COMPLEX NUMBERS
PRACTICE
(part 2)
ROOTS OF COMPLEX NUMBERS
Question 1

\[ z^4 = -16, \ z \in \mathbb{C}. \]

a) Solve the above equation, giving the answers in the form \( a + bi \), where \( a \) and \( b \) are real numbers.

b) Plot the roots of the equation as points in an Argand diagram.

\[ z = \sqrt{2}(\pm 1 \pm i) \]
Question 2

\[ z^5 = i, \quad z \in \mathbb{C}. \]

a) Solve the equation, giving the roots in the form \( r e^{i\theta}, \quad r > 0, \quad -\pi < \theta \leq \pi \).

b) Plot the roots of the equation as points in an Argand diagram.

\[
\begin{align*}
z &= e^{\frac{i\pi}{5}}, \quad z = e^{\frac{3i\pi}{5}}, \quad z = e^{\frac{2i\pi}{5}}, \quad z = e^{-\frac{3i\pi}{5}}, \quad z = e^{-\frac{i\pi}{5}}
\end{align*}
\]
Question 3

\[ z = 4 + 4i. \]

a) Find the fifth roots of \( z \).
Give the answers in the form \( re^{i\theta}, \ r > 0, \ -\pi < \theta \leq \pi \).

b) Plot the roots as points in an Argand diagram.

\[ \sqrt[5]{2} e^{\frac{1\pi}{5}}, \sqrt[5]{2} e^{\frac{2\pi}{5}}, \sqrt[5]{2} e^{\frac{3\pi}{5}}, \sqrt[5]{2} e^{\frac{4\pi}{5}}, \sqrt[5]{2} e^{\frac{5\pi}{5}} \]
Question 4

\[ z = 4 - 4\sqrt{3}i. \]

a) Find the cube roots of \( z \).

Give the answers in polar form \( r(\cos \theta + i \sin \theta), \, r > 0, \, -\pi < \theta \leq \pi \).

b) Plot the roots as points in an Argand diagram.

\[ z = 2\left(\cos \frac{\pi}{9} - i \sin \frac{\pi}{9}\right), \quad z = 2\left(\cos \frac{5\pi}{9} + i \sin \frac{5\pi}{9}\right), \quad z = 2\left(\cos \frac{7\pi}{9} - i \sin \frac{7\pi}{9}\right) \]
Question 5
The following complex number relationships are given
\( w = -2 + 2\sqrt{3}i, \quad z^4 = w. \)

a) Express \( w \) in the form \( r(\cos \theta + i\sin \theta) \), where \( r > 0 \) and \( -\pi < \theta \leq \pi \).

b) Find the possible values of \( z \), giving the answers in the form \( x + iy \), where \( x \) and \( y \) are real numbers.

\[
\begin{align*}
w &= 2 \cos \left( \frac{2\pi}{3} \right) + i\sin \left( \frac{2\pi}{3} \right), \\
\theta &= \frac{\pi}{6}, \\
z &= \frac{1}{2}(\sqrt{6} + i\sqrt{2}), \quad z = \frac{1}{2}(-\sqrt{6} + i\sqrt{2}).
\end{align*}
\]

Question 6
Find the cube roots of the imaginary unit \( i \), giving the answers in the form \( a + bi \), where \( a \) and \( b \) are real numbers.

\[
\begin{align*}
z_1 &= \frac{\sqrt{3}}{2} + \frac{1}{2}i, \\
z_2 &= -\frac{\sqrt{3}}{2} + \frac{1}{2}i, \\
z_3 &= -i.
\end{align*}
\]
Question 7
Find the cube roots of the complex number $-8i$, giving the answers in the form $a+bi$, where $a$ and $b$ are real numbers.

\[ z_1 = \sqrt{3} - i, \quad z_2 = -\sqrt{3} - i, \quad z_3 = 2i \]

Question 8

\[ z^4 = -8 - 8\sqrt{3}i, \quad z \in \mathbb{C}. \]

Solve the above equation, giving the answers in the form $a+bi$, where $a$ and $b$ are real numbers.

\[ z = \sqrt{3} - i, \quad z = 1 + \sqrt{3}i, \quad z = -\sqrt{3} + i, \quad z = -1 - \sqrt{3}i \]
Question 9

\[ z^2 = (1 + i\sqrt{3})^3, \ z \in \mathbb{C}. \]

Solve the above equation, giving the answers in the form \( a + bi \), where \( a \) and \( b \) are real numbers.

\[ z = \pm 2i\sqrt{2} \]

Question 10

\[ z^3 = 32 + 32\sqrt{3}i, \ z \in \mathbb{C}. \]

a) Solve the above equation.

