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# COMPLEX NUMBERS PRACTICE (part 2)

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# ROOTS OF COMPLEX NUMBERS

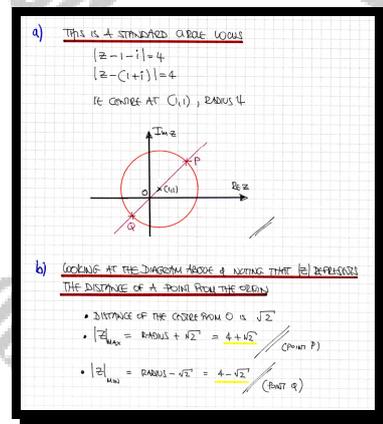
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Question 1

$$z^4 = -16, z \in \mathbb{C}.$$

- Solve the above equation, giving the answers in the form  $a + bi$ , where  $a$  and  $b$  are real numbers.
- Plot the roots of the equation as points in an Argand diagram.

$$z = \sqrt{2}(\pm 1 \pm i)$$



Question 2

$$z^5 = i, \quad z \in \mathbb{C}.$$

- a) Solve the equation, giving the roots in the form  $r e^{i\theta}$ ,  $r > 0$ ,  $-\pi < \theta \leq \pi$ .
- b) Plot the roots of the equation as points in an Argand diagram.

$$z = e^{i\frac{\pi}{10}}, \quad z = e^{i\frac{3\pi}{10}}, \quad z = e^{i\frac{5\pi}{10}}, \quad z = e^{i\frac{7\pi}{10}}, \quad z = e^{i\frac{9\pi}{10}}$$

a)  $z^5 = i$

WORKING IN EXPONENTIALS

$$\Rightarrow z^5 = 1 \times e^{i(\frac{\pi}{2} + 2k\pi)}; \quad z \in \mathbb{C}$$

$$\Rightarrow z^5 = e^{i\frac{\pi}{2}(1+4k)}$$

$$\Rightarrow (z^5)^{\frac{1}{5}} = [e^{i\frac{\pi}{2}(1+4k)}]^{\frac{1}{5}}$$

$$\Rightarrow z = e^{i\frac{\pi}{10}(1+4k)}$$

$k=0,1,2,3,4$  as we may have to go negative

$k=0 \quad z_0 = e^{i\frac{\pi}{10}}$   
 $k=1 \quad z_1 = e^{i\frac{5\pi}{10}}$   
 $k=2 \quad z_2 = e^{i\frac{9\pi}{10}}$   
 $k=3 \quad z_3 = e^{i\frac{13\pi}{10}} \quad (\text{or } k=2 \quad z_2 = e^{-i\frac{7\pi}{10}})$   
 $k=4 \quad z_4 = e^{i\frac{17\pi}{10}} \quad (\text{or } k=1 \quad z_1 = e^{-i\frac{3\pi}{10}})$

b) THE ROOTS ARE EQUALLY SPACED AND OF MODULUS 1

Question 3

$$z = 4 + 4i.$$

- a) Find the fifth roots of  $z$ .  
Give the answers in the form  $re^{i\theta}$ ,  $r > 0$ ,  $-\pi < \theta \leq \pi$ .
- b) Plot the roots as points in an Argand diagram.

$$\sqrt{2} e^{i\frac{\pi}{20}}, \sqrt{2} e^{i\frac{9\pi}{20}}, \sqrt{2} e^{i\frac{17\pi}{20}}, \sqrt{2} e^{-i\frac{7\pi}{20}}, \sqrt{2} e^{-i\frac{3\pi}{4}}$$

(a)  $(4+4i) = \sqrt{4^2+4^2} = \sqrt{32} = 4\sqrt{2} \left( \frac{2 \times 2^2}{2^2} = 2^2 \right)$   
 $\arg(4+4i) = \arctan\left(\frac{4}{4}\right) = \frac{\pi}{4}$

$\Rightarrow w^5 = 4\sqrt{2} e^{i\left(\frac{\pi}{4} + 2k\pi\right)}$   $\forall k \in \mathbb{Z}$

$\Rightarrow w^5 = 2^2 e^{i\left(\frac{\pi}{4} + 2k\pi\right)}$

$\Rightarrow (w^5)^{\frac{1}{5}} = \left[ 2^2 e^{i\left(\frac{\pi}{4} + 2k\pi\right)} \right]^{\frac{1}{5}}$

$\Rightarrow w = 2^{\frac{2}{5}} e^{i\frac{1}{5}\left(\frac{\pi}{4} + 2k\pi\right)}$

$\Rightarrow |w| = \sqrt{2} e^{i\frac{1}{5}\left(\frac{\pi}{4} + 2k\pi\right)}$

for  $k=0$   
 $w_0 = \sqrt{2} e^{i\frac{\pi}{20}}$   
 $w_1 = \sqrt{2} e^{i\frac{9\pi}{20}}$   
 $w_2 = \sqrt{2} e^{i\frac{17\pi}{20}}$   
 $w_3 = \sqrt{2} e^{-i\frac{7\pi}{20}}$   
 $w_4 = \sqrt{2} e^{-i\frac{3\pi}{4}}$

(b)

Question 4

$$z = 4 - 4\sqrt{3}i.$$

- a) Find the cube roots of  $z$ .

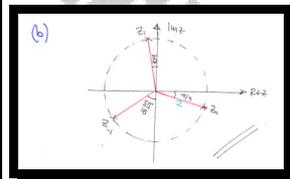
Give the answers in polar form  $r(\cos\theta + i\sin\theta)$ ,  $r > 0$ ,  $-\pi < \theta \leq \pi$ .

- b) Plot the roots as points in an Argand diagram.

$$z = 2\left(\cos\frac{\pi}{9} - i\sin\frac{\pi}{9}\right), \quad z = 2\left(\cos\frac{5\pi}{9} + i\sin\frac{5\pi}{9}\right), \quad z = 2\left(\cos\frac{7\pi}{9} - i\sin\frac{7\pi}{9}\right)$$

(a)  $4 - 4\sqrt{3}i = 8e^{-i\frac{\pi}{3}}$   
 OR IN Cartesian  
 $\rightarrow 4 - 4\sqrt{3}i = 8e^{i\left(-\frac{\pi}{3} + 2m\pi\right)}$   
 $\rightarrow 4 - 4\sqrt{3}i = 8e^{i\left(-\frac{\pi}{3}\right)}$   
 $\Rightarrow [4 - 4\sqrt{3}i]^{\frac{1}{3}} = [8e^{i\left(-\frac{\pi}{3}\right)}]^{\frac{1}{3}}$   
 $\rightarrow [4 - 4\sqrt{3}i]^{\frac{1}{3}} = 2e^{i\left(-\frac{\pi}{9}\right)}$   
 $\rightarrow [4 - 4\sqrt{3}i]^{\frac{1}{3}} = 2e^{-i\frac{\pi}{9}}$   
 Hence  $z_0 = 2e^{-i\frac{\pi}{9}} = 2\left(\cos\frac{\pi}{9} - i\sin\frac{\pi}{9}\right)$   
 $z_1 = 2e^{i\frac{5\pi}{9}} = 2\left(\cos\frac{5\pi}{9} + i\sin\frac{5\pi}{9}\right)$   
 $z_2 = 2e^{i\frac{7\pi}{9}} = 2\left(\cos\frac{7\pi}{9} - i\sin\frac{7\pi}{9}\right)$

$|4 - 4\sqrt{3}i| = \sqrt{16 + 48} = 8$   
 $\arg(4 - 4\sqrt{3}i) = \arctan\left(\frac{-4\sqrt{3}}{4}\right) = -\frac{\pi}{3}$



**Question 5**

The following complex number relationships are given

$$w = -2 + 2\sqrt{3}i, \quad z^4 = w.$$

- a) Express  $w$  in the form  $r(\cos\theta + i\sin\theta)$ , where  $r > 0$  and  $-\pi < \theta \leq \pi$ .
- b) Find the possible values of  $z$ , giving the answers in the form  $x + iy$ , where  $x$  and  $y$  are real numbers.

$$w = 2 \left[ \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) \right],$$

$$z = \frac{1}{2}(\sqrt{6} + i\sqrt{2}), \quad z = \frac{1}{2}(-\sqrt{2} + i\sqrt{6}), \quad z = \frac{1}{2}(\sqrt{2} - i\sqrt{6}), \quad z = \frac{1}{2}(-\sqrt{6} - i\sqrt{2})$$

(a)  $|-2 + 2\sqrt{3}i| = \sqrt{4 + 12} = 4$   
 $\arg(-2 + 2\sqrt{3}i) = \pi + \arctan\left(\frac{2\sqrt{3}}{-2}\right) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$   
 $\therefore -2 + 2\sqrt{3}i = 4 \left[ \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} \right]$

(b)  $z^4 = -2 + 2\sqrt{3}i$   
 $z^4 = 4 \left[ \cos\left(\frac{2\pi}{3} + 2\pi k\right) + i\sin\left(\frac{2\pi}{3} + 2\pi k\right) \right]$   
 $z = 4^{\frac{1}{4}} \left[ \cos\left(\frac{\pi}{6} + \frac{2\pi k}{4}\right) + i\sin\left(\frac{\pi}{6} + \frac{2\pi k}{4}\right) \right]^{\frac{1}{2}}$   
 $z = \sqrt{2} \left[ \cos\left(\frac{\pi}{12} + \frac{\pi k}{2}\right) + i\sin\left(\frac{\pi}{12} + \frac{\pi k}{2}\right) \right]$

$z_0 = \sqrt{2} \left[ \cos\left(\frac{\pi}{12}\right) + i\sin\left(\frac{\pi}{12}\right) \right] = \frac{\sqrt{6}}{2} + \frac{\sqrt{2}}{2}i$   
 $z_1 = \sqrt{2} \left[ \cos\left(\frac{5\pi}{12}\right) + i\sin\left(\frac{5\pi}{12}\right) \right] = -\frac{\sqrt{6}}{2} + \frac{\sqrt{2}}{2}i$   
 $z_2 = \sqrt{2} \left[ \cos\left(\frac{13\pi}{12}\right) + i\sin\left(\frac{13\pi}{12}\right) \right] = \frac{\sqrt{6}}{2} - \frac{\sqrt{2}}{2}i$   
 $z_3 = \sqrt{2} \left[ \cos\left(\frac{17\pi}{12}\right) + i\sin\left(\frac{17\pi}{12}\right) \right] = -\frac{\sqrt{6}}{2} - \frac{\sqrt{2}}{2}i$

**Question 6**

Find the cube roots of the imaginary unit  $i$ , giving the answers in the form  $a + bi$ , where  $a$  and  $b$  are real numbers.

$$z_1 = \frac{\sqrt{3}}{2} + \frac{1}{2}i, \quad z_2 = -\frac{\sqrt{3}}{2} + \frac{1}{2}i, \quad z_3 = -i$$

$z^3 = i = e^{i\left(\frac{\pi}{2} + 2\pi k\right)}, \quad k \in \mathbb{Z}$  (Note:  $|i| = 1, \arg i = \frac{\pi}{2}$ )

$\Rightarrow z = e^{\frac{i(\pi/2 + 2\pi k)}{3}}$

$\Rightarrow z^3 = e^{i(\pi/2 + 2\pi k)}$

$\Rightarrow z = e^{\frac{i(\pi/2 + 2\pi k)}{3}}$

$z_0 = e^{\frac{i\pi}{6}} = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + \frac{1}{2}i$   
 $z_1 = e^{\frac{5i\pi}{6}} = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6} = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$   
 $z_2 = e^{\frac{3i\pi}{2}} = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = -i$

**Question 7**

Find the cube roots of the complex number  $-8i$ , giving the answers in the form  $a + bi$ , where  $a$  and  $b$  are real numbers.

$$z_1 = \sqrt{3} - i, \quad z_2 = -\sqrt{3} - i, \quad z_3 = 2i$$

$z^3 = -8i$   
 $\rightarrow z^3 = 8 \times e^{i(\frac{3\pi}{2} + 2k\pi)}$   $k \in \mathbb{Z}$   $|-8i| = 8$   
 $\rightarrow z^3 = 8 e^{i\frac{3\pi}{2}(k+1)}$   $\arg(-8i) = \frac{3\pi}{2}$   
 $\rightarrow (z^3)^{\frac{1}{3}} = [8 e^{i\frac{3\pi}{2}(k+1)}]^{\frac{1}{3}}$   
 $\rightarrow z = 2 e^{i\frac{\pi}{2}(k+1)}$   
 $z_0 = 2 e^{i\frac{\pi}{2}} = 2(\cos(\frac{\pi}{2}) + i\sin(\frac{\pi}{2})) = 2(\frac{0}{2} - \frac{1}{2}i) = -i$   
 $z_1 = 2 e^{i\frac{3\pi}{2}} = 2(\cos(\frac{3\pi}{2}) + i\sin(\frac{3\pi}{2})) = 2(0 + i) = 2i$   
 $z_2 = 2 e^{i\frac{5\pi}{2}} = 2(\cos(\frac{5\pi}{2}) + i\sin(\frac{5\pi}{2})) = 2(\frac{0}{2} - \frac{1}{2}i) = -i$

**Question 8**

$$z^4 = -8 - 8\sqrt{3}i, \quad z \in \mathbb{C}.$$

Solve the above equation, giving the answers in the form  $a + bi$ , where  $a$  and  $b$  are real numbers.

$$z = \sqrt{3} - i, \quad z = 1 + \sqrt{3}i, \quad z = -\sqrt{3} + i, \quad z = -1 - \sqrt{3}i$$

$z^4 = -8 - 8\sqrt{3}i$   
 $\rightarrow z^4 = 16 e^{i(\frac{4\pi}{3} + 2k\pi)}$   $k \in \mathbb{Z}$   
 $\rightarrow z^4 = 16 e^{i\frac{4\pi}{3}(k+1)}$   
 $\rightarrow (z^4)^{\frac{1}{4}} = [16 e^{i\frac{4\pi}{3}(k+1)}]^{\frac{1}{4}}$   
 $\rightarrow z = 2 e^{i\frac{\pi}{3}(k+1)}$   
 $\arg(-8 - 8\sqrt{3}i) = \frac{4\pi}{3}$   
 $z_0 = 2 e^{i\frac{\pi}{3}} = 2(\cos(\frac{\pi}{3}) + i\sin(\frac{\pi}{3})) = 1 + i\sqrt{3}$   
 $z_1 = 2 e^{i\frac{2\pi}{3}} = 2(\cos(\frac{2\pi}{3}) + i\sin(\frac{2\pi}{3})) = -1 + i\sqrt{3}$   
 $z_2 = 2 e^{i\frac{4\pi}{3}} = 2(\cos(\frac{4\pi}{3}) + i\sin(\frac{4\pi}{3})) = -1 - i\sqrt{3}$   
 $z_3 = 2 e^{i\frac{5\pi}{3}} = 2(\cos(\frac{5\pi}{3}) + i\sin(\frac{5\pi}{3})) = 1 - i\sqrt{3}$

Question 9

$$z^2 = (1+i\sqrt{3})^3, z \in \mathbb{C}.$$

Solve the above equation, giving the answers in the form  $a + bi$ , where  $a$  and  $b$  are real numbers.

$$z = \pm i2\sqrt{2}$$

Handwritten solution for Question 9:

- $z^2 = (1+i\sqrt{3})^3$
- $z^2 = (2e^{i\frac{\pi}{3}})^3$
- $z^2 = 8e^{i\pi}$
- $z^2 = 8e^{i(\pi+2\pi)}$
- $z^2 = 8e^{i\pi(1+2n)}$
- $(\frac{z}{8})^2 = \frac{1}{8} e^{i\pi(1+2n)}$
- $\frac{z}{8} = \sqrt[2]{\frac{1}{8} e^{i\pi(1+2n)}}$
- $z = \sqrt[2]{8} e^{i\frac{\pi(1+2n)}{2}}$
- $|1+i\sqrt{3}| = \sqrt{1+3} = 2$
- $\arg(1+i\sqrt{3}) = \arctan(\sqrt{3}) = \frac{\pi}{3}$
- $z_1 = \sqrt[2]{8} e^{i\frac{\pi}{2}} = \sqrt[2]{8} (\cos(\frac{\pi}{2}) + i\sin(\frac{\pi}{2})) = i\sqrt[2]{8}$
- $z_2 = \sqrt[2]{8} e^{i\frac{3\pi}{2}} = \sqrt[2]{8} (\cos(\frac{3\pi}{2}) + i\sin(\frac{3\pi}{2})) = -i\sqrt[2]{8}$
- $\therefore z = \pm i2\sqrt{2}$

Question 10

$$z^3 = 32 + 32\sqrt{3}i, z \in \mathbb{C}.$$

a) Solve the above equation.

