

Created by T. Madas

# PROOF BY INDUCTION

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# SUMMATION RESULTS

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**Question 1 (\*\*)**

Prove by induction that

$$\sum_{r=1}^n r(r+1) = \frac{1}{3}n(n+1)(n+2), \quad n \geq 1, \quad n \in \mathbb{N}.$$

□, proof

$\sum_{r=1}^n r(r+1) = \frac{1}{3}n(n+1)(n+2) \quad n \geq 1$

**BASE CASE; IF  $n=1$**

- LHS =  $1 \times 2 = 2$
- RHS =  $\frac{1}{3} \times 1 \times 2 \times 3 = 2$

$\therefore$  RESULT HELDS FOR  $n=1$

**INDUCTIVE HYPOTHESIS**

SUPPOSE THE RESULT HELDS FOR  $n=k, k \in \mathbb{N}$

$\Rightarrow \sum_{r=1}^k r(r+1) = \frac{1}{3}k(k+1)(k+2)$

$\Rightarrow \sum_{r=1}^{k+1} r(r+1) + (k+1)(k+2) = \frac{1}{3}(k+1)(k+2) + (k+1)(k+2)$

$\Rightarrow \sum_{r=1}^{k+1} r(r+1) = \frac{1}{3}(k+1)(k+2)[k+3]$

$\Rightarrow \sum_{r=1}^{k+1} r(r+1) = \frac{1}{3}(k+1)(k+1+1)(k+1+2)$

**CONCLUSION**

- IF THE RESULT HELDS FOR  $n=k \in \mathbb{N}$ , THEN IT ALSO HELDS FOR  $n=k+1$
- SINCE THE RESULT HELDS FOR  $n=1$ , THEN IT MUST HOLD FOR ALL  $n \in \mathbb{N}$

**Question 2 (\*\*)**

Prove by induction that

$$\sum_{r=1}^n r(r+3) = \frac{1}{3}n(n+1)(n+5), \quad n \geq 1, \quad n \in \mathbb{N}.$$

proof

$\sum_{r=1}^n r(r+3) = \frac{1}{3}n(n+1)(n+5) \quad n \in \mathbb{N}$

**BASE CASE  $n=1$**

- LHS =  $\sum_{r=1}^1 r(r+3) = 1 \times 4 = 4$
- RHS =  $\frac{1}{3} \times 1 \times 2 \times 6 = 4$

$\therefore$  RESULT HELDS FOR  $n=1$

**SUPPOSE THAT THE RESULT HELDS FOR  $n=k \in \mathbb{N}$**

$\sum_{r=1}^k r(r+3) = \frac{1}{3}k(k+1)(k+5)$

$\Rightarrow \left[ \sum_{r=1}^k r(r+3) \right] + (k+1)(k+4) = \frac{1}{3}k(k+1)(k+5) + (k+1)(k+4)$

$\Rightarrow \sum_{r=1}^{k+1} r(r+3) = \frac{1}{3}(k+1)[k(k+5) + 3(k+4)]$

$\Rightarrow \sum_{r=1}^{k+1} r(r+3) = \frac{1}{3}(k+1)(k^2 + 8k + 16)$

$\Rightarrow \sum_{r=1}^{k+1} r(r+3) = \frac{1}{3}(k+1)(k+2)(k+4)$

$\Rightarrow \sum_{r=1}^{k+1} r(r+3) = \frac{1}{3}(k+1)(k+1+1)(k+1+5)$

**IF THE RESULT HELDS FOR  $n=k \in \mathbb{N}$ , THEN THE RESULT HELDS FOR  $n=k+1$**

**SINCE THE RESULT HELDS FOR  $n=1$ , THEN IT MUST HOLD FOR ALL  $n \in \mathbb{N}$**

Question 3 (\*\*\*)

Prove by induction that

$$\sum_{r=1}^n (r-1)(r+1) = \frac{1}{6}n(n-1)(2n+5), \quad n \geq 1, n \in \mathbb{N}.$$

proof

$\sum_{r=1}^n (r-1)(r+1) = \frac{1}{6}n(n-1)(2n+5)$   
 • If  $n=1$  LHS = 0  
 RHS =  $\frac{1}{6} \times 1 \times 0 \times 7 = 0$  ) it's true for  $n=1$   
 • Suppose the result holds for  $n=k \in \mathbb{N}$   
 $\sum_{r=1}^k (r-1)(r+1) = \frac{1}{6}k(k-1)(2k+5)$   
 $\sum_{r=1}^{k+1} (r-1)(r+1) = (k-1)(k+1) + \frac{1}{6}k(k-1)(2k+5) + (k-1)(k+1)$   
 $\sum_{r=1}^{k+1} (r-1)(r+1) = \frac{1}{6}k(k-1)(2k+5) + k(k+2)$   
 $\sum_{r=1}^{k+1} (r-1)(r+1) = \frac{1}{6}k[(k-1)(2k+5) + 6(k+2)]$   
 $\sum_{r=1}^{k+1} (r-1)(r+1) = \frac{1}{6}k[2k^2 + 3k - 5 + 6k + 12]$   
 $\sum_{r=1}^{k+1} (r-1)(r+1) = \frac{1}{6}k(2k^2 + 9k + 7)$   
 $\sum_{r=1}^{k+1} (r-1)(r+1) = \frac{1}{6}k(2k+7)(k+1)$   
 $\sum_{r=1}^{k+1} (r-1)(r+1) = \frac{1}{6}(2k+7)(k+1)(k+1)$   
 • If the result holds for  $n=k \in \mathbb{N} \Rightarrow$  the result also holds for  $n=k+1$   
 since the result holds for  $n=1 \Rightarrow$  it's true for  $\forall n \in \mathbb{N}$

Question 4 (\*\*\*)

Prove by induction that

$$\sum_{r=2}^n r^2(r-1) = \frac{1}{12}n(n-1)(n+1)(3n+2), \quad n \geq 2, \quad n \in \mathbb{N}.$$

□, proof

$\sum_{r=2}^n r^2(r-1) = \frac{1}{12}n(n-1)(n+1)(3n+2), \quad n \geq 2$

**BASE CASE;  $n=2$**

- LHS =  $2^2(2-1) = 4$
- RHS =  $\frac{1}{12} \times 2 \times 1 \times 3 \times 6 = 4$

$\therefore$  RESULT HELD FOR  $n=2$

**INDUCTIVE HYPOTHESIS**

SUPPOSE THAT THE RESULT HELD FOR  $n=k, k \in \mathbb{N}$

$\Rightarrow \sum_{r=2}^k r^2(r-1) = \frac{1}{12}k(k-1)(k+1)(3k+2)$

$\Rightarrow \sum_{r=2}^{k+1} r^2(r-1) + (k+1)^2(k+1) = \frac{1}{12}k(k-1)(k+1)(3k+2) + k(k+1)^2$

$\Rightarrow \sum_{r=2}^{k+1} r^2(r-1) = \frac{1}{12}k(k+1)[(k-1)(3k+2) + 12(k+1)]$

$\Rightarrow \sum_{r=2}^{k+1} r^2(r-1) = \frac{1}{12}k(k+1)(3k^2 - k - 2 + 12k + 12)$

$\Rightarrow \sum_{r=2}^{k+1} r^2(r-1) = \frac{1}{12}k(k+1)(3k^2 + 11k + 10)$

$\Rightarrow \sum_{r=2}^{k+1} r^2(r-1) = \frac{1}{12}k(k+1)(3k+5)(k+2)$

$\Rightarrow \sum_{r=2}^{k+1} r^2(r-1) = \frac{1}{12}(k+1)(k+1)(k+1)(3(k+1)+2)$

**CONCLUSION**

- IF THE RESULT HELD FOR  $n=k \in \mathbb{N}$ , THEN IT ALSO HELD FOR  $n=k+1$
- SINCE THE RESULT HELD FOR  $n=2$ , THEN IT MUST HOLD FOR ALL  $n \in \mathbb{N}$

**Question 5 (\*\*+)**

Prove by induction that

$$1 + 8 + 27 + 64 + \dots + n^3 = \frac{1}{4}n^2(n+1)^2, \quad n \geq 1, n \in \mathbb{N}.$$

proof

- If  $n=1$   $\frac{1}{4} \times 1^2 \times (1+1)^2 = 1$  ✓ Result holds for  $n=1$
- Suppose the result holds for  $n \in \mathbb{N}$   
 $1+8+27+\dots+n^3 = \frac{1}{4}n^2(n+1)^2$   
 $1+8+27+\dots+n^3+(n+1)^3 = \frac{1}{4}n^2(n+1)^2 + (n+1)^3$   
 $1+8+27+\dots+n^3+(n+1)^3 = \frac{1}{4}n^2(n+1)^2 + (n+1)^3$   
 $1+8+27+\dots+n^3+(n+1)^3 = \frac{1}{4}n^2(n+1)^2 + (n+1)^3$   
 $1+8+27+\dots+n^3+(n+1)^3 = \frac{1}{4}(n+1)^2(n+3)^2$   
 $1+8+27+\dots+n^3+(n+1)^3 = \frac{1}{4}(n+1)^2(n+3)^2$
- If the result holds for  $n \in \mathbb{N} \Rightarrow$  the result holds for  $n+1$   
 Since the result holds for  $n=1 \Rightarrow$  the result holds  $\forall n \in \mathbb{N}$  ✓

**Question 6 (\*\*\*)**

Prove by induction that

$$\sum_{r=1}^n (2r-1)^2 = \frac{1}{3}n(2n-1)(2n+1), \quad n \geq 1, n \in \mathbb{N}.$$

proof

$\sum_{r=1}^n (2r-1)^2 = \frac{1}{3}n(2n-1)(2n+1)$   

- If  $n=1$   $\left. \begin{aligned} LHS &= \sum_{r=1}^1 (2r-1)^2 = 1^2 = 1 \\ RHS &= \frac{1}{3} \times 1 \times 1 \times 3 = 1 \end{aligned} \right\}$  ✓ Result holds for  $n=1$
- Suppose the result holds for  $n \in \mathbb{N}$   
 $\sum_{r=1}^n (2r-1)^2 = \frac{1}{3}n(2n-1)(2n+1)$   
 $\sum_{r=1}^n (2r-1)^2 + (2(n+1)-1)^2 = \frac{1}{3}n(2n-1)(2n+1) + (2(n+1)-1)^2$   
 $\sum_{r=1}^n (2r-1)^2 = \frac{1}{3}n(2n-1)(2n+1) + (2n+1)^2$   
 $\sum_{r=1}^n (2r-1)^2 = \frac{1}{3}(n+1)[n(2n-1) + 3(2n+1)]$   
 $\sum_{r=1}^n (2r-1)^2 = \frac{1}{3}(n+1)(2n^2-1n+6n+3)$   
 $\sum_{r=1}^n (2r-1)^2 = \frac{1}{3}(n+1)(2n^2+5n+3)$   
 $\sum_{r=1}^n (2r-1)^2 = \frac{1}{3}(n+1)(2n+1)(2n+3)$
- If the result holds for  $n \in \mathbb{N} \Rightarrow$  it also holds for  $n+1$   
 Since the result holds for  $n=1 \Rightarrow$  it must hold  $\forall n \in \mathbb{N}$  ✓

**Question 7 (\*\*\*)**

Prove by induction that

$$\sum_{r=1}^n r(3r-1) = n^2(n+1), \quad n \geq 1, n \in \mathbb{N}.$$

proof

$\sum_{r=1}^n r(3r-1) = n^2(n+1)$

- If  $n=1$  LHS =  $\sum_{r=1}^1 r(3r-1) = 1 \times 2 = 2$  RHS =  $1^2(1+1) = 2$  ✓ Both sides are equal
- Suppose the result holds for  $n=k \in \mathbb{N}$   
 $\sum_{r=1}^k r(3r-1) = k^2(k+1)$   
 $\sum_{r=1}^{k+1} r(3r-1) = \sum_{r=1}^k r(3r-1) + (k+1)(3(k+1)-1)$   
 $\sum_{r=1}^{k+1} r(3r-1) = k^2(k+1) + (k+1)(3k+2)$   
 $\sum_{r=1}^{k+1} r(3r-1) = (k+1)[k^2 + 3k + 2]$   
 $\sum_{r=1}^{k+1} r(3r-1) = (k+1)(k+1)(k+2)$   
 $\sum_{r=1}^{k+1} r(3r-1) = (k+1)^2(k+2)$
- If the result holds for  $n=k \in \mathbb{N} \Rightarrow$  result holds for  $n=k+1$   
 Since the result holds for  $n=1 \Rightarrow$  the result holds for  $n \in \mathbb{N}$

**Question 8 (\*\*\*)**

Prove by induction that

$$\sum_{r=1}^n \frac{1}{r(r+1)} = \frac{n}{n+1}, \quad n \geq 1, n \in \mathbb{N}.$$

proof

$\sum_{r=1}^n \frac{1}{r(r+1)} = \frac{n}{n+1}$

- If  $n=1$  LHS =  $\sum_{r=1}^1 \frac{1}{r(r+1)} = \frac{1}{1 \times 2} = \frac{1}{2}$  RHS =  $\frac{1}{1+1} = \frac{1}{2}$  ✓ Both sides are equal
- Suppose the result holds for  $n=k \in \mathbb{N}$   
 $\sum_{r=1}^k \frac{1}{r(r+1)} = \frac{k}{k+1}$   
 $\sum_{r=1}^{k+1} \frac{1}{r(r+1)} = \sum_{r=1}^k \frac{1}{r(r+1)} + \frac{1}{(k+1)(k+2)}$   
 $\sum_{r=1}^{k+1} \frac{1}{r(r+1)} = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$   
 $\sum_{r=1}^{k+1} \frac{1}{r(r+1)} = \frac{k(k+2) + 1}{(k+1)(k+2)}$   
 $\sum_{r=1}^{k+1} \frac{1}{r(r+1)} = \frac{k^2 + 2k + 1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}$   
 $\sum_{r=1}^{k+1} \frac{1}{r(r+1)} = \frac{k+1}{k+2}$
- If the result holds for  $n=k \in \mathbb{N} \Rightarrow$  it also holds for  $n=k+1$   
 Since the result holds for  $n=1 \Rightarrow$  it holds for  $n \in \mathbb{N}$

Question 9 (\*\*\*)

Prove by induction that

$$\sum_{r=1}^n (3^{r-1}) = \frac{3^n - 1}{2}, \quad n \geq 1, n \in \mathbb{N}.$$

proof

Handwritten proof for the induction problem:

$\sum_{r=1}^n (3^{r-1}) = \frac{3^n - 1}{2} \quad n \in \mathbb{N}$

- Base case,  $n=1$   
 $LHS = 3^{1-1} = 3^0 = 1$   
 $RHS = \frac{3^1 - 1}{2} = \frac{3 - 1}{2} = 1$  } Result holds for  $n=1$
- Suppose that the result holds for  $n=k \in \mathbb{N}$   
 $\Rightarrow \sum_{r=1}^k (3^{r-1}) = \frac{3^k - 1}{2}$   
 $\Rightarrow \left[ \sum_{r=1}^k 3^{r-1} \right] + 3^{k-1} = \frac{3^k - 1}{2} + 3^{k-1}$   
 $\Rightarrow \sum_{r=1}^k 3^{r-1} + 3^{k-1} = \frac{3^k - 1}{2} + \frac{2 \cdot 3^{k-1}}{2}$   
 $\Rightarrow \sum_{r=1}^k 3^{r-1} + 3^{k-1} = \frac{3^k - 1 + 2 \cdot 3^k - 2 \cdot 3^{k-1}}{2}$   
 $\Rightarrow \sum_{r=1}^k 3^{r-1} + 3^{k-1} = \frac{3^k - 1 + 2 \cdot 3^k - 2 \cdot 3^{k-1}}{2}$   
 $\Rightarrow \sum_{r=1}^k 3^{r-1} + 3^{k-1} = \frac{3^k - 1 + 2 \cdot 3^k - 2 \cdot 3^{k-1}}{2}$
- If the result holds for  $n=k \in \mathbb{N}$ , then it also holds for  $n=k+1$   
 Since the result holds for  $n=1$ , then it will hold for all  $n \in \mathbb{N}$

Question 10 (\*\*\*)

Prove by induction that

$$\sum_{r=1}^n \frac{r}{2^r} = 2 - \frac{n+2}{2^n}, \quad n \geq 1, n \in \mathbb{N}.$$

□, proof

SPM Unit 1: The Base Case, n=1

L.H.S. =  $\sum_{r=1}^1 \frac{r}{2^r} = \frac{1}{2} = 0.5$       R.H.S. =  $2 - \frac{1+2}{2^1} = 2 - \frac{3}{2} = 0.5$

∴ The result holds for n=1

Suppose that the result holds for n=k ∈ ℕ

→  $\sum_{r=1}^k \frac{r}{2^r} = 2 - \frac{k+2}{2^k}$

→  $\left[ \sum_{r=1}^k \frac{r}{2^r} \right] + \frac{k+1}{2^{k+1}} = 2 - \frac{k+2}{2^k} + \frac{k+1}{2^{k+1}}$

→  $\sum_{r=1}^{k+1} \frac{r}{2^r} = 2 + \left[ \frac{k+1}{2^{k+1}} - \frac{k+2}{2^k} \right] = 2 + \left[ \frac{(k+1) - 2(k+2)}{2^{k+1}} \right]$

→  $\sum_{r=1}^{k+1} \frac{r}{2^r} = 2 + \frac{k-3}{2^{k+1}} = 2 - \frac{k+2}{2^{k+1}}$

→  $\sum_{r=1}^{k+1} \frac{r}{2^r} = 2 - \frac{(k+1)+2}{2^{k+1}}$

If the result holds for n=k ∈ ℕ, then it must also hold for n=k+1  
Since the result holds for n=1, then it must hold for all n ∈ ℕ

**Question 11** (\*\*\*)

Prove by induction that

$$\sum_{r=1}^n \frac{1}{4r^2 - 1} = \frac{n}{2n+1}, \quad n \geq 1, n \in \mathbb{N}.$$

□, proof

$\sum_{r=1}^n \left( \frac{1}{4r^2-1} \right) = \frac{n}{2n+1}$

- TESTING THE BASE CASE, i.e.  $n=1$   
 $LHS = \sum_{r=1}^1 \frac{1}{4r^2-1} = \frac{1}{4(1)^2-1} = \frac{1}{3}$   
 $RHS = \frac{1}{2(1)+1} = \frac{1}{3}$   
 $\therefore$  THE RESULT HOLDS FOR  $n=1$
- SUPPOSE THAT THE RESULT HOLDS FOR  $n=k \in \mathbb{N}$   
 $\sum_{r=1}^k \left( \frac{1}{4r^2-1} \right) = \frac{k}{2k+1}$   
 $\sum_{r=1}^{k+1} \left( \frac{1}{4r^2-1} \right) = \frac{k}{2k+1} + \frac{1}{4(k+1)^2-1}$   
 $\sum_{r=1}^{k+1} \left( \frac{1}{4r^2-1} \right) = \frac{k}{2k+1} + \frac{1}{[2(k+1)][2(k+1)-1]}$   
 $\sum_{r=1}^{k+1} \left( \frac{1}{4r^2-1} \right) = \frac{k}{2k+1} + \frac{1}{(2k+2)(2k+1)}$   
 $\sum_{r=1}^{k+1} \left( \frac{1}{4r^2-1} \right) = \frac{k(2k+2)+1}{(2k+1)(2k+2)} = \frac{2k^2+2k+1}{(2k+1)(2k+2)}$   
 $\sum_{r=1}^{k+1} \left( \frac{1}{4r^2-1} \right) = \frac{k+1}{2k+3} = \frac{k+1}{2(k+1)+1}$
- IF THE RESULT HOLDS FOR  $n=k \in \mathbb{N}$ , THEN IT ALSO HOLDS FOR  $n=k+1$   
 SINCE THE RESULT HOLDS FOR  $n=1$ , THEN IT MUST HOLD  $\forall n \in \mathbb{N}$

**Question 12** (\*\*\*)

Prove by induction that

$$\sum_{r=1}^n r \times 2^r = 2 + (n-1)2^{n+1}, \quad n \geq 1, n \in \mathbb{N}.$$

proof

$\sum_{r=1}^n r \times 2^r = 2 + (n-1)2^{n+1}$

- BASE CASE  $n=1$   
 $LHS = 1 \times 2^1 = 2$   
 $RHS = 2 + (1-1) \times 2^2 = 2$  ✓
- SUPPOSE THE RESULT HOLDS FOR  $n=k \in \mathbb{N}$   
 $\sum_{r=1}^k r \times 2^r = 2 + (k-1) \times 2^{k+1}$   
 $\sum_{r=1}^{k+1} r \times 2^r = 2 + (k-1) \times 2^{k+1} + (k+1) \times 2^{k+1}$   
 $\sum_{r=1}^{k+1} r \times 2^r = 2 + 2^k \times (k-1 + k+1)$   
 $\sum_{r=1}^{k+1} r \times 2^r = 2 + 2^k \times 2k$   
 $\sum_{r=1}^{k+1} r \times 2^r = 2 + 2 \times 2^k \times k$
- IF THE RESULT HOLDS FOR  $n=k \in \mathbb{N}$ , THEN IT ALSO HOLDS FOR  $n=k+1$   
 SINCE IT HOLDS FOR  $n=1$ , THEN IT MUST HOLD FOR ALL  $n$

**Question 13 (\*\*\*)**

Prove by induction that

$$\sum_{r=1}^n [(r+1) \times 2^r] = n \times 2^n, \quad n \geq 1, n \in \mathbb{N}.$$

proof

• IF  $n=1$  LHS =  $(1+1) \times 2^1 = 2 \times 1 = 2$  DERIV AXIS FOR  $n=1$   
 RHS =  $1 \times 2^1 = 2$   
 • SUPPOSE THE RESULT HOLDS FOR  $n=k \in \mathbb{N}$   

$$\sum_{r=1}^k (r+1) 2^r = k \times 2^k$$

$$\sum_{r=1}^{k+1} (r+1) 2^r = \sum_{r=1}^k (r+1) 2^r + (k+1) 2^{k+1}$$

$$\sum_{r=1}^{k+1} (r+1) 2^r = k \times 2^k + (k+1) 2^{k+1}$$

$$\sum_{r=1}^{k+1} (r+1) 2^r = 2^k [k + (k+1) \times 2]$$

$$\sum_{r=1}^{k+1} (r+1) 2^r = 2^k (2k+2) = 2 \times 2^k (k+1)$$

$$\sum_{r=1}^{k+1} (r+1) 2^r = 2^{k+1} (k+1) = (k+1) \times 2^{k+1}$$
 • IF THE RESULT HOLDS FOR  $k \in \mathbb{N}$ , THEN IT MUST ALSO HOLD FOR  $n=k+1$  SINCE THE RESULT HOLDS FOR  $n=1$ , THEN IT MUST HOLD FOR ALL  $n$

**Question 14 (\*\*\*)**

If  $n \geq 1, n \in \mathbb{N}$ , prove by induction that

$$1 \times 1! + 2 \times 2! + 3 \times 3! + \dots + n \times n! = (n+1)! - 1.$$

proof

$$\sum_{r=1}^n (r \times r!) + (3 \times 3!) + \dots + (n \times n!) = (n+1)! - 1$$

$$\sum_{r=1}^n r \times r! = (n+1)! - 1$$
 • IF  $n=1$  LHS =  $1 \times 1! = 1$   
 RHS =  $(1+1)! - 1 = 2! - 1 = 2 - 1 = 1$   
 • SUPPOSE THE RESULT HOLDS FOR  $n=k \in \mathbb{N}$   

$$\sum_{r=1}^k r \times r! = (k+1)! - 1$$

$$\sum_{r=1}^{k+1} r \times r! = \sum_{r=1}^k r \times r! + (k+1) \times (k+1)! = (k+1)! - 1 + (k+1) \times (k+1)!$$

$$\sum_{r=1}^{k+1} r \times r! = (k+1)! [1 + (k+1)] - 1$$

$$\sum_{r=1}^{k+1} r \times r! = (k+1)! (k+2) - 1$$

$$\sum_{r=1}^{k+1} r \times r! = (k+2)! - 1$$

$$\sum_{r=1}^{k+1} r \times r! = (k+2)! - 1$$
 • IF THE RESULT HOLDS FOR  $n=k \in \mathbb{N}$ , THEN IT ALSO HOLDS FOR  $n=k+1$  SINCE THE RESULT HOLDS FOR  $n=1$ , THEN IT MUST HOLD FOR ALL  $n$

