

Created by T. Madas

ADVANCED MENSURATION

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Question 1 ()**

A water tank has a square base of length 48 cm and height 35 cm.

It is filled with water to a height of 25 cm.

When a solid sphere is placed in the tank the water level rises by π cm.

Assuming the sphere is fully submerged determine the radius of the sphere.

$$r = 12 \text{ cm}$$

The diagram shows a rectangular tank with a square base of side length 48 cm and a total height of 35 cm. The initial water level is 25 cm high. When a sphere is placed in the tank, the water level rises to 35 cm. The diagram also shows the sphere with radius r .

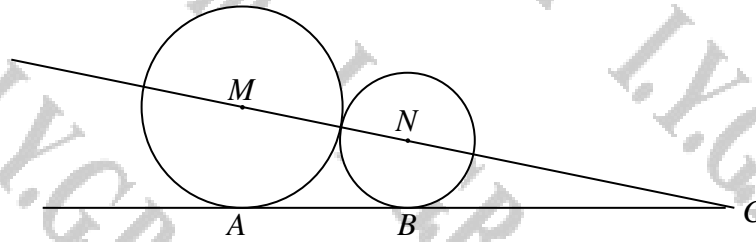
The solution is as follows:

- THE VOLUMES OF 10 & 25 ARE EQUAL
- VOLUME OF WATER DISPLACED MUST EQUAL THE VOLUME OF THE SPHERE

$$48 \times 48 \times (35 - 25) = \frac{4}{3} \pi r^3$$
$$48 \times 48 = \frac{4}{3} \pi r^3$$
$$\frac{3 \times 48 \times 48}{4} = \pi r^3$$
$$3 \times 12 \times 48 = \pi r^3$$
$$3 \times 12 \times 12 \times 4 = \pi r^3$$
$$12 \times 12 \times 12 = \pi r^3$$
$$r = 12$$

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Question 2 (**+)



The figure above shows two circles, with centres at M and N , with respective radii 7 cm and 3 cm touch each other. The points M , N and C lie in a straight line and the straight line ABC is a common tangent to the circles.

Determine the length of MC .

$$|MC| = \frac{35}{2}$$

• LET $|AC| = x$
 $\frac{7}{x} = \frac{3-10}{2}$
 $2y = 7x - 70$
 $70 = 4y$
 $45 = 2y$
 $y = \frac{35}{2}$

Question 3 (***)

A circular cylinder and a sphere both have radius r cm.

The total surface area of the cylinder is twice as large as the surface area of the sphere.

Determine the ratio of the volume of the cylinder to the volume of the sphere.

9:4

The handwritten solution is as follows:

Diagram of a cylinder with radius r and height h . The surface area is labeled C . Diagram of a sphere with radius r . The surface area is labeled S .

Surface of C = 2
Surface of S

$$\frac{(\pi r^2) + (2\pi r h)}{4\pi r^2} = 2$$
$$\Rightarrow \frac{2\pi r^2 + 2\pi r h}{4\pi r^2} = 2$$
$$\Rightarrow \frac{2\pi r^2 + 2\pi r h}{4\pi r^2} = 2$$
$$\Rightarrow 2\pi r^2 + 2\pi r h = 8\pi r^2$$
$$\Rightarrow r + h = 4r$$
$$\Rightarrow h = 3r$$

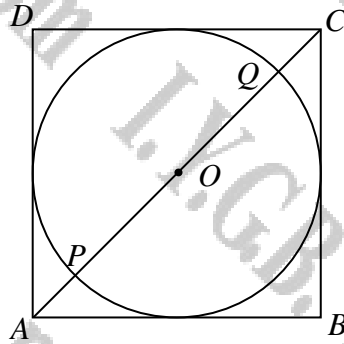
Volume of $C = \pi r^2 h$
 $= \pi r^2 (3r)$
 $= 3\pi r^3$

Volume of $S = \frac{4}{3}\pi r^3$

$$\therefore \frac{\text{Volume of } C}{\text{Volume of } S} = \frac{3\pi r^3}{\frac{4}{3}\pi r^3}$$
$$= \frac{3}{\frac{4}{3}}$$
$$= \frac{9}{4}$$

$\therefore 9:4$

Question 4 (****)

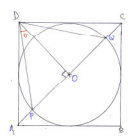


The figure above shows a circle with centre at O , inscribed in a square $ABCD$. The diagonal AC meets the circle at the points P and Q .