Give the answers in exponential form  $z = re^{i\theta}$ ,  $r > 0$ ,  $-\pi < \theta \leq \pi$ .

b) Show that these roots satisfy the equation

$$w^9 + 2^{18} = 0.$$

$$z = 4e^{i\frac{\pi}{9}}, 4e^{i\frac{7\pi}{9}}, 4e^{-i\frac{5\pi}{9}}$$

Handwritten solution for Question 10:

(a)  $z^3 = 32 + 32\sqrt{3}i$

- $\Rightarrow z^3 = 64 e^{i(\frac{\pi}{3} + 2\pi n)}$ ,  $n \in \mathbb{Z}$
- $\Rightarrow z = \sqrt[3]{64} e^{i\frac{\pi}{3}(1+2n)}$
- $\Rightarrow z = 4 e^{i\frac{\pi}{3}(1+2n)}$
- $z_0 = 4e^{i\frac{\pi}{3}}$
- $z_1 = 4e^{i\frac{7\pi}{3}}$
- $z_2 = 4e^{-i\frac{5\pi}{3}}$

(b)  $z^9 + 2^{18} = [4e^{i\frac{\pi}{3}(1+2n)}]^9 + 2^{18} = 4^9 e^{i\pi(1+2n)} + 2^{18}$

- $= 2^{18} e^{i\pi(1+2n)} + 2^{18} = 2^{18} [e^{i\pi(1+2n)} + 1]$
- $= 2^{18} [\cos(\pi(1+2n)) + i\sin(\pi(1+2n)) + 1]$
- $= 2^{18} \times [\cos(\pi(1+2n)) + 1]$
- $= 2^{18} \times [-1 + 1]$  (odd multiples of  $\pi$ )
- $= 0$

Question 11

$$z^7 - 1 = 0, z \in \mathbb{C}.$$

One of the roots of the above equation is denoted by  $\omega$ , where  $0 < \arg \omega < \frac{\pi}{3}$ .

a) Find  $\omega$  in the form  $\omega = re^{i\theta}$ ,  $r > 0$ ,  $0 < \theta \leq \frac{\pi}{3}$ .

b) Show clearly that

$$1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 = 0.$$

c) Show further that

$$\omega^2 + \omega^5 = 2 \cos\left(\frac{4\pi}{7}\right).$$

d) Hence, using the results from the previous parts deduce that

$$\cos\left(\frac{2\pi}{7}\right) + \cos\left(\frac{4\pi}{7}\right) + \cos\left(\frac{6\pi}{7}\right) = -\frac{1}{2}.$$

$$\omega = e^{i\frac{2\pi}{7}}$$

Handwritten solution for Question 11:

(a)  $z^7 - 1 = 0$   
 $\Rightarrow z^7 = 1$   
 $\Rightarrow z^7 = 1 \times e^{i(0+2m\pi)}$   
 $\Rightarrow z^7 = e^{i2m\pi}$   
 $\Rightarrow z = e^{i\frac{2m\pi}{7}}$   
 $\omega = z_1 = e^{i\frac{2\pi}{7}}$

(b)  $z^7 - 1 = 0$   
 $(z-1)(z^6 + z^5 + z^4 + z^3 + z^2 + z + 1) = 0$   
 Either  $z=1$  or  $z^6 + z^5 + z^4 + z^3 + z^2 + z + 1 = 0$   
 Let  $\omega$  is a solution ( $\in z = \omega$ )  
 $\therefore \omega^6 + \omega^5 + \omega^4 + \omega^3 + \omega^2 + \omega + 1 = 0$

(c)  $\omega^2 + \omega^5 = \left(e^{i\frac{4\pi}{7}}\right) + \left(e^{i\frac{10\pi}{7}}\right) = e^{i\frac{4\pi}{7}} + e^{-i\frac{4\pi}{7}}$   
 $= e^{i\frac{4\pi}{7}} + e^{-i\frac{4\pi}{7}} = 2\cos\left(\frac{4\pi}{7}\right) = 2\cos\frac{4\pi}{7}$

(d) Similarly  $\omega + \omega^6 = e^{i\frac{2\pi}{7}} + \left(e^{i\frac{12\pi}{7}}\right) = e^{i\frac{2\pi}{7}} + e^{-i\frac{2\pi}{7}}$   
 $= 2\cos\frac{2\pi}{7} = 2\cos\frac{2\pi}{7}$   
 $\omega^3 + \omega^4 = \left(e^{i\frac{6\pi}{7}}\right) + \left(e^{i\frac{8\pi}{7}}\right) = e^{i\frac{6\pi}{7}} + e^{-i\frac{6\pi}{7}}$   
 $= 2\cos\frac{6\pi}{7} = 2\cos\frac{6\pi}{7}$

So  $1 + (\omega + \omega^6) + (\omega^2 + \omega^5) + (\omega^3 + \omega^4) = 0$   
 $1 + 2\cos\frac{2\pi}{7} + 2\cos\frac{4\pi}{7} + 2\cos\frac{6\pi}{7} = 0$   
 $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = -\frac{1}{2}$

Question 12

$$z^3 = (1+i\sqrt{3})^8 (1-i)^5, z \in \mathbb{C}.$$

Find the three roots of the above equation, giving the answers in the form  $k\sqrt{2}e^{i\theta}$ , where  $-\pi < \theta \leq \pi$ ,  $k \in \mathbb{Z}$ .

$$z = 8\sqrt{2}e^{i\theta}, \quad \theta = -\frac{31\pi}{36}, -\frac{7\pi}{36}, \frac{17\pi}{36}$$

Handwritten solution showing the conversion of complex numbers to polar form and the calculation of the three roots:

$$z^3 = (1+i\sqrt{3})^8 (1-i)^5$$

$$\Rightarrow z^3 = (2e^{i\frac{\pi}{3}})^8 (2e^{-i\frac{\pi}{4}})^5$$

$$\Rightarrow z^3 = 256e^{i\frac{8\pi}{3}} \cdot 32e^{-i\frac{5\pi}{4}}$$

$$\Rightarrow z^3 = 8192e^{i(\frac{8\pi}{3} - \frac{5\pi}{4})}$$

$$\Rightarrow z^3 = 8192e^{i(\frac{32\pi}{12} - \frac{15\pi}{12})}$$

$$\Rightarrow z^3 = 8192e^{i\frac{17\pi}{12}}$$

$$\Rightarrow z = \sqrt[3]{8192} e^{i\frac{17\pi}{36}}$$

$$z_0 = 8\sqrt{2} e^{i\frac{17\pi}{36}}$$

$$z_1 = 8\sqrt{2} e^{-i\frac{7\pi}{36}}$$

$$z_2 = 8\sqrt{2} e^{-i\frac{31\pi}{36}}$$

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# TRIGONOMETRIC IDENTITIES QUESTIONS

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**Question 1**

If  $z = \cos\theta + i\sin\theta$ , show clearly that ...

a) ...  $z^n + \frac{1}{z^n} \equiv 2\cos n\theta$ .

b) ...  $16\cos^5\theta \equiv \cos 5\theta + 5\cos 3\theta + 10\cos\theta$ .

proof

(a)  $z = \cos\theta + i\sin\theta$   
 $z^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$   
 $z^{-n} = (\cos\theta + i\sin\theta)^{-n} = \cos(-n\theta) + i\sin(-n\theta) = \cos n\theta - i\sin n\theta$   
 $\therefore z^n + \frac{1}{z^n} = (\cos n\theta + i\sin n\theta) + (\cos n\theta - i\sin n\theta) = 2\cos n\theta$

(b)  $z + \frac{1}{z} = 2\cos\theta$   
 $(2\cos\theta)^5 = \left(z + \frac{1}{z}\right)^5$

Binomial expansion:

$$z^5 + 5z^4\left(\frac{1}{z}\right) + 10z^3\left(\frac{1}{z}\right)^2 + 10z^2\left(\frac{1}{z}\right)^3 + 5z\left(\frac{1}{z}\right)^4 + \frac{1}{z^5}$$

$$= z^5 + 5z^3 + 10z + 10\left(\frac{1}{z}\right) + 5\left(\frac{1}{z}\right)^3 + \frac{1}{z^5}$$

$$= z^5 + 5z^3 + 10z + 10\left(\frac{1}{z}\right) + 5\left(\frac{1}{z^3}\right) + \frac{1}{z^5}$$

$$= z^5 + 5z^3 + 10z + 10\left(\frac{1}{z}\right) + 5\left(\frac{1}{z^3}\right) + \frac{1}{z^5}$$

Identify:

$$= \cos 5\theta + 5\cos 3\theta + 10\cos\theta$$



**Question 3**

The complex number  $z$  is given by

$$z = e^{i\theta}, \quad -\pi < \theta \leq \pi.$$

a) Show clearly that

$$z^n + \frac{1}{z^n} \equiv 2 \cos n\theta.$$

b) Hence show further that

$$\cos^4 \theta \equiv \frac{1}{8} \cos 4\theta + \frac{1}{2} \cos 2\theta + \frac{3}{8}.$$

c) Solve the equation

$$2 \cos 4\theta + 8 \cos 2\theta + 5 = 0, \quad 0 \leq \theta < 2\pi.$$

$$\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$$

(a)  $z = e^{i\theta}$   
 $z^n = (e^{i\theta})^n = e^{in\theta}$   
 $\frac{1}{z^n} = (e^{i\theta})^{-n} = e^{-in\theta}$   
 $z^n + \frac{1}{z^n} = e^{in\theta} + e^{-in\theta} = \cos n\theta + i \sin n\theta + \cos(-n\theta) + i \sin(-n\theta)$   
 $= \cos n\theta + i \sin n\theta + \cos n\theta - i \sin n\theta = 2 \cos n\theta$   
 As required

(b)  $z^n + \frac{1}{z^n} = 2 \cos n\theta$   
 If  $n=1$   
 $z + \frac{1}{z} = 2 \cos \theta$   
 $(z + \frac{1}{z})^4 = (2 \cos \theta)^4$   
 $16 \cos^4 \theta = z^4 + 4z^2 + 6 + \frac{4}{z^2} + \frac{1}{z^4}$   
 $16 \cos^4 \theta = z^4 + 4z^2 + 6 + \frac{4}{z^2} + \frac{1}{z^4}$   
 $16 \cos^4 \theta = 2 \cos 4\theta + 4(2 \cos 2\theta) + 6$   
 $16 \cos^4 \theta = \frac{1}{2} \cos 4\theta + \frac{1}{2} \cos 2\theta + \frac{3}{8}$   
 As required

(c)  $2 \cos 4\theta + 8 \cos 2\theta + 5 = 0$   
 $\frac{1}{2} \cos 4\theta + \frac{1}{2} \cos 2\theta + \frac{5}{8} = 0$   
 $\cos^2 \theta = \frac{1}{8}$   
 $\cos \theta = \pm \frac{1}{2\sqrt{2}}$   
 $\cos \theta = \frac{1}{2}$   
 $\therefore \theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

**Question 4**

The complex number  $z$  is given by

$$z = e^{i\theta}, \quad -\pi < \theta \leq \pi.$$

a) Show clearly that

$$z^n + \frac{1}{z^n} \equiv 2 \cos n\theta.$$

b) Hence show further that

$$16 \cos^5 \theta \equiv \cos 5\theta + 5 \cos 3\theta + 10 \cos \theta.$$

c) Use the results of part (a) and (b) to solve the equation

$$\cos 5\theta + 5 \cos 3\theta + 6 \cos \theta = 0, \quad 0 \leq \theta < \pi.$$

$$\theta = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$$

Handwritten solution for Question 4:

(a) Let  $z = \cos\theta + i\sin\theta$   
 $z^n = (\cos\theta + i\sin\theta)^n$   
 $z^{-n} = \cos n\theta + i\sin n\theta$   
 $z^n + \frac{1}{z^n} = 2 \cos n\theta$  (a)

(b) Let  $n=1$  in (a)  
 $\Rightarrow 2 \cos\theta = z + \frac{1}{z}$   
 $\Rightarrow (2 \cos\theta)^5 = (z + \frac{1}{z})^5$   
 $\Rightarrow 32 \cos^5\theta = z^5 + 5z^3 + 10z + \frac{10}{z} + \frac{5}{z^3} + \frac{1}{z^5}$   
 $\Rightarrow 32 \cos^5\theta = (z^5 + \frac{1}{z^5}) + 5(z^3 + \frac{1}{z^3}) + 10(z + \frac{1}{z})$   
 $\Rightarrow 32 \cos^5\theta = (2 \cos 5\theta) + 5(2 \cos 3\theta) + 10(2 \cos \theta)$   
 $\Rightarrow 16 \cos^5\theta = \cos 5\theta + 5 \cos 3\theta + 10 \cos \theta$  (b)

(c)  $\cos 5\theta + 5 \cos 3\theta + 6 \cos \theta = 0$   
 $\Rightarrow \cos 5\theta + 5 \cos 3\theta + 10 \cos \theta = 4 \cos \theta$   
 $\Rightarrow 16 \cos^5\theta = 4 \cos \theta$   
 $\Rightarrow 4 \cos^4\theta = \cos \theta$   
 $\Rightarrow 4 \cos^4\theta - \cos \theta = 0$   
 $\Rightarrow \cos \theta (4 \cos^3\theta - 1) = 0$   
 $\Rightarrow \cos \theta = 0$  or  $\cos^3\theta = \frac{1}{4}$   
 $\cos \theta = \frac{1}{\sqrt[3]{4}}$   
 $\cos \theta = -\frac{1}{\sqrt[3]{4}}$   
 For  $0 \leq \theta < \pi$   
 $\theta = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$

Question 5

De Moivre's theorem asserts that

$$(\cos \theta + i \sin \theta)^n \equiv \cos n\theta + i \sin n\theta, \theta \in \mathbb{R}, n \in \mathbb{Q}.$$

- a) Use the theorem to prove the validity of the following trigonometric identity.

$$\cos 6\theta \equiv 32 \cos^6 \theta - 48 \cos^4 \theta + 18 \cos^2 \theta - 1.$$

- b) Use the result of part (a) to find, in exact form, the largest positive root of the equation

$$64x^6 - 96x^4 + 36x^2 - 1 = 0.$$

$$x = \cos\left(\frac{\pi}{9}\right)$$

(a) Let  $\cos \theta + i \sin \theta = c + is$   
 This  
 $(\cos \theta + i \sin \theta)^6 = (c + is)^6$   
 $(\cos \theta + i \sin \theta)^6 = c^6 + 6c^5is + 15c^4s^2 - 20c^3s^3 + 15c^2s^4 - 6cs^5 + s^6$   
 Equate Real Parts  
 $\Rightarrow \cos 6\theta = c^6 - 15c^4s^2 + 15c^2s^4 - s^6$   
 $\Rightarrow \cos 6\theta = c^6 - 15c^4(1-c^2) + 15c^2(1-c^2)^2 - (1-c^2)^3$   
 $\Rightarrow \cos 6\theta = c^6 - 15c^4 + 15c^6 + 15c^2(1-2c^2+c^4) - (1-3c^2+3c^4-c^6)$   
 $\Rightarrow \cos 6\theta = c^6 - 15c^4 + 15c^6 + 15c^2 - 30c^4 + 15c^6 - 1 + 3c^2 - 3c^4 + c^6$   
 $\Rightarrow \cos 6\theta = 32c^6 - 48c^4 + 18c^2 - 1$   
 $\therefore \cos 6\theta = 32 \cos^6 \theta - 48 \cos^4 \theta + 18 \cos^2 \theta - 1$  As required

(b)  $64x^6 - 96x^4 + 36x^2 - 1 = 0$   
 $\Rightarrow 32x^6 - 48x^4 + 18x^2 - \frac{1}{2} = 0$   
 $\Rightarrow 32x^6 - 48x^4 + 18x^2 - 1 = -\frac{1}{2}$   
 Let  $z = \cos \theta$   
 $\Rightarrow 32z^6 - 48z^4 + 18z^2 - 1 = -\frac{1}{2}$   
 $\Rightarrow 64z^6 - 96z^4 + 36z^2 - 1 = 0$   
 $\bullet \text{arc cos}\left(\frac{1}{2}\right) = \frac{\pi}{3}$   
 $\left(\theta = \frac{\pi}{3} \neq 2\pi\right)$   
 $\left(\theta = \frac{5\pi}{3} \neq 2\pi\right)$   
 $\therefore z = \cos \frac{\pi}{9}$  is the largest positive root of the equation