Give the answers in exponential form \( z = re^{i\theta}, \ r > 0, -\pi < \theta \leq \pi \).

b) Show that these roots satisfy the equation

\[ w^9 + 2^{18} = 0. \]

\[ z = 4e^{i\pi/9}, \ 4e^{i2\pi/9}, \ 4e^{-i2\pi/9} \]
Question 11

\[ z^7 - 1 = 0, \quad z \in \mathbb{C}. \]

One of the roots of the above equation is denoted by \( \omega \), where \( 0 < \arg(\omega) < \frac{\pi}{3} \).

a) Find \( \omega \) in the form \( \omega = re^{i\theta}, \quad r > 0, \quad 0 < \theta < \frac{\pi}{3} \).

b) Show clearly that

\[ 1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 = 0. \]

c) Show further that

\[ \omega^2 + \omega^5 = 2\cos\left(\frac{4\pi}{7}\right). \]

d) Hence, using the results from the previous parts deduce that

\[ \cos\left(\frac{2\pi}{7}\right) + \cos\left(\frac{4\pi}{7}\right) + \cos\left(\frac{6\pi}{7}\right) = -\frac{1}{2}. \]
Question 12

\[ z^3 = \left(1 + i\sqrt{3}\right)^8 (1-i)^5, \quad z \in \mathbb{C}. \]

Find the three roots of the above equation, giving the answers in the form \( k\sqrt{2} e^{i\theta} \), where \(-\pi < \theta \leq \pi, \quad k \in \mathbb{Z}. \)

\[ z = 8\sqrt{2} e^{i\theta}, \quad \theta = -\frac{31\pi}{36}, \quad \frac{7\pi}{36}, \quad \frac{17\pi}{36} \]
TRIGONOMETRIC IDENTITIES QUESTIONS
Question 1

If \( z = \cos \theta + i \sin \theta \), show clearly that …

a) \( \ldots z^n + \frac{1}{z^n} \equiv 2 \cos n \theta \).

b) \( \ldots 16 \cos^5 \theta \equiv 5 \cos 5 \theta + 5 \cos 3 \theta + 10 \cos \theta \).

proof
Question 2

It is given that

\[ \sin 5\theta = 16\sin^5 \theta - 20\sin^3 \theta + 5\sin \theta . \]

a) Use de Moivre’s theorem to prove the validity of the above trigonometric identity.

It is further given that

\[ \sin 3\theta = 3\sin \theta - 4\sin^3 \theta . \]

b) Solve the equation

\[ \sin 5\theta = 5\sin 3\theta \quad \text{for} \quad 0 \leq \theta < \pi , \]

giving the solutions correct to 3 decimal places.

\[ \theta = 0, \ 1.095, \ 2.046 \]
Question 3

The complex number \( z \) is given by

\[ z = e^{i\theta}, \quad -\pi < \theta \leq \pi. \]

a) Show clearly that

\[ z^n + \frac{1}{z^n} = 2 \cos n\theta. \]

b) Hence show further that

\[ \cos^4 \theta = \frac{1}{8} \cos 4\theta + \frac{1}{2} \cos 2\theta + \frac{3}{8}. \]

c) Solve the equation

\[ 2 \cos 4\theta + 8 \cos 2\theta + 5 = 0, \quad 0 \leq \theta < 2\pi. \]

\[ \theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}. \]
Question 4
The complex number \( z \) is given by
\[
z = e^{i\theta}, \quad -\pi < \theta \leq \pi.
\]

a) Show clearly that
\[
z^n + \frac{1}{z^n} \equiv 2 \cos n\theta.
\]

b) Hence show further that
\[
16 \cos^5 \theta \equiv \cos 5\theta + 5 \cos 3\theta + 10 \cos \theta.
\]

c) Use the results of part (a) and (b) to solve the equation
\[
\cos 5\theta + 5 \cos 3\theta + 6 \cos \theta = 0, \quad 0 \leq \theta < \pi.
\]
\[
\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \text{ or } \frac{3\pi}{2}.
\]
Question 5

De Moivre’s theorem asserts that

\[(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta, \; \theta \in \mathbb{R}, \; n \in \mathbb{Q}.\]

a) Use the theorem to prove the validity of the following trigonometric identity.

\[
\cos 6\theta \equiv 32 \cos^6 \theta - 48 \cos^4 \theta + 18 \cos^2 \theta - 1.
\]

b) Use the result of part (a) to find, in exact form, the largest positive root of the equation

\[
64x^6 - 96x^4 + 36x^2 - 1 = 0.
\]

\[
x = \cos \left(\frac{\pi}{9}\right)
\]
Question 6

Euler’s identity states

\[ e^{i\theta} = \cos \theta + i \sin \theta, \quad \theta \in \mathbb{R}. \]

a) Use the identity to show that

\[ e^{in\theta} + e^{-in\theta} = 2\cos n\theta. \]

b) Hence show further that

\[ 32\cos^6 \theta = \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10. \]

c) Use the fact that \( \cos \left( \frac{\pi}{2} - \theta \right) \equiv \sin \theta \) to find a similar expression for \( 32\sin^6 \theta \).

