

Created by T. Madas

SERIES

79 EXAM QUESTIONS

Created by T. Madas

Created by T. Madas

SUMMATIONS BY FORMULAS

17 BASIC QUESTIONS

Created by T. Madas

Question 1 (**)

Use standard results on summations to find the value of

$$\sum_{r=36}^{48} [(r-1)(3r-2)].$$

 , 66638

Find a simplified expression for the sum of the first n terms

$$\begin{aligned} \sum_{r=1}^n [(r-1)(3r-2)] &= \sum_{r=1}^n (3r^2 - 5r + 2) \\ &= 3 \sum_{r=1}^n r^2 - 5 \sum_{r=1}^n r + 2 \sum_{r=1}^n 1 \\ &= 3 \times \frac{1}{6} n(n+1)(2n+1) - 5 \times \frac{1}{2} n(n+1) + 2n \\ &= \frac{1}{2} n [(n+1)(2n+1) - 5(n+1) + 4] \\ &= \frac{1}{2} n [2n^2 + 3n + 1 - 5n - 5 + 4] \\ &= \frac{1}{2} n [2n^2 - 2n] \\ &= n^2 (n-1) \end{aligned}$$

You use this

$$\begin{aligned} \sum_{r=36}^{48} [(r-1)(3r-2)] &= \sum_{r=1}^{48} [(r-1)(3r-2)] - \sum_{r=1}^{35} [(r-1)(3r-2)] \\ &= 48^2 (48-1) - 35^2 (35-1) \\ &= 108288 - 41650 \\ &= \underline{66638} \end{aligned}$$

Question 2 (**)

Use standard results on summations to show that

$$\sum_{r=1}^n r(r+1)(r+5) = \frac{1}{4}n(n+a)(n+b)(n+c),$$

where a , b , and c are positive integers to be found.

$$\boxed{}, \quad \boxed{a=1, \quad b=2, \quad c=7}$$

EXPANDING AND SPLITTING

$$\sum_{r=1}^n (r+1)(r+5) = \sum_{r=1}^n (r^2 + 6r + 5) = \sum_{r=1}^n r^2 + 6 \sum_{r=1}^n r + 5 \sum_{r=1}^n 1$$

USING STANDARD RESULTS

$$\sum_{r=1}^n r = \frac{1}{2}n(n+1)$$

$$\sum_{r=1}^n r^2 = \frac{1}{6}n(n+1)(2n+1)$$

$$\sum_{r=1}^n 1 = n$$

$$\begin{aligned} \therefore &= \frac{1}{6}n(n+1)(2n+1) + 6 \times \frac{1}{2}n(n+1) + 5 \times n(n+1) \\ &= \frac{1}{6}n(n+1)(2n+1) + 3n(n+1) + 5n(n+1) \\ &= \frac{1}{6}n(n+1)[(2n+1) + 6(n+1) + 5(n+1)] \\ &= \frac{1}{6}n(n+1)[2n+1+6n+6+5n+5] \\ &= \frac{1}{6}n(n+1)[13n+12] \\ &= \frac{1}{6}n(n+1)(13n+12) \end{aligned}$$

$\therefore a=1, b=2, c=7$

Question 3 (**)

Use standard results on summations to show that

$$\sum_{r=1}^n \left[r^2(r-1) \right] = \frac{1}{12} n(n-1)(n+1)(3n+2) + m,$$

where m is an integer to be found.

$$\boxed{}, \boxed{m = -22}$$

EXPAND THE SUMMATION & USE STANDARD RESULTS

$$\sum_{r=1}^n r^2(r-1) = \sum_{r=1}^n (r^3 - r^2) = \left[\sum_{r=1}^n (r^3 + r^2) \right] - (0 + 0 + 0)$$

$$= \sum_{r=1}^n (r^3 + r^2) - 22$$

Using $\sum_{r=1}^n r^3 = \frac{1}{4}n^2(n+1)$ & $\sum_{r=1}^n r^2 = \frac{1}{6}n(n+1)(2n+1)$

$$\dots = \frac{1}{4}n^2(n+1) + \frac{1}{6}n(n+1)(2n+1) - 22$$

$$\dots = \frac{1}{12}n(n+1) [2n(2n+1) + 2(2n+1)] - 22$$

$$\dots = \frac{1}{12}n(n+1) [2n^2 + 2n + 4n + 2] - 22$$

$$\dots = \frac{1}{12}n(n+1) (2n^2 + 6n + 2) - 22$$

$$\dots = \frac{1}{12}n(n+1)(2n+1)(n+1) - 22$$

16. $m = -22$

Question 4 (**)

Use standard results on summations to show that

$$\sum_{r=1}^n \left[r^3(r+1)(r-1) \right] = \frac{1}{6} n^2 (n+1)^2 (n-1)(n+2).$$

You may assume without proof that $\sum_{r=1}^n r^5 = \frac{1}{12} n^2 (n+1)^2 (2n^2 + 2n - 1)$

, proof

EXPAND THE SUMMATION FIRST

$$\begin{aligned} \sum_{r=1}^n [r^3(r+1)(r-1)] &= \sum_{r=1}^n [r^3(r^2-1)] = \sum_{r=1}^n (r^5-r^3) \\ &= \sum_{r=1}^n r^5 - \sum_{r=1}^n r^3 \end{aligned}$$

USING STANDARD RESULTS

$$\begin{aligned} \dots &= \frac{1}{12} n^2 (n+1)^2 (2n^2+2n-1) - \frac{1}{4} n^2 (n+1)^2 \\ \dots &= \frac{1}{12} n^2 (n+1)^2 [2n^2+2n-1-3] \\ \dots &= \frac{1}{12} n^2 (n+1)^2 (2n^2-4) \\ \dots &= \frac{1}{12} n^2 (n+1)^2 (2n^2-4) \\ \dots &= \frac{1}{12} n^2 (n+1)^2 (2n^2-4) \\ \dots &= \frac{1}{12} n^2 (n+1)^2 (2n^2-4) \end{aligned}$$

* REQUIRED

Question 5 (**)

$$F(r) \equiv \sum_{n=1}^r [n(n-1)(n+2)].$$

Use standard results on summations express $F(n)$ in fully factorized form.

$$\boxed{}, \quad F(r) = \frac{1}{12} r(r+1)(r-1)(3r+10)$$

START BY EXPANDING THE SUMMATION

$$F(r) = \sum_{n=1}^r [n(n-1)(n+2)] = \sum_{n=1}^r n(n^2+n-2) = \sum_{n=1}^r (n^3+n^2-2n)$$

$$= \sum_{n=1}^r n^3 + \sum_{n=1}^r n^2 - 2 \sum_{n=1}^r n$$

USING STANDARD RESULTS ON SUMMATIONS

$$\Rightarrow F(r) = \frac{1}{4} r^2(r+1)^2 + \frac{1}{6} r(r+1)(2r+1) - 2 \times \frac{1}{2} r(r+1)$$

$$\Rightarrow F(r) = \frac{1}{4} r^2(r+1)^2 + \frac{1}{6} r(r+1)(2r+1) - r(r+1)$$

$$\Rightarrow F(r) = \frac{1}{12} r(r+1) [3r(r+1) + 2(2r+1) - 4]$$

$$\Rightarrow F(r) = \frac{1}{12} r(r+1) [3r^2 + 3r + 4r + 2 - 4]$$

$$\Rightarrow F(r) = \frac{1}{12} r(r+1) (3r^2 + 7r - 2)$$

$3r^2 + 7r - 2 = (3r-1)(r+2)$

$$F(r) = \frac{1}{12} r(r+1)(3r+10)(r-1)$$

Question 6 (**+)

Find, in fully simplified factorized form, an expression for the sum of the first n terms of the following series.

$$(5 \times 3) + (11 \times 7) + (17 \times 11) + (23 \times 15) + \dots$$

$$\boxed{}, \quad n^2(8n+7)$$

WRITE THE EXPRESSION IN COMPACT NOTATION

$$(5 \times 3) + (11 \times 7) + (17 \times 11) + \dots = \sum_{k=1}^n (6k-1)(4k+1)$$

USING STANDARD RESULTS

$$\sum_{k=1}^n (6k-1)(4k+1) = \sum_{k=1}^n (24k^2 - 10k + 1)$$

$$= 24 \sum_{k=1}^n k^2 - 10 \sum_{k=1}^n k + \sum_{k=1}^n 1$$

$$= 24 \times \frac{1}{6} n(n+1)(2n+1) - 10 \times \frac{1}{2} n(n+1) + n$$

$$= 4n(n+1)(2n+1) - 5n(n+1) + n$$

$$= n [4(n+1)(2n+1) - 5(n+1) + 1]$$

$$= n [8n^2 + 12n + 4 - 5n - 5 + 1]$$

$$= n [8n^2 + 7n]$$

$$= n^2 (8n+7)$$

Question 7 (**+)

Show by using standard summation results that ...

$$\text{a) } \dots \sum_{r=1}^n (r+1)(r+5) = \frac{1}{6}n(n+7)(2n+7).$$

$$\text{b) } \dots \sum_{r=11}^{40} (r+1)(r+5) = 26495.$$

 , proof

4) USING STANDARD FORMULAE TO SUM

$$\begin{aligned} \sum_{r=1}^n (r+1)(r+5) &= \sum_{r=1}^n (r^2 + 6r + 5) = \sum_{r=1}^n r^2 + 6 \sum_{r=1}^n r + 5 \sum_{r=1}^n 1 \\ &= \frac{1}{6}n(n+1)(2n+1) + 6 \times \frac{1}{2}n(n+1) + 5n \\ &= \frac{1}{6}n(n+1)(2n+1) + 3n(n+1) + 5n \\ &= \frac{1}{6}n [(n+1)(2n+1) + 6n(n+1) + 30] \\ &= \frac{1}{6}n [2n^2 + 3n + 1 + 6n^2 + 6n + 30] \\ &= \frac{1}{6}n [8n^2 + 9n + 31] \\ &= \frac{1}{6}n (2n+7)(n+7) \quad \text{As Required} \end{aligned}$$

b) Check the result of part (a)

$$\begin{aligned} \sum_{r=11}^{40} (r+1)(r+5) &= \sum_{r=1}^{40} (r+1)(r+5) - \sum_{r=1}^{10} (r+1)(r+5) \\ &= \frac{1}{6} \times 40 \times 41 \times 47 - \frac{1}{6} \times 10 \times 11 \times 17 \\ &= 27240 - 765 \\ &= 26475 \quad \text{As Required} \end{aligned}$$

Question 8 (**+)

Show by using standard summation results that ...

$$\text{a) } \dots \sum_{k=1}^n (k^2 - k - 1) = \frac{1}{3}n(n+2)(n-2).$$

$$\text{b) } \dots \sum_{k=10}^{40} (k^2 - k - 1) = 21049.$$

, proof

Using the Formulae

$$\sum_{k=1}^n k^2 = \frac{1}{6}n(n+1)(2n+1) \quad \& \quad \sum_{k=1}^n k = \frac{1}{2}n(n+1)$$

$$\sum_{k=1}^n (k^2 - k - 1) = \sum_{k=1}^n k^2 - \sum_{k=1}^n k - \sum_{k=1}^n 1$$

$$= \frac{1}{6}n(n+1)(2n+1) - \frac{1}{2}n(n+1) - n$$

$$= \frac{1}{6}n [(n+1)(2n+1) - 3(n+1) - 6]$$

$$= \frac{1}{6}n [2n^2 + 3n + 1 - 3n - 3 - 6]$$

$$= \frac{1}{6}n [2n^2 - 8]$$

$$= \frac{1}{3}n (n^2 - 4)$$

$$= \frac{1}{3}n (n-2)(n+2) \quad \text{As Required}$$

Using Part (a)

$$\sum_{k=10}^{40} (k^2 - k - 1) = \sum_{k=1}^{40} (k^2 - k - 1) - \sum_{k=1}^9 (k^2 - k - 1)$$

$$= \frac{1}{3} \times 40 \times 38 \times 42 - \frac{1}{3} \times 9 \times 7 \times 11$$

$$= 21280 - 231$$

$$= 21049$$

Question 9 (+)**

Find, in fully factorized form, an expression for the sum

$$\sum_{p=1}^k (p^3 + p^2).$$

$$\boxed{\text{11P}}, \quad \boxed{\frac{1}{12}k(k+1)(k+2)(3k+1)}$$

USING SUMMATION FORMULAE

$$\begin{aligned} \sum_{p=1}^k (p^3 + p^2) &= \frac{1}{6}k(k+1)(2k+1) + \frac{1}{4}k^2(k+1) \\ &= \frac{1}{12}k(k+1) [2(2k+1) + 3k(k+1)] \\ &= \frac{1}{12}k(k+1) (4k+2+3k^2+3k) \\ &= \frac{1}{12}k(k+1) (3k^2+7k+2) \\ &= \frac{1}{12}k(k+1)(3k+1)(k+2) \end{aligned}$$

Question 10 (+)**

Find, in fully factorized form, an expression for the sum

$$\sum_{r=1}^{2n} \left(3r^2 - \frac{1}{2} \right).$$

$$\boxed{\text{12P}}, \quad \boxed{2n^2(4n+3)}$$

USING STANDARD SUMMATION FORMULAE

$$\begin{aligned} \sum_{r=1}^{2n} \left(3r^2 - \frac{1}{2} \right) &= 3 \sum_{r=1}^{2n} r^2 - \frac{1}{2} \sum_{r=1}^{2n} 1 \\ &= 3 \times \frac{1}{6} (2n)(2n+1) [2(2n)+1] - \frac{1}{2} \times 2n \\ &= \frac{1}{2} (2n)(2n+1) [2(2n)+1] - n \\ &= n (2n+1) (4n+1) - n \\ &= n [(2n+1)(4n+1) - 1] \\ &= n [8n^2 + 6n + 1 - 1] \\ &= n (8n^2 + 6n) \\ &= 2n^2 (4n+3) \end{aligned}$$

Question 11 (***)

Use standard results on summations to show that

$$\sum_{r=1}^{n-2} r(r+1)^2 = \frac{1}{12}n(n-1)(n-2)(3n-1).$$

□, proof

GENUS: THE STANDARD SUMMATIONS "to n "

$$\begin{aligned} \sum_{r=1}^{n-2} r(r+1)^2 &= \sum_{r=1}^{n-2} (r^3 + 2r^2 + r) = \sum_{r=1}^{n-2} r^3 + 2\sum_{r=1}^{n-2} r^2 + \sum_{r=1}^{n-2} r \\ &= \frac{1}{4}n^2(n+1)^2 + 2 \times \frac{1}{6}n(n+1)(2n+1) + \frac{1}{2}n(n+1) \\ &= \frac{1}{4}n^2(n+1)^2 + \frac{1}{3}n(n+1)(2n+1) + \frac{1}{2}n(n+1) \\ &= \frac{1}{12}n(n+1) \left[3n(n+1)^2 + 4(2n+1) + 6 \right] \\ &= \frac{1}{12}n(n+1) (3n^3 + 3n^2 + 6n + 4 + 6) \\ &= \frac{1}{12}n(n+1) (3n^3 + 3n^2 + 11n + 10) \\ &= \frac{1}{12}n(n+1) (n+2)(3n+5) \end{aligned}$$

FINALLY: Now $n \rightarrow n-2$

$$\begin{aligned} \sum_{r=1}^{n-2} r(r+1)^2 &= \frac{1}{12}(n-2)(n-2+1)(n-2+2) \left[3(n-2) + 5 \right] \\ &= \frac{1}{12}(n-2)(n-1)n(3n-1) \\ &= \frac{1}{12}n(n-1)(n-2)(3n-1) \end{aligned}$$

Question 12 (***)

It is given that

$$\sum_{r=1}^n [(3r+a)(r+2)] \equiv n(n+2)(n+b).$$

Determine the values of each of the constants a and b .

$$\boxed{}, \boxed{a=1}, \boxed{b=3}$$

PROCEED AS FOLLOWS

$$\Rightarrow \sum_{r=1}^n (3r+a)(r+2) \equiv n(n+2)(n+b)$$

$$\Rightarrow \sum_{r=1}^n (3r^2 + (6+a)r + 2a) \equiv n^3 + (2+b)n^2 + 2bn$$

$$\Rightarrow 3 \sum_{r=1}^n r^2 + (6+a) \sum_{r=1}^n r + 2a \sum_{r=1}^n 1 \equiv n^3 + (2+b)n^2 + 2bn$$

$$\Rightarrow 3 \times \frac{1}{6}n(n+1)(n+1) + (6+a) \times \frac{1}{2}n(n+1) + 2an \equiv n^3 + (2+b)n^2 + 2bn$$

$$\Rightarrow \frac{1}{2}n(n+1)(n+1) + \frac{1}{2}(6+a)n(n+1) + 2an \equiv n^3 + (2+b)n^2 + 2bn$$

COMBINE THE LHS FULLY

$$\Rightarrow \frac{1}{2}n^3 + \frac{3}{2}n^2 + \frac{1}{2}n + \frac{1}{2}(6+a)n^2 + \frac{1}{2}(6+a)n + 2an \equiv n^3 + (2+b)n^2 + 2bn$$

LOOKING AT THE COEFFICIENTS OF n^2 & n

$\bullet \frac{3}{2} + \frac{1}{2}(6+a) = 2+b$ $3 + 6+a = 2+b$ $a = 2b-5$	$\bullet \frac{1}{2} + \frac{1}{2}(6+a) + 2a = 2b$ $1 + 6+a + 4a = 4b$ $7 + 5a = 4b$
---	--

\swarrow

$$\begin{aligned} 7 + 5(2b-5) &= 4b \\ 7 + 10b - 25 &= 4b \\ 6b &= 18 \\ b &= 3 \\ a &= 1 \end{aligned}$$

Question 13 (*)**

Show clearly that

$$(1 \times 3) + (2 \times 4) + (3 \times 5) + \dots + (n-5)(n-3) = \frac{1}{6}(n+6)(2n+11)(n+5).$$

, proof

USE THE SUMMATION OPERATOR

$$(1 \times 3) + (2 \times 4) + (3 \times 5) + \dots + (n-5)(n-3) = \sum_{r=1}^{n-5} r(r+2)$$

SIM TH. R. METHOD

$$\begin{aligned} \sum_{r=1}^{n-5} r(r+2) &= \sum_{r=1}^{n-5} (r^2 + 2r) = \sum_{r=1}^{n-5} r^2 + 2 \sum_{r=1}^{n-5} r \\ &= \frac{1}{6}n(n+1)(2n+1) + 2 \times \frac{1}{2}n(n+1) \\ &= \frac{1}{6}n(n+1)(2n+1) + n(n+1) \\ &= \frac{1}{6}n(n+1)(2n+1+6) \\ &= \frac{1}{6}n(n+1)(2n+7) \end{aligned}$$

NOW CALC. "RIGHT HAND" EXPRESSION

$$\begin{aligned} f(n) &= \sum_{r=1}^{n-5} r(r+2) = \frac{1}{6}n(n+1)(2n+7) \\ f(n-5) &= \sum_{r=1}^{n-5-1} r(r+2) = \frac{1}{6}(n-5)(n-4)(2n-3) \\ &= \frac{1}{6}(n-5)(n-4)(2n-3) \end{aligned}$$

As required

Question 14 (*)**

Use standard results on summations to show that

$$\sum_{r=1}^n (3r^2 + r - 1) \equiv n^2(n+2).$$

, proof

USE THE UNIFORM PROPERTY OF THE SUMMATION OPERATOR

$$\begin{aligned} \sum_{r=1}^n (3r^2 + r - 1) &= 3 \sum_{r=1}^n r^2 + \sum_{r=1}^n r - \sum_{r=1}^n 1 \\ &= 3 \times \frac{1}{6}n(n+1)(2n+1) + \frac{1}{2}n(n+1) - n \\ &= \frac{1}{2}n(n+1)(2n+1) + \frac{1}{2}n(n+1) - n \\ &= \frac{1}{2}n \left[(n+1)(2n+1) + (n+1) - 2 \right] \\ &= \frac{1}{2}n \left[2n^2 + 3n + 1 + n + 1 - 2 \right] \\ &= \frac{1}{2}n \left[2n^2 + 4n \right] \\ &= n \left(n^2 + 2n \right) \\ &= n^2(n+2) \end{aligned}$$