**Question 6**

Euler's identity states

$$e^{i\theta} \equiv \cos\theta + i\sin\theta, \theta \in \mathbb{R}.$$

a) Use the identity to show that

$$e^{in\theta} + e^{-in\theta} \equiv 2\cos n\theta.$$

b) Hence show further that

$$32\cos^6\theta \equiv \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10.$$

c) Use the fact that  $\cos\left(\frac{\pi}{2} - \theta\right) \equiv \sin\theta$  to find a similar expression for  $32\sin^6\theta$ .

d) Determine the exact value of

$$\int_0^{\frac{\pi}{4}} \sin^6\theta + \cos^6\theta \, d\theta.$$

$$32\sin^6\theta \equiv -\cos 6\theta + 6\cos 4\theta - 15\cos 2\theta + 10, \quad \frac{5\pi}{32}$$

a)  $e^{i\theta} = \cos\theta + i\sin\theta$   
 $(e^{i\theta})^n = e^{in\theta} = \cos n\theta + i\sin n\theta$   
 $(e^{-i\theta})^n = e^{-in\theta} = \cos n\theta - i\sin n\theta$   
 adding  $\frac{e^{in\theta} + e^{-in\theta}}{2} = 2\cos n\theta$

b) If  $n=1$   
 $\Rightarrow 2\cos\theta = e^{i\theta} + e^{-i\theta}$   
 $\Rightarrow (2\cos\theta)^6 = (e^{i\theta} + e^{-i\theta})^6$   
 $\Rightarrow 64\cos^6\theta = e^{i6\theta} + 6e^{i4\theta} + 15e^{i2\theta} + 20 + 15e^{-i2\theta} + 6e^{-i4\theta} + e^{-i6\theta}$   
 $\Rightarrow 64\cos^6\theta = (e^{i6\theta} + e^{-i6\theta}) + 6(e^{i4\theta} + e^{-i4\theta}) + 15(e^{i2\theta} + e^{-i2\theta}) + 20$   
 $\Rightarrow 64\cos^6\theta = 2\cos 6\theta + 6(2\cos 4\theta) + 15(2\cos 2\theta) + 20$   
 $\Rightarrow 32\cos^6\theta = \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10$

c)  $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$   
 $\cos(\frac{6\pi}{2} - \theta) = \cos(3\pi - \theta) = \cos 3\pi \cos\theta + \sin 3\pi \sin\theta = -\cos\theta$   
 $\cos(\frac{4\pi}{2} - \theta) = \cos(2\pi - \theta) = \cos 2\pi \cos\theta + \sin 2\pi \sin\theta = \cos\theta$   
 $\cos(\frac{2\pi}{2} - \theta) = \cos(\pi - \theta) = \cos\pi \cos\theta + \sin\pi \sin\theta = -\cos\theta$   
 $\therefore 32\sin^6\theta = -\cos 6\theta + 6\cos 4\theta - 15\cos 2\theta + 10$

d)  $\int_0^{\frac{\pi}{4}} \sin^6\theta + \cos^6\theta \, d\theta = \frac{1}{32} \int_0^{\frac{\pi}{4}} (32\sin^6\theta + 32\cos^6\theta) \, d\theta$   
 $= \frac{1}{32} \int_0^{\frac{\pi}{4}} (2\cos 6\theta + 12\cos 4\theta - 30\cos 2\theta + 20) \, d\theta$   
 $= \frac{1}{32} \left[ \frac{2}{6} \sin 6\theta + \frac{12}{4} \sin 4\theta - \frac{30}{2} \sin 2\theta + 20\theta \right]_0^{\frac{\pi}{4}}$   
 $= \frac{1}{32} \left[ \frac{1}{3} \sin \frac{3\pi}{2} + 3 \sin \pi - 15 \sin \pi + 20 \cdot \frac{\pi}{4} \right] = \frac{5\pi}{32}$

**Question 7**

De Moivre's theorem asserts that

$$(\cos \theta + i \sin \theta)^n \equiv \cos n\theta + i \sin n\theta, \theta \in \mathbb{R}, n \in \mathbb{Q}.$$

- a) Use the theorem to prove validity of the following trigonometric identity

$$\sin 5\theta = \sin \theta (16 \cos^4 \theta - 12 \cos^2 \theta + 1).$$

- b) Hence, or otherwise, solve the equation

$$\sin 5\theta = 10 \cos \theta \sin 2\theta - 11 \sin \theta, 0 < \theta < \pi.$$

$$\theta = \frac{\pi}{4}, \frac{3\pi}{4}$$

(a) Let  $\cos \theta + i \sin \theta = C + iS$   
 $(\cos \theta + i \sin \theta)^5 = (C + iS)^5$   
 $\cos 5\theta + i \sin 5\theta = C^5 + 5C^4 iS - 10C^3 S^2 - 10iC^2 S^3 + 5CS^4 + iS^5$   
 Equate imaginary parts  
 $\Rightarrow \sin 5\theta = 5C^4 S - 10C^2 S^3 + S^5$   
 $\Rightarrow \sin 5\theta = S [5C^4 - 10C^2 S^2 + S^4]$   
 $\Rightarrow \sin 5\theta = S [5C^4 - 10C^2(1-C^2) + (1-C^2)^2]$   
 $\Rightarrow \sin 5\theta = S [5C^4 - 10C^2 + 10C^4 + 1 - 2C^2 + C^4]$   
 $\Rightarrow \sin 5\theta = S [6C^4 - 12C^2 + 1]$   
 i.e.  $\sin 5\theta = \sin \theta [6 \cos^4 \theta - 12 \cos^2 \theta + 1]$  ✓ as required

(b)  $\sin 5\theta = 10 \cos \theta \sin 2\theta - 11 \sin \theta$   
 $\sin \theta [6 \cos^4 \theta - 12 \cos^2 \theta + 1] = 10 \cos \theta (2 \sin \theta \cos \theta) - 11 \sin \theta$   
 As  $0 < \theta < \pi$ ,  $\sin \theta \neq 0$  Hence divide it  
 $6 \cos^4 \theta - 12 \cos^2 \theta + 1 = 20 \cos^2 \theta - 11$   
 $6 \cos^4 \theta - 20 \cos^2 \theta + 12 = 0$   
 $4 \cos^4 \theta - 8 \cos^2 \theta + 3 = 0$   
 $(2 \cos^2 \theta - 1)(2 \cos^2 \theta - 3) = 0$   
 $\cos^2 \theta = \frac{1}{2}$   
 $\cos \theta = \frac{1}{\sqrt{2}} \dots \dots \theta = \frac{\pi}{4} \text{ only}$   
 $\cos \theta = -\frac{1}{\sqrt{2}} \dots \dots \theta = \frac{3\pi}{4} \text{ only}$  ✓

**Question 8**

It is given that

$$\sin 5\theta \equiv \sin \theta (16\cos^4 \theta - 12\cos^2 \theta + 1).$$

- a) Use de Moivre's theorem to prove the validity of the above trigonometric identity.

Consider the general solution of the trigonometric equation

$$\sin 5\theta = 0.$$

- b) Find exact simplified expressions for

$$\cos^2\left(\frac{\pi}{5}\right) \text{ and } \cos^2\left(\frac{2\pi}{5}\right),$$

fully justifying each step in the workings.

$$\boxed{\cos^2\left(\frac{\pi}{5}\right) = \frac{3 + \sqrt{5}}{8}}, \quad \boxed{\cos^2\left(\frac{2\pi}{5}\right) = \frac{3 - \sqrt{5}}{8}}$$

$(\cos \theta + i \sin \theta)^5 = (\cos 5\theta + i \sin 5\theta)$   
 $\Rightarrow \cos^5 \theta + 5i \cos^4 \theta \sin \theta - 10 \cos^3 \theta \sin^2 \theta - 10i \cos^2 \theta \sin^3 \theta + 5 \cos \theta \sin^4 \theta + i \sin^5 \theta$   
 $\Rightarrow \cos^5 \theta + i 5 \cos^4 \theta \sin \theta - 10 \cos^3 \theta \sin^2 \theta - 10i \cos^2 \theta \sin^3 \theta + 5 \cos \theta \sin^4 \theta + i \sin^5 \theta$   
 $\Rightarrow \cos 5\theta + i \sin 5\theta = \cos^5 \theta + 5i \cos^4 \theta \sin \theta - 10 \cos^3 \theta \sin^2 \theta - 10i \cos^2 \theta \sin^3 \theta + 5 \cos \theta \sin^4 \theta + i \sin^5 \theta$   
 $\therefore \sin 5\theta = 5 \cos^4 \theta \sin \theta - 10 \cos^2 \theta \sin^3 \theta + \sin^5 \theta$   
 $\Rightarrow \sin 5\theta = \sin \theta [5 \cos^4 \theta - 10 \cos^2 \theta \sin^2 \theta + \sin^4 \theta]$   
 $\Rightarrow \sin 5\theta = \sin \theta [5 \cos^4 \theta - 10 \cos^2 \theta (1 - \cos^2 \theta) + (1 - \cos^2 \theta)^2]$   
 $\Rightarrow \sin 5\theta = \sin \theta [5 \cos^4 \theta - 10 \cos^2 \theta + 10 \cos^4 \theta - 2 \cos^2 \theta + 1 - 2 \cos^2 \theta + \cos^4 \theta]$   
 $\Rightarrow \sin 5\theta = \sin \theta [16 \cos^4 \theta - 12 \cos^2 \theta + 1]$   
 $\Rightarrow \sin 5\theta = \sin \theta [16 \cos^4 \theta - 12 \cos^2 \theta + 1]$

b)  $\sin 5\theta = 0$   
 $\cos 5\theta = 0$   
 $5\theta = 0 \pm 2\pi$   
 $5\theta = \pi \pm 2\pi$   
 $\theta = 0 \pm \frac{2\pi}{5}$   
 $\theta = \frac{\pi}{5} \pm \frac{2\pi}{5}$   
 $\theta = 0, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}, \pi, \dots$

$\sin \theta (16 \cos^4 \theta - 12 \cos^2 \theta + 1) = 0$   
 $\sin \theta = 0 \Rightarrow \theta = 0, \pi, 2\pi, 3\pi, \dots$   
 $16 \cos^4 \theta - 12 \cos^2 \theta + 1 = 0$   
 $\cos^2 \theta = \frac{12 \pm \sqrt{80}}{32}$   
 $\cos^2 \theta = \frac{12 \pm 4\sqrt{5}}{32}$   
 $\cos^2 \theta = \frac{3 \pm \sqrt{5}}{8}$

SOLUTION  
 $\frac{\pi}{5} > \frac{\pi}{2}$   
 $\cos \frac{\pi}{5} < \cos \frac{2\pi}{5}$   
 $\cos^2 \frac{\pi}{5} < \cos^2 \frac{2\pi}{5}$   
 $\cos^2 \frac{\pi}{5} < \frac{3}{8}$   
 $\therefore \cos^2 \frac{\pi}{5} = \frac{3 + \sqrt{5}}{8}$   
 $\therefore \cos^2 \frac{2\pi}{5} = \frac{3 - \sqrt{5}}{8}$

## Question 9

By considering the binomial expansion of  $(\cos \theta + i \sin \theta)^4$  show that

$$\tan 4\theta = \frac{4 \tan \theta - 4 \tan^3 \theta}{1 - 6 \tan^2 \theta + \tan^4 \theta}$$

proof

$(\cos \theta + i \sin \theta)^4 = C + iS$   
 $(\cos \theta + i \sin \theta)^4 = C^4 + 4iC^3S - 6C^2S^2 - 4iCS^3 + S^4$   
 $\therefore \tan 4\theta = \frac{4iC^3S - 4iCS^3}{C^4 - 6C^2S^2 + S^4}$   
 Divide top and bottom by  $C^4$   
 $\tan 4\theta = \frac{4iC^3S/C^4 - 4iCS^3/C^4}{C^4/C^4 - 6C^2S^2/C^4 + S^4/C^4}$   
 $\tan 4\theta = \frac{4i \tan \theta - 4i \tan^3 \theta}{1 - 6 \tan^2 \theta + \tan^4 \theta}$  // as required

**Question 10**

By using de Moivre's theorem followed by a suitable trigonometric identity, show clearly that ...

a) ...  $\cos 3\theta \equiv 4\cos^3 \theta - 3\cos \theta$ .

b) ...  $\cos 6\theta \equiv (2\cos^2 \theta - 1)(16\cos^4 \theta - 16\cos^2 \theta + 1)$

Consider the solutions of the equation.

$$\cos 6\theta = 0, 0 \leq \theta \leq \pi.$$

c) By fully justifying each step in the workings, find the exact value of

$$\cos \frac{\pi}{12} \cos \frac{5\pi}{12} \cos \frac{7\pi}{12} \cos \frac{11\pi}{12}.$$

Handwritten solution for Question 10c:

(c)  $\cos 6\theta = 0$   
 $6\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \dots$   
 $\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \dots$

• Now  $2\cos^2 \theta - 1 = 0$  or  $4\cos^2 \theta - 4\cos \theta + 1 = 0$   
 $\cos^2 \theta = \frac{1}{2}$  or  $4\cos^2 \theta - 4\cos \theta + 1 = 0$   
 $\cos \theta = \pm \frac{1}{\sqrt{2}}$  or  $(2\cos \theta - 1)^2 = 0$   
 $\theta = \frac{\pi}{4}, \frac{7\pi}{4}, \dots$  or  $2\cos \theta - 1 = 0$   
 $\cos \theta = \frac{1}{2}$

• Finally  
 $\cos \frac{\pi}{12} \cos \frac{5\pi}{12} \cos \frac{7\pi}{12} \cos \frac{11\pi}{12} = \cos \frac{\pi}{12} \cos \frac{\pi}{12} (-\cos \frac{\pi}{12})(-\cos \frac{\pi}{12})$   
 $= (-\cos^2 \frac{\pi}{12})(-\cos^2 \frac{\pi}{12})$   
 $= \cos^4 \frac{\pi}{12}$   
 $= \left(\frac{1}{2}\right)^2 \times \left(\frac{1}{2}\right)^2 = \frac{1}{16}$

Alternative: All the solutions of  $4\cos^2 \theta - 4\cos \theta + 1 = 0$  occur at the same time.  
 So product of roots  $\cos \theta = \frac{1}{4}$  or  $\cos \theta = \frac{1}{2}$

Created by T. Madas

# COMPLEX LOCI

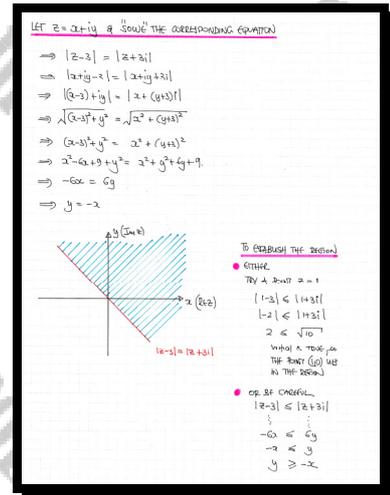
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**Question 1**

By finding a suitable Cartesian locus in the complex  $z$  plane, shade the region  $R$  that satisfies the inequality

$$|z - 3| \leq |z + 3i|$$

$$x + y \geq 0$$

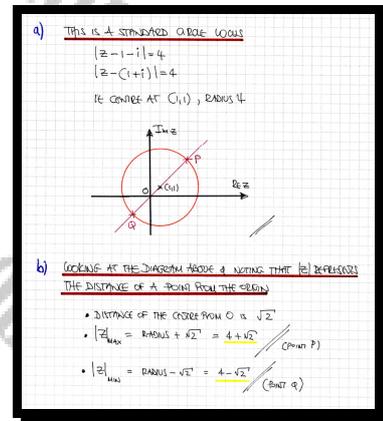


Question 2

$$|z-1-i|=4, z \in \mathbb{C}.$$

- a) Sketch the locus of the points that satisfy the above equation in a standard Argand diagram.
- b) Find the minimum and maximum values of  $|z|$  for points that lie on this locus.

$$z_{\min} = 4 - \sqrt{2}, \quad z_{\max} = 4 + \sqrt{2}$$



**Question 3**

The complex number  $z$  represents the point  $P(x, y)$  in the Argand diagram.