d) Determine the exact value of

\[
\int_{0}^{\frac{\pi}{2}} \sin^6 \theta + \cos^6 \theta \, d\theta.
\]

\[ 32\sin^6 \theta = -\cos 6\theta + 6\cos 4\theta - 15\cos 2\theta + 10. \]
Question 7

De Moivre’s theorem asserts that

\[(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta, \quad \theta \in \mathbb{R}, \quad n \in \mathbb{Q}.\]

a) Use the theorem to prove validity of the following trigonometric identity

\[
\sin 5\theta = \sin \theta \left(16\cos^4 \theta - 12\cos^2 \theta + 1\right).
\]

b) Hence, or otherwise, solve the equation

\[
\sin 5\theta = 10\cos\theta \sin 2\theta - 11\sin \theta, \quad 0 < \theta < \pi.
\]

\[
\theta = \frac{\pi}{4}, \quad \frac{3\pi}{4}, \quad \frac{5\pi}{4}, \quad \frac{7\pi}{4}
\]
Question 8

It is given that

\[ \sin 5\theta = \sin \theta (16\cos^4 \theta - 12\cos^2 \theta + 1). \]

a) Use de Moivre's theorem to prove the validity of the above trigonometric identity.

Consider the general solution of the trigonometric equation

\[ \sin 5\theta = 0. \]

b) Find exact simplified expressions for

\[ \cos^2 \left( \frac{\pi}{5} \right) \text{ and } \cos^2 \left( \frac{2\pi}{5} \right), \]

fully justifying each step in the workings.

\[ \cos^2 \left( \frac{\pi}{5} \right) = \frac{3 + \sqrt{5}}{8}, \quad \cos^2 \left( \frac{2\pi}{5} \right) = \frac{3 - \sqrt{5}}{8} \]
Question 9

By considering the binomial expansion of \((\cos \theta + i \sin \theta)^4\) show that

\[
\tan 4\theta \equiv \frac{4 \tan \theta - 4 \tan^3 \theta}{1 - 6 \tan^2 \theta + \tan^4 \theta}.
\]
Question 10

By using de Moivre’s theorem followed by a suitable trigonometric identity, show clearly that …

a) \[ \cos 3\theta \equiv 4\cos^3 \theta - 3\cos \theta . \]

b) \[ \cos 6\theta \equiv (2\cos^2 \theta - 1)(16\cos^4 \theta - 16\cos^2 \theta + 1) \]

Consider the solutions of the equation.

\[ \cos 6\theta = 0 , \ 0 \leq \theta \leq \pi . \]

c) By fully justifying each step in the workings, find the exact value of

\[ \cos \frac{\pi}{12} \cos \frac{5\pi}{12} \cos \frac{7\pi}{12} \cos \frac{11\pi}{12} . \]
COMPLEX LOCI
Question 1

By finding a suitable Cartesian locus in the complex $z$ plane, shade the region $R$ that satisfies the inequality

$$|z - 3| \leq |z + 3i|.$$
Question 2

\[ |z - 1 - i| = 4, \; z \in \mathbb{C}. \]

a) Sketch the locus of the points that satisfy the above equation in a standard Argand diagram.

b) Find the minimum and maximum values of \(|z|\) for points that lie on this locus.

\[ z_{\text{min}} = 4 - \sqrt{2}, \quad z_{\text{max}} = 4 + \sqrt{2} \]
Question 3

The complex number $z$ represents the point $P(x, y)$ in the Argand diagram.

Given that

$$|z - 1| = 2|z + 2|,$$

show that the locus of $P$ is given by

$$(x + 3)^2 + y^2 = 4.$$
Question 5

Sketch, on the same Argand diagram, the locus of the points satisfying each of the following equations.

a) \(|z - 3 + i| = 3\).

b) \(|z| = |z - 2i|\).

Give in each case a Cartesian equation for the locus.

c) Shade in the sketch the region that is satisfied by both these inequalities

\[
\begin{align*}
|z - 3 + i| &\leq 3 \\
|z| &\geq |z - 2i|
\end{align*}
\]

\[(x-3)^2 + (y+1)^2 = 9, \quad y = 1\]
Question 6

a) Sketch on the same Argand diagram the locus of the points satisfying each of the following equations.

i. \(|z - i| = |z - 2|\).

ii. \(\arg(z - 2) = \frac{\pi}{2}\).

b) Shade in the sketch the region that is satisfied by both these inequalities

\[ |z - i| \leq |z - 2| \quad \text{and} \quad \frac{\pi}{2} \leq \arg(z - 2) \leq \frac{2\pi}{3}. \]
Question 7

The complex number \( z \) represents the point \( P(x, y) \) in the Argand diagram.

