

Created by T. Madas

SYSTEMATIC CURVE SKETCHING

Created by T. Madas

Question 1 ()**

The curve C has equation

$$y = \frac{a}{x}, \quad x \neq 0,$$

where a is a positive constant.

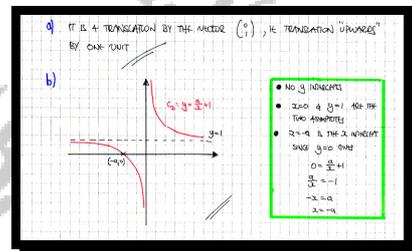
- a) Describe geometrically the transformation that maps the graph of $y = \frac{a}{x}$ onto the graph of $y = \frac{a}{x} + 1$.

- b) Sketch the graph of C .

The sketch must include the coordinates of ...

- ... the coordinates of all the points where the curve meets the coordinate axes.
- ... the equations of any asymptotes of the curve.

, translation "upwards" by 1 unit



Question 2 ()**

A curve C has equation

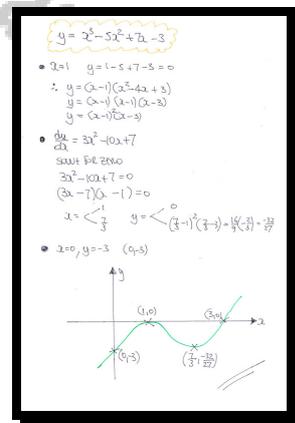
$$y = x^3 - 5x^2 + 7x - 3, \quad x \in \mathbb{R}.$$

Sketch the graph of C .

The sketch must include in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.

graph



Question 3 (**)

A curve C has equation

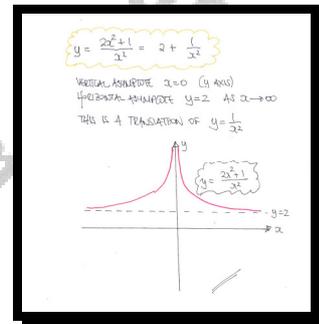
$$y = \frac{2x^2 + 1}{x^2}, x \in \mathbb{R}, x \neq 0.$$

Sketch the graph of C .

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the equations of any asymptotes.

graph



Question 4 (***)

A curve C has equation

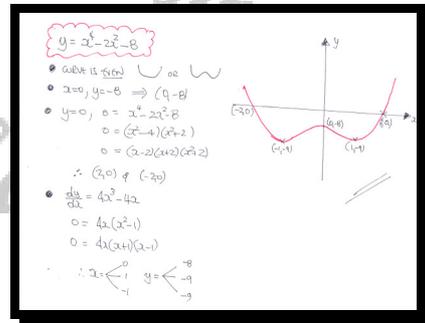
$$y = x^4 - 2x^2 - 8, \quad x \in \mathbb{R}.$$

Sketch the graph of C .

The sketch must include

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.

graph



Question 5 (*)**

A curve C has equation

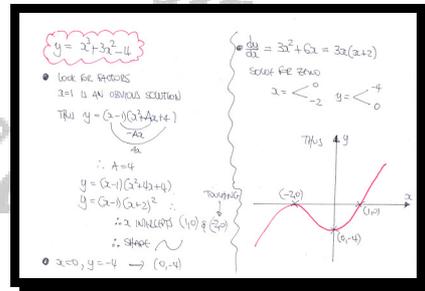
$$y = x^3 + 3x^2 - 4, \quad x \in \mathbb{R}.$$

Sketch the graph of C .

The sketch must include

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.

graph



Question 6 (***)

A curve C has equation

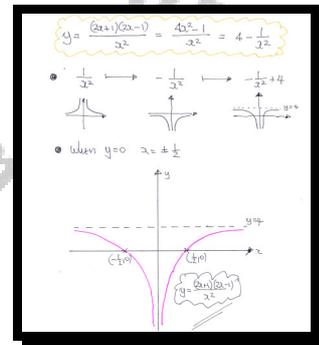
$$y = \frac{(2x+1)(2x-1)}{x^2}, x \in \mathbb{R}, x \neq 0.$$

Sketch the graph of C .

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the equations of any asymptotes.

graph



Question 7 (*)**

A curve C has equation

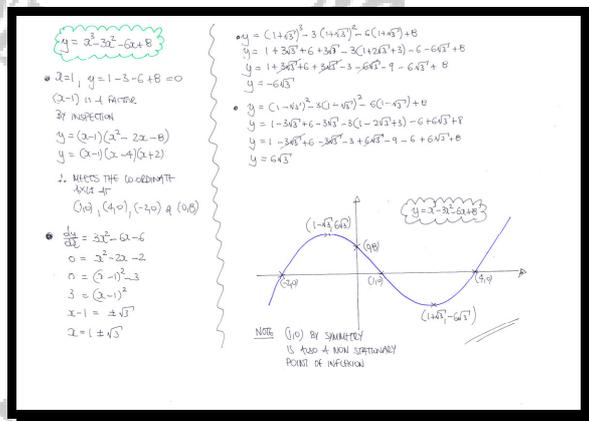
$$y = x^3 - 3x^2 - 6x + 8, \quad x \in \mathbb{R}.$$

Sketch the graph of C .

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the coordinates of any non stationary turning points.

graph



Question 8 (***)

$$f(x) = \frac{4x-13}{x-3}, \quad x \in \mathbb{R}, \quad x \neq 3.$$

a) Show that the equation of $f(x)$ can be written as

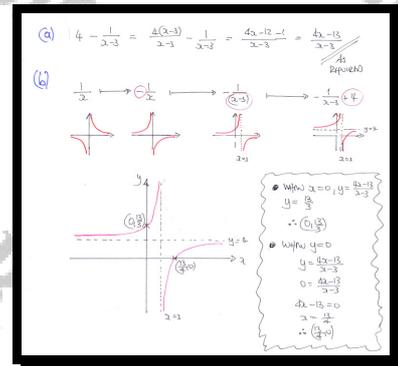
$$f(x) = 4 - \frac{1}{x-3}, \quad x \in \mathbb{R}, \quad x \neq 3.$$

b) Sketch the graph of $f(x)$.

The sketch must include ...

- ... the coordinates of the points where $f(x)$ meets the coordinate axes.
- ... the equations of any asymptotes of the curve.

graph



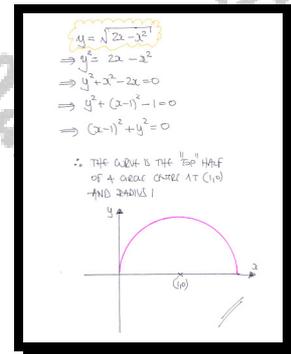
Question 10 (***)

A curve C has equation

$$y = \sqrt{2x - x^2}, \quad x \in \mathbb{R}, \quad 0 \leq x \leq 2.$$

Sketch the graph of C .

graph



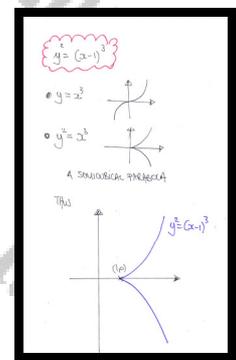
Question 11 (***)

A curve C has equation

$$y^2 = (x-1)^3, \quad x \in \mathbb{R}, \quad x \geq 1.$$

Sketch the graph of C .

graph



Question 13 (***)

A curve C has equation

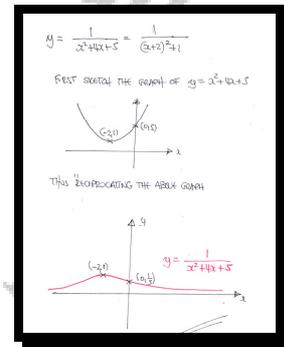
$$y = \frac{1}{x^2 + 4x + 5}, \quad x \in \mathbb{R}.$$

Sketch the graph of C .

The sketch must include,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the equations of any asymptotes.

graph



Question 14 (***)

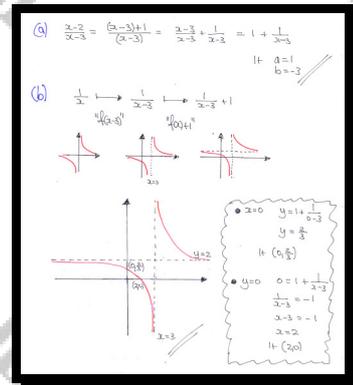
$$f(x) = \frac{x-2}{x-3}, \quad x \in \mathbb{R}, \quad x \neq 3.$$

- a) Express $f(x)$ in the form $f(x) = a + \frac{1}{x+b}$, where a and b are integers.
- b) By considering a series of transformations which map the graph of $\frac{1}{x}$ onto the graph of $f(x)$, sketch the graph of $f(x)$.

The sketch must include ...

- ... the coordinates of all the points where the curve meets the coordinate axes.
- ... the equations of the two asymptotes of the curve.

$$a = 1, \quad b = -3$$



Question 15 (***)

A curve C has equation

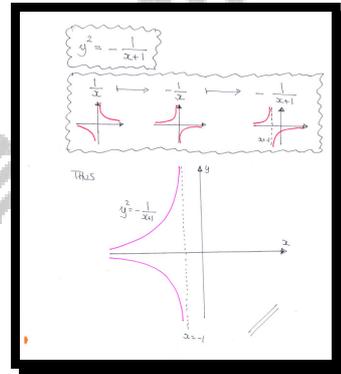
$$y^2 = -\frac{1}{x+1}, \quad x \in \mathbb{R}, \quad x \neq -1.$$

Sketch the graph of C .

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the equations of any asymptotes.

graph



Question 16 (***)

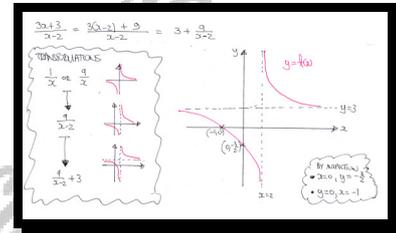
$$f(x) = \frac{3x+3}{x-2}, x \in \mathbb{R}, x \neq 2.$$

Sketch the graph of $f(x)$.

The sketch must include ...

- ... the coordinates of all the points where the curve meets the coordinate axes.
- ... the equations of the two asymptotes of the curve.

graph



Question 17 (***)

A curve has equation $y = f(x)$ given by

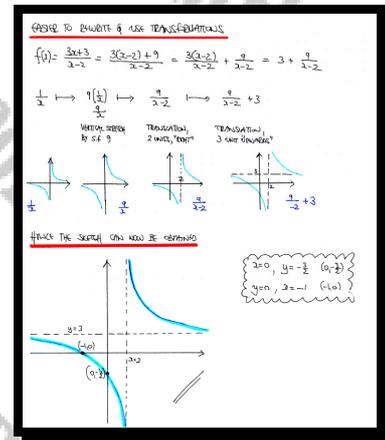
$$f(x) = \frac{3x-1}{x+2}, \quad x \in \mathbb{R}, \quad x \neq -2.$$

Sketch the graph of $f(x)$.

The sketch must include ...

- ... the coordinates of all the points where the curve meets the coordinate axes.
- ... the equations of the two asymptotes of the curve.

, graph



Question 18 (***)

A curve C has equation

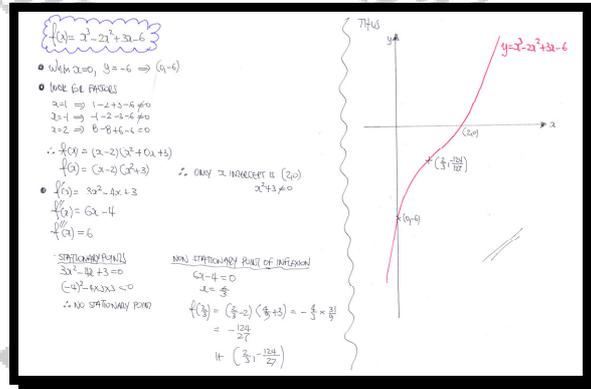
$$y = x^3 - 2x^2 + 3x - 6, \quad x \in \mathbb{R}.$$

Sketch the graph of C .

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the coordinates of any non stationary turning points.

graph



Question 20 (***)

A curve C has equation

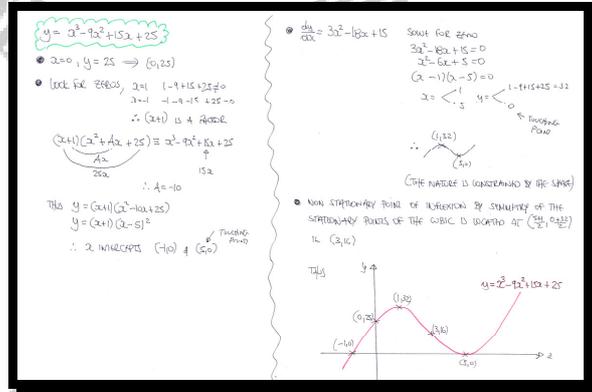
$$y = x^3 - 9x^2 + 15x + 25, \quad x \in \mathbb{R}.$$

Sketch the graph of C .

The sketch must include,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the coordinates of any non stationary turning points.

graph



Question 21 (***)

A curve C has equation

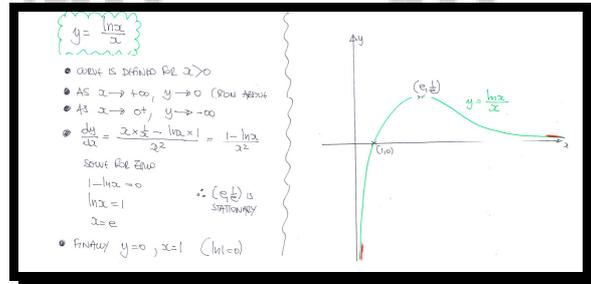
$$y = \frac{\ln x}{x}$$

Sketch the graph of C , for the largest possible domain.

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the equations of any asymptotes.

graph



Question 22 (***)

A curve C has equation

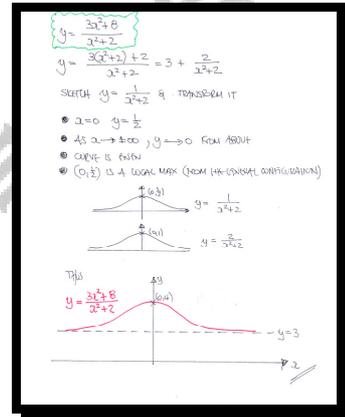
$$y = \frac{3x^2 + 8}{x^2 + 2}, \quad x \in \mathbb{R}.$$

Sketch the graph of C .

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the equations of any asymptotes.

graph



Question 23 (***)

A curve C has equation

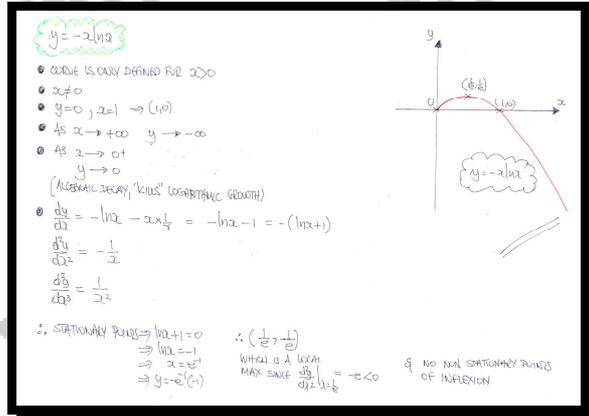
$$y = -x \ln x, \quad x \in \mathbb{R}.$$

Sketch the graph of C , for the largest possible domain.

