\,000$

$\Rightarrow k^3 - k - 511\,920 < -400\,000$

$\Rightarrow k(k+1)(k-1) < 111\,920$

$k \in \mathbb{N}$ BUT IF $k \in \mathbb{R}$

so $f(x) = k(k+1)(k-1)$ is INCREASING FOR $k > 1$

STEP 4: BY TRIAL OF VALUES NOTING THAT $f(x) \approx x^3$

$f(40) = 40 \times 41 \times 39 = 63\,960 < 111\,920$

$f(50) = 50 \times 51 \times 49 = 124\,950 > 111\,920$

$f(49) = 49 \times 50 \times 48 = 117\,600 > 111\,920$

$f(48) = 48 \times 49 \times 47 = 110\,544 < 111\,920$

$\therefore k = 48$

Question 5 (*****)

Use algebra to find the sum of the first 100 terms of the following sequence.

7, 12, 19, 28, 39, 52, 67, 84, 103, ...

, $f(n) = \frac{1}{12}n(n+2)(n+1)^2$

● INVESTIGATING THE PATTERNS FURTHER BY DIFFERENCING

7 + 12 + 19 + 28 + 39 + 52 + 67 + 84 + 103 + ...

5 7 9 11 13 15 17 19

2 2 2 2 2 2 2

As the second differences ARE constant, this is a QUADRATIC PATTERNS, whose QUADRATIC COEFFICIENT IS HALF THE CONSTANT SECOND DIFFERENCE, i.e. $u_n = n^2 + an + b$

● SUPPOSE THE n TH TERM OF THE SEQUENCE/CELLS WAS u_n

n^2	1	4	9	16	25	36
"our series"	7	12	19	28	39	52
	+6	+8	+10	+12	+14	+16

← $2n+4$

● Hence the required n th term is

$$u_n = n^2 + 2n + 4$$

● This we require to find

$$\sum_{n=1}^k (n^2 + 2n + 4) \quad \text{with } k=100$$

● USING THE STANDARD SUMMATION FORMULAE IN k

AND SUBSTITUTE $k=100$ AT THE END

$$\sum_{n=1}^k (n^2 + 2n + 4) = \sum_{n=1}^k n^2 + 2 \sum_{n=1}^k n + 4 \sum_{n=1}^k 1$$

$$= \frac{1}{6}k(k+1)(2k+1) + 2 \times \frac{1}{2}k(k+1) + 4 \times k$$

$$= \frac{1}{6}k(k+1)(2k+1) + k(k+1) + 4k$$

$$= \frac{1}{6}k(k+1)(2k+1) + 6k$$

● LET $k=100$ AND WE OBTAIN

$$\sum_{n=1}^{100} (n^2 + 2n + 4) = \frac{1}{6} \times 100 \times 101 \times 201 + 4 \times 100$$

$$= 348\,850$$

Question 6 (****)

Evaluate the following expression

$$\sum_{n=1}^9 \sum_{m=n+1}^{2n} [2m+n].$$

Detailed workings must be shown.

V, , 1185

PROCEED AS FOLLOWS

$$\sum_{n=1}^9 \left[\sum_{m=n+1}^{2n} (2m+n) \right] = \sum_{n=1}^9 \left[2 \sum_{m=n+1}^{2n} m + n \sum_{m=n+1}^{2n} 1 \right]$$
USE STANDARD SUMMATION FORMULAS

$$= \sum_{n=1}^9 \left[2 \left(\frac{1}{2} \times 4 \times (2n) \times (2n+1) \right) - 2 \left(\frac{1}{2} \times n \times (n+1) \right) + n(2n-n) \right]$$

$$= \sum_{n=1}^9 [4n^2 + 2n - n^2 - n + 4n^2] = \sum_{n=1}^9 (4n^2 + 4n)$$

$$= 4 \sum_{n=1}^9 n^2 + \sum_{n=1}^9 4n$$
FORMULA LINKS MORE STANDARD SUMMATION FORMULAS

$$= 4 \times \left(\frac{1}{6} \times 9 \times 10 \times 19 \right) + \frac{1}{2} \times 4 \times 10$$

$$= 60 \times 19 + 45$$

$$= 1140 + 45$$

$$= 1185$$

Question 7 (****)

Evaluate the following expression

$$\sum_{m=1}^{19} \sum_{n=m}^{19} [2m + n].$$

You may find reversing the order of summation useful in this question

V, ,

IT IS PREFERABLE TO REVERSE THE ORDER OF SUMMATION

$$\sum_{m=1}^{19} \sum_{n=m}^{19} (2m + n)$$

$$= \sum_{n=1}^{19} \sum_{m=1}^n (2m + n)$$

$$= \sum_{n=1}^{19} \left[2 \sum_{m=1}^n m + \sum_{m=1}^n n \right]$$

$$= \sum_{n=1}^{19} \left[2 \cdot \frac{1}{2} n^2 + n \cdot n \right]$$

$$= \sum_{n=1}^{19} [n^2 + n^2] = \sum_{n=1}^{19} 2n^2 = 2 \sum_{n=1}^{19} n^2$$

$$= 2 \left[\frac{1}{3} n^3 + \frac{1}{2} n \right]_{n=1}^{19} = 2 \left[\frac{1}{3} (19^3) + \frac{1}{2} (19) - \left(\frac{1}{3} (1^3) + \frac{1}{2} (1) \right) \right]$$

$$= 2 \left[\frac{1}{3} (6859) + \frac{1}{2} (19) - \left(\frac{1}{3} + \frac{1}{2} \right) \right]$$

$$= 2 \left[2286.33 + 9.5 - 0.5 \right] = 2 \times 2295.33 = 4590.66$$

TRY AND EVALUATE

$$= 19 \times 20 \times \left[\frac{1}{3} \times 19 + \frac{1}{2} \right]$$

$$= 19 \times 10 \times (26 + 1)$$

$$= 19 \times 10 \times 27$$

$$= 19 \times 270$$

$$= 5130 - 270$$

$$= 5130$$

Question 8 (****)

The function f is defined as

$$f(n, y) \equiv \sum_{x=1}^n \frac{x^2 y^x}{k}, \quad n \in \mathbb{N}, y \in \mathbb{R}$$

where $k = \sum_{r=1}^n r^2$.

Use standard results on series to show that

$$\left. \frac{d^2 f}{dy^2} \right|_{y=1} + \left. \frac{df}{dy} \right|_{y=1} - \left[\left. \frac{df}{dy} \right|_{y=1} \right]^2 = \frac{3n^4 + 6n^3 - n^2 - 4n - 4}{20(2n+1)^2}$$

You may assume without proof $\sum_{r=1}^n r^4 = \frac{1}{30}n(n+1)(6n^3 + 9n^2 + n - 1)$.

P, proof

Using standard summation results

$$k = \sum_{x=1}^n x^2 = \frac{1}{6}n(n+1)(2n+1)$$

k does not depend on x, so we can pull it out of the summation

$$\therefore f(n, y) = \sum_{x=1}^n \frac{x^2 y^x}{k} = \frac{1}{k} \sum_{x=1}^n x^2 y^x$$

Differentiate with respect to y - k does not depend on y either

$$\frac{df}{dy} = \frac{1}{k} \sum_{x=1}^n [2x^2 y^{x-1}] = \frac{2}{k} \sum_{x=1}^n x^2 y^{x-1}$$

$$\left. \frac{df}{dy} \right|_{y=1} = \frac{2}{k} \sum_{x=1}^n x^2 = \frac{2}{k} \sum_{x=1}^n x^2 \leftarrow \text{no need to simplify yet}$$

Another differentiation with respect to y is needed

$$\frac{d^2 f}{dy^2} = \frac{2}{k} \sum_{x=1}^n [2x^2 y^{x-2}] = \frac{4}{k} \sum_{x=1}^n x^2 y^{x-2}$$

$$\left. \frac{d^2 f}{dy^2} \right|_{y=1} = \frac{4}{k} \sum_{x=1}^n x^2 = \frac{4}{k} \sum_{x=1}^n x^2 \leftarrow \text{no need to simplify yet}$$

Substituting into the expression given

$$\left. \frac{d^2 f}{dy^2} \right|_{y=1} + \left. \frac{df}{dy} \right|_{y=1} - \left[\left. \frac{df}{dy} \right|_{y=1} \right]^2$$

$$= \frac{4}{k} \sum_{x=1}^n x^2 - \frac{2}{k} \sum_{x=1}^n x^2 + \frac{2}{k} \sum_{x=1}^n x^2 - \left[\frac{2}{k} \sum_{x=1}^n x^2 \right]^2$$

$$= \frac{4}{k} \sum_{x=1}^n x^2 - \frac{2}{k} \left[\sum_{x=1}^n x^2 \right]^2$$

Substitute k = 1/6 n(n+1)(2n+1) and simplify

$$= \frac{4}{\frac{1}{6}n(n+1)(2n+1)} \sum_{x=1}^n x^2 - \frac{2}{\left[\frac{1}{6}n(n+1)(2n+1) \right]^2} \left[\sum_{x=1}^n x^2 \right]^2$$

$$= \frac{24 \sum_{x=1}^n x^2}{n(n+1)(2n+1)} - \frac{2 \left[\sum_{x=1}^n x^2 \right]^2}{\left[\frac{1}{6}n(n+1)(2n+1) \right]^2}$$

$$= \frac{24 \sum_{x=1}^n x^2}{n(n+1)(2n+1)} - \frac{2 \left[\sum_{x=1}^n x^2 \right]^2}{\frac{1}{36} n^2 (n+1)^2 (2n+1)^2}$$

$$= \frac{24 \sum_{x=1}^n x^2}{n(n+1)(2n+1)} - \frac{72 \left[\sum_{x=1}^n x^2 \right]^2}{n^2 (n+1)^2 (2n+1)^2}$$