Given that

\[ |z - 1| = \sqrt{2}|z - i|. \]

show that the locus of \( P \) is a circle, stating its centre and radius.

\[ (x+1)^2 + (y-2)^2 = 4, \quad (-1, 2), \quad r = 2 \]

Question 8

\[ |z - 2i| = 1, \quad z \in \mathbb{C}. \]

a) In the Argand diagram, sketch the locus of the points that satisfy the above equation.

b) Find the minimum value and the maximum value of \( |z| \), and the minimum value and the maximum of \( \arg z \), for points that lie on this locus.

\[ |z|_{\min} = 1, \quad |z|_{\max} = 3, \quad \arg z_{\min} = \frac{\pi}{3}, \quad \arg z_{\max} = \frac{2\pi}{3} \]
Question 9

The complex number $z$ represents the point $P(x, y)$ in the Argand diagram.

Given that

$$|z + 1| = 2|z - 2i|,$$

show that the locus of $P$ is a circle and state its radius and the coordinates of its centre.

$$\left(\frac{4}{3}, \frac{8}{3}\right), \quad r = \frac{2}{3}\sqrt{5}$$
Question 10

The complex number $z = x + iy$ satisfies the relationship

$$2 \leq |z - 2 - 3i| < 3.$$ 

a) Shade **accurately** in an Argand diagram the region represented by the above relationship.

b) Determine algebraically whether the point that represents the number $4 + i$ lies inside or outside this region.

inside the region
Question 11

Two sets of loci in the Argand diagram are given by the following equations

\[ |z| = |z + 2| \quad \text{and} \quad |z| = 2, \quad z \in \mathbb{C}. \]

a) Sketch both these loci in the same Argand diagram.

The points \( P \) and \( Q \) in the Argand diagram satisfy both loci equations.

b) Write the complex numbers represented by \( P \) and \( Q \), in the form \( a + ib \), where \( a \) and \( b \) are real numbers.

c) Find a quadratic equation with real coefficients, whose solutions are the complex numbers represented by the points \( P \) and \( Q \).

\[ z = -1 \pm \sqrt{3}, \quad z^2 + 2z + 4 = 0 \]
Question 12

a) Sketch in the same Argand diagram the locus of the points satisfying each of the following equations

i. \(|z - 3 - 2i| = 2\).

ii. \(|z - 3 - 2i| = |z + 1 + 2i|\).

b) Show by a geometric calculation that no points lie on both loci.
Question 13

The point $A$ represents the complex number on the $z$ plane such that

$$|z - 6i| = 2|z - 3|,$$

and the point $B$ represents the complex number on the $z$ plane such that

$$\arg(z - 6) = -\frac{3\pi}{4}.$$

a) Show that the locus of $A$ as $z$ varies is a circle, stating its radius and the coordinates of its centre.

b) Sketch, on the same $z$ plane, the locus of $A$ and $B$ as $z$ varies.

c) Find the complex number $z$, so that the point $A$ coincides with the point $B$.

$$C(4, -2), \quad r = \sqrt{20}, \quad z = (4 - \sqrt{10}) + i(-2 - \sqrt{10})$$
Question 14

\[ |z - 2 + i| = 5. \]

\[ \arg(z - 2) = \frac{3\pi}{4}. \]

a) Sketch each of the above complex loci in the same Argand diagram.

b) Determine, in the form \( x + iy \), the complex number \( z_0 \) represented by the intersection of the two loci of part (a).

\[ z_0 = -2 - 4i \]
Question 15
The locus of the point \( z \) in the Argand diagram, satisfy the equation
\[ |z - 2 + i| = \sqrt{3}. \]

a) Sketch the locus represented by the above equation.

The half line \( L \) with equation
\[ y = mx - 1, \quad x \geq 0, \quad m > 0, \]

**touches** the locus described in part (a) at the point \( P \).

b) Find the value of \( m \).

c) Write the equation of \( L \), in the form
\[ \arg(z - z_0) = \theta, \quad z_0 \in \mathbb{C}, \quad -\pi < \theta \leq \pi. \]

d) Find the complex number \( w \), represented by the point \( P \).

\[
\begin{align*}
    m &= \sqrt{3}, & \arg(z + i) &= \frac{\pi}{3}, & w &= \frac{1}{2} + i \left(\frac{\sqrt{3}}{2} - 1\right)
\end{align*}
\]
Question 16

The complex numbers \( z_1 \) and \( z_2 \) are given by

\[
z_1 = 1 + i\sqrt{3} \quad \text{and} \quad z_2 = iz_1.
\]

a) Label accurately the points representing \( z_1 \) and \( z_2 \), in an Argand diagram.

b) On the same Argand diagram, sketch the locus of the points \( z \) satisfying ...

i. \( \ldots |z - z_1| = |z - z_2| \).

ii. \( \ldots \arg(z - z_1) = \arg(z_2) \).

c) Determine, in the form \( x + iy \), the complex number \( z_3 \) represented by the intersection of the two loci of part (b).

\[
z_3 = \left(1 - \sqrt{3}\right) + i\left(1 + \sqrt{3}\right)
\]
Question 17

The complex number $z$ lies in the region $R$ of an Argand diagram, defined by the inequalities

$$\frac{\pi}{3} \leq \arg(z-4) \leq \pi \quad \text{and} \quad 0 \leq \arg(z-12) \leq \frac{5\pi}{6}.$$

**a)** Sketch the region $R$, indicating clearly all the relevant details.

The complex number $w$ lies in $R$, so that $|w|$ is minimum.