As required

Question 15 (***)

Use standard results on summations to show that

$$\sum_{n=1}^k (18n^2 + 28n + 5) \equiv k(k+2)(6k+11).$$

 , **proof**

SPLIT THE SUM INTO INDIVIDUAL COMPONENTS

$$\sum_{n=1}^k (18n^2 + 28n + 5) = \sum_{n=1}^k (18n^2) + \sum_{n=1}^k (28n) + \sum_{n=1}^k 5$$

$$= 18 \sum_{n=1}^k n^2 + 28 \sum_{n=1}^k n + 5 \sum_{n=1}^k 1$$

USING STANDARD RESULTS

$$\sum_{n=1}^k n^2 = \frac{1}{6}k(k+1)(2k+1) \quad \sum_{n=1}^k n = \frac{1}{2}k(k+1) \quad \sum_{n=1}^k 1 = k$$

SUBSTITUTION

$$\begin{aligned} &= 18 \times \frac{1}{6}k(k+1)(2k+1) + 28 \times \frac{1}{2}k(k+1) + 5 \times k \\ &= 3k(k+1)(2k+1) + 14k(k+1) + 5k \\ &= k[3(k+1)(2k+1) + 14(k+1) + 5] \\ &= k[6k^2 + 10k + 3 + 14k + 14 + 5] \\ &= k[6k^2 + 24k + 22] \\ &= k(k+2)(6k+11) \end{aligned}$$

Question 16 (***)

Use standard results on summations to find the value of the following sum.

$$\sum_{k=2}^{16} [(k-1)(k+2)].$$

 , **1600**

SPLIT BY FINDING AN EXPRESSION FOR THE SUM OF THE FIRST N TERM
(CHECK THAT k=1 MEETS REQ)

$$\sum_{k=2}^n [(k-1)(k+2)] = \sum_{k=1}^n [(k-1)(k+2)] - [(1-1)(1+2)]$$

$$= \sum_{k=1}^n (k^2 + k - 2) = \sum_{k=1}^n k^2 + \sum_{k=1}^n k - 2 \sum_{k=1}^n 1$$

$$= \frac{1}{6}n(n+1)(2n+1) + \frac{1}{2}n(n+1) - 2 \times n$$

$$= \frac{1}{6}n[2n^2 + 3n + 1 + 3n + 2 - 12]$$

$$= \frac{1}{6}n[2n^2 + 6n - 9]$$

$$= \frac{1}{6}n(2n^2 + 6n - 9)$$

$$= \frac{1}{6}n(2n^2 + 6n - 9)$$

$$= \frac{1}{6}n(2n^2 + 6n - 9)$$

Now LETTING $n=16$

$$\sum_{k=2}^{16} [(k-1)(k+2)] = \frac{1}{6} \times 16 \times (2 \times 16^2 + 6 \times 16 - 9) = 1600$$

Question 17 (***)

Use standard results on summations to show that

$$\sum_{r=1}^{2n} r^3 - \sum_{r=1}^n (6r-3)^2 \equiv f(n),$$

where $f(n)$ is written as a product of 4 linear factors.

$$\boxed{}, \quad f(n) = n(n-1)(2n+1)(2n-3)$$

USING STANDARD SUMMATION RESULTS

- $\sum_{r=1}^n r = \frac{1}{2}n(n+1)$
- $\sum_{r=1}^n r^2 = \frac{1}{6}n(n+1)(2n+1)$
- $\sum_{r=1}^n r^3 = \frac{1}{4}n^2(n+1)^2$

$$\begin{aligned} \sum_{r=1}^{2n} r^3 - \sum_{r=1}^n (6r-3)^2 &= \sum_{r=1}^{2n} r^3 - \left[\sum_{r=1}^n (36r^2 - 36r + 9) \right] \\ &= \sum_{r=1}^{2n} r^3 - 36 \sum_{r=1}^n r^2 + 36 \sum_{r=1}^n r - 9 \sum_{r=1}^n 1 \\ &= \frac{1}{4}(2n)^2(2n+1)^2 - 36 \left[\frac{1}{6}n(n+1)(2n+1) \right] + 36 \left[\frac{1}{2}n(n+1) \right] - 9n \\ &= \frac{1}{4}(2n)^2(2n+1)^2 - 6n(n+1)(2n+1) + 18n(n+1) - 9n \end{aligned}$$

PROCEED IN LINES

$$\begin{aligned} &= n(2n+1) [n(2n+1) - 6(n+1) + 18(n+1) - 9] \\ &= n(2n+1) [2n^2 + n - 6n - 6 + 18n + 18 - 9] \\ &= n(2n+1) [2n^2 - 5n + 12] + 9n(2n+1) \end{aligned}$$

FACTORISE AGAIN

$$\begin{aligned} &= n(2n+1) [2n^2 - 5n + 12] \\ &= n(2n+1) (2n^2 - 5n + 3) \\ &= n(2n+1) (2n-3)(n+1) \end{aligned}$$

ALTERNATIVELY TRY UP

$$\begin{aligned} \dots &= \frac{1}{4}(2n)^2(2n+1)^2 - 6n(n+1)(2n+1) + 18n(n+1) - 9n \\ &= n [4n^2(2n+1)^2 - 24n(n+1)(2n+1) + 36n(n+1) - 9n] \\ &= n [4n^2(4n^2 + 8n + 4) - 24n^2 - 24n - 6 + 36n^2 + 36n - 9] \\ &= n [16n^4 + 32n^3 + 16n^2 - 24n^2 - 24n - 6 + 36n^2 + 36n - 9] \\ &= n [16n^4 + 32n^3 + 28n^2 + 12n - 15] \end{aligned}$$

q. This looks like the product of 4 linear factors. Hence a root of

Created by T. Madas

SUMMATIONS BY FORMULAS

15 STANDARD QUESTIONS

Created by T. Madas

Question 1 (***)

Find the sum of the first n terms of the series

$$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 5 + 3 \cdot 4 \cdot 7 + 4 \cdot 5 \cdot 9 + \dots$$

Express the answer as a product of linear factors.

1, proof

SUM BY ADDING THE SERIES IN CUBIC NOTATION

$$\underbrace{(1 \cdot 2 \cdot 3) + (2 \cdot 3 \cdot 5) + (3 \cdot 4 \cdot 7) + (4 \cdot 5 \cdot 9) + \dots}_{n \text{ terms}} = \sum_{r=1}^n [r(r+1)(r+2)]$$

EXPAND & SUMMATION

$$\sum_{r=1}^n [r(r+1)(r+2)] = \sum_{r=1}^n [r^3 + 3r^2 + 2r]$$

$$= 2 \sum_{r=1}^n r^3 + 3 \sum_{r=1}^n r^2 + \sum_{r=1}^n 2r$$

USING STANDARD RESULTS

$$= 2 \times \frac{1}{4} n^2(n+1)^2 + 3 \times \frac{1}{3} n(n+1)(n+2) + \frac{1}{2} n(n+1)$$

$$= \frac{1}{2} n^2(n+1)^2 + \frac{1}{2} n(n+1)(n+2) + \frac{1}{2} n(n+1)$$

$$= \frac{1}{2} n(n+1) [n(n+1) + (n+2) + 1]$$

$$= \frac{1}{2} n(n+1) [n^2 + 2n + 2]$$

$$= \frac{1}{2} n(n+1)(n+1)(n+2)$$

$$= \frac{1}{2} n(n+1)^2(n+2)$$

Question 2 (***)

By using standard results, show that

$$\sum_{r=n+1}^{4n} (2r-1)^2 \equiv n(84n^2-1).$$

□, proof

PROVED-AS-FOLLOWS

$$f(n) = \sum_{r=1}^n (2r-1)^2 = \sum_{r=1}^n [4r^2 - 4r + 1] = 4 \sum_{r=1}^n r^2 - 4 \sum_{r=1}^n r + \sum_{r=1}^n 1$$

USING STANDARD RESULTS

$$\begin{aligned} f(n) &= 4 \times \frac{1}{6} n(n+1)(2n+1) - 4 \times \frac{1}{2} n(n+1) + n \\ f(n) &= \frac{2}{3} n(n+1)(2n+1) - 2n(n+1) + n \\ f(n) &= \frac{1}{3} n [2(n+1)(2n+1) - 6(n+1) + 3] \\ f(n) &= \frac{1}{3} n [4n^2 + 6n + 2 - 6n - 6 + 3] \\ f(n) &= \frac{1}{3} n (4n^2 - 1) \end{aligned}$$

DO WE HAVE

$$\begin{aligned} \sum_{r=n+1}^{4n} (2r-1)^2 &= \sum_{r=1}^{4n} (2r-1)^2 - \sum_{r=1}^n (2r-1)^2 \\ &= f(4n) - f(n) \\ &= \frac{1}{3} (4n) (4n^2 - 1) - \frac{1}{3} n (4n^2 - 1) \\ &= \frac{1}{3} n (4n^2 - 1) - \frac{1}{3} n (4n^2 - 1) \\ &= \frac{1}{3} n [20n^2 - 4 - 4n^2 + 1] \\ &= \frac{1}{3} n (20n^2 - 3) \\ &= n(84n^2 - 1) \end{aligned}$$

□ PROVED

Question 3 (***)Determine the value of a and the value of b given that

$$\sum_{r=1}^n r(r+a)(r+b) \equiv \frac{1}{12}n(n+1)(n+2)(3n+17).$$

, $a=1, b=4$ or the other way round

USE STANDARD SUMS AND EQUIVOCAL COEFFICIENTS

$$\sum_{r=1}^n r(r+a)(r+b) = \frac{1}{2}n(n+1)(n+2)(3n+17)$$

$$\sum_{r=1}^n [r^3 + (a+b)r^2 + ab r] = \frac{1}{2}n(n+1)(n+2)(3n+17)$$

$$\sum_{r=1}^n r^3 + (a+b) \sum_{r=1}^n r^2 + ab \sum_{r=1}^n r = \frac{1}{2}n(n+1)(n+2)(3n+17)$$

$$\frac{1}{4}n^2(n+1)^2 + \frac{1}{2}(a+b)n(n+1)(n+2) + \frac{1}{2}abn(n+1) = \frac{1}{2}n(n+1)(n+2)(3n+17)$$

"SOLVE" A PORTION OF $n(n+1)$ ALL THE WAY THROUGH

$$\frac{1}{4}n(n+1) + \frac{1}{2}(a+b)(n+1) + \frac{1}{2}ab = \frac{1}{2}(n+2)(3n+17)$$

$$3n^2 + 3n + 4(a+b)n + 2(a+b) + 6ab = (n+2)(3n+17)$$

$$3n^2 + 3n + 4(a+b)n + 2(a+b) + 6ab = 3n^2 + 7n + 6n + 34$$

$$3n^2 + [3 + 4(a+b)]n + [2(a+b) + 6ab] = 3n^2 + 13n + 34$$

FOCUSING THE EQUATIONS

$$\begin{array}{l} 3 + 4(a+b) = 13 \\ 4(a+b) = 10 \\ a+b = \frac{5}{2} \end{array} \quad \begin{array}{l} 2(a+b) + 6ab = 34 \\ 10 + 6ab = 34 \\ 6ab = 24 \\ ab = 4 \end{array}$$

BY INSPECTION, SUBSTITUTION, OR RECOGNISING EXACT TRIANGLE

$a=1, b=4$ (OR THE OTHER WAY ROUND)

Question 4 (***)

Find, in fully factorized form, an expression for the following sum.

$$\sum_{r=n}^{2n} (r^3 - 2r).$$

$$\boxed{}, \quad \sum_{r=n}^{2n} (r^3 - 2r) = \frac{3}{4}n(5n-4)(n+1)^2$$

USING THE STANDARD SUMMATION FORMULAE

$$\sum_{k=1}^n k = \frac{1}{2}n(n+1)$$

$$\sum_{k=1}^n k^2 = \frac{1}{6}n(n+1)(2n+1)$$

HOWEVER WE NOW HAVE:

$$\sum_{k=1}^n (k^2 - 2) = \sum_{k=1}^n k^2 - 2 \sum_{k=1}^n 1$$

$$= \left[\frac{1}{6}n(n+1)(2n+1) - \frac{1}{2}n(n+1) \right] - 2 \left[\frac{1}{2}n(n+1) - \frac{1}{2}n(n+1) \right]$$

$$= \frac{1}{6}n(n+1)(2n+1) - \frac{1}{2}n(n+1) - 2(n(n+1) - \frac{1}{2}n(n+1))$$

$$= \frac{1}{6}n \left[4n(n+1)(2n+1) - 3n(n+1) - 8n(n+1) + 4(n+1) \right]$$

AS IT WILL BE A PESS TO EXPAND TO GET, WHILE IT BET TO FACTORISE

THE TERMS, INSIDE THE BRACKET, IS PARS:

$$= \frac{1}{6}n \left[n \left[4(2n+1)(n+1) - 3(n+1) - 8(n+1) + 4 \right] \right]$$

$$= \frac{1}{6}n \left[n \left[4(n+1)(2n+1) - 3(n+1) - 8(n+1) + 4 \right] \right]$$

DIFFERENTIATE

$$= \frac{1}{6}n \left[n \left[2(2n+1) - 3(n+1) + 4(n+1) - 8(n+1) \right] \right]$$

$$= \frac{1}{6}n \left[n \left[4n+2 - 3n-3 + 4n+4 - 8n-8 \right] \right]$$

$$= \frac{1}{6}n \left[n \left[4n^2 - 7n - 7 \right] \right]$$

$$= \frac{1}{6}n \left[3n(n+1)(2n+1) - 12(n+1) \right]$$

$$= \frac{1}{6}n \left[3n(n+1)(2n+1) - 12(n+1) \right]$$

ALTERNATIVELY BY OBSERVING A CUBIC, WHERE THE NTH TERMING

$$\dots = \frac{1}{6}n \left[4n(2n+1)(n+1) - 3(n+1) - 8(n+1) + 4(n+1) \right]$$

$$= \frac{1}{6}n \left[8n^3 + 12n^2 + 8n - 3n^2 - 3n - 8n - 8 + 4n + 4 \right]$$

$$= \frac{1}{6}n \left[8n^3 + 9n^2 - 9n - 12 \right]$$

$$= \frac{1}{6}n \left[3n^3 + 3n^2 - 3n - 4 \right]$$

WORKING FOR PROVERBS

$$n=1 \quad 546 - 3 - 4 = 0$$

$$n=1 \quad -546 + 3 - 4 = 0 \quad \text{Hence } (n+1) \text{ is a factor}$$

LONG DIVIDE

$\begin{array}{r} 3n^2 + 3n - 4 \\ 3n^3 + 9n^2 + 3n - 4 \\ \hline -6n^2 - 6n \\ \hline 9n^2 + 9n - 4 \\ \hline -9n^2 - 9n \\ \hline 0 \end{array}$	$\dots = \frac{1}{6}n(n+1)(3n^2 + 3n - 4)$ $= \frac{1}{6}n(n+1)(3n-4)(n+1)$ $= \frac{1}{6}n(n+1)(3n-4)$
--	---

360000

Question 5 (***)

It is thought that for some values of the constants p and q that

$$\sum_{r=1}^n r^2(r+p) \equiv n(n+1)(n+2)(3n+q).$$

Use a detailed method to show that there exist no such values of p and q .

 , proof

EXPAND THE LHS & COMPARE COEFFICIENTS

$$\Rightarrow \sum_{r=1}^n r^2(r+p) = qn(n+1)(n+2)(3n+1)$$

$$\Rightarrow \sum_{r=1}^n (r^3 + pr^2) = \sum_{r=1}^n r^3 + p \sum_{r=1}^n r^2 \equiv qn(n+1)(n+2)(3n+1)$$

$$\Rightarrow \frac{1}{4}n^2(n+1) + \frac{1}{6}pn(n+1)(2n+1) \equiv qn(n+1)(n+2)(3n+1)$$

$$\Rightarrow n(n+1) \left[\frac{1}{4}n(n+1) + \frac{1}{6}p(2n+1) \right] \equiv qn(n+1)(n+2)(3n+1)$$

$$\Rightarrow \frac{1}{4}n(n+1) + \frac{1}{6}p(2n+1) \equiv q(n+2)(3n+1)$$

$$\Rightarrow \frac{1}{4}n^2 + \frac{1}{4}n + \frac{1}{3}pn + \frac{1}{6}p \equiv (3q^2 + n + 6n + 2)q$$

$$\equiv \frac{1}{4}n^2 + \left(\frac{1}{4} + \frac{1}{3}p\right)n + \frac{1}{6}p \equiv 3qn^2 + 7qn + 2q$$

NOW LOOKING AT EACH POWER

$[n^2]: \frac{1}{4} = 3q$
 $q = \frac{1}{12}$

$[n^1]: \frac{1}{4} + \frac{1}{3}p = 7q$
 $\frac{1}{4} + \frac{1}{3}p = \frac{7}{12}$
 $3 + 4p = 7$
 $4p = 4$
 $p = 1$

BUT NOW $[n^0]$ YIELDS INCONSISTENCY SINCE $\frac{1}{6}p = 2q$
 $\frac{1}{6} \neq \frac{1}{6}$

Question 6 (****)

Use standard results on summations to solve the following equation.

$$\sum_{r=1}^k (r^3 - 1) = 89976.$$

$$\boxed{}, \quad k = 24$$

START BY GETTING A POSITIVE EXPRESSION USING STANDARD RESULTS

$$\begin{aligned} \sum_{r=1}^k (r^3 - 1) &= \sum_{r=1}^k r^3 - \sum_{r=1}^k 1 = \frac{1}{4}k^2(k+1)^2 - k \\ &= \frac{1}{4}k(k^2(k+1)^2 - 4) = \frac{1}{4}k(k^2 + 2k^2 + k - 4) \end{aligned}$$

NOW $k-1$ IS AN OBVIOUS ZERO OF THE CUBIC, SO $(k-1)$ IS A FACTOR

$$\begin{aligned} &= \frac{1}{4}k \left[k^3 - 1 + 2k(k-1) + 4(k-1) \right] \quad \text{no one attempted ALGEBRAIC DIVISION!} \\ &= \frac{1}{4}k(k-1)(k^2 + 3k + 4) \quad \text{IRREDUCIBLE} \end{aligned}$$

NOW ROUND BY TRYING 6 VALUES AS k IS A POSITIVE INTEGER

$$\begin{aligned} f(1) &= \frac{1}{4}k(k-1)(k^2 + 3k + 4) \\ f(2) &= \frac{1}{4} \times 2 \times 1 \times (4 + 6 + 4) = 2.015 < 89976 \\ f(3) &= \frac{1}{4} \times 3 \times 2 \times (9 + 9 + 4) = -4680 < 89976 \\ f(4) &= \frac{1}{4} \times 4 \times 3 \times (16 + 12 + 4) = 26195 > 89976 \\ f(5) &= \frac{1}{4} \times 5 \times 4 \times (25 + 15 + 4) = 105600 > 89976 \\ f(6) &= \frac{1}{4} \times 6 \times 5 \times (36 + 18 + 4) = 89976 \end{aligned}$$

$\therefore k = 24 //$

Question 7 (****)

It is given that

$$\sum_{r=1}^n (Ar^3 + Br^2 + Cr) = n(n+1)(n+2)(4n-5).$$

Use a detailed method to find the value of each of the integer constants, A , B and C .