Given that

$$|z-1| = 2|z+2|,$$

show that the locus of  $P$  is given by

$$(x+3)^2 + y^2 = 4.$$

**proof**

$|z-1| = 2|z+2|$   
 let  $z = a+iy$   
 $|a+iy-1| = 2|a+iy+2|$   
 $|(a-1)+iy| = 2|(a+2)+iy|$   
 $\Rightarrow \sqrt{(a-1)^2 + y^2} = 2\sqrt{(a+2)^2 + y^2}$   
 $\Rightarrow (a-1)^2 + y^2 = 4(a+2)^2 + 4y^2$   
 $\Rightarrow a^2 - 2a + 1 + y^2 = 4a^2 + 16a + 16 + 4y^2$   
 $\Rightarrow 0 = 3a^2 + 3y^2 + 14a + 15$   
 $\Rightarrow 0 = 3a^2 + 6a + 15 + 3y^2$   
 $\Rightarrow (a+3)^2 + y^2 = 4$

**Question 4**

The complex number  $z = x+iy$  represents the point  $P$  in the complex plane.

Given that

$$\bar{z} = \frac{1}{z}, \quad z \neq 0$$

determine a Cartesian equation for the locus of  $P$ .

$$x^2 + y^2 = 1$$

$\bar{z} = \frac{1}{z}$   
 let  $z = a+iy$   
 $(a-iy) = \frac{1}{(a+iy)}$   
 $(a-iy)(a+iy) = 1$   
 $a^2 + y^2 = 1$   
 A UNIT CIRCLE  
 CENTRE AT (0,0)

**Question 5**

Sketch, on the same Argand diagram, the locus of the points satisfying each of the following equations.

a)  $|z - 3 + i| = 3$ .

b)  $|z| = |z - 2i|$ .

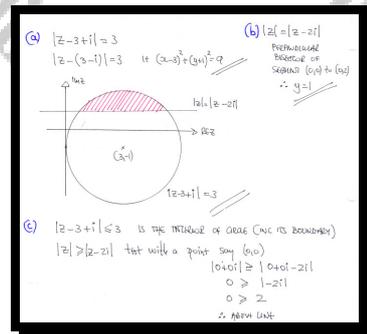
Give in each case a Cartesian equation for the locus.

c) Shade in the sketch the region that is satisfied by both these inequalities

$$|z - 3 + i| \leq 3$$

$$|z| \geq |z - 2i|$$

$$(x - 3)^2 + (y + 1)^2 = 9, \quad y = 1$$



Question 6

a) Sketch on the same Argand diagram the locus of the points satisfying each of the following equations.

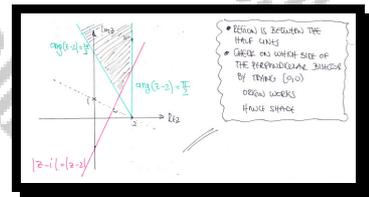
i.  $|z - i| = |z - 2|$ .

ii.  $\arg(z - 2) = \frac{\pi}{2}$ .

b) Shade in the sketch the region that is satisfied by both these inequalities

$$|z - i| \leq |z - 2| \quad \text{and} \quad \frac{\pi}{2} \leq \arg(z - 2) \leq \frac{2\pi}{3}$$

sketch



**Question 7**

The complex number  $z$  represents the point  $P(x, y)$  in the Argand diagram.

Given that

$$|z-1| = \sqrt{2}|z-i|,$$

show that the locus of  $P$  is a circle, stating its centre and radius.

$$(x+1)^2 + (y-2)^2 = 4, \quad (-1, 2), r=2$$

$|z-1| = \sqrt{2}|z-i|$   
 LET  $z = x+iy$   
 $\Rightarrow |x+iy-1| = \sqrt{2}|x+iy-i|$   
 $\Rightarrow \sqrt{(x-1)^2 + y^2} = \sqrt{2}\sqrt{x^2 + (y-1)^2}$   
 $\Rightarrow \sqrt{(x-1)^2 + y^2} = \sqrt{2}\sqrt{x^2 + (y-1)^2}$   
 $\Rightarrow (x-1)^2 + y^2 = 2(x^2 + (y-1)^2)$   
 $\Rightarrow x^2 - 2x + 1 + y^2 = 2x^2 + 2y^2 - 4y + 2$   
 $\Rightarrow 0 = x^2 + y^2 + 2x - 4y + 1$   
 $\Rightarrow x^2 + 2x + y^2 - 4y + 1 = 0$   
 $\Rightarrow (x+1)^2 + (y-2)^2 - 1 - 4 + 1 = 0$   
 $\Rightarrow (x+1)^2 + (y-2)^2 = 4$   
 CIRCLE  
 CENTRE  $(-1, 2)$   
 RADIUS 2

**Question 8**

$$|z-2i|=1, \quad z \in \mathbb{C}.$$

- In the Argand diagram, sketch the locus of the points that satisfy the above equation.
- Find the minimum value and the maximum value of  $|z|$ , and the minimum value and the maximum of  $\arg z$ , for points that lie on this locus.

$$|z|_{\min} = 1, \quad |z|_{\max} = 3, \quad \arg z_{\min} = \frac{\pi}{3}, \quad \arg z_{\max} = \frac{2\pi}{3}$$

(a)  $|z-2i|=1$   
 $|z-(0+2i)|=1$   
 CIRCLE CENTRE  $(0, 2)$   
 RADIUS 1  
 (b)  $|z|_{\min} = 1$  (DISTANCE OA)  
 $|z|_{\max} = 3$  (DISTANCE OB)  
 $\bullet$  SUB  $\frac{|z-2i|=1}{|z|}$   
 $\theta = \frac{\pi}{3}$   
 $\therefore \arg z_{\min} = \frac{\pi}{3}$   
 $\bullet$  SUB  $\frac{|z-2i|=1}{|z|}$   
 $\theta = \frac{2\pi}{3}$   
 $\therefore \arg z_{\max} = \frac{2\pi}{3}$  (BY SYMMETRY)

**Question 9**

The complex number  $z$  represents the point  $P(x, y)$  in the Argand diagram.

Given that

$$|z+1| = 2|z-2i|,$$

show that the locus of  $P$  is a circle and state its radius and the coordinates of its centre.

$$\left(\frac{1}{3}, \frac{8}{3}\right), \quad r = \frac{2}{3}\sqrt{5}$$

$|z+1| = 2|z-2i|$   
 $\Rightarrow |(x+1)+iy| = 2|x+iy-2i|$   
 $\Rightarrow |(x+1)+iy| = 2|x+(y-2)i|$   
 $\Rightarrow \sqrt{(x+1)^2 + y^2} = 2\sqrt{x^2 + (y-2)^2}$   
 $\Rightarrow (x+1)^2 + y^2 = 4(x^2 + (y-2)^2)$   
 $\Rightarrow x^2 + 2x + 1 + y^2 = 4x^2 + 4y^2 - 16y + 16$   
 $\Rightarrow 0 = 3x^2 - 2x + 3y^2 - 6y + 15$   
 $\Rightarrow x^2 - \frac{2}{3}x + y^2 - 2y + \frac{4}{3} = 0$   
 $\Rightarrow \left(x - \frac{1}{3}\right)^2 + \left(y - 1\right)^2 = \frac{5}{3}$   
 circle centre  $\left(\frac{1}{3}, 1\right)$  radius  $\frac{\sqrt{5}}{3}$

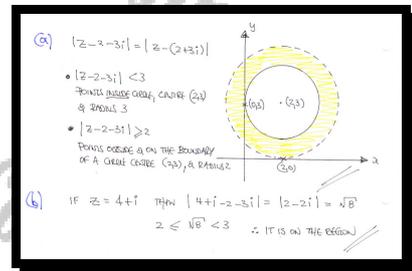
**Question 10**

The complex number  $z = x + iy$  satisfies the relationship

$$2 \leq |z - 2 - 3i| < 3.$$

- Shade **accurately** in an Argand diagram the region represented by the above relationship.
- Determine algebraically whether the point represents the number  $4 + i$  lies inside or outside this region.

inside the region



**Question 11**

Two sets of loci in the Argand diagram are given by the following equations

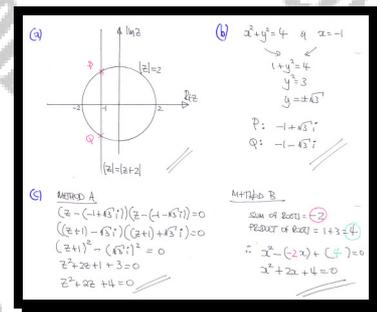
$$|z| = |z+2| \quad \text{and} \quad |z| = 2, \quad z \in \mathbb{C}.$$

- a) Sketch both these loci in the same Argand diagram.

The points  $P$  and  $Q$  in the Argand diagram satisfy both loci equations.

- b) Write the complex numbers represented by  $P$  and  $Q$ , in the form  $a+ib$ , where  $a$  and  $b$  are real numbers.  
 c) Find a quadratic equation with real coefficients, whose solutions are the complex numbers represented by the points  $P$  and  $Q$ .

$$z = -1 \pm \sqrt{3}i, \quad z^2 + 2z + 4 = 0$$



Question 12

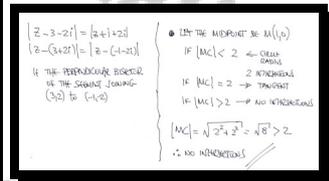
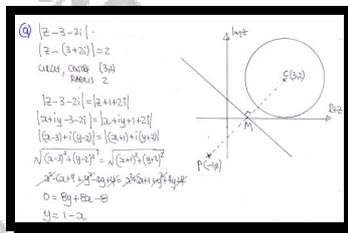
a) Sketch in the same Argand diagram the locus of the points satisfying each of the following equations

i.  $|z - 3 - 2i| = 2$ .

ii.  $|z - 3 - 2i| = |z + 1 + 2i|$ .

b) Show by a **geometric** calculation that no points lie on both loci.

proof



Question 13

The point  $A$  represents the complex number on the  $z$  plane such that

$$|z - 6i| = 2|z - 3|,$$

and the point  $B$  represents the complex number on the  $z$  plane such that

$$\arg(z - 6) = -\frac{3\pi}{4}.$$

- Show that the locus of  $A$  as  $z$  varies is a circle, stating its radius and the coordinates of its centre.
- Sketch, on the same  $z$  plane, the locus of  $A$  and  $B$  as  $z$  varies.
- Find the complex number  $z$ , so that the point  $A$  coincides with the point  $B$ .

$$C(4, -2), r = \sqrt{20}, \quad z = (4 - \sqrt{10}) + i(-2 - \sqrt{10})$$

(a)  $|z - 6i| = 2|z - 3|$   
 $\Rightarrow |x + iy - 6i| = 2|(x + iy) - 3|$   
 $\Rightarrow |x + i(y - 6)| = 2|(x - 3) + iy|$   
 $\Rightarrow \sqrt{x^2 + (y - 6)^2} = 2\sqrt{(x - 3)^2 + y^2}$   
 $\Rightarrow x^2 + (y - 6)^2 = 4[(x - 3)^2 + y^2]$   
 $\Rightarrow x^2 + y^2 - 12y + 36 = 4[x^2 - 6x + 9 + y^2]$   
 $\Rightarrow x^2 + y^2 - 12y + 36 = 4x^2 - 24x + 36 + 4y^2$   
 $\Rightarrow 0 = 3x^2 + 3y^2 + 12y - 24x$   
 $\Rightarrow x^2 + y^2 + 4y - 8x = 0$   
 $\Rightarrow (x - 4)^2 - 16 + (y + 2)^2 - 4 = 0$   
 $\Rightarrow (x - 4)^2 + (y + 2)^2 = 20$   
 (4, -2) CIRCLE CENTRE AT (4, -2) RADIUS  $\sqrt{20}$

(b)

GRABBIT OF THIS UNIT IS 1 A PASSING THROUGH (4, -2)

$$y - 0 = 1(x - 6)$$

$$y = x - 6 \quad (x \leq 6)$$

$$(y + 2)^2 + (x - 4)^2 = 20$$

$$(x - 6 + 2)^2 + (x - 4)^2 = 20$$

$$2(x - 4)^2 = 20$$

$$(x - 4)^2 = 10$$

$$x - 4 = \begin{matrix} \sqrt{10} \\ -\sqrt{10} \end{matrix}$$

$$x = \begin{matrix} 4 + \sqrt{10} \\ 4 - \sqrt{10} \end{matrix} > 6$$

$$\therefore x = 4 - \sqrt{10}$$

$$y = (4 - \sqrt{10}) - 6 = -2 - \sqrt{10}$$

$$\therefore (4 - \sqrt{10}) + i(-2 - \sqrt{10})$$

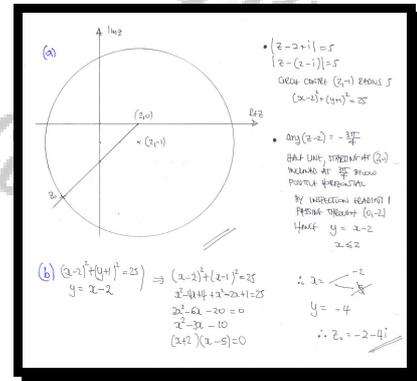
Question 14

$$|z - 2 + i| = 5.$$

$$\arg(z - 2) = -\frac{3\pi}{4}.$$

- a) Sketch each of the above complex loci in the same Argand diagram.
- b) Determine, in the form  $x + iy$ , the complex number  $z_0$  represented by the intersection of the two loci of part (a).

$$z_0 = -2 - 4i$$



**Question 15**

The locus of the point  $z$  in the Argand diagram, satisfy the equation

$$|z - 2 + i| = \sqrt{3}.$$

- a) Sketch the locus represented by the above equation.

The half line  $L$  with equation

$$y = mx - 1, \quad x \geq 0, \quad m > 0,$$

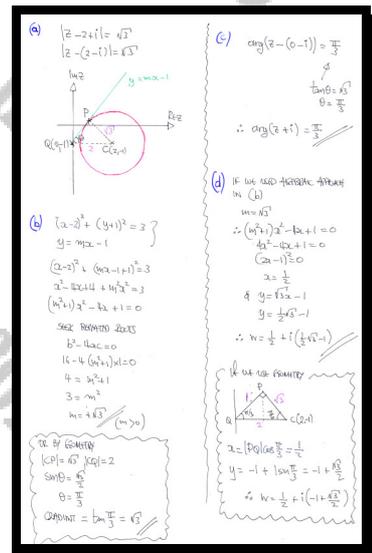
touches the locus described in part (a) at the point  $P$ .

- b) Find the value of  $m$ .
- c) Write the equation of  $L$ , in the form

$$\arg(z - z_0) = \theta, \quad z_0 \in \mathbb{C}, \quad -\pi < \theta \leq \pi.$$

- d) Find the complex number  $w$ , represented by the point  $P$ .

$$m = \sqrt{3}, \quad \arg(z + i) = \frac{\pi}{3}, \quad w = \frac{1}{2} + i \left( \frac{\sqrt{3}}{2} - 1 \right)$$



Question 16

The complex numbers  $z_1$  and  $z_2$  are given by

$$z_1 = 1 + i\sqrt{3} \quad \text{and} \quad z_2 = iz_1.$$

a) Label accurately the points representing  $z_1$  and  $z_2$ , in an Argand diagram.