Question 15 (\*\*\*)

Prove by induction that

$$\sum_{r=1}^n r \times 2^{-r} = 2 - (n+2)2^{-n}, \quad n \geq 1, n \in \mathbb{N}.$$

proof

Handwritten proof for the induction problem:

$\sum_{r=1}^n (r \times 2^{-r}) = 2 - (n+2)2^{-n}$

- If  $n=1$ , LHS =  $1 \times 2^{-1} = \frac{1}{2}$   
RHS =  $2 - (1+2) \times 2^{-1} = 2 - 3 \times \frac{1}{2} = \frac{1}{2} \Rightarrow$  RESULT FOR  $n=1$
- ASSUME THE RESULT FOR  $n=0$   $n=1$   $n \in \mathbb{N}$   
 $\Rightarrow \sum_{r=1}^k (r \times 2^{-r}) = 2 - (k+2)2^{-k}$   
 $\Rightarrow \sum_{r=1}^{k+1} (r \times 2^{-r}) + (k+1)2^{-(k+1)} = 2 - (k+2)2^{-k} + (k+1)2^{-(k+1)}$   
 $\Rightarrow \sum_{r=1}^{k+1} (r \times 2^{-r}) = 2 - (k+2)2^{-k} + (k+1)2^{-k-1}$   
 $\Rightarrow \sum_{r=1}^{k+1} (r \times 2^{-r}) = 2 + \frac{1}{2}2^{-k} [(k+1) - 2(k+2)]$   
 $\Rightarrow \sum_{r=1}^{k+1} (r \times 2^{-r}) = 2 + 2^{-k-1} [-k-3]$   
 $\Rightarrow \sum_{r=1}^{k+1} (r \times 2^{-r}) = 2 - 2^{-k-1} (k+3)$   
 $\Rightarrow \sum_{r=1}^{k+1} (r \times 2^{-r}) = 2 - 2^{-(k+1)} [(k+3)+2]$
- IF THE RESULT FOR  $n \in \mathbb{N}$ , THEN IT ALSO HOLDS FOR  $n+1$   
SINCE THE RESULT FOR  $n=1$  IS TRUE IT MUST HOLD  $\forall n \in \mathbb{N}$

Question 16 (\*\*\*)

Prove by induction that

$$\sum_{r=1}^n \frac{2r^2-1}{r^2(r+1)^2} = \frac{n^2}{(n+1)^2}, \quad n \geq 1, n \in \mathbb{N}.$$

Q.P., proof

$\sum_{r=1}^n \frac{2r^2-1}{r^2(r+1)^2} = \frac{n^2}{(n+1)^2}, n \geq 1$

BASE CASE; n=1

- LHS =  $\frac{2(1)^2-1}{1^2(1+1)^2} = \frac{1}{4}$
- RHS =  $\frac{1^2}{(1+1)^2} = \frac{1}{4}$

INDUCTIVE HYPOTHESIS

SUPPOSE THAT THE RESULT HELDS FOR  $n=k, k \in \mathbb{N}$

$\Rightarrow \sum_{r=1}^k \frac{2r^2-1}{r^2(r+1)^2} = \frac{k^2}{(k+1)^2}$

$\Rightarrow \sum_{r=1}^k \frac{2r^2-1}{r^2(r+1)^2} + \frac{2(k+1)^2-1}{(k+1)^2(k+2)^2} = \frac{k^2}{(k+1)^2} + \frac{2(k+1)^2-1}{(k+1)^2(k+2)^2}$

$\Rightarrow \sum_{r=1}^{k+1} \frac{2r^2-1}{r^2(r+1)^2} = \frac{k^2(k+2)^2 + 2(k+1)^2-1}{(k+1)^2(k+2)^2}$

$\Rightarrow \sum_{r=1}^{k+1} \frac{2r^2-1}{r^2(r+1)^2} = \frac{k^2(k+2)^2 + 2k^2 + 4k + 1}{(k+1)^2(k+2)^2}$

$\Rightarrow \sum_{r=1}^{k+1} \frac{2r^2-1}{r^2(r+1)^2} = \frac{k^4 + 4k^3 + 4k^2 + 2k^2 + 4k + 1}{(k+1)^2(k+2)^2}$

$\Rightarrow \sum_{r=1}^{k+1} \frac{2r^2-1}{r^2(r+1)^2} = \frac{k^4 + 4k^3 + 6k^2 + 4k + 1}{(k+1)^2(k+2)^2}$

BY INSPECTION

$(k+1)^4 = k^4 + 4k^3 + 6k^2 + 4k + 1$

RETURNING TO THE MAIN QN

$\Rightarrow \sum_{r=1}^{k+1} \frac{2r^2-1}{r^2(r+1)^2} = \frac{(k+1)^4}{(k+1)^2(k+2)^2}$

$\Rightarrow \sum_{r=1}^{k+1} \frac{2r^2-1}{r^2(r+1)^2} = \frac{(k+1)^2}{(k+2)^2}$

$\Rightarrow \sum_{r=1}^{k+1} \frac{2r^2-1}{r^2(r+1)^2} = \frac{(k+1)^2}{(k+2)^2}$

CONCLUSION

- IF THE RESULT HELDS FOR  $n=k \in \mathbb{N}$ , THEN IT ALSO HELDS FOR  $n=k+1$
- SINCE THE RESULT HELDS FOR  $n=2$ , THEN IT MUST HOLD FOR ALL  $n$

Question 17 (\*\*\*\*)

Prove by induction that

$$\sum_{r=1}^n [r(r-1)-1] = \frac{1}{3}n(n+2)(n-2), \quad n \geq 1, n \in \mathbb{N}.$$

proof

$\bullet$  LHS =  $((1-1)-1) = -1$      RHS =  $\frac{1}{3} \times 1 \times (1+2)(1-2) = -1$   
 $\therefore$  Both sides are  $-1$

$\bullet$  Suppose the result holds for  $n \in \mathbb{N}$ ;  
 $[r(r-1)-1] = \frac{1}{3}k(k+2)(k-2)$

$\therefore$  For  $n+1$   
 $[r(r-1)-1] + [(k+1)-1] = \frac{1}{3}k(k+2)(k-2) + (k+1)-1$

$\therefore$  For  $n+1$   
 $[r(r-1)-1] = \frac{1}{3}k(k+2)(k-2) + k^2 - 1$   
 $= \frac{1}{3}[k^3 - k + 2k^2 - 2k - 3]$   
 $= \frac{1}{3}[k^3 + 2k^2 - k - 3]$   
 $= \frac{1}{3}[k^3 + 2k^2 - k + 3]$   
 $= \frac{1}{3}(k+3)(k^2 - 1)$   
 $= \frac{1}{3}(k+3)(k+1)(k-1)$

$\therefore$  For  $n+1$   
 $[r(r-1)-1] = \frac{1}{3}(k+1)(k+2)(k+1-2)$

$\bullet$  If the result holds for  $n \in \mathbb{N}$ , then it also holds for  $n+1$  since it holds for  $n=1$ , then it must hold  $\forall n \in \mathbb{N}$

**Question 18** (\*\*\*)

Prove by induction that

$$\sum_{r=1}^n \frac{r \times 2^r}{(r+2)!} = 1 - \frac{2^{n+1}}{(n+2)!}, \quad n \geq 1, n \in \mathbb{N}.$$

□, proof

SUPPOSE THAT THE RESULT HOLDS FOR  $n-1$

$$L.H.S = \sum_{r=1}^n \frac{r \times 2^r}{(r+2)!} = \frac{1 \times 2^1}{3!} = \frac{2}{6} = \frac{1}{3}$$

$$R.H.S = 1 - \frac{2^2}{3!} = 1 - \frac{4}{6} = 1 - \frac{2}{3} = \frac{1}{3}$$

IE RESULT HOLDS FOR  $n=1$

NOW SUPPOSE THAT THE RESULT HOLDS FOR  $n=k, k \in \mathbb{N}$

$$\sum_{r=1}^k \frac{r \times 2^r}{(r+2)!} = 1 - \frac{2^{k+1}}{(k+2)!}$$

$$\sum_{r=1}^{k+1} \frac{r \times 2^r}{(r+2)!} = \left[ \sum_{r=1}^k \frac{r \times 2^r}{(r+2)!} \right] + \frac{(k+1) \times 2^{k+1}}{(k+3)!} = 1 - \frac{2^{k+1}}{(k+2)!} + \frac{(k+1) \times 2^{k+1}}{(k+3)!}$$

$$\sum_{r=1}^{k+1} \frac{r \times 2^r}{(r+2)!} = 1 - \frac{(k+2) \times 2^{k+1}}{(k+3)(k+2)!} + \frac{(k+1) \times 2^{k+1}}{(k+3)!}$$

$$\sum_{r=1}^{k+1} \frac{r \times 2^r}{(r+2)!} = 1 - \frac{(k+2) \times 2^{k+1}}{(k+3)!} + \frac{(k+1) \times 2^{k+1}}{(k+3)!}$$

$$\sum_{r=1}^{k+1} \frac{r \times 2^r}{(r+2)!} = 1 + \frac{(k+1) \times 2^{k+1} - (k+2) \times 2^{k+1}}{(k+3)!}$$

$$\sum_{r=1}^{k+1} \frac{r \times 2^r}{(r+2)!} = 1 + \frac{-2 \times 2^{k+1}}{(k+3)!}$$

$$\sum_{r=1}^{k+1} \frac{r \times 2^r}{(r+2)!} = 1 - \frac{2^{k+2}}{(k+3)!} = 1 - \frac{2^{(k+1)+1}}{((k+1)+2)!}$$

IF THE RESULT HOLDS FOR  $n=k \in \mathbb{N}$ , THEN IT MUST ALSO HOLD FOR  $n=k+1$   
SINCE THE RESULT HOLDS FOR  $n=1$ , THEN THE RESULT HOLDS FOR ALL  $n$

**Question 19** (\*\*\*)

Prove by induction that

$$1^2 + 3^2 + 5^2 + 7^2 + \dots + (2n-1)^2 = \frac{1}{3}n(4n^2 - 1), \quad n \geq 1, n \in \mathbb{N}.$$

proof

- If  $n=1$   $\frac{1}{3} \times 1 \times (4 \times 1^2 - 1) = \frac{1}{3} \times 1 \times (4-1) = 1$  which is  $1^2$   
 ie RESULT HOLDS FOR  $n=1$
- SUPPOSE THE RESULT HOLDS FOR  $n=k \in \mathbb{N}$   
 $\Rightarrow 1^2 + 3^2 + 5^2 + \dots + (2k-1)^2 = \frac{1}{3}k(4k^2 - 1)$   
 $\Rightarrow 1^2 + 3^2 + 5^2 + \dots + (2k-1)^2 + (2k+1)^2 = \frac{1}{3}k(4k^2 - 1) + (2k+1)^2$   
 $\Rightarrow 1^2 + 3^2 + 5^2 + \dots + (2k+1)^2 = \frac{1}{3}k(4k^2 - 1) + (2k+1)^2$   
 $\Rightarrow 1^2 + 3^2 + 5^2 + \dots + (2k+1)^2 = \frac{1}{3}(4k^3 - k + 12k^2 + 4k + 1)$   
 $\Rightarrow 1^2 + 3^2 + 5^2 + \dots + (2k+1)^2 = \frac{1}{3}(4k^3 + 12k^2 + 4k + 1)$   
 $\Rightarrow 1^2 + 3^2 + 5^2 + \dots + (2k+1)^2 = \frac{1}{3}(4k^3 + 12k^2 + 4k + 1)$   
 $\Rightarrow 1^2 + 3^2 + 5^2 + \dots + (2k+1)^2 = \frac{1}{3}(4k^3 + 12k^2 + 4k + 1)$   
 $\Rightarrow 1^2 + 3^2 + 5^2 + \dots + (2k+1)^2 = \frac{1}{3}(4k^3 + 12k^2 + 4k + 1)$
- IF THE RESULT HOLDS FOR  $n=k \in \mathbb{N}$   $\Rightarrow$  IT ALSO HOLDS FOR  $n=k+1$   
 SINCE THE RESULT HOLDS FOR  $n=1$   $\Rightarrow$  THE RESULT HOLDS FOR ALL  $n \in \mathbb{N}$

Question 20 (\*\*\*\*)

Prove by mathematical induction that if  $n$  is a positive integer then

$$\sum_{r=1}^n (3r-2)^2 = \frac{1}{2}n(6n^2 - 3n - 1).$$

You may not use other methods of proof in this question.

,  proof

ESTABLISH A BASE CASE FOR  $n=1$

- LHS =  $\sum_{r=1}^1 (3r-2)^2 = (3 \times 1 - 2)^2 = 1$
- RHS =  $\frac{1}{2} \times 1 \times (6 \times 1^2 - 3 \times 1 - 1) = \frac{1}{2} \times 1 \times 2 = 1$   
 $\therefore$  THE RESULT HOLDS FOR  $n=1$

SUPPOSE THAT THE RESULT HOLDS FOR  $n=k \in \mathbb{N}$

$\Rightarrow \sum_{r=1}^k (3r-2)^2 = \frac{1}{2}k(6k^2 - 3k - 1)$

$\Rightarrow \sum_{r=1}^{k+1} (3r-2)^2 = \sum_{r=1}^k (3r-2)^2 + [3(k+1)-2]^2$

$\Rightarrow \sum_{r=1}^{k+1} (3r-2)^2 = \frac{1}{2}k(6k^2 - 3k - 1) + 9(k+1)^2$

$\Rightarrow \sum_{r=1}^{k+1} (3r-2)^2 = \frac{1}{2}k(6k^2 - 3k - 1) + 9k^2 + 18k + 9$

$\Rightarrow \sum_{r=1}^{k+1} (3r-2)^2 = \frac{1}{2}(k+1) \left[ \frac{6k^3 - 3k^2 - k + 18k^2 + 36k + 18}{k+1} \right]$

$\Rightarrow \sum_{r=1}^{k+1} (3r-2)^2 = \frac{1}{2}(k+1) \left[ \frac{6k^3 + 33k^2 + 35k + 18}{k+1} \right]$

BY LONG DIVISION OR MULTIPLICATION

$\Rightarrow \sum_{r=1}^{k+1} (3r-2)^2 = \frac{1}{2}(k+1) \left[ \frac{6k^2(k+1) + 27k(k+1) + 18(k+1)}{k+1} \right]$

$\Rightarrow \sum_{r=1}^{k+1} (3r-2)^2 = \frac{1}{2}(k+1)(6k^2 + 33k + 18)$

MULTIPLIATE FLUENTLY

$\Rightarrow \sum_{r=1}^k (3r-2)^2 = \frac{1}{2}(k+1)(6k^2 + 33k + 18)$

$\Rightarrow \sum_{r=1}^k (3r-2)^2 = \frac{1}{2}(k+1)[6k^2 + 33k + 18]$

$\Rightarrow \sum_{r=1}^{k+1} (3r-2)^2 = \frac{1}{2}(k+1)[6(k+1)^2 - 3(k+1) - 1]$

$\Rightarrow \sum_{r=1}^{k+1} (3r-2)^2 = \frac{1}{2}(k+1)[6k^2 + 12k + 6 - 3k - 3 - 1]$

IF THE RESULT HOLDS FOR  $n=k \in \mathbb{N}$ , THEN IT MUST ALSO HOLD FOR  $n=k+1$

SINCE THE RESULT HOLDS FOR  $n=1$ , THEN IT MUST ALSO HOLD FOR ALL  $n$

Question 21 (\*\*\*\*)

Prove by mathematical induction that if  $n$  is a positive integer then

$$\sum_{r=1}^n \frac{3r+2}{r(r+1)(r+2)} = \frac{n(2n+3)}{(n+1)(n+2)}$$

You may not use other methods of proof in this question.

, proof

STATE THE BASE CASE

$P(1) = \sum_{r=1}^1 \frac{3r+2}{r(r+1)(r+2)} = \frac{3(1)+2}{1(2)(3)} = \frac{5}{6}$   
 $P(1) = \frac{1(2(1)+3)}{(1+1)(1+2)} = \frac{1(5)}{2(3)} = \frac{5}{6}$  } THE RESULT HOLDS FOR  $n=1$

SUPPOSE THAT THE RESULT HOLDS FOR  $n=k \in \mathbb{N}$

$\sum_{r=1}^k \frac{3r+2}{r(r+1)(r+2)} = \frac{5(2k+3)}{(k+1)(k+2)}$  ✓  $P(k)$  ✓

$\sum_{r=1}^{k+1} \frac{3r+2}{r(r+1)(r+2)} = \frac{5(2k+3)}{(k+1)(k+2)} + \frac{3(k+1)+2}{(k+1)(k+2)(k+3)}$

$\sum_{r=1}^{k+1} \frac{3r+2}{r(r+1)(r+2)} = \frac{5(2k+3)}{(k+1)(k+2)} + \frac{3k+5}{(k+1)(k+2)(k+3)}$

$\sum_{r=1}^{k+1} \frac{3r+2}{r(r+1)(r+2)} = \frac{5(2k+3)(k+3) + (3k+5)(k+2)}{(k+1)(k+2)(k+3)}$

$\sum_{r=1}^{k+1} \frac{3r+2}{r(r+1)(r+2)} = \frac{2k^3 + 9k^2 + 9k + 5k^2 + 10k + 5}{(k+1)(k+2)(k+3)}$

$\sum_{r=1}^{k+1} \frac{3r+2}{r(r+1)(r+2)} = \frac{2k^3 + 14k^2 + 19k + 5}{(k+1)(k+2)(k+3)}$

YOU WOULD EXPECT THAT  $P(k+1)$  IS A FACTOR FOR THE INDICATOR TO WORK

$\sum_{r=1}^{k+1} \frac{3r+2}{r(r+1)(r+2)} = \frac{2k^3 + 14k^2 + 19k + 5}{(k+1)(k+2)(k+3)}$

$\sum_{r=1}^{k+1} \frac{3r+2}{r(r+1)(r+2)} = \frac{(k+1)(2k^2 + 7k + 5)}{(k+1)(k+2)(k+3)}$

$\sum_{r=1}^k \frac{3r+2}{r(r+1)(r+2)} = \frac{2k^2 + 7k + 5}{(k+2)(k+3)}$

$\sum_{r=1}^{k+1} \frac{3r+2}{r(r+1)(r+2)} = \frac{(2k+3)(k+1)}{(k+2)(k+3)}$

$\sum_{r=1}^{k+1} \frac{3r+2}{r(r+1)(r+2)} = \frac{(k+1)(2k+3)}{(k+2)(k+3)}$

IF THE RESULT DOES FOR  $n=k \in \mathbb{N}$ , THEN IT MUST ALSO HOLD FOR  $n=k+1$   
 SINCE THE RESULT HOLDS FOR  $n=1$ , THEN IT MUST HOLD FOR ALL  $n$

Question 22 (\*\*\*)

Prove by induction that

$$\sum_{r=1}^n \left[ r(r+1) \left( \frac{1}{2} \right)^{r-1} \right] = 16 - \left( \frac{1}{2} \right)^{n-1} (n^2 + 5n + 8), \quad n \geq 1, n \in \mathbb{N}.$$

1/2, proof

BASE CASE n=1

- $\sum_{r=1}^1 [r(r+1) \left(\frac{1}{2}\right)^{r-1}] = 1 \times 2 \times \left(\frac{1}{2}\right)^0 = 2$
- $16 - \left(\frac{1}{2}\right)^{1-1} (1^2 + 5 \times 1 + 8) = 16 - \left(\frac{1}{2}\right)^0 (1 + 5 + 8) = 16 - 1 \times 14 = 2$

IS THE RESULT TRUE FOR n=1

SUPPOSE THAT THE RESULT HOLDS FOR n=k, k ∈ ℕ

→  $\sum_{r=1}^k [r(r+1) \left(\frac{1}{2}\right)^{r-1}] = 16 - \left(\frac{1}{2}\right)^{k-1} (k^2 + 5k + 8)$

→  $\sum_{r=1}^k [r(r+1) \left(\frac{1}{2}\right)^{r-1}] + \frac{(k+1)(k+2) \left(\frac{1}{2}\right)^k}{1} = 16 - \left(\frac{1}{2}\right)^{k-1} (k^2 + 5k + 8) + (k+1)(k+2) \left(\frac{1}{2}\right)^k$

→  $\sum_{r=1}^{k+1} [r(r+1) \left(\frac{1}{2}\right)^{r-1}] = 16 + \left(\frac{1}{2}\right)^k [(k+1)(k+2) - (k^2 + 5k + 8)]$

→  $\sum_{r=1}^{k+1} [r(r+1) \left(\frac{1}{2}\right)^{r-1}] = 16 + \left(\frac{1}{2}\right)^k [(k^2 + 3k + 2) - (k^2 + 5k + 8)]$

→  $\sum_{r=1}^{k+1} [r(r+1) \left(\frac{1}{2}\right)^{r-1}] = 16 + \left(\frac{1}{2}\right)^k [k^2 + 3k + 2 - k^2 - 5k - 8]$

→  $\sum_{r=1}^{k+1} [r(r+1) \left(\frac{1}{2}\right)^{r-1}] = 16 + \left(\frac{1}{2}\right)^k [-2k - 6]$

→  $\sum_{r=1}^{k+1} [r(r+1) \left(\frac{1}{2}\right)^{r-1}] = 16 - \left(\frac{1}{2}\right)^k (k^2 + 7k + 4)$

→  $\sum_{r=1}^{k+1} [r(r+1) \left(\frac{1}{2}\right)^{r-1}] = 16 - \left(\frac{1}{2}\right)^k [(k+1)(k+2) + 5(k+1) + 8]$

→  $\sum_{r=1}^{k+1} [r(r+1) \left(\frac{1}{2}\right)^{r-1}] = 16 - \left(\frac{1}{2}\right)^k [k^2 + 7k + 4 + 5k + 5 + 8]$

→  $\sum_{r=1}^{k+1} [r(r+1) \left(\frac{1}{2}\right)^{r-1}] = 16 - \left(\frac{1}{2}\right)^k [k^2 + 12k + 17]$

IF THE RESULT HOLDS FOR n=k ∈ ℕ, THEN IT MUST ALSO HOLD FOR n=k+1 — SINCE THE RESULT HOLDS FOR n=1 THEN IT MUST HOLD FOR ALL n ∈ ℕ

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# DIVISIBILITY RESULTS

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Question 1 (\*\*)

$$f(n) = 7^n + 5, n \in \mathbb{N}.$$

Prove by induction that  $f(n)$  is divisible by 6, for all  $n \in \mathbb{N}$ .

proof

$$f(n) = 7^n + 5$$

- $f(1) = 7 + 5 = 12$  which is divisible by 6
- suppose the result holds for  $n = k \in \mathbb{N}$ , i.e.  $f(k) = 6m$ ,  $m \in \mathbb{N}$

$$f(k+1) - f(k) = [7^{k+1} + 5] - [7^k + 5]$$

$$f(k+1) - 6m = 7^{k+1} - 7^k$$

$$f(k+1) - 6m = 7^k \cdot 7 - 7^k$$

$$f(k+1) = 6m + 6 \cdot 7^k$$

$$f(k+1) = 6[m + 7^k]$$

- if the result holds for  $n = k \in \mathbb{N}$ , then it must hold for  $n = k+1$  since the result holds for  $n=1$ , then it must hold for all  $n \in \mathbb{N}$

Question 2 (\*\*)

$$f(n) = 6^n + 4, n \in \mathbb{N}.$$

Prove by induction that  $f(n)$  is divisible by 5, for all  $n \in \mathbb{N}$ .

proof

$$f(n) = 6^n + 4$$

- $f(1) = 6 + 4 = 10 = 5 \times 2$   
i.e. result holds for  $n=1$
- suppose the result holds for  $n = k \in \mathbb{N}$ , i.e.  $f(k) = 5m$ , where  $m \in \mathbb{N}$

$$f(k+1) - f(k) = [6^{k+1} + 4] - [6^k + 4]$$

$$\Rightarrow f(k+1) - 5m = 6^{k+1} - 6^k$$

$$\Rightarrow f(k+1) - 5m = 6 \times 6^k - 6^k$$

$$\Rightarrow f(k+1) = 5m + 5 \times 6^k$$

$$\Rightarrow f(k+1) = 5[m + 6^k]$$

- if the result holds for  $n = k \in \mathbb{N}$ , then it also holds for  $n = k+1$ .
- since it holds for  $n=1$ , then it must hold for all  $n \in \mathbb{N}$