**b)** Find $|w|$, further giving $w$ in the form $u+iv$, where $u$ and $v$ are real numbers.

$$|w| = 3, \quad w = 3 + 3\sqrt{3}i$$
Question 18

The point $P$ represents the number $z = x + iy$ in an Argand diagram and further satisfies the equation

$$\text{arg}\left(\frac{1-iz}{1-z}\right) = \frac{\pi}{4}, \quad z \neq -i.$$ 

Use an algebraic method to find an equation of the locus of $P$ and sketch this locus accurately in an Argand diagram.

$$x^2 + y^2 = 1, \quad \text{such that } y > x - 1$$
Question 19

The complex number \( x + iy \) in the \( z \) plane of an Argand diagram satisfies the inequality

\[
x^2 + y^2 + x > 0.
\]

a) Sketch the region represented by this inequality.

A locus in the \( z \) plane of an Argand diagram is given by the equation

\[
\arg\left(\frac{z + 1}{z}\right) = \frac{\pi}{4}.
\]

b) Sketch the locus represented by this equation.
Question 20

The complex number \( z \) satisfies the relationship

\[
\arg(z - 2) - \arg(z + 2) = \frac{\pi}{4}.
\]

Show that the locus of \( z \) is a circular arc, stating …

- … the coordinates of its endpoints.
- … the coordinates of its centre.
- … the length of its radius.

\([-2, 0], (2, 0), (0, 2), r = 2\sqrt{2}\)
Question 1
A transformation from the $z$ plane to the $w$ plane is defined by the complex function

$$w = \frac{3 - z}{z + 1}, \quad z \neq -1.$$

The locus of the points represented by the complex number $z = x + iy$ is transformed to the circle with equation $|w| = 1$ in the $w$ plane.

Find, in Cartesian form, an equation of the locus of the points represented by the complex number $z$.

\[
x = 1
\]

Question 2
Find an equation of the locus of the points which lie on the half line with equation

$$\arg z = \frac{\pi}{4}, \quad z \neq 0$$

after it has been transformed by the complex function

$$w = \frac{1}{z}.$$

$$\arg w = -\frac{\pi}{4}$$
Question 3

The complex function

\[ w = \frac{1}{z-1}, \quad z \neq 1, \quad z \in \mathbb{C}, \quad z \neq 1 \]

transforms the point represented by \( z = x + iy \) in the \( z \) plane into the point represented by \( w = u + iv \) in the \( w \) plane.

Given that \( z \) satisfies the equation \( |z| = 1 \), find a Cartesian locus for \( w \).

\[ u = -\frac{1}{2} \]
Question 4

The complex function \( w = f(z) \) is given by

\[
    w = \frac{3 - z}{z + 1} \quad \text{where} \quad z \in \mathbb{C}, \quad z \neq -1.
\]

A point \( P \) in the \( z \) plane gets mapped onto a point \( Q \) in the \( w \) plane.

The point \( Q \) traces the circle with equation \( |w| = 3 \).

Show that the locus of \( P \) in the \( z \) plane is also a circle, stating its centre and its radius.

\[
    \text{centre } \left(- \frac{3}{2}, 0\right), \quad \text{radius } = \frac{3}{2}
\]
Question 5

The general point $P(x, y)$ which is represented by the complex number $z = x + iy$ in the $z$-plane, lies on the locus of $|z| = 1$.

A transformation from the $z$-plane to the $w$-plane is defined by

$$w = \frac{z + 3}{z + 1}, \quad z \neq -1,$$

and maps the point $P(x, y)$ onto the point $Q(u, v)$.

Find, in Cartesian form, the equation of the locus of the point $Q$ in the $w$-plane.
Question 6

The point $P$ represented by $z = x + iy$ in the $z$ plane is transformed into the point $Q$ represented by $w = u + iv$ in the $w$ plane, by the complex transformation

$$w = \frac{2z}{z - 1}, \quad z \neq 1.$$ 

The point $P$ traces a circle of radius 2, centred at the origin $O$.

Find a Cartesian equation of the locus of the point $Q$.

$$\left(u - \frac{8}{3}\right)^2 + v^2 = \frac{16}{9}$$
Question 7

The complex numbers \( z = x + iy \) and \( w = u + iv \) are represented by the points \( P \) and \( Q \), respectively, in separate Argand diagrams.