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the coordinates of any non stationary turning points.
- the equations of any asymptotes.

graph



Question 24 (***)

A curve C has equation

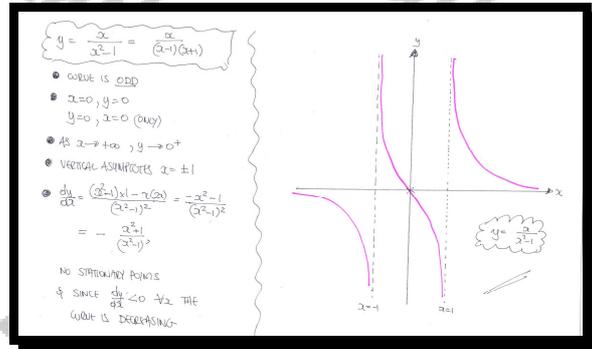
$$y = \frac{x}{x^2 - 1}, \quad x \in \mathbb{R}.$$

Sketch the graph of C , for the largest possible domain.

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the equations of any asymptotes.

graph



Question 25 (***)

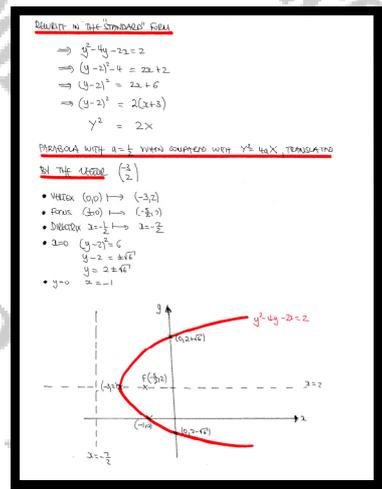
Sketch the parabola with equation

$$y^2 - 4y - 2x = 2.$$

The sketch must include the ...

- a) ... coordinates of points of intersection with the coordinate axes.
- b) ... coordinates of the vertex of the parabola.
- c) ... coordinates of the focus of the parabola.
- d) ... equation of the directrix of the parabola.

, graph



Question 26 (****)

A curve C has equation

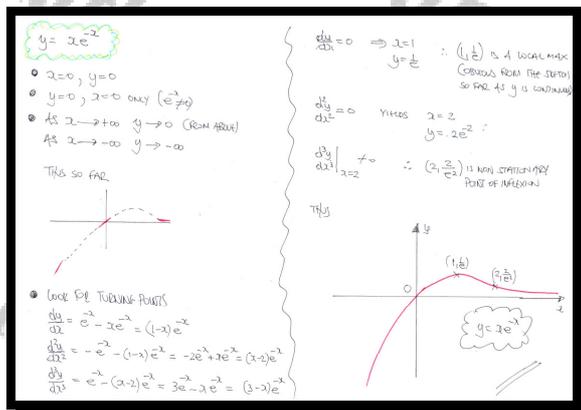
$$y = xe^{-x}, \quad x \in \mathbb{R}.$$

Sketch the graph of C .

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the coordinates of any non stationary turning points.
- the equations of any asymptotes.

graph



Question 27 (****)

A curve C has equation

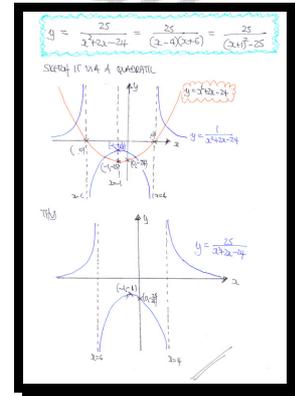
$$y = \frac{25}{x^2 + 2x - 24}, \quad x \in \mathbb{R}.$$

Sketch the graph of C , for the largest possible domain.

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the equations of any asymptotes.

graph



Question 28 (***)

A curve C has equation

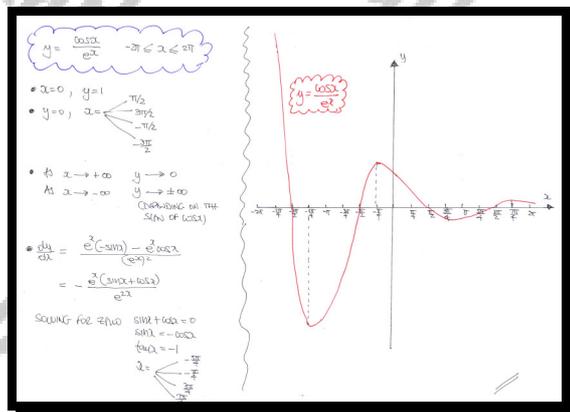
$$y = \frac{\cos x}{e^x}, \quad -2\pi \leq x \leq 2\pi.$$

Sketch the graph of C .

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the x coordinates of any stationary points.
- the equations of any asymptotes.

graph



Question 29 (***)

A curve C has equation

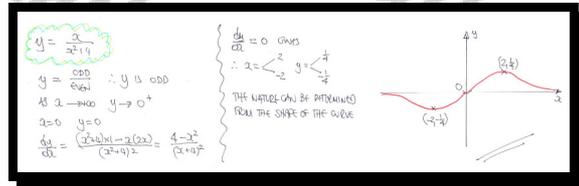
$$y = \frac{x}{x^2 + 4}, \quad x \in \mathbb{R}.$$

Sketch the graph of C .

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the equations of any asymptotes.

graph



Question 30 (***)

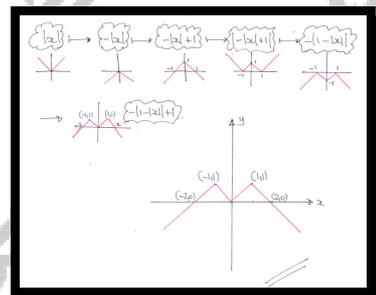
Sketch the graph of

$$y = 1 - |1 - |x||, \quad x \in \mathbb{R}.$$

The sketch must include the coordinates ...

- ... of any points where the graph meets the coordinate axes
- ... of any cusps of the graph.

graph



Question 31 (****)

A curve C has equation

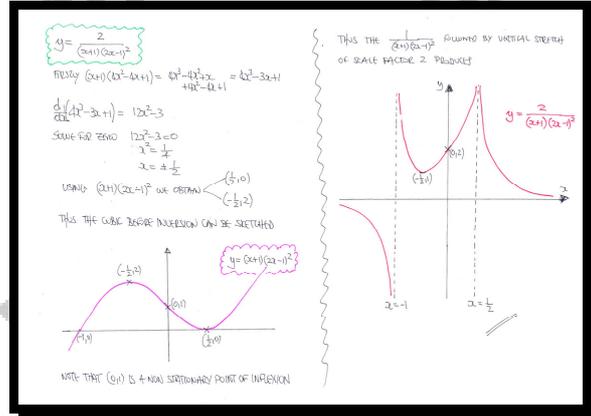
$$y = \frac{2}{(x+1)(2x-1)^2}, \quad x \in \mathbb{R}, \quad x \neq -1, \quad x \neq \frac{1}{2}.$$

Sketch the graph of C .

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the equations of any asymptotes.

graph



Question 32 (***)

A curve C has equation

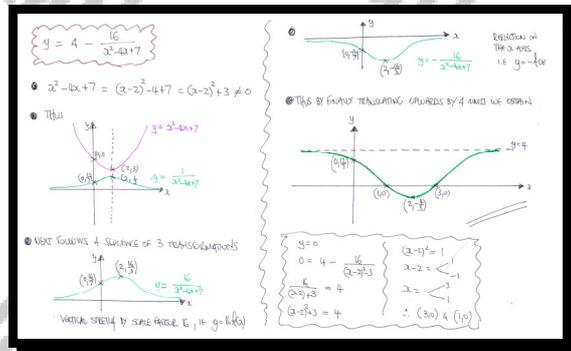
$$y = 4 - \frac{16}{x^2 - 4x + 7}, \quad x \in \mathbb{R}.$$

Sketch the graph of C .

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the equations of any asymptotes.

graph



Question 33 (****)

A curve C has equation

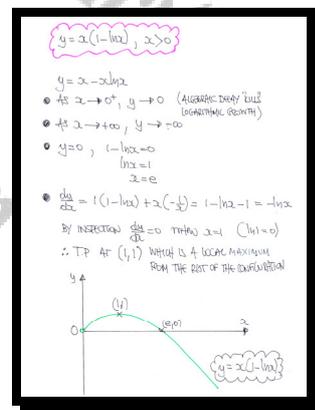
$$y = x(1 - \ln x), \quad x \in \mathbb{R}, \quad x > 0.$$

Sketch the graph of C .

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the equations of any asymptotes.

graph



Question 34 (****)

A curve C has equation

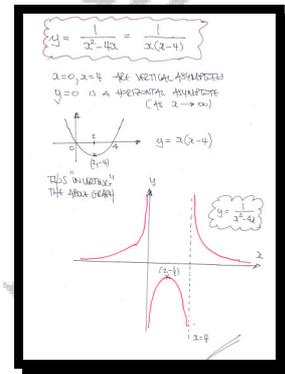
$$y = \frac{1}{x^2 - 4x}, \quad x \in \mathbb{R}, \quad x \neq 0.$$

Sketch the graph of C .

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the equations of any asymptotes.

graph



Question 35 (****)

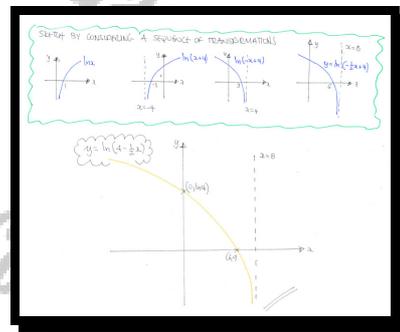
The function f is defined as

$$f : x \mapsto \ln\left(4 - \frac{1}{2}x\right), \quad x \in \mathbb{R}, \quad x < 8.$$

Sketch the graph of f .

Indicate clearly any intersections with the axes and the equation of its asymptote.

graph



Question 36 (***)

A curve C has equation

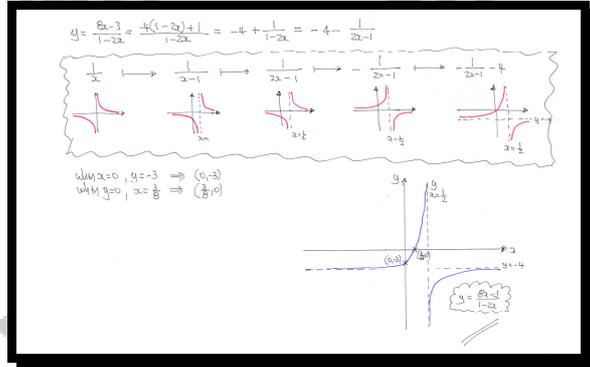
$$y = \frac{8x-3}{1-2x}, x \in \mathbb{R}, x \neq \frac{1}{2}.$$

Sketch the graph of C .

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the equations of any asymptotes.

graph



Question 37 (***)

A curve C has equation

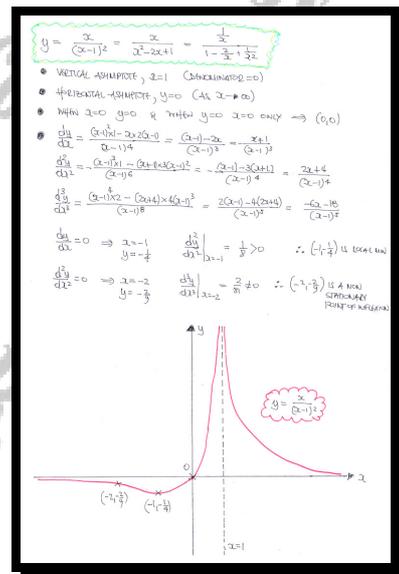
$$y = \frac{x}{(x-1)^2}, \quad x \in \mathbb{R}, \quad x \neq 1.$$

Sketch the graph of C .

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the coordinates of any non stationary turning points.
- the equations of any asymptotes.

graph



Question 38 (***)

A curve C has equation

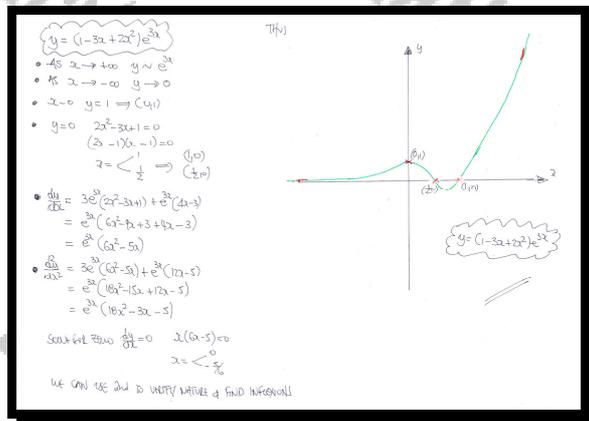
$$y = (1 - 3x + 2x^2)e^{3x}, \quad x \in \mathbb{R}.$$

Sketch the graph of C .

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the equations of any asymptotes.

graph



Question 39 (***)

A curve C has equation

$$y = \frac{1}{x^3 - 9x^2 + 24x}, \quad x \in \mathbb{R}, \quad x \neq 0.$$

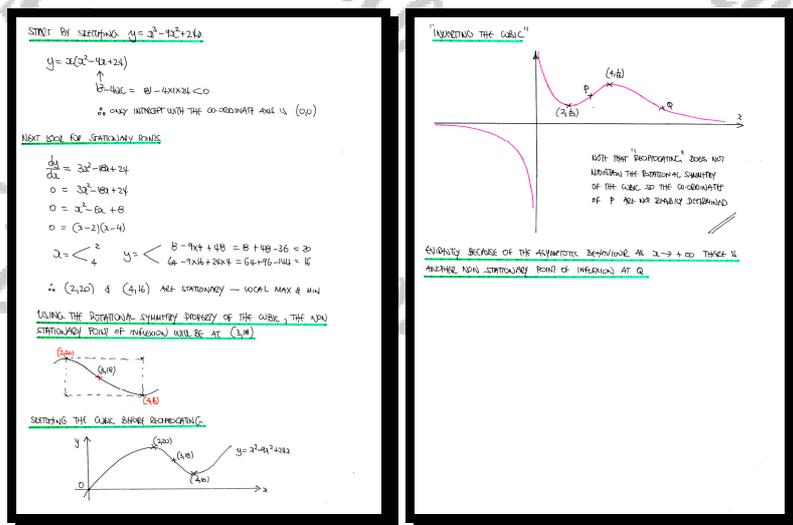
Sketch the graph of C .

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the equations of any asymptotes.

You must further label any non stationary turning points, without explicitly giving their coordinates.

graph



Question 40 (****)

A curve C has equation

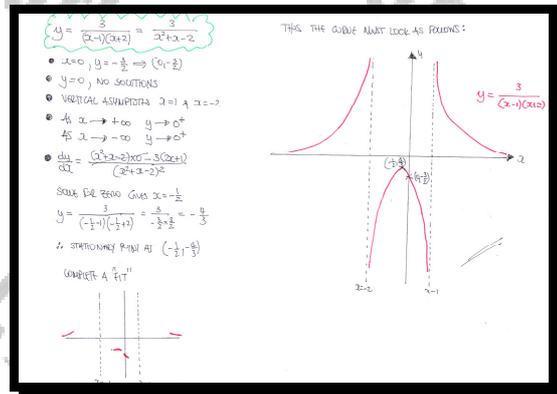
$$y = \frac{3}{(x-1)(x+2)}, \quad x \in \mathbb{R}, \quad x \neq -2, \quad x \neq 1.$$

Sketch the graph of C .

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the equations of any asymptotes.

graph



Question 41 (***)

A curve C has equation

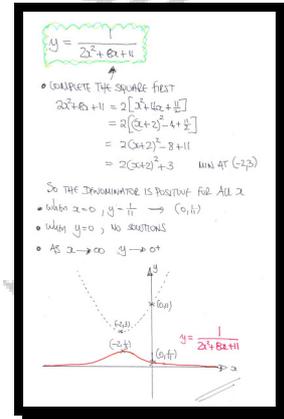
$$y = \frac{1}{2x^2 + 8x + 11}, \quad x \in \mathbb{R}.$$

Sketch the graph of C .