The denominator is now what we require so they're the same

$$= \frac{24 \sum_{x=1}^n x^2}{n(n+1)(2n+1)} - \frac{72 \left[\sum_{x=1}^n x^2 \right]^2}{n^2 (n+1)^2 (2n+1)^2}$$

$$= \frac{24 \sum_{x=1}^n x^2}{n(n+1)(2n+1)} - \frac{72 \left[\sum_{x=1}^n x^2 \right]^2}{n^2 (n+1)^2 (2n+1)^2}$$

$$= \frac{24 \sum_{x=1}^n x^2}{n(n+1)(2n+1)} - \frac{72 \left[\sum_{x=1}^n x^2 \right]^2}{n^2 (n+1)^2 (2n+1)^2}$$

Let the required numerator

$$\therefore \left. \frac{d^2 f}{dy^2} \right|_{y=1} + \left. \frac{df}{dy} \right|_{y=1} - \left[\left. \frac{df}{dy} \right|_{y=1} \right]^2 = \frac{3n^4 + 6n^3 - n^2 - 4n - 4}{20(2n+1)^2} \quad \text{As required}$$

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SUMMATIONS

METHOD OF DIFFERENCES

8 BASIC QUESTIONS

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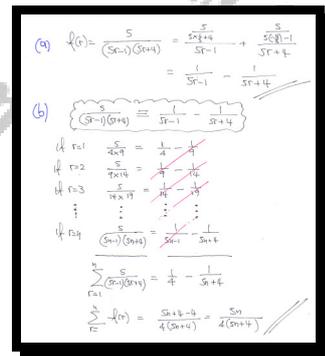
Question 1 (**)

$$f(r) = \frac{5}{(5r-1)(5r+4)}, r \in \mathbb{N}$$

- a) Express $f(r)$ into partial fractions
- b) Hence show that

$$\sum_{r=1}^n f(r) = \frac{5n}{4(5n+4)}$$

$$f(r) = \frac{1}{5r-1} - \frac{1}{5r+4}$$



Question 2 (**)

a) Show carefully that

$$\frac{1}{r^2} - \frac{1}{(r+1)^2} = \frac{2r+1}{r^2(r+1)^2}$$

b) Hence use the method of differences to find

$$\sum_{r=1}^n \frac{2r+1}{r^2(r+1)^2}$$

$$1 - \frac{1}{(n+1)^2}$$

(a) LHS = $\frac{1}{r^2} - \frac{1}{(r+1)^2} = \frac{(r+1)^2 - r^2}{r^2(r+1)^2} = \frac{r^2 + 2r + 1 - r^2}{r^2(r+1)^2} = \frac{2r+1}{r^2(r+1)^2}$

(b) $\sum_{r=1}^n \frac{2r+1}{r^2(r+1)^2}$

r=1: $\frac{3}{1^2 \cdot 2^2} = \frac{1}{1^2} - \frac{1}{2^2}$

r=2: $\frac{5}{2^2 \cdot 3^2} = \frac{1}{2^2} - \frac{1}{3^2}$

r=3: $\frac{7}{3^2 \cdot 4^2} = \frac{1}{3^2} - \frac{1}{4^2}$

...

r=n: $\frac{2n+1}{n^2(n+1)^2} = \frac{1}{n^2} - \frac{1}{(n+1)^2}$

$\sum_{r=1}^n \frac{2r+1}{r^2(r+1)^2} = 1 - \frac{1}{(n+1)^2}$

Question 3 (**)

a) Show carefully that

$$\frac{1}{r!} - \frac{1}{(r+1)!} = \frac{r}{(r+1)!}$$

b) Hence find

$$\sum_{r=1}^n \frac{r}{(r+1)!}$$

$$\boxed{1 - \frac{1}{(n+1)!}}$$

$\frac{1}{r!} - \frac{1}{(r+1)!} = \frac{r+1}{(r+1)!} - \frac{1}{(r+1)!} = \frac{r+1-1}{(r+1)!} = \frac{r}{(r+1)!}$
 $\frac{r}{(r+1)!} = \frac{1}{r!} - \frac{1}{(r+1)!}$
 $r=1 \Rightarrow \frac{1}{2!} = \frac{1}{1!} - \frac{1}{2!}$
 $r=2 \Rightarrow \frac{2}{3!} = \frac{1}{2!} - \frac{1}{3!}$
 $r=3 \Rightarrow \frac{3}{4!} = \frac{1}{3!} - \frac{1}{4!}$
 \vdots
 $r=n \Rightarrow \frac{n}{(n+1)!} = \frac{1}{n!} - \frac{1}{(n+1)!}$
 $\text{Ans } \sum_{r=1}^n \frac{r}{(r+1)!} = 1 - \frac{1}{(n+1)!}$

Question 4 (***)

$$f(r) = \frac{1}{r(r+2)}, r \in \mathbb{N}$$

- a) Express $f(r)$ into partial fractions.
 b) Hence show that

$$\sum_{r=1}^{30} f(r) = \frac{1425}{1984}$$

$$f(r) = \frac{1}{2r} - \frac{1}{2(r+2)}$$

Handwritten solution for Question 4:

(a) $f(r) = \frac{1}{r(r+2)} = \frac{1}{r} - \frac{1}{r+2}$

(b) $\sum_{r=1}^{30} f(r) = \sum_{r=1}^{30} \left(\frac{1}{r} - \frac{1}{r+2} \right)$

• $r=1: \frac{1}{1 \times 3} = \frac{1}{1} - \frac{1}{3}$

• $r=2: \frac{1}{2 \times 4} = \frac{1}{2} - \frac{1}{4}$

• $r=3: \frac{1}{3 \times 5} = \frac{1}{3} - \frac{1}{5}$

• $r=4: \frac{1}{4 \times 6} = \frac{1}{4} - \frac{1}{6}$

• \vdots

• $r=29: \frac{1}{29 \times 31} = \frac{1}{29} - \frac{1}{31}$

• $r=30: \frac{1}{30 \times 32} = \frac{1}{30} - \frac{1}{32}$

$\Rightarrow \sum_{r=1}^{30} \frac{1}{r(r+2)} = 1 + \frac{1}{2} - \frac{1}{31} - \frac{1}{32}$

$\Rightarrow \sum_{r=1}^{30} f(r) = \frac{1425}{1984}$

Question 6 (***)

a) Simplify $\frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)}$ into a single fraction.

b) Hence show that

$$\sum_{r=1}^{20} \left[\frac{1}{r(r+1)(r+2)} \right] = \frac{115}{462}$$

$$\frac{2}{r(r+1)(r+2)}$$

a) COMMON + COMMON DENOMINATOR AND +ve

$$\frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)} = \frac{(r+2) - r}{r(r+1)(r+2)} = \frac{2}{r(r+1)(r+2)}$$

b) USE THE IDENTITY FROM (a)

$$\frac{2}{r(r+1)(r+2)} = \frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)}$$

• IF r=1: $\frac{2}{1 \times 2 \times 3} = \frac{1}{1 \times 2} - \frac{1}{2 \times 3}$

• IF r=2: $\frac{2}{2 \times 3 \times 4} = \frac{1}{2 \times 3} - \frac{1}{3 \times 4}$

• IF r=3: $\frac{2}{3 \times 4 \times 5} = \frac{1}{3 \times 4} - \frac{1}{4 \times 5}$

• IF r=20: $\frac{2}{20 \times 21 \times 22} = \frac{1}{20 \times 21} - \frac{1}{21 \times 22}$

→ $\sum_{r=1}^{20} \left[\frac{2}{r(r+1)(r+2)} \right] = \frac{1}{2} - \frac{1}{21 \times 22}$ (ADDING)

→ $\sum_{r=1}^{20} \left[\frac{1}{r(r+1)(r+2)} \right] = \frac{1}{2} - \frac{1}{462}$

→ $\sum_{r=1}^{20} \left[\frac{1}{r(r+1)(r+2)} \right] = \frac{1}{2} \left(\frac{1}{1} - \frac{1}{462} \right) = \frac{115}{462}$ ✓

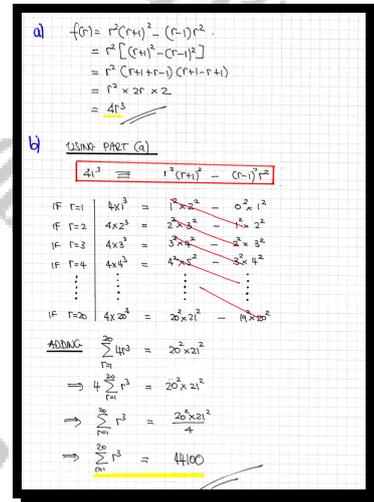
Question 7 (***)

$$f(r) \equiv r^2(r+1)^2 - (r-1)^2 r^2, r \in \mathbb{N}.$$

- a) Simplify $f(r)$ as far as possible.
- b) Use the method of differences to show that

$$\sum_{r=1}^{20} r^3 = 44100.$$

44100, $f(r) = 4r^3$



Question 8 (***)

$$f(r) = \frac{1}{r(r+2)}, \quad r \in \mathbb{N}.$$

- a) Express $f(r)$ in partial fractions.
 b) Hence prove, by the method of differences, that

$$\sum_{r=1}^n f(r) = \frac{n(An+B)}{4(n+1)(n+2)},$$

where A and B are constants to be found.

