NUMBER
THEORY
Question 1 (**)
Use Euclid’s algorithm to find the Highest common factor of 560 and 1169:

\[ \text{?} \]

Question 2 (**)

\[ f(n) = n^2 + n + 2, \ n \in \mathbb{N}. \]

Show that \( f(n) \) is always even.

\[ \text{?}, \text{ proof} \]

Question 3 (**)

Prove that when the square of a positive odd integer is divided by 4 the remainder is always 1.

\[ \text{?}, \text{ proof} \]
Question 4  (**)
Use Euclid’s algorithm to find the Highest common factor of 3059 and 7728.

\[
\text{(EUCLID’S ALGORITHM FOR \text{HCF})}
\]

\begin{align*}
\text{Trial 1: } & \quad 3059 = 1 \times 7728 + 2801 \\
\text{Trial 2: } & \quad 7728 = 2 \times 2801 + 1125 \\
\text{Trial 3: } & \quad 2801 = 2 \times 1125 + 586 \\
\text{Trial 4: } & \quad 1125 = 1 \times 586 + 539 \\
\text{Trial 5: } & \quad 586 = 1 \times 539 + 47 \\
\text{Trial 6: } & \quad 539 = 11 \times 47 + 22 \\
\text{Trial 7: } & \quad 47 = 2 \times 22 + 13 \\
\text{Trial 8: } & \quad 22 = 1 \times 13 + 9 \\
\text{Trial 9: } & \quad 13 = 1 \times 9 + 4 \\
\text{Trial 10: } & \quad 9 = 2 \times 4 + 1 \\
\text{Trial 11: } & \quad 4 = 4 \times 1 + 0
\end{align*}

The H.C.F. of 3059 and 7728 is 1.

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Question 5  (***)
Prove that the square of a positive integer can never be of the form \(3k + 2\), \(k \in \mathbb{N}\).

\[
\text{proof}
\]
Question 6 (***)

Show that $a^3 - a + 1$ is odd for all positive integer values of $a$. 

[Proof]
Question 7  (**+)**

When $a, a \in \mathbb{N}$, is divided by $b, b \in \mathbb{N}$, the quotient is 20 and the remainder is 17.

a) Find the remainder when $a$ is divided by 5.

Suppose that when a positive integer is divided by 8 the remainder is 6, and when the same positive integer is divided by 18 the remainder is 3.

b) Determine whether the positive integer of part (b) exists.

\[ \begin{array}{c} 2 \\ \text{no such integer} \end{array} \]
Question 8  (**8+)**

\[ f(n) \equiv n^2 + 4n + 3, \ n \in \mathbb{N}. \]

a) Given that \( n \) is odd show that \( f(n) \) is a multiple of 8.

\[ g(n) \equiv (n^2 + 15)(n^2 + 7), \ n \in \mathbb{N}. \]

b) Given that \( n \) is odd show that \( g(n) \) is a multiple of 128.

You may assume that the square of an odd integer is of the form \( 8k + 1, \ k \in \mathbb{N}. \)

\[ \text{proof} \]
Question 9  (***)

\[ f(n) = 5^{2n} - 1, \quad n \in \mathbb{N}. \]

Without using proof by induction, show that \( f(n) \) is a multiple of 8.

**Proof**

Question 10  (***)

Bernoulli's inequality asserts that if \( a \in \mathbb{R}, \quad a > -1 \) and \( n \in \mathbb{N}, \quad n \geq 2 \), then

\[ (1+a)^n > 1+an. \]

Prove, by induction, the validity of Bernoulli's identity.

**Proof**

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Question 11  (***)
When some positive integer $N$ is divided by 4, the quotient is 3 times as large as the remainder.

Determine the possible values of $N$.

\[ N = \{13, 26, 39\}\]

Question 12  (***)
Use proof by exhaustion to show that if $m \in \mathbb{N}$ and $n \in \mathbb{N}$, then

\[ m^2 - n^2 \neq 102. \]
Question 13  (**+)**

It is given that for \( a, b, c \in \mathbb{N} \),

\[ a^2 + b^2 + c^2 = 116. \]

a) Prove that \( a, b \) and \( c \) are all even.

You may assume that the square of an odd integer is of the form \( 8k + 1, \ k \in \mathbb{N} \).

b) Determine the values of \( a, b \) and \( c \).

\[ a = 8, \ b = 6, \ c = 4 \quad \text{in any order} \]
Question 14 (***+)

The figure above shows two right angled triangles.

- The triangle, on the left section of the figure, has side lengths of
  \[ a, b \text{ and } c, \]
  where \( c \) is the length of its hypotenuse.

- The triangle, on the right section of the figure, has side lengths of
  \[ a+1, b+1 \text{ and } c+1, \]
  where \( c+1 \) is the length of its hypotenuse.

Show that \( a, b \) and \( c \) cannot all be integers.
Question 15  (***)

When 165 is divided by some integer the quotient is 7 and the remainder is \( R \).

Determine the possible values of \( R \).

\[ R = \{4, 11, 18\} \]
Question 16 (***+)  
It is given that $a$ and $b$ are positive integers, with $a > b$.  

Use proof by contradiction to show that if $a + b$ is a multiple of 4, then $a - b$ cannot be a multiple of 4.
Question 17  (***)

When a positive integer \( N \) is divided by 4 the remainder is 3.

When \( N \) is divided by 5 the remainder is 2.