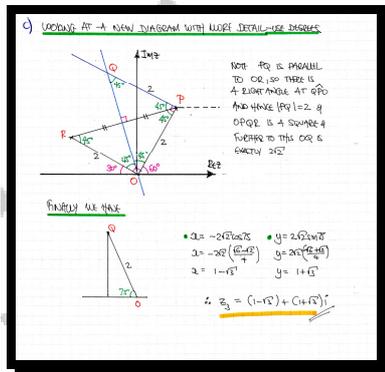
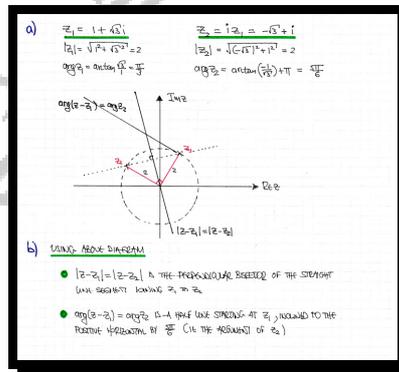
b) On the same Argand diagram, sketch the locus of the points  $z$  satisfying ...

i. ...  $|z - z_1| = |z - z_2|$ .

ii. ...  $\arg(z - z_1) = \arg z_2$ .

c) Determine, in the form  $x + iy$ , the complex number  $z_3$  represented by the intersection of the two loci of part (b).

$$z_3 = (1 - \sqrt{3}) + i(1 + \sqrt{3})$$



**Question 17**

The complex number  $z$  lies in the region  $R$  of an Argand diagram, defined by the inequalities

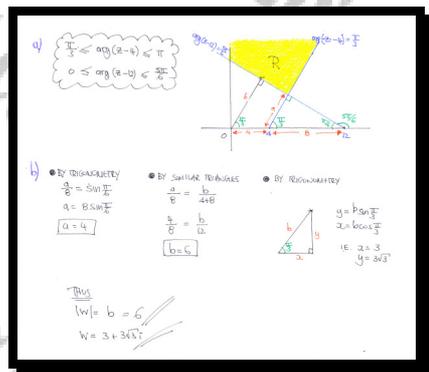
$$\frac{\pi}{3} \leq \arg(z-4) \leq \pi \quad \text{and} \quad 0 \leq \arg(z-12) \leq \frac{5\pi}{6}$$

- a) Sketch the region  $R$ , indicating clearly all the relevant details.

The complex number  $w$  lies in  $R$ , so that  $|w|$  is minimum.

- b) Find  $|w|$ , further giving  $w$  in the form  $u+iv$ , where  $u$  and  $v$  are real numbers.

$$|w| = 3, \quad w = 3 + 3\sqrt{3}i$$



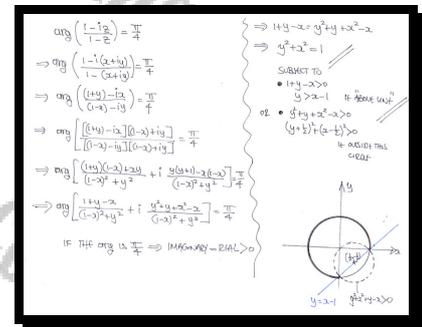
**Question 18**

The point  $P$  represents the number  $z = x + iy$  in an Argand diagram and further satisfies the equation

$$\arg\left(\frac{1-iz}{1-z}\right) = \frac{\pi}{4}, \quad z \neq -i.$$

Use an algebraic method to find an equation of the locus of  $P$  and sketch this locus accurately in an Argand diagram.

$$x^2 + y^2 = 1, \quad \text{such that } y > x - 1$$



**Question 19**

The complex number  $x+iy$  in the  $z$  plane of an Argand diagram satisfies the inequality

$$x^2 + y^2 + x > 0.$$

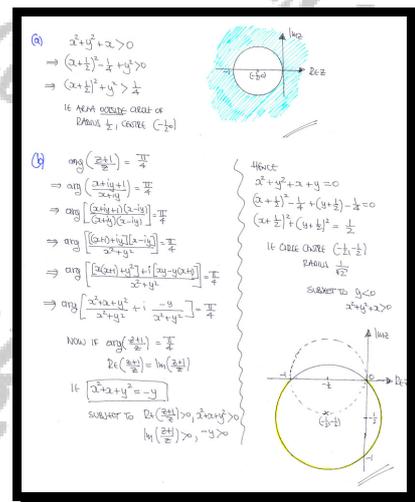
- a) Sketch the region represented by this inequality.

A locus in the  $z$  plane of an Argand diagram is given by the equation

$$\arg\left(\frac{z+1}{z}\right) = \frac{\pi}{4}.$$

- b) Sketch the locus represented by this equation.

sketch



**Question 20**

The complex number  $z$  satisfies the relationship

$$\arg(z-2) - \arg(z+2) = \frac{\pi}{4}$$

Show that the locus of  $z$  is a circular arc, stating ...

- ... the coordinates of its endpoints.
- ... the coordinates of its centre.
- ... the length of its radius.

$$\boxed{(-2,0), (2,0)}, \quad \boxed{(0,2)}, \quad \boxed{r = 2\sqrt{2}}$$

**GEOMETRICAL APPROACH**

$\arg(z-2) - \arg(z+2) = \frac{\pi}{4}$

•  $\theta - \phi = \frac{\pi}{4}$   
 $(\theta = \frac{\pi}{4} + \phi)$

• So  $z$  lies on the ARC OF A CIRCLE, whose CHORDS LIES BETWEEN  $(-2,0)$  &  $(2,0)$  AND INSIDE THE MAJOR SEGMENT

• CHORD MUST LIE ON THE  $y$ -axis (PERPENDICULAR BISECTOR OF THE CHORD)

• BY GEOMETRY THE CHORD IS AT  $(0,2)$  & RADIUS  $2\sqrt{2}$

**ALGEBRAIC APPROACH**

$\rightarrow \arg(z-2) - \arg(z+2) = \frac{\pi}{4}$

$\rightarrow \arg\left(\frac{z-2}{z+2}\right) = \frac{\pi}{4}$  &  $\Re\left(\frac{z-2}{z+2}\right) > 0$  &  $\Im\left(\frac{z-2}{z+2}\right) > 0$

$\rightarrow \arg\left(\frac{x+iy-2}{x+iy+2}\right) = \frac{\pi}{4}$

$\rightarrow \arg\left(\frac{(x-2)+iy}{(x+2)+iy}\right) = \frac{\pi}{4}$

$\rightarrow \arg\left[\frac{(x-2+iy)(x+2-iy)}{(x+2+iy)(x-2-iy)}\right] = \frac{\pi}{4}$

$\rightarrow \arg\left[\frac{(x-2)(x+2) + y^2 + i(y(x+2) - y(x-2))}{(x+2)^2 + y^2}\right] = \frac{\pi}{4}$

$\rightarrow \arg\left[\frac{x^2-4+y^2 + i(4y)}{(x+2)^2 + y^2}\right] = \frac{\pi}{4}$

$\rightarrow \arg\left[\frac{x^2+y^2-4 + i(4y)}{(x+2)^2 + y^2}\right] = \frac{\pi}{4}$

SINCE THE  $\arg\left(\frac{z-2}{z+2}\right) = \frac{\pi}{4} \rightarrow \Re(z) > 0$

$x^2 + y^2 - 4 = 4y$

$x^2 + y^2 - 4y = 4$

$x^2 + (y-2)^2 = 8$

SUBJECT TO

$x^2 + y^2 > 4$

&  $4y > 0$

$y > 0$

$\therefore$  CIRCULAR ARC FROM THE ORIGIN ON THE  $(y)$  AXIS,  $(y > 0)$  WHICH HAS POSITIVE  $x$  AND LIES OUTSIDE THE CIRCLE  $x^2 + y^2 = 4$

As Required

Created by T. Madas

# COMPLEX FUNCTIONS

Created by T. Madas

**Question 1**

A transformation from the  $z$  plane to the  $w$  plane is defined by the complex function

$$w = \frac{3-z}{z+1}, \quad z \neq -1.$$

The locus of the points represented by the complex number  $z = x + iy$  is transformed to the circle with equation  $|w| = 1$  in the  $w$  plane.

Find, in Cartesian form, an equation of the locus of the points represented by the complex number  $z$ .

$$\boxed{x = 1}$$

Handwritten solution for Question 1:

$$w = \frac{3-z}{z+1}$$

$$\Rightarrow |w| = \frac{|3-z|}{|z+1|}$$

$$\Rightarrow |3-z| = |z+1|$$

Let  $z = x + iy$

$$\Rightarrow |(3-x-iy)| = |(x+1+iy)|$$

$$\Rightarrow \sqrt{(3-x)^2 + (-y)^2} = \sqrt{(x+1)^2 + y^2}$$

$$\Rightarrow (3-x)^2 + y^2 = (x+1)^2 + y^2$$

$$\Rightarrow 9 - 6x + x^2 + y^2 = x^2 + 2x + 1 + y^2$$

$$\Rightarrow 8 - 8x = 0$$

$$\Rightarrow x = 1$$

**Question 2**

Find an equation of the locus of the points which lie on the half line with equation

$$\arg z = \frac{\pi}{4}, \quad z \neq 0$$

after it has been transformed by the complex function

$$w = \frac{1}{z}.$$

$$\boxed{\arg w = -\frac{\pi}{4}}$$

Handwritten solution for Question 2:

$$w = \frac{1}{z} \Rightarrow z = \frac{1}{w}$$

$$\Rightarrow \arg z = \arg\left(\frac{1}{w}\right)$$

$$\Rightarrow \frac{\pi}{4} = \arg 1 - \arg w$$

$$\Rightarrow \arg w = -\frac{\pi}{4}$$

if  $y = -x \quad x > 0$

**Question 3**

The complex function

$$w = \frac{1}{z-1}, \quad z \neq 1, z \in \mathbb{C}, z \neq 1$$

transforms the point represented by  $z = x + iy$  in the  $z$  plane into the point represented by  $w = u + iv$  in the  $w$  plane.

Given that  $z$  satisfies the equation  $|z| = 1$ , find a Cartesian locus for  $w$ .

$$u = -\frac{1}{2}$$

Handwritten solution showing the derivation of the Cartesian locus for  $w$  from the condition  $|z| = 1$ .

$$\begin{aligned} \therefore w &= \frac{1}{z-1} \\ \Rightarrow z-1 &= \frac{1}{w} \\ \Rightarrow z &= \frac{w+1}{w} \\ \Rightarrow |z| &= \left| \frac{w+1}{w} \right| \\ \Rightarrow 1 &= \frac{|w+1|}{|w|} \\ \Rightarrow |w| &= |w+1| \end{aligned} \quad \begin{aligned} \Rightarrow |u+iv| &= |(u+1)+iv| \\ \Rightarrow |u+iv|^2 &= |(u+1)+iv|^2 \\ \Rightarrow \sqrt{u^2+v^2} &= \sqrt{(u+1)^2+v^2} \\ \Rightarrow u^2+v^2 &= (u+1)^2+v^2 \\ \Rightarrow u^2-v^2 &= u^2+2u+1+v^2 \\ \Rightarrow 2u &= -1 \\ \Rightarrow u &= -\frac{1}{2} \quad \left( \text{+ the line } z=1 \right) \end{aligned}$$

Question 4

The complex function  $w = f(z)$  is given by

$$w = \frac{3-z}{z+1} \quad \text{where } z \in \mathbb{C}, \quad z \neq -1.$$

A point  $P$  in the  $z$  plane gets mapped onto a point  $Q$  in the  $w$  plane.

The point  $Q$  traces the circle with equation  $|w| = 3$ .

Show that the locus of  $P$  in the  $z$  plane is also a circle, stating its centre and its radius.

centre  $\left(-\frac{3}{2}, 0\right)$ , radius  $= \frac{3}{2}$

$$\bullet w = \frac{3-z}{z+1}$$

$$\Rightarrow |w| = \left| \frac{3-z}{z+1} \right|$$

$$\Rightarrow 3 = \frac{|3-z|}{|z+1|}$$

$$\Rightarrow 3|z+1| = |3-z|$$

$$\Rightarrow 3|(x+iy)+1| = |(3-x)-iy|$$

$$\Rightarrow 3\sqrt{(x+1)^2+y^2} = \sqrt{(3-x)^2+y^2}$$

$$\Rightarrow 3\sqrt{(x+1)^2+y^2} = \sqrt{(3-x)^2+y^2}$$

$$\Rightarrow 9\sqrt{(x+1)^2+y^2} = \sqrt{(3-x)^2+y^2}$$

$$\Rightarrow 9^2(x+1)^2 + 9^2y^2 = (3-x)^2 + y^2$$

$$\Rightarrow 81x^2 + 182x + 81y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 80x^2 + 188x + 80y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 79x^2 + 194x + 79y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 78x^2 + 200x + 78y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 77x^2 + 206x + 77y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 76x^2 + 212x + 76y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 75x^2 + 218x + 75y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 74x^2 + 224x + 74y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 73x^2 + 230x + 73y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 72x^2 + 236x + 72y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 71x^2 + 242x + 71y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 70x^2 + 248x + 70y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 69x^2 + 254x + 69y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 68x^2 + 260x + 68y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 67x^2 + 266x + 67y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 66x^2 + 272x + 66y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 65x^2 + 278x + 65y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 64x^2 + 284x + 64y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 63x^2 + 290x + 63y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 62x^2 + 296x + 62y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 61x^2 + 302x + 61y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 60x^2 + 308x + 60y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 59x^2 + 314x + 59y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 58x^2 + 320x + 58y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 57x^2 + 326x + 57y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 56x^2 + 332x + 56y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 55x^2 + 338x + 55y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 54x^2 + 344x + 54y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 53x^2 + 350x + 53y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 52x^2 + 356x + 52y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 51x^2 + 362x + 51y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 50x^2 + 368x + 50y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 49x^2 + 374x + 49y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 48x^2 + 380x + 48y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 47x^2 + 386x + 47y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 46x^2 + 392x + 46y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 45x^2 + 398x + 45y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 44x^2 + 404x + 44y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 43x^2 + 410x + 43y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 42x^2 + 416x + 42y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 41x^2 + 422x + 41y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 40x^2 + 428x + 40y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 39x^2 + 434x + 39y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 38x^2 + 440x + 38y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 37x^2 + 446x + 37y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 36x^2 + 452x + 36y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 35x^2 + 458x + 35y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 34x^2 + 464x + 34y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 33x^2 + 470x + 33y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 32x^2 + 476x + 32y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 31x^2 + 482x + 31y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 30x^2 + 488x + 30y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 29x^2 + 494x + 29y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 28x^2 + 500x + 28y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 27x^2 + 506x + 27y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 26x^2 + 512x + 26y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 25x^2 + 518x + 25y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 24x^2 + 524x + 24y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 23x^2 + 530x + 23y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 22x^2 + 536x + 22y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 21x^2 + 542x + 21y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 20x^2 + 548x + 20y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 19x^2 + 554x + 19y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 18x^2 + 560x + 18y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 17x^2 + 566x + 17y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 16x^2 + 572x + 16y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 15x^2 + 578x + 15y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 14x^2 + 584x + 14y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 13x^2 + 590x + 13y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 12x^2 + 596x + 12y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 11x^2 + 602x + 11y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 10x^2 + 608x + 10y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 9x^2 + 614x + 9y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 8x^2 + 620x + 8y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 7x^2 + 626x + 7y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 6x^2 + 632x + 6y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 5x^2 + 638x + 5y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 4x^2 + 644x + 4y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 3x^2 + 650x + 3y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 2x^2 + 656x + 2y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow x^2 + 662x + y^2 = 9 - 6x + x^2 + y^2$$

$$\Rightarrow 0 = 9 - 6x + 0 + 0$$

$$\Rightarrow 6x = 9$$

$$\Rightarrow x = \frac{3}{2}$$

$$\Rightarrow \text{INDICES + COEFF. CONSTANT} = \left(-\frac{3}{2}, 0\right)$$

$$\text{RADIUS} = \frac{3}{2}$$

**Question 5**

The general point  $P(x, y)$  which is represented by the complex number  $z = x + iy$  in the  $z$  plane, lies on the locus of

$$|z| = 1.$$

A transformation from the  $z$  plane to the  $w$  plane is defined by

$$w = \frac{z+3}{z+1}, \quad z \neq -1,$$

and maps the point  $P(x, y)$  onto the point  $Q(u, v)$ .

Find, in Cartesian form, the equation of the locus of the point  $Q$  in the  $w$  plane.

$$u = 2$$

Handwritten solution showing the derivation of the locus equation  $u = 2$  in the  $w$  plane.

Left side of the solution:

- $w = \frac{z+3}{z+1}$
- $\Rightarrow wz + w = z + 3$
- $\Rightarrow wz - z = 3 - w$
- $\Rightarrow z(w-1) = 3-w$
- $\Rightarrow z = \frac{3-w}{w-1}$
- $\Rightarrow |z| = \left| \frac{3-w}{w-1} \right|$
- $\Rightarrow 1 = \frac{|w-3|}{|w-1|}$
- $\Rightarrow |w-1| = |w-3|$

Right side of the solution:

- LET  $w = u + iv$
- $\Rightarrow |u+iv-1| = |u+iv-3|$
- $\Rightarrow |(u-1)+iv| = |(u-3)+iv|$
- $\Rightarrow \sqrt{(u-1)^2 + v^2} = \sqrt{(u-3)^2 + v^2}$
- $\Rightarrow (u-1)^2 + v^2 = (u-3)^2 + v^2$
- $\Rightarrow u^2 - 2u + 1 + v^2 = u^2 - 6u + 9 + v^2$
- $\Rightarrow 4u = 8$
- $\Rightarrow u = 2$
- (4 marks)

**Question 6**

The point  $P$  represented by  $z = x + iy$  in the  $z$  plane is transformed into the point  $Q$  represented by  $w = u + iv$  in the  $w$  plane, by the complex transformation

$$w = \frac{2z}{z-1}, \quad z \neq 1.$$

The point  $P$  traces a circle of radius 2, centred at the origin  $O$ .