, ,

a) BY INSPECTION (CHECK FOR METHOD OF SIMILAR)

$$f(r) = \frac{1}{r(r+2)} = \frac{A}{r} + \frac{B}{r+2} = \frac{A(r+2) + Br}{r(r+2)}$$

$$= \frac{Ar + 2A + Br}{r(r+2)} = \frac{(A+B)r + 2A}{r(r+2)}$$

b) SETTING UP (a) AS AN EQUATION

$$\frac{2}{r(r+2)} = \frac{A}{r} - \frac{B}{r+2}$$

- $r=1$: $\frac{2}{1 \cdot 3} = \frac{A}{1} - \frac{B}{3} \Rightarrow \frac{2}{3} = A - \frac{B}{3}$
- $r=2$: $\frac{2}{2 \cdot 4} = \frac{A}{2} - \frac{B}{4} \Rightarrow \frac{1}{2} = \frac{A}{2} - \frac{B}{4}$
- $r=3$: $\frac{2}{3 \cdot 5} = \frac{A}{3} - \frac{B}{5} \Rightarrow \frac{2}{15} = \frac{A}{3} - \frac{B}{5}$
- $r=4$: $\frac{2}{4 \cdot 6} = \frac{A}{4} - \frac{B}{6} \Rightarrow \frac{1}{6} = \frac{A}{4} - \frac{B}{6}$
- \vdots
- $r=n+1$: $\frac{2}{(n+1)(n+2)} = \frac{A}{n+1} - \frac{B}{n+2}$
- $r=n$: $\frac{2}{n(n+2)} = \frac{A}{n} - \frac{B}{n+2}$

$\Rightarrow \sum_{r=1}^n \frac{2}{r(r+2)} = \frac{2}{2} = \frac{2}{1} = \frac{2}{1} - \frac{1}{n+2}$ **ADDING**

$$\Rightarrow \sum_{r=1}^n \frac{1}{r(r+2)} = \frac{2(n+1)(n+2) - 2(n+1)}{2(n+1)(n+2)}$$

$$\Rightarrow \sum_{r=1}^n \frac{1}{r(r+2)} = \frac{2(n^2+2n+2) - 2n - 2}{2(n+1)(n+2)}$$

Alternative method on the right page:

$$\Rightarrow \sum_{r=1}^n \frac{1}{r(r+2)} = \frac{3n^2 + 9n + 6 - 4n - 6}{2(n+1)(n+2)}$$

$$\Rightarrow \sum_{r=1}^n \frac{1}{r(r+2)} = \frac{3n^2 + 5n}{2(n+1)(n+2)}$$

$$\Rightarrow \sum_{r=1}^n \frac{1}{r(r+2)} = \frac{n(3n+5)}{2(n+1)(n+2)}$$

16 $A=3$
16 $B=5$

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SUMMATIONS

METHOD OF DIFFERENCES

8 STANDARD QUESTIONS

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Question 1 (***)

Use the method of differences to show that

$$\frac{1}{1 \times 2 \times 3} + \frac{1}{2 \times 3 \times 4} + \frac{1}{3 \times 4 \times 5} + \dots + \frac{1}{n(n+1)(n+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$$

Q.E.D., proof

Using Partial Fractions

$$\frac{1}{k(k+1)(k+2)} = \frac{1}{2k} - \frac{1}{k+1} + \frac{1}{k+2}$$

$$\frac{1}{k(k+1)(k+2)} = \frac{1}{2k} - \frac{1}{k+1} + \frac{1}{k+2}$$

Double the above identity & simplify

$$\frac{2}{k(k+1)(k+2)} = \frac{1}{k} - \frac{2}{k+1} + \frac{1}{k+2}$$

- $k=1$ $\frac{2}{1 \times 2 \times 3} = \frac{1}{1} - \frac{2}{2} + \frac{1}{3}$
- $k=2$ $\frac{2}{2 \times 3 \times 4} = \frac{1}{2} - \frac{2}{3} + \frac{1}{4}$
- $k=3$ $\frac{2}{3 \times 4 \times 5} = \frac{1}{3} - \frac{2}{4} + \frac{1}{5}$
- $k=4$ $\frac{2}{4 \times 5 \times 6} = \frac{1}{4} - \frac{2}{5} + \frac{1}{6}$
- \vdots
- $k=n-1$ $\frac{2}{(n-1)n(n+1)} = \frac{1}{n-1} - \frac{2}{n} + \frac{1}{n+1}$
- $k=n$ $\frac{2}{n(n+1)(n+2)} = \frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2}$

$$\rightarrow \sum_{k=1}^n \frac{2}{k(k+1)(k+2)} = \frac{1}{1} - \frac{2}{n+1} + \frac{1}{n+2}$$

$$\rightarrow 2 \sum_{k=1}^n \frac{1}{k(k+1)(k+2)} = \frac{(n+1)(n+2) - 2(n+2) + 2(n+1)}{2(n+1)(n+2)}$$

Question 2 (***)

$$u_r = \frac{1}{6}r(r+1)(4r+11), r \in \mathbb{N}$$

- a) Simplify $u_r - u_{r-1}$ as far as possible.
- b) By using the method of differences, or otherwise, find the sum of the first 100 terms of the following series.

$$(1 \times 5) + (2 \times 7) + (3 \times 9) + (4 \times 11) + \dots$$

$$\boxed{}, \boxed{r(2r+3)}, \boxed{691850}$$

a) $u_r - u_{r-1} = \frac{1}{6}r(r+1)(4r+11) - \frac{1}{6}(r-1)r[4(r-1)+11]$
 $= \frac{1}{6}r(r+1)(4r+11) - \frac{1}{6}r(r-1)(4r+7)$
 $= \frac{1}{6}r [(r+1)(4r+11) - (r-1)(4r+7)]$
 $= \frac{1}{6}r [4r^2 + 15r + 11 - 4r^2 - 2r + 7]$
 $= \frac{1}{6}r (12r + 18)$
 $= r(2r+3)$

b) Method of Differences
 $(1 \times 5) + (2 \times 7) + (3 \times 9) + (4 \times 11) + \dots + (100 \times 203)$
 $\Rightarrow u_r - u_{r-1} = r(2r+3)$

• r=1	$u_1 = 1 \times 5$
• r=2	$u_2 = 2 \times 7$
• r=3	$u_3 = 3 \times 9$
• r=4	$u_4 = 4 \times 11$
• ...	• ...
• r=100	$u_{100} = 100 \times 203$

$\Rightarrow u_{100} - u_0 = (1 \times 5) + (2 \times 7) + (3 \times 9) + \dots + (100 \times 203)$
 $\Rightarrow \frac{1}{6} \times 100 \times 101 \times 41 = 0 = \sum_{r=1}^{100} r(2r+3)$
 $\Rightarrow \sum_{r=1}^{100} r(2r+3) = 691850$

Question 3 (***)

$$f(r) = \frac{1}{(r+1)(r-1)}, r \in \mathbb{N}.$$

- a) Express $f(r)$ into partial fractions.
 b) Hence show that

$$\sum_{r=2}^n \frac{1}{r^2-1} = \frac{3}{4} - \frac{1}{2n} - \frac{1}{2(n+1)}.$$

- c) State the value of

$$\sum_{r=2}^{\infty} \frac{1}{r^2-1}$$

$$\boxed{}, \quad f(r) = \frac{1}{2(r-1)} - \frac{1}{2(r+1)}, \quad \boxed{\frac{3}{4}}$$

Handwritten solution for Question 3:

a) $f(r) = \frac{1}{(r+1)(r-1)} = \frac{\frac{1}{2}}{r-1} - \frac{\frac{1}{2}}{r+1} = \frac{1}{2(r-1)} - \frac{1}{2(r+1)}$
By using the "double" rule

b) Check that (a) is "double" for similarity
 $\frac{1}{r^2-1} = \frac{1}{(r+1)(r-1)} = \frac{1}{r-1} - \frac{1}{r+1}$

• $r=2$: $\frac{1}{2^2-1} = \frac{1}{1} - \frac{1}{3}$
 • $r=3$: $\frac{1}{3^2-1} = \frac{1}{2} - \frac{1}{4}$
 • $r=4$: $\frac{1}{4^2-1} = \frac{1}{3} - \frac{1}{5}$
 • $r=5$: $\frac{1}{5^2-1} = \frac{1}{4} - \frac{1}{6}$
 • $r=6$: $\frac{1}{6^2-1} = \frac{1}{5} - \frac{1}{7}$
 • ...
 • $r=n-1$: $\frac{1}{(n-1)^2-1} = \frac{1}{n-2} - \frac{1}{n}$
 • $r=n$: $\frac{1}{n^2-1} = \frac{1}{n-1} - \frac{1}{n+1}$

$\sum_{r=2}^n \frac{1}{r^2-1} = 1 + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1}$
 $= \frac{3}{2} - \frac{1}{n} - \frac{1}{n+1}$
 $= \frac{3}{2} - \frac{1}{2n} - \frac{1}{2(n+1)}$

c) As $n \rightarrow \infty$ the sum tends to $\frac{3}{2}$

Question 5 (***)

Use the method of differences to show that

$$\frac{1}{1 \times 2 \times 3} + \frac{4}{2 \times 3 \times 4} + \frac{7}{3 \times 4 \times 5} + \dots + \frac{3n-2}{n(n+1)(n+2)} = \frac{n^2}{(n+1)(n+2)}$$