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the equations of any asymptotes.

graph



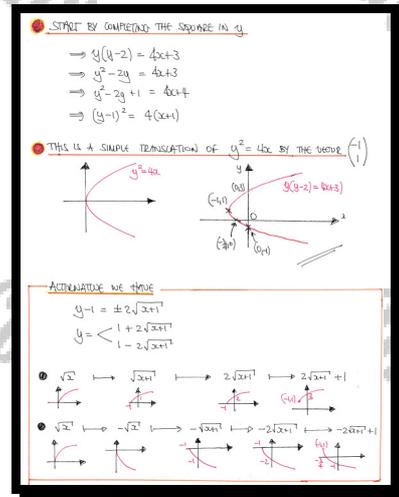
Question 42 (***)

Sketch the graph of the curve with equation

$$y(y-2) = 4x+3.$$

The sketch must include the coordinates of any intersections with the axes and the coordinates of the point where the tangent to the curve is parallel to the y axis.

, graph



Question 43 (***)

A curve C has equation

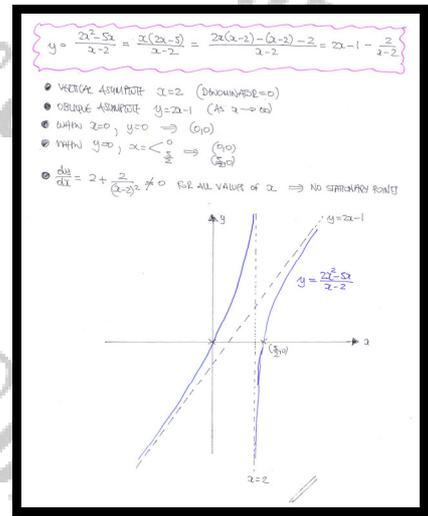
$$y = \frac{2x^2 - 5x}{x - 2}, \quad x \in \mathbb{R}, \quad x \neq 2.$$

Sketch the graph of C .

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the equations of any asymptotes.

graph



Question 44 (***)

A curve C has equation

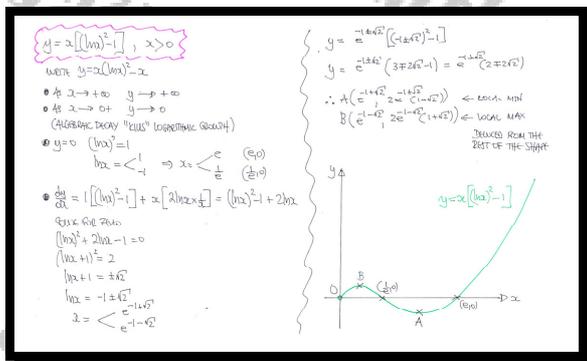
$$y = x \left[(\ln x)^2 - 1 \right], \quad x \in \mathbb{R}, \quad x > 0.$$

Sketch the graph of C .

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the equations of any asymptotes.

graph



Question 45 (****+)

A curve C has equation

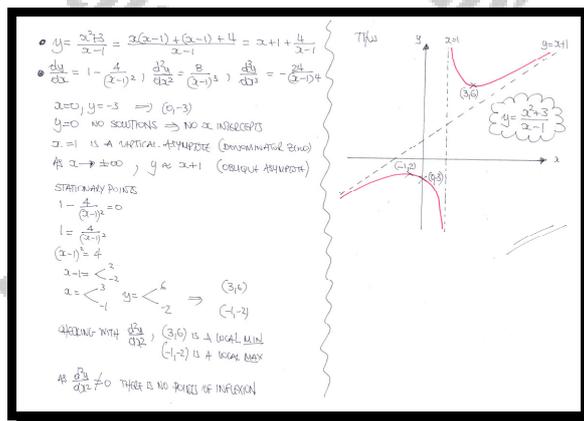
$$y = \frac{x^2 + 3}{x - 1}, \quad x \in \mathbb{R}, \quad x \neq 1.$$

Sketch the graph of C .

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the coordinates of any non stationary turning points.
- the equations of any asymptotes.

graph



Question 46 (****+)

A curve C has equation $y = f(x)$

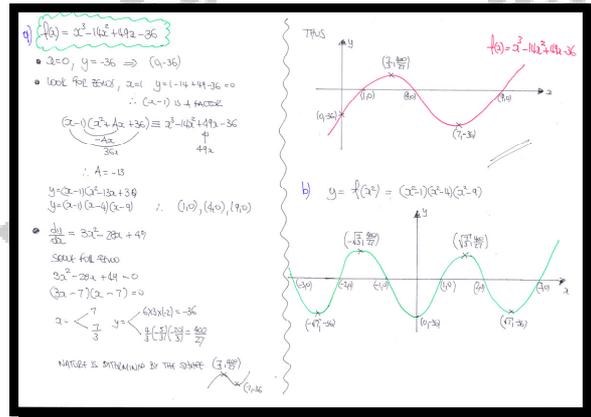
$$f(x) = x^3 - 14x^2 + 49x - 36, \quad x \in \mathbb{R}.$$

- Sketch the graph of C .
- Use the sketch of part (a) to deduce the graph of $y = f(x^2)$

Each of the sketches must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the coordinates of any non stationary turning points.

graph



Question 47 (****+)

A curve C has equation

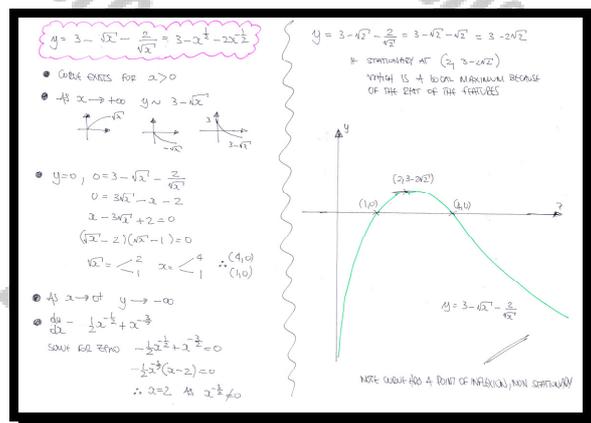
$$y = 3 - \sqrt{x} - \frac{2}{\sqrt{x}}$$

Sketch the graph of C , for the largest possible domain.

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the equations of any asymptotes.

graph



Question 48 (****+)

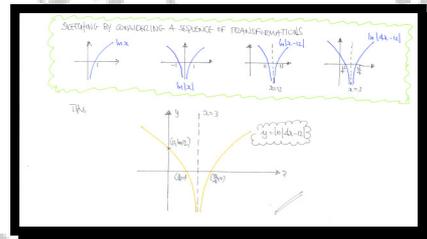
The function f is defined as

$$f : x \mapsto \ln|4x-12|, \quad x \in \mathbb{R}, x \neq 3.$$

Sketch the graph of f .

Indicate clearly any intersections with the axes and the equation of its asymptote.

graph



Question 49 (****+)

A curve C has equation

$$y = 2x - 1 + 4e^{-2x}, \quad x \in \mathbb{R}.$$

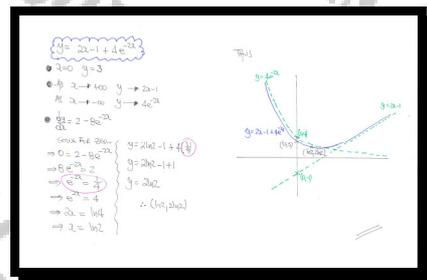
Sketch the graph of C , indicating clearly the behaviour for large positive and large negative values of x .

The graph must also include the exact coordinates, where appropriate, of ...

... any points where the graph of C meets the coordinate axes.

... any turning points of C .

graph



Question 50 (***)

A curve C has equation

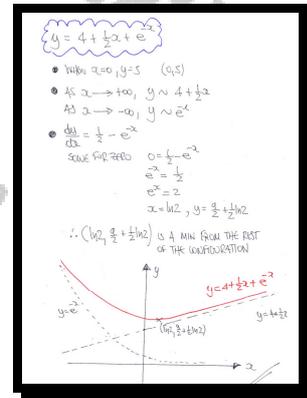
$$y = 4 + \frac{1}{2}x + e^{-x}, \quad x \in \mathbb{R}.$$

Sketch the graph of C , indicating clearly the behaviour for large positive and large negative values of x .

The graph must also include the exact coordinates, where appropriate, ...

- ... of any points where the graph of C meets the coordinate axes.
- ... of any turning points of C .

graph



Question 51 (***)

A curve C has equation

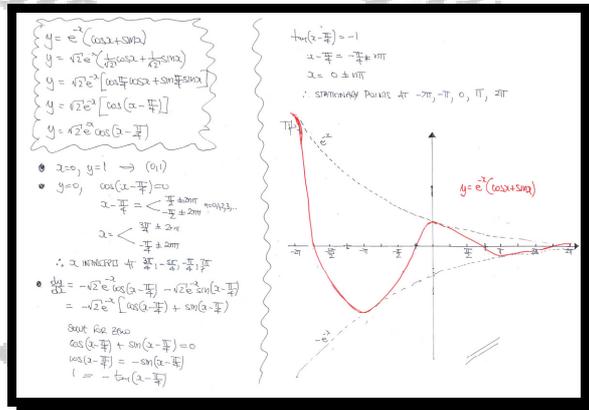
$$y = e^{-x}(\cos x + \sin x), \quad x \in \mathbb{R}.$$

Sketch the graph of C , indicating clearly the behaviour for large positive and large negative values of x .

The graph must also include the exact coordinates, where appropriate, ...

- ... of any points where the graph of C meets the coordinate axes.
- ... of any turning points of C .

graph



Question 52 (***)

A curve C has equation

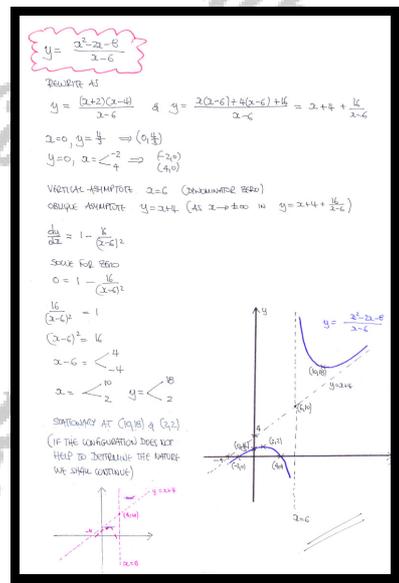
$$y = \frac{x^2 - 2x - 8}{x - 6}, \quad x \in \mathbb{R}, \quad x \neq 6.$$

Sketch the graph of C .

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the equations of any asymptotes.

graph



Question 53 (****+)

A curve C has equation

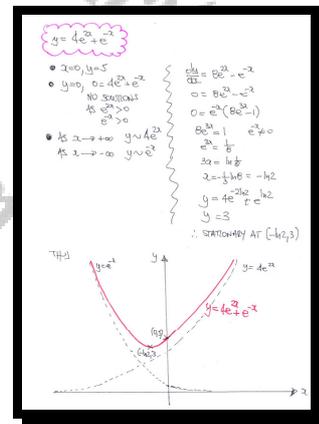
$$y = 4e^{2x} + e^{-x}, \quad x \in \mathbb{R}.$$

Sketch the graph of C .

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the equations of any asymptotes.

graph



Question 54 (***)

A curve C has equation

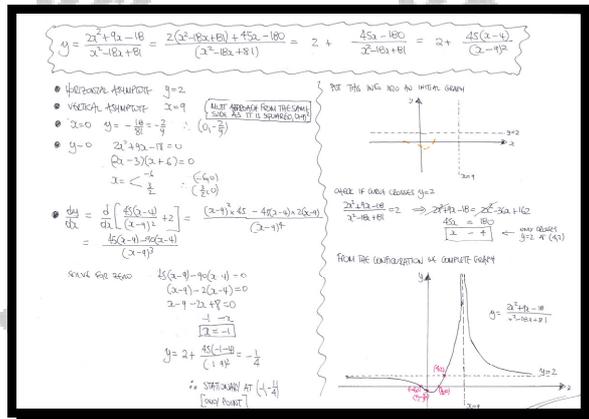
$$y = \frac{2x^2 + 9x - 18}{x^2 - 18x + 81}$$

Sketch the graph of C , for the largest possible domain.

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the equations of any asymptotes.

graph



Question 55 (****+)

A curve C has equation

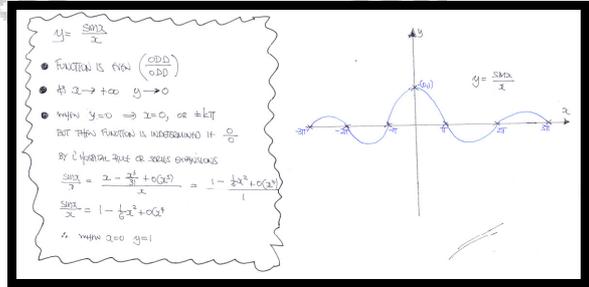
$$y = \frac{\sin x}{x}, \quad x \in \mathbb{R}.$$

Sketch the graph of C .

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points, for $-3\pi \leq x \leq 3\pi$.

graph



Question 56 (****+)

A curve C has equation

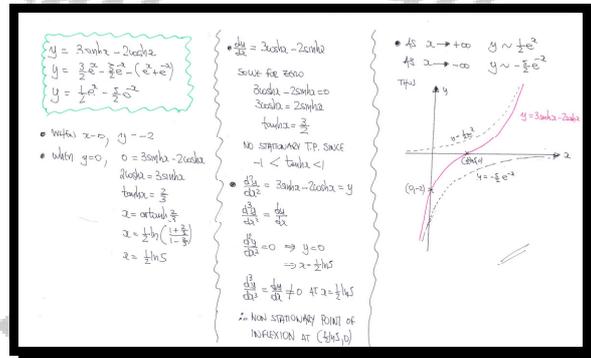
$$y = 3\sinh x - 2\cosh x, \quad x \in \mathbb{R}.$$

Sketch the graph of C .

The sketch must include ...

- ... the coordinates of any points where the graph of C meets the coordinates axes.
- ... the coordinates of any stationary or non stationary turning points.
- ... the behaviour of the curve for large positive and large negative values of x

graph



Question 57 (****+)

A curve C has equation

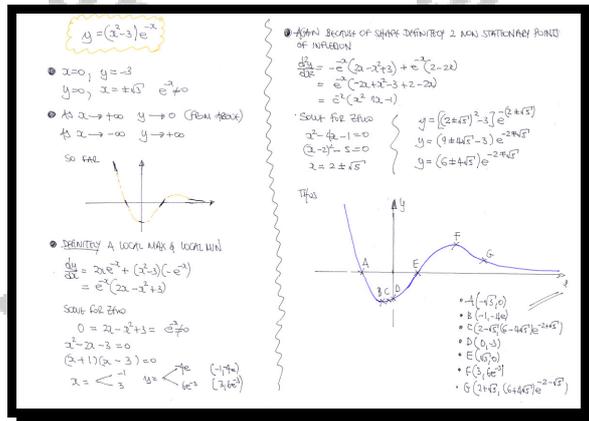
$$y = (x^2 - 3)e^{-x}, \quad x \in \mathbb{R}.$$

Sketch the graph of C .