Find a Cartesian equation of the locus of the point  $Q$ .

$$\left(u - \frac{8}{3}\right)^2 + v^2 = \frac{16}{9}$$

$z = \frac{u+iv}{u-iv}$   
 $|z|=2 \Rightarrow \frac{|u+iv|}{|u-iv|} = 2$   
 $\Rightarrow \frac{\sqrt{u^2+v^2}}{\sqrt{u^2+v^2}} = 2$   
 $\Rightarrow \frac{u^2+v^2}{u^2+v^2} = 4$   
 If circle centre  $(\frac{8}{3}, 0)$   
 Radius  $\frac{4}{3}$

**Question 7**

The complex numbers  $z = x + iy$  and  $w = u + iv$  are represented by the points  $P$  and  $Q$ , respectively, in separate Argand diagrams.

The two numbers are related by the equation

$$w = \frac{1}{z+1}, \quad z \neq -1.$$

If  $P$  is moving along the circle with equation

$$(x+1)^2 + y^2 = 4,$$

find in Cartesian form an equation of the locus of the point  $Q$ .

$$u^2 + v^2 = \frac{1}{4}$$

Handwritten solution for the locus of point  $Q$ :

- Given:  $(x+1)^2 + y^2 = 4$  (Circle:  $(-1, 0)$  radius 2)
- $\Rightarrow |z - (-1)| = 2$
- $\Rightarrow |z+1| = 2$
- Since  $w = \frac{1}{z+1}$
- $\Rightarrow z+1 = \frac{1}{w}$
- $\Rightarrow |z+1| = \left| \frac{1}{w} \right|$
- Therefore:  $2 = \frac{1}{|w|}$
- $\Rightarrow |w| = \frac{1}{2}$
- $\Rightarrow |u+iv| = \frac{1}{2}$
- $\Rightarrow \sqrt{u^2+v^2} = \frac{1}{2}$
- $\Rightarrow u^2+v^2 = \frac{1}{4}$  ✓



**Question 10**

The complex numbers  $z = x + iy$  and  $w = u + iv$  are represented by the points  $P$  and  $Q$ , respectively, in separate Argand diagrams.

The two numbers are related by the equation

$$w = \frac{1}{z}, \quad z \neq 0.$$

If  $P$  is moving along the circle with equation

$$x^2 + y^2 = 2,$$

find in Cartesian form an equation for the locus of the point  $Q$ .

$$u^2 + v^2 = \frac{1}{2}$$

Handwritten solution showing the derivation of the locus equation for point Q. The solution starts with the given equation  $x^2 + y^2 = 2$  and uses the relationship  $w = \frac{1}{z}$  to find the locus in terms of  $u$  and  $v$ .

$\bullet x^2 + y^2 = 2 \Leftrightarrow |z| = \sqrt{2}$   
 $\Rightarrow w = \frac{1}{z}$   
 $\Rightarrow |w| = \frac{1}{|z|}$   
 $\Rightarrow |w| = \frac{1}{\sqrt{2}}$   
 $\Rightarrow |u + iv| = \frac{1}{\sqrt{2}}$   
 $\Rightarrow \sqrt{u^2 + v^2} = \frac{1}{\sqrt{2}}$   
 $\Rightarrow u^2 + v^2 = \frac{1}{2}$

**ALTERNATIVE**  
 $w = \frac{1}{z}$   
 $\Rightarrow u + iv = \frac{1}{x + iy} = \frac{x - iy}{(x + iy)(x - iy)}$   
 $\Rightarrow u + iv = \frac{x - iy}{x^2 + y^2}$   
 $\Rightarrow u + iv = \frac{x - iy}{2}$   
 $\Rightarrow u + iv = \frac{x}{2} - i\frac{y}{2}$   
 $\Rightarrow \begin{cases} u = \frac{x}{2} \\ v = -\frac{y}{2} \end{cases} \Rightarrow \begin{cases} x = 2u \\ y = -2v \end{cases}$   
 $4u^2 = x^2$  AND  $4v^2 = y^2$   
 $\Rightarrow 4u^2 + 4v^2 = x^2 + y^2$   
 $\Rightarrow 4u^2 + 4v^2 = 2$   
 $\Rightarrow u^2 + v^2 = \frac{1}{2}$

**Question 11**

The complex numbers  $z = x + iy$  and  $w = u + iv$  are represented by the points  $P$  and  $Q$  on separate Argand diagrams.

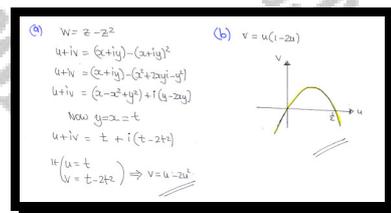
In the  $z$  plane, the point  $P$  is tracing the line with equation  $y = x$ .

The complex numbers  $z$  and  $w$  are related by

$$w = z - z^2.$$

- Find, in Cartesian form, the equation of the locus of  $Q$  in the  $w$  plane.
- Sketch the locus traced by  $Q$ .

$$v = u - 2u^2 \quad \text{or} \quad y = x - 2x^2$$



**Question 12**

The complex numbers  $z = x + iy$  and  $w = u + iv$  are represented by the points  $P$  and  $Q$  on separate Argand diagrams.

In the  $z$  plane, the point  $P$  is tracing the line with equation  $y = 2x$ .

Given that the complex numbers  $z$  and  $w$  are related by

$$w = z^2 + 1$$

find, in Cartesian form, the locus of  $Q$  in the  $w$  plane.

$$4u + 3v = 4 \quad \text{or} \quad 4x + 3y = 4$$

Handwritten solution showing the derivation of the locus equation  $4u + 3v = 4$  from the given relation  $w = z^2 + 1$  and the condition  $y = 2x$  in the  $z$ -plane.

$$\begin{aligned}
 w &= z^2 + 1 \\
 \Rightarrow u + iv &= (x + iy)^2 + 1 \\
 \Rightarrow u + iv &= x^2 + 2ixy - y^2 + 1 \\
 \Rightarrow u + iv &= (x^2 - y^2 + 1) + i(2xy) \\
 \text{Now } y &= 2x \\
 \Rightarrow u + iv &= (x^2 - 4x^2 + 1) + i(4x^2) \\
 \Rightarrow u + iv &= (1 - 3x^2) + 4ix^2 \\
 \text{is } \begin{cases} u = 1 - 3x^2 \\ v = 4x^2 \end{cases} & \quad \begin{cases} 3x^2 = 1 - u \\ 4x^2 = v \end{cases} \times 3 \\
 & \quad \begin{cases} 12x^2 = 3 - 4u \\ 12x^2 = 3v \end{cases} \\
 & \quad \therefore 3v = 4 - 4u \\
 & \quad 3v + 4u = 4
 \end{aligned}$$



Question 14

The complex function  $w = f(z)$  is given by

$$w = \frac{1}{z}, \quad z \in \mathbb{C}, \quad z \neq 0.$$

This function maps a general point  $P(x, y)$  in the  $z$  plane onto the point  $Q(u, v)$  in the  $w$  plane.

Given that  $P$  lies on the line with Cartesian equation  $y = 1$ , show that the locus of  $Q$  is given by

$$\left| w + \frac{1}{2}i \right| = \frac{1}{2}.$$

proof

Handwritten solution 1:

$$w = \frac{1}{z}$$

$$\Rightarrow z = \frac{1}{w}$$

$$\Rightarrow x+iy = \frac{1}{u+iv} \quad (\text{conjugate})$$

$$\Rightarrow x+iy = \frac{u-iv}{u^2+v^2}$$

$$\Rightarrow x+iy = \frac{u}{u^2+v^2} - i \frac{v}{u^2+v^2}$$

or  $y=1$

$$\therefore -\frac{v}{u^2+v^2} = 1$$

$$\Rightarrow u^2+v^2 = -v$$

$$\Rightarrow u^2+v^2+v = 0$$

$$\Rightarrow u^2 + (v+\frac{1}{2})^2 - \frac{1}{4} = 0$$

$$\Rightarrow u^2 + (v+\frac{1}{2})^2 = \frac{1}{4}$$

it's great circle  $(0, -\frac{1}{2})$   
radius  $\frac{1}{2}$

$$\therefore |w - (0 - \frac{1}{2}i)| = \frac{1}{2}$$

$$|w + \frac{1}{2}i| = \frac{1}{2}$$

as required

Handwritten solution 2:

ALTERNATIVE

- $P$  lies on  $y=1$
- $\therefore z = x+i$

$$\Rightarrow w = \frac{1}{x+i} \quad (\text{conjugate})$$

$$\Rightarrow w = \frac{x-i}{x^2+1}$$

$$\Rightarrow u+iv = \frac{x-i}{x^2+1}$$

if  $u = \frac{x}{x^2+1}$   
 $v = \frac{-1}{x^2+1}$

Divide both sides by  $x^2+1$  to eliminate

$$\Rightarrow \frac{u}{v} = -x$$

THIS  $v = -\frac{1}{x^2+1}$

$$\Rightarrow v = -\frac{1}{x^2+1}$$

Always obtain  $x^2+1$  by  $v$

$$\Rightarrow 1 = -\frac{v}{u^2+v^2}$$

$$\Rightarrow u^2+v^2 = -v$$

$$\Rightarrow u^2 + (v+\frac{1}{2})^2 - \frac{1}{4} = 0$$

$$\Rightarrow u^2 + (v+\frac{1}{2})^2 = \frac{1}{4}$$

- circle, centre  $(0, -\frac{1}{2})$  radius  $\frac{1}{2}$

$$\therefore |w - (0 - \frac{1}{2}i)| = \frac{1}{2}$$

$$\Rightarrow |w + \frac{1}{2}i| = \frac{1}{2}$$

as required



**Question 16**

A transformation of the  $z$  plane to the  $w$  plane is given by

$$w = \frac{1}{z-2}, \quad z \in \mathbb{C}, \quad z \neq 2$$

where  $z = x + iy$  and  $w = u + iv$ .

The line with equation

$$2x + y = 3$$

is mapped in the  $w$  plane onto a curve  $C$ .

- a) Show that  $C$  represents a circle and determine the coordinates of its centre and the size of its radius.

The points of a region  $R$  in the  $z$  plane are mapped onto the points which lie inside  $C$  in the  $w$  plane.

- b) Sketch and shade  $R$  in a suitable labelled Argand diagram, fully justifying the choice of region.

centre at  $\left(-1, \frac{1}{2}\right)$ , radius =  $\frac{\sqrt{5}}{2}$

**(a)**

$$w = \frac{1}{z-2} \Rightarrow w(z-2) = 1$$

$$\Rightarrow wz - 2w = 1$$

$$\Rightarrow wz = 2w + 1$$

$$\Rightarrow z = \frac{2w+1}{w}$$

$$\Rightarrow z = \frac{2(u+iv)+1}{u+iv} = \frac{(2u+1)+2iv}{u+iv}$$

$$\Rightarrow z = \frac{(2u+1)+2iv}{(u+iv)(u-iv)} = \frac{(2u+1)+2iv}{u^2+v^2}$$

$$\Rightarrow z = \frac{(2u+1)+2iv}{u^2+v^2} = \frac{2u+1}{u^2+v^2} + i \frac{2v}{u^2+v^2}$$

$$\Rightarrow x + iy = \frac{2u+1}{u^2+v^2} + i \frac{2v}{u^2+v^2}$$

$$\Rightarrow x = \frac{2u+1}{u^2+v^2}, \quad y = \frac{2v}{u^2+v^2}$$

$$\Rightarrow 2x + y = 3 \Rightarrow 2 \left( \frac{2u+1}{u^2+v^2} \right) + \frac{2v}{u^2+v^2} = 3$$

$$\Rightarrow \frac{4u+2+2v}{u^2+v^2} = 3$$

$$\Rightarrow 4u+2+2v = 3(u^2+v^2)$$

$$\Rightarrow 4u+2+2v = 3u^2+3v^2$$

$$\Rightarrow 3u^2+3v^2-4u-2v-2 = 0$$

$$\Rightarrow (u^2+v^2) - \frac{4}{3}u - \frac{2}{3}v - \frac{2}{3} = 0$$

$$\Rightarrow (u+\frac{2}{3})^2 + (v-\frac{1}{3})^2 = \frac{5}{3}$$

$\therefore$  circle, centre  $(-\frac{2}{3}, \frac{1}{3})$ , radius  $\frac{\sqrt{5}}{3}$

**(b)**

The required region is one of the two sides of the line  $2x+y=3$ .  
 If  $z=0$ ,  $w = \frac{1}{0-2} = -\frac{1}{2}$   
 which lies inside the circle.  
 $\therefore$  the region is in the  $w$  plane  $R$ ,  
 hence the choice of shading is correct.

**Question 17**

A transformation of the  $z$  plane to the  $w$  plane is given by

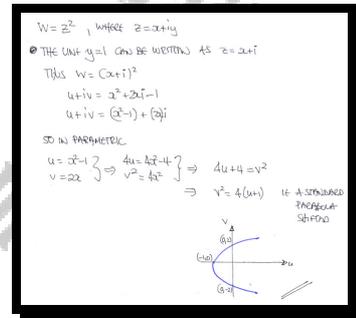
$$w = z^2, \quad z \in \mathbb{C},$$

where  $z = x + iy$  and  $w = u + iv$ .

The line with equation  $y = 1$  is mapped in the  $w$  plane onto a curve  $C$ .

Sketch the graph of  $C$ , marking clearly the coordinates of all points where the graph of  $C$  meets the coordinate axes.

sketch



## Question 18

A transformation of points from the  $z$  plane onto points in the  $w$  plane is given by the complex relationship

$$w = z^2, \quad z \in \mathbb{C},$$

where  $z = x + iy$  and  $w = u + iv$ .

Show that if the point  $P$  in the  $z$  plane lies on the line with equation

$$y = x - 1,$$

the locus of this point in the  $w$  plane satisfies the equation

$$v = \frac{1}{2}(u^2 - 1).$$

proof

Handwritten proof showing the derivation of the locus equation  $v = \frac{1}{2}(u^2 - 1)$  from the transformation  $w = z^2$  and the line  $y = x - 1$  in the  $z$ -plane.

Let  $z = x + iy$   
 $\Rightarrow w = z^2$   
 $\Rightarrow u + iv = (x + iy)^2$   
 $\Rightarrow u + iv = x^2 + 2xyi - y^2$   
 $\begin{cases} u = x^2 - y^2 \\ v = 2xy \end{cases}$   
 Now  $y = x - 1$   
 $\begin{cases} u = x^2 - (x - 1)^2 \\ v = 2x(x - 1) \end{cases}$

$\begin{cases} u = 2x - 1 \\ v = 2x^2 - 2x \quad (x^2) \end{cases}$   
 $2x = u + 1$   
 $2v = 4x^2 - 4x$   
 Hence eliminate  $x$   
 $\Rightarrow 2v = (u + 1)^2 - 2(u + 1)$   
 $\Rightarrow 2v = u^2 + 2u + 1 - 2u - 2$   
 $\Rightarrow 2v = u^2 - 1$   
 $\Rightarrow v = \frac{1}{2}(u^2 - 1)$  ✓

**Question 19**

A complex transformation from the  $z$  plane to the  $w$  plane is defined by

$$w = \frac{z+i}{3+iz}, \quad z \in \mathbb{C}, \quad z \neq 3i.$$

The point  $P(x, y)$  is mapped by this transformation into the point  $Q(u, v)$ .

It is further given that  $Q$  lies on the real axis for all the possible positions of  $P$ .