**Question 3 (\*\*)**

$$f(n) = 5^n + 3, n \in \mathbb{N}.$$

Prove by induction that  $f(n)$  is divisible by 4, for all  $n \in \mathbb{N}$ .

proof

$f(n) = 5^n + 3$   
 $f(1) = 5^1 + 3 = 8$  i.e. divisible by 4  
 • SUPPOSE THE RESULT HOLDS FOR  $n = k \in \mathbb{N}$ , SO  $f(k) = 4m, m \in \mathbb{Z}$   
 $\rightarrow f(k+1) - f(k) = [5^{k+1} + 3] - [5^k + 3]$   
 $\rightarrow f(k+1) - f(k) = 5^{k+1} - 5^k$   
 $\rightarrow f(k+1) - f(k) = 4m + 4 \times 5^k - 5^k$   
 $\rightarrow f(k+1) - f(k) = 4m + 3 \times 5^k$   
 $\rightarrow f(k+1) = 4(m + 3 \times 5^k)$   
 • IF THE RESULT HOLDS FOR  $n = k \in \mathbb{N}$ , THEN IT MUST ALSO HOLD FOR  $n = k+1$   
 SINCE THE RESULT HOLDS FOR  $n=1$ , THEN IT MUST HOLD FOR ALL  $n \in \mathbb{N}$

**Question 4 (\*\*)**

Prove by induction that for all natural numbers  $n$ ,

$$4^{2n} - 1$$

is divisible by 15.

proof

$f(n) = 4^{2n} - 1, n \in \mathbb{N}$   
BASE CASE;  $n=1$   
 $f(1) = 4^{2 \times 1} - 1 = 15$ , i.e. THE RESULT HOLDS FOR  $n=1$   
INDUCTIVE HYPOTHESIS  
 SUPPOSE THAT THE RESULT HOLDS FOR  $n=k, k \in \mathbb{N}$ , i.e.  $f(k) = 15m$   
 WHERE  $m \in \mathbb{N}$   
 $\Rightarrow f(k+1) - f(k) = [4^{2(k+1)} - 1] - [4^{2k} - 1]$   
 $\Rightarrow f(k+1) - f(k) = 4^{2(k+1)} - 4^{2k}$   
 $\Rightarrow f(k+1) - 15m = 4^{2k+2} - 4^{2k}$   
 $\rightarrow f(k+1) - 15m = 4^{2k} \times 4^2 - 4^{2k}$   
 $\rightarrow f(k+1) - 15m = 15m + 16 \times 4^{2k} - 4^{2k}$   
 $\Rightarrow f(k+1) = 15m + 15 \times 4^{2k}$   
 $\rightarrow f(k+1) = 15[m + 4^{2k}]$   
CONCLUSION  
 • IF THE RESULT HOLDS FOR  $n = k \in \mathbb{N}$ , THEN IT ALSO HOLDS FOR  $n = k+1$   
 • SINCE THE RESULT HOLDS FOR  $n=1$ , THEN IT MUST HOLD FOR ALL  $n \in \mathbb{N}$

**Question 5 (\*\*)**

Prove by induction that for all natural numbers  $n$ ,

$$7^{2n-1} + 1$$

is divisible by 8.

proof

Let  $f(n) = 7^{2n-1} + 1$

- $f(1) = 7^1 + 1 = 8$  i.e. divisible by 8
- Suppose the result holds for  $n \in \mathbb{N}$ , i.e.  $f(n) = 8m$ ,  $m \in \mathbb{N}$   
 $f(n+1) - f(n) = [7^{2(n+1)-1} + 1] - [7^{2n-1} + 1]$   
 $f(n+1) - 8m = 7^{2n+1} - 7^{2n-1}$   
 $f(n+1) = 8m + 7^2 \cdot 7^{2n-1} - 7^{2n-1}$   
 $f(n+1) = 8m + 48 \cdot 7^{2n-1} - 7^{2n-1}$   
 $f(n+1) = 8m + 47 \cdot 7^{2n-1}$   
 $f(n+1) = 8[m + 6 \cdot 7^{2n-1}]$
- If the result holds for  $n \in \mathbb{N}$ , then it also holds for  $n=k+1$   
 Since the result holds for  $n=1$ , then it must hold for  $n \in \mathbb{N}$

**Question 6 (\*\*)**

Prove by induction that for all natural numbers  $n$ ,

$$3^{2n} + 7 \text{ is divisible by } 8.$$

proof

Let  $f(n) = 3^{2n} + 7$

- $f(1) = 3^2 + 7 = 16$  is divisible by 8
- Suppose that the result holds for  $n \in \mathbb{N}$ , i.e.  $f(n) = 8m$  for  $m \in \mathbb{Z}$   
 $f(n+1) - f(n) = [3^{2(n+1)} + 7] - [3^{2n} + 7]$   
 $f(n+1) - 8m = 3^{2n+2} - 3^{2n}$   
 $f(n+1) = 8m + 3^2 \cdot 3^{2n} - 3^{2n}$   
 $f(n+1) = 8m + 9 \cdot 3^{2n} - 3^{2n}$   
 $f(n+1) = 8m + 8 \cdot 3^{2n}$   
 $f(n+1) = 8[m + 3^{2n}]$  is a multiple of 8
- If the result holds for  $n \in \mathbb{N}$ , then it also holds for  $n=k+1$   
 Since the result holds for  $n=1$ , then it must hold for  $n \in \mathbb{N}$

Question 7 (\*\*\*)

$$f(n) = 3^{2n} - 1, n \in \mathbb{N}.$$

Prove by induction that  $f(n)$  is a multiple of 8, for all  $n \in \mathbb{N}$ .

proof

$f(n) = 3^{2n} - 1$   
 •  $f(1) = 3^2 - 1 = 8 = 8 \times 1$  is multiple of 8  
 • suppose the result holds for  $n=k \in \mathbb{N}$ , i.e.  $f(k) = 8m, m \in \mathbb{N}$   
 $\Rightarrow f(k+1) - f(k) = [3^{2(k+1)} - 1] - [3^{2k} - 1]$   
 $\Rightarrow f(k+1) - 8m = 3^{2k+2} - 3^{2k}$   
 $\Rightarrow f(k+1) - 8m = 3^{2k} \cdot 3^2 - 3^{2k}$   
 $\Rightarrow f(k+1) - 8m = 9 \times 3^{2k} - 3^{2k}$   
 $\Rightarrow f(k+1) - 8m = 8 \times 3^{2k}$   
 $\Rightarrow f(k+1) = 8m + 8 \times 3^{2k} = 8[m + 3^{2k}]$  is multiple of 8  
 • if the result holds for  $n=k \in \mathbb{N} \Rightarrow$  it also holds for  $n=k+1$   
 since the result holds for  $n=1 \Rightarrow$  it must hold  $\forall n \in \mathbb{N}$

Question 8 (\*\*\*)

$$f(n) = 4^n + 6n - 1, n \in \mathbb{N}.$$

Prove by induction that  $f(n)$  is divisible by 3, for all  $n \in \mathbb{N}$ .

proof

$f(n) = 4^n + 6n - 1$   
 •  $f(1) = 4^1 + 6 \times 1 - 1 = 9 = 3 \times 3$   
 is result holds for  $n=1$   
 • suppose the result holds for  $n=k \in \mathbb{N}$   
 i.e.  $f(k) = 3m, m \in \mathbb{N}$   
 $\Rightarrow f(k+1) - f(k) = [4^{k+1} + 6(k+1) - 1] - [4^k + 6k - 1]$   
 $\Rightarrow f(k+1) - 3m = 4^{k+1} + 6k + 6 - 4^k - 6k + 1$   
 $\Rightarrow f(k+1) - 3m = 4 \times 4^k - 4^k + 6$   
 $\Rightarrow f(k+1) - 3m = 3 \times 4^k + 6$   
 $\Rightarrow f(k+1) = 3[m + 4^k + 2]$   
 • as the result holds for  $n=k \in \mathbb{N}$ , then it  
 also holds for  $n=k+1$   
 since it holds for  $n=1$ , then it must hold  
 $\forall n \in \mathbb{N}$

Question 9 (\*\*\*)

$$f(n) = 5^n + 8n + 3, n \in \mathbb{N}.$$

Prove by induction that  $f(n)$  is divisible by 4, for all  $n \in \mathbb{N}$ .

proof

$f(n) = 5^n + 8n + 3$

- $f(1) = 5^1 + 8 \times 1 + 3 = 16$ , i.e. divisible by 4
- SUPPOSE THE RESULT HOLDS FOR  $n = k \in \mathbb{N}$ , i.e.  $f(k) = 4m$  for  $m \in \mathbb{N}$
- $\Rightarrow f(k+1) - f(k) = [5^{k+1} + 8(k+1) + 3] - [5^k + 8k + 3]$
- $\Rightarrow f(k+1) - 4m = 5^{k+1} + 8k + 11 - 5^k - 8k - 3$
- $\Rightarrow f(k+1) - 4m = 5^{k+1} - 5^k + 8$
- $\Rightarrow f(k+1) = 4m + 5 \times 5^k - 5^k + 8$
- $\Rightarrow f(k+1) = 4m + 4 \times 5^k + 8$
- $\Rightarrow f(k+1) = 4[m + 5^k + 2]$ , i.e. divisible by 4
- IF THE RESULT HOLDS FOR  $n = k \in \mathbb{N}$ , THEN THE RESULT HOLDS FOR  $n = k+1$ .  
SINCE THE RESULT HOLDS FOR  $n=1$ , THEN IT MUST ALSO HOLD FOR  $n \in \mathbb{N}$

Question 10 (\*\*\*)

$$f(n) = 3^{4n} + 2^{4n+2}, n \in \mathbb{N}$$

Prove by induction that  $f(n)$  is divisible by 5, for all  $n \in \mathbb{N}$ .

proof

$f(n) = 3^{4n} + 2^{4n+2}$

- $f(1) = 3^4 + 2^6 = 81 + 64 = 145$ , i.e. divisible by 5
- SUPPOSE THAT THE RESULT HOLDS FOR  $n = k \in \mathbb{N}$ , i.e.  $f(k) = 5m$ ,  $m \in \mathbb{N}$
- $\Rightarrow f(k+1) - f(k) = [3^{4(k+1)} + 2^{4(k+1)+2}] - [3^{4k} + 2^{4k+2}]$
- $\Rightarrow f(k+1) - 5m = 3^{4k+4} + 2^{4k+6} - 3^{4k} - 2^{4k+2}$
- $\Rightarrow f(k+1) - 5m = 3^4 \times 3^{4k} - 3^4 \times 3^{4k} + 2^4 \times 2^{4k} - 2^{4k+2}$
- $\Rightarrow f(k+1) = 5m + 80 \times 3^{4k} + 15 \times 2^{4k+2}$
- $\Rightarrow f(k+1) = 5[m + 16 \times 3^{4k} + 3 \times 2^{4k+2}]$ , i.e. divisible by 5
- IF THE RESULT HOLDS FOR  $n = k \in \mathbb{N}$ , THEN THE RESULT HOLDS FOR  $n = k+1$ .  
SINCE THE RESULT HOLDS FOR  $n=1$ , THEN IT MUST ALSO HOLD FOR  $n \in \mathbb{N}$

**Question 11** (\*\*\*)

Prove by induction that for all natural numbers  $n$ ,

$$9^n - 5^n$$

is divisible by 4.

proof

Let  $f(n) = 9^n - 5^n$

- $f(1) = 9^1 - 5^1 = 9 - 5 = 4$ , i.e. the result holds for  $n=1$
- SUPPOSE THE RESULT HOLDS FOR  $n=k \in \mathbb{N}$ , i.e.  $f(k) = 4m$  for  $m \in \mathbb{N}$

$$f(k+1) - f(k) = (9^{k+1} - 5^{k+1}) - (9^k - 5^k)$$

$$f(k+1) - 4m = 9^{k+1} - 9^k + 5^k - 5^{k+1}$$

$$f(k+1) - 4m = 9(9^k) - 9^k + 5^k - 5(5^k)$$

$$f(k+1) = 4m + 8(9^k) - 4(5^k)$$

$$f(k+1) = 4[m + 2(9^k) - 5^k]$$

is also divisible by 4

- IF THE RESULT HOLDS FOR  $n=k \in \mathbb{N}$  THEN THE RESULT HOLDS FOR  $n=k+1$  SINCE THE RESULT HOLDS FOR  $n=1$ , THEN IT MUST HOLD FOR  $\forall n \in \mathbb{N}$

**Question 12** (\*\*\*)

$$f(n) = (4n+3)5^n - 3, n \in \mathbb{N}.$$

Prove by induction that  $f(n)$  is divisible by 16, for all  $n \in \mathbb{N}$ .

proof

$f(n) = (4n+3)5^n - 3$

- $f(1) = 7 \times 5 - 3 = 32$  which is divisible by 16
- SUPPOSE THE RESULT HOLDS FOR  $n=k \in \mathbb{N}$ , i.e.  $f(k) = 16m$ ,  $m \in \mathbb{N}$

$$f(k+1) - f(k) = [(4k+7)5^{k+1} - 3] - [(4k+3)5^k - 3]$$

$$f(k+1) - 16m = (4k+7)5^{k+1} - (4k+3)5^k$$

$$f(k+1) = 16m + 5(4k+7)5^k - (4k+3)5^k$$

$$f(k+1) = 16m + (4k+32)5^k$$

$$f(k+1) = 16[m + (k+32)5^k]$$

is divisible by 16

- IF THE RESULT HOLDS FOR  $n=k \in \mathbb{N}$ , THEN IT MUST HOLD FOR  $n=k+1$  SINCE THE RESULT HOLDS FOR  $n=1$ , THEN IT MUST HOLD FOR  $\forall n \in \mathbb{N}$

**Question 13 (\*\*\*)**

Prove by induction that the sum of the cubes of any three consecutive positive integers is always divisible by 9.

  , proof

$f(n) = n^3 + (n+1)^3 + (n+2)^3, n \in \mathbb{N}$

BASE CASE, ie f(0)  
 $f(0) = 0^3 + 1^3 + 2^3 = 1 + 8 + 27 = 36$  is divisible by 9

INDUCTIVE HYPOTHESIS  
 SUPPOSE THAT THE RESULT HOLDS FOR  $n=k, k \in \mathbb{N}$ , ie  $f(k) = 9A$  WHERE  $A \in \mathbb{N}$ .

$\Rightarrow f(k+1) - f(k) = [(k+1)^3 + (k+2)^3] - [k^3 + (k+1)^3]$   
 $\Rightarrow f(k+1) - 9A = (k+2)^3 - k^3$   
 $\Rightarrow f(k+1) - 9A = (k^3 + 6k^2 + 12k + 8) - k^3$   
 $\Rightarrow f(k+1) = 9A + 6k^2 + 12k + 8$   
 $\Rightarrow f(k+1) = 9[A + k^2 + 2k + 3]$

CONCLUSION  
 IF THE RESULT HOLDS FOR  $n=k, k \in \mathbb{N}$  THEN IT ALSO HOLDS FOR  $n=k+1$   
 SINCE THE RESULT HOLDS FOR  $n=1$ , THEN IT MUST HOLD FOR ALL  $n \in \mathbb{N}$

**Question 14 (\*\*\*)**

Prove by induction that for all natural numbers  $n$ , such that  $n \geq 2$ ,

$$15^n - 8^{n-2}$$

is divisible by 7.

proof

Let  $f(n) = 15^n - 8^{n-2}$

•  $f(2) = 15 - 8^0 = 225 - 1 = 224 = 7 \times 32$   
 ie result holds for  $n=2$

• SUPPOSE THE RESULT HOLDS FOR  $n=k \in \mathbb{N}$ ,  
 ie  $f(k) = 7M, M \in \mathbb{N}$

$\Rightarrow f(k+1) - f(k) = [15^{k+1} - 8^{k-1}] - [15^k - 8^{k-2}]$   
 $\Rightarrow f(k+1) - 7M = 15^k - 15^k - 8^{k-1} + 8^{k-2}$   
 $\Rightarrow f(k+1) - 7M = 15^k - 15^k - 8 \times 8^{k-2} + 8^{k-2}$   
 $\Rightarrow f(k+1) - 7M = 14 \times 15^k - 7 \times 8^{k-2}$   
 $\Rightarrow f(k+1) = 7M + 14 \times 15^k - 7 \times 8^{k-2}$   
 $\Rightarrow f(k+1) = 7[M + 2 \times 15^k - 8^{k-2}]$

• IF THE RESULT HOLDS FOR  $n=k \in \mathbb{N}$ , THEN IT ALSO HOLDS FOR  $n=k+1$   
 SINCE IT HOLDS FOR  $n=2$ , THEN IT MUST HOLD FOR ALL  $n \in \mathbb{N}, n \geq 2$

**Question 15 (\*\*\*)**

Prove by induction that for all natural numbers  $n$ ,

$$(2n+1)7^n + 11,$$

is divisible by 4.

**proof**

$f(n) = (2n+1)7^n + 11$

- $f(1) = (2(1)+1) \times 7^1 + 11 = 3 \times 7 + 11 = 21 + 11 = 32$   
is divisible by 4
- SUPPOSE THAT THE RESULT HOLDS FOR  $n = k \in \mathbb{N}$ , I.E.  $f(k) = 4m, m \in \mathbb{N}$
- $f(k+1) - f(k) = [(2(k+1)+1) \times 7^{k+1} + 11] - [(2k+1)7^k + 11]$
- $f(k+1) - 4m = (2k+3) \times 7^k \times 7 + 11 - (2k+1)7^k + 11$
- $f(k+1) - 4m = (2k+3) \times 7 \times 7^k - (2k+1)7^k + 22$
- $f(k+1) - 4m = 7^k [7(2k+3) - (2k+1)] + 22$
- $f(k+1) - 4m = 7^k [14k + 20]$
- $f(k+1) - 4m = 7^k [14k + 7 \times 4(3k+5)]$
- $f(k+1) - 4m = 4 [7^{k+1} + 7^k(3k+5)]$  is a multiple of 4
- IF THE RESULT HOLDS FOR  $n = k \in \mathbb{N}$ , THEN IT ALSO HOLDS FOR  $n = k+1$   
SINCE THE RESULT HOLDS FOR  $n=1$ , THEN IT MUST HOLD FOR ALL  $n \in \mathbb{N}$

**Question 16 (\*\*\*)**

$$f(n) = 24 \times 2^{4n} + 3^{4n}, n \in \mathbb{N}.$$

Prove by induction that  $f(n)$  is divisible by 5, for all  $n \in \mathbb{N}$ .

**proof**

$f(n) = 24 \times 2^{4n} + 3^{4n}$

- $f(1) = 24 \times 2^4 + 3^4 = 24 \times 16 + 81 = 384 + 81 = 465 = 5 \times 93$   
is a multiple of 5
- SUPPOSE THAT THE RESULT HOLDS FOR  $n = k \in \mathbb{N}$ , I.E.  $f(k) = 5m, m \in \mathbb{N}$
- $f(k+1) - f(k) = [24 \times 2^{4(k+1)} + 3^{4(k+1)}] - [24 \times 2^{4k} + 3^{4k}]$
- $f(k+1) - 5m = 24 \times 2^{4k} \times 2^4 - 24 \times 2^{4k} + 3^{4k} \times 3^4 - 3^{4k}$
- $f(k+1) - 5m = 16 \times 24 \times 2^{4k} - 24 \times 2^{4k} + 81 \times 3^{4k} - 3^{4k}$
- $f(k+1) - 5m = 15 \times 24 \times 2^{4k} + 80 \times 3^{4k}$
- $f(k+1) - 5m = 15 \times 24 \times 2^{4k} + 80 \times 3^{4k}$
- $f(k+1) - 5m = 5 [12 \times 24 \times 2^{4k} + 16 \times 3^{4k}]$  is a multiple of 5
- IF THE RESULT HOLDS FOR  $n = k \in \mathbb{N}$ , THEN IT ALSO HOLDS FOR  $n = k+1$   
SINCE THE RESULT HOLDS FOR  $n=1$ , THEN IT MUST HOLD FOR ALL  $n \in \mathbb{N}$

Question 17 (\*\*\*)

$$f(n) = 4 \times 7^n + 3 \times 5^n + 5, n \in \mathbb{N}.$$

Prove by induction that  $f(n)$  is divisible by 12, for all  $n \in \mathbb{N}$ .

proof

$f(n) = 4 \times 7^n + 3 \times 5^n + 5$

- $f(0) = 4 \times 7 + 3 \times 5 + 5 = 28 + 15 + 5 = 48 = 4 \times 12$  is divisible by 12
- suppose the result holds for  $n = k \in \mathbb{N}$ , i.e.  $f(k) = 12m$ ,  $m \in \mathbb{N}$
- $\rightarrow f(k+1) - f(k) = [4 \times 7^{k+1} + 3 \times 5^{k+1} + 5] - [4 \times 7^k + 3 \times 5^k + 5]$
- $\rightarrow f(k+1) - f(k) = 4 \times 7^k(7-1) + 3 \times 5^k(5-1)$
- $\rightarrow f(k+1) - f(k) = 28 \times 7^k + 12 \times 5^k = 4(7 \times 7^k + 3 \times 5^k)$
- $\rightarrow f(k+1) - f(k) = 4(7^{k+1} + 3 \times 5^k)$
- $\rightarrow f(k+1) = 4(7^{k+1} + 3 \times 5^k + 12m)$  is divisible by 12

if the result holds for  $n = k \in \mathbb{N}$  then it also holds for  $n = k+1$  since the result holds for  $n=0$  then it must hold for  $\forall n \in \mathbb{N}$

Question 18 (\*\*\*)

$$f(n) = (2n+1)7^n - 1, n \in \mathbb{N}$$

Prove by induction that  $f(n)$  is divisible by 4, for all  $n \in \mathbb{N}$ .

proof

$f(n) = (2n+1)7^n - 1$

- $f(0) = 3 \times 7^0 - 1 = 20 = 5 \times 4$  is divisible by 4
- suppose the result holds for  $n = k \in \mathbb{N}$ , i.e.  $f(k) = 4m$ ,  $m \in \mathbb{N}$
- $\rightarrow f(k+1) - f(k) = [(2(k+1))7^{k+1} - 1] - [(2k+1)7^k - 1]$
- $\rightarrow f(k+1) - f(k) = (2k+2)7^{k+1} - (2k+1)7^k$
- $\rightarrow f(k+1) - f(k) = (2k+2) \times 7 \times 7^k - (2k+1)7^k$
- $\rightarrow f(k+1) - f(k) = (14k+14)7^k - (2k+1)7^k$
- $\rightarrow f(k+1) - f(k) = (12k+13)7^k$
- $\rightarrow f(k+1) = 4m + 4(3k+3)7^k$
- $\rightarrow f(k+1) = 4[m + (3k+3)7^k]$

the result holds for  $n = k \in \mathbb{N}$ , then it also holds for  $n = k+1$  since it holds for  $n=0$ , then it must hold for  $\forall n \in \mathbb{N}$

**Question 19** (\*\*\*)

Prove by induction that for all natural numbers  $n$ ,

$$4^n + 6n - 1$$

is divisible by 9.

□, proof

$f(n) = 4^n + 6n - 1 \quad n \in \mathbb{N}$

BASE CASE  
 $f(1) = 4^1 + 6(1) - 1 = 4 + 6 - 1 = 9$ , it is a multiple of 9 for  $n=1$

INDUCTIVE HYPOTHESIS  
 SUPPOSE THAT THE RESULT HOLDS FOR  $n=k, k \in \mathbb{N}$  i.e.  $f(k) = 9m$   
 where  $m \in \mathbb{N}$

$\Rightarrow f(k+1) - f(k) = [4^{k+1} + 6(k+1) - 1] - [4^k + 6k - 1]$   
 $\Rightarrow f(k+1) - 9m = 4^{k+1} + 6k + 6 - 4^k - 6k + 1$   
 $\Rightarrow f(k+1) - 9m = 4^{k+1} - 4^k + 6 + 1$   
 $\Rightarrow f(k+1) - 9m = 4 \times 4^k - 4^k + 6 + 1$   
 $\Rightarrow f(k+1) - 9m = 3 \times 4^k + 6 + 1$

$\Rightarrow f(k+1) = 9m + 6 + 3[4^k - 6k + 1]$   
 $\Rightarrow f(k+1) = 9m + 6 + 3(9m - 18k + 3)$   
 $\Rightarrow f(k+1) = 9m - 18k + 9 + 3(9m)$   
 $\Rightarrow f(k+1) = 36m - 18k + 9$   
 $\Rightarrow f(k+1) = 9[4m - 2k + 1]$

CONCLUSION  
 IF THE RESULT HOLDS FOR  $n=k, k \in \mathbb{N}$ , THEN IT ALSO HOLDS FOR  $n=k+1$   
 SINCE THE RESULT HOLDS FOR  $n=1$ , THEN IT MUST HOLD FOR ALL  $n$

Question 20 (\*\*\*)

Prove by induction that for all natural numbers  $n$ ,

$$4^{n+1} + 5^{2n-1}$$

is divisible by 21.