$$\boxed{}, \boxed{A=16}, \boxed{B=-3}, \boxed{C=-19}$$

Method 1: Rounding

$$\sum_{r=1}^n (Ar^3 + Br^2 + Cr) = n(n+1)(n+2)(4n-5)$$

$$f(n) = n(n+1)(n+2)(4n-5)$$

$$f(n) = (n-1)n(n+1)(4(n-1)-5) = n(n-1)(n+1)(4n-9)$$

SUBSTITUTE USE DIFFERENT VALUES OF n

$$\rightarrow f(1) = 4(1-1) = n(n+1)(n+2)(4n-9) = n(n-1)(n+1)(4n-9)$$

$$\rightarrow U_1 = n(n+1)(n+2)(4n-9) - (n-1)n(n+1)(4n-9)$$

$$\rightarrow U_1 = n(n+1)(4n^2 + 3n - 10 - (4n^2 - 3n + 4))$$

$$\rightarrow U_1 = n(n+1)(3n-14)$$

$$\rightarrow U_1 = n(16n^2 - 13n - 14)$$

$$\rightarrow U_1 = 16n^3 - 13n^2 - 14n$$

$\therefore A=16, B=-13, C=-14$

Method 2: By Equating Coefficients

$$\sum_{r=1}^n (Ar^3 + Br^2 + Cr) = n(n+1)(n+2)(4n-5)$$

$$\rightarrow A \sum_{r=1}^n r^3 + B \sum_{r=1}^n r^2 + C \sum_{r=1}^n r = n(n+1)(n+2)(4n-5)$$

$$\rightarrow \frac{1}{4}An(n+1)^2 + \frac{1}{2}Bn(n+1)(2n+1) + \frac{1}{2}Cn(n+1) = n(n+1)(n+2)(4n-5)$$

$$\rightarrow n(n+1) \left[\frac{1}{4}An(n+1) + \frac{1}{2}B(2n+1) + \frac{1}{2}C \right] = n(n+1)(n+2)(4n-5)$$

Question 8 (****)

Show by a detailed method that

$$\sum_{r=0}^n \left[2r(2r^2 - 3r - 1) + n + 1 \right] = (n^2 - 1)^2.$$

□, proof

EXPAND THE SUMMATION OR USE ONS USE STANDARD RESULTS

$$\begin{aligned} & \sum_{r=0}^n [2r(2r^2 - 3r - 1) + n + 1] \\ &= \sum_{r=0}^n [4r^3 - 6r^2 - 2r + n + 1] \end{aligned}$$

THE SUMMATION HAS AN INDEPENDENCE ON r , SO IT MAY BE PRE-INDEPENDENT

$$= 4 \sum_{r=0}^n r^3 - 6 \sum_{r=0}^n r^2 - 2 \sum_{r=0}^n r + \sum_{r=0}^n (n + 1)$$

NOTE THE THE FIRST THREE IN THE FIRST 3 SUMMATIONS IS ZERO
SO WE MAY START THESE SUMMATIONS FROM $r=1$

$$= 4 \sum_{r=1}^n r^3 - 6 \sum_{r=1}^n r^2 - 2 \sum_{r=1}^n r + (n + 1) \sum_{r=0}^n 1$$

USING STANDARD RESULTS

$$\begin{aligned} &= 4 \times \frac{1}{4} n(n+1)^2 - 6 \times \frac{1}{6} n(n+1)(2n+1) - 2 \times \frac{1}{2} n(n+1) + (n+1) \times (n+1) \\ &= n^2(n+1)^2 - n(n+1)(2n+1) - n(n+1) + (n+1)^2 \\ &= n(n+1) [n(n+1) - (2n+1) - 1] + (n+1)^2 \\ &= n(n+1) [n^2 - n - 2] + (n+1)^2 \\ &= n(n+1)(n+1)(n-2) + (n+1)^2 \\ &= (n+1)^2 [n(n-2) + 1] \\ &= (n+1)^2 (n^2 - 2n + 1) \\ &= (n+1)^2 (n-1)^2 = [(n+1)(n-1)]^2 \\ &= (n^2 - 1)^2 \end{aligned}$$

AS REQUIRED

Question 9 (***)

The sum, S_n , of the first n terms of a series whose general term is denoted by u_n is given by the following expression.

$$S_n = n^2(n+1)(n+2).$$

a) Find the first term of the series.

b) Show clearly that ...

i. ... $u_n = n(n+1)(4n-1)$

ii. ... $\sum_{r=n+1}^{2n} u_r = 3n^2(n+1)(5n+2).$

$$\boxed{}, \boxed{u_1 = 6}$$

a) TECHNIQUE WE HAVE

$$u_1 = S_1 = 1^2(1+1)(1+2) = 1 \times 2 \times 3 = 6 //$$

b) i) USING $S_n - S_{n-1} = u_n$

$$\begin{aligned} \Rightarrow u_n &= n^2(n+1)(n+2) - (n-1)^2[(n-1)+1][(n-1)+2] \\ \Rightarrow u_n &= n^2(n+1)(n+2) - (n-1)^2 n(n+1) \\ \Rightarrow u_n &= n(n+1)[n(n+2) - (n-1)^2] \\ \Rightarrow u_n &= n(n+1)[n^2 + 2n - (n^2 - 2n + 1)] \\ \Rightarrow u_n &= n(n+1)(4n-1) // \text{As required} \end{aligned}$$

ii) $\sum_{r=n+1}^{2n} u_r = S_{2n} - S_n$

$$\begin{aligned} &= (2n)^2(2n+1)(2n+2) - n^2(n+1)(n+2) \\ &= 4n^2(2n+1) \times 2(n+1) - n^2(n+1)(n+2) \\ &= n^2(n+1)[8(2n+1) - (n+2)] \\ &= n^2(n+1)(16n+8-n-2) \\ &= n^2(n+1)(15n+6) \\ &= 3n^2(n+1)(5n+2) // \text{As required} \end{aligned}$$

Question 10 (****)

Use standard summation results to prove that

$$\sum_{r=1}^n (n-r)^2 = \frac{1}{2}n(n-1)(2n-1).$$

□, proof

FIRSTLY LET US NOTE THAT $\sum_{r=1}^n r^2 = \frac{1}{6}n(n+1)(2n+1)$
 NEXT PROCEED AS FOLLOWS
 $\Rightarrow \sum_{r=1}^n (n-r)^2 = \sum_{r=1}^n (n^2 - 2nr + r^2)$
 $\Rightarrow \sum_{r=1}^n (n-r)^2 = n^2 \sum_{r=1}^n 1 - 2n \sum_{r=1}^n r + \sum_{r=1}^n r^2$
 $\Rightarrow \sum_{r=1}^n (n-r)^2 = n^2 \times n - 2n \times \frac{1}{2}n(n+1) + \frac{1}{6}n(n+1)(2n+1)$
 $\Rightarrow \sum_{r=1}^n (n-r)^2 = n^3 - n^2(n+1) + \frac{1}{6}n(n+1)(2n+1)$
 $\Rightarrow \sum_{r=1}^n (n-r)^2 = \frac{1}{6}n \left[(n+1)(2n+1) - 6n \right]$
 $\Rightarrow \sum_{r=1}^n (n-r)^2 = \frac{1}{6}n \left[2n^2 + 3n + 1 - 6n \right]$
 $\Rightarrow \sum_{r=1}^n (n-r)^2 = \frac{1}{6}n \left[2n^2 - 3n + 1 \right]$
 $\Rightarrow \sum_{r=1}^n (n-r)^2 = \frac{1}{6}n (2n-1)(n-1)$

Question 11 (****)

Use standard results on summations to solve the following equation

$$\sum_{r=3}^9 \left[\left(\frac{r}{k} \right)^3 + (r-1)(r+1) \right] = 304.5.$$

$$\boxed{k = 4}$$

WRITE IN SECTIONS

$$\sum_{r=3}^9 \left[\left(\frac{r}{k} \right)^3 + (r-1)(r+1) \right] = \sum_{r=3}^9 \left[\frac{r^3}{k^3} + r^2 - 1 \right] = \sum_{r=3}^9 \frac{r^3}{k^3} + \sum_{r=3}^9 (r^2 - 1)$$

EVALUATE AND SIMPLIFY EACH TERM USING $\sum_{r=1}^n r^3 = \frac{1}{4}n^2(n+1)^2$

$$\begin{aligned} \sum_{r=3}^9 \frac{r^3}{k^3} &= \frac{1}{k^3} \sum_{r=3}^9 r^3 = \frac{1}{k^3} \left[\sum_{r=1}^9 r^3 - \sum_{r=1}^2 r^3 \right] \\ &= \frac{1}{k^3} \left[\frac{1}{4}9^2(9+1)^2 - \frac{1}{4}2^2(2+1)^2 \right] \\ &= \frac{1}{k^3} \times 2016 \end{aligned}$$

SIMILARLY USING $\sum_{r=1}^n r^2 = \frac{1}{6}n(n+1)(2n+1)$

$$\begin{aligned} \sum_{r=3}^9 (r^2 - 1) &= \sum_{r=3}^9 r^2 - \sum_{r=3}^9 1 \\ &= \sum_{r=1}^9 r^2 - \sum_{r=1}^2 r^2 - \sum_{r=3}^9 1 \\ &= \frac{1}{6}9 \times 10 \times 19 - (1^2 + 2^2) - 7 \\ &= 285 - 5 - 7 \\ &= 273 \end{aligned}$$

FINALLY PUT TOGETHER

$$\begin{aligned} \frac{2016}{k^3} + 273 &= 304.5 \Rightarrow \frac{2016}{k^3} = 31.5 \\ \Rightarrow 31.5 k^3 &= 2016 \\ \Rightarrow k^3 &= 64 \\ \Rightarrow k &= 4 \end{aligned}$$

Question 12 (****)

$$\sum_{r=1}^n (ar^2 + br + c) \equiv n^3 + 5n^2 + 6n,$$

where a , b and c are integer constants.

Determine the value of a , b and c .

$$\boxed{a=3}, \boxed{b=7}, \boxed{c=2}$$

$$\sum_{i=1}^n a_i^2 b_i + c = a \sum_{i=1}^n a_i^2 + b \sum_{i=1}^n a_i^2 + c \sum_{i=1}^n 1$$

$$= 2a \cdot n(n+1)/2 + \frac{1}{2}n(n+1) + cn$$

$$= \frac{1}{2}b \cdot (a(n+1)(2n+1) + 2(n+1) + 6c)$$

$$= \frac{1}{2}n \cdot [2n^2 + 3n + a + 2a + 3b + 6c]$$

$$= \frac{1}{2}n \cdot [2n^2 + 3(a+b) + (a+3b+6c)]$$

$$= \frac{1}{2}n \cdot [2n^2 + \frac{1}{2}(a+b)^2 + \frac{1}{2}(a+3b+6c)n]$$

$$\text{Now } \sqrt{n(n+1)(2n+3)} = \sqrt{n^2 + 3n^2 + 6n} \quad | \text{ ignore}$$

$$\frac{1}{2}n = 1 \quad \frac{1}{2}(a+b) = 5 \quad \frac{1}{2}(a+3b+6c) = 6$$

$$: a=3 \quad a+b=10 \quad a+3b+6c=36$$

$$b=7 \quad 3+21+6c=36$$

$$c=2$$

Question 13 (****)

The variance $\text{Var}(n)$ of the first n natural numbers is given by

$$\text{Var}(n) = \frac{1}{n} \sum_{r=1}^n r^2 - \left[\frac{1}{n} \sum_{r=1}^n r \right]^2.$$

Determine a simplified expression for $\text{Var}(n)$ and hence evaluate $\text{Var}(61)$.

$$\boxed{\text{Var}(n) = \frac{1}{12}(n^2 - 1)}, \quad \boxed{\text{Var}(61) = 310}$$

Handwritten derivation of the variance formula for the first n natural numbers:

$$\begin{aligned} \text{Variance} &= \frac{\sum_{r=1}^n r^2}{n} - \left(\frac{\sum_{r=1}^n r}{n} \right)^2 \\ &= \frac{\frac{1}{6}n(n+1)(2n+1)}{n} - \left(\frac{\frac{1}{2}n(n+1)}{n} \right)^2 \\ &= \frac{1}{6}n(n+1)(2n+1) - \frac{1}{4}(n+1)^2 \\ &= \frac{1}{12}n(n+1)[2(2n+1) - 3(n+1)] \\ &= \frac{1}{12}n(n+1)(4n+2-3n-3) \\ &= \frac{1}{12}n(n+1)(n-1) \end{aligned}$$

If $n=61$

$$\text{Variance} = \frac{1}{12} \times 61 \times 60 = 61 \times 5 = 310 //$$

Question 14 (****)

$$f(n) = \sum_{r=1}^n [r^3 - r], \quad n \in \mathbb{N}.$$

- a) Use standard summation results to find a fully factorized expression for $f(n)$.
- b) Hence solve the equation

$$\sum_{r=5}^{10} [r^3 - r + 6k] - \sum_{r=1}^{12} [r^2 + k^2] = 70$$

$$\boxed{}, \quad \boxed{f(n) = \frac{1}{4}n(n-1)(n+1)(n+2)}, \quad \boxed{k = -12, k = 15}$$

a) $\sum_{r=1}^n (r^3 - r) = \sum_{r=1}^n r^3 - \sum_{r=1}^n r = \frac{1}{4}n(n+1)^2 - \frac{1}{2}n(n+1)$
 $= \frac{1}{4}n(n+1)[n(n+1) - 2] = \frac{1}{4}n(n+1)(n^2 + n - 2)$
 $= \frac{1}{4}n(n+1)(n-1)(n+2)$

b) EVALUATE IN SECTIONS
 $\Rightarrow \sum_{r=5}^{10} [r^3 - r + 6k] - \sum_{r=1}^{12} [r^2 + k^2] = 70$
 $\Rightarrow \sum_{r=5}^{10} (r^3 - r) + 6k \sum_{r=5}^{10} 1 - \sum_{r=1}^{12} r^2 - k^2 \sum_{r=1}^{12} 1 = 70$
 $\Rightarrow \left[\sum_{r=1}^{10} (r^3 - r) - \sum_{r=1}^4 (r^3 - r) \right] + 6k(1+1+1+1+1) - \left[\sum_{r=1}^{12} r^2 - \sum_{r=1}^4 r^2 \right] - k^2(12) = 70$
 $\Rightarrow \frac{1}{4} \times 10 \times 11 \times 12 - \frac{1}{4} \times 4 \times 5 \times 6 + 6k \times 5 - \frac{1}{12} \times 12 \times 13 \times 14 - k^2 \times 12 = 70$
 $\Rightarrow 2870 - 90 + 36k - 650 - 12k^2 = 70$
 $\Rightarrow 0 = 12k^2 - 36k - 2460$
 $\Rightarrow k^2 - 3k - 190 = 0$
 $\Rightarrow (k-15)(k+12) = 0$
 $\Rightarrow k = \begin{matrix} 15 \\ -12 \end{matrix}$

Question 15 (****)

The function $F(n)$ is defined as

$$F(n) = \sum_{r=1}^n [r(r-1)(n-2)(r+1)] \quad n \in \mathbb{N}.$$

Show with detailed workings that

$$F(2n) - F(n) = \frac{1}{2}n(n^2 - 1)(31n^2 - 4).$$

□, proof

Solve the series using standard devices

$$F(n) = \sum_{r=1}^n [(r-1)(r-2)(r+1)] = (n-2) \sum_{r=1}^n [r(r-1)(r+1)]$$

$$= (n-2) \sum_{r=1}^n (r^3 - r) = (n-2) \left[\sum_{r=1}^n r^3 - \sum_{r=1}^n r \right]$$

Using $\sum_{r=1}^n r = \frac{1}{2}n(n+1)$ and $\sum_{r=1}^n r^3 = \frac{1}{4}n^2(n+1)^2$

$$\Rightarrow F(n) = (n-2) \left[\frac{1}{4}n^2(n+1)^2 - \frac{1}{2}n(n+1) \right]$$

$$\Rightarrow F(n) = (n-2) \times \frac{1}{4}n(n+1) [n(n+1) - 2]$$

$$\Rightarrow F(n) = \frac{1}{4}(n-2)n(n+1)(n^2+n-2)$$

$$\Rightarrow F(n) = \frac{1}{4}n(n+1)(n-2)(n-1)(n+2)$$

$$\Rightarrow F(n) = \frac{1}{4}n(n^2-1)(n^2-4)$$

Substituting into the equation

$$F(2n) - F(n) = \frac{1}{4}(2n)(2n^2-1)[(2n)^2-4] - \frac{1}{4}n(n^2-1)(n^2-4)$$

$$= \frac{1}{4} [2n(4n^2-1)(4n^2-4) - n(n^2-1)(n^2-4)]$$

$$= \frac{1}{4} [8n(4n^2-1)(n^2-1) - n(n^2-1)(n^2-4)]$$

$$= \frac{1}{4}n(n^2-1) [8(4n^2-1) - (n^2-4)]$$

$$= \frac{1}{4}n(n^2-1) [32n^2-8-n^2+4]$$

$$= \frac{1}{4}n(n^2-1)(31n^2-4)$$

A2 20/04/20

Created by T. Madas

SUMMATIONS

BY FORMULAS

5 HARD QUESTIONS

Created by T. Madas

Question 1 (****+)

It is given that

$$\sum_{r=1}^{20} (r-10) = 200 \quad \text{and} \quad \sum_{r=1}^{20} (r-10)^2 = 2800.$$

Find the value of

$$\sum_{r=1}^{20} r^2.$$

$$\boxed{}, \quad \sum_{r=1}^{20} r^2 = 8800$$

USING THE LINEARITY OF THE SIGMA OPERATOR

$$\sum_{r=k}^n [af(r) + bg(r)] = a \sum_{r=k}^n f(r) + b \sum_{r=k}^n g(r)$$

MANIPULATING AS FOLLOWS

$$\sum_{r=1}^{20} (r-10) = 200$$

$$\sum_{r=1}^{20} r - 10 \sum_{r=1}^{20} 1 = 200$$

$$S - 10 \times 20 = 200$$

$$S = 400$$

FINALLY USING THE SECOND FACT

$$\sum_{r=1}^{20} (r-10)^2 = 2800$$

$$\sum_{r=1}^{20} (r^2 - 20r + 100) = 2800$$

$$\sum_{r=1}^{20} r^2 - 20 \sum_{r=1}^{20} r + 100 \sum_{r=1}^{20} 1 = 2800$$

$$S^2 - 20 \times 400 + 100 \times 20 = 2800$$

$$S^2 - 8000 + 2000 = 2800$$

$$S^2 = 8800$$

Question 2 (****+)

$$\sum_{r=1}^n (r+a)(r+b) \equiv \frac{1}{3}n(n-1)(n+4),$$

where a and b are integer constants.

Use a clear algebraic method to determine the value of a and the value of b .

, 2 and -1 (in any order)

WIMP-STANDARD: PROVES A THE UNIFORMITY OF THE SIGMA OPERATOR

$$\begin{aligned} \Rightarrow \sum_{r=1}^n (r+a)(r+b) &\equiv \frac{1}{3}n(n-1)(n+4) \\ &\Rightarrow \sum_{r=1}^n [r^2 + (a+b)r + ab] \equiv \frac{1}{3}n(n-1)(n+4) \\ &\Rightarrow \sum_{r=1}^n r^2 + (a+b) \sum_{r=1}^n r + ab \sum_{r=1}^n 1 \equiv \frac{1}{3}n(n-1)(n+4) \\ &\Rightarrow \frac{1}{6}n(n+1)(2n+1) + (a+b) \frac{1}{2}n(n+1) + abn \equiv \frac{1}{3}n(n-1)(n+4) \\ &\Rightarrow n(n+1)(2n+1) + 3(a+b)n(n+1) + 6abn \equiv 2n(n-1)(n+4) \end{aligned}$$

DIVIDING BY $n > 0$, AND EXPANDING BOTH SIDES

$$\begin{aligned} \Rightarrow (n+1)(2n+1) + 3(a+b)(n+1) + 6ab &\equiv 2(n-1)(n+4) \\ \Rightarrow 2n^2 + 3n + 1 + 3(a+b)(n+1) + 6ab &\equiv 2n^2 + 6n - 8 \\ \Rightarrow 3(a+b)(n+1) + 6ab &\equiv 3n - 9 \\ \Rightarrow 3(a+b)n + 3(a+b) + 6ab &\equiv 3n - 9 \end{aligned}$$

$$\begin{aligned} \therefore \begin{cases} 3(a+b) = 3 \\ 3(a+b) + 6ab = -9 \end{cases} &\quad \begin{cases} 3(a+b) = 3 \\ a+b = 1 \end{cases} \\ \begin{cases} a+b = 1 \\ 2ab = -3 \end{cases} &\quad \begin{cases} a+b = 1 \\ 2ab = -3 \end{cases} \\ \begin{cases} a+b = 1 \\ ab = -2 \end{cases} &\quad \begin{cases} a+b = 1 \\ ab = -2 \end{cases} \end{aligned}$$

BY INSPECTION OR SOLVING WE OBTAIN $a=2$ $b=-1$ OR $a=-1$ $b=2$
ANY ORDER, AS EQUATIONS ARE SYMMETRIC

By using an algebraic method, find the value of

,

MITU 8.