- a) Show clearly that $\angle PDQ = \arccos \frac{1}{3}$.
- b) Given that the triangle PDQ is of unit area, determine the exact area of the triangle APD .

$$\text{area} = \frac{1}{2}(\sqrt{2} - 1)$$

a)



- let the radius of the circle be r
- $|AB| = |BC| = 2r$, so by Pythagoras $|AC| = \sqrt{(2r)^2 + (2r)^2} = \sqrt{8r^2} = 2\sqrt{2}r$
- since $|OC| = \sqrt{2}r$, so looking at $\triangle OQC$, $\frac{1}{2}\pi = \frac{|OC|}{|AC|} = \frac{r}{\sqrt{2}r} = \frac{1}{\sqrt{2}}$



So $\sin \alpha = \frac{1}{\sqrt{2}}$
 $\alpha = \frac{\pi}{4}$

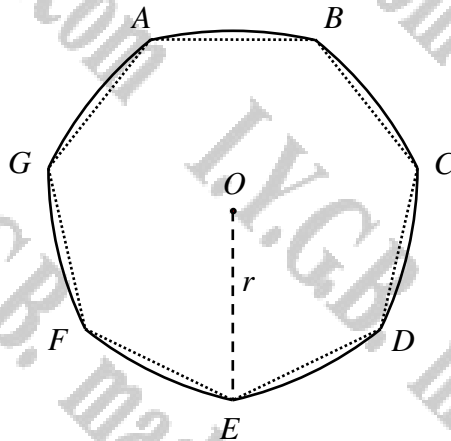
- Now let $\alpha = \angle PDQ$
- $\Rightarrow \alpha = 2\theta$
- $\Rightarrow \cos \alpha = \cos 2\theta$
- $\Rightarrow \cos \alpha = 2\cos^2 \theta - 1$
- $\Rightarrow \cos \alpha = 2\left(\frac{1}{\sqrt{2}}\right)^2 - 1$
- $\Rightarrow \cos \alpha = \frac{1}{2} - 1 = -\frac{1}{2}$
- $\Rightarrow \alpha = \angle PDQ = \arccos \left(-\frac{1}{2}\right)$

b)

Now Area $\triangle PDQ$ is 1
 Area $\triangle OQC$ is $\frac{1}{2}$
 $\frac{1}{2}|PQ||OC| = \frac{1}{2}$
 $|PQ||OC| = 1$
 $r \times \sqrt{2}r = 1$
 $r^2 = \frac{1}{\sqrt{2}}$

But the area of $\triangle APD$ is
 $\frac{1}{2} \times (2r)^2 = 2r^2 = 2 \times \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$
 Hence the required area is $\frac{1}{2}(\sqrt{2} - 1)$

Question 5 (*****)



The figure above shows a Reuleaux heptagon, $ABCDEFG$, constructed as follows.

Firstly a regular heptagon $ABCDEFG$ with centre at O and radius r is constructed. This is shown dotted in the figure.

A circular arc \widehat{AB} is drawn with centre at E and radius EA . A second circular arc \widehat{BC} is drawn with centre at F and radius FB .

A third circular arc \widehat{CD} is drawn with centre at G and radius GC and the process is repeated, forming a curved heptagon known as a Reuleaux heptagon.

Show that the area of this Reuleaux heptagon is

$$r^2 \left[2\pi \cos^2\left(\frac{\pi}{14}\right) - \sin\left(\frac{\pi}{7}\right) \right]$$

, proof

• LOCATING AT ONE SECTOR OF THE HEPTAGON
 HAD SCALE OF THE ANGLE IN THE DIAPHRAM
 $\angle AOB = \frac{2\pi}{7}$
 $A_{\text{sector}} = \frac{1}{2} \times \frac{2\pi}{7} = \frac{\pi}{7}$ (ORAL THEOREM)