, proof

$f(n) = 4^{n+1} + 5^{2n-1}$

- THE BASE CASE, i.e.  $n=1$   
 $f(1) = 4^2 + 5^1 = 16 + 5 = 21$  is divisible by 21
- INDUCTIVE HYPOTHESIS  
 SUPPOSE THAT  $f(n)$  IS DIVISIBLE BY 21 FOR  $n=k \in \mathbb{N}$ , i.e.  $f(k) = 21m$   
 FOR SOME  $m \in \mathbb{N}$

THEN  $f(k+1) - f(k) = (4^{k+2} + 5^{2k+1}) - (4^{k+1} + 5^{2k-1})$

$$f(k+1) - 21m = 4^k 4^2 - 4^{k+1} + 5^2 5^{2k-1} - 5^{2k-1}$$

$$f(k+1) - 21m = 4 \times 4^k - 4^{k+1} + 25 \times 5^{2k-1} - 5^{2k-1}$$

$$f(k+1) - 21m = 3 \times 4^k + 24 \times 5^{2k-1}$$

BOX  $f(k) = 4^{k+1} + 5^{2k-1} = 21m$

$$f(k+1) - 21m = (3 \times 4^k + 3 \times 5^{2k-1}) + 21 \times 5^{2k-1}$$

$$f(k+1) - 21m = 3 \times f(k) + 21 \times 5^{2k-1}$$

$$f(k+1) = 84m + 21 \times 5^{2k-1}$$

$$f(k+1) = 21 \times [4m + 5^{2k-1}]$$

- CONCLUSION  
 IF  $f(k)$  IS DIVISIBLE BY 21 FOR  $k \in \mathbb{N}$ , SO IS  $f(k+1)$ . SINCE  $f(1)$   
 IS DIVISIBLE BY 21 FOR ALL  $n \in \mathbb{N}$

**Question 21** (\*\*\*)

$$f(n) = 5^{2n} + 3n - 1, n \in \mathbb{N}.$$

Prove by induction that  $f(n)$  is divisible by 9, for all  $n \in \mathbb{N}$ .

proof

Let  $f(n) = 5^{2n} + 3n - 1$   
 •  $f(1) = 5^2 + 3(1) - 1 = 25 + 3 - 1 = 27$ , it is divisible by 9  
 • suppose the result holds for  $n = k \in \mathbb{N}$ , i.e.  $f(k) = 9m$ ,  $m \in \mathbb{N}$   
 $\Rightarrow f(k+1) - f(k) = [5^{2(k+1)} + 3(k+1) - 1] - [5^{2k} + 3k - 1]$   
 $\Rightarrow f(k+1) - 9m = 5^{2k+2} + 3k + 3 - 5^{2k} - 3k + 1$   
 $\Rightarrow f(k+1) - 9m = 5^{2k} \cdot 5^2 + 3k + 3 - 5^{2k} - 3k + 1$   
 $\Rightarrow f(k+1) - 9m = 25 \cdot 5^{2k} - 5^{2k} + 3$   
 $\Rightarrow f(k+1) - 9m = 24 \cdot 5^{2k} + 3$   
 But  $f(k) = 5^{2k} + 3k - 1$   
 $9m = 5^{2k} + 3k - 1$   
 $5^{2k} = 9m - 3k + 1$   
 $\Rightarrow f(k+1) - 9m = 24 \cdot [9m - 3k + 1] + 3$   
 $\Rightarrow f(k+1) = 9m + 9m \cdot 24 - 72k + 24 + 3$   
 $\Rightarrow f(k+1) = 9m \cdot 25 - 72k + 27$   
 $\Rightarrow f(k+1) = 9[25m - 8k + 3]$  is a multiple of 9  
 • if the result holds for  $n = k \in \mathbb{N}$ , then it must also hold for  $n = k+1$   
 since the result holds for  $n=1$ , then it must hold for all  $n \in \mathbb{N}$

**Question 22** (\*\*\*)

Prove by induction that 18 is a factor of  $4^n + 6n + 8$ , for all  $n \in \mathbb{N}$ .

proof

Let  $f(n) = 4^n + 6n + 8$   
 •  $f(1) = 4 + 6 + 8 = 18$  is divisible by 18  
 • suppose the result holds for  $n = k \in \mathbb{N}$ , i.e.  $f(k) = 18m$ , where  $m \in \mathbb{N}$   
 $f(k+1) - f(k) = [4^{k+1} + 6(k+1) + 8] - [4^k + 6k + 8]$   
 $f(k+1) - 18m = 4^{k+1} + 6k + 6 + 8 - 4^k - 6k - 8$   
 $f(k+1) - 18m = 4(4^k) - 4^k + 6$   
 $f(k+1) - 18m = 3 \cdot 4^k + 6$   
 $f(k+1) = 18m + 3 \cdot 4^k + 6$   
 $f(k+1) = 18m + 3(4^k + 2)$   
 $f(k+1) = 18[4m - k - 1]$  is a multiple of 18  
 if the result holds for  $n = k \in \mathbb{N}$ , then it must also hold for  $n = k+1$   
 since the result holds for  $n=1$ , then it must hold for all  $n \in \mathbb{N}$

Question 23 (\*\*\*\*)

Prove by induction that for all natural numbers  $n$ ,

$$2^n + 6^n$$

is divisible by 8.

, proof

$f(n) = 2^n + 6^n, n \in \mathbb{N}$

BASE CASE  
 $f(1) = 2^1 + 6^1 = 2 + 6 = 8$ , is divisible by 8

INDUCTIVE HYPOTHESIS  
 SUPPOSE THAT THE RESULT HOLDS FOR  $n = k, k \in \mathbb{N}$ , IF  $f(k) = 8m$ ,  $m \in \mathbb{N}$ .

$\Rightarrow f(k+1) - f(k) = [2^{k+1} + 6^{k+1}] - [2^k + 6^k]$   
 $\Rightarrow f(k+1) - 8m = 2^{k+1} - 2^k + 6^{k+1} - 6^k$   
 $\Rightarrow f(k+1) - 8m = 2 \times 2^k - 2^k + 6 \times 6^k - 6^k$   
 $\Rightarrow f(k+1) - 8m = 2^k + 5 \times 6^k$   
 $\Rightarrow f(k+1) - 8m = [2^k - 6^k] + 5 \times 6^k$   $f(k) = 2^k + 6^k$   
 $2^k = f(k) - 6^k$   
 $\Rightarrow f(k+1) - 8m = f(k) + 4 \times 6^k$   
 $\Rightarrow f(k+1) - 8m = 8m + 4 \times 6 \times 6^{k-1}$   
 $\Rightarrow f(k+1) = 16m + 24 \times 6^{k-1}$   
 $\Rightarrow f(k+1) = 8[2m + 3 \times 6^{k-1}]$

CONCLUSION  
 IF THE RESULT HOLDS FOR  $n = k, k \in \mathbb{N}$ , THEN IT ALSO HOLDS FOR  $n = k+1$   
 SINCE THE RESULT HOLDS FOR  $n=1$ , THEN IT HOLDS

Question 24 (\*\*\*)

Prove by mathematical induction that if  $n$  is a positive integer then  $3^{2n+3} + 2^{n+3}$  is always divisible by 7.

, proof

Let  $f(n) = 3^{2n+3} + 2^{n+3}, n \in \mathbb{N}$

EXHIBIT A BASE CASE

$$f(0) = 3^3 + 2^3 = 27 + 8 = 35 = 5 \times 7$$

is the exact value for  $n=0$

SUPPOSE THAT THE RESULT HOLDS FOR  $n \in \mathbb{N}$ , i.e.  $f(n) = 7A, A \in \mathbb{N}$

$$\Rightarrow f(n) - f(n) = [3^{2(n+1)+3} + 2^{(n+1)+3}] - [3^{2n+3} + 2^{n+3}]$$

$$\Rightarrow f(n) - 7A = 3^{2n+5} + 2^{n+4} - 3^{2n+3} - 2^{n+3}$$

$$\Rightarrow f(n) - 7A = 3^{2n+3} \cdot 3^2 + 2^{n+3} \cdot 2^1 - 3^{2n+3} - 2^{n+3}$$

$$\Rightarrow f(n) - 7A = 9 \times 3^{2n+3} - 3^{2n+3} + 2 \times 2^{n+3} - 2^{n+3}$$

$$\Rightarrow f(n) - 7A = 8 \times 3^{2n+3} + 2^{n+3}$$

BP  $f(n) = 7A$

$$\Rightarrow f(n) - 7A = 8 \times 3^{2n+3} + 2^{n+3} - 7A = 3^{2n+3}$$

$$\Rightarrow f(n) = 11A + 7 \times 3^{2n+3}$$

$$\Rightarrow f(n) = 7[2A + 3^{2n+3}]$$

is the exact value for  $n \in \mathbb{N}$ , then it must also hold for  $n+1$

SINCE THE RESULT HOLDS FOR  $n=0$ , THEN IT MUST HOLD FOR ALL  $n \in \mathbb{N}$

Question 25 (\*\*\*)

Prove by mathematical induction that if  $n$  is a positive integer then  $5^{n-1} + 11^n$  is always divisible by 6.

 , proof

LET  $f(n) = 5^{n-1} + 11^n, n \in \mathbb{N}$

ESTABLISH A BASE CASE  
 $f(1) = 5^{1-1} + 11^1 = 1 + 11 = 12 = 2 \times 6$

LET THE RESULT HOLD FOR  $n-1$

SUPPOSE THAT THE RESULT HOLDS FOR  $n \in \mathbb{N}$ , i.e.  $f(n) = 6A, A \in \mathbb{N}$

$\rightarrow f(n+1) - f(n) = [5^{n+1} + 11^{n+1}] - [5^n + 11^n]$   
 $\rightarrow f(n+1) - 6A = 5^n + 11^{n+1} - 5^n - 11^n$   
 $\rightarrow f(n+1) - 6A = 5^n \cdot 5 + 11 \cdot 11^n - 5^n - 11^n$   
 $\rightarrow f(n+1) - 6A = 5 \cdot 5^n + 11 \cdot 11^n - 5^n - 11^n$   
 $\rightarrow f(n+1) - 6A = 4 \cdot 5^n + 10 \cdot 11^n$

BUT WE ALSO HAVE  $f(n) = 6A$   
 $5^{n-1} + 11^n = 6A$   
 $11^n = 6A - 5^{n-1}$

$\rightarrow f(n+1) - 6A = 4 \cdot 5^n + 10[6A - 5^{n-1}]$   
 $\rightarrow f(n+1) - 6A = 4 \cdot 5^n + 60A - 10 \cdot 5^{n-1}$   
 $\rightarrow f(n+1) = 60A - 6 \cdot 5^{n-1} + 4 \cdot 5^n$   
 $\rightarrow f(n+1) = 6[10A - 5^{n-1} + 5^n]$

IF THE RESULT HOLDS FOR  $n \in \mathbb{N}$ , THEN IT MUST ALSO HOLD FOR  $n+1$   
HENCE THE RESULT HOLDS FOR ALL  $n \in \mathbb{N}$

**Question 26 (\*\*\*)**

Prove by the method of induction that

$$3^{3n-2} + 2^{4n-1}, n \in \mathbb{N},$$

is divisible by 11.

11, proof

$f(n) = 3^{3n-2} + 2^{4n-1}, n \in \mathbb{N}$

BASE CASE,  $n=1$   
 $f(1) = 3^1 + 2^1 = 3 + 2 = 11$ , is the result for  $n=1$

INDUCTIVE HYPOTHESIS  
 SUPPOSE THAT THE RESULT IS TRUE FOR  $n=k, k \in \mathbb{N}$ , i.e.  $f(k) = 11m, m \in \mathbb{N}$

$\Rightarrow f(k+1) - f(k) = [3^{3(k+1)-2} + 2^{4(k+1)-1}] - [3^{3k-2} + 2^{4k-1}]$   
 $\Rightarrow f(k+1) - 11m = 3^{3k+1} + 2^{4k+1} - 3^{3k-2} - 2^{4k-1}$   
 $\Rightarrow f(k+1) - 11m = 3^3 \times 3^{3k-2} + 2^4 \times 2^{4k-1} - 3^{3k-2} - 2^{4k-1}$   
 $\Rightarrow f(k+1) - 11m = 27 \times 3^{3k-2} - 3^{3k-2} + 16 \times 2^{4k-1} - 2^{4k-1}$   
 $\Rightarrow f(k+1) - 11m = 26 \times 3^{3k-2} + 15 \times 2^{4k-1}$

$f(k) = 3^{3k-2} + 2^{4k-1}$   
 $11m = 3^{3k-2} + 2^{4k-1}$   
 $2 \times 11m = 2 \times 3^{3k-2} + 2 \times 2^{4k-1}$

$\Rightarrow f(k+1) - 11m = 26 \times 3^{3k-2} + 15(2 \times 3^{3k-2} + 2 \times 2^{4k-1})$   
 $\Rightarrow f(k+1) - 11m = 26 \times 3^{3k-2} + 15 \times 2 \times 3^{3k-2} + 15 \times 2 \times 2^{4k-1}$   
 $\Rightarrow f(k+1) - 11m = 11 \times 3^{3k-2} + 176m$   
 $\Rightarrow f(k+1) - 11m = 11[3^{3k-2} + 16m]$ , is divisible by 11

CONCLUSION  
 IF THE RESULT IS TRUE FOR  $n=k, k \in \mathbb{N}$ , THEN IT ALSO HOLDS FOR  $n=k+1$   
 SINCE THE RESULT HOLDS FOR  $n=1$ , THEREFORE IT ALSO HOLDS FOR ALL  $n \in \mathbb{N}$ .

Question 27 (\*\*\*)

$$f(n) = 8^n - 2^n, n \in \mathbb{N}$$

Prove by induction that  $f(n)$  is divisible by 6, for all  $n \in \mathbb{N}$ .

proof

Handwritten proof for Question 27:

- $f(0) = 8^0 - 2^0 = 1 - 1 = 0$  is divisible by 6
- SUPPOSE THE RESULT HOLDS FOR  $n = k \in \mathbb{N}$ , i.e.  $f(k) = 6m, m \in \mathbb{N}$
- $f(k+1) - f(k) = (8^{k+1} - 2^{k+1}) - (8^k - 2^k)$
- $f(k+1) - 6m = 8^{k+1} - 2^{k+1} - 8^k + 2^k$
- $f(k+1) = 6m + 8 \times 8^k - 8^k - 2 \times 2^k + 2^k$
- $f(k+1) = 6m + 7 \times 8^k - 2^k$
- Now  $f(k) = 8^k - 2^k$
- $6m = 8^k - 2^k$
- $6m - 8^k = -2^k$
- $-2^k = 6m - 8^k$
- $f(k+1) = 6m + 7 \times 8^k + 6m - 8^k$
- $f(k+1) = 12m + 6 \times 8^k$
- $f(k+1) = 6(2m + 8^k)$
- IF THE RESULT HOLDS FOR  $n = k \in \mathbb{N}$ , THEN IT MUST ALSO HOLD FOR  $n = k+1$ . SINCE THE RESULT HOLDS FOR  $n = 1$ , THEN THE RESULT MUST HOLD FOR ALL  $n$ .

Question 28 (\*\*\*)

$$f(n) = 7^n - 2^n, n \in \mathbb{N}$$

Prove by induction that  $f(n)$  is divisible by 5, for all  $n \in \mathbb{N}$ .

proof

Handwritten proof for Question 28:

- $f(0) = 7^0 - 2^0 = 1 - 1 = 0$  is divisible by 5
- SUPPOSE THE RESULT HOLDS FOR  $n = k \in \mathbb{N}$ , i.e.  $f(k) = 5m, m \in \mathbb{N}$
- $f(k+1) - f(k) = (7^{k+1} - 2^{k+1}) - (7^k - 2^k)$
- $f(k+1) - 5m = 7^{k+1} - 7^k - 2^{k+1} + 2^k$
- $f(k+1) - 5m = 7 \times 7^k - 7^k - 2 \times 2^k + 2^k$
- $f(k+1) - 5m = 6 \times 7^k - 2^k$
- $f(k+1) = 5m + 6 \times 7^k - 2^k$
- $f(k+1) = 5m + 5 \times 7^k + 7^k - 2^k$
- $f(k+1) = 5m + 5 \times 7^k + f(k)$
- $f(k+1) = 5m + 5 \times 7^k + 5m$
- $f(k+1) = 5(2m + 7^k)$
- IF THE RESULT HOLDS FOR  $n = k \in \mathbb{N}$ , THEN IT MUST ALSO HOLD FOR  $n = k+1$ . SINCE THE RESULT HOLDS FOR  $n = 1$ , THEN IT MUST HOLD FOR ALL  $n$ .

Question 29 (\*\*\*)

$$f(n) = n^3 + 5n, n \in \mathbb{N}.$$

- a) Show that  $n^2 + n + 2$  is always even for all  $n \in \mathbb{N}$ .
- b) Hence, prove by induction that  $f(n)$  is divisible by 6, for all  $n \in \mathbb{N}$ .

proof

a)  $n^2 + n + 2 = n(n+1) + 2$  IF  $n$  IS EVEN  $n(n+1)$  IS EVEN  
 $n(n+1) + 2$  IS ALSO EVEN  
 IF  $n$  IS ODD  
 $n+1$  IS EVEN  $n(n+1)$  IS EVEN  
 $n(n+1) + 2$  IS EVEN  
 $\therefore n^2 + n + 2$  IS EVEN FOR ALL  $n \in \mathbb{N}$

b)  $f(n) = n^3 + 5n$

- $f(n) = n^3 + 5n = 6$ , IT IS DIVISIBLE BY 6
- ASSUME THAT THE RESULT HOLDS FOR  $n=k \in \mathbb{N}$ , i.e.  $f(k) = 6m, m \in \mathbb{N}$
- $\rightarrow f(k+1) - f(k) = [(k+1)^3 + 5(k+1)] - [k^3 + 5k]$
- $\Rightarrow f(k+1) - 6m = (k^3 + 3k^2 + 3k + 1 + 5k + 5) - (k^3 + 5k)$
- $\Rightarrow f(k+1) = 6m + 3k^2 + 3k + 1 + 5k + 5 - 5k$
- $\Rightarrow f(k+1) = 6m + 3(k^2 + k + 2)$
- $\rightarrow f(k+1) = 6m + 6p$  (FROM PART a)  $k^2 + k + 2$  IS EVEN
- $\rightarrow f(k+1) = 6(m+p)$  IS DIVISIBLE BY 6
- IF THE RESULT HOLDS FOR  $n=k \in \mathbb{N}$ , THEN IT MUST ALSO HOLD FOR  $n=k+1$   
 SINCE THE RESULT HOLDS FOR  $n=1$ , THEN IT MUST HOLD FOR ALL  $n \in \mathbb{N}$

Question 30 (\*\*\*)

A sequence of positive numbers is given by

$$a_n = 12^{n+1} + 2 \times 5^n, n \in \mathbb{N}.$$

Prove by induction that every term of the sequence is a multiple of 7

proof

$a_1 = 12^2 + 2 \times 5^1$

- $a_1 = 12^2 + 2 \times 5^1 = 144 + 10 = 154 = 7 \times 22$  IS A MULTIPLE OF 7
- ASSUME THE RESULT HOLDS FOR  $n=k \in \mathbb{N}$ , i.e.  $a_k = 7m, m \in \mathbb{N}$
- $\Rightarrow a_{k+1} - a_k = (12^{k+1} + 2 \times 5^{k+1}) - (12^k + 2 \times 5^k)$
- $\Rightarrow a_{k+1} - 7m = 12^k(12 - 1) + 2 \times 5^k(5 - 1)$
- $\Rightarrow a_{k+1} - 7m = 11 \times 12^k + 8 \times 5^k$
- $\Rightarrow a_{k+1} - 7m = 11 \times 12^k + 4(2 \times 5^k)$
- $\Rightarrow a_{k+1} - 7m = 7 \times 12^k + 4a_k$
- $\Rightarrow a_{k+1} = 7[12^k + m] + 4(7m) = 7m'$
- $\Rightarrow a_{k+1} = 7[12^k + 4m]$  IS A MULTIPLE OF 7
- IF THE RESULT HOLDS FOR  $n=k \in \mathbb{N}$ , THEN IT MUST ALSO HOLD FOR  $n=k+1$   
 SINCE THE RESULT HOLDS FOR  $n=1$ , THEN IT MUST HOLD FOR ALL  $n \in \mathbb{N}$

**Question 31** (\*\*\*)

$$f(r) = 4 + 6^r, \quad r \in \mathbb{N}.$$

Prove by induction that  $f(r)$  is divisible by 10

proof

$f(r) = 4 + 6^r$   
 $f(1) = 4 + 6 = 10$ , is divisible by 10  
 Suppose the result holds for  $r = k \in \mathbb{N}$ , i.e.  $f(k) = 10n, n \in \mathbb{N}$   
 $f(k+1) - f(k) = (4 + 6^{k+1}) - (4 + 6^k)$   
 $f(k+1) - 10n = 6^{k+1} - 6^k$   
 $f(k+1) - 10n = 6 \times 6^k - 6^k$   
 $f(k+1) - 10n = 5 \times 6^k$   
 $f(k+1) - 10n = 5[6^k]$   
 $f(k+1) = 10n + 5 \times 6^k = 5(2n + 6^k)$   
 $f(k+1) = 10[2n + 6^k]$ , is divisible by 10  
 If the result holds for  $r = k \Rightarrow$  it also holds for  $r = k+1$   
 Since it holds for  $r = 1 \Rightarrow$  it must hold  $\forall n \in \mathbb{N}$

**Question 32** (\*\*\*)

Prove by induction that for all natural numbers  $n$ , the following expression

$$7^n + 4^n + 1$$

is divisible by 6.

proof

Let  $f(n) = 7^n + 4^n + 1$   
 $f(1) = 7^1 + 4^1 + 1 = 12 = 2 \times 6$  is result holds for  $n = 1$   
 Suppose the result holds for  $n = k \in \mathbb{N}$ , i.e.  $f(k) = 6m, m \in \mathbb{N}$   
 $f(k+1) - f(k) = [7^{k+1} + 4^{k+1} + 1] - [7^k + 4^k + 1]$   
 $f(k+1) - 6m = 7^{k+1} - 7^k + 4^{k+1} - 4^k$   
 $f(k+1) - 6m = 7 \times 7^k - 7^k + 4 \times 4^k - 4^k$   
 $f(k+1) - 6m = 6 \times 7^k + 3 \times 4^k$   
 $f(k+1) - 6m = 6 \times 7^k + 3 \times (2^2)^k$   
 $f(k+1) - 6m = 6 \times 7^k + 3 \times 2^{2k}$   
 $f(k+1) - 6m = 6 \times 7^k + 6 \times 2^{2k-1}$   
 $f(k+1) = 6[7^k + 7^k + 2^{2k-1}]$   
 If the result holds for  $n = k \in \mathbb{N}$ , then it also holds for  $n = k+1$   
 Since the result holds for  $n = 1$ , then it must hold  $\forall n \in \mathbb{N}$

**Question 33** (\*\*\*)

A sequence of positive numbers is given by

$$u_n = 7^n + 3n + 8, n \in \mathbb{N}.$$

Prove by induction that every term of the sequence is a multiple of 9

proof

$u_k = 7^k + 3k + 8$

- $u_1 = 7^1 + 3 + 8 = 18 = 9 \times 2$  is multiple of 9
- SUPPOSE THE RESULT HOLDS FOR  $n=k \in \mathbb{N}$ ,  $u_k = 9m, m \in \mathbb{N}$   
 $\Rightarrow u_{k+1} - u_k = [7^{k+1} + 3(k+1) + 8] - [7^k + 3k + 8]$   
 $\Rightarrow u_{k+1} - u_k = 7^{k+1} - 7^k + 3k + 3 + 3 - 3k - 8 + 8 - 8$   
 $\Rightarrow u_{k+1} - u_k = 7^{k+1} - 7^k + 3$   
 $\Rightarrow u_{k+1} - 9m = (7^k \cdot 7 - 7^k) + 3$   
 $\Rightarrow u_{k+1} - 9m = 6 \cdot 7^k + 3$