PROBLEM 2. THE THIRDS ARE REMOVED

$= 9^4 - 9^3 + 9^2 - 9^1 + \dots + 2^2 - 1^2$

$= (9^4 - 9^3) + (9^2 - 9^1) + (1^2 - 0^1) + \dots + (2^2 - 1^2)$

$= (9^4 - 9^3) + (9^2 - 9^1) + (1^2 - 0^1) + \dots + (1^2 - 0^1) + \dots + (1^2 - 0^1)$

$= 2(9^4) + 2(9^2) + 2(1^2) + \dots + 2(1^2)$

$= 2[4 + 12 + 20 + \dots + 180 + 108 + 16]$

$= 2 \times 4 [1 + 3 + 5 + \dots + 45 + 47 + 49]$

$\Rightarrow V \sim$ Arithmetic Progression with $a=1$
 $d=2$
 $n=26$

$u_n = a + (n-1)d$
 $49 = 1 + (n-1)2$
 $48 = 1, 2n-2$
 $50 = 2n$
 $n=26$

$S_n = \frac{n}{2}[2a + (n-1)d]$

$= 8 \times \frac{26}{2} [1 + 49]$

$= 8 \times \frac{26 \times 50}{2}$

$= 5200$

Ans: 5200

Question 4 (****+)

Show clearly that

$$1^3 - 2^3 + 3^3 - 4^3 + \dots - 40^3 = -33200.$$

5⁵

,

proof

Method A

$$\begin{aligned}
 1^3 - 2^3 + 3^3 - 4^3 + \dots - 40^3 &= (1^3 + 3^3 + 5^3 + \dots + 39^3) - (2^3 + 4^3 + 6^3 + \dots + 40^3) \\
 &= \sum_{r=1}^{20} (2r-1)^3 - \sum_{r=1}^{20} (2r)^3 \\
 &= \sum_{r=1}^{20} [(2r-1)^3 - (2r)^3] \\
 &= \sum_{r=1}^{20} [8r^3 - 12r^2 + 6r - 1 - 8r^3] \\
 &= -12 \sum_{r=1}^{20} r^2 + 6 \sum_{r=1}^{20} r - \sum_{r=1}^{20} 1
 \end{aligned}$$

USING STANDARD SUMMATION RESULTS

$$\sum_{r=1}^n r^2 = \frac{1}{6}n(n+1)(2n+1) \quad \text{and} \quad \sum_{r=1}^n r = \frac{1}{2}n(n+1)$$

$$\begin{aligned}
 \therefore &= -12 \times \frac{1}{6} \times 20 \times 21 \times 41 + 6 \times \frac{1}{2} \times 20 \times 21 - 20 \\
 &= -2 \times 20 \times 21 \times 41 + 3 \times 20 \times 21 - 20 \\
 &= 20 [-2 \times 21 \times 41 + 3 \times 21 - 1] \\
 &= 20 [21 [-82 + 3] - 1] \\
 &= 20 [21 \times (-79) - 1] \\
 &= 20 [-1660] \\
 &= -33200
 \end{aligned}$$

Method B

$$\begin{aligned}
 1^3 - 2^3 + 3^3 - 4^3 + \dots - 40^3 &= (1^3 + 2^3 + 3^3 + \dots + 40^3) - 2(2^3 + 4^3 + 6^3 + \dots + 40^3) \\
 &= \sum_{r=1}^{40} r^3 - 2 \sum_{r=1}^{20} (2r)^3 \\
 &= \sum_{r=1}^{40} r^3 - 16 \sum_{r=1}^{20} r^3
 \end{aligned}$$

USING THE STANDARD SUMMATION FORMULA

$$\sum_{r=1}^n r^3 = \frac{1}{4}n^2(n+1)^2$$

$$\begin{aligned}
 \therefore &= \frac{1}{4} \times 40^2 \times 41^2 - 16 \times \frac{1}{4} \times 20^2 \times 21^2 \\
 &= \frac{1}{4} (1600 \times 1681) - 4 \times 20^2 \times 21^2 \\
 &= 20^2 \times 41^2 - 4 \times 20^2 \times 21^2 \\
 &= 20^2 [41^2 - 4 \times 21^2]
 \end{aligned}$$

QUICK CALCULATIONS

41	21	441	1764	83
$\times 41$	$\times 21$	$\times 41$	$\times 21$	$\times 41$
1681	441	1764	83	33200

$$\begin{aligned}
 \therefore &= 400 (1681 - 4 \times 441) \\
 &= 400 (1681 - 1764) \\
 &= 400 \times (-83) \\
 &= -33200
 \end{aligned}$$

Question 5 (****+)

The positive integer functions f and g are defined as

$$f(n) = \sum_{r=1}^n r^3 \quad \text{and} \quad g(n) = 1 + \sum_{r=1}^n (2r+1).$$

Evaluate

$$\sum_{n=1}^{39} \left[\frac{f(n)}{g(n)} \right].$$

, 5135

$f(n) = \sum_{r=1}^n r^3$, $g(n) = 1 + \sum_{r=1}^n (2r+1)$

• Define the "individual common/23" is simplified form

$f(n) = \sum_{r=1}^n r^3 = \frac{1}{4}n^2(n+1)^2$

$g(n) = 1 + \sum_{r=1}^n (2r+1) = 1 + 2 \sum_{r=1}^n r + \sum_{r=1}^n 1$

$= 1 + 2 \times \frac{1}{2}n(n+1) + n$

$= 1 + n(n+1) + n = 1 + n^2 + n + n$

$= n^2 + 2n + 1 = (n+1)^2$

• Hence use above

$\sum_{n=1}^{39} \frac{f(n)}{g(n)} = \sum_{n=1}^{39} \frac{\frac{1}{4}n^2(n+1)^2}{(n+1)^2} = \sum_{n=1}^{39} \frac{1}{4}n^2$

$= \frac{1}{4} \times \frac{1}{6} \times (39+1)(39+1)$

$= \frac{1}{24} \times 39 \times 40 \times 41$

$= 5135$

Created by T. Madas

SUMMATIONS BY FORMULAS

8 ENRICHMENT QUESTIONS

Created by T. Madas

Question 1 (****)

Use standard summation results to prove that

$$\sum_{r=n}^{2n} (n-r)^2 = \sum_{r=1}^n r^2.$$

□, proof

EXPAND AND TRY

$$\begin{aligned} \sum_{r=n}^{2n} (n-r)^2 &= \sum_{r=n}^{2n} (r^2 - 2nr + n^2) \\ &= \sum_{r=n}^{2n} r^2 - 2n \sum_{r=n}^{2n} r + \sum_{r=n}^{2n} n^2 \\ &= n^2 [2n - (n-1)] - 2n \left[\frac{1}{2}(2n)(2n+1) - \frac{1}{2}(n-1)(n-1) \right] \\ &\quad + \frac{1}{2}(2n)(2n+1)(2n+1) - \frac{1}{2}(n-1)(n-1)(n-1) \\ &= n^2(n+1) - 2n \left[n(2n+1) - \frac{1}{2}n(n-1) \right] + \frac{1}{2}n(2n+1)(2n+1) - \frac{1}{2}n(n-1)(n-1) \\ &= n^2(n+1) - 2n^2(2n+1) + n^2(n-1) + \frac{1}{2}n(2n+1)(2n+1) - \frac{1}{2}n(n-1)(n-1) \\ &= n^2(n+1) - 2n^2(2n+1) + n^2(n-1) + \frac{1}{2}n(2n+1)(2n+1) - \frac{1}{2}n(n-1)(n-1) \end{aligned}$$

FACTORISE +1, -1 FIRST

$$\begin{aligned} &= \frac{1}{2}n \left[2n(n+1) - 4n(2n+1) + 2n(n-1) + (2n+1)(2n+1) - (n-1)(n-1) \right] \\ &= \frac{1}{2}n \left[2n^2 + 2n - 8n^2 - 4n + 2n^2 + 2n - 1 + 4n^2 + 4n + 2n - 1 \right] \\ &= \frac{1}{2}n \left[2n^2 + 2n + 1 \right] \\ &= \frac{1}{2}n(2n+1)(n+1) \\ &= \sum_{r=1}^n r^2 \end{aligned}$$

✓ Q.E.D.

Question 2 (****)

Find the sum of the first 16 terms of the following series.

$$\frac{1^3}{1} + \frac{1^3 + 2^3}{1 + 3} + \frac{1^3 + 2^3 + 3^3}{1 + 3 + 5} + \frac{1^3 + 2^3 + 3^3 + 4^3}{1 + 3 + 5 + 7} + \dots$$

SP-M, 446

$$\frac{1}{1} + \frac{1^2+2^2}{1 \cdot 3} + \frac{1^3+2^3+3^3}{1 \cdot 3 \cdot 5} + \frac{1^4+2^4+3^4+4^4}{1 \cdot 3 \cdot 5 \cdot 7} + \dots$$

START BY WRITING THE ABOVE EXPRESSION COMPACTLY

$$\sum_{n=1}^{16} \left[\frac{\frac{n}{2} \cdot n^3}{\frac{n}{2} \cdot (n+1)} \right] = \sum_{n=1}^{16} \left[\frac{\frac{n}{2} \cdot n^3}{\frac{n^2}{2} \cdot (n+1)} \right]$$

USING STANDARD SUMMATION FORMULAS

$$= \sum_{n=1}^{16} \left[\frac{\frac{1}{2} n^2 (n+1)^2}{2 \cdot \frac{1}{2} n (n+1) \cdot n} \right] = \sum_{n=1}^{16} \frac{\frac{1}{2} n^2 (n+1)^2}{n^2 (n+1)}$$

$$= \sum_{n=1}^{16} \left[\frac{\frac{1}{2} n^2 (n+1)^2}{n^2} \right] = \frac{1}{2} \cdot \sum_{n=1}^{16} (n+1)^2$$

OTHER OPTION AND USE FORMULAS AGAIN OR "TRANSLATE" 1 STEP

$$\dots = \frac{1}{2} \sum_{k=2}^{17} k^2 = \frac{1}{2} \times \frac{1}{2} \times k \cdot (k+1) \cdot (k+1) \Big|_{k=17} - \frac{1}{2} \times$$

$$= \frac{1}{2} \times \frac{1}{2} \times 17 \times 18 \times 19 - \frac{1}{2}$$

$$= \frac{1}{2} \left[\frac{1}{2} \times 17 \times 18 \times 19 - 1 \right]$$

$$= \underline{\underline{444}}$$

Easy now

Question 3 (****)

The function f is defined for $n \in \mathbb{N}$ as

$$f(n) \equiv 1 \times n^2 + 2(n-1)^2 + 3(n-2)^2 + 4(n-3)^2 + \dots + (n-1) \times 2^2 + n \times 1^2.$$

Determine a simplified expression for the sum of $f(n)$, giving the final answer in fully factorized form.

$$\boxed{}, f(n) = \frac{1}{12}n(n+2)(n+1)^2$$

Handwritten solution for Question 3:

Given: $f(n) = 1 \times n^2 + 2(n-1)^2 + 3(n-2)^2 + \dots + (n-1) \times 2^2 + n \times 1^2$

Method 1: Using the difference of squares

$$\sum_{r=1}^n [r(n+1-r)^2] = \sum_{r=1}^n [r(n+1)^2 - 2(n+1)r^2 + r^3]$$

$$= \sum_{r=1}^n [r(n+1)^2 - 2(n+1)r^2 + r^3]$$

$$= (n+1)^2 \sum_{r=1}^n r - 2(n+1) \sum_{r=1}^n r^2 + \sum_{r=1}^n r^3$$

Method 2: Using standard summation formulae

$$\sum_{r=1}^n [r(n+1-r)^2] = \sum_{r=1}^n [r(n+1)^2 - 2(n+1)r^2 + r^3]$$

$$= (n+1)^2 \sum_{r=1}^n r - 2(n+1) \sum_{r=1}^n r^2 + \sum_{r=1}^n r^3$$

$$= (n+1)^2 \left[\frac{n(n+1)}{2} \right] - 2(n+1) \left[\frac{n(n+1)(2n+1)}{6} \right] + \left[\frac{n(n+1)^2(n+2)}{4} \right]$$

$$= \frac{1}{12}n(n+1)^2(n+2)$$

Question 4 (****)

Use an algebraic method justifying each step, to find the greatest value of k , $k \in \mathbb{N}$, which satisfies the following inequality.

$$\sum_{r=k+1}^{80} \left[\frac{r-1}{\log_{8^r}(16)} \right] > 100\,000.$$

$$\boxed{}, \boxed{k=48}$$

1. STRIKE BY MANIPULATING THE LOGS

$$\frac{1}{\log_{8^r} 16} = \log_{8^r} 8 = r \log_8 8 = r \times \frac{3}{4}$$

SINCE $16 = 2^4$ AND $8 = 2^3$

2. SUMMING FROM $r=1$ TO n , TO GET A GENERAL EXPRESSION

$$\begin{aligned} \sum_{r=1}^n \left[\frac{r-1}{\log_{8^r} 16} \right] &= \sum_{r=1}^n \left[\frac{3}{4} r(r-1) \right] \\ &= \frac{3}{4} \sum_{r=1}^n (r^2 - r) \\ &= \frac{3}{4} \left[\frac{1}{6} n(n+1)(2n+1) - \frac{1}{2} n(n+1) \right] \\ &= \frac{3}{4} \left[\frac{1}{6} n(n+1)(2n+1 - 3) \right] \\ &= \frac{1}{8} n(n+1)(2n-2) \\ &= \frac{1}{4} n(n+1)(n-1) \\ &= \frac{1}{4} n(n^2-1) \end{aligned}$$

3. RETURNING TO THE INEQUALITY

$$\begin{aligned} \Rightarrow \sum_{r=k+1}^{80} \left[\frac{r-1}{\log_{8^r} 16} \right] &> 100\,000 \\ \Rightarrow \frac{1}{4} [80(80^2-1)] - \frac{1}{4} k(k^2-1) &> 100\,000 \end{aligned}$$

4. BY TOTAL OF TERMS NOTING THAT $f(x) \approx x^3$

$$\begin{aligned} f(40) &= 40 \times 39 \times 38 = 5940 < 111920 \\ f(50) &= 50 \times 49 \times 48 = 117600 > 111920 \\ f(49) &= 49 \times 48 \times 47 = 110544 < 111920 \end{aligned} \quad \therefore k=48$$

so $f(x) = k(k+1)(k-1)$ IS INCREASING FOR $k > 1$

Question 5 (*****)

Use algebra to find the sum of the first 100 terms of the following sequence.

7, 12, 19, 28, 39, 52, 67, 84, 103, ...

$$\boxed{}, \quad f(n) = \frac{1}{12}n(n+2)(n+1)^2$$

● INVESTIGATING THE PATTERN FURTHER BY DIFFERENCING

7 + 12 + 19 + 28 + 39 + 52 + 67 + 84 + 103 + ...

5 7 9 11 13 15 17 19

2 2 2 2 2 2 2 2

As the second differences are constant, this is a quadratic pattern, where quadratic coefficient is half the constant second difference, i.e. $u_n = n^2 + cn + d$

● SUPPOSE THE n^{th} TERM OF THE SEQUENCE/ SERIES WAS u_n

n^2	1	4	9	16	25	36
"our series"	7	12	19	28	39	52
	+6	+8	+10	+12	+14	+16

← $2n+4$

● Hence the required n^{th} term is

$$u_n = n^2 + 2n + 4$$

● This we require to find

$$\sum_{n=1}^k (n^2 + 2n + 4) \quad \text{with } k=100$$

● USING THE STANDARD SUMMATION FORMULAE IN k

AND SUBSTITUTE $k=100$ AT THE END

$$\sum_{n=1}^k (n^2 + 2n + 4) = \sum_{n=1}^k n^2 + 2 \sum_{n=1}^k n + 4 \sum_{n=1}^k 1$$

$$= \frac{1}{6}k(k+1)(2k+1) + 2 \times \frac{1}{2}k(k+1) + 4 \times k$$

$$= \frac{1}{6}k(k+1)(2k+1) + k(k+1) + 4k$$

$$= \frac{1}{6}k(k+1)(2k+1+6) + 4k$$

$$= \frac{1}{6}k(k+1)(2k+7) + 4k$$

● LET $k=100$ AND WE OBTAIN

$$\sum_{n=1}^{100} (n^2 + 2n + 4) = \frac{1}{6} \times 100 \times 101 \times 207 + 4 \times 100$$

$$= 348\,850$$

Question 6 (****)

Evaluate the following expression

$$\sum_{n=1}^9 \sum_{m=n+1}^{2n} [2m+n].$$

Detailed workings must be shown.

V, , 1185

PROCEED AS FOLLOWS

$$\sum_{n=1}^9 \left[\sum_{m=n+1}^{2n} (2m+n) \right] = \sum_{n=1}^9 \left[2 \sum_{m=n+1}^{2n} m + n \sum_{m=n+1}^{2n} 1 \right]$$

ONLY STUDIED SUMMATION RESULTS

$$= \sum_{n=1}^9 \left[2 \times \frac{1}{2} (2n)(2n+1) - 2 \times \frac{1}{2} n(n+1) + n(2n-n+1) \right]$$

$$= \sum_{n=1}^9 [4n^2 + 2n - n^2 - n + 4n^2] = \sum_{n=1}^9 (4n^2 + 4n)$$

$$= 4 \sum_{n=1}^9 n^2 + \sum_{n=1}^9 4n$$

FINALLY USE MORE STUDIED SUMMATION RESULTS

$$= 4 \times \left(\frac{1}{6} \times 9 \times 10 \times 19 \right) + \frac{1}{2} \times 9 \times 10$$

$$= 60 \times 19 + 45$$

$$= 1140 + 45$$

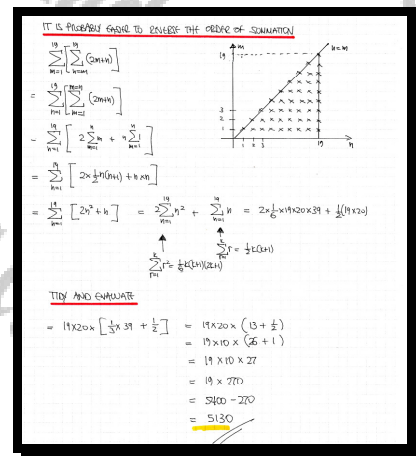
$$= 1185$$

Question 7 (**)**

Evaluate the following expression

$$\sum_{m=1}^{19} \sum_{n=m}^{19} [2m+n].$$

You may find reversing the order of summation useful in this question

 ,


Question 8 (**)**

The function f is defined as

$$f(n, y) \equiv \sum_{x=1}^n \frac{x^2 y^x}{k}, \quad n \in \mathbb{N}, \quad y \in \mathbb{R}$$

where $k = \sum_{r=1}^n r^2$.

Use standard results on series to show that

$$\left. \frac{d^2 f}{dy^2} \right|_{y=1} + \left. \frac{df}{dy} \right|_{y=1} - \left[\left. \frac{df}{dy} \right|_{y=1} \right]^2 = \frac{3n^4 + 6n^3 - n^2 - 4n - 4}{20(2n+1)^2}.$$

You may assume without proof $\sum_{r=1}^n r^4 = \frac{1}{30} n(n+1)(6n^3 + 9n^2 + n - 1)$.