□, proof

PROOF BY PARTIAL FRACTIONS (CONSIDER OR FOLLOW METHOD)

$$\frac{3n-2}{n(n+1)(n+2)} = \frac{\frac{-1}{1 \times 2} + \frac{\frac{5}{1 \times 1} + \frac{-8}{2 \times 2}}{n+1} + \frac{\frac{4}{1 \times 2}}{n+2}$$

$$= \frac{-1}{n} + \frac{5}{n+1} - \frac{4}{n+2}$$

SETTING UP THE METHOD OF DIFFERENCES BASED ON THE ABOVE RESULT

$$\frac{3r-2}{r(r+1)(r+2)} = -\frac{1}{r} + \frac{5}{r+1} - \frac{4}{r+2}$$

- If $r=1$: $\frac{1}{1 \times 2 \times 3} = -\frac{1}{1} + \frac{5}{2} - \frac{4}{3}$
- If $r=2$: $\frac{4}{2 \times 3 \times 4} = -\frac{1}{2} + \frac{5}{3} - \frac{4}{4}$
- If $r=3$: $\frac{7}{3 \times 4 \times 5} = -\frac{1}{3} + \frac{5}{4} - \frac{4}{5}$
- If $r=4$: $\frac{10}{4 \times 5 \times 6} = -\frac{1}{4} + \frac{5}{5} - \frac{4}{6}$
- If $r=n$: $\frac{3n-2}{n(n+1)(n+2)} = -\frac{1}{n} + \frac{5}{n+1} - \frac{4}{n+2}$

$$\sum_{r=1}^n \frac{3r-2}{r(r+1)(r+2)} = \left(-1 + \frac{5}{2} - \frac{4}{3}\right) + \left(\frac{1}{2} - \frac{5}{3} + \frac{4}{4}\right) + \dots + \left(-\frac{1}{n} + \frac{5}{n+1} - \frac{4}{n+2}\right)$$

$$= 1 + \frac{1}{n+1} - \frac{4}{n+2}$$

$$= \frac{(n+1)(n+2) + (n+1) - 4(n+1)}{(n+1)(n+2)}$$

$$= \frac{n^2 + 3n + 2 + n + 1 - 4n - 4}{(n+1)(n+2)}$$

$$= \frac{n^2}{(n+1)(n+2)}$$

Question 6 (*)**

It is given that

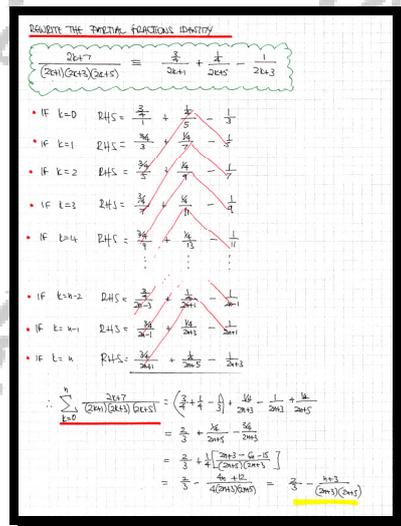
$$\frac{2k+7}{(2k+1)(2k+3)(2k+5)} \equiv \frac{3}{4(2k+1)} - \frac{1}{(2k+3)} + \frac{1}{4(2k+5)}$$

Use the method of differences to find a simplified expression for

$$\frac{7}{1 \times 3 \times 5} + \frac{9}{3 \times 5 \times 7} + \frac{11}{5 \times 7 \times 9} + \dots + \frac{2n+7}{(2n+1)(2n+3)(2n+5)}$$

Give your answer in the form $\frac{2}{3} - f(n)$, where $f(n)$ is a single simplified fraction.

, $f(n) = -\frac{n+3}{(2n+3)(2n+5)}$



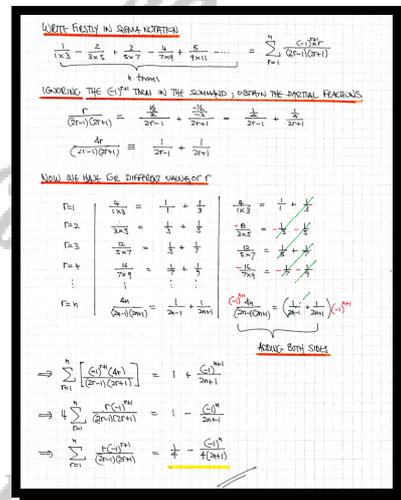
Question 7 (****)

Use the method of differences to find a simplified expression for the first n terms of the following series.

$$\frac{1}{1 \times 3} + \frac{2}{3 \times 5} + \frac{3}{5 \times 7} + \frac{4}{7 \times 9} + \dots$$

Give your answer in the form $\frac{1}{4} - f(n)$, where $f(n)$ is a single simplified fraction.

$$, f(n) = \frac{(-1)^n}{4(2n+1)}$$



Question 8 (***)

$$f(r) = \frac{1}{\sqrt{r+2} + \sqrt{r}}, \quad r \geq 0.$$

- a) Rationalize the denominator of $f(r)$.
- b) Find an expression for

$$\sum_{r=1}^n f(r).$$

- c) Show clearly that

$$\sum_{r=1}^{48} f(r) = 3 + 2\sqrt{2}$$

$$\boxed{}, \quad \boxed{f(r) = \frac{\sqrt{r+2} - \sqrt{r}}{2}}, \quad \boxed{\sum_{r=1}^n f(r) = \frac{1}{2}(\sqrt{n+2} + \sqrt{n+1} - \sqrt{2} - 1)}$$

a) USING STANDARD STEPS

$$\frac{1}{\sqrt{r+2} + \sqrt{r}} = \frac{\sqrt{r+2} - \sqrt{r}}{(\sqrt{r+2} + \sqrt{r})(\sqrt{r+2} - \sqrt{r})} = \frac{\sqrt{r+2} - \sqrt{r}}{r+2 - r} = \frac{\sqrt{r+2} - \sqrt{r}}{2}$$

b) USING PART (a)

$$\frac{2}{\sqrt{r+2} + \sqrt{r}} \equiv \sqrt{r+2} - \sqrt{r}$$

$r=1: \frac{2}{\sqrt{1+2} + \sqrt{1}} = \sqrt{1+2} - \sqrt{1}$
 $r=2: \frac{2}{\sqrt{2+2} + \sqrt{2}} = \sqrt{2+2} - \sqrt{2}$
 $r=3: \frac{2}{\sqrt{3+2} + \sqrt{3}} = \sqrt{3+2} - \sqrt{3}$
 $r=4: \frac{2}{\sqrt{4+2} + \sqrt{4}} = \sqrt{4+2} - \sqrt{4}$
 \vdots
 $r=n-1: \frac{2}{\sqrt{(n-1)+2} + \sqrt{(n-1)}} = \sqrt{(n-1)+2} - \sqrt{(n-1)}$
 $r=n: \frac{2}{\sqrt{n+2} + \sqrt{n}} = \sqrt{n+2} - \sqrt{n}$

$\Rightarrow \sum_{r=1}^n \frac{2}{\sqrt{r+2} + \sqrt{r}} = \sqrt{n+2} + \sqrt{n+1} - \sqrt{2} - 1$ (ADDING BOTH SIDES)
 $\Rightarrow \sum_{r=1}^n \frac{1}{\sqrt{r+2} + \sqrt{r}} = \frac{1}{2}(\sqrt{n+2} + \sqrt{n+1} - \sqrt{2} - 1)$

c) LET $n=48$ IN PART (b)

$$\sum_{r=1}^{48} f(r) = \frac{1}{2}[\sqrt{48+2} + \sqrt{48+1} - \sqrt{2} - 1]$$

$$= \frac{1}{2}[\sqrt{50} + \sqrt{49} - \sqrt{2} - 1]$$

$$= \frac{1}{2}[5\sqrt{2} + 7 - \sqrt{2} - 1]$$

$$= \frac{1}{2}[4\sqrt{2} + 6]$$

$$= 2 + 2\sqrt{2} \quad \checkmark$$

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SUMMATIONS

METHOD OF DIFFERENCES

3 HARD QUESTIONS

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Question 1 (****+)

Consider the following infinite convergent series.

$$\frac{3}{1 \times 2} - \frac{5}{2 \times 3} + \frac{7}{3 \times 4} - \frac{9}{4 \times 5} + \frac{11}{5 \times 6} - \dots$$

- a) Use the method of differences, to find the sum of this series.
- b) Verify the answer of part (a) by using a method based on the Maclaurin expansion of $\ln(1+x)$.