Show that the  $P$  traces the curve with equation

$$|z - i| = 2.$$

proof

$w = \frac{z+i}{3+iz}$   
 $\Rightarrow u+iv = \frac{z+i}{3+iz}$   
 $\Rightarrow u+iv = \frac{z+i}{(3-y)+iz}$   
 CONJUGATE RHS  
 $\Rightarrow u+iv = \frac{(z+i)(3-y-iz)}{(3-y)^2+z^2}$   
 $\Rightarrow u+iv = \frac{(x+iy)(3-y-iz)}{(3-y)^2+z^2}$

NOW Q NON-ON REAL AXIS  
 $\therefore v=0$   
 THIS  $(y+i)(3-y) - z^2 = 0$   
 $\Rightarrow 3y-iy^2+3-y^2-z^2=0$   
 $\Rightarrow 0=y^2-2y+z^2-3$   
 $\Rightarrow 0=z^2+(y-1)^2-4$   
 $\Rightarrow z^2+(y-1)^2=4$   
 it's a circle centre (0,1)  
 radius 2  
 $|z - (0+i)| = 2$   
 $|z - i| = 2$

TAKE ALTERNATE  
 $w = \frac{z+i}{3+iz}$   
 $\Rightarrow 3w+izw = z+i$   
 $\Rightarrow 3w-1 = z-izw$   
 $\Rightarrow 3w-1 = z(1-iw)$   
 $\Rightarrow z = \frac{3w-1}{1-iw}$   
 NOW W LIES ON REAL AXIS  
 $w = t+iv$   
 $w = t+0i$   
 $w = t$   
 $\Rightarrow z = \frac{3t-1}{1-it}$   
 $\Rightarrow z = \frac{(3t-1)(1+it)}{(1-it)(1+it)}$   
 $\Rightarrow z = \frac{3t+3t^2-i+it}{1+t^2}$   
 $\Rightarrow x+iy = \frac{3t+3t^2-i+it}{1+t^2}$

$x+iy = \frac{3t^2+3t-i+it}{1+t^2}$   
 $x = \frac{3t^2+3t}{1+t^2}$   
 $y = \frac{-1+t}{1+t^2}$

$x^2+y^2 = 3t^2-1$   
 $y+1 = 3t^2-y^2$   
 $y+1 = t^2(3-y)$   
 $t^2 = \frac{y+1}{3-y}$

$x^2 = \frac{16(\frac{y+1}{3-y})}{(1+\frac{y+1}{3-y})^2}$   
 $\Rightarrow x^2 = \frac{16(\frac{y+1}{3-y})}{(\frac{3-y+y+1}{3-y})^2}$   
 $\Rightarrow x^2 = \frac{16(\frac{y+1}{3-y})}{(\frac{4-y}{3-y})^2}$

MULTIPLY TOP BOTTOM BY  $(3-y)^2$   
 $\Rightarrow x^2 = \frac{16(y+1)(3-y)}{(4-y)^2}$   
 $\Rightarrow x^2 = \frac{3y^2-y^2+3-9}{16}$   
 $\Rightarrow x^2 = -y^2+2y+3$   
 $\Rightarrow x^2+y^2+2y = 3$   
 $\Rightarrow x^2+(y+1)^2 = 4$



**Question 21**

A transformation of the  $z$  plane to the  $w$  plane is given by

$$w = z + \frac{1}{z}, \quad z \in \mathbb{C}, \quad z \neq 0,$$

where  $z = x + iy$  and  $w = u + iv$ .

The locus of the points in the  $z$  plane that satisfy the equation  $|z| = 2$  are mapped in the  $w$  plane onto a curve  $C$ .

By considering the equation of the locus  $|z| = 2$  in exponential form, or otherwise, show that a Cartesian equation of  $C$  is

$$36u^2 + 100v^2 = 225.$$

proof

$|z| = 2$  can be written as  $z = 2e^{i\theta}$  in exponential form  
 so  
 $w = z + \frac{1}{z} = 2e^{i\theta} + \frac{1}{2e^{i\theta}} = 2e^{i\theta} + \frac{1}{2}e^{-i\theta}$   
 $= 2(\cos\theta + i\sin\theta) + \frac{1}{2}(\cos\theta - i\sin\theta) = \frac{5}{2}\cos\theta + \frac{3}{2}i\sin\theta$   
 so  $u + iv = \frac{5}{2}\cos\theta + \frac{3}{2}i\sin\theta$   
 $\left. \begin{array}{l} u = \frac{5}{2}\cos\theta \\ v = \frac{3}{2}\sin\theta \end{array} \right\} \Rightarrow \left. \begin{array}{l} \frac{2}{5}u = \cos\theta \\ \frac{2}{3}v = \sin\theta \end{array} \right\} \Rightarrow \left. \begin{array}{l} \cos\theta + i\sin\theta = 1 \\ \frac{4}{25}u^2 + \frac{4}{9}v^2 = 1 \end{array} \right\} \Rightarrow 36u^2 + 100v^2 = 225$   
 $\Rightarrow$  R4p(10)

Question 22

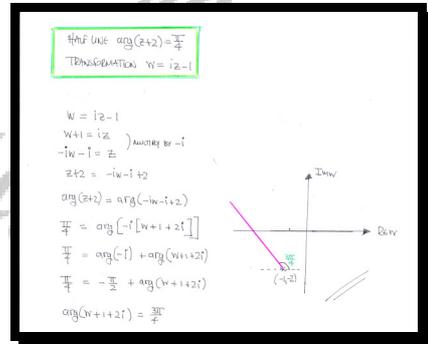
A transformation from the  $z$  plane to the  $w$  plane is defined by the equation

$$w = iz - 1, z \in \mathbb{C}.$$

Sketch in the  $w$  plane, in Cartesian form, the equation of the image of the half line with equation

$$\arg(z+2) = \frac{\pi}{4}, z \in \mathbb{C}.$$

graph



Question 23

A transformation from the  $z$  plane to the  $w$  plane is defined by the equation

$$f(z) = \frac{iz}{z-i}, \quad z \in \mathbb{C}.$$

Find, in Cartesian form, the equation of the image of straight line with equation

$$|z-i| = |z-2|, \quad z \in \mathbb{C}.$$

$$\left(u + \frac{2}{5}\right)^2 + \left(v - \frac{4}{5}\right)^2 = \frac{1}{5}$$

Solution:  $|z-i| = |z-2|$      $f(z) = w = \frac{iz}{z-i}$   
 $w = \frac{iz}{z-i}$   
 $wz - iw = iz$   
 $wz - iz = iw$   
 $z(w-i) = iw$   
 $z = \frac{iw}{w-i}$

Now  $|z-i| = |z-2|$   
 $\left| \frac{iw}{w-i} - i \right| = \left| \frac{iw}{w-i} - 2 \right|$   
 $\left| \frac{iw - iw + i^2}{w-i} \right| = \left| \frac{iw - 2w + 2i}{w-i} \right|$   
 $\left| \frac{-i(w-2i)}{w-i} \right| = \left| \frac{i(w-2i)}{w-i} \right|$   
 $|-1| = \left| \frac{i(w-2i)}{w-i} \right|$   
 $|i(w-2i+2i)| = 1$   
 $|w(i-2)+2i| = 1$

Now proceed by letting  $w = u+iv$  in the above equation  
 $|(-2+i)(u+iv) + 2i| = 1$   
 $|-2u - 2iv + iu - v + 2i| = 1$   
 $|(-2u-v) + i(-2v+u+2)| = 1$   
 $\sqrt{(-2u-v)^2 + (-2v+u+2)^2} = 1$   
 $4u^2 + v^2 + 4uv + 4v^2 + u^2 + 4u = -3$   
 $5u^2 + 5v^2 - 2v + 4u = -3$   
 $u^2 + v^2 - \frac{2}{5}v + \frac{4}{5}u = -\frac{3}{5}$   
 $(u + \frac{2}{5})^2 + (v - \frac{1}{5})^2 = \frac{1}{5}$   
 $\therefore$  circle centre  $(-\frac{2}{5}, \frac{1}{5})$  radius  $\frac{1}{\sqrt{5}}$

Alternatively:  
 $|w(-2+i) + 2i| = 1$   
 $|(C+i)(u + \frac{2i}{5})| = 1$   
 $|1+i||w + \frac{2i}{5}| = 1$   
 $\sqrt{2} \left| w + \frac{2i}{5} \right| = 1$   
 $\left| w + \frac{2i}{5} \right| = \frac{1}{\sqrt{2}}$

Question 24

The complex function  $w = f(z)$  is given by

$$w = \frac{1}{1-z}, \quad z \neq 1.$$

The point  $P(x, y)$  in the  $z$  plane traces the line with Cartesian equation

$$y + x = 1.$$

Show that the locus of the **image** of  $P$  in the  $w$  plane traces the line with equation

$$v = u.$$

proof

Handwritten proof steps:

- $w = \frac{1}{1-z}$
- $\Rightarrow 1-z = \frac{1}{w}$
- $\Rightarrow 1 - \frac{1}{w} = z$
- $\Rightarrow z = \frac{w-1}{w}$
- $\Rightarrow z = \frac{(u+iv)-1}{u+iv} = \frac{(u-1)+iv}{u+iv}$
- Cancel (1+i) RHS
- $\Rightarrow z = \frac{(u-1)+iv}{(u+iv)(u-iv)}$
- $\Rightarrow z = \frac{u(u-1)+v^2+i(v(u-1))}{u^2+v^2}$
- $\Rightarrow x+iy = \frac{u^2-v^2+u}{u^2+v^2} + i \frac{v}{u^2+v^2}$

Now  $y+x=1$

THIS

$$\frac{u^2-v^2+u}{u^2+v^2} + \frac{v}{u^2+v^2} = 1$$

$$u^2-v^2+u+v = u^2+v^2$$

$$u^2-v^2+u+v-u^2-v^2 = 0$$

$$-2v^2+u+v = 0$$

$$v = u$$

As required

Question 25

The complex function  $w = f(z)$  satisfies

$$w = \frac{1}{z}, \quad z \in \mathbb{C}, \quad z \neq 0.$$

This function maps the point  $P(x, y)$  in the  $z$  plane onto the point  $Q(u, v)$  in the  $w$  plane.

It is further given that  $P$  traces the curve with equation

$$\left| z + \frac{1}{2}i \right| = \frac{1}{2}.$$

Find, in Cartesian form, the equation of the locus of  $Q$ .

$$v = 1$$

SOEEL AL FOONEL

$$w = \frac{1}{z} \Rightarrow z = \frac{1}{w}$$

$$\Rightarrow z + \frac{1}{2}i = \frac{1}{w} + \frac{1}{2}i$$

$$\Rightarrow z + \frac{1}{2}i = \frac{z + \frac{1}{2}i}{w}$$

TRENS MOODU ON BOTH SIDES

$$\Rightarrow |z + \frac{1}{2}i| = \left| \frac{z + \frac{1}{2}i}{w} \right|$$

$$\Rightarrow \frac{1}{2} = \frac{|z + \frac{1}{2}i|}{|w|}$$

$$\Rightarrow |w| = |z + \frac{1}{2}i|$$

LET  $w = u + iv$

$$\Rightarrow |u + iv| = |z + \frac{1}{2}i|$$

$$\Rightarrow |u + iv| = |z + \frac{1}{2}i - \frac{1}{2}i|$$

$$\Rightarrow |u + iv| = |(z - \frac{1}{2}i) + \frac{1}{2}i|$$

$$\Rightarrow \sqrt{u^2 + v^2} = \sqrt{(x - \frac{1}{2})^2 + (y + \frac{1}{2})^2}$$

$$\Rightarrow x^2 + y^2 = 4 - 4xy + 4x^2$$

$$\Rightarrow 4v = 4$$

$$\Rightarrow v = 1$$

[OR  $y = 1$ ]

Question 26

$$z = \cos \theta + i \sin \theta, \quad -\pi < \theta \leq \pi.$$

a) Show clearly that

$$\frac{2}{1+z} = 1 - i \tan \frac{\theta}{2}.$$

The complex function  $w = f(z)$  is defined by

$$w = \frac{2}{1+z}, \quad z \in \mathbb{C}, \quad z \neq -1.$$

The circular arc  $|z|=1$ , for which  $0 \leq \arg z < \frac{\pi}{2}$ , is transformed by this function.

b) Sketch the image of this circular arc in a suitably labelled Argand diagram.

proof/sketch

(a) 
$$\frac{2}{1+z} = \frac{2}{1+(\cos \theta + i \sin \theta)} = \frac{2}{(\cos \theta + 1) + i \sin \theta}$$

$$= \frac{2(\cos \theta + 1 - i \sin \theta)}{[(\cos \theta + 1) + i \sin \theta][(\cos \theta + 1) - i \sin \theta]} = \frac{2(\cos \theta + 1) - 2i \sin \theta}{(\cos \theta + 1)^2 + \sin^2 \theta}$$

$$= \frac{2(\cos \theta + 1) - 2i \sin \theta}{\cos^2 \theta + 2\cos \theta + 1 + \sin^2 \theta} = \frac{2(\cos \theta + 1) - 2i \sin \theta}{2 + 2\cos \theta}$$

$$= \frac{2\cos \theta + 2}{2 + 2\cos \theta} - \frac{2i \sin \theta}{2 + 2\cos \theta} = 1 - i \frac{\sin \theta}{1 + \cos \theta}$$

$$= 1 - i \frac{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{1 + 2 \cos^2 \frac{\theta}{2} - 1} = 1 - i \frac{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{2 \cos^2 \frac{\theta}{2}} = 1 - i \tan \frac{\theta}{2} //$$

(b)  $|z|=1, \quad 0 \leq \arg z < \frac{\pi}{2}$   
 $z = \cos \theta + i \sin \theta, \quad 0 \leq \theta < \frac{\pi}{2}$   
 $\therefore w = 1 - i \tan \frac{\theta}{2}$   
 $u = 1$   
 $v = \tan \frac{\theta}{2}$  is PARAMETRIC EQUATIONS  
 $0 \leq \theta < \frac{\pi}{2}$

**Question 27**

The complex function with equation

$$f(z) = \frac{1}{z^2}, \quad z \in \mathbb{C}, \quad z \neq 0$$

maps the complex number  $x + iy$  from the  $z$  plane onto the complex number  $u + iv$  in the  $w$  plane.

The line with equation

$$y = mx, \quad x \neq 0,$$

is mapped onto the line with equation

$$v = Mu,$$

where  $m$  and  $M$  are the respective gradients of the two lines.

Given that  $m = M$ , determine the three possible values of  $m$ .

$$m = 0, \pm\sqrt{3}$$

Handwritten solution showing the mapping of the complex function  $f(z) = \frac{1}{z^2}$  from the  $z$  plane to the  $w$  plane. The solution involves expressing  $z = x + iy$  and  $w = u + iv$ , and then using the condition  $m = M$  to solve for  $m$ . The final result is  $m = 0, \pm\sqrt{3}$ .

**Question 28**

A complex transformation of points from the  $z$  plane onto points in the  $w$  plane is defined by the equation

$$w = z^2, \quad z \in \mathbb{C}.$$

The point represented by  $z = x + iy$  is mapped onto the point represented by  $w = u + iv$ .

Show that if  $z$  traces the curve with Cartesian equation

$$y^2 = 2x^2 - 1,$$

the locus of  $w$  satisfies the equation

$$v^2 = 4(u-1)(2u-1).$$

proof

The handwritten proof is as follows:

$$\begin{aligned} \bullet w &= z^2 \\ \Rightarrow (u+iv) &= (x+iy)^2 \\ \Rightarrow u+iv &= x^2+2xyi-y^2 \\ \Rightarrow \begin{cases} u = x^2-y^2 \\ v = 2xy \end{cases} & \text{subject to } y^2 = 2x^2-1 \\ \Rightarrow \begin{cases} u = x^2-(2x^2-1) \\ \Rightarrow v^2 = 4x^2 \end{cases} & \\ \Rightarrow \begin{cases} u = 1-x^2 \\ v^2 = 4x^2(2x^2-1) \end{cases} & \\ \Rightarrow \begin{cases} u = 1-x^2 \\ v^2 = 4(1-u)(2(1-u)-1) \end{cases} & \\ \Rightarrow \begin{cases} u = 1-x^2 \\ v^2 = 4(1-u)(2-2u-1) \end{cases} & \\ \Rightarrow \begin{cases} u = 1-x^2 \\ v^2 = 4(1-u)(1-2u) \end{cases} & \\ \Rightarrow \begin{cases} u = 1-x^2 \\ v^2 = 4(1-u)(2u-1) \end{cases} & \\ \Rightarrow v^2 = 4(u-1)(2u-1) & \text{is required} \end{aligned}$$

Question 29

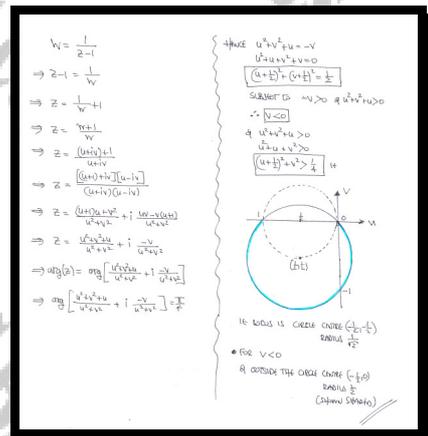
The complex function  $w = f(z)$  is defined by

$$w = \frac{1}{z-1}, \quad z \in \mathbb{C}, \quad z \neq 1.$$

The half line with equation  $\arg z = \frac{\pi}{4}$  is transformed by this function.