BUT  $u_k = 7^k + 3k + 8$   
 $7^k = u_k - 3k - 8$   
 $6 \cdot 7^k = 6u_k - 18k - 48$

- $\Rightarrow u_{k+1} - 9m = 6u_k - 18k - 48 + 3$   
 $\Rightarrow u_{k+1} - 9m = 6(9m) - 18k - 45$   
 $\Rightarrow u_{k+1} = 54m - 18k - 45$   
 $\Rightarrow u_{k+1} = 9[7m - 2k - 5]$  is multiple of 9

IF THE RESULT HOLDS FOR  $n=k \in \mathbb{N} \Rightarrow$  IT ALSO HOLDS FOR  $n=k+1$   
 SINCE THE RESULT HOLDS FOR  $n=1 \Rightarrow$  IT HOLDS FOR  $\forall n \in \mathbb{N}$

**Question 34** (\*\*\*)

$$f(n) = 5^{n+1} - 4n - 5, n \in \mathbb{N}.$$

Prove by induction that  $f(n)$  is divisible by 16

proof

$f(n) = 5^{n+1} - 4n - 5$

- $f(1) = 5^2 - 4(1) - 5 = 25 - 4 - 5 = 16$  is divisible by 16
- SUPPOSE THE RESULT HOLDS FOR  $n=k \in \mathbb{N}$ , i.e.  $f(k) = 16m, m \in \mathbb{N}$   
 $\Rightarrow f(k+1) - f(k) = [5^{k+2} - 4(k+1) - 5] - [5^{k+1} - 4k - 5]$   
 $\Rightarrow f(k+1) - 16m = 5^{k+2} - 4k - 4 - 5 - 5^{k+1} + 4k + 5$   
 $\Rightarrow f(k+1) - 16m = 5^{k+2} - 5^{k+1} - 4$   
 $\Rightarrow f(k+1) - 16m = 5 \cdot 5^{k+1} - 5^{k+1} - 4$   
 $\Rightarrow f(k+1) - 16m = 4 \cdot 5^{k+1} - 4$

BUT  $f(k) = 5^{k+1} - 4k - 5$   
 $16m = 5^{k+1} - 4k - 5 \Rightarrow 4 \cdot 5^{k+1} = 4m + 4k + 20$   
 $4m = 4 \cdot 5^{k+1} - 4k - 20$

- $\Rightarrow f(k+1) - 16m = (4m + 4k + 20) - 4$   
 $\Rightarrow f(k+1) = 4m + 4k + 16$   
 $\Rightarrow f(k+1) = 16[5m + k + 1]$  is divisible by 16

IF THE RESULT HOLDS FOR  $n=k \in \mathbb{N}$ , THEN IT ALSO HOLDS FOR  $n=k+1$   
 SINCE THE RESULT HOLDS FOR  $n=1$ , THEN THE RESULT HOLDS FOR ALL  $n$

**Question 35** (\*\*\*)

A sequence of positive numbers is given by

$$u_n = 2^{3n+2} + 5^{n+1}, \quad n \in \mathbb{N}.$$

Prove by induction that every term of the sequence is a multiple of 3.

proof

$u_1 = 2^{3 \cdot 1 + 2} + 5^{1+1} = 2^5 + 5^2 = 32 + 25 = 57 = 3 \times 19$  is divisible by 3.  
 • SUPPOSE THE RESULT HOLDS FOR  $n = k \in \mathbb{N}$ , i.e.  $u_k = 3m$ ,  $m \in \mathbb{N}$   
 $u_{k+1} - u_k = [2^{3(k+1)+2} + 5^{(k+1)+1}] - [2^{3k+2} + 5^{k+1}]$   
 $u_{k+1} - u_k = 2^{3k+5} + 5^{k+2} - 2^{3k+2} - 5^{k+1}$   
 $u_{k+1} - 3m = 2^{3k+2}(2^3 - 1) + 5^{k+1}(5 - 1)$   
 $u_{k+1} - 3m = 7 \times 2^{3k+2} + 4 \times 5^{k+1}$   
 $u_{k+1} - 3m = 7 \times 2^{3k+2} + 3 \times 5^{k+1} + 5^{k+1}$   
 • BIF  $u_k = 3m = 2^{3k+2} + 5^{k+1}$   
 $u_{k+1} - 3m = 7 \times 2^{3k+2} + 3 \times 5^{k+1} + u_k - 2^{3k+2} - 5^{k+1}$   
 $u_{k+1} - 3m = 6 \times 2^{3k+2} + 3 \times 5^{k+1} + 3m$   
 $u_{k+1} = 6 \times 2^{3k+2} + 3 \times 5^{k+1} + 3m$   
 $u_{k+1} = 3[2 \times 2^{3k+2} + 5^{k+1} + m]$   
 • IF THE RESULT HOLDS FOR  $n = k \in \mathbb{N}$   $\Rightarrow$  IT ALSO HOLDS FOR  $n = k+1$   
 SINCE THE RESULT HOLDS FOR  $n=1 \Rightarrow$  IT HOLDS  $\forall n \in \mathbb{N}$

**Question 36** (\*\*\*)

$$f(n) = 3^{2n+4} - 2^{2n}, \quad n \in \mathbb{N}$$

Prove by induction that  $f(n)$  is divisible by 5, for all  $n \in \mathbb{N}$ .

proof

$f(1) = 3^{2 \cdot 1 + 4} - 2^{2 \cdot 1} = 3^6 - 2^2 = 729 - 4 = 725$  which is divisible by 5.  
 • SUPPOSE THAT THE RESULT HOLDS FOR  $n = k \in \mathbb{N}$ , i.e.  $f(k) = 5m$  for  $m \in \mathbb{N}$   
 $f(k+1) - f(k) = [3^{2(k+1)+4} - 2^{2(k+1)}] - [3^{2k+4} - 2^{2k}]$   
 $f(k+1) - 5m = 3^{2k+6} - 2^{2k+2} - 3^{2k+4} + 2^{2k}$   
 $f(k+1) = 5m + 3^{2k+4}(3^2 - 1) - 2^{2k}(2^2 - 1)$   
 $f(k+1) = 5m + 8(3^{2k+4}) - 3(2^{2k})$   
 $f(k+1) = 5m + 8(3^{2k+4}) - 3(2^{2k})$   
 $f(k+1) = 5m + 5(3^{2k+4}) + 3(3^{2k+4}) - 3(2^{2k})$   
 $f(k+1) = 5[4m + 3^{2k+4}]$  is divisible by 5.  
 • IF THE RESULT HOLDS FOR  $n = k \in \mathbb{N}$ , THEN IT ALSO HOLDS FOR  $n = k+1$ . SINCE THE RESULT HOLDS FOR  $n=1$ , THEN THE RESULT HOLDS  $\forall n \in \mathbb{N}$

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# RECURRENCE RELATIONS

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**Question 1** (\*\*)

A sequence of integers is defined recursively by the relation

$$a_{n+1} = a_n - 4, \quad a_1 = 3, \quad n = 1, 2, 3, \dots$$

Prove by induction that its  $n^{\text{th}}$  term is given by

$$a_n = 7 - 4n, \quad n = 1, 2, 3, \dots$$

proof

The image shows a handwritten proof for the induction problem. It starts with the formula  $a_n = 7 - 4n$  circled in pink. The proof is organized into two main bullet points. The first bullet point is for the base case: "if  $n=1$ ,  $a_1 = 7 - 4(1) = 3$  is the result of the relation". Below this, it shows the calculation:  $a_1 = 7 - 4(1)$ ,  $a_1 - 4 = (7 - 4(1)) - 4$ , and  $a_1 - 4 = 3 - 4(1)$ , leading to  $a_{1+1} = 7 - 4(1+1)$ . The second bullet point is for the inductive step: "if the result holds for  $n=k \in \mathbb{N}$ , then it also holds for  $n=k+1$  since it holds for  $n=1$ , then it must hold for  $n$ ".

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Question 2 (\*\*)

A sequence of integers  $t_1, t_2, t_3, \dots$  is given by the recurrence relation

$$t_{n+1} = 3t_n + 2, \quad t_1 = 1, \quad n \in \mathbb{N}.$$

Prove by induction that its  $n^{\text{th}}$  term of the sequence is given by

$$t_n = 2 \times 3^{n-1} - 1, \quad n \in \mathbb{N}.$$

,  proof

The image shows a handwritten mathematical proof for the induction problem. It is organized into several sections:

- Initial Equations:** At the top, two equations are written in blue ink and enclosed in cloud-like shapes. The first is  $t_{n+1} = 3t_n + 2$  for  $n \in \mathbb{N}$ . The second is  $t_n = 2 \times 3^{n-1} - 1$  for  $n \in \mathbb{N}$ . A double-headed arrow connects these two equations.
- BASE CASE:** Below the equations, it states  $t_1 = 1$  and  $t_1 = 2 \times 3^{1-1} - 1 = 2 \times 1 - 1 = 1$ . A bracket on the right side of the second equation is labeled "Result holds for  $n=1$ ".
- INDUCTION HYPOTHESIS:** It says "SUPPOSE THAT THE RESULT HOLDS FOR  $n=k, k \in \mathbb{N}$ ".
- Inductive Step:** A series of steps are shown with arrows:  $t_k = 2 \times 3^{k-1} - 1$ ;  $3t_k = 3[2 \times 3^{k-1} - 1]$ ;  $3t_k = 2 \times 3^k - 3$ ;  $3t_k + 2 = 2 \times 3^k - 3 + 2$ ;  $t_{k+1} = 2 \times 3^k - 1$ .
- CONCLUSION:** It states "IF THE RESULT HOLDS FOR  $n=k, k \in \mathbb{N}$ , THEN IT ALSO HOLDS FOR  $n=k+1$  SINCE THE RESULT HOLDS FOR  $n=1$ , THEN IT MUST HOLD FOR ALL  $n$ ".

**Question 3 (\*\*)**

A sequence of integers is defined inductively by the relation

$$a_{n+1} = 3a_n + 4, \quad a_1 = 3, \quad n = 1, 2, 3, \dots$$

Prove by induction that its  $n^{\text{th}}$  term is given by

$$a_n = 5 \times 3^{n-1} - 2, \quad n = 1, 2, 3, \dots$$

proof

$a_{n+1} = 3a_n + 4, a_1 = 3$  is the same as  $a_n = 5 \times 3^{n-1} - 2$   
 \* IF  $n=1$   $a_1 = 3$   
 $a_1 = 5 \times 3^0 - 2 = 3$  } BOTH AGREE ON THE FIRST TERM  
 • SUPPOSE THE RESULT HOLDS FOR  $n=k \in \mathbb{N}$   
 $\Rightarrow a_k = 5 \times 3^{k-1} - 2$   
 $\Rightarrow 3a_k = 3(5 \times 3^{k-1}) - 6$   
 $\Rightarrow 3a_k = 5 \times 3^k - 6$   
 $\Rightarrow 3a_k + 4 = 5 \times 3^k - 2$   
 $\Rightarrow a_{k+1} = 5 \times 3^{k+1} - 2$   
 • IF THE RESULT HOLDS FOR  $n=k \in \mathbb{N}$ , THEN IT MUST ALSO HOLD FOR  $n=k+1$   
 SINCE THE RESULT HOLDS FOR  $n=1$ , THEN IT MUST HOLD  $\forall n \in \mathbb{N}$

**Question 4 (\*\*)**

The terms of a sequence can be generated by the recurrence relation

$$b_{n+1} = 4b_n + 2, \quad b_1 = 2, \quad n = 1, 2, 3, \dots$$

Prove by induction that the  $n^{\text{th}}$  term of the sequence is given by

$$b_n = \frac{2}{3}(4^n - 1), \quad n = 1, 2, 3, \dots$$

proof

$b_{n+1} = 4b_n + 2, b_1 = 2$  and  $b_n = \frac{2}{3}(4^n - 1)$   
 • IF  $n=1$   $b_1 = 2$   
 $b_1 = \frac{2}{3}(4^1 - 1) = \frac{2}{3} \times 3 = 2$  } I.E. RESULT HOLDS FOR  $n=1$   
 • SUPPOSE THE RESULT HOLDS FOR  $n=k \in \mathbb{N}$   
 $b_k = \frac{2}{3}(4^k - 1)$   
 $4b_k = 4 \times \frac{2}{3}(4^k - 1) = \frac{8}{3}(4^k - 1)$   
 $4b_k + 2 = \frac{8}{3}(4^k - 1) + 2 = \frac{8}{3}(4^k) - \frac{8}{3} + 2 = \frac{8}{3}(4^k) - \frac{2}{3}$   
 $b_{k+1} = \frac{2}{3}(4^{k+1} - 1)$   
 $b_{k+1} = \frac{2}{3}(4^{k+1} - 1)$   
 • IF THE RESULT HOLDS FOR  $n=k \in \mathbb{N}$ , THEN IT MUST ALSO HOLD FOR  $n=k+1$   
 SINCE THE RESULT HOLDS FOR  $n=1$ , THEN IT MUST HOLD  $\forall n \in \mathbb{N}$

Question 5 (\*\*)

A sequence is defined by the recurrence relation

$$u_{n+1} = 7u_n - 3, \quad u_1 = 7, \quad n = 1, 2, 3, \dots$$

Prove by induction that its  $n^{\text{th}}$  term is given by

$$u_n = \frac{1}{2}(13 \times 7^{n-1} + 1), \quad n = 1, 2, 3, \dots$$

proof

Handwritten mathematical proof for the induction step:

- $u_n = 7u_{n-1} - 3, u_1 = 7$  and  $u_n = \frac{1}{2}(13 \times 7^{n-1} + 1)$
- $u_1 = 7$   
 $u_1 = \frac{1}{2}(13 \times 7^0 + 1) = \frac{1}{2}(14) = 7$  ✓ if  $n=1$  then formula proves the first term
- SUPPOSE THAT THE  $n^{\text{th}}$  TERM FORMULA PROVES CORRECTLY THE  $k^{\text{th}}$  TERM,  $k \in \mathbb{N}$   
 $u_k = \frac{1}{2}(13 \times 7^{k-1} + 1)$   
 $7u_k - 3 = 7 \times \frac{1}{2}(13 \times 7^{k-1} + 1) - 3 = \frac{1}{2}(13 \times 7^k + 7) - 3$   
 $7u_k - 3 = \frac{1}{2}(13 \times 7^k + 7) - 3$   
 $u_{k+1} = \frac{1}{2}(13 \times 7^k + 7) - 3$   
 $u_{k+1} = \frac{1}{2}(13 \times 7^k + 1)$
- IF THE RESULT HOLDS FOR  $n=k \in \mathbb{N}$ , THEN IT ALSO HOLDS FOR  $n=k+1$ , SINCE THE RESULT HOLDS FOR  $n=1$ , THEN THE RESULT MUST HOLD  $\forall n \in \mathbb{N}$

**Question 6 (\*\*)**

A sequence of integers  $a_1, a_2, a_3, a_4, \dots$  is given by

$$a_{n+1} = 3a_n + 2, \quad a_1 = 2, \quad n = 1, 2, 3, \dots$$

Prove by induction that its  $n^{\text{th}}$  term is given by

$$a_n = 3 \times 3^{n-1} - 1, \quad n = 1, 2, 3, \dots$$

proof

The handwritten proof is enclosed in a black box and contains the following text:

$a_{n+1} = 3a_n + 2, a_1 = 2$  is the same as  $a_n = 3 \times 3^{n-1} - 1$

- If  $n=1$ ,  $a_1 = 2$   
 $a_1 = 3 \times 3^{1-1} - 1 = 2$  ✓ is true since on the first time
- Suppose that the result holds for  $n = k \in \mathbb{N}$ ,  
 $a_k = 3 \times 3^{k-1} - 1$   
 $3a_k = 3[3^k - 1]$   
 $3a_k + 2 = 3[3^k - 1] + 2$   
 $a_{k+1} = 3 \times 3^k - 3 + 2$   
 $a_{k+1} = 3 \times 3^{k+1-1} - 1$
- If the result holds for  $n = k \in \mathbb{N}$ , then it must also hold for  $n = k+1$  since it holds for  $n=1$ , thus it must hold  $\forall n \in \mathbb{N}$

Question 7 (\*\*\*)

A certain sequence can be generated by the recurrence relation

$$u_{n+1} = \frac{1}{3}(2u_n - 1), \quad u_1 = 1, \quad n = 1, 2, 3, \dots$$

Prove by induction that the  $n^{\text{th}}$  term of the sequence is given by

$$u_n = 3\left(\frac{2}{3}\right)^n - 1, \quad n = 1, 2, 3, \dots$$

proof

Handwritten mathematical proof for the induction step of the sequence problem. The proof shows the base case  $u_1 = 3\left(\frac{2}{3}\right)^1 - 1 = 2 - 1 = 1$ . It then assumes the formula holds for  $n=k$  and shows that  $u_{k+1} = 3\left(\frac{2}{3}\right)^{k+1} - 1$ . The steps are:  $u_k = 3\left(\frac{2}{3}\right)^k - 1$ ,  $\Rightarrow 2u_k = 6\left(\frac{2}{3}\right)^k - 2$ ,  $\Rightarrow 2u_k - 1 = 6\left(\frac{2}{3}\right)^k - 3$ ,  $\Rightarrow \frac{1}{3}(2u_k - 1) = 2\left(\frac{2}{3}\right)^k - 1$ ,  $\Rightarrow u_{k+1} = 3 \times \frac{2}{3} \times \left(\frac{2}{3}\right)^k - 1$ ,  $\Rightarrow u_{k+1} = 3 \times \left(\frac{2}{3}\right)^{k+1} - 1$ . The proof concludes by stating that if the formula holds for  $n=k \in \mathbb{N}$ , then it also holds for  $n=k+1$ , and since it holds for  $n=1$ , it holds for all  $n \in \mathbb{N}$ .

**Question 8** (\*\*\*)

A sequence is defined recursively by

$$u_{n+1} = \frac{3}{4 - u_n}, \quad u_1 = \frac{3}{4}, \quad n = 1, 2, 3, \dots$$

Prove by induction that

$$u_n = \frac{3^{n+1} - 3}{3^{n+1} - 1}, \quad n = 1, 2, 3, \dots$$

,  proof

The image shows two pages of handwritten mathematical work. The left page contains the following text:

$u_{n+1} = \frac{3}{4 - u_n}$   
 $u_1 = \frac{3}{4}$   
 $n = 1, 2, 3, \dots$

BASE CASE  
 $u_1 = \frac{3}{4}$   
 $u_1 = \frac{3^{1+1} - 3}{3^{1+1} - 1} = \frac{9 - 3}{9 - 1} = \frac{6}{8} = \frac{3}{4}$  } 16 RESULT HAS FOR  $n=1$

INDUCTIVE HYPOTHESIS  
 SUPPOSE THAT THE RESULT HAS FOR  $n=k, k \in \mathbb{N}$

$\Rightarrow u_k = \frac{3^{k+1} - 3}{3^{k+1} - 1}$   
 $\Rightarrow -u_k = -\frac{3^{k+1} - 3}{3^{k+1} - 1}$   
 $\Rightarrow 4 - u_k = 4 - \frac{3^{k+1} - 3}{3^{k+1} - 1}$   
 $\Rightarrow 4 - u_k = \frac{4(3^{k+1} - 1) - (3^{k+1} - 3)}{3^{k+1} - 1}$   
 $\Rightarrow 4 - u_k = \frac{3 \times 3^{k+1} - 1}{3^{k+1} - 1}$   
 $\Rightarrow \frac{1}{4 - u_k} = \frac{3^{k+1} - 1}{3 \times 3^{k+1} - 1}$   
 $\Rightarrow \frac{3}{4 - u_k} = 3 \left[ \frac{3^{k+1} - 1}{3 \times 3^{k+1} - 1} \right]$

The right page contains the following text:

$\Rightarrow u_{k+1} = \frac{3^{k+2} - 3}{3^{k+2} - 1}$   
 $\Rightarrow u_{k+1} = \frac{3^{k+1} - 3}{3^{k+1} - 1}$

CONCLUSION  
 IF THE RESULT HAS FOR  $n=k, k \in \mathbb{N}$  THEN IT ALSO HAS FOR  $n=k+1$   
 SINCE THE RESULT HAS FOR  $n=1$ , THEN IT MUST HOLD FOR ALL  $n \in \mathbb{N}$

**Question 9 (\*\*\*)**

A sequence is defined recursively by

$$u_{n+1} = u_n + 3k - 2, \quad u_1 = 3, \quad n = 1, 2, 3, \dots$$

Prove by induction that

$$u_n = \frac{1}{2}(3n-1)(n-2) + 4, \quad n = 1, 2, 3, \dots$$

 , proof

PROVE TO (42) THAT THE WH. TERM OF THE RECURSIVE RELATION

IS GIVEN BY  $u_n = \frac{1}{2}(3n-1)(n-2) + 4$

PROVE IF  $n=1$   
 $u_1 = \frac{1}{2} \times 2 \times (-1) + 4 = -1 + 4 = 3$  ∴ THE RESULT HAS BE.  $n=1$

SUPPOSE THAT THE RESULT HELDS FOR  $n=k \in \mathbb{N}$

→  $u_k = \frac{1}{2}(3k-1)(k-2) + 4$   
 $u_k + 3k - 2 = \frac{1}{2}(3k-1)(k-2) + 4 + 3k - 2$   
 $u_{k+1} = \frac{1}{2}(3k-1)(k-2) + 3k + 2$   
 $u_{k+1} = \frac{1}{2}[(3k-1)(k-2) + 6k + 4]$   
 $u_{k+1} = \frac{1}{2}[3k^2 - 5k + 2 + 6k + 4]$   
 $u_{k+1} = \frac{1}{2}[3k^2 - k + 6]$   
 $u_{k+1} = \frac{1}{2}[(3k+2)(k-1) + 6]$   
 $u_{k+1} = \frac{1}{2}(3k+2)(k-1) + 4$   
 $u_{k+1} = \frac{1}{2}(3(k+1)-1)[(k+1)-2] + 4$

IF THE RESULT HELDS FOR  $n=k \in \mathbb{N}$ , THEN IT MUST ALSO HOLD FOR  $n=k+1$   
SINCE THE RESULT HELDS FOR  $n=1$ , THEN IT MUST HOLD FOR ALL  $n \in \mathbb{N}$

**Question 10 (\*\*\*)**

A sequence is defined recursively by

$$u_{n+1} = \frac{u_n}{u_n + 1}, u_1 = 2, n \in \mathbb{N}.$$

By writing the above recurrence relation in the form

$$u_{n+1} = A + \frac{B}{u_n + 1},$$

where  $A$  and  $B$  are integers, use proof by induction to show that

$$u_n = \frac{2}{2n-1}, n \in \mathbb{N}.$$

,  proof

$$u_{n+1} = \frac{u_n}{u_n + 1}, n \in \mathbb{N} \iff u_n = \frac{2}{2n-1}, n \in \mathbb{N}$$

$$u_1 = 2$$

SIMPLY BY WRITING THE RECURSION RELATION  

$$u_{n+1} = \frac{u_n}{u_n + 1} = \frac{(u_n + 1) - 1}{(u_n + 1)} = 1 - \frac{1}{u_n + 1}$$

BASE CASE, i.e.  $n=1$   
 $u_1 = 2$   
 $u_1 = \frac{2}{2 \times 1 - 1} = 2$  } i.e. the result holds for  $n=1$

INDUCTIVE HYPOTHESIS  
 SUPPOSE THAT THE RESULT HOLDS FOR  $n=k, k \in \mathbb{N}$   
 $\Rightarrow u_k = \frac{2}{2k-1}$   
 $\Rightarrow u_k + 1 = \frac{2}{2k-1} + 1 = \frac{2 + (2k-1)}{2k-1} = \frac{2k+1}{2k-1}$   
 $\Rightarrow \frac{1}{u_k + 1} = \frac{2k-1}{2k+1}$   
 $\Rightarrow 1 - \frac{1}{u_k + 1} = 1 - \frac{2k-1}{2k+1} = \frac{(2k+1) - (2k-1)}{2k+1} = \frac{2}{2k+1}$   
 $\Rightarrow u_{k+1} = \frac{2}{2(k+1)-1}$

CONCLUSION  
 IF THE RESULT HOLDS FOR  $n=k, k \in \mathbb{N}$  THEN IT ALSO HOLDS FOR  $n=k+1$   
 SINCE THE RESULT HOLDS FOR  $n=1$ , THEN IT MUST HOLD FOR ALL  $n \in \mathbb{N}$

Question 11 (\*\*\*)

A sequence is generated by the recurrence relation

$$u_{n+2} = 5u_{n+1} - 6u_n, \quad u_1 = 5, \quad u_2 = 13, \quad n = 1, 2, 3, \dots$$

Prove by induction that  $n^{\text{th}}$  term of this sequence is given by

$$u_n = 2^n + 3^n, \quad n = 1, 2, 3, \dots$$

proof

Handwritten mathematical proof for the recurrence relation problem. The proof is written in a notebook style with various annotations and corrections.