V, ,

a) SPLIT BY OBTAINING THE GENERAL TERM IN SIGMA NOTATION

$$\frac{3}{1 \times 2} - \frac{5}{2 \times 3} + \frac{7}{3 \times 4} - \frac{9}{4 \times 5} + \frac{11}{5 \times 6} - \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (2n+1)}{n(n+1)}$$

INDICATES $(-1)^{n+1}$ EXPRESS THE FIRST TWO PARTIAL FRACTIONS BY GEORGE

$$\frac{2n+1}{n(n+1)} = \frac{1}{n} + \frac{1}{n+1}$$

WIND UP HAVE

$n=1$ $\frac{3}{1 \times 2} = \frac{1}{1} + \frac{1}{2}$

$n=2$ $-\frac{5}{2 \times 3} = -\frac{1}{2} - \frac{1}{3}$

$n=3$ $+\frac{7}{3 \times 4} = \frac{1}{3} + \frac{1}{4}$

$n=4$ $-\frac{9}{4 \times 5} = -\frac{1}{4} - \frac{1}{5}$

\vdots

$n=n$ $\frac{(-1)^{n+1} (2n+1)}{n(n+1)} = \frac{(-1)^{n+1}}{n} + \frac{(-1)^{n+1}}{n+1}$

$$\sum_{n=1}^{\infty} \left[\frac{(-1)^{n+1}}{n} + \frac{(-1)^{n+1}}{n+1} \right] = 1 + (-1)^{n+1} \frac{1}{n+1}$$

\therefore As $n \rightarrow \infty$ THE SUM TO INFINITY IS $\underline{1}$

b) LOOKING AT THE EXPANSION OF $\ln(1+x)$, VALID FOR $-1 < x \leq 1$

- $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}$
- LET $x=1$
- $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

USING THE PARTIAL FRACTIONS FROM PART (a)

$$\sum_{n=1}^{\infty} \left[\frac{(-1)^{n+1}}{n} + \frac{(-1)^{n+1}}{n+1} \right] = \sum_{n=1}^{\infty} (-1)^{n+1} \left[\frac{1}{n} + \frac{1}{n+1} \right]$$

$$= \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1}$$

RE-INDEXING AND MANIPULATING

$$= \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1}$$

$$= \ln 2 + \left[1 - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \right]$$

$$= \ln 2 + \left[1 - \ln 2 \right]$$

$$= \underline{1}$$

ALTERNATIVE TO RE-INDEXING & MANIPULATING

$$S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$$

$$-S = -\frac{1}{1} + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \dots$$

$$1 - S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$$

$$1 - S = \ln 2$$

$$S = 1 - \ln 2$$

At Result

Question 2 (***)

Use partial fractions to sum the following series.

$$\sum_{n=1}^{\infty} \frac{2n+1}{n^4 + 2n^3 + n^2}$$

You may assume that the series converges.

1

START BY TRYING TO DECOMPOSE THE DENOMINATOR

$$\sum_{n=1}^{\infty} \frac{2n+1}{n^4 + 2n^3 + n^2} = \sum_{n=1}^{\infty} \frac{2n+1}{n^2(n^2 + 2n + 1)} = \sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}$$

ALTHOUGH WE HAVE DECOMPOSED THE DENOMINATOR INTO FACTORS THE PARTIAL FRACTIONS CAN EASILY BE DONE BY INSPECTION

$$= \sum_{n=1}^{\infty} \left[\frac{1}{n^2} - \frac{1}{(n+1)^2} \right]$$
$$= \left(\frac{1}{1^2} - \frac{1}{2^2} \right) + \left(\frac{1}{2^2} - \frac{1}{3^2} \right) + \left(\frac{1}{3^2} - \frac{1}{4^2} \right) + \left(\frac{1}{4^2} - \frac{1}{5^2} \right) + \dots$$
$$= 1$$

Question 3 (***)

It is given that

$$f(r) = \frac{6r^4 + 6r^3 - ar^2 - ar + 1}{r(r+1)}, \quad r \in \mathbb{N},$$

where a is a non zero constant.

It is further given that

$$\sum_{r=1}^n f(r) = \frac{n^2(n+2)(2n+1)}{n+1}.$$

Determine the value of a .

, $a = 2$

MANIPULATE $f(r)$ AS FOLLOWS

$$f(r) = \frac{6r^4 + 6r^3 - ar^2 - ar + 1}{r(r+1)} = \frac{6r^2(r+1) - ar(r+1) + 1}{r(r+1)}$$

$$= \frac{6r^2 - a + \frac{1}{r(r+1)}}{1} \quad \text{PARTIAL FRACTIONS BY INSPECTION}$$

$$= 6r^2 - a + \frac{1}{r} - \frac{1}{r+1}$$

NOW PROCEED BY THE METHOD OF DIFFERENCES

$$f(r) \equiv 6r^2 - a + \frac{1}{r} - \frac{1}{r+1}$$

r=1 $f(1) = 6(1)^2 - a + \frac{1}{1} - \frac{1}{1+1}$

r=2 $f(2) = 6(2)^2 - a + \frac{1}{2} - \frac{1}{2+1}$

r=3 $f(3) = 6(3)^2 - a + \frac{1}{3} - \frac{1}{3+1}$

...

r=n $f(n) = 6(n)^2 - a + \frac{1}{n} - \frac{1}{n+1}$

ADD

$$\sum_{r=1}^n f(r) = 6 \sum_{r=1}^n r^2 - na + 1 - \frac{1}{n+1} \quad \text{ADD}$$

$$= 6 \times \frac{1}{6} n(n+1)(2n+1) - an + \frac{n+1-1}{n+1}$$

$$= n(n+1)(2n+1) - an + \frac{n}{n+1}$$

$$= \frac{n(n+1)(2n+1) - an(n+1) + n}{n+1}$$

COMPARING NUMERATORS

$$\frac{n^2(n+2)(2n+1)}{n+1} \equiv \frac{n(n+1)(2n+1) - an(n+1) + n}{n+1}$$

$$n^2(2n^2+5n+2) \equiv n(2n+1)(2n+1) - an(n+1) + n$$

$$n(2n^2+5n+2) \equiv (2n+1)(2n+1) - a(n+1) + 1$$

$$2n^3 + 5n^2 + 2n \equiv 2n^3 + 4n^2 + 2n + 1 - an - a + 1$$

$$2n^3 + 5n^2 + 2n \equiv 2n^3 + 5n^2 + (2-a)n + (2-a)$$

$$\begin{cases} 4-a=2 & a=2 \\ a=2 & a=2 \end{cases}$$

$\therefore a = 2$

Created by T. Madas

SUMMATIONS

METHOD OF DIFFERENCES

15 ENRICHMENT QUESTIONS

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Question 1 (****)

Determine the exact value of the following sum.

$$\sum_{n=2}^{20} \left[\frac{n^3 - n^2 + 1}{n^2 - n} \right]$$

$$\boxed{\text{N/A}}, \quad \boxed{\frac{4199}{20}}$$

• SPAT MANIPULATING BY DIVISION EXCEPT BY PARTIAL FRACTIONS

$$\frac{n^3 - n^2 + 1}{n^2 - n} = \frac{n(n^2 - n) + 1}{n^2 - n} = n + \frac{1}{n^2 - n} = n + \frac{1}{n(n-1)}$$

$$= n + \frac{-1}{n} + \frac{1}{n-1} = n + \frac{1}{n-1} - \frac{1}{n}$$

• THIS USES THE

$$\frac{n^3 - n^2 + 1}{n^2 - n} = n + \frac{1}{n-1} - \frac{1}{n}$$

• ADDING

$$\sum_{n=2}^{20} \left[\frac{n^3 - n^2 + 1}{n^2 - n} \right] = \left[\sum_{n=2}^{20} n \right] + 1 - \frac{1}{20}$$

$$= \frac{11(21+2)}{2} + 1 - \frac{1}{20}$$

$$= (9 \times 11) + 1 - \frac{1}{20}$$

$$= 240 - \frac{1}{20}$$

$$= \frac{4199}{20}$$

(Note: A small box contains the calculation 11(21+2) = 242, 242 - 1 = 241, 241 - 1/20 = 240 - 1/20)

Question 2 (****)

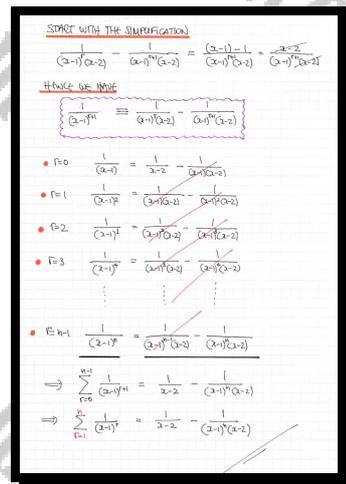
$$f(x, n) = \sum_{r=1}^n \left[\frac{1}{(x-1)^r} \right], \quad x \in \mathbb{R}, n \in \mathbb{N}.$$

By observing the simplification of

$$\frac{1}{(x-2)(x-1)^r} - \frac{1}{(x-2)(x-1)^{r+1}}$$

find a simplified expression for $f(x, n)$.

$$\boxed{\frac{1}{x-2} - \frac{1}{(x-2)(x-1)^n}}$$



Question 3 (****)

Determine, in terms of k and n , a simplified expression for

$$\sum_{r=2}^n \left[\frac{r(1-k)-1}{r(r-1)k^r} \right]$$

, $\frac{1}{n} \left(\frac{1}{k} \right)^n - \frac{1}{k}$

• SPLIT BY PARTIAL FRACTIONS

$$\frac{r(1-k)-1}{r(r-1)} = \frac{A}{r} + \frac{B}{r-1}$$

$$\frac{r(1-k)-1}{r(r-1)} = \frac{A(r-1) + Br}{r(r-1)}$$

if $r=0 \Rightarrow -1 = -A \Rightarrow A=1$
if $r=1 \Rightarrow -k = B \Rightarrow B=-k$

• Hence we have

$$\frac{r(1-k)-1}{r(r-1)} = \frac{1}{r} - k \left(\frac{1}{r-1} \right)$$

• $r=2$ $\left(\frac{1}{2} \right) - k \left(\frac{1}{1} \right) = \left(\frac{1}{2} \right) - k$
 • $r=3$ $\left(\frac{1}{3} \right) - k \left(\frac{1}{2} \right) = \left(\frac{1}{3} \right) - \left(\frac{k}{2} \right)$
 • $r=4$ $\left(\frac{1}{4} \right) - k \left(\frac{1}{3} \right) = \left(\frac{1}{4} \right) - \left(\frac{k}{3} \right)$
 • $r=5$ $\left(\frac{1}{5} \right) - k \left(\frac{1}{4} \right) = \left(\frac{1}{5} \right) - \left(\frac{k}{4} \right)$
 • ...
 • $r=n$ $\left(\frac{1}{n} \right) - k \left(\frac{1}{n-1} \right) = \left(\frac{1}{n} \right) - \left(\frac{k}{n-1} \right)$