- Find a Cartesian equation of the locus of the **image** of the half line.
- Sketch the **image** of the locus in an Argand diagram.

$$\left(u + \frac{1}{2}\right)^2 + \left(v + \frac{1}{2}\right)^2 = \frac{1}{2}, \quad v < 0, \quad u^2 + v^2 + u > 0$$



Question 30

The complex function  $w = f(z)$  is defined by

$$w = \frac{3z+i}{1-z}, \quad z \in \mathbb{C}, \quad z \neq 1.$$

The half line with equation  $\arg z = \frac{3\pi}{4}$  is transformed by this function.

- Find a Cartesian equation of the locus of the **image** of the half line.
- Sketch the **image** of the locus in an Argand diagram.

$$(u+1)^2 + (v+1)^2 = 5, \quad v > \frac{1}{3}u + 1$$

The handwritten solution shows the following steps:

- Given:**  $w = \frac{3z+i}{1-z}$ , locus  $\arg z = \frac{3\pi}{4}$ .
- Let:**  $w = u+iv$ .
- Equation:**  $w-1 = \frac{3z+i}{1-z} - 1 = \frac{3z+i-1+z}{1-z} = \frac{4z-1+i}{1-z}$ .
- Substitution:**  $z = \frac{1-i}{\sqrt{2}}$  (since  $\arg z = \frac{3\pi}{4}$ ).
- Algebraic manipulation:**

$$w-1 = \frac{4\left(\frac{1-i}{\sqrt{2}}\right) - 1 + i}{1 - \frac{1-i}{\sqrt{2}}} = \frac{2\sqrt{2}(1-i) - 1 + i}{\sqrt{2} - 1 + i}$$
- Final Cartesian Equation:**  $(u+1)^2 + (v+1)^2 = 5$ .
- Domain:**  $v > \frac{1}{3}u + 1$ .
- Sketch:** An Argand diagram showing a circle centered at  $(-1, -1)$  with radius  $\sqrt{5}$ . A dashed line  $v = \frac{1}{3}u + 1$  is also shown. The locus is the part of the circle above this line.

Created by T. Madas

# COMPLEX SERIES

Created by T. Madas

**Question 1**

The following convergent series  $C$  and  $S$  are given by

$$C = 1 + \frac{1}{2} \cos \theta + \frac{1}{4} \cos 2\theta + \frac{1}{8} \cos 3\theta \dots$$

$$S = \frac{1}{2} \sin \theta + \frac{1}{4} \sin 2\theta + \frac{1}{8} \sin 3\theta \dots$$

a) Show clearly that

$$C + iS = \frac{2}{2 - e^{i\theta}}$$

b) Hence show further that

$$C = \frac{4 - 2 \cos \theta}{5 - 4 \cos \theta}$$

and find a similar expression for  $S$ .

$$S = \frac{2 \sin \theta}{5 - 4 \cos \theta}$$

(a)  $C + iS = 1 + \frac{1}{2}(e^{i\theta} + ie^{i2\theta}) + \frac{1}{4}(e^{i2\theta} + ie^{i4\theta}) + \frac{1}{8}(e^{i3\theta} + ie^{i6\theta}) + \dots$   
 $= 1 + \frac{1}{2}e^{i\theta} + \frac{1}{4}e^{i2\theta} + \frac{1}{8}e^{i3\theta} + \dots$   
 $= \left(1 + \frac{1}{2}e^{i\theta} + \left(\frac{1}{2}\right)^2 e^{i2\theta} + \left(\frac{1}{2}\right)^3 e^{i3\theta} + \dots\right)$   
 G.P. with  $a=1$   
 $r = \frac{e^{i\theta}}{2}$   
 $\Rightarrow \frac{1}{1 - \frac{e^{i\theta}}{2}} = \frac{2}{2 - e^{i\theta}}$

(b)  $C + iS = \frac{2}{2 - e^{i\theta}} = \frac{2(2 - e^{-i\theta})}{(2 - e^{i\theta})(2 - e^{-i\theta})} = \frac{2(2 - (2\cos\theta - isin\theta))}{4 - 2e^{i\theta}2e^{-i\theta} + 1}$   
 $= \frac{2(2 - 2\cos\theta + isin\theta)}{5 - 2(e^{i\theta} + e^{-i\theta})} = \frac{4 - 2\cos\theta + 2isin\theta}{5 - 4\cos\theta}$   
 $= \frac{(4 - 2\cos\theta) + i(2\sin\theta)}{5 - 4\cos\theta}$   
 $\therefore C = \frac{4 - 2\cos\theta}{5 - 4\cos\theta}$      $S = \frac{2\sin\theta}{5 - 4\cos\theta}$

**Question 2**

The following finite sums,  $C$  and  $S$ , are given by

$$C = 1 + 5 \cos 2\theta + 10 \cos 4\theta + 10 \cos 6\theta + 5 \cos 8\theta + \cos 10\theta$$

$$S = 5 \sin 2\theta + 10 \sin 4\theta + 10 \sin 6\theta + 5 \sin 8\theta + \sin 10\theta$$

By considering the binomial expansion of  $(1 + A)^5$ , show clearly that

$$C = 32 \cos^5 \theta \cos 5\theta,$$

and find a similar expression for  $S$

$$S = 32 \cos^5 \theta \sin 5\theta$$

$C = 1 + 5 \cos 2\theta + 10 \cos 4\theta + 10 \cos 6\theta + 5 \cos 8\theta + \cos 10\theta$   
 $S = 5 \sin 2\theta + 10 \sin 4\theta + 10 \sin 6\theta + 5 \sin 8\theta + \sin 10\theta$   
 THIS  
 $C + iS = 1 + 5e^{i2\theta} + 10e^{i4\theta} + 10e^{i6\theta} + 5e^{i8\theta} + e^{i10\theta}$   
 WHICH IS THE BINOMIAL EXPANSION:  
 $= (1 + e^{i2\theta})^5$   
 $= (1 + \cos 2\theta + i \sin 2\theta)^5$   
 $= (2 \cos^2 \theta - 1 + 2i \sin \theta \cos \theta)^5$   
 $= (2 \cos^2 \theta + 2i \sin \theta \cos \theta)^5$   
 $= [2 \cos^2 \theta (\cos \theta + i \sin \theta)]^5$   
 $= 32 \cos^6 \theta (\cos \theta + i \sin \theta)^5$   
 $= 32 \cos^6 \theta (\cos 5\theta + i \sin 5\theta)$   
 $= (32 \cos^6 \theta \cos 5\theta) + i(32 \cos^6 \theta \sin 5\theta)$   
 $\therefore C = 32 \cos^6 \theta \cos 5\theta$   
 $S = 32 \cos^6 \theta \sin 5\theta$

**Question 3**

The following convergent series  $S$  is given below

$$S = \sin \theta - \frac{1}{3} \sin 2\theta + \frac{1}{9} \sin 3\theta - \frac{1}{27} \sin 4\theta \dots$$

By considering the sum to infinity of a suitable geometric series involving the complex exponential function, show that

$$S = \frac{9 \sin \theta}{10 + 6 \cos \theta}$$

proof

$\sin \theta - \frac{1}{3} \sin 2\theta + \frac{1}{9} \sin 3\theta - \frac{1}{27} \sin 4\theta + \frac{1}{81} \sin 5\theta - \dots$

$C = \cos \theta - \frac{1}{3} \cos 2\theta + \frac{1}{9} \cos 3\theta - \frac{1}{27} \cos 4\theta + \dots$   
 $S = \sin \theta - \frac{1}{3} \sin 2\theta + \frac{1}{9} \sin 3\theta - \frac{1}{27} \sin 4\theta + \dots$

This  
 $C + iS = [\cos \theta + i \sin \theta] - \frac{1}{3} [\cos 2\theta + i \sin 2\theta] + \frac{1}{9} [\cos 3\theta + i \sin 3\theta] - \frac{1}{27} [\cos 4\theta + i \sin 4\theta] + \dots$   
 $C + iS = e^{i\theta} - \frac{1}{3} e^{i2\theta} + \frac{1}{9} e^{i3\theta} - \frac{1}{27} e^{i4\theta} + \dots$

This is a geometric progression with first term  $e^{i\theta}$  of common ratio  $(-\frac{1}{3}e^{i\theta})$

$\sum_{n=0}^{\infty} a r^n = \frac{a}{1-r}$   
 $\sum_{n=0}^{\infty} e^{i(n+1)\theta} (-\frac{1}{3})^n = \frac{e^{i\theta}}{1 - (-\frac{1}{3}e^{i\theta})} = \frac{3e^{i\theta}}{3 + e^{i\theta}} = \frac{3e^{i\theta}(3 + e^{-i\theta})}{(3 + e^{i\theta})(3 + e^{-i\theta})} = \frac{9e^{i\theta} + 3}{9 + 3e^{i\theta} + 3e^{-i\theta} + 1}$   
 $= \frac{9(\cos \theta + i \sin \theta) + 3}{10 + 6 \cos \theta} = \frac{[9 \cos \theta + 3] + i[9 \sin \theta]}{10 + 6 \cos \theta}$

The required part is the imaginary part of the expression, i.e.  $\sum_{n=1}^{\infty} (-\frac{1}{3})^{n-1} \sin n\theta = \frac{9 \sin \theta}{10 + 6 \cos \theta}$

**Question 4**

The sum  $C$  is given below

$$C = 1 - \binom{n}{1} \cos \theta \cos \theta + \binom{n}{2} \cos^2 \theta \cos 2\theta - \binom{n}{3} \cos^3 \theta \cos 3\theta + \dots + (-1)^n \cos^n \theta \cos n\theta$$

Given that  $n \in \mathbb{N}$  determine the 4 possible expressions for  $C$ .

Give the answers in exact simplified form.

$$n = 4k, k \in \mathbb{N} : C = \cos n\theta \sin^n \theta$$

$$n = 4k + 1, k \in \mathbb{N} : C = \sin n\theta \sin^n \theta$$

$$n = 4k + 2, k \in \mathbb{N} : C = -\cos n\theta \sin^n \theta$$

$$n = 4k + 3, k \in \mathbb{N} : C = -\sin n\theta \sin^n \theta$$

Handwritten solution showing the derivation of the sum  $C$  using complex numbers and binomial expansion. The derivation starts with the expression for  $C$  and uses the binomial theorem to expand  $(1 - e^{i2\theta})^n$ . It then separates the real and imaginary parts to find the final expression for  $C$  based on the value of  $n$  modulo 4.

$$C = 1 - \binom{n}{1} \cos \theta \cos \theta + \binom{n}{2} \cos^2 \theta \cos 2\theta - \binom{n}{3} \cos^3 \theta \cos 3\theta + \dots + (-1)^n \cos^n \theta \cos n\theta$$

$$S = -\binom{n}{1} \cos \theta \sin \theta + \binom{n}{2} \cos^2 \theta \sin 2\theta - \binom{n}{3} \cos^3 \theta \sin 3\theta + \dots + (-1)^n \cos^n \theta \sin n\theta$$

$$C + iS = 1 - \binom{n}{1} \cos \theta (\cos \theta + i \sin \theta) + \binom{n}{2} \cos^2 \theta (\cos 2\theta + i \sin 2\theta) - \binom{n}{3} \cos^3 \theta (\cos 3\theta + i \sin 3\theta) + \dots + (-1)^n \cos^n \theta (\cos n\theta + i \sin n\theta)$$

$$= 1 - \binom{n}{1} e^{i\theta} \cos \theta + \binom{n}{2} e^{i2\theta} \cos^2 \theta - \binom{n}{3} e^{i3\theta} \cos^3 \theta + \dots + (-1)^n e^{in\theta} \cos^n \theta$$

$$= (1 - e^{i2\theta})^n = (1 - \cos 2\theta - i \sin 2\theta)^n = (1 - \cos 2\theta - i \sin 2\theta)^n$$

$$= (\cos^2 \theta - i \sin 2\theta)^n = \cos^n \theta (\cos \theta - i \sin \theta)^n = (-1)^n \cos^n \theta (\cos \theta + i \sin \theta)^n$$

$$= (-1)^n \cos^n \theta e^{in\theta} = (-1)^n \cos^n \theta (\cos n\theta + i \sin n\theta)$$

• If  $n = 4k, k \in \mathbb{N} \Rightarrow (-1)^n = 1 \Rightarrow C + iS = \cos^n \theta \cos n\theta + i \sin^n \theta \sin n\theta \Rightarrow C = \cos^n \theta \cos n\theta$   
 • If  $n = 4k + 1, k \in \mathbb{N} \Rightarrow (-1)^n = -1 \Rightarrow C + iS = \sin^n \theta \sin n\theta - i \cos^n \theta \cos n\theta \Rightarrow C = \sin^n \theta \sin n\theta$   
 • If  $n = 4k + 2, k \in \mathbb{N} \Rightarrow (-1)^n = 1 \Rightarrow C + iS = -\cos^n \theta \cos n\theta - i \sin^n \theta \sin n\theta \Rightarrow C = -\cos^n \theta \cos n\theta$   
 • If  $n = 4k + 3, k \in \mathbb{N} \Rightarrow (-1)^n = -1 \Rightarrow C + iS = -\sin^n \theta \sin n\theta + i \cos^n \theta \cos n\theta \Rightarrow C = -\sin^n \theta \sin n\theta$

**Question 5**

The following convergent series  $S$  is given below

$$S = \frac{\sin \theta}{1!} - \frac{\sin 2\theta}{2!} + \frac{\sin 3\theta}{3!} - \frac{\sin 4\theta}{4!} + \dots$$

By considering the sum to infinity of a suitable series involving the complex exponential function, show that

$$S = e^{-\cos \theta} \sin(\sin \theta).$$

proof

$$S = \frac{\sin \theta}{1!} - \frac{\sin 2\theta}{2!} + \frac{\sin 3\theta}{3!} - \frac{\sin 4\theta}{4!} + \dots$$

$$C = \frac{\cos \theta}{1!} - \frac{\cos 2\theta}{2!} + \frac{\cos 3\theta}{3!} - \frac{\cos 4\theta}{4!} + \dots$$

$$C + iS = \frac{1}{1!}(\cos \theta + i \sin \theta) - \frac{1}{2!}(\cos 2\theta + i \sin 2\theta) + \frac{1}{3!}(\cos 3\theta + i \sin 3\theta) - \frac{1}{4!}(\cos 4\theta + i \sin 4\theta) + \dots$$

$$= \frac{1}{1!} e^{-i\theta} - \frac{1}{2!} e^{-2i\theta} + \frac{1}{3!} e^{-3i\theta} - \frac{1}{4!} e^{-4i\theta} + \dots$$

$$= e^{-i\theta} = 1 - i\theta - \frac{\theta^2}{2!} + \frac{i\theta^3}{3!} - \frac{\theta^4}{4!} + \dots$$

$$1 - e^{-i\theta} = i\theta - \frac{\theta^2}{2!} + \frac{i\theta^3}{3!} - \frac{\theta^4}{4!} + \dots$$

$$= 1 - e^{-i\theta} = 1 - e^{-(\cos \theta + i \sin \theta)} = 1 - [e^{-\cos \theta} \times e^{-i \sin \theta}] = 1 - e^{-\cos \theta} \times e^{-i \sin \theta}$$

$$= 1 - e^{-\cos \theta} [\cos(\sin \theta) - i \sin(\sin \theta)] = [1 - e^{-\cos \theta} \cos(\sin \theta)] + i [e^{-\cos \theta} \sin(\sin \theta)]$$

∴ AS WE REQUIRE THE IMAGINARY PART, THE ANSWER IS  $\frac{-\cos \theta}{e^{-\cos \theta}} \sin(\sin \theta)$