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the coordinates of any non stationary turning points.
- the equations of any asymptotes.

graph



Question 58 (****+)

A curve C has equation

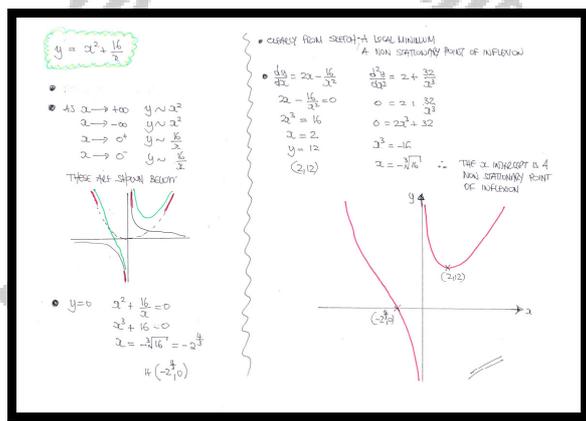
$$y = x^2 + \frac{16}{x}, \quad x \in \mathbb{R}, \quad x \neq 0.$$

Sketch the graph of C .

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the coordinates of any non stationary turning points.
- the equations of any asymptotes.

graph



Question 59 (***)

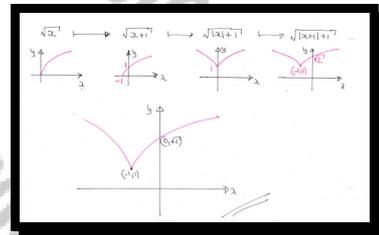
By considering the following sequence of transformations T_1 , T_2 and T_3

$$\sqrt{x} \xrightarrow{T_1} \sqrt{x+1} \xrightarrow{T_2} \sqrt{|x|+1} \xrightarrow{T_3} \sqrt{|x+1|+1}$$

sketch the graph of $y = \sqrt{|x+1|+1}$.

Indicate the coordinates of any intersections with the axes, and the coordinates of the cusp of the curve.

graph



Question 60 (***)

A curve C has equation

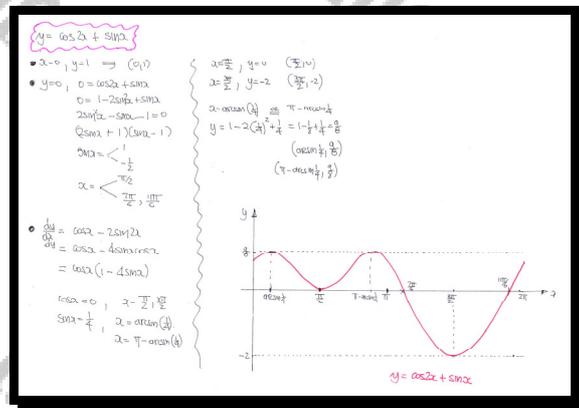
$$y = \cos 2x + \sin x, \quad 0 \leq x \leq 2\pi.$$

Sketch the graph of C , for the largest possible domain.

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.

graph



Question 61 (****+)

A curve C has equation

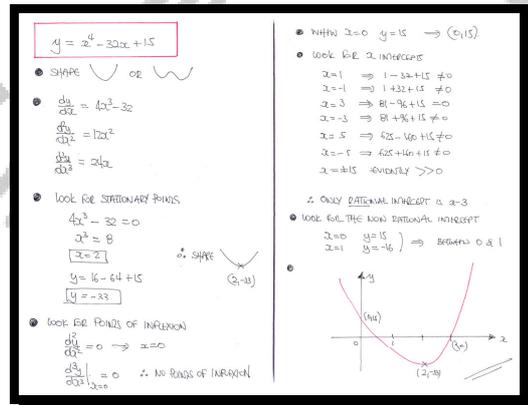
$$y = x^4 - 32x + 15, \quad x \in \mathbb{R}.$$

Sketch the graph of C .

The sketch must include

- the coordinates of any points where the graph of C meets the coordinate axes.
- If a coordinate is not rational indicate suitably on the axis an interval of consecutive integers in which the graph meets that particular coordinate axis.
- the coordinates of any stationary points.
- the coordinates of any non stationary turning points.
- the equations of any asymptotes.

graph



Question 62 (***)

A curve C has equation

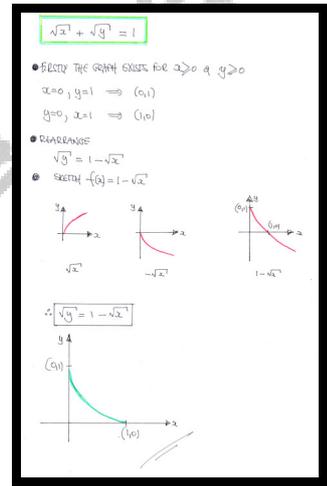
$$\sqrt{y} + \sqrt{x} = 1.$$

Sketch the graph of C , for the largest possible domain.

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the coordinates of any non stationary turning points.
- the equations of any asymptotes.

, graph



Question 63 (*****)

A curve C has equation

$$y^3 - y^2 = x, \quad x \in \mathbb{R}, \quad y \in \mathbb{R}.$$

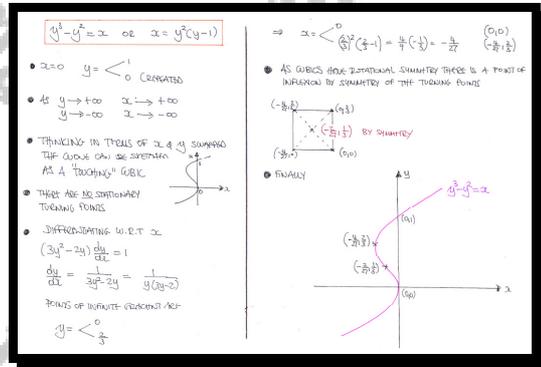
Sketch the graph of C .

The graph must include the coordinates ...

... of any points where the graph of C meets the coordinate axes.

... of the three turning points of C , of which one is a point of inflection.

, graph



Question 64 (*****)

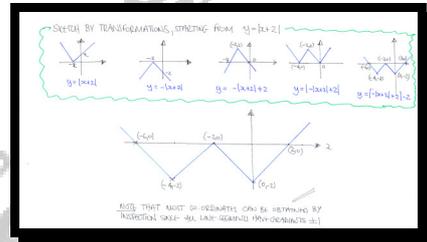
$$f(x) = |2 - |x + 2|| - 2, \quad x \in \mathbb{R}.$$

Sketch the graph of $f(x)$

The sketch must include the coordinates ...

- ... of any points where the graph meets the coordinate axes
- ... of any cusps of the graph.

graph



Question 65 (*****)

A curve C is defined in the largest real domain by the equation

$$x = \sqrt{\frac{y}{y+1}}$$

Sketch a detailed graph of C , fully justifying its key features.

, graph

APPROACH THE PROBLEM AS

- SKETCH $y = \frac{x^2}{x+1}$
- INCLUDE $y^2 = \frac{x^2}{x+1}$
- TAKE THE "TOP HALF" $y = +\sqrt{\frac{x^2}{x+1}}$
- SWAP x & y (CHECK IN $y=x$) TO OBTAIN $x = \sqrt{\frac{y}{y+1}}$

$y = \frac{x^2}{x+1} = \frac{(x+1)-1}{x+1} = 1 - \frac{1}{x+1}$

$y^2 = \frac{x^2}{x+1}$

- GRAPH FOR WHAT $y < 0$ VALUES
- x CO-ORDS INTERCHANGED
- y CO-ORDS SQUARE ROOT
- FORM FOR WHAT $y > 0$ SKETCH
- EXPLECTS IN THE x AXIS
- AS x INCREASES USE "THE INFINITE BEARING"

FURTHER WE HAVE $y = +\sqrt{\frac{x^2}{x+1}}$

REFLECTING IN $y=x$ YIELDS THE GRAPH OF $x = \sqrt{\frac{y}{y+1}}$

ALTERNATIVE APPROACH

$x = \sqrt{\frac{y}{y+1}} \Rightarrow x^2 \geq 0$

$\Rightarrow x^2 = \frac{y}{y+1}$

$\Rightarrow x^2 y + x^2 = y$

$\Rightarrow x^2 = y - x^2 y$

$\Rightarrow x^2 = (1-x^2)y$

$\Rightarrow y = \frac{x^2}{1-x^2}, x^2 \geq 0$

- CURVE IS EVEN
- AS $x \rightarrow \pm\infty, y \rightarrow -1$ (HORIZONTAL)
- $\frac{dy}{dx} = \frac{(1-x^2)(2x) - x^2(-2x)}{(1-x^2)^2} = \frac{2x(-2x^2+2x^2)}{(1-x^2)^2} = \frac{2x}{(1-x^2)^2}$ \therefore STATIONARY AT THE ORIGIN
- VERTICAL ASYMPTOTES AT $x = \pm 1$

CHECK VIA THE DISCRIMINANT THAT THE CURVE DOES NOT OBTAIN POSITIVE $-1 < y < 0$

$x^2 y + x^2 = y$

$x^2(y+1) - y = 0$

$b^2 - 4ac < 0$

$0^2 - 4(y+1)(y) < 0$

$4y(y+1) < 0$

$-1 < y < 0$

HENCE THE DRAWN CURVE IS THE CURVE IN THE PREVIOUS

SKETCH FOR WHICH $x^2 \geq 0$

Created by T. Madas

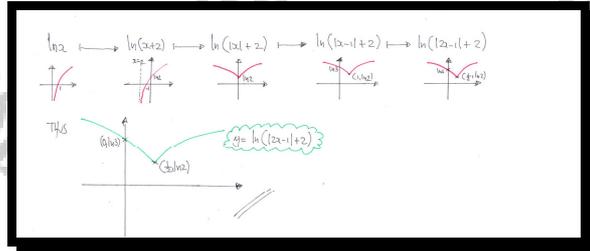
Question 66 (*****)

By considering a sequence of transformations, or otherwise, sketch the graph of

$$y = \ln(|2x-1|+2), \quad x \in \mathbb{R}.$$

Indicate the coordinates of any intersections with the axes, and the coordinates of the cusp of the curve.

graph



Created by T. Madas

Question 67 (*****)

Sketch the graph of the curve with equation

$$x = y^2 \ln y.$$

The sketch must include ...

- ... the coordinates of any intersections with the axes.
- ... the coordinates of any points where the tangent to the curve is parallel to the coordinate axes.
- ... the coordinates of any points of inflexion.

Sketch, graph

● WE START FROM SKETCHING THE GRAPH OF $y = x^2 \ln x$

- THE CURVE IS ONLY DRAWN FOR $x > 0$
- AS $x \rightarrow 0^+$, $y \rightarrow 0^-$ (As x TENDS TO ZERO FROM THE RIGHT, $\ln x$ TENDS TO $-\infty$)
- AS $x \rightarrow +\infty$, $y \sim x^2$
- WHEN $x=1$, $y=0$

THIS IS OUR STARTING POINT

● FROM THE INITIAL CONFIGURATION THE CURVE HAS A MINIMUM STATIONARY POINT.

$$\frac{dy}{dx} = 2x \ln x + x^2 \left(\frac{1}{x}\right)$$

$$\frac{dy}{dx} = 2x \ln x + x$$

$$\frac{dy}{dx} = x(2 \ln x + 1)$$

SETTING $\frac{dy}{dx} = 0$

$$x=0, \quad 2 \ln x + 1 = 0$$

$$y \rightarrow 0, \quad \ln x = -\frac{1}{2}$$

$$x = e^{-\frac{1}{2}}$$

$$y = e^{-\frac{1}{2}} \left(-\frac{1}{2}\right)$$

$$y = -\frac{1}{2e}$$

FROM THE INITIAL CONFIGURATION WE DECIDE THE SHAPE OF THE CURVE CLOSE TO THE ORIGIN, & DECIDE WHETHER THAT THERE IS A POINT OF INFLEXION BETWEEN 'O' AND THE STATIONARY POINT.

● LOCATING FOR THE CO-ORDINATE OF THE POINT OF INFLEXION

$$\frac{d^2y}{dx^2} = (2 \ln x + 1) + x \left(\frac{1}{x}\right) = 3 + 2 \ln x$$

SEARCHING FOR ZERO

$$3 + 2 \ln x = 0$$

$$\ln x = -\frac{3}{2}$$

$$x = e^{-\frac{3}{2}}$$

$$y = \left(e^{-\frac{3}{2}}\right)^2 \left(-\frac{3}{2}\right) = -\frac{3}{2} e^{-3}$$

$$\therefore \left(e^{-\frac{3}{2}}, -\frac{3}{2} e^{-3}\right)$$

● FINALLY THE DEVIATED CURVE WILL BE A REFLECTION ABOUT THE LINE $y=x$, i.e. SWAPPING x & y .

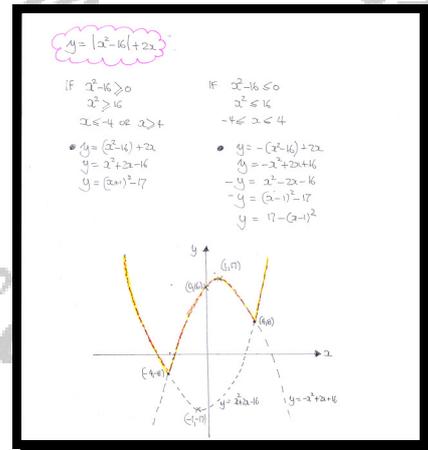
Question 68 (*****)

Sketch the graph of

$$y = |x^2 - 16| + 2x, \quad x \in \mathbb{R}.$$

The sketch must include the coordinates of any cusps or any stationary points

graph



Question 69 (*****)

A curve C is defined in the largest possible real number domain and has equation

$$y = \frac{x^3}{1-x^4}$$

Sketch the graph of C .