• Adding

$$\sum_{r=2}^n \left[\frac{r(1-k)-1}{r(r-1)k^r} \right] = \left(\frac{1}{2} \right) - k$$

Question 4 (****)

Determine the value of the following infinite convergent sum.

$$\sum_{r=2}^{\infty} \left[\frac{4r-1}{r(r-1)} \left(-\frac{1}{3} \right)^r \right]$$

, $\frac{1}{3}$

● **Split by partial fractions (by addition)**
 $\frac{4r-1}{r(r-1)} = \frac{1}{r} + \frac{3}{r-1}$

● **Since we know that**
 $\frac{4r-1}{r(r-1)} \left(-\frac{1}{3} \right)^r = \frac{1}{r} \left(-\frac{1}{3} \right)^r + \frac{3}{r-1} \left(-\frac{1}{3} \right)^r$

● $r=2$ $\frac{1}{2r} \left(-\frac{1}{3} \right)^2 = \frac{1}{2} \left(-\frac{1}{3} \right)^2 + \frac{3}{2} \left(-\frac{1}{3} \right)^2 = \frac{1}{2} \left(-\frac{1}{3} \right)^2 + \frac{3}{2} \left(-\frac{1}{3} \right)^2$
● $r=3$ $\frac{1}{3r} \left(-\frac{1}{3} \right)^3 = \frac{1}{3} \left(-\frac{1}{3} \right)^3 + \frac{3}{3} \left(-\frac{1}{3} \right)^3 = \frac{1}{3} \left(-\frac{1}{3} \right)^3 + \frac{3}{3} \left(-\frac{1}{3} \right)^3$
● $r=4$ $\frac{1}{4r} \left(-\frac{1}{3} \right)^4 = \frac{1}{4} \left(-\frac{1}{3} \right)^4 + \frac{3}{4} \left(-\frac{1}{3} \right)^4 = \frac{1}{4} \left(-\frac{1}{3} \right)^4 + \frac{3}{4} \left(-\frac{1}{3} \right)^4$
● $r=5$ $\frac{1}{5r} \left(-\frac{1}{3} \right)^5 = \frac{1}{5} \left(-\frac{1}{3} \right)^5 + \frac{3}{5} \left(-\frac{1}{3} \right)^5 = \frac{1}{5} \left(-\frac{1}{3} \right)^5 + \frac{3}{5} \left(-\frac{1}{3} \right)^5$
● \dots
● $r=n$ $\frac{1}{nr} \left(-\frac{1}{3} \right)^n = \frac{1}{n} \left(-\frac{1}{3} \right)^n + \frac{3}{n} \left(-\frac{1}{3} \right)^n$

● **Adding**
 $\sum_{r=2}^{\infty} \left[\frac{4r-1}{r(r-1)} \left(-\frac{1}{3} \right)^r \right] = \frac{1}{3} + \frac{1}{3}$
 $\Rightarrow \sum_{r=2}^{\infty} \left[\frac{4r-1}{r(r-1)} \left(-\frac{1}{3} \right)^r \right] = \lim_{n \rightarrow \infty} \left[\frac{1}{3} + \frac{1}{3} \right]$
 $\Rightarrow \sum_{r=2}^{\infty} \left[\frac{4r-1}{r(r-1)} \left(-\frac{1}{3} \right)^r \right] = \frac{1}{3}$

Question 5 (****)

Determine a simplified expression, in the form $\ln[f(n)]$, for the following sum.

$$\sum_{r=2}^N \left[\int_2^r \frac{2}{x^2-1} dx \right]$$

, $\ln \left[\frac{2 \times 3^{N-1}}{N(N+1)} \right]$

• STRIKE BY PARTIAL FRACTIONS IN THE INTEGRAND (BY INVERTING)

$$\sum_{r=2}^N \left[\int_2^r \frac{2}{x^2-1} dx \right] = \sum_{r=2}^N \left[\int_2^r \frac{2}{(x-1)(x+1)} dx \right]$$

$$= \sum_{r=2}^N \left[\int_2^r \left(\frac{1}{x-1} - \frac{1}{x+1} \right) dx \right] = \sum_{r=2}^N \left[\ln|2-1| - \ln|2+1| \right]_{2+1}^{2+r}$$

• WRITING THE TERMS EXPLICITLY, LOOKING FOR PATTERNS

$$= \sum_{r=2}^N \left[\ln|r-1| - \ln|r+1| - \left(\ln|1| - \ln|3| \right) \right]$$

$$= \sum_{r=2}^N \left[\ln(r-1) - \ln(r+1) + \ln 3 \right]$$

$\ln 1 - \ln 3 + \ln 3$	$\leftarrow r=2$	} (N-1) TERMS
$\ln 2 - \ln 4 + \ln 3$	$\leftarrow r=3$	
$\ln 3 - \ln 5 + \ln 3$	$\leftarrow r=4$	
$\ln 4 - \ln 6 + \ln 3$	$\leftarrow r=5$	
\vdots		
$\ln(N-2) - \ln N + \ln 3$	$\leftarrow r=N-1$	
$\ln(N-1) - \ln(N+1) + \ln 3$	$\leftarrow r=N$	

• ADDING:

$$\sum_{r=2}^N \left[\int_2^r \frac{2}{x^2-1} dx \right] = \ln 2 - \ln N - \ln(N+1) + (N-1)\ln 3$$

$$= \ln 2 + \ln 3^{N-1} - \ln(N+1)$$

$$= \ln \left[\frac{2 \times 3^{N-1}}{N(N+1)} \right]$$

Question 7 (****)

$$\frac{3}{1^2} + \frac{5}{1^2+2^2} + \frac{7}{1^2+2^2+3^2} + \frac{9}{1^2+2^2+3^2+4^2} + \frac{11}{1^2+2^2+3^2+4^2+5^2} + \dots$$

Show, by a detailed method, that the sum of the first 40 terms of this series shown above is $\frac{240}{41}$.

, proof

The handwritten solution shows the following steps:

$$\begin{aligned}
 S_{40} &= \frac{3}{1^2} + \frac{5}{1^2+2^2} + \frac{7}{1^2+2^2+3^2} + \frac{9}{1^2+2^2+3^2+4^2} + \frac{11}{1^2+2^2+3^2+4^2+5^2} + \dots \\
 S_{40} &= \sum_{n=1}^{40} \left[\frac{2n+1}{\sum_{r=1}^n r^2} \right] = \sum_{n=1}^{40} \left[\frac{2n+1}{\frac{1}{6}n(n+1)(2n+1)} \right] \\
 &= 6 \sum_{n=1}^{40} \frac{1}{n(n+1)} = 6 \sum_{n=1}^{40} \left[\frac{1}{n} - \frac{1}{n+1} \right] \\
 &= 6 \left[(1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + \dots + (\frac{1}{40} - \frac{1}{41}) \right] \\
 &= 6 \left[1 - \frac{1}{41} \right] = 6 \times \frac{41-1}{41} = 6 \times \frac{40}{41} = \frac{240}{41}
 \end{aligned}$$

Question 8 (*****)

By considering the simplification of

$$\arctan(2n+1) - \arctan(2n-1),$$

determine the exact value of

$$\sum_{n=1}^{\infty} \left[\arctan\left(\frac{1}{2n^2}\right) \right].$$

$$\boxed{}, \frac{\pi}{4}$$

Handwritten solution for Question 8:

$\arctan(2n+1) - \arctan(2n-1) = \psi$
 • TAKE TANGENTS ON BOTH SIDES
 $\tan[\arctan(2n+1) - \arctan(2n-1)] = \tan \psi$
 $\frac{(2n+1) - (2n-1)}{1 + (2n+1)(2n-1)} = \tan \psi$
 $\tan \psi = \frac{2}{1+4n^2-1} = \frac{1}{2n^2}$
 $\psi = \arctan\left(\frac{1}{2n^2}\right)$
 • Hence $\arctan\left(\frac{1}{2n^2}\right) = \arctan(2n+1) - \arctan(2n-1)$
 $n=1: \arctan\left(\frac{1}{2}\right) = \arctan 3 - \arctan 1$
 $n=2: \arctan\left(\frac{1}{8}\right) = \arctan 5 - \arctan 3$
 $n=3: \arctan\left(\frac{1}{18}\right) = \arctan 7 - \arctan 5$
 \vdots
 $n=k: \arctan\left(\frac{1}{2k^2}\right) = \arctan(2k+1) - \arctan(2k-1)$
 • SUMMATION:
 $\sum_{n=1}^k \arctan\left(\frac{1}{2n^2}\right) = \arctan(2k+1) - \arctan 1$
 $\sum_{n=1}^{\infty} \arctan\left(\frac{1}{2n^2}\right) = \lim_{k \rightarrow \infty} [\arctan(2k+1) - \arctan 1]$
 $= \frac{\pi}{2} - \frac{\pi}{4}$
 $= \frac{\pi}{4}$

Question 9 (****)

$$S_n = (2 \times 1!) + (5 \times 2!) + (10 \times 3!) + (17 \times 4!) + \dots + (n^2 + 1)n!$$

Use an appropriate method to show that

$$S_n = n(n+1)!$$

, proof

START BY WRITING THE SERIES IN SIGMA NOTATION

$$(2 \times 1!) + (5 \times 2!) + (10 \times 3!) + (17 \times 4!) + \dots + (n^2 + 1)n! = \sum_{r=1}^n [(r^2 + 1)r!]$$

