GENERAL PROOF
Question 1  (**)

\[ f(n) = n^2 + n + 2, \ n \in \mathbb{N}. \]

Show that \( f(n) \) is always even.

Question 2  (**)

Prove that when the square of a positive odd integer is divided by 4 the remainder is always 1.
Question 3  (**)
Show that \(a^3 - a + 1\) is odd for all positive integer values of \(a\).

Question 4  (**)
Prove that the square of a positive integer can never be of the form \(3k + 2\), \(k \in \mathbb{N}\).
Question 5 (***+)

It is asserted that

\[ |2x+1| \leq 5 \implies |x| \leq 2. \]

Disprove this assertion by a **counter-example**.

Question 6 (***+)

Prove by **contradiction** that for all real \( \theta \)

\[ \cos \theta + \sin \theta \leq \sqrt{2}. \]
Question 7  (**+)
Prove by contradiction that if $p$ and $q$ are positive integers, then

\[ \frac{p}{q} + \frac{q}{p} \geq 2. \]

proof

Question 8  (***)

\[ f(n) = 5^{2n} - 1, \quad n \in \mathbb{N}. \]

Without using proof by induction, show that $f(n)$ is a multiple of 8.

proof
Question 9  (***)
Prove by contradiction that for all real $x$

$$(13x+1)^2 + 3 > (5x-1)^2.$$  

Question 10  (***)
It is given that

$$N = k^2 - 1 \quad \text{and} \quad k = 2^n - 1, \quad n \in \mathbb{N}.$$  

Use direct proof to show that $2^{n+1}$ is a factor of $N$.  

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Question 11  (***)
Prove by exhaustion that if $n$ is a positive integer that is not divisible by 3, then $n^2 - 1$ is divisible by 3.

**proof**

Question 12  (***)
Prove that if we subtract 1 from a positive odd square number, the answer is always divisible by 8.

**proof**
Question 13  (***)

Given that \( k > 0 \), use algebra to show that

\[
\frac{k + 1}{\sqrt{k}} \geq 2.
\]

Proof:

\[
\frac{k + 1}{\sqrt{k}} = \frac{k^{1/2} + 1}{\sqrt{k}} \geq \frac{2}{\sqrt{k}}
\]

Since \( k > 0 \), \( \sqrt{k} > 0 \), and \( k + 1 > k \), it follows that

\[
\frac{k + 1}{\sqrt{k}} \geq 2.
\]

Question 14  (***)

Prove by the method of contradiction that there are no integers \( n \) and \( m \) which satisfy the following equation.

\[
3n + 21m = 137
\]

Proof:

Assume that there exist integers \( n \) and \( m \) such that

\[
3n + 21m = 137
\]

This implies

\[
3n + 21m \equiv 0 \pmod{21}
\]

Since 137 is not divisible by 21, we have a contradiction. Therefore, there are no integers \( n \) and \( m \) which satisfy the equation.

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Question 15 (***)
Use the method of **proof by contradiction** to show that if \( x \) then
\[
\left| x + \frac{1}{x} \right| \geq 2.
\]

\[\text{NG}^2, \text{ proof}\]

Question 16 (***)
Prove that the sum of two even consecutive powers of 2 is always a multiple of 20.

\[\text{proof}\]
Question 17 (***+)
Prove by the method of contradiction that there are no integers $a$ and $b$ which satisfy the following equation.

\[ a^2 - 8b = 7 \]
Question 18  (***)

Use proof by exhaustion to show that if \( m \in \mathbb{N} \) and \( n \in \mathbb{N} \), then

\[
m^2 - n^2 \neq 102.
\]

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**Proof**

1. Assume for contradiction that \( m^2 - n^2 = 102 \).
2. Since \( 102 = 2 \times 3 \times 17 \), we can consider the cases for the prime factors of \( m^2 - n^2 \).
3. If \( m^2 - n^2 \) is divisible by 2, then both \( m^2 \) and \( n^2 \) must be even, implying that both \( m \) and \( n \) are even.
4. If \( m \) and \( n \) are even, then there exist integers \( k \) and \( l \) such that \( m = 2k \) and \( n = 2l \).
5. Substituting these expressions into the equation, we get \( (2k)^2 - (2l)^2 = 102 \), which simplifies to \( 4k^2 - 4l^2 = 102 \), and further to \( k^2 - l^2 = 25.5 \).
6. However, \( k^2 - l^2 \) is an integer, which contradicts the assumption that \( m^2 - n^2 = 102 \).

Therefore, our initial assumption is false, and \( m^2 - n^2 \neq 102 \).
Question 19 (***+)

Use a calculus method to prove that if $x \in \mathbb{R}$, $x > 0$, then

$$x^4 + x^{-4} \geq 2.$$
The figure above shows two right angled triangles.

- The triangle, on the left section of the figure, has side lengths of \(a\), \(b\) and \(c\), where \(c\) is the length of its hypotenuse.

- The triangle, on the right section of the figure, has side lengths of \(a+1\), \(b+1\) and \(c+1\), where \(c+1\) is the length of its hypotenuse.

Show that \(a\), \(b\) and \(c\) cannot all be integers.
Question 21 (***+)

It is given that $x \in \mathbb{R}$ and $y \in \mathbb{R}$ such that $x + y = 1$.

Prove that

\[ x^2 + y = y^2 + x. \]
Question 22  (***)

It is given that $a$ and $b$ are positive odd integers, with $a > b$.