The sketch must include

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the coordinates of any non stationary turning points.
- the equations of any asymptotes.

graph

• FIRSTLY WRITE THE EQUATION IN EXPANDED/FACORIZED FORM

$$y = \frac{x^3}{1-x^4} = \frac{x^3}{(1-x)(1+x)(1+x^2)}$$

• NOTICE THE EQUATION IS ODD, SO IT HAS ROTATIONAL SYMMETRY ABOUT THE ORIGIN SINCE

$$\frac{-x^3}{1-x^4} = \frac{-x^3}{(1-x)(1+x)(1+x^2)} = \frac{-x^3}{(1-x)(1+x)(1+x^2)}$$

SO WE NEED ONLY CONSIDER $x > 0$ & REPEAT IT

• THE ONLY x/y INTERCEPT IS THE ORIGIN

• LOOKING AT THE ZEROS OF THE DENOMINATOR, WE ONLY HAVE TWO VERTICAL ASYMPTOTES AT $x=1$ & $x=-1$ ($x^2 \neq 0$)

AS $x \rightarrow +\infty$ $y \rightarrow 0^-$ IE FROM BELOW
 AS $x \rightarrow -\infty$ $y \rightarrow 0^+$ IE FROM ABOVE (CONSIDER FROM COORDINATES)

• WORK FOR STATIONARITY (TURNING POINTS)

$$\frac{dy}{dx} = \frac{(1-x^2)(3x^2) - x^3(-4x^3)}{(1-x^4)^2} = \frac{3x^2 - 3x^2 + 4x^6}{(1-x^4)^2} = \frac{4x^6}{(1-x^4)^2}$$

$$\frac{dy}{dx} = \frac{2x^2(2x^2)}{(1-x^4)^2}$$

SETTING TO ZERO YIELDS $2x=0$ (GIVEN $x^2 \neq 0$)

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left[\frac{2x^2(2x^2)}{(1-x^4)^2} \right] = \frac{(6x^3+4x)(1-x^4)^2 - 2x^2(2x^2)(-2)(1-x^4)(-4x^3)}{(1-x^4)^4}$$

$$= \frac{(1-x^4)(6x^3+4x) + 8x^4(1-x^4)}{(1-x^4)^3} = \frac{6x^3+4x+8x^4-8x^7}{(1-x^4)^3} = \frac{2x^3+24x^4+4x}{(1-x^4)^3}$$

$\frac{dy}{dx} \Big|_{x=0} = 0$ SO POSSIBLY INFLECTION AT THE ORIGIN

INDETERMINATE FORM

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left[\frac{2x^2(2x^2)}{(1-x^4)^2} \right] = \frac{(6x^3+4x)(1-x^4)^2 + 8x^4(1-x^4)}{(1-x^4)^4}$$

$$\frac{d^2y}{dx^2} \Big|_{x=0} = \frac{4x+12x^4}{1-x^4} = 4 \neq 0 \therefore \text{ORIGIN IS A STATIONARY POINT OF INFLECTION}$$

• THE BEHAVIOUR OF THE CURVE AS IT APPROACHES THE VERTICAL ASYMPTOTES CAN BE DETERMINED FROM THE REMAINDER (COMBINATION) / OBSERVATION WE HAVE SO FAR

OR AS $x \rightarrow 1^-$ $y \rightarrow +\infty$
 AS $x \rightarrow 1^+$ $y \rightarrow -\infty$
 AS $x \rightarrow -1^-$ $y \rightarrow -\infty$
 AS $x \rightarrow -1^+$ $y \rightarrow +\infty$

• THEN WE CAN NOW SKETCH

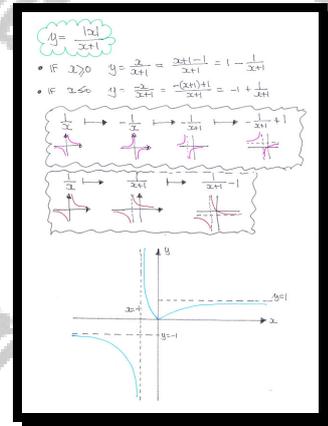
Question 70 (*****)

Sketch the graph of

$$y = \frac{|x|}{x+1}, \quad x \in \mathbb{R}.$$

The sketch must include the equations of any asymptotes of the curve, and the coordinates of any points where the curve meets the coordinate axes.

graph



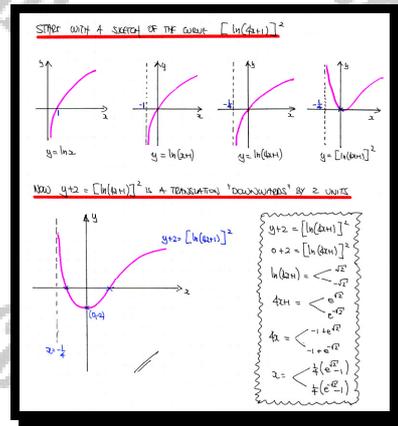
Question 71 (*****)

The curve C has equation

$$y + 2 = [\ln(4x + 1)]^2, \quad x \in \mathbb{R}, \quad x \geq -\frac{1}{4}.$$

Sketch a detailed graph of C .

graph



Question 72 (*****)

A curve C is defined in the largest real domain by the parametric equations

$$x = \sin \theta, \quad y = \tan \theta.$$

Sketch a detailed graph of C , fully justifying its key features.

The sketch must include the range of values of θ , which produces each section of C .

, graph

START BY ELIMINATING θ AND OBTAINING AN EQUATION

$$x = \sin \theta \quad y = \tan \theta$$

$$x^2 = \sin^2 \theta \quad y^2 = \tan^2 \theta$$

$$\frac{1}{x^2} = \csc^2 \theta \quad \frac{1}{y^2} = \sec^2 \theta$$

NOW USES THE IDENTITY $1 + \cos^2 \theta \equiv \sec^2 \theta$

$$\frac{1}{x^2} + 1 = \frac{1}{x^2} + \frac{1}{x^2}$$

$$\frac{1}{y^2} = \frac{1}{x^2} + 1$$

$$\frac{1}{y^2} = \frac{1+x^2}{x^2}$$

$$y^2 = \frac{x^2}{1+x^2}$$

NEXT WE SKETCH $y = \frac{x^2}{1+x^2}$

$$y = \frac{x^2}{1+x^2} = \frac{1-(1-x^2)}{1+x^2} = \frac{1}{1+x^2} - 1 = \frac{1}{(1+i)(1-i)} - 1$$

$$= \frac{1}{1+i} + \frac{1}{1-i} - 1$$

• $\frac{dy}{dx} = \frac{(1-x^2)(2x) - x^2(2x)}{(1-x^2)^2} = \frac{2x-2x^3+2x^3}{(1-x^2)^2} = \frac{2x}{(1-x^2)^2}$ \therefore STATIONARY AT THE ORIGIN

- AS $x \rightarrow \infty$ $y \rightarrow -1$ (FROM BELOW)
- FUNCTION IS EVEN

• IT HAS OBVIOUS ASYMPTOTES AT $x = \pm 1$

WE DRAW THE FOLLOWING SKETCH

VERIFICATION THAT THE GRAPHS DO NOT MEET BETWEEN $x = -1$ & 0

$$y - y^2 = x^2$$

$$x^2 y^2 - y = 0$$

$$x^2(y-1) - y = 0$$

$$b^2 - 4ac < 0$$

$$(0)^2 - 4(x^2)(-1) < 0$$

$$4x^2 < 0$$

HENCE THE GRAPH OF $y = \frac{x^2}{1+x^2}$ CAN BE DEDUCED

	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin \theta$	+	+	-	-	+
$\tan \theta$	+	+	-	-	+

Question 73 (*****)

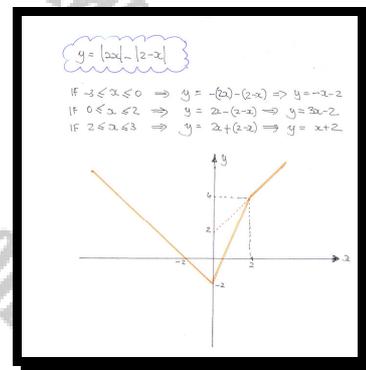
Sketch the graph of

$$y = |2x| - |2 - x|, \quad x \in \mathbb{R}.$$

The sketch must include the coordinates of any points where the curve meets the coordinate axes.

[No credit will be given to non analytical sketches based on plotting coordinates]

graph



Question 74 (*****)

A curve C has equation

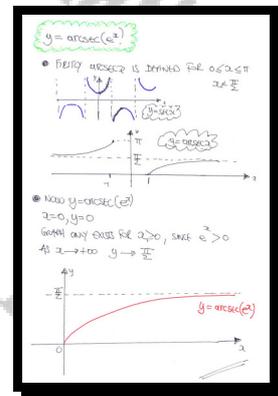
$$y = \operatorname{arcsec}(e^x).$$

Sketch the graph of C , for the largest possible domain.

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the equations of any asymptotes.

graph



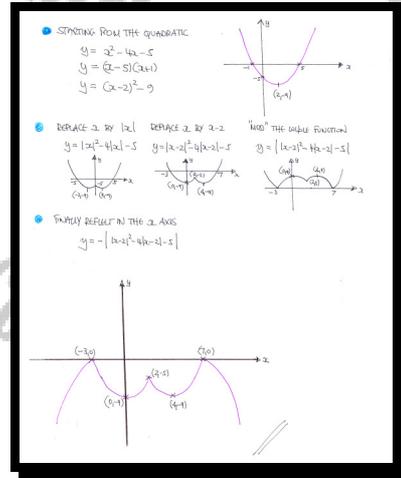
Question 75 (*****)

By considering a sequence of four transformations, or otherwise, sketch the graph of

$$y = -|x-2|^2 - 4|x-2| - 5$$

Indicate the coordinates of any intersections with the axes, and the coordinates of the cusp of the curve.

, (-3,0), (7,0), (0,-9), (2,-5)



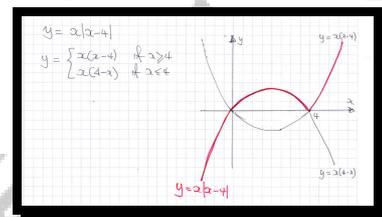
Question 76 (*****)

By considering the graphs of two separate curves, or otherwise, sketch the graph of

$$y = x|x-4|$$

Indicate the coordinates of any intersections with the axes, and the coordinates of the cusp of the curve.

graph



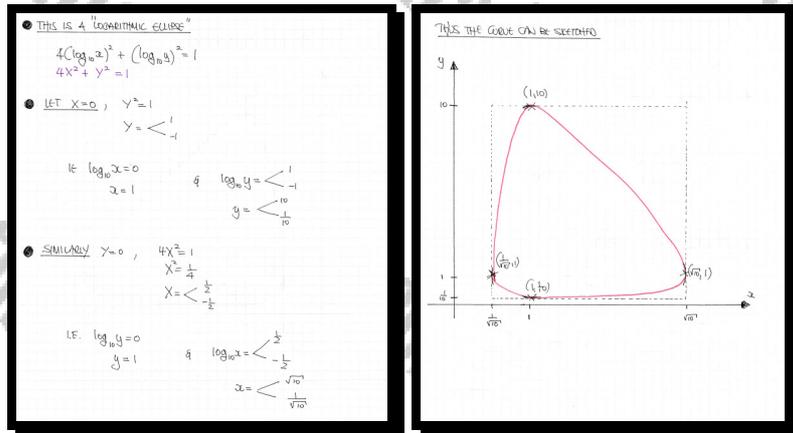
Question 77 (*****)

Sketch the graph of the curve with equation

$$4[\log_{10} x]^2 + [\log_{10} y]^2 = 1, \quad x > 0, \quad y > 0.$$

The sketch must include the coordinates of any points where the tangent to the curve is parallel to the coordinate axes.

, graph



Question 78 (*****)

Sketch the graph of

$$\left[x + \sqrt{x^2 + 4} \right] \left[y + \sqrt{y^2 + 1} \right] = 2, \quad x \in (-\infty, \infty), \quad y \in (-\infty, \infty)$$

You must show a detailed method in this question

P, proof

LOOKING AT THE EQUATION

- y - THEN IS THE HYPERBOLA OF A LOG. THE RESULT
- z - THEN ALSO CHECKS LIKE A SIMILAR LOG ARGUMENT

$\rightarrow (x + \sqrt{x^2 + 4})(y + \sqrt{y^2 + 1}) = 2$
 $\rightarrow \ln[(x + \sqrt{x^2 + 4})(y + \sqrt{y^2 + 1})] = \ln 2$
 $\rightarrow \ln(x + \sqrt{x^2 + 4}) + \ln(y + \sqrt{y^2 + 1}) = \ln 2$
 $\rightarrow \ln(x + \sqrt{x^2 + 4}) + \operatorname{arsinh} y = \ln 2$

MANIPULATE THE LOGS/EXP, SO THE RADICAL THING IS INSTEAD OF 4

$\rightarrow \ln[2 + 2\sqrt{(x^2 + 4)}] + \operatorname{arsinh} y = \ln 2$
 $\rightarrow \ln[2(x + \sqrt{x^2 + 4})] + \operatorname{arsinh} y = \ln 2$
 $\rightarrow \ln 2 + \ln(x + \sqrt{x^2 + 4}) + \operatorname{arsinh} y = \ln 2$
 $\rightarrow \operatorname{arsinh}(x + \sqrt{x^2 + 4}) + \operatorname{arsinh} y = 0$
 $\rightarrow \operatorname{arsinh}(x + \sqrt{x^2 + 4}) = -\operatorname{arsinh} y$

BUT ARGUMENT IS AN ODD FUNCTION

$\rightarrow \operatorname{arsinh}(x + \sqrt{x^2 + 4}) = \operatorname{arsinh}(y)$
BUT THIS IS A ONE TO ONE MAPPING
 $\rightarrow x + \sqrt{x^2 + 4} = y$
 $\rightarrow y = -x + \sqrt{x^2 + 4}$

ALTERNATIVE WITHOUT HYPERBOLICS

$[2 + \sqrt{x^2 + 4}][y + \sqrt{y^2 + 1}] = 2$
 LET $u = 2 + \sqrt{x^2 + 4}$ | BUT $v = 2 + \sqrt{y^2 + 1}$
 $\rightarrow u(y + \sqrt{y^2 + 1}) = 2$ | $\rightarrow \frac{1}{v} = \frac{1}{2 + \sqrt{y^2 + 1}}$
 $\rightarrow y + \sqrt{y^2 + 1} = \frac{2}{u}$ | $\rightarrow \frac{1}{v} = \frac{2 - \sqrt{y^2 + 1}}{2 + \sqrt{y^2 + 1}} \cdot \frac{2 + \sqrt{y^2 + 1}}{2 + \sqrt{y^2 + 1}}$
 $\rightarrow \sqrt{y^2 + 1} = \frac{2}{u} - y$ | $\rightarrow \frac{1}{v} = \frac{2 - \sqrt{y^2 + 1}}{2 - (y^2 + 1)}$
 $\rightarrow y^2 + 1 = \frac{4}{u^2} - 2y$ | $\rightarrow \frac{1}{v} = \frac{2 - \sqrt{y^2 + 1}}{1 - y^2}$
 $\rightarrow 4y^2 + 4 - u^2 = 0$ | $\rightarrow \frac{1}{v} = \frac{1}{1 - y^2} - \frac{1}{1 + y^2}$
 $\rightarrow y = \frac{1}{u} - \frac{2}{u^2}$

CONTINUING BRACES

$y = \frac{1}{u} - \frac{2}{u^2} = -\frac{1}{2} + \frac{1}{2}\sqrt{x^2 + 4} - \frac{1}{2}(2 + \sqrt{x^2 + 4})$
 $= -\frac{1}{2} + \frac{1}{2}\sqrt{x^2 + 4} - 1 - \frac{1}{2}\sqrt{x^2 + 4}$
 $= -\frac{3}{2}$
 $\therefore y = -\frac{3}{2}$ IS BRANCH AND THE OTHER BRANCH

Question 79 (*****)

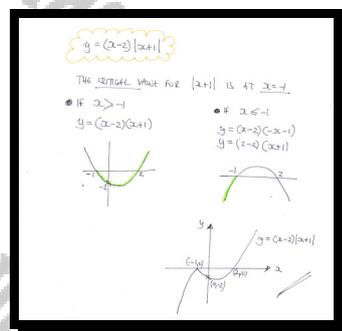
By considering the graphs of two separate curves, or otherwise, sketch the graph of

$$y = (x-2)|x+1|.$$

Indicate the coordinates of any intersections with the axes, and the coordinates of the cusp of the curve.

[No credit will be given to non analytical sketches based on plotting coordinates]

graph



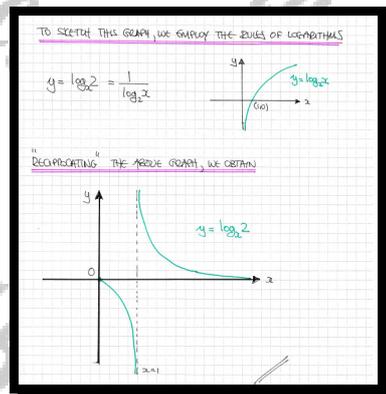
Question 80 (*****)

A curve C is defined in the largest real domain by the equation

$$y = \log_x 2.$$

Sketch a detailed graph of C .

graph



Question 81 (*****)

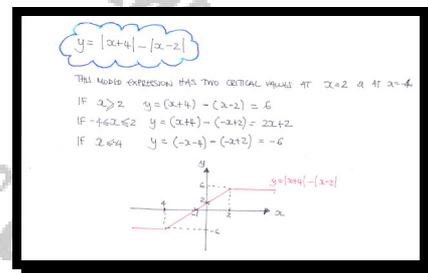
By considering the graphs of three separate lines, or otherwise, sketch the graph of

$$y = |x+4| - |x-2|$$

Indicate the coordinates of any intersections with the axes, and the coordinates of the cusp of the curve.