TRY SOME DIFFERENCES INVOLVING PERIODICALLY (TRYING TO OBTAIN AN OFF SETTING)

$$(r+1)! - r! = (r+1)r! - r! = r \times r!$$

AS THIS DOES NOT PRODUCE A QUANTITATIVE TERM IN R USE MAY TRY

$$(r+2)! - r! = (r+2)(r+1)r! - r!$$

$$(r+2)! - r! = (r^2 + 3r + 2)r! - r!$$

$$(r+2)! - r! = (r^2 + 3r + 1)r!$$

$$(r+2)! - r! = (r^2 + 0)r! + 3r \times r!$$

↑
MAX ABOVE $r \times r! = (r+1)! - r!$

$$(r+2)! - r! = (r+1)r! + 3[(r+1)! - r!]$$

$$(r+2)! - r! = (r+1)r! + 3(r+1)! - 3r!$$

$$(r+2)! = (r+1)r! + 3(r+1)! - 2r!$$

$$(r+2)! - 3(r+1)! + 2r! = (r+1)r!$$

HENCE WE HAVE

$$(r+1)r! = (r+2)! - 3(r+1)! + 2r!$$

WRITING THE IDENTITY JUST OBTAINED

$$(r+1)r! = (r+2)! - 3(r+1)! + 2r!$$

r=1	$2 \times 1! = 0! - 3 \times 2! + 2 \times 1!$
r=2	$5 \times 2! = 4! - 3 \times 3! + 2 \times 2!$
r=3	$10 \times 3! = 5! - 3 \times 4! + 2 \times 3!$
r=4	$17 \times 4! = 6! - 3 \times 5! + 2 \times 4!$
⋮	⋮
r=n-1	$(n-1)r! = n! - 3(n-1)! + 2(n-2)!$
r=n	$(n)r! = (n+1)! - 3(n+1)! + 2n \times n!$

$$\sum_{r=1}^n [(r+1)r!] = (n+2)! - 2(n+1)! - 3 \times 2! + 2 \times 1!$$

$$= (n+2)(n+1)! - 2(n+1)! - 6 + 2 + 4$$

$$= (n+2-2)(n+1)!$$

$$= n(n+1)!$$

Question 10 (****)

By considering the trigonometric identity for $\tan(A-B)$, with $A = \arctan(n+1)$ and $B = \arctan(n)$, sum the following series

$$\sum_{n=1}^{\infty} \arctan\left(\frac{1}{n^2 + n + 1}\right).$$

You may assume the series converges.

, , $\frac{\pi}{4}$

CONSIDER THE COMPOUND ANGLE IDENTITY FOR $\tan(A-B)$

$$\begin{aligned} \rightarrow \tan(A-B) &= \frac{\tan A - \tan B}{1 + \tan A \tan B} \\ \rightarrow \tan(\arctan(n+1) - \arctan n) &= \frac{\tan(\arctan(n+1)) - \tan(\arctan n)}{1 + \tan(\arctan(n+1))\tan(\arctan n)} \\ \Rightarrow \tan(\arctan(n+1) - \arctan n) &= \frac{(n+1) - n}{1 + n(n+1)} \\ \rightarrow \tan(\arctan(n+1) - \arctan n) &= \frac{1}{n^2 + n + 1} \\ \Rightarrow \arctan[\tan(\arctan(n+1) - \arctan n)] &= \arctan\left(\frac{1}{n^2 + n + 1}\right) \\ \Rightarrow \arctan(n+1) - \arctan n &= \arctan\left(\frac{1}{n^2 + n + 1}\right) \end{aligned}$$

REWRITE THE SUMMATION NOW AS

$$\begin{aligned} \sum_{n=1}^{\infty} \arctan\left(\frac{1}{n^2 + n + 1}\right) &= \sum_{n=1}^{\infty} [\arctan(n+1) - \arctan n] \\ &= \sum_{n=1}^{\infty} \arctan(n+1) - \sum_{n=1}^{\infty} \arctan n \\ &= \lim_{k \rightarrow \infty} \left[\sum_{n=1}^k \arctan(n+1) - \sum_{n=1}^k \arctan n \right] \end{aligned}$$

PROCEED AS BEFORES - WRITE THE "COSE"

$$\begin{aligned} \dots \lim_{k \rightarrow \infty} & \begin{bmatrix} \arctan(k+1) & - & \arctan k \\ \arctan k & - & \arctan(k-1) \\ \arctan(k-1) & - & \arctan(k-2) \\ + & & + \\ \vdots & & \vdots \\ \arctan 5 & - & \arctan 4 \\ \arctan 2 & - & \arctan 1 \end{bmatrix} \\ &= \lim_{k \rightarrow \infty} [\arctan(k+1) - \arctan 1] \\ &= \frac{\pi}{4} - \frac{\pi}{4} \\ &= \frac{\pi}{4} \end{aligned}$$

Question 11 (*****)

Determine, in terms of n , a simplified expression

$$\sum_{r=1}^n \left[\frac{r^2 + 9r + 19}{(r+5)!} \right],$$

and hence, or otherwise, deduce the value of

$$\sum_{r=1}^{\infty} \left[\frac{r^2 + 7r + 11}{(r+4)!} \right].$$

$$\boxed{\frac{5}{24}}, \quad \sum_{r=1}^n \left[\frac{r^2 + 9r + 19}{(r+5)!} \right] = \frac{1}{6} - \frac{n+5}{(n+5)!}, \quad \sum_{r=1}^{\infty} \left[\frac{r^2 + 7r + 11}{(r+4)!} \right] = \frac{5}{24}$$

• SIMPLIFY WITH ORIGNAL FRACTIONS — NOTE THAT THE NUMERATOR IS A QUADRATIC IN r , SO USE ANS TO TRY 2 FRACTIONS METHOD

i.e. $\frac{r^2 + 9r + 19}{(r+5)!} = \frac{A}{(r+5)!} + \frac{B}{(r+3)!}$

$\Rightarrow r^2 + 9r + 19 = A + 5(r+3)(r+4)$
 $\Rightarrow r^2 + 9r + 19 = Br^2 + 9Br + (20B+A)$
 $\therefore B=1 \quad A=1$

• HENCE BY THE METHOD OF DIFFERENCES

$\frac{r^2 + 9r + 19}{(r+5)!} = \frac{1}{(r+5)!} - \frac{1}{(r+3)!}$

r=1: $\frac{1+9+19}{6!} = \frac{1}{6!} - \frac{1}{4!}$
 r=2: $\frac{4+18+19}{7!} = \frac{1}{7!} - \frac{1}{5!}$
 r=3: $\frac{9+27+19}{8!} = \frac{1}{8!} - \frac{1}{6!}$
 r=4: $\frac{16+36+19}{9!} = \frac{1}{9!} - \frac{1}{7!}$
 ...
 r=n-1: $\frac{(n-1)^2 + 9(n-1) + 19}{(n+4)!} = \frac{1}{(n+4)!} - \frac{1}{(n+2)!}$
 r=n: $\frac{n^2 + 9n + 19}{(n+5)!} = \frac{1}{(n+5)!} - \frac{1}{(n+3)!}$

• FINALLY $\sum_{r=1}^n \left[\frac{r^2 + 9r + 19}{(r+5)!} \right] = \frac{1}{4!} + \frac{1}{5!} - \left[\frac{1}{(n+4)!} + \frac{1}{(n+3)!} \right]$
 $= \frac{5}{24} + \frac{1}{5!} - \left[\frac{1}{(n+4)!} + \frac{1}{(n+3)!} \right]$

$\therefore \sum_{r=1}^{\infty} \left[\frac{r^2 + 9r + 19}{(r+5)!} \right] = \frac{5}{24} - \frac{1}{5!} = \frac{1}{6} - \frac{1}{120} = \frac{19}{120}$

• NOW PROCEED AS FOLLOWS

$\Rightarrow \sum_{r=1}^n \left[\frac{r^2 + 7r + 11}{(r+4)!} \right] = \frac{5}{24} - \frac{1}{(n+4)!}$
 (BOTH ARE THE SAME)
 $\Rightarrow \sum_{r=1}^{\infty} \left[\frac{r^2 + 7r + 11}{(r+4)!} \right] = \frac{5}{24} - \frac{1}{(n+4)!}$
 $\Rightarrow \sum_{r=1}^{\infty} \left[\frac{r^2 + 7r + 11}{(r+4)!} \right] = \frac{5}{24} - \frac{1}{(n+4)!}$
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 $\Rightarrow \sum_{r=1}^{\infty} \left[\frac{r^2 + 7r + 11}{(r+4)!} \right] = \frac{5}{24} - \frac{1}{(n+4)!}$
 $\Rightarrow \sum_{r=1}^{\infty} \left[\frac{r^2 + 7r + 11}{(r+4)!} \right] = \frac{5}{24} - \frac{1}{(n+4)!}$
 $\Rightarrow \sum_{r=1}^{\infty} \left[\frac{r^2 + 7r + 11}{(r+4)!} \right] = \frac{5}{24}$

Question 12 (*****)

A sequence is defined as

$$u_{r+1} = u_r + \frac{2r}{r^4 + r^2 + 1}, \quad u_1 = 0, \quad r \in \mathbb{N}.$$

Determine the exact value of u_{61} .