Use proof by contradiction to show that if $a + b$ is a multiple of 4, then $a - b$ cannot be a multiple of 4.

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Question 23  (***)

Prove by contradiction that $\log_{10} 5$ is an irrational number.
Let $a \in \mathbb{N}$ with $\frac{1}{5}a \notin \mathbb{N}$.

a) Show that the remainder of the division of $a^2$ by 5 is either 1 or 4.

b) Given further that $b \in \mathbb{N}$ with $\frac{1}{5}b \notin \mathbb{N}$, deduce that $\frac{1}{5}(a^4 - b^4) \in \mathbb{N}$. 

Proof,
Question 25  (****)

It is asserted that

“The difference of the squares of two non consecutive positive integers can never be a prime number”.

a) Prove the validity of the above assertion.

The difference between two consecutive square numbers is 163.

b) Given further that 163 is a prime number find the above mentioned consecutive square numbers.

6561, 6724
Question 26  (***)

By considering \((\sqrt{2})^{\sqrt{2}}\), or otherwise, prove that an irrational number raised to the power of an irrational number can be a rational number.

\[
(\sqrt{2})^{\sqrt{2}}
\]

**Proof**


Question 27  (***)

It is given that

\[a^2 + b^2 = c^2, \quad a \in \mathbb{N}, \quad b \in \mathbb{N}.
\]

Show that \(a\) and \(b\) cannot both be odd.

**Proof**


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Question 28  (****)
Given that \( k \in \mathbb{N} \), use algebra to prove that
\[
\frac{2k + 2}{2k + 3} > \frac{2k}{2k + 1}.
\]

proof

Question 29  (****)
\[
f(a) = a^3 + 5a, \ a \in \mathbb{N}.
\]
Without using proof by induction, show that \( f(a) \) is a multiple of 6.

proof
Question 30  (****)

\[ f(k) = k^3 + 2k, \quad k \in \mathbb{N}. \]

Without using proof by induction, show that \( f(k) \) is always a multiple of 3.

Question 31  (****)

Consider the following sequence

\[ 3, 8, 15, 24, 35, 48, \ldots \]

Prove that the product of any two consecutive terms of the above sequence can be written as the product of 4 consecutive integers.
Question 32  (***)
Prove that if 1 is added to the product of any 4 consecutive positive integers, the resulting number will always be a square number.

\[
\text{proof}
\]

Question 33  (****+)
Show that for all positive real numbers \(a\) and \(b\)

\[
a^3 + b^3 \geq a^2b + ab^2.
\]

\[
\text{proof}
\]
Question 34  (***)
Show clearly that for all real numbers \( \alpha \), \( \beta \) and \( \gamma \)
\[
\alpha^2 + \beta^2 + \gamma^2 \geq \alpha \beta + \beta \gamma + \gamma \alpha.
\]

proof

Question 35  (**++)
Show, without using proof by induction, that the sum of cubes of any 3 consecutive positive integers is a multiple of 9.

proof
Question 36  (***)

Use a detailed method to show that

\[ \sqrt{1000 \times 1001 \times 1002 \times 1003 + 1} = 1003001 \]

You may NOT use a calculating aid in this question.
Question 37 \( \text{****} \)

Show that the square of an odd positive integer greater than 1 is of the form \( 8T + 1 \),

where \( T \) is a triangular number.
Question 38 (*****)

It is given that

\[ f(m,n) = 2m(m^2 + 3n^2), \]

where \( m \) and \( n \) are distinct positive integers, with \( m > n \).

By using the expansion of \((A \pm B)^3\), prove that \( f(m,n) \) can always be written as the sum of two cubes.

\[ \boxed{\text{proof}} \]
It is given that

\[ f(k) \equiv (k^3 - k)(2k^2 + 5k - 3), \]

where \( k \) is a positive integer.

Prove that \( f(k) \) is divisible by \( 5 \).

You may not use proof by induction in this question.
Prove that for all real numbers, $a$ and $b$,

$$\sqrt{a^2 + b^2} \leq \frac{\sqrt{4a^2 + b^2} + \sqrt{a^2 + 4b^2}}{3}.$$
Question 41  (*****)

Show that for all positive real numbers \( a \) and \( b \)

\[
a^3 + 2b^3 \geq 3ab^2.
\]
Question 42 (*****)

It is given that $x$, $a$ and $b$ are positive real numbers, with $a > b$ and $x^2 > ab$.

Use proof by contradiction to show that

$$\frac{x+a}{\sqrt{x^2+a^2}} + \frac{x+b}{\sqrt{x^2+b^2}} > 0.$$
Question 43     (*****)

Prove that the sum of the squares of two distinct positive integers, when doubled, it can be written as the sum of two distinct square numbers

proof
Question 44  (*****)

The Rational Zero Theorem asserts that if the polynomial

\[ f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots + a_1 x + a_0 \]

has integer coefficients, then every rational zero of \( f(x) \) has the form \( \frac{p}{q} \), where \( p \) is a factor of the constant term \( a_0 \) and \( q \) is a factor of the leading coefficient \( a_n \).

Use this result to show that \( \sin \left( \frac{\pi}{18} \right) \) is irrational.

\[ \square \text{, proof} \]
Question 45 (*****)
By using the definition of $e$ as an infinite convergent series, prove by contradiction that $e$ is irrational.