[No credit will be given to non analytical sketches based on plotting coordinates]

graph



Question 82 (*****)

A curve C is defined in the largest real domain by the equation

$$y = \frac{4x^2 - 25}{(2x-1)(x-2)(x+3)}$$

a) Sketch the graph of C .

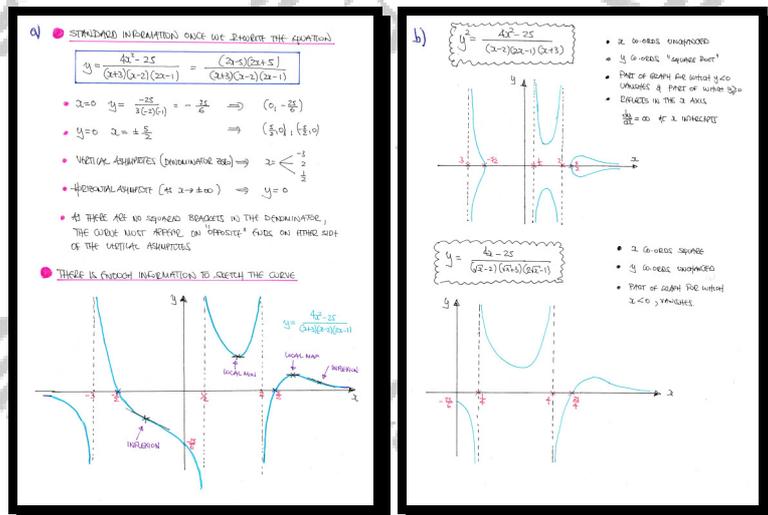
The sketch must include the equations of any asymptotes of C and the coordinates of any point where C meets the coordinate axes. Any turning points, including points of inflexion, must be clearly indicated but their coordinates need **not** be found.

b) Hence sketch on separate set of axes the graph of ...

a) ... $y^2 = \frac{4x^2 - 25}{(2x-1)(x-2)(x+3)}$

b) ... $y = \frac{4x - 25}{(2\sqrt{x} - 1)(\sqrt{x} - 2)(\sqrt{x} + 3)}$

graph



Question 83 (****)

A curve is defined in the largest real domain by the equation

$$f(x) = -x^2 + 8x - 12.$$

Sketch on separate set of axes detailed graph of ...

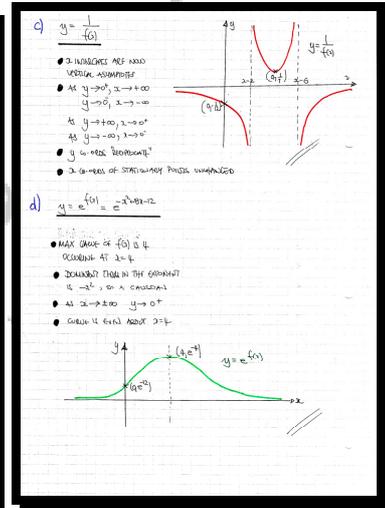
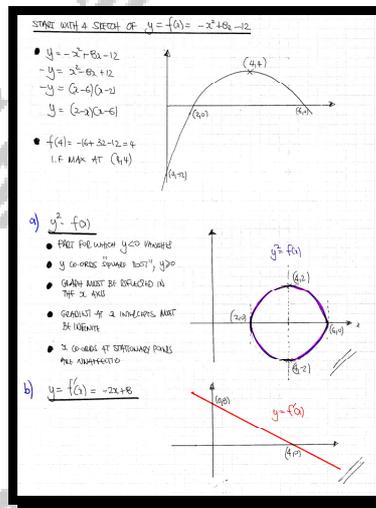
a) ... $y^2 = f(x)$.

b) ... $y = f'(x)$.

c) ... $y = \frac{1}{f(x)}$.

d) ... $y^2 = e^{f(x)}$.

, graph



Question 84 (*****)

On a clearly labelled set of axes, draw a detailed sketch of the graph of

$$y = (\arcsin x)^2 \arccos x, \quad -1 \leq x \leq 1.$$

graph

Question 85 (*****)

By considering the graphs of three separate lines, or otherwise, sketch the graph of

$$y = |x - 4| + |x + 1|$$

Indicate the coordinates of any intersections with the axes, and the coordinates of the cusp of the curve.

[No credit will be given to non analytical sketches based on plotting coordinates]

graph

Question 86 (*****)

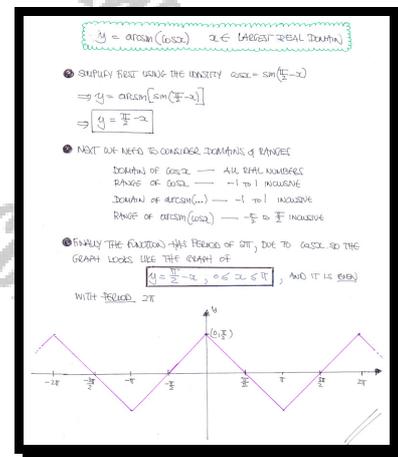
Sketch the graph of

$$f(x) = \arcsin(\cos x),$$

in the largest domain that the function is defined.

Indicate the coordinates of any intersections with the axes, and the coordinates of the cusps of the curve.

, graph



Question 87 (*****)

The curve C has equation

$$y = |x^2 - 16| + 2(x - 4), \quad x \in \mathbb{R}.$$

Sketch a detailed graph of C and hence show that the area of the finite region bounded by C and the x axis, for which $y < 0$, is 32 square units.

, proof

$y = |x^2 - 16| + 2(x - 4), \quad x \in \mathbb{R}$

THE CRITICAL VALUES OF THIS GRAPH (DUE TO THE ABSOLUTE) ARE DETERMINED BY THE INEQUALITY

$\Rightarrow x^2 - 16 \geq 0$
 $\Rightarrow x^2 \geq 16$
 $\Rightarrow x \leq -4$ or $x \geq 4$

HENCE WE HAVE TWO CASES TO CONSIDER

IF $x^2 - 16 \geq 0$ $x \leq -4$ or $x \geq 4$	IF $x^2 - 16 < 0$ $-4 < x < 4$
THE GRAPH REDUCES TO $\Rightarrow y = (x^2 - 16) + 2(x - 4)$ $\Rightarrow y = (x - 4)(x + 4) + 2(x - 4)$ $\Rightarrow y = (x - 4)(x + 4 + 2)$ $\Rightarrow y = (x - 4)(x + 6)$ $x \leq -4$ or $x \geq 4$	THE GRAPH REDUCES TO $\Rightarrow y = -(x^2 - 16) + 2(x - 4)$ $\Rightarrow y = -(x - 4)(x + 4) + 2(x - 4)$ $\Rightarrow y = (x - 4)[-x - 4 + 2]$ $\Rightarrow y = (x - 4)(-x - 2)$ $\Rightarrow y = (x + 2)(4 - x)$ $-4 < x < 4$

SOLVING SIMULTANEOUSLY TO LOCATE CUSPS

$x = 4, y = 0 \Rightarrow (4, 0)$
 $x = -4, y = 16 \Rightarrow (-4, 16)$

THE SKETCH CAN NOW BE PRODUCED

THE REQUIRED AREA CAN BE FOUND AS

$$\begin{aligned} \text{Area} &= \int_{-4}^{-2} (x+6)(x-4) dx + \int_{-2}^4 (x+2)(4-x) dx \\ &= \int_{-4}^{-2} x^2 + 2x - 24 dx + \int_{-2}^4 -x^2 + 2x + 8 dx \\ &= \left[\frac{1}{3}x^3 + x^2 - 24x \right]_{-4}^{-2} + \left[-\frac{1}{3}x^3 + x^2 + 8x \right]_{-2}^4 \\ &= (-22 + 36 + 48) - (-\frac{128}{3} + 16 + 96) + (-\frac{64}{3} + 16 - 32) - (-\frac{8}{3} + 4 - 16) \\ &= 108 - \frac{272}{3} + \frac{8}{3} + \frac{24}{3} \\ &= 32 \end{aligned}$$

Question 88 (*****)

A curve C is defined in the largest real domain by the equation

$$y = \frac{2 - \sqrt{x}}{x - 9\sqrt{x} + 18}$$

Sketch the graph of C .

The sketch must include

- ... the equations of any asymptotes of C .
- ... the coordinates of any point where C meets the coordinate axes.
- ... the coordinates any stationary points of C .

, graph

• START BY SKETCHING THE CURVE WITH EQUATION $y = \frac{2 - \sqrt{x}}{x^2 - 9x + 18}$

$$y = \frac{2 - \sqrt{x}}{(x-3)(x-6)} = \frac{1}{x-3} - \frac{1}{x-6} \quad (\text{BY INSPECTION})$$

• FROM THE ABOVE RESULTS WE HAVE

$x=0$; $y = \frac{1}{3}$ $(0, \frac{1}{3})$
 $y=0$; $x=2$ $(2, 0)$
 VERTICAL ASYMPTOTES $x=3, x=6$
 HORIZONTAL ASYMPTOTE $y=0$ (As $x \rightarrow \pm\infty$)

• WORK FOR STATIONARY POINTS

$$\frac{dy}{dx} = -\frac{1}{2} \frac{1}{(x-3)^2} + \frac{1}{2} \frac{1}{(x-6)^2} = \frac{1}{2} \left[\frac{1}{(x-6)^2} - \frac{1}{(x-3)^2} \right]$$

$$= \frac{1}{2} \left[\frac{4(x-3)^2 - (x-6)^2}{(x-6)^2(x-3)^2} \right]$$

$$= \frac{1}{2} \left[\frac{[2(x-3) - (x-6)][2(x-3) + (x-6)]}{(x-6)^2(x-3)^2} \right]$$

$$= \frac{1}{2} \left[\frac{2(3x-12)}{(x-6)^2(x-3)^2} \right] = \frac{3(x-4)}{(x-6)^2(x-3)^2}$$

SEARCHING FOR ZERO DERIVATIVE

$$x = \begin{cases} 0 \\ 4 \end{cases} \quad y = \begin{cases} \frac{1}{3} \\ -\frac{2-4}{(4-3)(4-6)} = -\frac{-2}{-2} = 1 \end{cases}$$

$\therefore (0, \frac{1}{3})$ & $(4, 1)$ ARE STATIONARY

• PUT THE INFORMATION SO FAR INTO A PRELIMINARY GRAPH

• THERE IS ONLY ONE x-INTERCEPT AND NO OTHERS BECAUSE IN THE DENOMINATOR FOR THE CURVE HAVE 'FACTORED' IN 2 TERMS AND DENOMINATOR ON THE OTHER ASYMPTOTES; THE CURVE CAN BE SKETCHED

• THIS WE CAN NOW SKETCH ITS TRANSFORMATION $x \rightarrow \sqrt{x}$

NOTE ONLY CHANGING THE x TO \sqrt{x} DOES NOT CHANGE THE CO-ORDINATES

$y = \frac{2 - \sqrt{x}}{x^2 - 9x + 18}$

Question 89 (*****)

A curve C is defined parametrically by

$$x = \frac{1}{t} + \arctan t, \quad y = \frac{1}{t} - \arctan t, \quad t \in \mathbb{R}, t \neq 0.$$

Sketch the graph of C .

Indicate the equations of any asymptotes, stationary points and any endpoints.

You need not mark the coordinates of any intersections with the axes.

, graph

• FIRSTLY SOLVE LIMITING PROCESSES

As $t \rightarrow 0$ $x \sim \frac{1}{t}, y \sim \frac{1}{t} \Rightarrow$ ASYMPTOTE $y=x$
 As $t \rightarrow \infty$ $x \rightarrow \frac{1}{t}, y \rightarrow -\frac{1}{t} \Rightarrow (\frac{1}{t}, -\frac{1}{t})$
 As $t \rightarrow -\infty$ $x \rightarrow -\frac{1}{t}, y \rightarrow \frac{1}{t} \Rightarrow (-\frac{1}{t}, \frac{1}{t})$

• NEXT WE NOTICE THAT THE COORD IS ODD, AS FOLLOWS

REPLACE x FOR $-x \Rightarrow -x = -\frac{1}{t} - \arctan t$
 $= \frac{1}{t} + \arctan(-t)$
 $\therefore t \rightarrow -t$

AND NOW $y = \frac{1}{t} - \arctan(-t)$
 $= -\frac{1}{t} + \arctan t$
 $= -[\frac{1}{t} - \arctan t]$
 $= -y$

\therefore IF $x \rightarrow -x$ THEN $y \rightarrow -y$

• NEXT WE LOOK FOR TURNING POINTS (MIN/MAX)

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-\frac{1}{t^2} - \frac{1}{1+t^2}}{\frac{1}{t^2} + \frac{1}{1+t^2}} = \frac{-\frac{1}{t^2} - \frac{1}{1+t^2}}{\frac{1}{t^2} + \frac{1}{1+t^2}} \times \frac{t^2(1+t^2)}{t^2(1+t^2)}$$

$$= \frac{-(1+t^2) + t^2}{(1+t^2) + t^2} = \frac{-1 - t^2 + t^2}{1 + 2t^2} = \frac{-1}{1 + 2t^2}$$

NO STATIONARY POINTS
 VALUE IS DECREASING

• LOOKING FOR Y INTERCEPTS

$x=0$
 $\frac{1}{t} + \arctan t = 0$
 $\arctan t = -\frac{1}{t}$
 NO REAL SOLUTIONS

• LOOKING FOR X INTERCEPTS

$y=0$
 $\frac{1}{t} - \arctan t = 0$
 $\arctan t = \frac{1}{t}$
 2 NUMERICAL SOLUTIONS

• FINALLY WE CAN USE THIS INFORMATION TO SKETCH

Question 90 (*****)

It is required to sketch the curve with equation $y = f(x)$, defined over the set of real numbers, in the greatest domain.

The curve has the property that at every point on the curve, the second derivative equals to the first derivative **squared**.

Showing all the relevant details, sketch the graph of $y = f(x)$, given further that the curve passes through the point $(0,2)$ and the gradient at that point is 1.

, graph

WE REQUIRE TO SOLVE:
 $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$ WITH $x=0, y=2, \frac{dy}{dx}=1$

LET $v = \frac{dy}{dx}$
 $\frac{dv}{dx} = v^2$
 $\frac{1}{v^2} dv = 1 dx$
 $-\frac{1}{v} = x + C$
 $\frac{1}{v} = A - x$
 $v = \frac{1}{A-x}$
 $\therefore \frac{dy}{dx} = \frac{1}{A-x}$

APPLY CONDITION
 $1 = \frac{1}{A}$
 $A = 1$
 $\therefore \frac{dy}{dx} = \frac{1}{1-x}$

INTEGRATE ONCE MORE
 $y = -\ln|1-x| + B$

APPLY CONDITION
 $2 = -\ln|1-0| + B$
 $B = 2$
 $\therefore y = 2 - \ln|1-x|$
 $y = 2 - \ln|x-1|$

NOW PROCEED WITH SKETCH
 • $x=0, y=2 - \ln 1$
 IF $(0,2)$ GRAD IS 1
 • $y=0$
 $0 = 2 - \ln|x-1|$
 $\ln|x-1| = 2$
 $|x-1| = e^2$
 $x-1 = e^2$
 $x-1 = -e^2$
 $\therefore (e^2+1, 0)$ & $(-e^2-1, 0)$

FINALLY THE SHAPE BY TRANSFORMATIONS
 $y = kx$ $y = h|kx|$ $y = h|kx-1|$ $y = -h|kx-1|$

FINALLY TRANSLATE THE GRAPH LOGONIC BY 2 UNITS
 $y = 2 - \ln|x-1|$

Question 91 (*****)

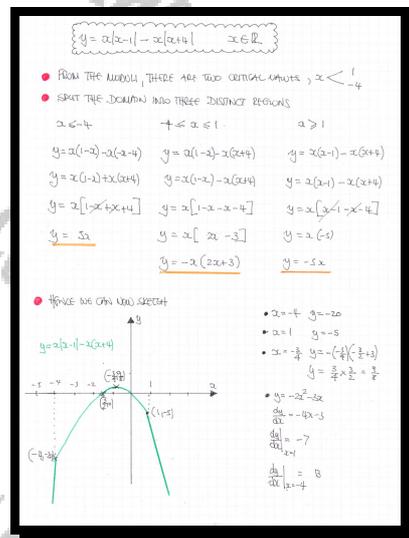
Sketch the graph of

$$y = x|x-1| - x|x+4|, \quad x \in \mathbb{R}.$$

Indicate the coordinates of any intersections with the axes, and the coordinates of any cusps of the curve.