Initial conditions:  $u_1 = 5$ ,  $u_2 = 13$ . The goal is to prove  $u_n = 2^n + 3^n$ .

Step 1: Check for  $n=1$  and  $n=2$ .  
 $u_1 = 2^1 + 3^1 = 2 + 3 = 5$   
 $u_2 = 2^2 + 3^2 = 4 + 9 = 13$

Step 2: Assume the result holds for two consecutive terms  $n=k$  and  $n=k+1$ .  
 $u_k = 2^k + 3^k$   
 $u_{k+1} = 2^{k+1} + 3^{k+1}$

Step 3: Calculate  $5u_{k+1} - 6u_k$  and show it equals  $2^{k+2} + 3^{k+2}$ .  
 $5u_{k+1} - 6u_k = 5(2^{k+1} + 3^{k+1}) - 6(2^k + 3^k)$   
 $= 5 \times 2^{k+1} + 5 \times 3^{k+1} - 6 \times 2^k - 6 \times 3^k$   
 $= (5 \times 2^{k+1} - 6 \times 2^k) + (5 \times 3^{k+1} - 6 \times 3^k)$   
 $= (10 \times 2^k - 6 \times 2^k) + (15 \times 3^k - 6 \times 3^k)$   
 $= 4 \times 2^k + 9 \times 3^k$   
 $= 2^2 \times 2^k + 3^2 \times 3^k$   
 $= 2^{k+2} + 3^{k+2}$

Step 4: Conclude the proof by induction.  
 If the result holds for  $n=k \in \mathbb{N}$  and  $n=k+1$ , then it must also hold for  $n=k+2$ .  
 Since the result holds for  $n=1$  and  $n=2$ , then it must hold  $\forall n \in \mathbb{N}$ .

Question 12 (\*\*\*\*)

A sequence is generated by the recurrence relation

$$u_{n+2} = 6u_{n+1} - 8u_n, \quad u_1 = 0, \quad u_2 = 32, \quad n = 1, 2, 3, \dots$$

Prove by induction that  $n^{\text{th}}$  term of this sequence is given by

$$u_n = 4^{n+1} - 2^{n+3}, \quad n = 1, 2, 3, \dots$$

proof

$u_{n+2} = 6u_{n+1} - 8u_n$  IS THE SAME AS  $u_n = 4^{n+1} - 2^{n+3}$   
 $u_1 = 0$   $u_2 = 32$   
 IF  $n=1$   $u_1 = 4^{1+1} - 2^{1+3} = 16 - 8 = 8 \neq 0$  IT DOESN'T WORK  
 IF  $n=2$   $u_2 = 4^{2+1} - 2^{2+3} = 64 - 32 = 32$  IT DOES WORK  
 • SHOW THAT THE  $n^{\text{th}}$  TERM FORMULA GENERATES CORRECTLY THE  
 SUBSEQUENT TERMS OF THE SEQUENCE I.E.  $u_n$  &  $u_{n+1}$  WORKS!  
 THEN  
 $6u_{n+1} - 8u_n = 6(4^{n+2} - 2^{n+4}) - 8(4^{n+1} - 2^{n+3})$   
 $= 6 \times 4^{n+2} - 6 \times 2^{n+4} - 8 \times 4^{n+1} + 8 \times 2^{n+3}$   
 $= 3 \times 2 \times 2^{2n+4} - 3 \times 2^{n+4} - 2 \times 2^{n+4} + 2 \times 2^{n+3}$   
 $= 6 \times 2^{2n+3} - 3 \times 2^{n+4} - 2 \times 2^{n+4} + 2 \times 2^{n+3}$   
 $= 6 \times 2^{2n+3} - 5 \times 2^{n+4} + 2 \times 2^{n+3}$   
 $= 4^{n+2} - 2^{n+3}$   
 COMPARE WITH  $u_{n+2} = 4^{n+2} - 2^{n+3}$   
 • IF THE  $n^{\text{th}}$  TERM FORMULA PRODUCES CORRECTLY ANY TWO CONSECUTIVE  
 TERMS OF THE SEQUENCE, THEN IT PRODUCES CORRECTLY THE NEXT  
 ONE ALSO.  
 SINCE THE  $n^{\text{th}}$  TERM FORMULA PRODUCES CORRECTLY THE FIRST TWO  
 TERMS THEN IT PRODUCES CORRECTLY EVERY TERM OF THE SEQUENCE.

Question 13 (\*\*\*\*)

A sequence is generated by the recurrence relation

$$u_{n+2} = u_{n+1} + u_n, \quad u_1 = 0, \quad u_2 = 1, \quad n = 1, 2, 3, \dots$$

Prove by induction that  $u_{5m}$  is a multiple of 5, for all  $m \in \mathbb{N}$ .

,  proof

BASE CASE FOR  $u_{5k} = 5m$   
 $u_1 = 0, u_2 = 1, u_3 = 2, u_4 = 3, u_5 = 5$   
INDUCTION STEP: AS  $u_5$  IS A MULTIPLE OF 5  
SUPPOSE THAT THE RESULT HOLDS, i.e.  $u_{5k}$  IS A MULTIPLE OF 5, i.e.  $u_{5k} = 5m$   
FOR  $m \in \mathbb{N}$   
 $\Rightarrow u_{5k} = u_{5k-1} + u_{5k-2} = 5m$   
 $\Rightarrow u_{5(k+1)} = (u_{5k} + u_{5k-1}) + u_{5k}$   
 $= 2u_{5k} + u_{5k-1}$   
 $\Rightarrow u_{5(k+1)} = 2(5m) + u_{5k-1} + u_{5k-2}$   
 $= 3(u_{5k}) + 2u_{5k-1}$   
 $\Rightarrow u_{5(k+1)} = 3(5m) + 2u_{5k-1}$   
 $= 15m + 2u_{5k-1}$   
 $= 5(3m + 2u_{5k-1})$   
IS A MULTIPLE OF 5  
IF THE RESULT HOLDS FOR  $n = 5k$  THEN IT HOLDS FOR  $n = 5(k+1)$   
SINCE IT HOLDS FOR  $n = 5$  THEN IT HOLDS FOR ALL MULTIPLES OF 5

Question 14 (\*\*\*\*+)

A sequence of numbers is given by the recurrence relation

$$u_{n+1} = \frac{5u_n - 1}{4u_n + 1}, u_1 = 1, n \in \mathbb{N}, n \geq 1.$$

Prove by induction that the  $n^{\text{th}}$  term of the sequence is given by

$$u_n = \frac{n+2}{2n+1}.$$

,  proof

The image shows two pages of handwritten mathematical work on grid paper. The left page contains the following text:

$u_{n+1} = \frac{5u_n - 1}{4u_n + 1}, u_1 = 1 \implies u_n = \frac{n+2}{2n+1}$

• STATE THE PROPOS BY REWRITING THE RECURRENCE

$$u_{n+1} = \frac{5(\frac{n+2}{2n+1}) - 1}{4(\frac{n+2}{2n+1}) + 1} = \frac{5}{4} - \frac{1}{4(n+1)}$$

$$\therefore u_{n+1} = \frac{5}{4} - \frac{1}{4(n+1)}$$

• BASE CASE  
 $n=1, u_1 = \frac{1+2}{2(1)+1} = \frac{3}{3} = 1$ , THE RESULT HOLDS FOR  $n=1$

• INDUCTIVE HYPOTHESIS  
 SUPPOSE THAT THE RESULT HOLDS FOR  $n=k, k \in \mathbb{N}$

$$\implies u_k = \frac{k+2}{2k+1}$$

$$\implies 4u_{k+1} = 4\left(\frac{k+2}{2k+1}\right) + 1 = \frac{4k+8}{2k+1} + 1 = \frac{4k+8+2k+1}{2k+1}$$

$$= \frac{6k+9}{2k+1}$$

$$\implies \frac{1}{4u_{k+1}} = \frac{2k+1}{6k+9}$$

$$\implies \frac{5}{4} - \frac{1}{4u_{k+1}} = \frac{5}{4} - \frac{2k+1}{6k+9} = \frac{5(6k+9) - 4(2k+1)}{2k+1} = \frac{30k+45-8k-4}{2k+1} = \frac{22k+41}{2k+1}$$

The right page contains the following text:

$$\implies \frac{5}{4} - \frac{1}{4u_{k+1}} = \frac{5}{4} - \frac{6k+9}{6k+9} = \frac{5(6k+9) - (6k+9)}{6k+9} = \frac{5(6k+9) - (6k+9)}{6k+9}$$

$$\implies u_{k+1} = \frac{k+3}{2(k+1)} = \frac{k+3}{2k+2}$$

$$\implies u_{k+1} = \frac{(k+1)+2}{2(k+1)+1}$$

• CONCLUSION OF THE PROOF  
 IF THE RESULT HOLDS FOR  $n=k, k \in \mathbb{N}$ , THEN IT ALSO HOLDS FOR  $n=k+1$ . SINCE THE RESULT HOLDS FOR  $n=1$ , THEN IT HOLDS FOR ALL  $n \in \mathbb{N}$

Question 15 (\*\*\*\*+)

A sequence of numbers is given by the recurrence relation

$$u_{n+1} = \frac{u_n - 5}{3u_n - 7}, u_1 = -1, n \in \mathbb{N}, n \geq 1.$$

Prove by induction that the  $n^{\text{th}}$  term of the sequence is given by

$$u_n = \frac{2^{n+1} - 5}{2^{n+1} - 3}.$$

, proof

REMOVE THE RECURRENCE FIRST

$$u_{k+1} = \frac{u_k - 5}{3u_k - 7} = \frac{1}{3} \left( \frac{u_k - 5}{u_k - \frac{7}{3}} \right) = \frac{1}{3} \left( \frac{u_k - \frac{7}{3} - \frac{8}{3}}{u_k - \frac{7}{3}} \right)$$

$$u_{k+1} = \frac{1}{3} \left[ 1 - \frac{\frac{8}{3}}{u_k - \frac{7}{3}} \right]$$

$$u_{k+1} = \frac{1}{3} - \frac{8}{9(u_k - \frac{7}{3})}$$

$u_{k+1} = \frac{1}{3} - \frac{8}{9(u_k - \frac{7}{3})}$

SUPPOSE THAT THE RESULT HOLDS FOR  $n=k \in \mathbb{N}$

$$\rightarrow u_k = \frac{2^{k+1} - 5}{2^{k+1} - 3}$$

$$\rightarrow 9(u_k - \frac{7}{3}) = 9 \left( \frac{2^{k+1} - 5}{2^{k+1} - 3} - \frac{7}{3} \right) - 21$$

$$\rightarrow 9(u_k - \frac{7}{3}) = \frac{9(2^{k+1} - 5) - 7(2^{k+1} - 3) - 63}{2^{k+1} - 3} - 21$$

$$\rightarrow 9(u_k - \frac{7}{3}) = \frac{9 \times 2^{k+1} - 45 - 7 \times 2^{k+1} + 21 - 63}{2^{k+1} - 3}$$

$$\rightarrow 9(u_k - \frac{7}{3}) = \frac{-2 \times 2^{k+1} - 87}{2^{k+1} - 3}$$

$$\rightarrow \frac{1}{9(u_k - \frac{7}{3})} = \frac{-2 \times 2^{k+1} - 87}{-2 \times 2^{k+1} + 10}$$

$$\rightarrow \frac{1}{9(u_k - \frac{7}{3})} = \frac{2^{k+1} + 43.5}{2^{k+1} - 5}$$

$$\rightarrow \frac{1}{3} - \frac{8}{9(u_k - \frac{7}{3})} = \frac{1}{3} + \frac{8 \times (2^{k+1} + 43.5)}{12 \times (2^{k+1} - 5)}$$

$$\rightarrow u_{k+1} = \frac{1}{3} + \frac{4 \times 2^{k+1} - 12}{6 \times 2^{k+1} - 9}$$

$$\rightarrow u_{k+1} = \frac{6 \times 2^{k+1} - 9 + 4 \times 2^{k+1} - 12}{3(6 \times 2^{k+1} - 9)}$$

$$\rightarrow u_{k+1} = \frac{10 \times 2^{k+1} - 21}{3 \times (6 \times 2^{k+1} - 9)}$$

$$\rightarrow u_{k+1} = \frac{2 \times 2^{k+1} - 5}{2 \times 2^{k+1} - 3}$$

$$\rightarrow u_{k+1} = \frac{2^{k+2} - 5}{2^{k+2} - 3}$$

COMPARE WITH  $u_n = \frac{2^{n+1} - 5}{2^{n+1} - 3}$

IF THE RESULT HOLDS FOR  $n=k \in \mathbb{N}$ , THEN IT ALSO HOLDS FOR  $n=k+1$

TEST IF  $n=1$   $u_1 = \frac{2^{1+1} - 5}{2^{1+1} - 3} = \frac{4 - 5}{4 - 3} = \frac{-1}{1} = -1$ , IF RESULT HOLDS FOR  $n=1$ , ALSO

∴ THE RESULT HOLDS FOR ALL  $n \in \mathbb{N}$

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# POWERS OF MATRICES

Created by T. Madas

**Question 1 (\*\*)**

Prove by induction that

$$\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}^n = \begin{pmatrix} 1 & 2^n - 1 \\ 0 & 2^n \end{pmatrix}, \quad n \geq 1, n \in \mathbb{N}$$

proof

• TAKE CASE  $n=1$

$$\text{LHS} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}^1 = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$

$$\text{RHS} = \begin{pmatrix} 1 & 2^1 - 1 \\ 0 & 2^1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$

∴ RESULT HOLDS FOR  $n=1$

• SUPPOSE THE RESULT HOLDS FOR  $n=k \in \mathbb{N}$

$$\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}^k = \begin{pmatrix} 1 & 2^k - 1 \\ 0 & 2^k \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}^k = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2^k - 1 \\ 0 & 2^k \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}^{k+1} = \begin{pmatrix} 1+0 & 2^k - 1 + 2^k \\ 0+0 & 0+2 \times 2^k \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & 2 \times 2^k - 1 \\ 0 & 2 \times 2^k \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & 2^{k+1} - 1 \\ 0 & 2^{k+1} \end{pmatrix}$$

• IF THE RESULT HOLDS FOR  $n=k \in \mathbb{N}$ , THEN IT ALSO HOLDS FOR  $n=k+1$   
 SINCE THE RESULT HOLDS FOR  $n=1$ , THEN IT HOLDS FOR ALL  $n \in \mathbb{N}$

Question 2 (\*\*)

A transformation where  $\mathbb{R}^2 \mapsto \mathbb{R}^2$  is defined by

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

- a) Find the elements of the matrices,  $\mathbf{A}^2$  and  $\mathbf{A}^3$ .
- b) Write down a suitable form for  $\mathbf{A}^n$  and use the method of proof by induction to prove it.

,  $\mathbf{A}^2 = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$  ,  $\mathbf{A}^3 = \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}$  ,  $\mathbf{A}^n = \begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix}$

a) CARRY OUT THE REQUIRED "MULTIPLICATIONS"

$$\mathbf{A}^2 = \mathbf{A}\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 \times 1 + 2 \times 0 & 1 \times 2 + 2 \times 1 \\ 0 \times 1 + 1 \times 0 & 0 \times 2 + 1 \times 1 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{A}^3 = \mathbf{A}^2\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 \times 1 + 4 \times 0 & 1 \times 2 + 4 \times 1 \\ 0 \times 1 + 1 \times 0 & 0 \times 2 + 1 \times 1 \end{pmatrix} = \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}$$

b) A POSSIBLE FORM OF  $\mathbf{A}^n$  MIGHT BE

$$\mathbf{A}^n = \begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix}$$

- IF  $n=1$ ,  $\mathbf{A}^1 = \mathbf{A} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ , i.e. THE RESULT STANDS
- SUPPOSE THAT THE RESULT STANDS FOR  $n=k \in \mathbb{N}$ 

$$\Rightarrow \mathbf{A}^k = \begin{pmatrix} 1 & 2k \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \mathbf{A}^k \mathbf{A} = \begin{pmatrix} 1 & 2k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \mathbf{A}^{k+1} = \begin{pmatrix} 1 \times 1 + 2k \times 0 & 1 \times 2 + 2k \times 1 \\ 0 \times 1 + 1 \times 0 & 0 \times 2 + 1 \times 1 \end{pmatrix}$$

$$\Rightarrow \mathbf{A}^{k+1} = \begin{pmatrix} 1 & 2k+2 \\ 0 & 1 \end{pmatrix}$$

IF THE RESULT STANDS FOR  $n=k \in \mathbb{N}$ , THEN IT ALSO STANDS FOR  $n=k+1$   
 SINCE THE RESULT STANDS FOR  $n=1$ , THEN IT MUST ALSO BE TRUE FOR ALL  $n \in \mathbb{N}$

**Question 3 (\*\*)**

Prove by induction that if  $n \geq 1, n \in \mathbb{N}$ , then

$$\begin{pmatrix} -2 & -1 \\ 9 & 4 \end{pmatrix}^n = \begin{pmatrix} 1-3n & -n \\ 9n & 3n+1 \end{pmatrix}.$$

proof

The handwritten proof is as follows:

- If  $n=1$ ,  $\begin{pmatrix} -2 & -1 \\ 9 & 4 \end{pmatrix}^1 = \begin{pmatrix} -2 & -1 \\ 9 & 4 \end{pmatrix}$   
 $\begin{pmatrix} 1-3(1) & -1 \\ 9(1) & 3(1)+1 \end{pmatrix} = \begin{pmatrix} -2 & -1 \\ 9 & 4 \end{pmatrix}$   
 ✓ The result holds for  $n=1$
- Suppose the result holds for  $n=k \in \mathbb{N}$   
 $\begin{pmatrix} -2 & -1 \\ 9 & 4 \end{pmatrix}^k = \begin{pmatrix} 1-3k & -k \\ 9k & 3k+1 \end{pmatrix}$   
 $\begin{pmatrix} -2 & -1 \\ 9 & 4 \end{pmatrix} \begin{pmatrix} -2 & -1 \\ 9 & 4 \end{pmatrix}^k = \begin{pmatrix} -2 & -1 \\ 9 & 4 \end{pmatrix} \begin{pmatrix} 1-3k & -k \\ 9k & 3k+1 \end{pmatrix}$   
 $\begin{pmatrix} -2 & -1 \\ 9 & 4 \end{pmatrix}^{k+1} = \begin{pmatrix} -2(1-3k) - 1k & -2(-k) - 1(3k+1) \\ 9(1-3k) + 3k & 9(-k) + 4(3k+1) \end{pmatrix}$   
 $\begin{pmatrix} -2 & -1 \\ 9 & 4 \end{pmatrix}^{k+1} = \begin{pmatrix} -2+6k-k & 2k-3k-1 \\ 9-27k+3k & -9k+12k+4 \end{pmatrix}$   
 $\begin{pmatrix} -2 & -1 \\ 9 & 4 \end{pmatrix}^{k+1} = \begin{pmatrix} -3k-2 & -k-1 \\ 9(1-k) & 3k+4 \end{pmatrix}$
- If the result holds for  $n=k \in \mathbb{N}$ , then it also holds for  $n=k+1$
- Since the result holds for  $n=1$ , then it holds for  $\forall n \in \mathbb{N}$

Question 4 (\*\*)

$$A = \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}$$

Prove by induction that if  $n \geq 1, n \in \mathbb{N}$ , then

$$A^n = \begin{pmatrix} 3^n & 0 \\ 3(3^n - 1) & 1 \end{pmatrix}.$$

proof

$A = \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}$   
 • IF  $n=1$   $A^1 = \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 3(3^1-1) & 1 \end{pmatrix}$   
 $A^1 = \begin{pmatrix} 3(3^1-1) & 0 \\ 3(3^1-1) & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}$   
 • SUPPOSE THE RESULT HOLDS FOR  $n=k \in \mathbb{N}$   
 $A^k = \begin{pmatrix} 3^k & 0 \\ 3(3^k-1) & 1 \end{pmatrix}$   
 $A^{k+1} = \begin{pmatrix} 3^k & 0 \\ 3(3^k-1) & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix} = \begin{pmatrix} 3^k \cdot 3 & 0 \\ 3(3^k-1) \cdot 3 + 1 \cdot 3 & 3(3^k-1) \cdot 0 + 1 \cdot 1 \end{pmatrix}$   
 $A^{k+1} = \begin{pmatrix} 3^{k+1} & 0 \\ 3(3^{k+1}-1) & 1 \end{pmatrix}$   
 $A^{k+1} = \begin{pmatrix} 3^{k+1} & 0 \\ 3(3^{k+1}-1) & 1 \end{pmatrix}$   
 • IF THE RESULT HOLDS FOR  $n=k \in \mathbb{N}$ , THEN IT ALSO HOLDS FOR  $n=k+1$   
 • SINCE THE RESULT HOLDS FOR  $n=1$ , THEN IT MUST HOLD FOR ALL  $n \in \mathbb{N}$

**Question 5 (\*\*)**

Prove by induction that

$$\begin{pmatrix} 5 & 8 \\ -2 & -3 \end{pmatrix}^n = \begin{pmatrix} 1+4n & 8n \\ -2n & 1-4n \end{pmatrix}, \quad n \geq 1, n \in \mathbb{N}.$$

proof

• IF  $n=1$   
 $\begin{bmatrix} 5 & 8 \\ -2 & -3 \end{bmatrix}^1 = \begin{bmatrix} 5 & 8 \\ -2 & -3 \end{bmatrix}$   
 $\begin{bmatrix} 1+4(1) & 8(1) \\ -2(1) & 1-4(1) \end{bmatrix} = \begin{bmatrix} 5 & 8 \\ -2 & -3 \end{bmatrix}$  } THE RESULT HOLDS FOR  $n=1$

• SUPPOSE THE RESULT HOLDS FOR  $n=k \in \mathbb{N}$   
 $\begin{bmatrix} 5 & 8 \\ -2 & -3 \end{bmatrix}^k = \begin{bmatrix} 1+4k & 8k \\ -2k & 1-4k \end{bmatrix}$   
 $\begin{bmatrix} 5 & 8 \\ -2 & -3 \end{bmatrix}^{k+1} = \begin{bmatrix} 1+4k & 8k \\ -2k & 1-4k \end{bmatrix} \begin{bmatrix} 5 & 8 \\ -2 & -3 \end{bmatrix}$   
 $\begin{bmatrix} 5 & 8 \\ -2 & -3 \end{bmatrix}^{k+1} = \begin{bmatrix} 5+20k-4k & 8+32k-24k \\ -2k-2+4k & -2k-3+12k \end{bmatrix}$   
 $\begin{bmatrix} 5 & 8 \\ -2 & -3 \end{bmatrix}^{k+1} = \begin{bmatrix} 4k+5 & 8k+8 \\ -2k-2 & -4k-3 \end{bmatrix}$   
 $\begin{bmatrix} 5 & 8 \\ -2 & -3 \end{bmatrix}^{k+1} = \begin{bmatrix} 1+4(k+1) & 8(k+1) \\ -2(k+1) & 1-4(k+1) \end{bmatrix}$

• IF THE RESULT HOLDS FOR  $k \in \mathbb{N}$ , THEN IT MUST ALSO HOLD FOR  $k+1$ .  
 SINCE THE RESULT HOLDS FOR  $n=1$ , THEN IT MUST HOLD FOR  $n \in \mathbb{N}$ .