P, proof

USE STANDARD SUMMATION RESULTS

$$k = \sum_{x=1}^n x^2 = \frac{1}{6} n(n+1)(2n+1) \quad \text{K DOES NOT DEPEND ON Y, SO WE CAN PULL IT OUT OF THE SUMMATION}$$

$$\therefore f(n, y) = \sum_{x=1}^n \frac{x^2 y^x}{k} = \frac{1}{k} \sum_{x=1}^n x^2 y^x$$

DIFFERENTIATE WITH RESPECT TO Y - K DOES NOT DEPEND ON Y, ENTER

$$\frac{df}{dy} = \frac{1}{k} \sum_{x=1}^n [x(2y^{x-1})] = \frac{1}{k} \sum_{x=1}^n (x^2 y^{x-1})$$

$$\left. \frac{df}{dy} \right|_{y=1} = \frac{1}{k} \sum_{x=1}^n (x^2 \times 1^{x-1}) = \frac{1}{k} \sum_{x=1}^n x^2 \quad \leftarrow \text{NO NEED TO SUMMATE YET}$$

ANOTHER DIFFERENTIATION WITH RESPECT TO Y IS NEEDED

$$\frac{d^2 f}{dy^2} = \frac{1}{k} \sum_{x=1}^n (x^2 y^{x-2})$$

$$\frac{d^2 f}{dy^2} = \frac{1}{k} \sum_{x=1}^n [x(x-1)y^{x-2}] = \frac{1}{k} \sum_{x=1}^n (x^2 - x)y^{x-2}$$

$$\frac{d^2 f}{dy^2} = \frac{1}{k} \sum_{x=1}^n x^2 y^{x-2} - \frac{1}{k} \sum_{x=1}^n x y^{x-2} \quad \leftarrow \text{NO NEED TO SUMMATE YET}$$

SUBSTITUTED INTO THE EXPRESSION GIVEN

$$\left. \frac{d^2 f}{dy^2} \right|_{y=1} + \left. \frac{df}{dy} \right|_{y=1} - \left[\left. \frac{df}{dy} \right|_{y=1} \right]^2$$

$$= \frac{1}{k} \sum_{x=1}^n x^2 - \frac{1}{k} \sum_{x=1}^n x^2 + \frac{1}{k} \sum_{x=1}^n x^2 - \left[\frac{1}{k} \sum_{x=1}^n x^2 \right]^2$$

$$= \frac{1}{k} \sum_{x=1}^n x^2 - \frac{1}{k^2} \left[\sum_{x=1}^n x^2 \right]^2$$

SUBSTITUTE k = 1/6 n(n+1)(2n+1) & SIMPLIFY

$$\dots = \frac{1}{\frac{1}{6} n(n+1)(2n+1)} \times \frac{1}{6} n(n+1)(2n+1) - \left[\frac{1}{\frac{1}{6} n(n+1)(2n+1)} \times \frac{1}{6} n(n+1)(2n+1) \right]^2$$

$$= \frac{1}{\frac{1}{6} n(n+1)(2n+1)} - \frac{1}{\frac{1}{6} n(n+1)(2n+1)^2}$$

$$= \frac{6n^3 + 9n^2 + n - 1}{5n(n+1)} - \frac{9n^2(n+1)^2}{4(2n+1)^2}$$

$$= \frac{4(6n^3 + 9n^2 + n - 1) - 5 \times 9n^2(n+1)^2}{20(n+1)^2}$$

THE DENOMINATOR IS NOW WHAT WE REQUIRE SO THEY 'THE NUMERATOR'

$$(8n^4 + 12n^3 + 9n^2 - 4) - 45n^2(n^2 + 2n + 1)$$

$$= \{ 8n^4 + 12n^3 + 9n^2 - 4n - 45n^4 - 90n^3 - 45n^2 \}$$

$$= 40n^4 + 96n^3 + 24n^2 - 4n - 4 - 45n^4 - 90n^3 - 45n^2$$

$$= 3n^4 + 6n^3 - n^2 - 4n - 4$$

IS THE REQUIRED NUMERATOR

$$\therefore \left. \frac{d^2 f}{dy^2} \right|_{y=1} + \left. \frac{df}{dy} \right|_{y=1} - \left[\left. \frac{df}{dy} \right|_{y=1} \right]^2 = \frac{3n^4 + 6n^3 - n^2 - 4n - 4}{20(2n+1)^2} \quad \text{K.E. REQUIRED}$$

Created by T. Madas

SUMMATIONS

METHOD OF DIFFERENCES

8 BASIC QUESTIONS

Created by T. Madas

Question 1 (**)

$$f(r) = \frac{5}{(5r-1)(5r+4)}, \quad r \in \mathbb{N}$$

a) Express $f(r)$ into partial fractions

b) Hence show that

$$\sum_{r=1}^n f(r) = \frac{5n}{4(5n+4)}.$$

$$f(r) = \frac{1}{5r-1} - \frac{1}{5r+4}$$

(a) $f(r) = \frac{5}{(5r-1)(5r+4)} = \frac{5r+4}{5r-1} \cdot \frac{1}{5r+4} = \frac{1}{5r-1} - \frac{1}{5r+4}$

(b) $\sum_{r=1}^n \frac{5}{(5r-1)(5r+4)} = \sum_{r=1}^n \left(\frac{1}{5r-1} - \frac{1}{5r+4} \right)$

$\frac{1}{4} - \frac{1}{9}$
 $\frac{1}{9} - \frac{1}{14}$
 \vdots
 $\frac{1}{5n-1} - \frac{1}{5n+4}$

$\sum_{r=1}^n \frac{5}{(5r-1)(5r+4)} = \frac{1}{4} - \frac{1}{5n+4}$

$\sum_{r=1}^n f(r) = \frac{5n+4-4}{4(5n+4)} = \frac{5n}{4(5n+4)}$

Question 2 (**)

a) Show carefully that

$$\frac{1}{r^2} - \frac{1}{(r+1)^2} = \frac{2r+1}{r^2(r+1)^2}.$$

b) Hence use the method of differences to find

$$\sum_{r=1}^n \frac{2r+1}{r^2(r+1)^2}.$$

$$1 - \frac{1}{(n+1)^2}$$

Handwritten solution for Question 2b using the method of differences:

(a) LHS = $\frac{1}{r^2} - \frac{1}{(r+1)^2} = \frac{(r+1)^2 - r^2}{r^2(r+1)^2} = \frac{r^2 + 2r + 1 - r^2}{r^2(r+1)^2} = \frac{2r+1}{r^2(r+1)^2}$

(b) $\frac{2r+1}{r^2(r+1)^2} = \frac{1}{r^2} - \frac{1}{(r+1)^2}$

For $r=1$: $\frac{3}{1^2 \cdot 2^2} = \frac{1}{1^2} - \frac{1}{2^2}$

For $r=2$: $\frac{5}{2^2 \cdot 3^2} = \frac{1}{2^2} - \frac{1}{3^2}$

For $r=3$: $\frac{7}{3^2 \cdot 4^2} = \frac{1}{3^2} - \frac{1}{4^2}$

...

For $r=n$: $\frac{2n+1}{n^2(n+1)^2} = \frac{1}{n^2} - \frac{1}{(n+1)^2}$

Summing from $r=1$ to n : $\sum_{r=1}^n \frac{2r+1}{r^2(r+1)^2} = 1 - \frac{1}{(n+1)^2}$

Question 3 (**)

a) Show carefully that

$$\frac{1}{r!} - \frac{1}{(r+1)!} = \frac{r}{(r+1)!}.$$

b) Hence find

$$\sum_{r=1}^n \frac{r}{(r+1)!}.$$

$$1 - \frac{1}{(n+1)!}$$

Handwritten solution for Question 3b:

Q) $\frac{1}{r!} - \frac{1}{(r+1)!} = \frac{r+1}{(r+1)r!} - \frac{1}{(r+1)!} = \frac{r+1}{(r+1)!} - \frac{1}{(r+1)!} = \frac{r}{(r+1)!}$

Q) $\frac{r}{(r+1)!} = \frac{1}{r!} - \frac{1}{(r+1)!}$

$r=1 \Rightarrow \frac{1}{2!} = \frac{1}{1!} - \frac{1}{2!}$
 $r=2 \Rightarrow \frac{2}{3!} = \frac{1}{2!} - \frac{1}{3!}$
 $r=3 \Rightarrow \frac{3}{4!} = \frac{1}{3!} - \frac{1}{4!}$
 \vdots
 $r=n \Rightarrow \frac{n}{(n+1)!} = \frac{1}{n!} - \frac{1}{(n+1)!}$

Ans $\sum_{r=1}^n \frac{r}{(r+1)!} = 1 - \frac{1}{(n+1)!}$

Question 4 (**+)

$$f(r) = \frac{1}{r(r+2)}, \quad r \in \mathbb{N}$$

- a) Express $f(r)$ into partial fractions.
- b) Hence show that

$$\sum_{r=1}^{30} f(r) = \frac{1425}{1984}.$$

$$f(r) = \frac{1}{2r} - \frac{1}{2(r+2)}$$

Handwritten solution for Question 4:

(a) $f(r) = \frac{1}{r(r+2)} = \frac{1}{2r} - \frac{1}{2(r+2)}$

(b) $\sum_{r=1}^{30} f(r) = \sum_{r=1}^{30} \left(\frac{1}{2r} - \frac{1}{2(r+2)} \right)$

Using the decomposition, the sum telescopes:

$\sum_{r=1}^{30} \left(\frac{1}{2r} - \frac{1}{2(r+2)} \right) = \frac{1}{2} \left(\frac{1}{1} - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \dots + \frac{1}{30} - \frac{1}{32} \right)$

The terms cancel out, leaving:

$\frac{1}{2} \left(\frac{1}{1} + \frac{1}{2} - \frac{1}{31} - \frac{1}{32} \right)$

Calculating the sum:

$\frac{1}{2} \left(1 + \frac{1}{2} - \frac{1}{31} - \frac{1}{32} \right) = \frac{1}{2} \left(\frac{3}{2} - \frac{1}{31} - \frac{1}{32} \right)$

$= \frac{1}{2} \left(\frac{3 \cdot 992 - 32 - 64}{1984} \right) = \frac{1}{2} \left(\frac{2996}{1984} \right) = \frac{1425}{1984}$

Question 5 (**+)

$$f(r) = \frac{2}{(r+1)(r+3)}, \quad r \in \mathbb{N}$$

- a) Express $f(r)$ into partial fractions
- b) Use the method of differences to find

$$\sum_{r=1}^n f(r).$$

- c) Hence evaluate

$$\sum_{r=8}^{\infty} f(r).$$

$$f(r) = \frac{1}{r+1} - \frac{1}{r+3}, \quad \sum_{r=1}^n f(r) = \frac{5}{6} - \frac{1}{n+2} - \frac{1}{n+3}, \quad \sum_{r=8}^{\infty} f(r) = \frac{19}{90}$$

(a) $\frac{2}{(r+1)(r+3)} = \frac{A}{r+1} + \frac{B}{r+3}$
 $2 = A(r+3) + B(r+1)$
 $\bullet \frac{1}{2} r = -1 \quad 2 = 2A \Rightarrow A = 1$
 $\bullet \frac{1}{2} r = -3 \quad 2 = -2B \Rightarrow B = -1$
 $\therefore \frac{2}{(r+1)(r+3)} = \frac{1}{r+1} - \frac{1}{r+3}$

(b) $\frac{2}{(r+1)(r+3)} = \frac{1}{r+1} - \frac{1}{r+3}$
 $\bullet r=1 \quad \frac{2}{2 \times 4} = \frac{1}{2} - \frac{1}{4}$
 $\bullet r=2 \quad \frac{2}{3 \times 5} = \frac{1}{3} - \frac{1}{5}$
 $\bullet r=3 \quad \frac{2}{4 \times 6} = \frac{1}{4} - \frac{1}{6}$
 $\bullet r=4 \quad \frac{2}{5 \times 7} = \frac{1}{5} - \frac{1}{7}$
 \vdots
 $\bullet r=n-1 \quad \frac{2}{n(n+2)} = \frac{1}{n} - \frac{1}{n+2}$
 $\bullet r=n \quad \frac{2}{(n+1)(n+3)} = \frac{1}{n+1} - \frac{1}{n+3}$
 $\sum_{r=1}^n \frac{2}{(r+1)(r+3)} = \frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3}$
 $= \frac{5}{6} - \frac{1}{n+2} - \frac{1}{n+3}$

(c) $\sum_{r=8}^{\infty} \frac{2}{(r+1)(r+3)} = \sum_{r=8}^{\infty} \left(\frac{1}{r+1} - \frac{1}{r+3} \right)$
 $= \frac{5}{6} - \left[\frac{1}{9} - \frac{1}{10} \right]$
 $= \frac{1}{3} + \frac{1}{10}$
 $= \frac{19}{30}$

Question 6 (***)

a) Simplify $\frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)}$ into a single fraction.

b) Hence show that

$$\sum_{r=1}^{20} \left[\frac{1}{r(r+1)(r+2)} \right] = \frac{115}{462}.$$

$$\frac{2}{r(r+1)(r+2)}$$

a) Obtain a common denominator and add

$$\frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)} = \frac{(r+2) - r}{r(r+1)(r+2)} = \frac{2}{r(r+1)(r+2)}$$

b) Using the identity obtained in part (a)

$$\frac{2}{r(r+1)(r+2)} = \frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)}$$

• If $r=1$: $\frac{2}{1 \times 2 \times 3} = \frac{1}{1 \times 2} - \frac{1}{2 \times 3}$

• If $r=2$: $\frac{2}{2 \times 3 \times 4} = \frac{1}{2 \times 3} - \frac{1}{3 \times 4}$

• If $r=3$: $\frac{2}{3 \times 4 \times 5} = \frac{1}{3 \times 4} - \frac{1}{4 \times 5}$

• If $r=20$: $\frac{2}{20 \times 21 \times 22} = \frac{1}{20 \times 21} - \frac{1}{21 \times 22}$

Adding

$$\Rightarrow \sum_{r=1}^{20} \left[\frac{2}{r(r+1)(r+2)} \right] = \frac{1}{1 \times 2} - \frac{1}{21 \times 22}$$

$$\Rightarrow 2 \sum_{r=1}^{20} \left[\frac{1}{r(r+1)(r+2)} \right] = \frac{1}{2} - \frac{1}{462}$$

$$\Rightarrow \sum_{r=1}^{20} \left[\frac{1}{r(r+1)(r+2)} \right] = \frac{1}{2} \left(\frac{1}{2} - \frac{1}{462} \right) = \frac{115}{462}$$

Question 7 (***)

$$f(r) \equiv r^2(r+1)^2 - (r-1)^2 r^2, \quad r \in \mathbb{N}.$$

- a) Simplify $f(r)$ as far as possible.
- b) Use the method of differences to show that

$$\sum_{r=1}^{20} r^3 = 44100.$$

$$\boxed{4r^3}, \quad \boxed{f(r) = 4r^3}$$

a) $f(r) = r^2(r+1)^2 - (r-1)^2 r^2$
 $= r^2[(r+1)^2 - (r-1)^2]$
 $= r^2(r+1+r-1)(r+1-r+1)$
 $= r^2 \times 2r \times 2$
 $= 4r^3$

b) USING PART (a)
 $4r^3 \equiv r^2(r+1)^2 - (r-1)^2 r^2$

IF $r=1$	$4 \times 1^3 = 1^2 \times 2^2 - 0^2 \times 1^2$
IF $r=2$	$4 \times 2^3 = 2^2 \times 3^2 - 1^2 \times 2^2$
IF $r=3$	$4 \times 3^3 = 3^2 \times 4^2 - 2^2 \times 3^2$
IF $r=4$	$4 \times 4^3 = 4^2 \times 5^2 - 3^2 \times 4^2$
...	...
IF $r=20$	$4 \times 20^3 = 20^2 \times 21^2 - 19^2 \times 20^2$

ADDING
 \downarrow
 $\frac{4 \times 20^3}{4} = \frac{20^2 \times 21^2}{4}$
 \downarrow
 $\frac{20^2 \times 21^2}{4} = \frac{20^2 \times 21^2}{4}$
 \downarrow
 $\frac{20^2 \times 21^2}{4} = 44100$

Question 8 (***)

$$f(r) = \frac{1}{r(r+2)}, \quad r \in \mathbb{N}.$$

- Express $f(r)$ in partial fractions.
- Hence prove, by the method of differences, that

$$\sum_{r=1}^n f(r) = \frac{n(An+B)}{4(n+1)(n+2)},$$

where A and B are constants to be found.

$$\square, A=3, B=5$$

a) BY INSPECTION (COOROP METHOD OR SIMILAR)

$$f(r) = \frac{1}{r(r+2)} = \frac{1}{r} + \frac{-1}{r+2} = \frac{1}{r} - \frac{1}{r+2}$$

$$= \frac{1}{r} - \frac{1}{2(r+1)}$$

b) SETTING PART (a) AS AN IDENTITY

$$\frac{1}{r(r+2)} = \frac{1}{r} - \frac{1}{r+2}$$

- $r=1$ $\frac{1}{1 \times 3} = \frac{1}{1} - \frac{1}{3}$
- $r=2$ $\frac{1}{2 \times 4} = \frac{1}{2} - \frac{1}{4}$
- $r=3$ $\frac{1}{3 \times 5} = \frac{1}{3} - \frac{1}{5}$
- $r=4$ $\frac{1}{4 \times 6} = \frac{1}{4} - \frac{1}{6}$
- $r=5$ $\frac{1}{5 \times 7} = \frac{1}{5} - \frac{1}{7}$
- \vdots
- $r=n-1$ $\frac{1}{(n-1)(n)} = \frac{1}{n-1} - \frac{1}{n}$
- $r=n$ $\frac{1}{n(n+2)} = \frac{1}{n} - \frac{1}{n+2}$

$$\Rightarrow \sum_{r=1}^n \frac{1}{r(r+2)} = \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}$$

ADDING

$$\Rightarrow \sum_{r=1}^n \frac{1}{r(r+2)} = \frac{3(n+1)(n+2) - 2(n+1) - 2(n+2)}{2(n+1)(n+2)}$$

$$\Rightarrow \sum_{r=1}^n \frac{1}{r(r+2)} = \frac{3(n^2 + 3n + 2) - 2n - 4 - 2n - 2}{2(n+1)(n+2)}$$

Created by T. Madas

SUMMATIONS

METHOD OF DIFFERENCES

8 STANDARD QUESTIONS

Created by T. Madas

Question 1 (***)

Use the method of differences to show that

$$\frac{1}{1 \times 2 \times 3} + \frac{1}{2 \times 3 \times 4} + \frac{1}{3 \times 4 \times 5} + \dots + \frac{1}{n(n+1)(n+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$$

4P, proof

Using Partial Fractions

$$\frac{1}{k(k+1)(k+2)} = \frac{1}{2k} - \frac{1}{k+1} + \frac{1}{2(k+2)}$$

$$\frac{1}{k(k+1)(k+2)} = \frac{1}{2k} - \frac{1}{k+1} + \frac{1}{2(k+2)}$$

Doubling the above identity we simplify

$$\frac{2}{k(k+1)(k+2)} = \frac{1}{k} - \frac{2}{k+1} + \frac{1}{k+2}$$

• $k=1$: $\frac{2}{1 \times 2 \times 3} = \frac{1}{1} - \frac{2}{2} + \frac{1}{3}$

• $k=2$: $\frac{2}{2 \times 3 \times 4} = \frac{1}{2} - \frac{2}{3} + \frac{1}{4}$

• $k=3$: $\frac{2}{3 \times 4 \times 5} = \frac{1}{3} - \frac{2}{4} + \frac{1}{5}$

• $k=4$: $\frac{2}{4 \times 5 \times 6} = \frac{1}{4} - \frac{2}{5} + \frac{1}{6}$

• \vdots

• $k=n-1$: $\frac{2}{(n-1)n(n+1)} = \frac{1}{n-1} - \frac{2}{n} + \frac{1}{n+1}$

• $k=n$: $\frac{2}{n(n+1)(n+2)} = \frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2}$