[No credit will be given to non analytical sketches based on plotting coordinates]

, graph



Question 92 (****)

A curve C has equation

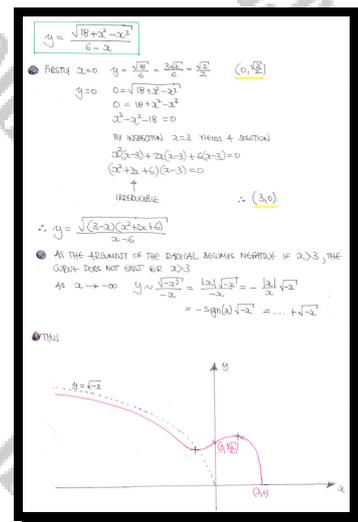
$$y = \frac{\sqrt{18+x^2-x^3}}{6-x}$$

It is given that C has two stationary points whose x coordinates have opposite signs.

Sketch the graph of C , for the largest possible domain.

- The sketch must include, in exact form where appropriate the coordinates of any points where the graph of C meets the coordinate axes the equations of any asymptotes.
- You need not find the coordinates of the stationary points of C .

5*, graph



Question 93 (*****)

Sketch the curve with equation

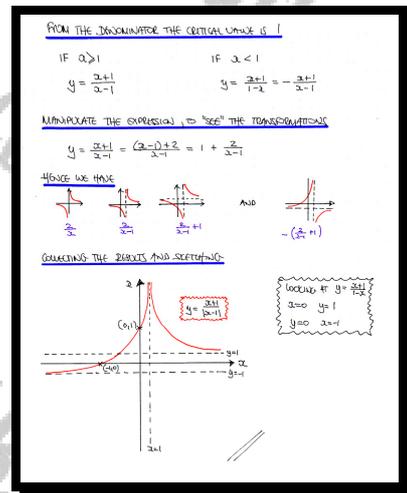
$$y = \frac{x+1}{|x-1|}, x \in \mathbb{R}, x \neq 1.$$

The sketch must include ...

- ... the coordinates of all the points where the curve meets the coordinate axes.
- ... the equations of the asymptotes of the curve.

[No credit will be given to non analytical sketches based on plotting coordinates]

, graph



Question 94 (*****)

Sketch in separate sets of axes detailed graphs of the following curves, fully justifying their key features.

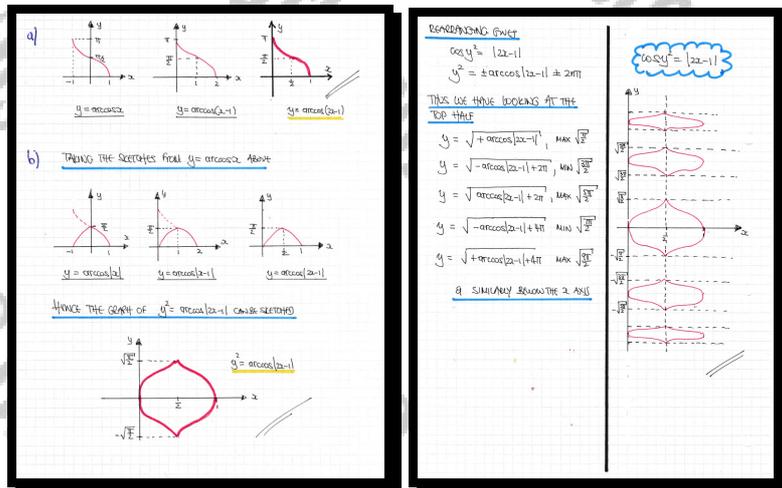
a) $y = \arccos(2x-1)$.

b) $y^2 = \arccos|2x-1|$.

c) $\cos y^2 = |2x-1|$

You may assume that each curve is defined in the largest real domain.

31, graph



Question 95 (*****)

Sketch the graph of the curve with equation

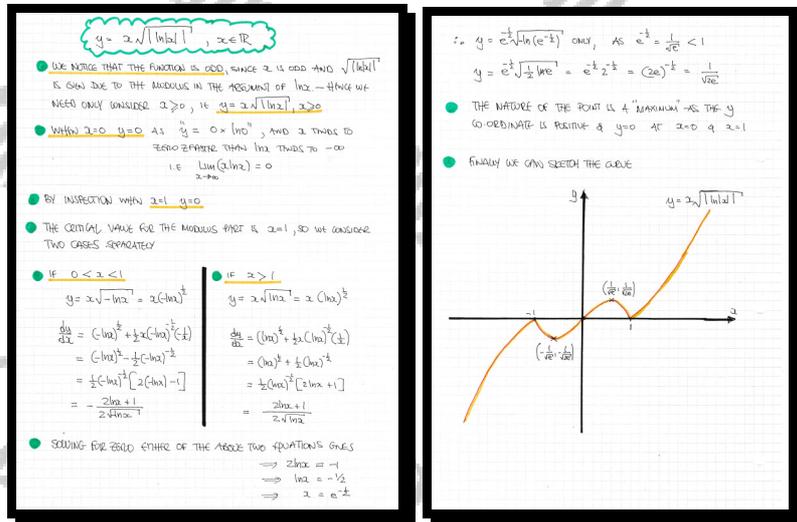
$$y = x\sqrt{\ln|x|}, \quad x \in \mathbb{R}.$$

The sketch must include the coordinates of ...

... any points where the curve meets the coordinate axes.

... any stationary points.

, graph



Question 96 (*****)

The distinct points A and B lie on the curve with equation

$$\ln(x-y) = \ln x + \ln y, \quad x \in (0, \infty), \quad y \in (0, \infty).$$

- Determine possible coordinates for A and B , further verifying that these coordinates indeed satisfy the above given equation.
- Sketch the curve, showing clearly all the relevant details.

~~$A(2, 2)$~~ , $A(4, 2)$, $B\left(\frac{9}{2}, \frac{3}{2}\right)$

a) EXPONENTIATING BOTH SIDES OF THE EQUATION

$$\ln(x-y) = \ln x + \ln y$$

$$e^{\ln(x-y)} = e^{\ln x + \ln y}$$

$$x-y = e^{\ln x} \times e^{\ln y}$$

$$x-y = x \times y$$

$$xy - y^2 = x$$

$$xy - x = y^2$$

$$x(y-1) = y^2$$

$$x = \frac{y^2}{y-1} \quad (\text{DIVIDED BY WITH ZEROES AT } 1)$$

PICKING SOME SENSIBLE VALUES OF y

$y=2 \quad x = \frac{2^2}{2-1} = 4 \quad \therefore A(4, 2)$

$\bullet \ln(x-y) = \ln(4-2) = \ln 2$
 $\bullet \ln x + \ln y = \ln 4 + \ln 2 = \ln 8 = \ln 2$

$y=3 \quad x = \frac{3^2}{3-1} = \frac{9}{2} = 4.5 \quad \therefore B\left(\frac{9}{2}, 3\right)$

$\bullet \ln(x-y) = \ln\left(4.5 - \frac{9}{2}\right) = \ln 3$
 $\bullet \ln x + \ln y = \ln\left(\frac{9}{2}\right) + \ln 3 = \ln\left(\frac{27}{2}\right) = \ln 3$

INDICATE THE ABOVE AS POSSIBLE COORDINATES

b) TO SKETCH THIS CURVE WE NEED TO CONSIDER THAT $x > y > 0$

$$x = \frac{y^2}{y-1} - y(y-1) = \frac{y^2}{y-1} - y^2 + y = \frac{y^2 - y^3 + y}{y-1}$$

NOW CONSIDER THE GRAPH OF $y = x + \frac{1}{x}$

- For $x > 1$, $y > 2$
- For $0 < x < 1$, $y > 2$
- As $x \rightarrow 1^+$, $y \rightarrow +\infty$
- As $x \rightarrow 1^-$, $y \rightarrow +\infty$
- $y' = 1 - \frac{1}{x^2}$
- $0 = 1 - \frac{1}{x^2}$
- $x = \pm 1$
- $x < -1$ or $x > 1$

NOW REFLECTING IN $y=2$ (SWAPPING x & y)

- $x > 0$
- $y > 0$
- $x > y$
- $\frac{y^2}{y-1} > y$
- $\frac{y^2}{y-1} > 1$
- $y > y-1$ not

\therefore SKETCH IS SHOWN IN RED IN THE FIRST COMMENT

Question 97 (****)

A curve C is defined in the largest real domain by the equation

$$y = -\sqrt{\frac{x(1-x)}{4-x^2}}$$

Sketch the graph of C .

The sketch must include

- ... the equations of any asymptotes of C .
- ... the coordinates of any point where C meets the coordinate axes.
- ... the coordinates of the stationary points of C , giving the answer in the form $\left[2k + k\sqrt{3}, -\frac{1}{2k}(\sqrt{3k} + \sqrt{k})\right]$, where k is a positive integer.

, graph

• START BY CONSIDERING THE GRAPH OF $y = \frac{x(1-x)}{4-x^2}$

$$y = \frac{x-x^2}{4-x^2} = \frac{x(1-x)}{(2-x)(2+x)} = \frac{x^2-x}{x^2-4}$$

$$= \frac{x^2-4}{x^2-4} + \frac{-x+4}{x^2-4} = 1 + \frac{-x+4}{x^2-4}$$

• FIND THE ASYMPTOTES OR BEHAVIOUR DIRECTLY

- $x=0, y=0$ is (0,0)
- $y=0, x=1$ is (1,0)
- VERTICAL ASYMPTOTES $x=2, x=-2$ (NON-MIXED SIGNS)
- HORIZONTAL ASYMPTOTES $y=1$ (As $x \rightarrow \pm\infty$)

• NEXT LOOK FOR STATIONARY POINTS

$$\frac{d}{dx} \left[1 + \frac{-x+4}{x^2-4} \right] = \frac{(x^2-4)(-1) - (-x+4)(2x)}{(x^2-4)^2} = \frac{-x^2-8x+2x^2}{(x^2-4)^2}$$

$$= \frac{x^2-8x+4}{(x^2-4)^2}$$

SOLVING FOR ZERO VALUES

$$x^2-8x+4=0$$

$$(x-4)^2-12=0$$

$$(x-4)^2=12$$

$$x-4=\pm 2\sqrt{3}$$

$$x=4\pm 2\sqrt{3}$$

• $x=4+2\sqrt{3}$
 $x=4-2\sqrt{3}$

THIS $\frac{x^2-x}{x^2-4} = \frac{(2\sqrt{3}+16\sqrt{3}) - (4\pm 2\sqrt{3})}{2\sqrt{3}\pm 16\sqrt{3}-4}$

$$= \frac{2\sqrt{3}+16\sqrt{3}-4\pm 2\sqrt{3}}{2\sqrt{3}\pm 16\sqrt{3}-4} = \frac{18\sqrt{3}\pm 2\sqrt{3}-4}{2\sqrt{3}\pm 16\sqrt{3}-4} = \frac{2\sqrt{3}(9\pm 1)-4}{2\sqrt{3}(1\pm 8)-4}$$

HENCE THE STATIONARY POINTS ARE $(4\pm 2\sqrt{3}, \frac{1}{2}(\pm 2\sqrt{3}))$

• THIS WE WANT TO FIND

- $y=0$ IS ONE OF SEVERAL POINTS
- PART OF CURVE FOR WHICH $y > 0$ REMAINS ABOVE THE x -AXIS
- THE PART OF THE CURVE FOR WHICH $y < 0$ REMAINS BELOW THE x -AXIS
- THE CURVE HAS VERTICAL ASYMPTOTES AT $x=2$ AND $x=-2$

NOTE THAT $\frac{1}{2}(2+2\sqrt{3}) = \frac{1}{2}(1+2\sqrt{3}) = \frac{1}{2}(1^2 + 2 \times 1 \times \sqrt{3} + (\sqrt{3})^2) = \frac{1}{2}(1+\sqrt{3})^2$

• SIMILARLY $\frac{1}{2}(2-2\sqrt{3}) = \frac{1}{2}(1-\sqrt{3})^2$

THIS $\sqrt{\frac{x(1-x)}{4-x^2}}$ IS THE UPPER HALF OF $y = \frac{x(1-x)}{4-x^2}$

AND SIMILARLY $\sqrt{\frac{x(1-x)}{4-x^2}}$ IS THE LOWER HALF

• FINALLY THE DEFINITIVE GRAPH IS THE UPPER HALF OF $y = \frac{x(1-x)}{4-x^2}$

• $(4-2\sqrt{3}, \frac{1}{2}(2\sqrt{3}-1))$
 $(4+2\sqrt{3}, \frac{1}{2}(2\sqrt{3}+1))$

$y = -\sqrt{\frac{x(1-x)}{4-x^2}}$

Question 98 (*****)

The curve C has equation

$$y = A \ln|x| + Bx^2 + x, \quad x \in \mathbb{R},$$

where A and B are non zero constants.

The curve has stationary points at $x = -1$ and at $x = 2$.

Sketch the graph of C .

The sketch must include ...

- ... the coordinates of all the stationary points.
- ... the equations of the asymptotes of the curve.

You need not find any intercepts with the coordinate axes.

, graph

$y = A \ln|x| + Bx^2 + x, \quad x \in \mathbb{R}$

- CONSIDER THE FORM IN TWO SEPARATE SECTIONS
- IF $a > 0$
 $y = A \ln|x| + Bx^2 + x$
 $\frac{dy}{dx} = \frac{A}{x} + 2Bx + 1$
- STATIONARY AT $x=2$
 $0 = \frac{A}{2} + 4B + 1$
 $0 = A + 8B + 2$
 $A + 8B = -2$
- IF $a < 0$
 $y = A \ln|x| + Bx^2 + x$
 $\frac{dy}{dx} = \frac{A}{|x|} \times \text{sign}(x) + 2Bx + 1$
- STATIONARY AT $x=-1$
 $0 = \frac{A}{|-1|} \times (-1) + 2B(-1) + 1$
 $0 = -A - 2B + 1$
 $A + 2B = 1$

or

$y = A \ln(-x) + Bx^2 + x$
 $\frac{dy}{dx} = \frac{A}{-x}(-1) + 2Bx + 1$
 $\frac{dy}{dx} = \frac{A}{x} + 2Bx + 1$
 At $x=-1$
 $0 = -A - 2B + 1$ for

- SOLVING SIMULTANEOUSLY THE TWO EQUATIONS WE OBTAIN ON SUBTRACTION

$\frac{A}{8} = -\frac{3}{8}$
 $A = -3$

$y = 2 \ln|x| - \frac{1}{2}x^2 + x$

- WE NOW HAVE THE EQUATION OF THE CURVE
- FURTHER FEATURES OF THE CURVE
- VERTICAL ASYMPTOTE $x=0$ (y-axis), TENDING TO $-\infty$
- AS $x \rightarrow \pm\infty$
 $y \sim -\frac{1}{2}x^2 + x$
 $\sim -\frac{1}{2}(x^2 - 2x)$
 $\sim -\frac{1}{2}(x^2 - 2x + 1 - 1)$
 $\sim -\frac{1}{2}(x-1)^2 + \frac{1}{2}$
as x to a large magnitude, $y \sim -\frac{1}{2}x^2$
- WHEN $x=2$, $y = 2 \ln 2 - 2 + 2 \Rightarrow (2, 2 \ln 2)$
- WHEN $x=-1$, $y = 2 \ln 1 - \frac{1}{2} + (-1) \Rightarrow (-1, -\frac{3}{2})$
- A SKETCH CAN NOW BE PRODUCED AS THE NATURE OF THE STATIONARY POINTS CAN BE DETERMINED BY THE 'SECOND DERIVATIVE' TESTS.