**Question 6 (\*\*)**

$$M = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$$

Prove by induction that

$$M^n = \begin{pmatrix} n+1 & n \\ -n & 1-n \end{pmatrix}, \quad n \geq 1, n \in \mathbb{N}.$$

proof

• IF  $n=1$   
 $M^1 = \begin{pmatrix} 1+1 & 1 \\ -1 & 1-1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} = M$   
 ∴ RESULT HOLDS FOR  $n=1$

• SUPPOSE THAT THE RESULT HOLDS FOR  $n=k \in \mathbb{N}$   
 $M^k = \begin{pmatrix} k+1 & k \\ -k & 1-k \end{pmatrix}$   
 $M^{k+1} = \begin{pmatrix} k+1 & k \\ -k & 1-k \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$   
 $M^{k+1} = \begin{pmatrix} 2k+2-k & 2k+1-k \\ -k-2+k & -k-1-k \end{pmatrix}$   
 $M^{k+1} = \begin{pmatrix} k+2 & k+1 \\ -k-1 & -k \end{pmatrix}$   
 $M^{k+1} = \begin{pmatrix} (k+1)+1 & (k+1) \\ -(k+1) & 1-(k+1) \end{pmatrix}$

• IF THE RESULT HOLDS FOR  $n=k \in \mathbb{N}$ , THEN IT MUST HOLD FOR  $n=k+1$ . SINCE THE RESULT HOLDS FOR  $n=1$ , THEN THE RESULT MUST HOLD FOR  $n \in \mathbb{N}$ .

Question 7 (\*\*\*)

$$A = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$$

Prove by induction that

$$A^n = \begin{pmatrix} 2^n & 3(2^n - 1) \\ 0 & 1 \end{pmatrix}, n \geq 1, n \in \mathbb{N}.$$

proof

• IF  $n=1$   
 $A^1 = \begin{bmatrix} 2^1 & 3(2^1-1) \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} = A$  (ie. result holds for  $n=1$ )

• suppose the result holds for  $n=k \in \mathbb{N}$   
 $A^k = \begin{bmatrix} 2^k & 3(2^k-1) \\ 0 & 1 \end{bmatrix}$   
 $A^{k+1} = \begin{bmatrix} 2 \times 2^k + 3 \times 0 & 2 \times 3(2^k-1) + 3 \times 1 \\ 0 \times 2^k + 1 \times 0 & 0 \times 3(2^k-1) + 1 \times 1 \end{bmatrix}$   
 $A^{k+1} = \begin{bmatrix} 2^{k+1} & 3[2^{k+1}-2+3] \\ 0 & 1 \end{bmatrix}$   
 $A^{k+1} = \begin{bmatrix} 2^{k+1} & 3[2^{k+1}-3+3] \\ 0 & 1 \end{bmatrix}$   
 $A^{k+1} = \begin{bmatrix} 2^{k+1} & 3[2^{k+1}-3+3] \\ 0 & 1 \end{bmatrix}$   
 $A^{k+1} = \begin{bmatrix} 2^{k+1} & 3(2^{k+1}-1) \\ 0 & 1 \end{bmatrix}$

• if the result holds for  $n=k \in \mathbb{N}$ , then it also holds for  $n=k+1$   
 Since the result holds for  $n=1$ , then it also holds for  $n \in \mathbb{N}$

Question 8 (\*\*\*)

Prove by induction that

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n & \frac{1}{2}n(n+1) \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}, n \geq 1, n \in \mathbb{N}$$

proof

• IF  $n=1$   $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$  and  $\begin{pmatrix} 1 & 1 & \frac{1}{2} \times 1 \times (1+1) \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$  ∴ BASE CASE FOR  $n=1$

• SUPPOSE THE RESULT HOLDS FOR  $n=k \in \mathbb{N}$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^k = \begin{pmatrix} 1 & k & \frac{1}{2}k(k+1) \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & k & \frac{1}{2}k(k+1) \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1+k & \frac{1}{2}k(k+1) + k + 1 \\ 0 & 1 & k+1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1+k & \frac{1}{2}k(k+1) + k + 1 \\ 0 & 1 & k+1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1+k & \frac{1}{2}(k+1)(k+2) \\ 0 & 1 & k+1 \\ 0 & 0 & 1 \end{pmatrix}$$

• IF THE RESULT HOLDS FOR  $n=k \in \mathbb{N}$ , THEN IT ALSO HOLDS FOR  $n=k+1$ .  
SINCE THE RESULT HOLDS FOR  $n=1$ , THEN IT HOLDS FOR  $n \in \mathbb{N}$

Question 9 (\*\*+)

Prove by induction that

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n & \frac{1}{2}(n^2 + 3n) \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}, \quad n \geq 1, n \in \mathbb{N}.$$

proof

• If  $n=1$ :  $A^1 = \begin{pmatrix} 1 & 1 & \frac{1}{2}(1^2+3 \cdot 1) \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = A$

• Suppose the result holds for  $n=k \in \mathbb{N}$

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^k = \begin{pmatrix} 1 & k & \frac{1}{2}(k^2+3k) \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & k & \frac{1}{2}(k^2+3k) \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 1+k & 1+k & \frac{1}{2}(k^2+3k) + 2k + \frac{1}{2}(k^2+3k) \\ 0 & 1+k & k+k \\ 0 & 0 & 1+k \end{pmatrix}$$

$$\Rightarrow A^{k+1} = \begin{pmatrix} 1 & k+1 & 2k + \frac{1}{2}(k^2+3k) \\ 0 & 1 & k+1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow A^{k+1} = \begin{pmatrix} 1 & k+1 & \frac{1}{2}(k^2+k+2) \\ 0 & 1 & k+1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow A^{k+1} = \begin{pmatrix} 1 & k+1 & \frac{1}{2}(k^2+3k+4) \\ 0 & 1 & k+1 \\ 0 & 0 & 1 \end{pmatrix}$$

But  $\frac{1}{2}(k^2+3k+4) = \frac{1}{2}(k^2+2k+1 + 2k+3) = \frac{1}{2}(k^2+3k+4)$

$$\Rightarrow A^{k+1} = \begin{pmatrix} 1 & k+1 & \frac{1}{2}(k^2+3k+4) \\ 0 & 1 & k+1 \\ 0 & 0 & 1 \end{pmatrix}$$

• If the result holds for  $n=k \in \mathbb{N} \Rightarrow$  it also holds for  $n=k+1$   
 Since it holds for  $n=1 \Rightarrow$  it now holds for  $\forall n \in \mathbb{N}$

Question 10 (\*\*\*\*\*)

$$A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

Prove by induction that

$$A^n = nA - (n-1)I, \quad n \geq 1, \quad n \in \mathbb{N} .$$

,  proof

$A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ , then  $A^n = nA - (n-1)I$

- CHECK THE RESULT FOR  $n=1$   
 $A^1 = (1 \times A - (1-1)I) = A$  ✓ THE RESULT HOLDS FOR  $n=1$
- SUPPOSE THE RESULT HOLDS FOR  $n=k, k \in \mathbb{N}$   
 $\Rightarrow A^k = kA - (k-1)I$   
 $\Rightarrow A^k A = kA A - (k-1)A$   
 $\Rightarrow A^{k+1} = kA^2 - (k-1)A$
- NOW IN ORDER TO COMPLETE THE MANIPULATION WE NEED TO REPLACE  $A^2$  WITH SOME (SIMPLE COMBINATION) OF  $A$  &  $I$   
 $\Rightarrow A^2 = \lambda A + \mu I$   
 $\Rightarrow \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \lambda \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} + \mu \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$   
 $\Rightarrow \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} \lambda + \mu & 0 \\ 2\lambda & \lambda + \mu \end{pmatrix}$   
 $\begin{matrix} 2\lambda = 4 & \lambda + \mu = 1 \\ \lambda = 2 & 2 + \mu = 1 \\ & \mu = -1 \end{matrix}$   
 $\therefore A^2 = 2A - I$

- RETURNING TO THE MANIPULATION OF THE INDUCTION  
 $\Rightarrow A^{k+1} = k[2A - I] - (k-1)A$   
 $\Rightarrow A^{k+1} = 2kA - kI - kA + A$   
 $\Rightarrow A^{k+1} = kA + A - kI$   
 $\Rightarrow A^{k+1} = (k+1)A - [(k+1)-1]I$
- IF THE RESULT HOLDS FOR  $n=k \in \mathbb{N}$ , THEN THE RESULT ALSO HOLDS FOR  $n=k+1$ . SINCE THE RESULT HOLDS FOR  $n=1$ , THEN THE RESULT HOLDS FOR ALL  $n \in \mathbb{N}$ .

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# MISCELLANEOUS RESULTS

Created by T. Madas

**Question 1** (\*\*\*)

De Moivre's theorem states

$$(\cos \theta + i \sin \theta)^n \equiv \cos n\theta + i \sin n\theta, \quad n \in \mathbb{N}.$$

Prove this theorem by induction.

proof

• If  $n=1$  LHS =  $(\cos \theta + i \sin \theta) = \cos \theta + i \sin \theta$  RHS =  $\cos(1\theta) + i \sin(1\theta) = \cos \theta + i \sin \theta$   $\therefore$  LHS = RHS  
 • Suppose that result holds for  $n=k \in \mathbb{N}$   
 $(\cos \theta + i \sin \theta)^k = \cos k\theta + i \sin k\theta$   
 $(\cos \theta + i \sin \theta)^{k+1} = (\cos \theta + i \sin \theta)(\cos k\theta + i \sin k\theta)$   
 $(\cos \theta + i \sin \theta)^{k+1} = (\cos \theta \cos k\theta - \sin \theta \sin k\theta) + i(\sin \theta \cos k\theta + \cos \theta \sin k\theta)$   
 $(\cos \theta + i \sin \theta)^{k+1} = \cos(k\theta + \theta) + i \sin(k\theta + \theta)$   
 $(\cos \theta + i \sin \theta)^{k+1} = \cos[(k+1)\theta] + i \sin[(k+1)\theta]$   
 • If the result holds for  $n=k \in \mathbb{N} \Rightarrow$  it also holds for  $n=k+1$   
 Since the result holds for  $n=1 \Rightarrow$  it will hold for all  $n$

**Question 2** (\*\*\*)

$$u_n = \frac{3}{7}(8^n - 1), \quad n \in \mathbb{N}.$$

Prove by induction that every term of this sequence is an integer.

proof

$u_n = \frac{3}{7}(8^n - 1)$   
 •  $u_1 = \frac{3}{7}(8^1 - 1) = \frac{3}{7} \times 7 = 3$  (is an integer)  
 • Suppose the result holds for  $n=k \in \mathbb{N}$ , i.e.  $u_k$  is an integer, say  $N$   
 Then  
 $u_{k+1} - u_k = \frac{3}{7}(8^{k+1} - 1) - \frac{3}{7}(8^k - 1)$   
 $u_{k+1} - N = \frac{3}{7}[8^{k+1} - 8^k + 1 - 1]$   
 $u_{k+1} - N = \frac{3}{7}[8 \times 8^k - 8^k]$   
 $u_{k+1} - N = \frac{3}{7} \times 7 \times 8^k$   
 $u_{k+1} = N + 8^k$  which is also an integer  
 • If the result holds for  $n=k \in \mathbb{N}$ , then it also holds for  $n=k+1$   
 Since the result holds for  $n=1$ , then the result will also hold for all  $n$

Question 3 (\*\*\*)

$$\sum_{r=1}^n (2r+1) = (n+1)^2, n \in \mathbb{N}$$

- a) Show that if the above result holds for  $n=k$ , then it also holds for  $n=k+1$ .
- b) Explain why the result is **not** true.

proof

Handwritten solution for part a):

$\sum_{r=1}^n (2r+1) = (n+1)^2$   
 Suppose the result holds for  $n=k \in \mathbb{N}$   
 $\sum_{r=1}^k (2r+1) = (k+1)^2$   
 $\sum_{r=1}^{k+1} (2r+1) = \sum_{r=1}^k (2r+1) + [2(k+1)+1] = (k+1)^2 + [2k+3]$   
 $\sum_{r=1}^{k+1} (2r+1) = (k+1)^2 + (2k+3)$   
 $\sum_{r=1}^{k+1} (2r+1) = k^2 + 2k + 1 + 2k + 3$   
 $\sum_{r=1}^{k+1} (2r+1) = (k+2)^2 = [(k+1)+1]^2$   
 If the result holds for  $n=k \in \mathbb{N}$ , then it must hold for  $n=k+1$

Handwritten solution for part b):

Result is NOT true because there is one case  
 e.g.  
 $n=1$     3  $\neq$  4  
 $n=2$     8  $\neq$  9  
 $n=3$     15  $\neq$  16  
 $n=4$     24  $\neq$  25

Question 4 (\*\*\*)

The distinct square matrices  $A$  and  $B$  have the properties

- $AB = B^5A$
- $B^6 = I$

where  $I$  is the identity matrix.

a) Show that  $BAB = A$ .

b) Hence prove by induction that  $B^nAB^n = A$ , for all  $n \in \mathbb{N}$ .

proof

$\hookrightarrow BAB = B(AB) = B(B^5A) = B^6A = IA = A$   
 $\hookrightarrow$  If  $n=1$   $B^1AB^1 = BAB = A$   
Let  $B$  have order  $n=k$

• Suppose the result holds for  $n=k \in \mathbb{N}$

$B^kAB^k = A$   
 $BB^kAB^k = BA$   
 $BB^kAB^k = BAB$   
 $B^{k+1}AB^{k+1} = A$

• If the result holds for  $n=k \in \mathbb{N}$ , then it also holds for  $n=k+1$ . Since the result holds for  $n=1$ , then it holds for all  $n \in \mathbb{N}$ .

Question 5 (\*\*\*)

$$xy + 3y = x.$$

Prove by induction

$$(x+3) \frac{d^n}{dx^n}(y) + (n+1) \frac{d^{n-1}}{dx^{n-1}}(y) = 0.$$

proof

$2y + 3y - 2 = 0$   
 Diff wrt  $x$   
 $y + 2 \frac{dy}{dx} + 3 \frac{dy}{dx} - 1 = 0$   
 $(2+3) \frac{dy}{dx} + y - 1 = 0$

- SUPPOSE THAT THE RESULT HOLDS FOR  $n=k \in \mathbb{N}$   
 $(2+3) \frac{d^k}{dx^k} y + k \frac{d^{k-1}}{dx^{k-1}} y = 0$
- DIFFERENTIATE WRIT  $x$  ONCE MORE  
 $1 \frac{d^k}{dx^k} y + (2+3) \frac{d^k}{dx^k} y + k \frac{d^k}{dx^k} y = 0$   
 $(2+3) \frac{d^k}{dx^k} y + (k+1) \frac{d^k}{dx^k} y = 0$   
 $(2+3) \frac{d^k}{dx^k} y + (k+1) \frac{d^k}{dx^k} y = 0$
- IF THE RESULT HOLDS FOR  $n=k \in \mathbb{N}$ , THEN IT ALSO HOLDS FOR  $n=k+1$ .  
 SINCE THE RESULT HOLDS FOR  $n=1$ ,  
 THEN IT MUST HOLD  $\forall n \in \mathbb{N}$ .

**Question 6 (\*\*\*)**

Bernoulli's inequality asserts that if  $a \in \mathbb{R}$ ,  $a > -1$  and  $n \in \mathbb{N}$ ,  $n \geq 2$ , then

$$(1+a)^n > 1+an.$$

Prove, by induction, the validity of Bernoulli's identity.

 , proof

BERNOULLI INEQUALITY

$(1+a)^n > 1+an$      $a \in \mathbb{R}, a > -1$   
 $n \in \mathbb{N}, n \geq 2$

PROVE BY INDUCTION

- IF  $n=2$     LHS =  $(1+a)^2 = 1^2 + 2a + a^2$   
RHS =  $1+2a$   
 $\therefore a^2 + 2a + 1 > 2a + 1$ , so the result holds for  $n=2$
- SUPPOSE THAT THE INEQUALITY HOLDS FOR  $n=k \in \mathbb{N}$ ,  $k \geq 2$   
 $\Rightarrow (1+a)^k > 1+ak$   
 $\Rightarrow (1+a)^k(1+a) > (1+a)(1+ak)$   
 $\Rightarrow (1+a)^{k+1} > 1+ak+a+a^2k$   
 $\Rightarrow (1+a)^{k+1} > 1+a(k+1) + a^2k$      $\uparrow$   
(positive)  
 $\Rightarrow (1+a)^{k+1} > 1+a(k+1)$
- IF THE INEQUALITY HOLDS FOR  $n=k \in \mathbb{N}$ ,  $k \geq 2$ , THEN IT WILL ALSO HOLD FOR  $n=k+1$ .

AS THE INEQUALITY HOLDS FOR  $n=2$ , THEN IT MUST HOLD FOR ALL POSITIVE INTEGERS GREATER THAN 2

**Question 7 (\*\*\*)**

Prove by induction that

$$\sum_{r=1}^n r > \frac{1}{2}n^2, \text{ for } n \geq 1, n \in \mathbb{N}.$$

proof

$\sum_{r=1}^n r > \frac{1}{2}n^2$

- If  $n=1$  LHS = 1  
RHS =  $\frac{1}{2} \times 1^2 = \frac{1}{2}$  ✓ The result holds for  $n=1$
- Assume the result holds for  $n=k$ ,  $k \in \mathbb{N}$   
 $\sum_{r=1}^k r > \frac{1}{2}k^2$   
 $\sum_{r=1}^{k+1} r = \sum_{r=1}^k r + (k+1) > \frac{1}{2}k^2 + k + 1$   
 $> \frac{1}{2}(k^2 + 2k + 2)$   
 $> \frac{1}{2}(k^2 + 2k + 1) + \frac{1}{2}$   
 $> \frac{1}{2}(k+1)^2 + \frac{1}{2}$   
 $> \frac{1}{2}(k+1)^2$

If the inequality holds for  $n=k \in \mathbb{N}$ , then it also holds for  $n=k+1$ .  
 Since the inequality holds for  $n=1$ , then the inequality must hold  $\forall n$ .

**Question 8 (\*\*\*)**

Prove by induction that

$$\sum_{r=1}^n r^2 \geq \frac{1}{4}n(n+1)^2, \text{ for } n \geq 1, n \in \mathbb{N}.$$

proof

- If  $n=1$  LHS =  $1^2 = 1$   
RHS =  $\frac{1}{4}(1+1)^2 = 1$   
∴ Result holds true for  $n=1$
- Assume the result holds for  $n=k$ ,  $k \in \mathbb{N}$   
 $\sum_{r=1}^k r^2 \geq \frac{k(k+1)^2}{4}$   
 $\sum_{r=1}^{k+1} r^2 = \sum_{r=1}^k r^2 + (k+1)^2 \geq \frac{k(k+1)^2}{4} + (k+1)^2$   
 $\geq \frac{k(k+1)^2 + 4(k+1)^2}{4}$   
 $\geq \frac{(k+1)(k+4)(k+1)}{4}$   
 $\geq \frac{(k+1)(k^2 + 4k + 4)}{4}$   
 $\geq \frac{(k+1)(k^2 + 4k + 4)}{4} \geq \frac{(k+1)(k+1)(k+4)}{4}$   
 $\geq \frac{(k+1)(k+1)^2}{4} = \frac{1}{4}(k+1)(k+1)^2$

• If the result holds for  $n=k \in \mathbb{N}$ , then it also holds for  $n=k+1$ .  
 Since the result holds for  $n=1$ , then it must hold for every  $n \in \mathbb{N}$ .

**Question 9** (\*\*\*)

Prove by induction that

$$2^n > 2n, \text{ for } n \geq 3, n \in \mathbb{N}.$$

proof

$2^n > 2n$  for  $n \in \mathbb{N}, n \geq 3$

- if  $n=3$   $2^3=8$   $2 \times 3=6$   $8 > 6$  is true for  $n=3$
- suppose the result holds for  $n=k \in \mathbb{N}, k \geq 3$ 

$$2^k > 2k \Rightarrow 2 \times 2^k > 2 \times 2k$$

$$2^{k+1} > 4k = 2k + 2k$$

$$2^{k+1} > 2k + 2 \quad (\text{LHS } k \geq 3)$$

$$2^{k+1} > 2k + 2 = 2(k+1)$$
- if the result holds for  $n=k \in \mathbb{N}, k \geq 3$ , then it also holds for  $k+1$ .  
Since the result holds for  $n=3$ , then it also holds for  $n \geq 3$ .

**Question 10** (\*\*\*)

Prove by induction that

$$2^n > n^2, \text{ for } n \geq 5, n \in \mathbb{N}.$$

proof

$2^n > n^2$

- if  $n=5$   $2^5=32$   $5^2=25$   $32 > 25$  is true for  $n=5$
- suppose the result holds for  $n=k, k \in \mathbb{N}, k \geq 5$ 

$$2^k > k^2$$

$$\Rightarrow 2 \times 2^k > 2 \times k^2$$

$$\Rightarrow 2^{k+1} > 2k^2$$

$$\Rightarrow 2^{k+1} > k^2 + k^2$$

$$\Rightarrow 2^{k+1} > k^2 + (2k+1) \quad \text{for } k \geq 5 \quad \{k^2 > 2k+1\}$$

$$\Rightarrow 2^{k+1} > (k+1)^2$$
- if the result holds for  $n=k, k \in \mathbb{N}, k \geq 5$ , then the result also holds for  $n=k+1$ .  
Since the result holds for  $n=5$ , then the result also holds for  $n \geq 5$ .

$k^2 > 2k+1$   
 $k^2 - 2k - 1 > 2$   
 $(k-1)^2 > 2$   
 $k-1 > \sqrt{2}$   
 $k > 1 + \sqrt{2}$   
 $k < 1 + \sqrt{2}$   
 $\therefore$  for  $k \geq 5, k^2 > 2k+1$

**Question 11** (\*\*\*\*)

Prove by induction that if  $n \in \mathbb{N}$ ,  $n \geq 3$ , then

$$3^n > (n+1)^2.$$

, proof

IF  $n \in \mathbb{N}$ ,  $n \geq 3$  THEN  $3^n > (n+1)^2$

BASE CASE,  $n=3$   
 LHS =  $3^3 = 27$   
 RHS =  $(3+1)^2 = 16$   $27 > 16$  SO THE INEQUALITY HOLDS FOR  $n=3$

INDUCTIVE HYPOTHESIS  
 SUPPOSE THAT THE INEQUALITY HOLDS FOR  $n=k \in \mathbb{N}$ ,  $k > 3$

$$\Rightarrow 3^k > (k+1)^2$$

$$\Rightarrow 3 \times 3^k > 3 \times (k+1)^2$$

$$\Rightarrow 3^{k+1} > 3k^2 + 6k + 3 > k^2 + 6k + 2$$

$$\Rightarrow 3^{k+1} > k^2 + 4k + (2k+2)$$

NOW AS  $k > 3$   $2k+2 > 0 > 4$

$$\Rightarrow 3^{k+1} > k^2 + 4k + 4$$

$$\Rightarrow 3^{k+1} > (k+2)^2 = [(k+1)+1]^2$$

CONCLUSION  
 IF THE INEQUALITY HOLDS FOR  $n=k \in \mathbb{N}$ ,  $k > 3$ , THEN IT ALSO HOLDS FOR  $n=k+1$ .  
 SINCE THE INEQUALITY HOLDS FOR  $n=3$ , THEN IT MUST HOLD FOR ALL  $n \in \mathbb{N}$ ,  $n \geq 3$

**Question 12** (\*\*\*\*)

Prove by induction that for all even natural numbers  $n$

$$\frac{d^n}{dx^n}(\sin 3x) = (-1)^{\frac{n}{2}} \times 3^n \times \sin 3x.$$

, proof

CHECK THE BASE CASE,  $n=2$

$$\frac{d^2}{dx^2}(\sin 3x) = \frac{d}{dx}(3 \cos 3x) = -9 \sin 3x$$

$$(-1)^{\frac{2}{2}} \times 3^2 \times \sin 3x = 1 \times 9 \times \sin 3x = 9 \sin 3x$$

IF THE RESULT HOLDS FOR  $n=2$

SUPPOSE THAT THE RESULT HOLDS FOR  $n=k=2m$ ,  $m \in \mathbb{N}$

$$\frac{d^{2m}}{dx^{2m}}(\sin 3x) = (-1)^m \times 3^{2m} \times \sin 3x$$

$$\frac{d^{2m+2}}{dx^{2m+2}}(\sin 3x) = \frac{d}{dx} [(-1)^m \times 3^{2m} \times \cos 3x] = (-1)^{m+1} \times 3^{2m+2} \times \sin 3x$$

$$\frac{d^{2m+2}}{dx^{2m+2}}(\sin 3x) = \frac{d}{dx} [(-1)^m \times 3^{2m} \times \sin 3x] = (-1)^m \times 3^{2m+2} \times \sin 3x$$

$$\frac{d^{2m+2}}{dx^{2m+2}}(\sin 3x) = (-1)^{m+1} \times 3^{2m+2} \times \sin 3x$$

$$\frac{d^{2m+2}}{dx^{2m+2}}(\sin 3x) = (-1)^{\frac{2m+2}{2}} \times 3^{2m+2} \times \sin 3x$$

