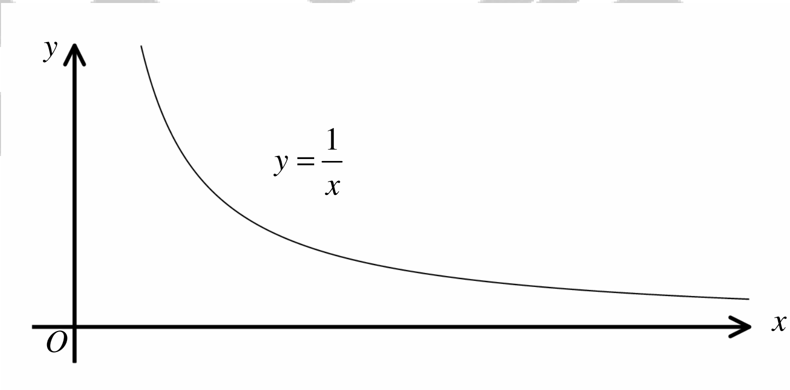
SERIES and INTEGRALS

Question 1 (***)

The figure below shows the curve C with equation $y = e^{-\frac{1}{x}}$, $0 < x \le 1$.



a) By using a two different sets of rectangles of unit width under the graph of C, show that

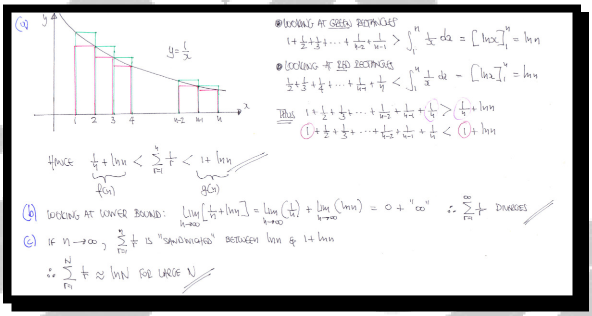
$$f(n) < \sum_{r=1}^{n} \frac{1}{r} < g(n),$$

where $f(n)$ and $g(n)$ are functions involving natural logarithms.

b) Determine whether $\displaystyle\sum_{r=1}^{\infty} \frac{1}{r}$ exists.

c) Write down an approximation for $\displaystyle\sum_{r=1}^{N} \frac{1}{r}$ if N is very large.

$\boxed{f(n) = \frac{1}{n} + \ln n}$, $\boxed{g(n) = 1 + \ln n}$, $\boxed{\sum_{r=1}^{\infty} \frac{1}{r} \text{ diverges}}$, $\boxed{\text{as } n \to \infty, \sum_{r=1}^{N} \frac{1}{r} \approx \ln N}$


(a)

- LOOKING AT GREEN RECTANGLES
$1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n-2} + \frac{1}{n-1} > \int_1^n \frac{1}{x}\,dx = [\ln x]_1^n = \ln n$
- LOOKING AT RED RECTANGLES
$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n-1} + \frac{1}{n} < \int_1^n \frac{1}{x}\,dx = [\ln x]_1^n = \ln n$

THUS $1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n-2} + \frac{1}{n-1} + \frac{1}{n} > \frac{1}{n} + \ln n$

$1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n-2} + \frac{1}{n-1} + \frac{1}{n} < 1 + \ln n$

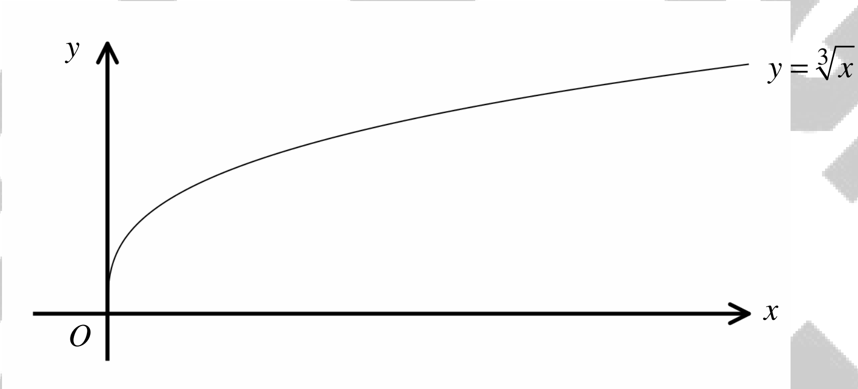
HENCE $\underbrace{\frac{1}{n} + \ln n}_{f(n)} < \sum_{r=1}^{n} \frac{1}{r} < \underbrace{1 + \ln n}_{g(n)}$