, $u_{61} = \frac{3660}{3661}$

Handwritten solution for Question 12:

$\therefore u_0 = 2, \quad u_{n+1} = u_n + \frac{2r}{r^4 + r^2 + 1}, \quad u_1 = 0$

• START BY FACTORING $r^4 + r^2 + 1$ INTO TWO QUADRATIC TERMS.

$$(r^2 + r + 1)(r^2 - r + 1) = r^4 - r^3 + r^2 + r^3 - r^2 + r + r^2 - r + 1 = r^4 + r^2 + 1$$

• BY INSPECTION.

$$\frac{1}{r^2 - r + 1} - \frac{1}{r^2 + r + 1} = \frac{(r^2 + r) - (r^2 - r)}{(r^2 + r + 1)(r^2 - r + 1)} = \frac{2r}{r^4 + r^2 + 1}$$

• WRITE THE ABOVE EXPRESSION AS FRACTIONS

$$u_{r+1} - u_r = \frac{1}{r^2 - r + 1} - \frac{1}{r^2 + r + 1}$$

TE1 $u_2 - u_1 = \frac{1}{1^2 - 1 + 1} - \frac{1}{1^2 + 1 + 1} = \frac{1}{1} - \frac{1}{3}$

TE2 $u_3 - u_2 = \frac{1}{2^2 - 2 + 1} - \frac{1}{2^2 + 2 + 1} = \frac{1}{3} - \frac{1}{7}$

TE3 $u_4 - u_3 = \frac{1}{3^2 - 3 + 1} - \frac{1}{3^2 + 3 + 1} = \frac{1}{7} - \frac{1}{13}$

...

TE9 $u_{10} - u_9 = \frac{1}{9^2 - 9 + 1} - \frac{1}{9^2 + 9 + 1} = \frac{1}{81 - 9 + 1} - \frac{1}{81 + 9 + 1}$

TE10 $u_{11} - u_{10} = \frac{1}{10^2 - 10 + 1} - \frac{1}{10^2 + 10 + 1} = \frac{1}{100 - 10 + 1} - \frac{1}{100 + 10 + 1}$

...

TE60 $u_{61} - u_{60} = \frac{1}{60^2 - 60 + 1} - \frac{1}{60^2 + 60 + 1}$

ADDING $u_{61} - 0 = 1 - \frac{1}{3661} \therefore u_{61} = \frac{3660}{3661}$

Question 13 (*****)

Find the value of

$$\sum_{r=0}^{\infty} \left[\frac{\sin^4(\pi \times 2^{r-2})}{4^r} \right]$$

Hint: Express $\sin^4 \theta$ in terms of $\sin^2 \theta$ and $\sin^2 2\theta$ only.

, $\frac{1}{2}$

• STARTING BY MANIPULATING THE SIN TO THE POWER 4

$$\begin{aligned} \sin^4 \theta &= (\sin^2 \theta)^2 = \left(\frac{1}{2} - \frac{1}{2} \cos 2\theta \right)^2 = \frac{1}{4} - \frac{1}{2} \cos 2\theta + \frac{1}{4} \cos^2 2\theta \\ &= \frac{1}{4} - \frac{1}{2} (1 - 2\sin^2 \theta) + \frac{1}{4} (1 - \sin^2 2\theta) \\ &= \frac{1}{4} - \frac{1}{2} + \sin^2 \theta + \frac{1}{4} - \frac{1}{4} \sin^2 2\theta \\ &= \sin^2 \theta - \frac{1}{4} \sin^2 2\theta \end{aligned}$$

• NOW WE FIND BY OBSERVING THE SUM OF THE FIRST N TERMS

$$\begin{aligned} \sum_{r=0}^{10} \frac{\sin^4(\pi \times 2^{r-2})}{4^r} &= \sum_{r=0}^{10} \left[\frac{1}{4^r} \left(\sin^2(\pi \times 2^{r-2}) - \frac{1}{4} \sin^2(\pi \times 2^{r-1}) \right) \right] \\ &= \sum_{r=0}^{10} \left[\frac{1}{4^r} \sin^2(\pi \times 2^{r-2}) - \frac{1}{4^{r+1}} \sin^2(\pi \times 2^{r-1}) \right] \\ &= \frac{\sin^2 \pi}{4^0} - \frac{1}{4^1} \sin^2 2\pi \quad \leftarrow r=0 \\ &\quad + \frac{1}{4^1} \sin^2 2\pi - \frac{1}{4^2} \sin^2 4\pi \quad \leftarrow r=1 \\ &\quad + \frac{1}{4^2} \sin^2 4\pi - \frac{1}{4^3} \sin^2 8\pi \quad \leftarrow r=2 \\ &\quad \vdots \\ &\quad + \frac{1}{4^9} \sin^2(\pi \times 2^8) - \frac{1}{4^{10}} \sin^2(\pi \times 2^9) \quad \leftarrow r=9 \\ &= \sin^2 \frac{\pi}{4} - \frac{1}{4^{10}} \sin^2(\pi \times 2^9) \end{aligned}$$

• HENCE WE HAVE

$$\sum_{r=0}^{10} \frac{\sin^4(\pi \times 2^{r-2})}{4^r} = \sin^2 \frac{\pi}{4} = \left(\frac{1}{\sqrt{2}} \right)^2 = \frac{1}{2}$$

Question 14 (*****)

Find the sum to infinity of the following convergent series.

$$\frac{1}{4 \times 2!} + \frac{1}{5 \times 3!} + \frac{1}{6 \times 4!} + \frac{1}{7 \times 5!} + \frac{1}{8 \times 6!} + \dots$$

p2 , $\frac{1}{6}$

WRITING THE SERIES IN SIGMA NOTATION

$$S_{\infty} = \sum_{r=1}^{\infty} \frac{1}{(r+3)(r+1)!}$$

ATTEMPT SUMMATION BY THE METHOD OF DIFFERENCES

• TRY

$$\frac{1}{(r+3)(r+1)!} \equiv \frac{A}{(r+3)!} + \frac{B}{(r+1)!}$$

$$1 \equiv A + B(r+3)(r+2)$$

NO A & B CAN SATISFY THE ABOVE

• TRY NEXT

$$\frac{1}{(r+3)(r+1)!} \equiv \frac{A}{(r+3)!} + \frac{B}{(r+2)!}$$

$$\rightarrow \frac{1}{(r+3)(r+1)!} \equiv \frac{A + B(r+3)}{(r+3)!}$$

$$\rightarrow \frac{r+2}{(r+3)!} \equiv \frac{A + B(r+3)}{(r+3)!}$$

$$\rightarrow r+2 \equiv (A + 3B) + Br$$

$\therefore B = 1$ & $A = -1$

HENCE WE NOW HAVE A SUITABLE IDENTITY

$$\frac{1}{(r+3)(r+1)!} \equiv \frac{1}{(r+3)!} - \frac{1}{(r+2)!}$$

• r=1 : $\frac{1}{4 \times 2!} \equiv \frac{1}{3!} - \frac{1}{2!}$

• r=2 : $\frac{1}{5 \times 3!} \equiv \frac{1}{4!} - \frac{1}{3!}$

• r=3 : $\frac{1}{6 \times 4!} \equiv \frac{1}{5!} - \frac{1}{4!}$

• r=4 : $\frac{1}{7 \times 5!} \equiv \frac{1}{6!} - \frac{1}{5!}$

• ...

• r=n : $\frac{1}{(n+3)(n+1)!} \equiv \frac{1}{(n+3)!} - \frac{1}{(n+2)!}$

$$\Rightarrow \sum_{r=1}^n \frac{1}{(r+3)(r+1)!} = \frac{1}{3!} - \frac{1}{(n+2)!}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \left[\sum_{r=1}^n \frac{1}{(r+3)(r+1)!} \right] = \lim_{n \rightarrow \infty} \left[\frac{1}{3!} - \frac{1}{(n+2)!} \right]$$

$$\Rightarrow \sum_{r=1}^{\infty} \frac{1}{(r+3)(r+1)!} = \frac{1}{3!} = \frac{1}{6}$$

Question 15 (*****)

Evaluate the following expression

$$\sum_{k=1}^{\infty} \left[\sum_{r=1}^k r \right]^{-1}$$

SP, 2

• REWRITE THE SUMMATION AS FRACTIONS

$$\sum_{k=1}^{\infty} \left[\sum_{r=1}^k r \right]^{-1} = \sum_{k=1}^{\infty} \left[\frac{1}{\sum_{r=1}^k r} \right]$$

$$= \frac{1}{1} + \frac{1}{1+2} + \frac{1}{1+2+3} + \frac{1}{1+2+3+4} + \dots$$

• INTRODUCE A LIMIT FOR THE SUMMATION, SAY n

$$= \lim_{n \rightarrow \infty} \left[\frac{1}{1} + \frac{1}{1+2} + \frac{1}{1+2+3} + \frac{1}{1+2+3+4} + \dots + \frac{1}{1+2+\dots+n} \right]$$

$$= \lim_{n \rightarrow \infty} \left[\sum_{r=1}^n \frac{1}{r(r+1)} \right]$$

$$= 2 \lim_{n \rightarrow \infty} \left[\sum_{r=1}^n \frac{1}{r(r+1)} \right]$$

• SPLIT INTO TWO FRACTIONS BY INSPECTION

$$= 2 \lim_{n \rightarrow \infty} \left[\sum_{r=1}^n \left(\frac{1}{r} - \frac{1}{r+1} \right) \right]$$

$$= 2 \lim_{n \rightarrow \infty} \left[\left(1 - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \dots + \left(\frac{1}{n} - \frac{1}{n+1} \right) \right]$$

$$= 2 \lim_{n \rightarrow \infty} \left[1 - \frac{1}{n+1} \right]$$

$$= 2$$