Show that the remainder of the division of \( N \) by 20 is 5.

\[
\text{proof}
\]
Question 18  (***)

a) Show that $9^{40} + 3^{40} + 6$ is a multiple of 8.

b) Show further that $3^{40} + 2$ divides $9^{40} + 3^{40} - 2$.
Question 19  (***)

Suppose that when a positive integer is divided by 6 the remainder is 4, and when the same positive integer is divided by 12 the remainder is 8.

a) Determine whether such positive exists.

Suppose next that when a positive integer is divided by 6, the quotient is \( q \) and the remainder is 1. When the square of the same positive integer is divided by \( q \), the quotient is 984 and the remainder is 1.

b) Determine whether the positive integer of part (b) exists.

\[
\text{no such integer, } 163
\]
Question 20 (***+)

In the following question \( A \), \( B \) and \( C \) are positive odd integers.

Show, using a clear method, that …

a) \( A^2 + B^2 + C^2 + 5 \) is a multiple of 8.

b) \( A^2(A^2 + 6) - 7 \) is divisible by 128.

c) \( A^4 - B^4 \) is a multiple of 16.
Question 21 (****)

It is given that

\[ a^2 + b^2 = c^2, \quad a \in \mathbb{N}, \ b \in \mathbb{N}. \]

Show that \( a \) and \( b \) cannot both be odd.
Question 22  (****)  
Let \( a \in \mathbb{N} \) with \( \frac{1}{5}a \notin \mathbb{N} \).

a) Show that the remainder of the division of \( a^2 \) by 5 is either 1 or 4.

b) Given further that \( b \in \mathbb{N} \) with \( \frac{1}{5}b \notin \mathbb{N} \), deduce that \( \frac{1}{5}(a^4 - b^4) \in \mathbb{N} \).
Question 23  (****)

It is given that $k$ is a positive integer.

a) If $k - 2$ divides $k^2 + 4$, determine the possible values of $k$.

It is further given that $a$ and $b$ are positive integers.

b) Show that $8a^2 - b^2$ cannot equal 2017.
Question 24  (****)
Prove by induction that if $n \in \mathbb{N}$, $n \geq 3$, then

$$3^n > (n+1)^2.$$
Question 25  (***)

i. The function \( f \) is defined as

\[
 f(n) \equiv n(n^3 + 2n + 1), \ n \in \mathbb{N}.
\]

Show that \( f \) is even for all \( n \in \mathbb{N} \).

ii. The positive integer \( k \) divides both \( 2a + 5b \) and \( 3a + 7b \), where \( a \in \mathbb{N}, \ b \in \mathbb{N} \).

Show that \( k \) must then divide both \( a \) and \( b \).
Question 26  (***)

It is given that \( k \) is a positive integer.

a) If \( 7^k \) divides \( k + 5 \), determine the possible values of \( k \).

The second part of this question is unrelated to the first part.

b) By showing a detailed method, find the remainder of the division of \( 6^{26} + 26^{6} \) by 5.

\[ k = 8, 9, 10, 11, 13, 19 \]
Question 27  (***)

i. The function $f$ is defined as
\[ f(n) \equiv (n^2 + n)(n+5), \quad n \in \mathbb{N}. \]
Show that $f$ is multiple of 6 for all $n \in \mathbb{N}$.

ii. The function $g$ is defined as
\[ g(m,n) = m^3 n - mn^3, \quad m \in \mathbb{N}, \quad n \in \mathbb{N}. \]
Show that $g$ is divisible by 3 for all $m \in \mathbb{N}, \quad n \in \mathbb{N}$.
Question 28  (*****)

It is given that

\[ f(m, n) = 2m(m^2 + 3n^2), \]

where \( m \) and \( n \) are distinct positive integers, with \( m > n \).

By using the expansion of \( (A \pm B)^3 \), prove that \( f(m, n) \) can always be written as the sum of two cubes.

\[
\text{proof}
\]
Question 29  (***)

Prove that the sum of the squares of two distinct positive integers, when doubled, it can be written as the sum of two distinct square numbers.
Question 30 (*****)

Show that the square of an odd positive integer greater than 1 is of the form

\[ 8T + 1, \]

where \( T \) is a triangular number.
Question 31 (*****)

The product operator $\prod$, is defined as

$$\prod_{r=1}^{k} [u_r] = u_1 \times u_2 \times u_3 \times u_4 \times \ldots \times u_{k-1} \times u_k.$$ 

The integer $Z$ is a square number and defined as

$$Z = \prod_{r=1}^{20} \left( \frac{r!}{n!} \right), \{ n \in \mathbb{N} : 1 \leq n \leq 20 \}.$$ 

By considering the terms inside the product operator in pairs, or otherwise, determine a possible value of $n$.

You must show a detailed method in this question.

\[ \boxed{, \ n = 10} \]
Question 32  (*****)

Prove by induction that if \( n \in \mathbb{N}, \ n \geq 3 \), then

\[ n^{n+1} > (n+1)^n, \]

and hence deduce that if \( n \in \mathbb{N}, \ n \geq 3 \), then

\[ \sqrt[n]{n} > \frac{n+1}{\sqrt[n]{n+1}}. \]

Proof
Question 33     (****)

It is given that $11a + 13b$ is a multiple of $13 - a$, where $a \in \mathbb{N}, b \in \mathbb{N}$.

It is then asserted that $(13 + a)(11 + b)$ is also a multiple of $13 - a$.

Prove the validity of this assertion.