(b) LOOKING AT LOWER BOUND: $\lim_{n \to \infty}\left[\frac{1}{n} + \ln n\right] = \lim_{n \to \infty}\left(\frac{1}{n}\right) + \lim_{n \to \infty}(\ln n) = 0 + \text{“}\infty\text{”}$ $\therefore \sum_{r=1}^{\infty} \frac{1}{r}$ DIVERGES

(c) IF $n \to \infty$, $\sum_{r=1}^{n} \frac{1}{r}$ IS "SANDWICHED" BETWEEN $\ln n$ & $1 + \ln n$

$\therefore \sum_{r=1}^{N} \frac{1}{r} \approx \ln N$ FOR LARGE N

Question 2 (***)



The figure above shows the curve C with equation $y = \sqrt[3]{x}$, $x \geq 0$.

a) By using a two different sets of rectangles of unit width under the graph of C, show that

$$\int_a^b \sqrt[3]{x}\ dx \ < \ \sqrt[3]{1} + \sqrt[3]{2} + \sqrt[3]{3} + \ldots + \sqrt[3]{n} \ < \ \int_c^d \sqrt[3]{x} \ ,$$

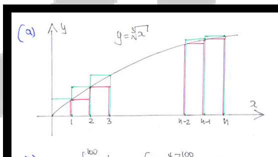
stating the limits in the integrals.

b) Hence show that

$$\sum_{n=1}^{100} \sqrt[3]{n} \approx 350 .$$

$\boxed{a = 0,\ b = n,\ c = 1,\ d = n+1}$

(a)



SUM OF GREEN RECTANGLES (AREA)

$\sqrt[3]{1} + \sqrt[3]{2} + \sqrt[3]{3} + \ldots + \sqrt[3]{n-1} + \sqrt[3]{n} > \int_0^n \sqrt[3]{x}\,dx$

SUM OF RED RECTANGLES (AREA)

$\sqrt[3]{1} + \sqrt[3]{2} + \sqrt[3]{3} + \ldots + \sqrt[3]{n-1} < \int_1^n \sqrt[3]{x}\,dx$

$\sqrt[3]{1} + \sqrt[3]{2} + \sqrt[3]{3} + \ldots + \sqrt[3]{n-1} + \sqrt[3]{n} < \int_1^{n+1} \sqrt[3]{x}$

$\int_0^n \sqrt[3]{x}\,dx < \sum_{r=1}^{n} \sqrt[3]{r} < \int_1^{n+1} \sqrt[3]{x}\,dx$ //

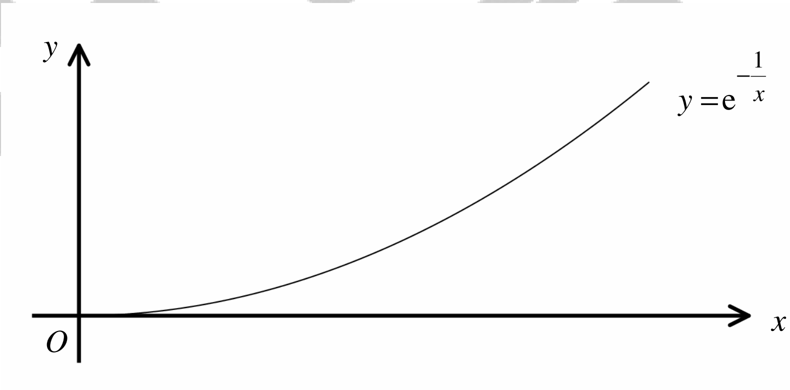
(b) L.B. $= \int_0^{100} x^{\frac{1}{3}}\,dx = \left[\frac{3}{4}x^{\frac{4}{3}}\right]_0^{100} \approx 348.1191625\ldots$

U.B. $= \int_1^{101} x^{\frac{1}{3}}\,dx = \left[\frac{3}{4}x^{\frac{4}{3}}\right]_1^{101} \simeq 352.0184702$

$\therefore \sum_{n=1}^{100} \sqrt[3]{n} \simeq 350$ // (2 s.f.)

Question 3 (*+)**

The figure below shows the curve C with equation $y = e^{-\frac{1}{x}}$, $0 < x \le 1$.



a) By using a two different sets of rectangles of width $\frac{1}{n}$ under the graph of C, show that

$$A < \int_0^1 e^{-\frac{1}{x}}\,dx < A + \frac{1}{2e},$$

where A is an exact finite series involving exponentials.

The above expression is to be used to approximate the area under C for $0 < x \le 1$. When $n \ge N$, the error is less than 10^{-5}.

b) Determine the least possible value of N.

$$\boxed{A = \frac{1}{n}\left[e^{-n} + e^{-\frac{1}{2}n} + e^{-\frac{1}{3}n} + e^{-\frac{1}{4}n} + \ldots + e^{-\frac{n}{n-1}}\right]}, \quad \boxed{N = 36788}$$