$\Rightarrow \sum_{k=1}^n \frac{2}{k(k+1)(k+2)} = \frac{1}{1} - \frac{2}{2} + \frac{1}{3} + \frac{1}{2} - \frac{2}{3} + \frac{1}{4} + \dots + \frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2}$

$\Rightarrow 2 \sum_{k=1}^n \frac{1}{k(k+1)(k+2)} = \frac{(n+1)(n+2) - 2(n+2) + 2(n+1)}{2(n+1)(n+2)}$

$\Rightarrow \sum_{k=1}^n \frac{1}{k(k+1)(k+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$

$\Rightarrow \sum_{k=1}^n \frac{1}{k(k+1)(k+2)} = \frac{n^2 + 3n}{4(n+1)(n+2)}$

$\Rightarrow \sum_{k=1}^n \frac{1}{k(k+1)(k+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$

Question 2 (***)

$$u_r = \frac{1}{6}r(r+1)(4r+11), \quad r \in \mathbb{N}.$$

- a) Simplify $u_r - u_{r-1}$ as far as possible.
- b) By using the method of differences, or otherwise, find the sum of the first 100 terms of the following series.

$$(1 \times 5) + (2 \times 7) + (3 \times 9) + (4 \times 11) + \dots$$

$$\boxed{}, \boxed{r(2r+3)}, \boxed{691850}$$

a)
$$\begin{aligned} u_r - u_{r-1} &= \frac{1}{6}r(r+1)(4r+11) - \frac{1}{6}(r-1)r[4(r-1)+11] \\ &= \frac{1}{6}r(r+1)(4r+11) - \frac{1}{6}r(r-1)(4r+7) \\ &= \frac{1}{6}r[(r+1)(4r+11) - (r-1)(4r+7)] \\ &= \frac{1}{6}r[4r^2 + 15r + 11 - 4r^2 - 5r + 7] \\ &= \frac{1}{6}r(10r + 18) \\ &= r(2r+3) \end{aligned}$$

b) Method of Differences

$$(1 \times 5) + (2 \times 7) + (3 \times 9) + \dots + (100 \times 203)$$

$$\Rightarrow u_r - u_{r-1} = r(2r+3)$$

• $r=1$	u_1	=	1×5
• $r=2$	u_2	=	2×7
• $r=3$	u_3	=	3×9
• $r=4$	u_4	=	4×11
• \vdots	\vdots		
• $r=100$	$u_{100} - u_{99}$	=	100×203

$$\Rightarrow u_{100} - u_0 = (1 \times 5) + (2 \times 7) + (3 \times 9) + \dots + (100 \times 203)$$

$$\Rightarrow \frac{1}{6} \times 100 \times (100+1) = 0 = \sum_{r=1}^{100} r(2r+3)$$

$$\Rightarrow \sum_{r=1}^{100} r(2r+3) = 691850$$

Question 3 (***)

$$f(r) = \frac{1}{(r+1)(r-1)}, \quad r \in \mathbb{N}.$$

a) Express $f(r)$ into partial fractions.

b) Hence show that

$$\sum_{r=2}^n \frac{1}{r^2-1} = \frac{3}{4} - \frac{1}{2n} - \frac{1}{2(n+1)}.$$

c) State the value of

$$\sum_{r=2}^{\infty} \frac{1}{r^2-1}$$

$$\boxed{}, \quad f(r) = \frac{1}{2(r-1)} - \frac{1}{2(r+1)}, \quad \boxed{\frac{3}{4}}$$

Handwritten solution for Question 3:

a) $f(r) = \frac{1}{(r+1)(r-1)} = \frac{\frac{1}{2}}{r-1} - \frac{\frac{1}{2}}{r+1} = \frac{1}{2(r-1)} - \frac{1}{2(r+1)}$
By cover up (multiplication)

b) ALWAYS CHECK (a) "DOUBLE" THE SIGN!
 $\frac{2}{r^2-1} = \frac{2}{(r+1)(r-1)} = \frac{1}{r-1} - \frac{1}{r+1}$

• $r=2$: $\frac{2}{2^2-1} = \frac{1}{1} - \frac{1}{3}$
 • $r=3$: $\frac{2}{3^2-1} = \frac{1}{2} - \frac{1}{4}$
 • $r=4$: $\frac{2}{4^2-1} = \frac{1}{3} - \frac{1}{5}$
 • $r=5$: $\frac{2}{5^2-1} = \frac{1}{4} - \frac{1}{6}$
 • $r=6$: $\frac{2}{6^2-1} = \frac{1}{5} - \frac{1}{7}$
 • \vdots
 • $r=n-1$: $\frac{2}{(n-1)^2-1} = \frac{1}{n-2} - \frac{1}{n}$
 • $r=n$: $\frac{2}{n^2-1} = \frac{1}{n-1} - \frac{1}{n+1}$

$\sum_{r=2}^n \frac{2}{r^2-1} = 1 + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1}$
 $2 \sum_{r=2}^n \frac{1}{r^2-1} = \frac{3}{2} - \frac{1}{n} - \frac{1}{n+1}$
 $\sum_{r=2}^n \frac{1}{r^2-1} = \frac{3}{4} - \frac{1}{2n} - \frac{1}{2(n+1)}$

c) As $n \rightarrow \infty$ THE SUM TENDS TO $\frac{3}{4}$

Question 4 (***)

$$f(r) = \frac{2}{r(r+1)(r+2)}, \quad r \in \mathbb{N}.$$

- Express $f(r)$ into partial fractions.
- Hence show that

$$\sum_{r=1}^n f(r) = \frac{1}{2} - \frac{1}{(n+1)(n+2)}.$$

- c) Find the value of the convergent infinite sum

$$\frac{1}{5 \times 6 \times 7} + \frac{1}{6 \times 7 \times 8} + \frac{1}{7 \times 8 \times 9} + \dots$$

$$\boxed{}, \quad \boxed{f(r) = \frac{1}{r} - \frac{2}{r+1} + \frac{1}{r+2}}, \quad \boxed{\frac{1}{60}}$$

[illegible]

Question 5 (***)

Use the method of differences to show that

$$\frac{1}{1 \times 2 \times 3} + \frac{4}{2 \times 3 \times 4} + \frac{7}{3 \times 4 \times 5} + \dots + \frac{3n-2}{n(n+1)(n+2)} = \frac{n^2}{(n+1)(n+2)}$$

Q.E.D., **proof**

PROCEED BY FRACTIONAL FRACTIONS (COLUMN OR ROW METHOD)

$$\frac{3n-2}{n(n+1)(n+2)} = \frac{-\frac{2}{1 \times 2}}{n} + \frac{-\frac{5}{1 \times 1}}{n+1} + \frac{-\frac{8}{-2 \times 1}}{n+2}$$

$$= -\frac{1}{n} + \frac{5}{n+1} - \frac{4}{n+2}$$

SETTING UP THE METHOD OF DIFFERENCES BASED ON THE ABOVE RESULT

$$\frac{3n-2}{n(n+1)(n+2)} = -\frac{1}{n} + \frac{5}{n+1} - \frac{4}{n+2}$$

- IF $n=1$: $\frac{1}{1 \times 2 \times 3} = -\frac{1}{1} + \frac{5}{2} - \frac{4}{3}$
- IF $n=2$: $\frac{4}{2 \times 3 \times 4} = -\frac{1}{2} + \frac{5}{3} - \frac{4}{4}$
- IF $n=3$: $\frac{7}{3 \times 4 \times 5} = -\frac{1}{3} + \frac{5}{4} - \frac{4}{5}$
- IF $n=4$: $\frac{10}{4 \times 5 \times 6} = -\frac{1}{4} + \frac{5}{5} - \frac{4}{6}$
- IF $n=n$: $\frac{3n-2}{n(n+1)(n+2)} = -\frac{1}{n} + \frac{5}{n+1} - \frac{4}{n+2}$

$$\sum_{n=1}^n \frac{3n-2}{n(n+1)(n+2)} = \left(-1 + \frac{5}{2} - \frac{4}{3}\right) + \left(\frac{1}{2} - \frac{5}{3} + \frac{4}{4}\right) + \dots + \left(-\frac{1}{n} + \frac{5}{n+1} - \frac{4}{n+2}\right)$$

$$= 1 + \frac{1}{n+1} - \frac{4}{n+2}$$

Question 6 (***)

It is given that

$$\frac{2k+7}{(2k+1)(2k+3)(2k+5)} \equiv \frac{3}{4(2k+1)} - \frac{1}{(2k+3)} + \frac{1}{4(2k+5)}.$$

Use the method of differences to find a simplified expression for

$$\frac{7}{1 \times 3 \times 5} + \frac{9}{3 \times 5 \times 7} + \frac{11}{5 \times 7 \times 9} + \dots + \frac{2n+7}{(2n+1)(2n+3)(2n+5)}.$$

Give your answer in the form $\frac{2}{3} - f(n)$, where $f(n)$ is a single simplified fraction.

$$\boxed{}, \quad f(n) = -\frac{n+3}{(2n+3)(2n+5)}$$

REGRETT THE TRIANGLE IDENTITIES

$$\frac{2k+7}{(2k+1)(2k+3)(2k+5)} \equiv \frac{3}{4(2k+1)} - \frac{1}{(2k+3)} + \frac{1}{4(2k+5)}$$

- If $k=0$ RHS = $\frac{7}{1 \times 3 \times 5} = \frac{7}{15}$
- If $k=1$ RHS = $\frac{9}{3 \times 5 \times 7} = \frac{9}{105}$
- If $k=2$ RHS = $\frac{11}{5 \times 7 \times 9} = \frac{11}{315}$
- If $k=3$ RHS = $\frac{13}{7 \times 9 \times 11} = \frac{13}{693}$
- If $k=4$ RHS = $\frac{15}{9 \times 11 \times 13} = \frac{15}{1287}$
- If $k=n-2$ RHS = $\frac{2n-3}{(2n-3)(2n-1)(2n+1)}$
- If $k=n-1$ RHS = $\frac{2n-1}{(2n-1)(2n+1)(2n+3)}$
- If $k=n$ RHS = $\frac{2n+7}{(2n+1)(2n+3)(2n+5)}$

$$\therefore \sum_{k=0}^n \frac{2k+7}{(2k+1)(2k+3)(2k+5)} = \left(\frac{3}{4} + \frac{1}{4} - \frac{1}{3} \right) + \frac{16}{243} - \frac{1}{243} + \frac{16}{243}$$

$$= \frac{3}{4} + \frac{16}{243} - \frac{1}{243}$$

$$= \frac{3}{4} + \frac{1}{4} \left[\frac{243-1}{243} \right]$$

$$= \frac{3}{4} + \frac{1}{4} \left[\frac{242}{243} \right] = \frac{3}{4} + \frac{242}{972} = \frac{3}{4} + \frac{121}{486}$$

Question 7 (**)**

Use the method of differences to find a simplified expression for the first n terms of the following series.

$$\frac{1}{1 \times 3} + \frac{2}{3 \times 5} + \frac{3}{5 \times 7} + \frac{4}{7 \times 9} + \dots$$

Give your answer in the form $\frac{1}{4} - f(n)$, where $f(n)$ is a single simplified fraction.

$\frac{1}{4} - f(n) = \frac{(-1)^n}{4(2n+1)}$

WRITE FIRSTLY IN SUMMATION FORM

$$\frac{1}{1 \times 3} + \frac{2}{3 \times 5} + \frac{3}{5 \times 7} + \frac{4}{7 \times 9} + \dots = \sum_{r=1}^n \frac{r \cdot 2r}{(2r-1)(2r+1)}$$

EXPRESS THE $(2r)^{th}$ TERM IN THE SUMMATION, IN TERMS OF THE FACTORIAL FRACIONS

$$\frac{r \cdot 2r}{(2r-1)(2r+1)} = \frac{2r}{2r-1} - \frac{2r}{2r+1} = \frac{2r}{2r-1} + \frac{1}{2r+1}$$

OR

$$\frac{r \cdot 2r}{(2r-1)(2r+1)} = \frac{1}{2r-1} + \frac{1}{2r+1}$$

NOW USE MAX. OF DIFFERENT VALUES OF r

$r=1$	$\frac{1 \cdot 2}{1 \times 3} = \frac{1}{1} + \frac{1}{3}$	$\frac{1 \cdot 2}{1 \times 3} = \frac{1}{1} + \frac{1}{3}$
$r=2$	$\frac{2 \cdot 4}{3 \times 5} = \frac{1}{3} + \frac{1}{5}$	$\frac{2 \cdot 4}{3 \times 5} = \frac{1}{3} + \frac{1}{5}$
$r=3$	$\frac{3 \cdot 6}{5 \times 7} = \frac{1}{5} + \frac{1}{7}$	$\frac{3 \cdot 6}{5 \times 7} = \frac{1}{5} + \frac{1}{7}$
$r=4$	$\frac{4 \cdot 8}{7 \times 9} = \frac{1}{7} + \frac{1}{9}$	$\frac{4 \cdot 8}{7 \times 9} = \frac{1}{7} + \frac{1}{9}$
\vdots	\vdots	\vdots
$r=n$	$\frac{n \cdot 2n}{(2n-1)(2n+1)} = \frac{1}{2n-1} + \frac{1}{2n+1}$	$\frac{n \cdot 2n}{(2n-1)(2n+1)} = \frac{1}{2n-1} + \frac{1}{2n+1}$

ALTERNATIVE METHOD

$$\Rightarrow \sum_{r=1}^n \left[\frac{1}{2r-1} - \frac{1}{2r+1} \right] = 1 - \frac{1}{2n+1}$$

$$\Rightarrow \sum_{r=1}^n \frac{r \cdot 2r}{(2r-1)(2r+1)} = 1 - \frac{1}{2n+1}$$

$$\Rightarrow \sum_{r=1}^n \frac{r \cdot 2r}{(2r-1)(2r+1)} = \frac{1}{4} - \frac{1}{4(2n+1)}$$

Question 8 (***)

$$f(r) = \frac{1}{\sqrt{r+2} + \sqrt{r}}, \quad r \geq 0.$$

a) Rationalize the denominator of $f(r)$.

b) Find an expression for

$$\sum_{r=1}^n f(r).$$

c) Show clearly that

$$\sum_{r=1}^{48} f(r) = 3 + 2\sqrt{2}$$

$$\boxed{}, \quad f(r) = \frac{\sqrt{r+2} - \sqrt{r}}{2}, \quad \sum_{r=1}^n f(r) = \frac{1}{2}(\sqrt{n+2} + \sqrt{n+1} - \sqrt{2} - 1)$$

a) USING STANDARD SURDS

$$\frac{1}{\sqrt{r+2} + \sqrt{r}} = \frac{\sqrt{r+2} - \sqrt{r}}{(\sqrt{r+2} + \sqrt{r})(\sqrt{r+2} - \sqrt{r})} = \frac{\sqrt{r+2} - \sqrt{r}}{r+2 - r} = \frac{\sqrt{r+2} - \sqrt{r}}{2}$$

b) USING PART (a)

$$\frac{1}{\sqrt{r+2} + \sqrt{r}} = \frac{\sqrt{r+2} - \sqrt{r}}{2}$$

Let $r=1$: $\frac{1}{\sqrt{3} + \sqrt{1}} = \frac{\sqrt{3} - \sqrt{1}}{2}$
 $r=2$: $\frac{1}{\sqrt{4} + \sqrt{2}} = \frac{\sqrt{4} - \sqrt{2}}{2}$
 $r=3$: $\frac{1}{\sqrt{5} + \sqrt{3}} = \frac{\sqrt{5} - \sqrt{3}}{2}$
 $r=4$: $\frac{1}{\sqrt{6} + \sqrt{4}} = \frac{\sqrt{6} - \sqrt{4}}{2}$
 \vdots
 $r=n-1$: $\frac{1}{\sqrt{n} + \sqrt{n-1}} = \frac{\sqrt{n} - \sqrt{n-1}}{2}$
 $r=n$: $\frac{1}{\sqrt{n+2} + \sqrt{n+1}} = \frac{\sqrt{n+2} - \sqrt{n+1}}{2}$

Adding both sides

$$\sum_{r=1}^n \frac{1}{\sqrt{r+2} + \sqrt{r}} = \frac{1}{2}(\sqrt{n+2} + \sqrt{n+1} - \sqrt{2} - 1)$$

c) LET $n=48$ IN PART (b)

$$\sum_{r=1}^{48} f(r) = \frac{1}{2}(\sqrt{48+2} + \sqrt{48+1} - \sqrt{2} - 1)$$

$$= \frac{1}{2}(\sqrt{50} + \sqrt{49} - \sqrt{2} - 1)$$

$$= \frac{1}{2}(5\sqrt{2} + 7 - \sqrt{2} - 1)$$

$$= \frac{1}{2}(4\sqrt{2} + 6)$$

$$= 2 + 2\sqrt{2}$$

Created by T. Madas

SUMMATIONS

METHOD OF DIFFERENCES

3 HARD QUESTIONS

Created by T. Madas

Question 1 (****+)

Consider the following infinite convergent series.

$$\frac{3}{1 \times 2} - \frac{5}{2 \times 3} + \frac{7}{3 \times 4} - \frac{9}{4 \times 5} + \frac{11}{5 \times 6} - \dots$$

- Use the method of differences, to find the sum of this series.
- Verify the answer of part (a) by using a method based on the Maclaurin expansion of $\ln(1+x)$.

V, ,

a) METHOD OF DIFFERENCES

Start by determining the general term in sigma notation

$$\frac{3}{1 \times 2} - \frac{5}{2 \times 3} + \frac{7}{3 \times 4} - \frac{9}{4 \times 5} + \frac{11}{5 \times 6} - \dots = \sum_{n=1}^{\infty} \left[(-1)^{n+1} \frac{2n+1}{n(n+1)} \right]$$

INDICATES $(-1)^{n+1}$ GIVES THE FIRST TWO PARTIAL FRACTIONS BY GORE OF

$$\frac{2n+1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

WIND UP HERE

n=1: $\frac{1}{1} - \frac{1}{2}$
n=2: $-\frac{1}{2} + \frac{1}{3}$
n=3: $\frac{1}{3} - \frac{1}{4}$
n=4: $-\frac{1}{4} + \frac{1}{5}$
...
n=n: $\frac{1}{n} - \frac{1}{n+1}$

$$\sum_{n=1}^{\infty} \left[(-1)^{n+1} \frac{2n+1}{n(n+1)} \right] = 1 - \frac{1}{n+1}$$

As $n \rightarrow \infty$ THE SUM TO INFINITY IS

b) LOOKING AT THE EXPANSION OF $\ln(1+x)$, VALID FOR $-1 < x \leq 1$

- $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \dots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$
- LET $x=1$
- $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$

USING THE PARTIAL FRACTIONS FROM PART (a)

$$\sum_{n=1}^{\infty} \left[(-1)^{n+1} \frac{2n+1}{n(n+1)} \right] = \sum_{n=1}^{\infty} \left[(-1)^{n+1} \left[\frac{1}{n} + \frac{1}{n+1} \right] \right]$$

$$= \ln 2 + \sum_{n=1}^{\infty} \left[(-1)^{n+1} \frac{1}{n+1} \right]$$

RE-INDEXING AND MANIPULATING

$$= \ln 2 + \left[1 - \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \right]$$

$$= \ln 2 + \left[1 - \ln 2 \right]$$

$$= 1$$

ALTERNATIVE TO RE-INDEXING & MANIPULATING

$$S = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$$

$$-S = -\frac{1}{1} + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \dots$$

$$1 - S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$$

$$1 - S = \ln 2$$

$$S = 1 - \ln 2$$

As above

Question 2 (****+)

Use partial fractions to sum the following series.

$$\sum_{n=1}^{\infty} \frac{2n+1}{n^4 + 2n^3 + n^2}.$$

You may assume that the series converges.