• NOW USE HAD
 $H = R = 2 \times \frac{R}{2} = 2 \times R \cos \frac{\pi}{7}$

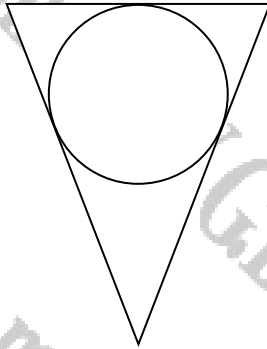
• AREA OF SECTOR ABC
 $\frac{1}{2} R^2 \left(\frac{2\pi}{7}\right) = \frac{1}{2} (2R \cos \frac{\pi}{7})^2 \frac{\pi}{7}$
 $= \frac{2}{7} \pi R^2 \cos^2 \frac{\pi}{7}$

• AREA OF TRIANGLE ABC
 $\frac{1}{2} |AB| |OC| \sin \frac{\pi}{7} = \frac{1}{2} R \cdot R \sin \frac{\pi}{7}$
 $= \frac{1}{2} \times 2R \cos \frac{\pi}{7} \times R \sin \frac{\pi}{7}$
 $= R^2 \cos \frac{\pi}{7} \sin \frac{\pi}{7}$

• "RECESS" REGION = $\frac{2}{7} \pi R^2 \cos^2 \frac{\pi}{7} - 2 \times \left[\frac{1}{2} R^2 \cos \frac{\pi}{7} \sin \frac{\pi}{7} \right]$
 $= \frac{2}{7} \pi R^2 \cos^2 \frac{\pi}{7} - R^2 \sin \frac{\pi}{7} \cos \frac{\pi}{7}$

• REULEUX AREA = $7 \left(\frac{2}{7} \pi R^2 \cos^2 \frac{\pi}{7} - R^2 \sin \frac{\pi}{7} \cos \frac{\pi}{7} \right)$
 $= R^2 \left[2\pi \cos^2 \frac{\pi}{7} - \sin \frac{\pi}{7} \right]$

Question 6 (*****)



The figure above shows the cross sectional view of a solid sphere that **just fits** inside a right circular conical shell of radius 6 cm and height h cm.

If the sphere occupies $\frac{3}{8}$ of the volume of the conical shell determine the two possible values of h .

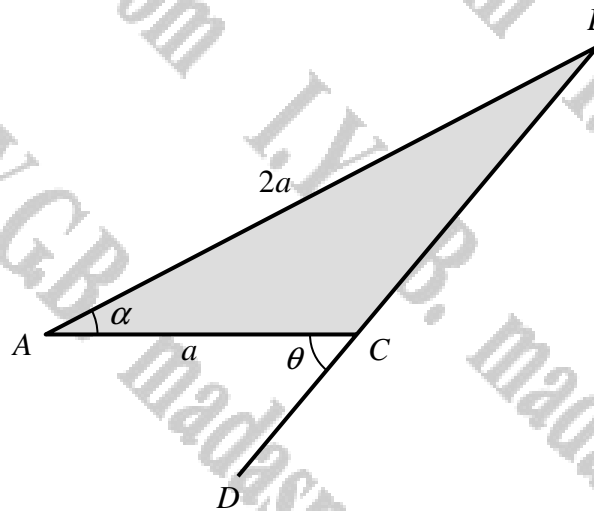
$$h = 8, 24\sqrt{3}$$

\bullet BY SIMILAR TRIANGLES
 $\frac{6}{r} = \frac{h}{\sqrt{h^2 - r^2}}$
 $\Rightarrow \frac{6}{\sqrt{h^2 - r^2}} = \frac{h}{\sqrt{h^2 - r^2}}$
 $\Rightarrow \frac{36}{(h^2 - r^2)} = \frac{h^2}{h^2 - r^2}$ $y \neq 0$
 $\Rightarrow \frac{36}{h^2 - r^2} = \frac{h^2}{h^2 - r^2}$
 $\Rightarrow \frac{36}{h^2 - r^2} = \frac{h^2}{h^2 - r^2}$
 $\Rightarrow 36y - 36r = y^2 + 36r$
 $\Rightarrow 36y - y^2 = r^2 + 36r$
 $\Rightarrow y(36 - y) = r^2 + 36r$
 $\Rightarrow y = \frac{r^2 + 36r}{36 - r^2}$