The two numbers are related by the equation

\[
w = \frac{1}{z + 1}, \quad z \neq -1.\]

If \( P \) is moving along the circle with equation

\[(x+1)^2 + y^2 = 4,\]

find in Cartesian form an equation of the locus of the point \( Q \).
Question 8
A transformation from the $z$ plane to the $w$ plane is defined by the equation

$$w = \frac{z + 2i}{z - 2}, \quad z \neq 2.$$

Find in the $w$ plane, in Cartesian form, the equation of the image of the circle with equation $|z| = 1$, $z \in \mathbb{C}$.

$$\left(u + \frac{1}{3}\right)^2 + \left(v + \frac{4}{3}\right)^2 = \frac{8}{9}$$

Question 9
A transformation from the $z$ plane to the $w$ plane is given by the equation

$$w = \frac{1 + 2z}{3 - z}, \quad z \neq 3.$$

Show that in the $w$ plane, the image of the circle with equation $|z| = 1$, $z \in \mathbb{C}$, is another circle, stating its centre and its radius.

$$\left(u - \frac{5}{8}\right)^2 + v^2 = \frac{49}{64}. \quad \text{centre} \left(\frac{5}{8}, 0\right), \quad r = \frac{7}{8}$$
Question 10

The complex numbers $z = x + iy$ and $w = u + iv$ are represented by the points $P$ and $Q$, respectively, in separate Argand diagrams.

The two numbers are related by the equation

$$w = \frac{1}{z}, \quad z \neq 0.$$ 

If $P$ is moving along the circle with equation

$$x^2 + y^2 = 2,$$

find in Cartesian form an equation for the locus of the point $Q$.

$$u^2 + v^2 = \frac{1}{2}$$
Question 11
The complex numbers \( z = x + iy \) and \( w = u + iv \) are represented by the points \( P \) and \( Q \) on separate Argand diagrams.

In the \( z \) plane, the point \( P \) is tracing the line with equation \( y = x \).

The complex numbers \( z \) and \( w \) are related by
\[
2w = z - z^2.
\]

a) Find, in Cartesian form, the equation of the locus of \( Q \) in the \( w \) plane.

b) Sketch the locus traced by \( Q \).

\[
v = u - 2u^2 \quad \text{or} \quad y = x - 2x^2
\]
Question 12
The complex numbers \( z = x + iy \) and \( w = u + iv \) are represented by the points \( P \) and \( Q \) on separate Argand diagrams.

In the \( z \) plane, the point \( P \) is tracing the line with equation \( y = 2x \).

Given that the complex numbers \( z \) and \( w \) are related by
\[
    w = z^2 + 1
\]
find, in Cartesian form, the locus of \( Q \) in the \( w \) plane.

\[
4u + 3v = 4 \quad \text{or} \quad 4x + 3y = 4
\]
Question 13

A transformation of the \( z \) plane to the \( w \) plane is given by

\[
w = \frac{1 + 3z}{1 - z}, \quad z \in \mathbb{C}, \quad z \neq 1,
\]

where \( z = x + iy \) and \( w = u + iv \).

The set of points that lie on the \( y \) axis of the \( z \) plane, are mapped in the \( w \) plane onto a curve \( C \).

Show that a Cartesian equation of \( C \) is

\[
(u + 1)^2 + v^2 = 4.
\]
Question 14

The complex function \( w = f(z) \) is given by

\[
    w = \frac{1}{z}, \quad z \in \mathbb{C}, \quad z \neq 0.
\]

This function maps a general point \( P(x, y) \) in the \( z \) plane onto the point \( Q(u, v) \) in the \( w \) plane.

Given that \( P \) lies on the line with Cartesian equation \( y = 1 \), show that the locus of \( Q \) is given by

\[
    \left| w + \frac{1}{2} \right| = \frac{1}{2}.
\]

\[ \text{proof} \]
Question 15

A transformation of the $z$ plane onto the $w$ plane is given by

$$w = \frac{az + b}{z + c}, \quad z \in \mathbb{C}, \quad z \neq -c$$

where $a$, $b$ and $c$ are real constants.

Under this transformation the point represented by the number $1 + 2i$ gets mapped to its complex conjugate and the origin remains invariant.

a) Find the value of $a$, the value of $b$ and the value of $c$.

b) Find the number, other than the number represented by the origin, which remains invariant under this transformation.

\[
\begin{align*}
a &= \frac{5}{2}, & b &= 0, & c &= -\frac{5}{2}, & z &= 5
\end{align*}
\]
Question 16

A transformation of the \( z \) plane to the \( w \) plane is given by

\[
    w = \frac{1}{z - 2}, \quad z \in \mathbb{C}, \quad z \neq 2
\]

where \( z = x + iy \) and \( w = u + iv \).

The line with equation

\[
    2x + y = 3
\]

is mapped in the \( w \) plane onto a curve \( C \).

a) Show that \( C \) represents a circle and determine the coordinates of its centre and the size of its radius.

b) Sketch and shade \( R \) in a suitable labelled Argand diagram, fully justifying the choice of region.

\[ \text{centre at } (-1, \frac{1}{2}), \text{ radius } = \frac{\sqrt{5}}{2} \]
Question 17

A transformation of the $z$ plane to the $w$ plane is given by

$$w = z^2, \ z \in \mathbb{C},$$

where $z = x + iy$ and $w = u + iv$.

The line with equation $y = 1$ is mapped in the $w$ plane onto a curve $C$.