1

• SOMETIMES BY TRYING OF THE SUMMATION

$$\sum_{n=1}^{\infty} \frac{2n+1}{n^4 + 2n^3 + n^2} = \sum_{n=1}^{\infty} \frac{2n+1}{n^2(n^2 + 2n + 1)} = \sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}$$

• ALTHOUGH WE HAVE REPEATED FACTORS THE PARTIAL FRACTIONS
 CAN EASILY BE DONE BY INSPECTION

$$= \sum_{n=1}^{\infty} \left[\frac{1}{n^3} - \frac{1}{(n+1)^3} \right]$$

$$= \left(\frac{1}{1^3} - \frac{1}{2^3} \right) + \left(\frac{1}{2^3} - \frac{1}{3^3} \right) + \left(\frac{1}{3^3} - \frac{1}{4^3} \right) + \left(\frac{1}{4^3} - \frac{1}{5^3} \right) + \dots$$

$$= 1$$

Question 3 (***)

It is given that

$$f(r) = \frac{6r^4 + 6r^3 - ar^2 - ar + 1}{r(r+1)}, \quad r \in \mathbb{N},$$

where a is a non zero constant.

It is further given that

$$\sum_{r=1}^n f(r) = \frac{n^2(n+2)(2n+1)}{n+1}.$$

Determine the value of a .

$$\boxed{}, \quad a = 2$$

MANIPULATE $f(r)$ AS FOLLOWS

$$f(r) = \frac{6r^4 + 6r^3 - ar^2 - ar + 1}{r(r+1)} = \frac{6r^3(r+1) - ar(r+1) + 1}{r(r+1)}$$

$$= 6r^2 - a + \frac{1}{r(r+1)}$$

PARTIAL FRACTIONS BY INSPECTION

$$= 6r^2 - a + \frac{1}{r} - \frac{1}{r+1}$$

NOW PROCEED BY THE METHOD OF DIFFERENCES

$f(r)$	\equiv	$6r^2 - a + \frac{1}{r} - \frac{1}{r+1}$
--------	----------	--

r=1 $f(1) = 6(1)^2 - a + \frac{1}{1} - \frac{1}{1+1}$

r=2 $f(2) = 6(2)^2 - a + \frac{1}{2} - \frac{1}{2+1}$

r=3 $f(3) = 6(3)^2 - a + \frac{1}{3} - \frac{1}{3+1}$

r=4 $f(4) = 6(4)^2 - a + \frac{1}{4} - \frac{1}{4+1}$

\vdots

r=n $f(n) = 6(n)^2 - a + \frac{1}{n} - \frac{1}{n+1}$

ADD $\sum_{r=1}^n f(r) = 6 \sum_{r=1}^n r^2 - na + 1 - \frac{1}{n+1}$ **ADD**

$$= 6 \times \frac{1}{6} n(n+1)(2n+1) - an + 1 - \frac{1}{n+1}$$

$$= n(n+1)(2n+1) - an + \frac{n}{n+1}$$

$$= \frac{n(n+1)(2n+1) - an(n+1) + n}{n+1}$$

COMPARING NUMERATORS

$$\frac{n^2(n+2)(2n+1)}{n+1} \equiv \frac{n(n+1)(2n+1) - an(n+1) + n}{n+1}$$

$$n^2(2n^2 + 5n + 2) \equiv n(2n+1)(n^2 + 2n + 1) - a(n(n+1) + 1)$$

$$2n^3 + 5n^2 + 2n \equiv 2n^3 + 4n^2 + 2n + n^2 + 2n + 1 - a(n^2 + n + 1)$$

$$2n^3 + 5n^2 + 2n \equiv 2n^3 + 5n^2 + (1-a)n + (2-a)$$

$\therefore 4-a=2 \quad \text{and} \quad 2-a=0$

$a=2 \quad \quad \quad a=2$

$\therefore a=2$

Created by T. Madas

SUMMATIONS

METHOD OF DIFFERENCES

15 ENRICHMENT QUESTIONS

Created by T. Madas

Question 1 (****)

Determine the exact value of the following sum.

$$\sum_{n=2}^{20} \left[\frac{n^3 - n^2 + 1}{n^2 - n} \right].$$

$$\boxed{\text{Ans}}, \quad \boxed{\frac{4199}{20}}$$

• SPAT MANIPULATING BY DIVISION EXPLORE BY PARTIAL FRACTIONS

$$\frac{n^3 - n^2 + 1}{n^2 - n} = \frac{n(n^2 - n) + 1}{n^2 - n} = n + \frac{1}{n^2 - n} = n + \frac{1}{n(n-1)}$$

$$= n + \frac{-1}{n-1} + \frac{1}{n-1} = n + \frac{1}{n-1} - \frac{1}{n}$$

• THIS WE GET

$$\frac{n^3 - n^2 + 1}{n^2 - n} \equiv n + \frac{1}{n-1} - \frac{1}{n}$$

IF $n=2$ $\frac{2^3 - 2^2 + 1}{2^2 - 2} = 2 + \frac{1}{2-1} - \frac{1}{2}$

IF $n=3$ $\frac{3^3 - 3^2 + 1}{3^2 - 3} = 3 + \frac{1}{3-1} - \frac{1}{3}$

IF $n=4$ $\frac{4^3 - 4^2 + 1}{4^2 - 4} = 4 + \frac{1}{4-1} - \frac{1}{4}$

IF $n=5$ $\frac{5^3 - 5^2 + 1}{5^2 - 5} = 5 + \frac{1}{5-1} - \frac{1}{5}$

\vdots

IF $n=20$ $\frac{20^3 - 20^2 + 1}{20^2 - 20} = 20 + \frac{1}{20-1} - \frac{1}{20}$

• ADDING

$$\sum_{n=2}^{20} \left[\frac{n^3 - n^2 + 1}{n^2 - n} \right] = \left[\sum_{n=2}^{20} n \right] + 1 - \frac{1}{20}$$

$$= \frac{11(21+20)}{2} + 1 - \frac{1}{20}$$

$$= 240 - \frac{1}{20}$$

$$= \frac{4199}{20}$$

$11 \times 11 = 121$
 $11 \times 20 = 220$
 $121 + 220 = 341$
 $341 \times 11 = 3751$
 $3751 + 1 = 3752$
 $3752 - 1 = 3751$

Question 2 (****)

$$f(x, n) = \sum_{r=1}^n \left[\frac{1}{(x-1)^r} \right], \quad x \in \mathbb{R}, \quad n \in \mathbb{N}.$$

By observing the simplification of

$$\frac{1}{(x-2)(x-1)^r} - \frac{1}{(x-2)(x-1)^{r+1}}$$

find a simplified expression for $f(x, n)$.



$$f(x, n) = \frac{1}{x-2} - \frac{1}{(x-2)(x-1)^n}$$

START WITH THE SIMPLIFICATION

$$\frac{1}{(x-1)^r(x-2)} - \frac{1}{(x-1)^{r+1}(x-2)} = \frac{(x-1) - 1}{(x-1)^{r+1}(x-2)} = \frac{x-2}{(x-1)^{r+1}(x-2)}$$

THENCE WE HAVE

$$\frac{1}{(x-1)^r} \equiv \frac{1}{(x-1)^r(x-2)} - \frac{1}{(x-1)^{r+1}(x-2)}$$

- For $r=0$: $\frac{1}{(x-1)^0} = \frac{1}{x-2} - \frac{1}{(x-1)(x-2)}$
- For $r=1$: $\frac{1}{(x-1)^1} = \frac{1}{(x-1)(x-2)} - \frac{1}{(x-1)^2(x-2)}$
- For $r=2$: $\frac{1}{(x-1)^2} = \frac{1}{(x-1)^2(x-2)} - \frac{1}{(x-1)^3(x-2)}$
- For $r=3$: $\frac{1}{(x-1)^3} = \frac{1}{(x-1)^3(x-2)} - \frac{1}{(x-1)^4(x-2)}$
- For $r=n-1$: $\frac{1}{(x-1)^{n-1}} = \frac{1}{(x-1)^{n-1}(x-2)} - \frac{1}{(x-1)^n(x-2)}$

$$\Rightarrow \sum_{r=0}^{n-1} \frac{1}{(x-1)^r} = \frac{1}{x-2} - \frac{1}{(x-1)^n(x-2)}$$

$$\Rightarrow \sum_{r=1}^n \frac{1}{(x-1)^r} = \frac{1}{x-2} - \frac{1}{(x-1)^n(x-2)}$$

Question 3 (****)Determine, in terms of k and n , a simplified expression for

$$\sum_{r=2}^n \left[\frac{r(1-k)-1}{r(r-1)k^r} \right].$$

$$\boxed{}, \quad \frac{1}{n} \left(\frac{1}{k} \right)^n - \frac{1}{k}$$

• SPLIT BY PARTIAL FRACTIONS

$$\frac{r(1-k)-1}{r(r-1)} = \frac{A}{r} + \frac{B}{r-1}$$

$$\frac{r(1-k)-1}{r(r-1)} = \frac{A(r-1) + Br}{r(r-1)}$$

$$\begin{aligned} \text{At } r=0 &\Rightarrow -1 = -A \Rightarrow A=1 \\ \text{At } r=1 &\Rightarrow -k = B \Rightarrow B=-k \end{aligned}$$

• WRITE AS A SUM

$$\left(\frac{1}{k} \right)^r \frac{r(1-k)-1}{r(r-1)} = \left(\frac{1}{k} \right)^r \frac{1}{r} - \left(\frac{1}{k} \right)^r \frac{k}{r-1}$$

• $r=2$ $\left(\frac{1}{k} \right)^2 \frac{2(1-k)-1}{2(2-1)} = \left(\frac{1}{k} \right)^2 \times \frac{1}{2} - \left(\frac{1}{k} \right)^2 \times k$

• $r=3$ $\left(\frac{1}{k} \right)^3 \frac{3(1-k)-1}{3(3-1)} = \left(\frac{1}{k} \right)^3 \times \frac{1}{2} - \left(\frac{1}{k} \right)^3 \times k$

• $r=4$ $\left(\frac{1}{k} \right)^4 \frac{4(1-k)-1}{4(4-1)} = \left(\frac{1}{k} \right)^4 \times \frac{1}{3} - \left(\frac{1}{k} \right)^4 \times k$

• $r=5$ $\left(\frac{1}{k} \right)^5 \frac{5(1-k)-1}{5(5-1)} = \left(\frac{1}{k} \right)^5 \times \frac{1}{4} - \left(\frac{1}{k} \right)^5 \times k$

• \vdots

• $r=n$ $\left(\frac{1}{k} \right)^n \frac{n(1-k)-1}{n(n-1)} = \left(\frac{1}{k} \right)^n \times \frac{1}{n-1} - \left(\frac{1}{k} \right)^n \times k$

• ADDING

$$\sum_{r=2}^n \left(\frac{1}{k} \right)^r \frac{r(1-k)-1}{r(r-1)} = \left(\frac{1}{k} \right)^2 \times \frac{1}{2} - \frac{1}{k}$$

Question 4 (****)

Determine the value of the following infinite convergent sum.

$$\sum_{r=2}^{\infty} \left[\frac{4r-1}{r(r-1)} \left(-\frac{1}{3} \right)^r \right]$$

$$\boxed{\frac{1}{3}}$$

● START BY PARTIAL FRACTIONS (BY ADDITION)

$$\frac{4r-1}{r(r-1)} = \frac{1}{r-1} + \frac{3}{r}$$

● THEREFORE WE KNOW THAT

$$\frac{4r-1}{r(r-1)} \left(-\frac{1}{3} \right)^r = \frac{1}{r-1} \left(-\frac{1}{3} \right)^r + \frac{3}{r} \left(-\frac{1}{3} \right)^r$$

• $r=2$ $\frac{4(2)-1}{2(2-1)} \left(-\frac{1}{3} \right)^2 = \frac{1}{2-1} \left(-\frac{1}{3} \right)^2 + \frac{3}{2} \left(-\frac{1}{3} \right)^2 = \frac{1}{2} \left(-\frac{1}{3} \right)^2 + \frac{1}{2} \left(-\frac{1}{3} \right)^2$

• $r=3$ $\frac{4(3)-1}{3(3-1)} \left(-\frac{1}{3} \right)^3 = \frac{1}{3-1} \left(-\frac{1}{3} \right)^3 + \frac{3}{3} \left(-\frac{1}{3} \right)^3 = \frac{1}{2} \left(-\frac{1}{3} \right)^3 - \frac{1}{3} \left(-\frac{1}{3} \right)^3$

• $r=4$ $\frac{4(4)-1}{4(4-1)} \left(-\frac{1}{3} \right)^4 = \frac{1}{4-1} \left(-\frac{1}{3} \right)^4 + \frac{3}{4} \left(-\frac{1}{3} \right)^4 = \frac{1}{3} \left(-\frac{1}{3} \right)^4 - \frac{1}{4} \left(-\frac{1}{3} \right)^4$

• $r=5$ $\frac{4(5)-1}{5(5-1)} \left(-\frac{1}{3} \right)^5 = \frac{1}{5-1} \left(-\frac{1}{3} \right)^5 + \frac{3}{5} \left(-\frac{1}{3} \right)^5 = \frac{1}{4} \left(-\frac{1}{3} \right)^5 - \frac{1}{5} \left(-\frac{1}{3} \right)^5$

• \vdots

• $r=n$ $\frac{4n-1}{n(n-1)} \left(-\frac{1}{3} \right)^n = \frac{1}{n-1} \left(-\frac{1}{3} \right)^n - \frac{1}{n} \left(-\frac{1}{3} \right)^n$

● THEREFORE

$$\sum_{r=2}^{\infty} \left[\frac{4r-1}{r(r-1)} \left(-\frac{1}{3} \right)^r \right] = \lim_{n \rightarrow \infty} \left[\frac{1}{n-1} \left(-\frac{1}{3} \right)^n + \frac{1}{n} \left(-\frac{1}{3} \right)^n \right]$$

$$\Rightarrow \sum_{r=2}^{\infty} \left[\frac{4r-1}{r(r-1)} \left(-\frac{1}{3} \right)^r \right] = \frac{1}{3}$$

Question 5 (****)

Determine a simplified expression, in the form $\ln[f(n)]$, for the following sum.

$$\sum_{r=2}^N \left[\int_2^r \frac{2}{x^2-1} dx \right].$$

$$\boxed{}, \ln \left[\frac{2 \times 3^{N-1}}{N(N+1)} \right]$$

• STRIKE BY PARTIAL FRACTIONS IN THE INTEGRAND (BY INSPECTION)

$$\sum_{r=2}^N \left[\int_2^r \frac{2}{x^2-1} dx \right] = \sum_{r=2}^N \left[\int_2^r \frac{2}{(x-1)(x+1)} dx \right]$$

$$= \sum_{r=2}^N \left[\int_2^r \frac{1}{x-1} - \frac{1}{x+1} dx \right] = \sum_{r=2}^N \left[\ln|2-1| - \ln|2+1| \right]_{2=1}^{2=r}$$

• WRITING THE TERMS EXPLICITLY, LOOKING FOR PATTERNS

$$= \sum_{r=2}^N \left[\ln|r-1| - \ln|r+1| \right] - \left[\ln|1| - \ln|3| \right]$$

$$= \sum_{r=2}^N \left[\ln(r-1) - \ln(r+1) + \ln 3 \right]$$

$$= \begin{array}{l} \ln 1 - \ln 3 + \ln 3 \quad \leftarrow r=2 \\ \ln 2 - \ln 4 + \ln 3 \quad \leftarrow r=3 \\ \ln 3 - \ln 5 + \ln 3 \quad \leftarrow r=4 \\ \ln 4 - \ln 6 + \ln 3 \quad \leftarrow r=5 \\ \vdots \\ \ln(N-2) - \ln N + \ln 3 \quad \leftarrow r=N-1 \\ \ln(N-1) - \ln(N+1) + \ln 3 \quad \leftarrow r=N \end{array} \quad \left. \vphantom{\sum_{r=2}^N} \right\} (N-1) \text{ TERMS}$$

• ADDING:

$$\sum_{r=2}^N \left[\int_2^r \frac{2}{x^2-1} dx \right] = \ln 2 - \ln N - \ln(N+1) + (N-1)\ln 3$$

$$= \ln 2 + (N-1)\ln 3 - (\ln N + \ln(N+1))$$

$$= \ln \left[\frac{2 \times 3^{N-1}}{N(N+1)} \right]$$

Question 6 (****)

Show, by a detailed method, that

$$\frac{48}{2 \times 3} + \frac{47}{3 \times 4} + \frac{46}{4 \times 5} \dots + \frac{2}{48 \times 49} + \frac{1}{49 \times 50} = A + B \sum_{r=1}^{50} \frac{1}{r},$$

where A and B are constants to be found.

$$\boxed{}, \quad A = \frac{51}{2}, \quad B = -1$$

$$\frac{48}{2 \times 3} + \frac{47}{3 \times 4} + \frac{46}{4 \times 5} + \dots + \frac{2}{48 \times 49} + \frac{1}{49 \times 50} = A + B \sum_{r=1}^{50} \frac{1}{r}$$

METHOD: USES AN ALGEBRAIC IDENTIFICATION

$$\sum_{k=1}^{49} \frac{49-k}{(k+1)(k+2)}$$

REWRITE IN PARTIAL FRACTIONS

$$\sum_{k=1}^{49} \frac{49-k}{(k+1)(k+2)} = \sum_{k=1}^{49} \left(\frac{50}{k+1} - \frac{51}{k+2} \right)$$

$$= \left(\frac{50}{2} - \frac{51}{3} \right) + \left(\frac{50}{3} - \frac{51}{4} \right) + \dots + \left(\frac{50}{48} - \frac{51}{49} \right)$$

$$= 25 - \left[\frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{49} \right] - \frac{51}{49}$$

$$= 25 - \left[\frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{49} \right] - 1 - \frac{2}{49}$$

$$= 25 - \left[1 + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{49} \right] - \frac{2}{49}$$

$$= 25 + \frac{1}{2} - \left[1 + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{49} + \frac{1}{50} \right]$$

$$= \frac{51}{2} - \sum_{r=1}^{50} \frac{1}{r}$$

Hence $A = \frac{51}{2}$
 $B = -1$

Question 7 (****)

$$\frac{3}{1^2} + \frac{5}{1^2+2^2} + \frac{7}{1^2+2^2+3^2} + \frac{9}{1^2+2^2+3^2+4^2} + \frac{11}{1^2+2^2+3^2+4^2+5^2} + \dots$$

Show, by a detailed method, that the sum of the first 40 terms of this series shown above is $\frac{240}{41}$.