Question 99 (*****)

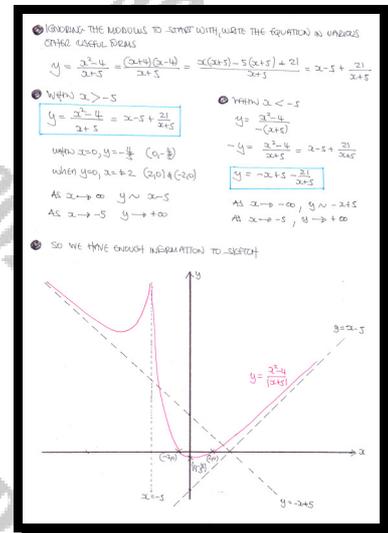
Sketch the curve with equation

$$y = \frac{x^2 - 4}{|x + 5|}, \quad x \in \mathbb{R}, \quad x \neq -5.$$

The sketch must include ...

- ... the coordinates of all the points where the curve meets the coordinate axes.
- ... the equations of the asymptotes of the curve.

graph



Question 100 (****)

A general curve C has equation

$$y = x^m(x-1)^n,$$

where $x \in \mathbb{R}$, $m \in \mathbb{N}$, $m \geq 2$, $n \in \mathbb{N}$, $n \geq 2$.

Sketch in four separate of axes, the 4 separate shapes which C can take, $m \geq 2$.

The sketches must contain the coordinates of any stationary points.

, graph

$y = x^m(x-1)^n$ $m, n \in \mathbb{N}, m \geq 2, n \geq 2$

- THE CURVE IS NEITHER ODD, NOR EVEN
- COLLECT INFORMATION ABOUT THE FEATURES OF THE CURVE DEPENDING ON WHETHER m & n ARE ODD OR EVEN

	n IS ODD	n IS EVEN
m IS ODD	<p>INFLEXION AT $x=0$ INFLEXION AT $x=1$</p>	<p>INFLEXION AT $x=0$ LOCAL MIN [MAX] AT $x=1$</p>
m IS EVEN	<p>LOCAL MIN [MAX] AT $x=0$ INFLEXION AT $x=1$</p>	<p>LOCAL MIN [MAX] AT $x=0$ LOCAL MIN [MAX] AT $x=1$</p>

- DIFFERENTIATE TO LOOK FOR MORE STATIONARY VALUES

$$\Rightarrow \frac{dy}{dx} = m x^{m-1} (x-1)^n + n x^m (x-1)^{n-1}$$

$$\Rightarrow \frac{dy}{dx} = x^{m-1} (x-1)^{n-1} [m(x-1) + n x]$$

- STATIONS OF ZERO VELOCITY $x=0$
 $x=1$
- OR $m(x-1) + nx = 0$
 $m(x-1) + nx = 0$
 $(m+n)x = m$
 $x = \frac{m}{m+n}$

$$\therefore y = \left(\frac{m}{m+n}\right)^m \left(\frac{m}{m+n}\right)^n = \left(\frac{m}{m+n}\right)^{m+n}$$

- AS THE CURVE IS QUADRATIC, THE NATURE OF THE POINTS CAN BE DEDUCED BY THE GENERAL CONSIDERATION. THE FOUR CASES ARE SHOWN

IN EVERY CASE $P \left[\frac{m}{m+n}, \left(\frac{m}{m+n}\right)^{m+n} \right]$

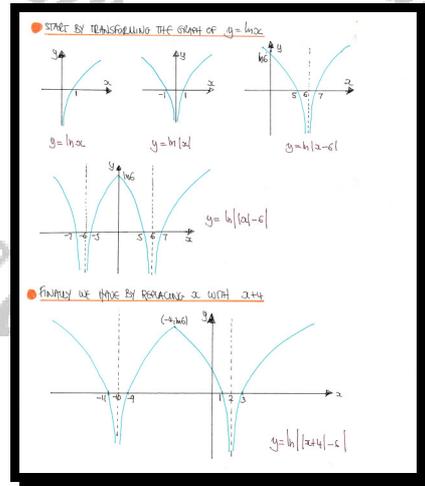
Question 101 (****)

Sketch, in the largest real domain, the graph of

$$y = \ln ||x+4|-6|.$$

Indicate the coordinates of any intersections with the axes, the equations of any asymptotes and the coordinates of any cusps of the curve.

, graph



Question 102 (****)

A curve C is defined, in the largest possible real domain, by the Cartesian equation

$$2y - 1 = (x - 1)(y - 1)^2.$$

By expressing the above equation in the form $y = f(x)$, sketch the graph of C .

Indicate the equations of any asymptotes, stationary points and any intersections with the coordinate axes.

, graph

● START BY REARRANGING THE EQUATION FOR y AS FOLLOWS

$$\begin{aligned} \Rightarrow 2y - 1 &= (x - 1)(y - 1)^2 \\ \Rightarrow 2y - 1 &= (x - 1)(y^2 - 2y + 1) \\ \Rightarrow 2y - 1 + (y - 1)^2 &= (x - 1)(y - 1)^2 \\ \Rightarrow \cancel{2y} + \cancel{y^2} - \cancel{2y} + \cancel{y^2} - \cancel{2y} + \cancel{1} &= (x - 1)(y - 1)^2 \\ \Rightarrow \frac{y^2}{(y - 1)^2} &= x \end{aligned}$$

$$\Rightarrow \frac{y}{y - 1} = \pm \sqrt{x}$$

$$\Rightarrow \frac{y - 1}{y} = \frac{1}{\pm \sqrt{x}}$$

$$\Rightarrow 1 - \frac{1}{y} = \frac{1}{\pm \sqrt{x}}$$

$$\Rightarrow 1 \pm \frac{1}{\sqrt{x}} = \frac{1}{y}$$

$$\Rightarrow \frac{\sqrt{x} \pm 1}{\sqrt{x}} = \frac{1}{y}$$

$$\Rightarrow y = \frac{\sqrt{x}}{\sqrt{x} \pm 1} = \begin{cases} \frac{\sqrt{x}}{\sqrt{x} + 1} \\ \frac{\sqrt{x}}{\sqrt{x} - 1} \end{cases}$$

● NEXT CONSIDER THE GRAPHS OF $y = \frac{\sqrt{x}}{\sqrt{x} \pm 1}$

$$y = \frac{\sqrt{x}}{\sqrt{x} + 1} = \frac{\sqrt{x} + 1 - 1}{\sqrt{x} + 1} = 1 - \frac{1}{\sqrt{x} + 1}$$

$$y = \frac{\sqrt{x}}{\sqrt{x} - 1} = \frac{\sqrt{x} - 1 + 1}{\sqrt{x} - 1} = 1 + \frac{1}{\sqrt{x} - 1}$$

● NEXT WE SKETCH GRAPHS OF THESE GRAPHS

● FINALLY REPLACE x WITH \sqrt{x}

- PLOT ON THE CURVES FOR WHICH $x < 0$ UNWANTED
- HORIZONTAL ASYMPTOTE APPROACHED AS $x \rightarrow \infty$
- VERTICAL ASYMPTOTE IS DRAWN (i.e. $\sqrt{x} = 0$), HIDE UNWANTED
- AT x INTERCEPTS, POINTS IS MARKED

$2y - 1 = (x - 1)(y - 1)^2$

Question 103 (****)

A finite region in the x - y plane is defined by the inequalities

$$|x-1| + |y-1| < 1 \quad \text{and} \quad |x(y-2)| > 1.$$

Sketch in detail this region, showing clearly any relevant coordinates.

, graph

● SKETCH BY SKETCHING $|x-1| + |y-1| = 1$ VIA TRANSFORMATIONS

● THIS $|x-1| + |y-1| = 1$ IS A FINITE TRANSLATION BY 1 UNIT, UPWARDS OF SPACE (1,1) WHICH IS MADE THE 'SPINE' SATISFIES THE INEQUALITY, WE HAVE

● IN A SIMILAR FASHION WE HAVE

● TESTING THE REQUIRED REGION WITH (1,2)

$$|x(y-2)| > 1$$

$$0 > 1$$

∴ REQUIRED REGION IS ON THE 'OUTSIDE'

● HENCE WE HAVE A FINAL SKETCH (REQUIRED REGION IN YELLOW)

● SIMILARLY

$$y = 2 - \frac{1}{x} \Rightarrow x + (2 - \frac{1}{x}) = 1$$

$$\Rightarrow x + 2 - \frac{1}{x} = 1 \Rightarrow x^2 + 2x - 1 = 0$$

$$\Rightarrow (x + \frac{1}{2})^2 = \frac{5}{4} \Rightarrow x = -\frac{1}{2} \pm \frac{\sqrt{5}}{2}$$

∴ $x < -\frac{1}{2} + \frac{\sqrt{5}}{2}$ or $x < -\frac{1}{2} - \frac{\sqrt{5}}{2}$

$$y = 2 - \frac{1}{x} = \frac{2x - 1}{x} = \frac{2(-\frac{1}{2} + \frac{\sqrt{5}}{2}) - 1}{-\frac{1}{2} + \frac{\sqrt{5}}{2}} = \frac{-1 + \sqrt{5} - 1}{-\frac{1}{2} + \frac{\sqrt{5}}{2}} = \frac{\sqrt{5} - 2}{-\frac{1}{2} + \frac{\sqrt{5}}{2}}$$

∴ $P(-\frac{1}{2} + \frac{\sqrt{5}}{2}, \frac{\sqrt{5} - 2}{-\frac{1}{2} + \frac{\sqrt{5}}{2}})$ & $Q(-\frac{1}{2} - \frac{\sqrt{5}}{2}, \frac{\sqrt{5} - 2}{-\frac{1}{2} - \frac{\sqrt{5}}{2}})$

Question 104 (****)

The curve C is defined in the greatest real domain by the equation

$$y = \frac{x}{(y-2)(y+1)(y-3)}$$

a) Show that

$$\frac{dy}{dx} = \frac{1}{2(y-1)(ay^2 + by + c)}$$

where a , b and c are integers to be found.

b) Determine the exact value of the gradient at the points on C , where $x = 40$.

c) Sketch the graph of C .

The sketch must include the coordinates of any points where C meets the coordinate axes, the coordinates of the points of infinite gradient. You must also find, with a full algebraic method, the line of symmetry of C .

$$\boxed{}, \quad a = 2, \quad b = -4, \quad c = -3, \quad \boxed{\pm \frac{1}{78}}$$

$y = \frac{x}{(y-2)(y+1)(y-3)}$

a) MANIPULATE THE EQUATION AS USUALS

$$\Rightarrow y(y-2)(y+1)(y-3) = x$$

$$\Rightarrow x = (y^2-2y)(y^2-2y-3)$$

$$\Rightarrow x = (y^2-2y)^2 - 3(y^2-2y)$$

DIFFERENTIATE WITH RESPECT TO y

$$\Rightarrow \frac{dx}{dy} = 2(y^2-2y)(2y-2) - 3(2y-2)$$

$$\Rightarrow \frac{dx}{dy} = (2y-2)[2(y^2-2y)-3]$$

$$\Rightarrow \frac{dx}{dy} = 2(y-1)(2y^2-4y-3)$$

$$\Rightarrow \frac{dx}{dy} = \frac{1}{2(y-1)(2y^2-4y-3)}$$

b) LOOKING AT THE EXPRESSION FROM ABOVE WITH $x=40$

$$\Rightarrow 40 = (y^2-2y)^2 - 3(y^2-2y)$$

$$\Rightarrow 0 = (y^2-2y)^2 - 3(y^2-2y) - 40$$

$$\Rightarrow 0 = [(y^2-2y) - 8][(y^2-2y) + 5]$$

$$\Rightarrow (y^2-2y-8)(y^2-2y+5) = 0$$

$$\Rightarrow (y+2)(y-4)(y^2-2y+5) = 0$$

RECOGNISE AS $b^2-4ac=0$

$$\therefore y < \begin{matrix} 4 \\ -2 \end{matrix}$$

USING THE RESULT FROM PART (a)

$$\frac{dy}{dx} \Big|_{y=4} = \frac{1}{2(3)(32-16-3)} = \frac{1}{6 \times 15} = \frac{1}{78}$$

$$\frac{dy}{dx} \Big|_{y=-2} = \frac{1}{2(-3)(8+4-3)} = \frac{1}{-4 \times 15} = -\frac{1}{78}$$

(c) COLLECTING ALL THE INFORMATION FOR THE SKETCH

- $x=0 \Rightarrow y=0, -1, 2, 3$
- $y=0 \Rightarrow x=0$
- $\frac{dy}{dx} = 0 \Rightarrow$ NO SOLUTIONS
- $\frac{dx}{dy} = 0 \Rightarrow y=1$ OR $2y^2-4y-3=0$
 $y^2-2y-\frac{3}{2}=0$
 $(y-1)^2 = \frac{5}{4} \Rightarrow y = 1 \pm \sqrt{\frac{5}{4}}$

USING $x = (y^2-2y)^2 - 3(y^2-2y)$

- IF $y=1 \Rightarrow x = (-1)^2 - 3(-1-2) = 1+3 = 4$ (4,1)
- IF $y=1 \pm \sqrt{\frac{5}{4}} = 1 \pm \frac{\sqrt{5}}{2} \Rightarrow x = \left(\frac{5}{4}\right)^2 - 3\left(\frac{5}{4}\right) = \frac{25}{16} - \frac{45}{16} = -\frac{20}{16} = -\frac{5}{4}$

$$x = (y^2-2y)^2 - 3(y^2-2y)$$

$$x = \left(\frac{5}{4}\right)^2 - 3\left(\frac{5}{4}\right)$$

$$x = \frac{25}{16} - \frac{45}{16} = -\frac{20}{16} = -\frac{5}{4}$$

$$\therefore \left(\frac{5}{4}, 1 + \frac{\sqrt{5}}{2}\right) \text{ \& } \left(\frac{5}{4}, 1 - \frac{\sqrt{5}}{2}\right)$$

• WRITE THE EQUATION

$$x = y(y-2)(y+1)(y-3)$$

THE CURVE IS EVEN ABOUT THE LINE $y=1$ SINCE

$$x = (y-1)(y-3)(y+1)(y-1) = (y-1)^2(y-3)(y+1)$$

$$x = (y-1)(y-3)(y+1)$$

ALTERNATIVE IN 3 STEPS

$$x = y(y-2)(y+1)(y-3)$$

$$x = (y+1)(y-2)(y+1)(y-3)$$

$$x = (y+1)(y-1)(y+2)(y-2)$$

$$x = (y+1)(y-1)(y+2)(y-2)$$

$$x = [(y+1)(y-1)][(y+2)(y-2)] = (y^2-1)(y^2-4)$$

$$x = (y^2-1)(y^2-4)$$

$$x = (y-1)(y+1)(y-2)(y+2)$$

$$x = y(y-2)(y+1)(y-3)$$

FINALLY A SKETCH CAN BE DRAWN

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