Sketch the graph of $C$, marking clearly the coordinates of all points where the graph of $C$ meets the coordinate axes.
Question 18

A transformation of points from the \( z \) plane onto points in the \( w \) plane is given by the complex relationship

\[ w = z^2, \quad z \in \mathbb{C}, \]

where \( z = x + iy \) and \( w = u + iv \).

Show that if the point \( P \) in the \( z \) plane lies on the line with equation

\[ y = x - 1, \]

the locus of this point in the \( w \) plane satisfies the equation

\[ v = \frac{1}{2}(u^2 - 1). \]
Question 19

A complex transformation from the \( z \) plane to the \( w \) plane is defined by

\[
    w = \frac{z + i}{3 + iz}, \quad z \in \mathbb{C}, \quad z \neq 3i.
\]

The point \( P(x, y) \) is mapped by this transformation into the point \( Q(u, v) \).

It is further given that \( Q \) lies on the real axis for all the possible positions of \( P \).

Show that the \( P \) traces the curve with equation

\[
    |x - i| = 2.
\]
Question 20

A transformation of the $z$ plane to the $w$ plane is given by

$$w = \frac{2z + 1}{z}, \quad z \in \mathbb{C}, \quad z \neq 0$$

where $z = x + iy$ and $w = u + iv$.

The circle $C_1$ with centre at $(1, -\frac{1}{2})$ and radius $\frac{\sqrt{2}}{2}$ in the $z$ plane is mapped in the $w$ plane onto another curve $C_2$.

a) Show that $C_2$ is also a circle and determine the coordinates of its centre and the size of its radius.

The points inside $C_1$ in the $z$ plane are mapped onto points of a region $R$ in the $w$ plane.

b) Sketch and shade $R$ in a suitably labelled Argand diagram, fully justifying the choice of the region.

centre at $\left(\frac{3}{2}, 0\right)$, radius $= \frac{1}{\sqrt{2}}$
**Question 21**

A transformation of the $z$ plane to the $w$ plane is given by

$$w = z + \frac{1}{z}, \quad z \in \mathbb{C}, \quad z \neq 0,$$

where $z = x + iy$ and $w = u + iv$.

The locus of the points in the $z$ plane that satisfy the equation $|z| = 2$ are mapped in the $w$ plane onto a curve $C$.

By considering the equation of the locus $|z| = 2$ in exponential form, or otherwise, show that a Cartesian equation of $C$ is

$$36u^2 + 100v^2 = 225.$$
Question 22

A transformation from the $z$ plane to the $w$ plane is defined by the equation

$$w = iz - 1, \quad z \in \mathbb{C}.$$ 

Sketch in the $w$ plane, in Cartesian form, the equation of the image of the half line with equation

$$\arg(z + 2) = \frac{\pi}{4}, \quad z \in \mathbb{C}.$$
Question 23

A transformation from the $z$ plane to the $w$ plane is defined by the equation

$$f(z) = \frac{iz}{z-i}, \ z \in \mathbb{C}.$$ 

Find, in Cartesian form, the equation of the image of straight line with equation

$$|z-i|=|z-2|, \ z \in \mathbb{C}.$$ 

\[
(u + \frac{2}{3})^2 + (v - \frac{4}{3})^2 = \frac{1}{5}
\]
Question 24

The complex function \( w = f(z) \) is given by

\[
   w = \frac{1}{1-z}, \quad z \neq 1.
\]

The point \( P(x, y) \) in the \( z \) plane traces the line with Cartesian equation

\[ y + x = 1. \]

Show that the locus of the image of \( P \) in the \( w \) plane traces the line with equation

\[ v = u. \]
Question 25

The complex function \( w = f(z) \) satisfies

\[ w = \frac{1}{z}, \quad z \in \mathbb{C}, \quad z \neq 0. \]

This function maps the point \( P(x, y) \) in the \( z \) plane onto the point \( Q(u, v) \) in the \( w \) plane.

It is further given that \( P \) traces the curve with equation

\[ |z + \frac{1}{2}i| = \frac{1}{2}. \]

Find, in Cartesian form, the equation of the locus of \( Q \).
Question 26

\[ z = \cos \theta + i \sin \theta, \quad -\pi < \theta \leq \pi. \]

a) Show clearly that

\[ \frac{2}{1+z} = 1 - i \tan \frac{\theta}{2}. \]

The complex function \( w = f(z) \) is defined by

\[ w = \frac{2}{1+z}, \quad z \in \mathbb{C}, \quad z \neq -1. \]

The circular arc \(|z| = 1\), for which \(0 \leq \arg z < \frac{\pi}{2}\), is transformed by this function.

b) Sketch the image of this circular arc in a suitably labelled Argand diagram.
Question 27

The complex function with equation

\[ f(z) = \frac{1}{z}, \quad z \in \mathbb{C}, \quad z \neq 0 \]

maps the complex number \( x + iy \) from the \( z \) plane onto the complex number \( u + iv \) in the \( w \) plane.