, proof

Handwritten solution for Question 7:

$$\begin{aligned}
 S_{40} &= \frac{3}{1^2} + \frac{5}{1^2+2^2} + \frac{7}{1^2+2^2+3^2} + \frac{9}{1^2+2^2+3^2+4^2} + \frac{11}{1^2+2^2+3^2+4^2+5^2} + \dots \\
 S_{40} &= \sum_{n=1}^{40} \left[\frac{2n+1}{\sum_{k=1}^n k^2} \right] = \sum_{n=1}^{40} \left[\frac{2n+1}{\frac{1}{6}n(n+1)(2n+1)} \right] \\
 &= 6 \sum_{n=1}^{40} \frac{1}{n(n+1)} = 6 \sum_{n=1}^{40} \left[\frac{1}{n} - \frac{1}{n+1} \right] \\
 &= 6 \left[\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots + \left(\frac{1}{40} - \frac{1}{41}\right) \right] \\
 &= 6 \left[1 - \frac{1}{41} \right] = 6 \times \frac{41-1}{41} = 6 \times \frac{40}{41} = \frac{240}{41}
 \end{aligned}$$

Question 8 (****)

By considering the simplification of

$$\arctan(2n+1) - \arctan(2n-1),$$

determine the exact value of

$$\sum_{n=1}^{\infty} \left[\arctan\left(\frac{1}{2n^2}\right) \right].$$

$$\boxed{}, \quad \frac{\pi}{4}$$

$\arctan(2n+1) - \arctan(2n-1) = \psi$
 • TAKE TANGENTS ON BOTH SIDES
 $\tan[\arctan(2n+1) - \arctan(2n-1)] = \tan \psi$
 $\frac{(2n+1) - (2n-1)}{1 + (2n+1)(2n-1)} = \tan \psi$
 $\tan \psi = \frac{2}{1 + 4n^2 - 1} = \frac{1}{2n^2}$
 $\psi = \arctan\left(\frac{1}{2n^2}\right)$
 • HENCE $\arctan\left(\frac{1}{2n^2}\right) = \arctan(2n+1) - \arctan(2n-1)$
 $n=1: \arctan\left(\frac{1}{2}\right) = \arctan 3 - \arctan 1$
 $n=2: \arctan\left(\frac{1}{8}\right) = \arctan 5 - \arctan 3$
 $n=3: \arctan\left(\frac{1}{18}\right) = \arctan 7 - \arctan 5$
 \vdots
 $n=k: \arctan\left(\frac{1}{2k^2}\right) = \arctan(2k+1) - \arctan(2k-1)$
 • SUMMATION:
 $\sum_{n=1}^k \arctan\left(\frac{1}{2n^2}\right) = \arctan(2k+1) - \arctan 1$
 $\sum_{n=1}^{\infty} \arctan\left(\frac{1}{2n^2}\right) = \lim_{k \rightarrow \infty} [\arctan(2k+1) - \arctan 1]$
 $= \frac{\pi}{2} - \frac{\pi}{4}$
 $= \frac{\pi}{4}$

Question 9 (****)

$$S_n = (2 \times 1!) + (5 \times 2!) + (10 \times 3!) + (17 \times 4!) + \dots + (n^2 + 1)n!$$

Use an appropriate method to show that

$$S_n = n(n+1)!$$

, proof

START BY WRITING THE SERIES IN SUMMATION NOTATION

$$(2 \times 1!) + (5 \times 2!) + (10 \times 3!) + (17 \times 4!) + \dots + (n^2 + 1)n! = \sum_{r=1}^n [(r^2 + 1)r!]$$

TRY SOME DIFFERENCES INVOLVING FACTORIALS, TRYING TO OBTAIN AN EASY SUMMATION

$$[(r+1)! - r!] = (r+1)r! - r! = r \times r!$$

AS THIS DOES NOT PRODUCE A QUANTITATIVE TERM IN R, WE MAY TRY

$$\begin{aligned} (r+2)! - r! &= (r+2)(r+1)r! - r! \\ (r+2)! - r! &= (r^2 + 3r + 2)r! - r! \\ (r+2)! - r! &= (r^2 + 3r + 1)r! \\ (r+2)! - r! &= (r^2 + 1)r! + 2r \times r! \end{aligned}$$

↑
REAR RANGE $r \times r! = (r+1)! - r!$

$$\begin{aligned} (r+2)! - r! &= (r^2 + 1)r! + 2[(r+1)! - r!] \\ (r+2)! - r! &= (r^2 + 1)r! + 2(r+1)! - 2r! \\ (r+2)! &= (r^2 + 1)r! + 2(r+1)! - 2r! \\ (r+2)! - 3(r+1)! + 2r! &= (r^2 + 1)r! \end{aligned}$$

HENCE WE HAVE

$$[(r^2 + 1)r! = (r+2)! - 3(r+1)! + 2r!]$$

WRITING THE SERIES JUST OBTAINED

$$[(r^2 + 1)r! = (r+2)! - 3(r+1)! + 2r!]$$

r=1	$2 \times 1! =$	$3! - 3 \times 2! + 2 \times 1!$
r=2	$5 \times 2! =$	$4! - 3 \times 3! + 2 \times 2!$
r=3	$10 \times 3! =$	$5! - 3 \times 4! + 2 \times 3!$
r=4	$17 \times 4! =$	$6! - 3 \times 5! + 2 \times 4!$
⋮		
r=n	$(n^2 + 1)n! =$	$(n+2)! - 3(n+1)! + 2n \times n!$
r=n	$(n^2 + 1)n! =$	$(n+2)! - 3(n+1)! + 2n \times n!$

$$\begin{aligned} \sum_{r=1}^n [(r^2 + 1)r!] &= (n+2)! - 3(n+1)! - 3 \times 2! + 2 \times 1! + 2n \times n! \\ &= (n+2)(n+1)! - 3(n+1)! - 6 + 2 + 2n \times n! \\ &= (n+2-3)(n+1)! \\ &= (n-1)(n+1)! \\ &= n(n+1)! \end{aligned}$$

Question 10 (****)

By considering the trigonometric identity for $\tan(A-B)$, with $A = \arctan(n+1)$ and $B = \arctan(n)$, sum the following series

$$\sum_{n=1}^{\infty} \arctan\left(\frac{1}{n^2 + n + 1}\right).$$

You may assume the series converges.

$$\boxed{}, \boxed{}, \boxed{\frac{\pi}{4}}$$

CONSIDER THE COMPOUND ANGLE IDENTITY FOR $\tan(A-B)$

$$\Rightarrow \tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$\Rightarrow \tan(\arctan(n+1) - \arctan n) = \frac{\tan(\arctan(n+1)) - \tan(\arctan n)}{1 + \tan(\arctan(n+1))\tan(\arctan n)}$$

$$\Rightarrow \tan(\arctan(n+1) - \arctan n) = \frac{(n+1) - n}{1 + (n+1)n}$$

$$\Rightarrow \tan(\arctan(n+1) - \arctan n) = \frac{1}{n^2 + n + 1}$$

$$\Rightarrow \arctan[\tan(\arctan(n+1) - \arctan n)] = \arctan\left(\frac{1}{n^2 + n + 1}\right)$$

$$\Rightarrow \arctan(n+1) - \arctan n = \arctan\left(\frac{1}{n^2 + n + 1}\right)$$

REWRITE THE SUMMATION NOW AS

$$\sum_{n=1}^{\infty} \arctan\left(\frac{1}{n^2 + n + 1}\right) = \sum_{n=1}^{\infty} [\arctan(n+1) - \arctan n]$$

$$= \sum_{n=1}^{\infty} \arctan(n+1) - \sum_{n=1}^{\infty} \arctan n$$

$$= \lim_{k \rightarrow \infty} \left[\sum_{n=1}^k \arctan(n+1) - \sum_{n=1}^k \arctan n \right]$$

PROCEED AS BEFORE - NOTE THE "CANCELL"

$$\lim_{k \rightarrow \infty} \begin{bmatrix} \arctan(k+1) & - & \arctan k \\ \arctan k & - & \arctan(k-1) \\ \arctan(k-1) & - & \arctan(k-2) \\ + & & + \\ + & & + \\ \arctan 3 & - & \arctan 2 \\ \arctan 2 & - & \arctan 1 \end{bmatrix}$$

$$= \lim_{k \rightarrow \infty} [\arctan(k+1) - \arctan 1]$$

$$= \frac{\pi}{2} - \frac{\pi}{4}$$

$$= \frac{\pi}{4}$$

Question 11 (****)

Determine, in terms of n , a simplified expression

$$\sum_{r=1}^n \left[\frac{r^2 + 9r + 19}{(r+5)!} \right],$$

and hence, or otherwise, deduce the value of

$$\sum_{r=1}^{\infty} \left[\frac{r^2 + 7r + 11}{(r+4)!} \right].$$

$$\boxed{\frac{5}{24}}, \quad \sum_{r=1}^n \left[\frac{r^2 + 9r + 19}{(r+5)!} \right] = \frac{1}{6} - \frac{n+5}{(n+5)!}, \quad \sum_{r=1}^{\infty} \left[\frac{r^2 + 7r + 11}{(r+4)!} \right] = \frac{5}{24}$$

• SIMPLIFY WITH ORIGNAL FRACTIONS — NOTE THAT THE NUMERATOR IS A QUADRATIC IN r , SO USE PARTIAL FRACTIONS METHOD

i.e. $\frac{r^2 + 9r + 19}{(r+5)!} = \frac{A}{(r+5)} + \frac{B}{(r+5)!}$

$\Rightarrow r^2 + 9r + 19 = A(r+5) + B$

$\Rightarrow r^2 + 9r + 19 = Ar + 5A + B$

$\therefore B = 14 \quad A = -1$

• HENCE BY THE METHOD OF DIFFERENCES

$\frac{r^2 + 9r + 19}{(r+5)!} = \frac{-1}{(r+5)} + \frac{14}{(r+5)!}$

$r=1: \frac{1+9+19}{6!} = \frac{-1}{6!} + \frac{14}{6!}$

$r=2: \frac{4+18+19}{7!} = \frac{-1}{7!} + \frac{14}{7!}$

$r=3: \frac{9+27+19}{8!} = \frac{-1}{8!} + \frac{14}{8!}$

$r=4: \frac{16+36+19}{9!} = \frac{-1}{9!} + \frac{14}{9!}$

\vdots

$r=n-1: \frac{(n-1)^2 + 9(n-1) + 19}{n!} = \frac{-1}{n!} + \frac{14}{n!}$

$r=n: \frac{n^2 + 9n + 19}{(n+1)!} = \frac{-1}{(n+1)!} + \frac{14}{(n+1)!}$

• ADDING

$\sum_{r=1}^n \left[\frac{r^2 + 9r + 19}{(r+5)!} \right] = \left[\frac{-1}{4!} + \frac{14}{4!} \right] - \left[\frac{-1}{(n+1)!} + \frac{14}{(n+1)!} \right]$

$= \frac{13}{24} - \frac{13}{(n+1)!}$

$\therefore \sum_{r=1}^{\infty} \left[\frac{r^2 + 9r + 19}{(r+5)!} \right] = \frac{13}{24} - \lim_{n \rightarrow \infty} \frac{13}{(n+1)!} = \frac{13}{24}$

• NOW PROCEED AS FOLLOWS

$\Rightarrow \sum_{r=1}^{\infty} \left[\frac{r^2 + 7r + 11}{(r+4)!} \right] = \frac{5}{24} - \frac{11}{24}$

$\Rightarrow \sum_{r=1}^{\infty} \left[\frac{(r-1)^2 + 9(r-1) + 19}{(r+4)!} \right] = \frac{5}{24} - \frac{11}{24}$

$\Rightarrow \sum_{r=1}^{\infty} \left[\frac{r^2 - 2r + 1 + 9r - 9 + 19}{(r+4)!} \right] = \frac{5}{24} - \frac{11}{24}$

$\Rightarrow \sum_{r=1}^{\infty} \left[\frac{r^2 + 7r + 11}{(r+4)!} \right] = \frac{5}{24} - \frac{11}{24}$

$\Rightarrow \sum_{r=1}^{\infty} \left[\frac{r^2 + 7r + 11}{(r+4)!} \right] = \frac{5}{24}$

Question 12 (****)

A sequence is defined as

$$u_{r+1} = u_r + \frac{2r}{r^4 + r^2 + 1}, \quad u_1 = 0, \quad r \in \mathbb{N}.$$

Determine the exact value of u_{61} .

$$\boxed{}, \quad u_{61} = \frac{3660}{3661}$$

Handwritten solution for Question 12:

Given: $u_{r+1} = u_r + \frac{2r}{r^4 + r^2 + 1}$, $u_1 = 0$

Start by factoring: $r^4 + r^2 + 1$ into two quadratic terms:

$$(r^2 + r + 1)(r^2 - r + 1) = r^4 - r^3 + r^2 + r^3 - r^2 + r + 1 = r^4 + r + 1$$

By inspection:

$$\frac{1}{r^2 - r + 1} - \frac{1}{r^2 + r + 1} = \frac{(r^2 + r + 1) - (r^2 - r + 1)}{(r^2 - r + 1)(r^2 + r + 1)} = \frac{2r}{r^4 + r^2 + 1}$$

Rewrite the above expression as follows:

$$u_{r+1} - u_r = \frac{1}{r^2 - r + 1} - \frac{1}{r^2 + r + 1}$$

Telescoping series:

$$\begin{aligned} u_2 - u_1 &= \frac{1}{1^2 - 1 + 1} - \frac{1}{1^2 + 1 + 1} \\ u_3 - u_2 &= \frac{1}{2^2 - 2 + 1} - \frac{1}{2^2 + 2 + 1} \\ u_4 - u_3 &= \frac{1}{3^2 - 3 + 1} - \frac{1}{3^2 + 3 + 1} \\ &\vdots \\ u_{61} - u_{60} &= \frac{1}{60^2 - 60 + 1} - \frac{1}{60^2 + 60 + 1} \\ \text{Adding: } u_1 - u_0 &= \frac{1}{1} - \frac{1}{3661} \end{aligned}$$

$\therefore u_{61} = \frac{3660}{3661}$

Question 13 (****)

Find the value of

$$\sum_{r=0}^{\infty} \left[\frac{\sin^4(\pi \times 2^{r-2})}{4^r} \right].$$

Hint: Express $\sin^4 \theta$ in terms of $\sin^2 \theta$ and $\sin^2 2\theta$ only.

$$\boxed{}, \boxed{\frac{1}{2}}$$

● **STARTING BY MANIPULATING THE SINE TO THE POWER 4**

$$\begin{aligned} \sin^4 \theta &= (\sin^2 \theta)^2 = \left(\frac{1}{2} - \frac{1}{2} \cos 2\theta \right)^2 = \frac{1}{4} - \frac{1}{2} \cos 2\theta + \frac{1}{4} \cos^2 2\theta \\ &= \frac{1}{4} - \frac{1}{2} (1 - 2\sin^2 \theta) + \frac{1}{4} (1 - \sin^2 2\theta) \\ &= \cancel{\frac{1}{4}} - \cancel{\frac{1}{2}} + \sin^2 \theta + \cancel{\frac{1}{4}} - \frac{1}{4} \sin^2 2\theta \\ &= \sin^2 \theta - \frac{1}{4} \sin^2 2\theta \end{aligned}$$

● **NOW WE HAVE BY CONSIDERING THE SUM OF THE FIRST N TERMS**

$$\begin{aligned} \sum_{r=0}^n \frac{\sin^4(\pi \times 2^{r-2})}{4^r} &= \sum_{r=0}^n \left[\frac{1}{4^r} \left(\sin^2(\pi \times 2^{r-2}) - \frac{1}{4} \sin^2(\pi \times 2^{r-1}) \right) \right] \\ &= \sum_{r=0}^n \left[\frac{1}{4^r} \sin^2(\pi \times 2^{r-2}) - \frac{1}{4^{r+1}} \sin^2(\pi \times 2^{r-1}) \right] \\ &= \begin{aligned} &\frac{1}{4^0} \sin^2 \frac{\pi}{4} - \frac{1}{4^1} \sin^2 \frac{\pi}{2} \quad \leftarrow r=0 \\ &\frac{1}{4^1} \sin^2 \frac{\pi}{2} - \frac{1}{4^2} \sin^2 \pi \quad \leftarrow r=1 \\ &\frac{1}{4^2} \sin^2 \pi - \frac{1}{4^3} \sin^2 2\pi \quad \leftarrow r=2 \\ &\vdots \\ &\frac{1}{4^n} \sin^2(\pi \times 2^{n-2}) - \frac{1}{4^{n+1}} \sin^2(\pi \times 2^{n-1}) \quad \leftarrow r=n \end{aligned} \\ &= \sin^2 \frac{\pi}{4} - \frac{1}{4^{n+1}} \sin^2(\pi \times 2^{n-1}) \end{aligned}$$

● **THUS WE HAVE**

$$\sum_{r=0}^{\infty} \frac{\sin^4(\pi \times 2^{r-2})}{4^r} = \sin^2 \frac{\pi}{4} = \left(\frac{1}{\sqrt{2}} \right)^2 = \frac{1}{2}$$

Question 14 (****)

Find the sum to infinity of the following convergent series.

$$\frac{1}{4 \times 2!} + \frac{1}{5 \times 3!} + \frac{1}{6 \times 4!} + \frac{1}{7 \times 5!} + \frac{1}{8 \times 6!} + \dots$$

$$\boxed{\frac{1}{6}}$$

WRITING THE SERIES IN SIGMA NOTATION

$$S_{\infty} = \sum_{r=1}^{\infty} \frac{1}{(r+3)(r+1)!}$$

ATTEMPT SUMMATION BY THE METHOD OF DIFFERENCES

• TRY

$$\frac{1}{(r+3)(r+1)!} \equiv \frac{A}{(r+3)!} + \frac{B}{(r+1)!}$$

$$1 \equiv A + B(r+3)(r+2)$$

NO A & B CAN SATISFY THE ABOVE

• TRY NEXT

$$\frac{1}{(r+3)(r+1)!} \equiv \frac{A}{(r+3)!} + \frac{B}{(r+2)!}$$

$$\Rightarrow \frac{1}{(r+3)(r+1)!} \equiv \frac{A + B(r+3)}{(r+3)!}$$

$$\Rightarrow \frac{r+2}{(r+3)!} \equiv \frac{A + B(r+3)}{(r+3)!}$$

$$\Rightarrow \frac{r+2}{(r+3)!} \equiv \frac{A + B(r+3)}{(r+3)!}$$

$$\Rightarrow r+2 \equiv (A+3B) + Br$$

$\therefore B=1$ & $A=-1$

HOWEVER WE NOW HAVE A SUITABLE IDENTITY

$$\frac{1}{(r+3)(r+1)!} \equiv \frac{1}{(r+2)!} - \frac{1}{(r+3)!}$$

• $r=1$: $\frac{1}{4 \times 2!} = \frac{1}{3!} - \frac{1}{4!}$

• $r=2$: $\frac{1}{5 \times 3!} = \frac{1}{4!} - \frac{1}{5!}$

• $r=3$: $\frac{1}{6 \times 4!} = \frac{1}{5!} - \frac{1}{6!}$

• $r=4$: $\frac{1}{7 \times 5!} = \frac{1}{6!} - \frac{1}{7!}$

• $r=5$: $\frac{1}{8 \times 6!} = \frac{1}{7!} - \frac{1}{8!}$

• $r=N$: $\frac{1}{(N+3)(N+1)!} = \frac{1}{(N+2)!} - \frac{1}{(N+3)!}$

$$\Rightarrow \sum_{r=1}^N \frac{1}{(r+3)(r+1)!} = \frac{1}{3!} - \frac{1}{(N+3)!}$$

$$\Rightarrow \lim_{N \rightarrow \infty} \left[\sum_{r=1}^N \frac{1}{(r+3)(r+1)!} \right] = \lim_{N \rightarrow \infty} \left[\frac{1}{3!} - \frac{1}{(N+3)!} \right]$$

$$\Rightarrow \sum_{r=1}^{\infty} \frac{1}{(r+3)(r+1)!} = \frac{1}{3!} - \frac{1}{\infty} = \frac{1}{6}$$

Question 15 (****)

Evaluate the following expression

$$\sum_{k=1}^{\infty} \left[\sum_{r=1}^k r \right]^{-1}.$$

SP, 2

• Rewrite for simplicity as follows

$$\sum_{k=1}^{\infty} \left[\sum_{r=1}^k r \right]^{-1} = \sum_{k=1}^{\infty} \left[\frac{1}{\sum_{r=1}^k r} \right]$$

$$= \frac{1}{1} + \frac{1}{1+2} + \frac{1}{1+2+3} + \frac{1}{1+2+3+4} + \dots$$

• Introduce a finite limit for the summation, say n

$$= \lim_{n \rightarrow \infty} \left[\frac{1}{1} + \frac{1}{1+2} + \frac{1}{1+2+3} + \frac{1}{1+2+3+4} + \dots + \frac{1}{1+2+\dots+n} \right]$$

$$= \lim_{n \rightarrow \infty} \left[\sum_{r=1}^n \frac{1}{r(r+1)} \right]$$

$$= 2 \lim_{n \rightarrow \infty} \left[\sum_{r=1}^n \frac{1}{r(r+1)} \right]$$

• Split into two fractions by inspection

$$= 2 \lim_{n \rightarrow \infty} \left[\sum_{r=1}^n \frac{1}{r} - \frac{1}{r+1} \right]$$

$$= 2 \lim_{n \rightarrow \infty} \left[\left(1 - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \dots + \left(\frac{1}{n} - \frac{1}{n+1} \right) \right]$$

$$= 2 \lim_{n \rightarrow \infty} \left[1 - \frac{1}{n+1} \right]$$

$$= 2$$