\bullet Now $\frac{V_{\text{sphere}}}{V_{\text{cone}}} = \frac{3}{8}$
 $\Rightarrow \frac{\frac{4}{3}\pi r^3}{\frac{1}{3}\pi r^2 h} = \frac{3}{8}$
 $\Rightarrow \frac{4r^3}{3\pi r^2 h} = \frac{3}{8}$
 $\Rightarrow \frac{4r}{3h} = \frac{3}{8}$
 $\Rightarrow \frac{4r}{3} = \frac{3h}{8}$
 $\Rightarrow \frac{4r}{3} = \frac{3h}{8}$
 $\Rightarrow \frac{4r}{3} = \frac{3h}{8}$
 $\Rightarrow y = \frac{r^2 + 36r}{36 - r^2}$

Thus $\frac{r^2 + 36r + 36r}{36 - r^2} = \frac{r^2 + 36r}{36 - r^2}$ $(r \neq 0)$
 $\Rightarrow \frac{r^2 + 36r}{36 - r^2} = \frac{r^2 + 36r}{36 - r^2}$
 $\Rightarrow \frac{228r + 21r^2}{36 - r^2} = \frac{r^2 + 36r}{36 - r^2}$
 $\Rightarrow 228r + 21r^2 = (r^2 + 36r)(36 - r^2)$
 $\Rightarrow 228r + 21r^2 = 36r^2 + 1296r - r^4 - 36r^3$
 $\Rightarrow r^4 - 36r^3 + 21r^2 - 1296r = 0$
 $\Rightarrow r^3 - 36r^2 + 21r - 1296 = 0$
 $\Rightarrow (r^2 - 9)(r^2 - 27) = 0$
 $\Rightarrow r^2 < 27 \Rightarrow r < \frac{3\sqrt{3}}{1}$
 $y = \frac{3(36+r)}{36-r} = \frac{3(36+8)}{36-8} = 5$
 $\frac{3(36+27)}{36-27} = \frac{3(63)}{9} = 21$
 $h < \frac{3+5}{3(3)+2(8)} = \frac{8}{24\sqrt{3}}$

Question 7 (****)



The figure above shows a triangle ABC , where $|AB| = a$ and $|AC| = 2a$.

The angle BAC is α , where $\tan \alpha = \frac{3}{4}$.

The side BC is extended to the point D so that the angle ACD is denoted by θ .

Show clearly that $\theta = \arctan 2$

proof

$\tan \alpha = \frac{3}{4}$
 $\sin \alpha = \frac{3}{5}$
 $\cos \alpha = \frac{4}{5}$

- BY THE COSINE RULE
 - $\Rightarrow |BC|^2 = a^2 + (2a)^2 - 2 \times a \times 2a \times \cos \alpha$
 - $\Rightarrow |BC|^2 = a^2 + 4a^2 - 4a^2 \times \frac{4}{5}$
 - $\Rightarrow |BC|^2 = 5a^2 - \frac{16}{5}a^2$
 - $\Rightarrow |BC|^2 = \frac{9}{5}a^2$
 - $\Rightarrow |BC| = \frac{3}{\sqrt{5}}a$
- NOW BY THE SINE RULE
 - $\Rightarrow \frac{\sin \alpha}{\frac{3}{\sqrt{5}}a} = \frac{\sin \theta}{a}$
 - $\Rightarrow \sin \theta = \frac{\sqrt{5}}{3} \sin \alpha$
 - $\Rightarrow \sin \theta = \frac{\sqrt{5}}{3} \times \frac{3}{5}$
 - $\Rightarrow \sin \theta = \frac{\sqrt{5}}{5}$
- FINALLY
 - $\Rightarrow \theta = \alpha + \beta$
 - $\Rightarrow \tan \theta = \tan(\alpha + \beta)$
 - $\Rightarrow \tan \theta = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$
 - $\Rightarrow \tan \theta = \frac{\frac{3}{4} + \frac{1}{2}}{1 - \frac{3}{4} \times \frac{1}{2}} = \frac{\frac{5}{4}}{\frac{5}{8}} = 2$
 - $\Rightarrow \theta = \arctan 2$