The line with equation

\[ y = mx, \quad x \neq 0, \]

is mapped onto the line with equation

\[ v = Mu, \]

where \( m \) and \( M \) are the respective gradients of the two lines.

Given that \( m = M \), determine the three possible values of \( m \).

\[ m = 0, \pm \sqrt{3} \]
Question 28

A complex transformation of points from the $z$ plane onto points in the $w$ plane is defined by the equation

$$w = z^2, \; z \in \mathbb{C}.$$

The point represented by $z = x + iy$ is mapped onto the point represented by $w = u + iv$.

Show that if $z$ traces the curve with Cartesian equation

$$y^2 = 2x^2 - 1,$$

the locus of $w$ satisfies the equation

$$v^2 = 4(u - 1)(2u - 1).$$
Question 29

The complex function \( w = f(z) \) is defined by

\[
w = \frac{1}{z - 1}, \quad z \in \mathbb{C}, \quad z \neq 1.
\]

The half line with equation \( \arg z = \frac{\pi}{4} \) is transformed by this function.

a) Find a Cartesian equation of the locus of the image of the half line.

b) Sketch the image of the locus in an Argand diagram.

\[
(u + \frac{1}{2})^2 + (v + \frac{1}{2})^2 = \frac{1}{2}, \quad v < 0, \quad u^2 + v^2 + u > 0
\]
Question 30

The complex function $w = f(z)$ is defined by

$$w = \frac{3z + i}{1 - z}, \quad z \in \mathbb{C}, \quad z \neq 1.$$ 

The half line with equation $\arg z = \frac{3\pi}{4}$ is transformed by this function.

a) Find a Cartesian equation of the locus of the image of the half line.

b) Sketch the image of the locus in an Argand diagram.

$$(u + 1)^2 + (v + 1)^2 = 5, \quad v > \frac{1}{3}u + 1$$
COMPLEX SERIES
Question 1

The following convergent series $C$ and $S$ are given by

$$C = 1 + \frac{1}{2}\cos \theta + \frac{1}{4}\cos 2\theta + \frac{1}{8}\cos 3\theta...$$

$$S = \frac{1}{2}\sin \theta + \frac{1}{4}\sin 2\theta + \frac{1}{8}\sin 3\theta...$$

a) Show clearly that

$$C + iS = \frac{2}{2 - e^{i\theta}}.$$ 

b) Hence show further that

$$C = \frac{4 - 2\cos \theta}{5 - 4\cos \theta},$$

and find a similar expression for $S$.

$$S = \frac{2\sin \theta}{5 - 4\cos \theta}.$$
Question 2

The following finite sums, \( C \) and \( S \), are given by

\[
C = 1 + 5 \cos 2\theta + 10 \cos 4\theta + 10 \cos 6\theta + 5 \cos 8\theta + \cos 10\theta
\]
\[
S = 5 \sin 2\theta + 10 \sin 4\theta + 10 \sin 6\theta + 5 \sin 8\theta + \sin 10\theta
\]

By considering the binomial expansion of \((1 + A)^5\), show clearly that

\[
C = 32 \cos^5 \theta \cos 5\theta,
\]

and find a similar expression for \( S \)

\[
S = 32 \cos^5 \theta \sin 5\theta
\]
Question 3

The following convergent series $S$ is given below:

$$S = \sin \theta - \frac{1}{3} \sin 2\theta + \frac{1}{9} \sin 3\theta - \frac{1}{27} \sin 4\theta \ldots$$

By considering the sum to infinity of a suitable geometric series involving the complex exponential function, show that

$$S = \frac{9\sin \theta}{10 + 6\cos \theta}.$$

proof
Question 4

The sum $C$ is given below

$$C = 1 - \binom{n}{1} \cos \theta \cos \theta + \binom{n}{2} \cos^2 \theta \cos 2\theta - \binom{n}{1} \cos^3 \theta \cos 3\theta + \ldots + (-1)^n \cos^n \theta \cos n\theta$$

Given that $n \in \mathbb{N}$ determine the 4 possible expressions for $C$.

Give the answers in exact simplified form.

- For $n = 4k$, $k \in \mathbb{N}$: $C = \cos n\theta \sin^n \theta$
- For $n = 4k + 1$, $k \in \mathbb{N}$: $C = \sin n\theta \sin^n \theta$
- For $n = 4k + 2$, $k \in \mathbb{N}$: $C = -\cos n\theta \sin^n \theta$
- For $n = 4k + 3$, $k \in \mathbb{N}$: $C = -\sin n\theta \sin^n \theta$
Question 5

The following convergent series $S$ is given below

$$S = \frac{\sin \theta}{1!} - \frac{\sin 2\theta}{2!} + \frac{\sin 3\theta}{3!} - \frac{\sin 4\theta}{4!} + \ldots$$

By considering the sum to infinity of a suitable series involving the complex exponential function, show that

$$S = e^{-\cos \theta} \sin (\sin \theta).$$

proof