IF THE RESULT HOLDS FOR  $n=k=2m$ , THEN IT MUST HOLD FOR  $n=k+2=2(m+1)$

AS THE RESULT HOLDS FOR  $n=2$ , THEN IT MUST HOLD FOR ALL EVEN NATURAL NUMBERS

Question 13 (\*\*\*)

Prove by induction that for  $n \geq 1, n \in \mathbb{N}$

$$\prod_{r=1}^n (\cos(2^{r-1}x)) = \frac{\sin(2^n x)}{2^n \sin x}$$

, proof

WRITE THE  $\square$  STATEMENT EXPLICITLY

$$\prod_{r=1}^n [\cos(2^{r-1}x)] = \cos x \cos 2x \cos 4x \dots \cos(2^{n-1}x)$$

CHECK THE BASE CASE, i.e. if  $n=1$

L.H.S =  $\prod_{r=1}^1 \cos(2^{r-1}x) = \cos x$   
 R.H.S =  $\frac{\sin(2^1 x)}{2^1 \sin x} = \frac{\sin 2x}{2 \sin x} = \frac{2 \sin x \cos x}{2 \sin x} = \cos x$  (i.e. THE RESULT IS TRUE FOR  $n=1$ )

SUPPOSE THAT THE RESULT HOLDS FOR  $n=k \in \mathbb{N}$

$$\rightarrow \prod_{r=1}^k [\cos(2^{r-1}x)] = \frac{\sin(2^k x)}{2^k \sin x}$$

$$\rightarrow \prod_{r=1}^{k+1} [\cos(2^{r-1}x)] \times \cos(2^k x) = \frac{\sin(2^k x)}{2^k \sin x} \times \cos(2^k x)$$

$$\rightarrow \prod_{r=1}^{k+1} [\cos(2^{r-1}x)] = \frac{\sin(2^k x) \cos(2^k x)}{2^k \sin x} = \frac{2 \sin(2^{k-1}x) \cos(2^k x)}{2^k \sin x}$$

$$\rightarrow \prod_{r=1}^{k+1} [\cos(2^{r-1}x)] = \frac{\sin[2 \times 2^{k-1}x]}{2^{k+1} \sin x}$$

$$\rightarrow \prod_{r=1}^{k+1} [\cos(2^{r-1}x)] = \frac{\sin(2^k x)}{2^{k+1} \sin x}$$

IF THE RESULT HOLDS FOR  $n=k \in \mathbb{N}$ , THEN IT MUST ALSO HOLD FOR  $n=k+1$   
SINCE THE RESULT HOLDS FOR  $n=1$ , THEN IT MUST HOLD FOR ALL  $n \in \mathbb{N}$

**Question 14** (\*\*\*\*+)

Prove by induction that

$$\cos x + \cos 3x + \cos 5x + \dots + \cos [(2n-1)x] \equiv \frac{\sin(2nx)}{2 \sin x}$$

□, proof

CHECK THE BASE CASE, n=1

LHS =  $\cos(2(1)-1) = \cos x$   
 RHS =  $\frac{\sin(2(1)x)}{2 \sin x} = \frac{\sin 2x}{2 \sin x} = \frac{2 \sin x \cos x}{2 \sin x} = \cos x$   
 $\therefore$  THE RESULT HOLDS FOR  $n=1$

SUPPOSE THAT THE RESULT HOLDS FOR  $n=k \in \mathbb{N}$

$$\sum_{r=1}^k \cos(2r-1)x = \frac{\sin(2kx)}{2 \sin x}$$

$$\cos[(2(k+1)-1)x] + \sum_{r=1}^k \cos(2r-1)x = \frac{\sin(2(k+1)x)}{2 \sin x} + \cos[(2k-1)x]$$

$$\sum_{r=1}^{k+1} \cos(2r-1)x = \frac{\sin(2kx)}{2 \sin x} + \cos(2kx)$$

$$\sum_{r=1}^{k+1} \cos(2r-1)x = \frac{\sin(2kx) + 2 \sin x \cos(2kx)}{2 \sin x}$$

NOW WE NEED TO DERIVE SOME IDENTITIES

$$\begin{aligned} \sin(A+B) &= \sin A \cos B + \cos A \sin B \\ \sin(A-B) &= \sin A \cos B - \cos A \sin B \end{aligned} \quad \left. \vphantom{\begin{aligned} \sin(A+B) &= \sin A \cos B + \cos A \sin B \\ \sin(A-B) &= \sin A \cos B - \cos A \sin B \end{aligned}} \right\} \text{Adding}$$

$$\sin(A+B) + \sin(A-B) = 2 \sin A \cos B$$

$$2 \sin A \cos B = \sin(A+B) + \sin(A-B)$$

$$2 \sin x \cos(2kx) = \sin(2kx + x) + \sin(2kx - x)$$

$$2 \sin x \cos(2kx) = \sin(2k+1)x + \sin(2k-1)x$$

RETURNING TO THE INDUCTION, USING  $\sin(-A) = -\sin A$

$$\sum_{r=1}^{k+1} \cos(2r-1)x = \frac{\sin(2kx) + \sin(2(k+1)x)}{2 \sin x} - \frac{\sin(2kx)}{2 \sin x}$$

$$\sum_{r=1}^{k+1} \cos(2r-1)x = \frac{\sin(2(k+1)x)}{2 \sin x}$$

COMPARE THE RESULT WITH THE STATEMENT OF THE INDUCTION

IF THE RESULT HOLDS FOR  $n=k \in \mathbb{N}$ , THEN IT MUST ALSO HOLD FOR  $n=k+1$

SINCE THE RESULT HOLDS FOR  $n=1$ , THEN THE RESULT HOLDS  $\forall n \in \mathbb{N}$

**Question 15** (\*\*\*\*+)

Prove by induction that every positive integer power of 5 can be written as the sum of squares of two distinct positive integers.

□, proof

START BY INVESTIGATING SMALLER CASES

IF  $n=1$   $5^1 = 2^2 + 1^2$  IF RESULT HOLDS FOR  $n=1$   
 IF  $n=2$   $5^2 = 3^2 + 4^2$  IF RESULT HOLDS FOR  $n=2$

SUPPOSE THE RESULT HOLDS FOR  $n=k, k \in \mathbb{N}$

$$\Rightarrow a^2 + b^2 = 5^k \quad \text{WHERE } a, b \text{ ARE DISTINCT POSITIVE INTEGERS}$$

$$\Rightarrow 25(a^2 + b^2) = 25 \times 5^k$$

$$\Rightarrow 25a^2 + 25b^2 = 5^{k+2}$$

$$\Rightarrow (5a)^2 + (5b)^2 = 5^{k+2}$$

[AS  $a, b$  ARE DISTINCT POSITIVE INTEGERS, SO WOULD  $5a$  AND  $5b$ ]

IF THE RESULT HOLDS FOR  $n=k$ , IT MUST ALSO HOLD FOR  $n=k+2$   
BUT THE RESULT HOLDS FOR  $n=1$ , SO IT MUST HOLD FOR ALL ODD POSITIVE POWERS OF 5  
AND AS THE RESULT HOLDS FOR  $n=2$ , IT MUST ALSO HOLD FOR ALL EVEN POSITIVE POWERS OF 5

$\therefore$  THE RESULT HOLDS FOR ALL  $n \in \mathbb{N}$

Question 16 (\*\*\*\*\*)

Prove by induction that

$$\frac{d^n}{dx^n} (e^x \cos x) = 2^{\frac{1}{2}n} e^x \cos\left(x + \frac{n\pi}{4}\right), \quad n \geq 1, n \in \mathbb{N}.$$

SP2D, proof

$\frac{d^n}{dx^n} [e^x \cos x] = 2^{\frac{1}{2}n} e^x \cos\left(x + \frac{n\pi}{4}\right), \quad n \geq 1, n \in \mathbb{N}$

BASE CASE, n=1

- $\frac{d}{dx} [e^x \cos x] = e^x \cos x + e^x (-\sin x) = e^x [\cos x - \sin x]$
- R.H.S. =  $2^{\frac{1}{2} \cdot 1} e^x \cos\left(x + \frac{1\pi}{4}\right) = 2^{\frac{1}{2}} e^x \cos\left(x + \frac{\pi}{4}\right)$   
 $= \sqrt{2} e^x \left[ \cos x \cos \frac{\pi}{4} - \sin x \sin \frac{\pi}{4} \right]$   
 $= \sqrt{2} e^x \left[ \cos x \times \frac{1}{\sqrt{2}} - \sin x \times \frac{1}{\sqrt{2}} \right]$   
 $= e^x [\cos x - \sin x]$   
i.e. RESULT HOLDS FOR n=1

INDUCTIVE HYPOTHESIS

SUPPOSE THAT THE RESULT HOLDS FOR n=k, k ∈ ℕ

$\Rightarrow \frac{d^k}{dx^k} [e^x \cos x] = 2^{\frac{1}{2}k} e^x \cos\left(x + \frac{k\pi}{4}\right)$

$\Rightarrow \frac{d}{dx} \left[ \frac{d^k}{dx^k} (e^x \cos x) \right] = \frac{d}{dx} \left[ 2^{\frac{1}{2}k} e^x \cos\left(x + \frac{k\pi}{4}\right) \right]$

$\Rightarrow \frac{d^{k+1}}{dx^{k+1}} (e^x \cos x) = 2^{\frac{1}{2}k} \left[ e^x \cos\left(x + \frac{k\pi}{4}\right) - e^x \sin\left(x + \frac{k\pi}{4}\right) \right]$

$\Rightarrow \frac{d^{k+1}}{dx^{k+1}} (e^x \cos x) = 2^{\frac{1}{2}k} e^x \left[ \cos\left(x + \frac{k\pi}{4}\right) - \sin\left(x + \frac{k\pi}{4}\right) \right]$

$\Rightarrow \frac{d^{k+1}}{dx^{k+1}} (e^x \cos x) = 2^{\frac{1}{2}k} e^x \sqrt{2} \times \left[ \frac{1}{\sqrt{2}} \cos\left(x + \frac{k\pi}{4}\right) - \frac{1}{\sqrt{2}} \sin\left(x + \frac{k\pi}{4}\right) \right]$

$\Rightarrow \frac{d^{k+1}}{dx^{k+1}} (e^x \cos x) = 2^{\frac{1}{2}k} e^x \sqrt{2} \left[ \cos\left(x + \frac{k\pi}{4}\right) - \sin\left(x + \frac{k\pi}{4}\right) \right]$

$\Rightarrow \frac{d^{k+1}}{dx^{k+1}} (e^x \cos x) = 2^{\frac{1}{2}(k+1)} e^x \left[ \cos\left(x + \frac{k\pi}{4} + \frac{\pi}{4}\right) \right]$

$\Rightarrow \frac{d^{k+1}}{dx^{k+1}} (e^x \cos x) = 2^{\frac{1}{2}(k+1)} e^x \cos\left[x + \frac{(k+1)\pi}{4}\right]$

CONCLUSION

IF THE RESULT HOLDS FOR n=k, k ∈ ℕ, THEN IT ALSO HOLDS FOR n=k+1  
 SINCE THE RESULT HOLDS FOR n=1, THEN IT MUST HOLD FOR ALL n ∈ ℕ

**Question 17** (\*\*\*\*\*)

It is given that for  $n \in \mathbb{N}$

$$u_{n+1} = \frac{7u_n + 12}{u_n + 3}, \quad u_1 = 7.$$

Prove by induction that

$$u_n > 6.$$

,  proof

$$u_{n+1} = \frac{7u_n + 12}{u_n + 3}, \quad u_1 = 7$$

- START BY IDENTIFYING THE RECURRENT RELATION AS FOLLOWS  

$$u_{n+1} = \frac{7(u_n + 3) - 9}{u_n + 3} = 7 - \frac{9}{u_n + 3}$$
- SUPPOSE THAT THE RESULT HOLDS FOR  $n=k, k \in \mathbb{N}$   

$$\Rightarrow u_k > 6$$

$$\Rightarrow u_k + 3 > 9$$

$$\Rightarrow \frac{1}{u_k + 3} < \frac{1}{9}$$

$$\Rightarrow \frac{9}{u_k + 3} < 1$$

$$\Rightarrow 7 - \frac{9}{u_k + 3} > 6$$

$$\Rightarrow u_{k+1} > 6$$
- IF THE RESULT HOLDS FOR  $n=k, k \in \mathbb{N}$ , THEN IT ALSO HOLDS FOR  $k+1$   
 AS THE RESULT HOLDS FOR  $n=1$  ( $u_1=7$ ), THEN IT MUST HOLD  $\forall n \in \mathbb{N}$

**Question 18** (\*\*\*\*\*)

Prove by induction that every positive integer power of 14 can be written as the sum of squares of three distinct positive integers.

,  proof

- IF  $n=1$   $1^2 + 2^2 + 3^2 = 14 = 14^1$ , 16 RESULT HOLDS FOR  $n=1$   
 $n=2$   $4^2 + 6^2 + 12^2 = 196 = 14^2$ , 16 RESULT HOLDS FOR  $n=2$
- SUPPOSE THE RESULT HOLDS FOR  $n=k \in \mathbb{N}$   

$$\Rightarrow 2a^2 + b^2 + c^2 = 14^k$$
where  $a, b, c$  ARE DISTINCT POSITIVE INTEGERS  

$$\Rightarrow 14^k (2a^2 + b^2 + c^2) = 14^k \times 14^k$$
  

$$\Rightarrow 14^{k+1} = 14^k (2a^2 + b^2 + c^2) = 14^{k+2}$$
  

$$\Rightarrow (14a)^2 + (14b)^2 + (14c)^2 = 14^{k+2}$$
IF  $2, 3, 2$  ARE DISTINCT THEN  $14a, 14b, 14c$  ARE ALSO DISTINCT
- IF THE RESULT HOLDS FOR  $n=k$ , THEN IT MUST ALSO HOLD FOR  $n=k+2$   
 SINCE THE RESULT HOLDS FOR  $n=1$ , THEN IT MUST HOLD FOR ALL POSITIVE INTEGERS  
 SINCE THE RESULT HOLDS FOR  $n=2$ , THEN IT MUST HOLD FOR ALL POSITIVE INTEGERS  
 THEN THE RESULT HOLDS  $\forall n \in \mathbb{N}$

**Question 19** (\*\*\*\*)

It is given that for  $n \in \mathbb{N}$

$$U_n = \frac{2n}{2n+1} U_{n-1}, \quad U_1 = \frac{2}{3}.$$

Prove by induction that

$$U_n \leq \left( \frac{2n}{2n+1} \right)^n.$$

 , proof

BASE CASE (TO  $n=1$  &  $n=2$  FOR THE STRICT INEQUALITY)

IF  $n=1$   $U_1 = \frac{2}{3}$   $U_1 = \frac{2 \times 1}{2 \times 1 + 1} = \frac{2}{3}$

IF  $n=2$   $U_2 = \frac{4}{5} U_1$   $U_2 = \left( \frac{2 \times 2}{2 \times 2 + 1} \right)^2$   
 $U_2 = \frac{4}{5} \times \frac{2}{3}$   $U_2 = \left( \frac{4}{5} \right)^2$   
 $U_2 = \frac{8}{15}$   $U_2 = \frac{8}{15}$

$\therefore$  THE RESULT HOLDS FOR  $n=1$  &  $n=2$ .

NOW SUPPOSE THAT THE RESULT HOLDS FOR  $n=k \in \mathbb{N}$

$U_k \leq \left( \frac{2k}{2k+1} \right)^k$

WE HAVE AN INEQUALITY

$U_{k+1} = \frac{2(k+1)}{2(k+1)+1} U_k = \frac{2k+2}{2k+3} U_k \leq \frac{2k+2}{2k+3} \left( \frac{2k}{2k+1} \right)^k$

FOR THE INEQUALITY TO HOLD

NOW WE HAVE TO PROVE THAT

$\frac{2k}{2k+1} < \frac{2k+2}{2k+3}$

DEFN:  $f(x) = \frac{2k+2}{2k+3} - \frac{2k}{2k+1}$ ,  $k \in \mathbb{N}$

$f(x) = \frac{(2k+2)(2k+1) - 2k(2k+3)}{(2k+3)(2k+1)} = \frac{4k^2 + 4k + 2 - 4k^2 - 6k}{(2k+3)(2k+1)} = \frac{-2k+2}{(2k+3)(2k+1)} > 0$

$f(x) > 0 \Rightarrow \frac{2k+2}{2k+3} > \frac{2k}{2k+1}$

RETURNING TO THE MAIN LINE OF THE INDUCTION

$U_{k+1} = \dots = \frac{2k+2}{2k+3} \left( \frac{2k}{2k+1} \right)^k$

$< \frac{2k+2}{2k+3} \left( \frac{2k+2}{2k+3} \right)^k$

$= \left( \frac{2k+2}{2k+3} \right)^{k+1}$

$= \left( \frac{2(k+1)}{2(k+1)+1} \right)^{k+1}$

$\therefore U_{k+1} \leq \left( \frac{2(k+1)}{2(k+1)+1} \right)^{k+1}$

IF THE RESULT HOLDS FOR  $n=k \in \mathbb{N}$ , THEN IT MUST HOLD FOR  $n=k+1$   
SINCE THE RESULT HOLDS FOR  $n=1$ , THEN THE RESULT HOLDS  $\forall n \in \mathbb{N}$

Question 20 (\*\*\*\*\*)

Prove by induction that

$$\frac{d^n}{dx^n} \left( e^x \sin(\sqrt{3}x) \right) = 2^n e^x \sin\left(\sqrt{3}x + \frac{n\pi}{3}\right), \quad n \geq 1, n \in \mathbb{N}.$$

P.P., proof

$\frac{d^1}{dx^1} [e^x \sin(\sqrt{3}x)] = 2^1 e^x \sin(\sqrt{3}x + \frac{\pi}{3})$

- IF  $n=1$   $\frac{d^1}{dx^1} [e^x \sin(\sqrt{3}x)] = e^x \sin \sqrt{3}x + \sqrt{3} e^x \cos \sqrt{3}x$   
 $= e^x [\sin \sqrt{3}x + \sqrt{3} \cos \sqrt{3}x]$   
 $= 2e^x \left[ \frac{1}{2} \sin \sqrt{3}x + \frac{\sqrt{3}}{2} \cos \sqrt{3}x \right]$   
 $= 2e^x [\sin(\sqrt{3}x + \frac{\pi}{3})]$   
 $= 2^1 e^x \sin(\sqrt{3}x + \frac{\pi}{3})$

IF THE RESULT HOLDS FOR  $n=1$

- SUPPOSE THE RESULT HOLDS FOR  $n=k \in \mathbb{N}$   
 $\frac{d^k}{dx^k} [e^x \sin(\sqrt{3}x)] = 2^k e^x \sin(\sqrt{3}x + \frac{k\pi}{3})$

DIFFERENTIATE AGAIN w.r.t  $x$

$$\frac{d^{k+1}}{dx^{k+1}} [e^x \sin(\sqrt{3}x)] = 2^k e^x \sin(\sqrt{3}x + \frac{k\pi}{3}) + 2^k e^x \times \sqrt{3} \cos(\sqrt{3}x + \frac{k\pi}{3})$$

$$= 2^k e^x [\sin(\sqrt{3}x + \frac{k\pi}{3}) + \sqrt{3} \cos(\sqrt{3}x + \frac{k\pi}{3})]$$

$$= 2^k e^x \times 2 \left[ \frac{1}{2} \sin(\sqrt{3}x + \frac{k\pi}{3}) + \frac{\sqrt{3}}{2} \cos(\sqrt{3}x + \frac{k\pi}{3}) \right]$$

$$= 2^{k+1} e^x [\sin(\sqrt{3}x + \frac{k\pi}{3}) + \sqrt{3} \cos(\sqrt{3}x + \frac{k\pi}{3})]$$

$$= 2^{k+1} e^x \sin(\sqrt{3}x + \frac{(k+1)\pi}{3})$$

IF THE RESULT HOLDS FOR  $n=k \in \mathbb{N} \Rightarrow$  THE RESULT ALSO HOLDS FOR  $n=k+1$   
 SINCE THE RESULT HOLDS FOR  $n=1 \Rightarrow$  THE RESULT HOLDS  $\forall n \in \mathbb{N}$

**Question 21** (\*\*\*\*\*)

The function  $f(x)$  is defined by

$$f(x) = 2 - \frac{1}{x}, \quad x \in \mathbb{R}, x \neq 0.$$

a) Prove that

$$f^n(x) = \frac{(n+1)x - n}{nx - (n-1)}, \quad n \geq 1,$$

where  $f^n(x)$  denotes the  $n^{\text{th}}$  composition of  $f(x)$  by itself.

b) State an expression for the domain of  $f^n(x)$ .

$$\boxed{\phantom{000000}}, \quad x \in \mathbb{R}, x \neq \frac{n-1}{n}$$

$$\text{(a)} \bullet f^1(x) = \frac{(1+1)x - 1}{1x - (1-1)} = \frac{2x-1}{x} = 2 - \frac{1}{x} = f(x)$$

$$\bullet f^2(x) = f(f(x)) = f\left(2 - \frac{1}{x}\right) = 2 - \frac{1}{2 - \frac{1}{x}} = 2 - \frac{x}{2x-1} = \frac{4x-2-x}{2x-1} = \frac{3x-2}{2x-1}$$
 Also  

$$f^2(x) = \frac{(2+1)x - 2}{2x - (2-1)} = \frac{3x-2}{2x-1} \rightarrow \text{RECUR RECUR FOR } n=1,2$$
 SUPPOSE THE RESULT HOLDS FOR  $n=k \in \mathbb{N}$   

$$\bullet f^k(x) = \frac{(k+1)x - k}{kx - (k-1)}$$

$$\bullet f^{k+1}(x) = f\left[\frac{(k+1)x - k}{kx - (k-1)}\right] = 2 - \frac{1}{\frac{(k+1)x - k}{kx - (k-1)}} = 2 - \frac{kx - (k-1)}{(k+1)x - k}$$

$$= \frac{2kx - 2x - kx + (k-1)}{(k+1)x - k} = \frac{(k+2)x - k - 1}{(k+1)x - k} = \frac{(k+1)x - (k-1)}{(k+1)x - k}$$
 THIS IS THE RESULT HOLDS FOR  $n=k \in \mathbb{N} \Rightarrow$  THE RESULT ALSO HOLDS FOR  $n=k+1$   
 SAVE THE RESULT HOLDS FOR  $n=1,2 \Rightarrow$  THE RESULT HOLDS FOR  $n \in \mathbb{N}$

$$\text{(b)} \text{ RESTRICTION IN DOMAIN OF } f(x) \text{ IS 'NATURAL'}$$

$$\therefore nx - (n-1) \neq 0$$

$$x \neq \frac{n-1}{n} \quad \therefore x \in \mathbb{R}, x \neq \frac{n-1}{n}$$

**Question 22** (\*\*\*\*)

Prove by induction that if  $n \in \mathbb{N}$ ,  $n \geq 3$ , then

$$n^{n+1} > (n+1)^n,$$

and hence deduce that if  $n \in \mathbb{N}$ ,  $n \geq 3$ , then

$$\sqrt[n]{n} > \sqrt[n+1]{n+1}$$

,  proof

The image shows two pages of handwritten mathematical work. The left page is titled "IF  $n \in \mathbb{N}$ ,  $n \geq 3$  THEN  $n^{n+1} > (n+1)^n$ ". It includes a base case for  $n=3$  where L.H.S. =  $3^4 = 81$  and R.H.S. =  $4^3 = 64$ , and an inductive hypothesis where it is assumed the result holds for  $n=k \geq 3$ . The proof then shows  $k^{k+1} > (k+1)^k$  and manipulates this to show  $(k+1)^{k+2} > \frac{(k+1)^{2k+2}}{k^{k+1}}$ . The right page is titled "RETURNING TO THE MAIN LINE OF THE INDUCTIVE-HYPOTHESIS". It shows the inductive step: if  $k^{k+1} > (k+1)^k$ , then  $(k+1)^{k+2} > \frac{(k+1)^{2k+2}}{k^{k+1}} > (k+2)^{k+1}$ . It concludes that the result holds for  $n=k+1$  and thus for all  $n \in \mathbb{N}$  with  $n \geq 3$ . A final note says "FINALLY WE HAVE  $n^{n+1} > (n+1)^n$   $n \in \mathbb{N}$ ,  $n \geq 3$ " and then shows the deduced inequality  $\sqrt[n]{n} > \sqrt[n+1]{n+1}$ .