ADVANCED MENSURATION

Question 1 (**)

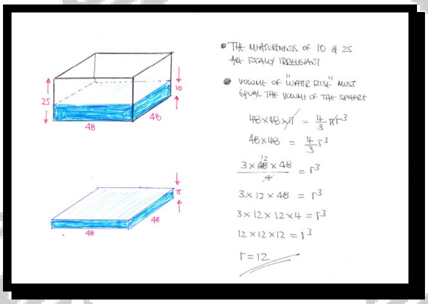
A water tank has a square base of length 48 cm and height 35 cm.

It is filled with water to a height of 25 cm.

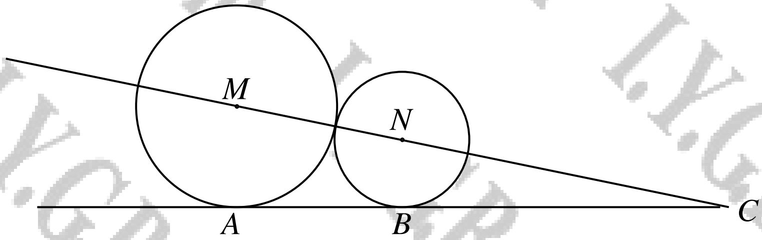
When a solid sphere is placed in the tank the water level rises by π cm.

Assuming the sphere is fully submerged determine the radius of the sphere.

$\boxed{r = 12 \text{ cm}}$



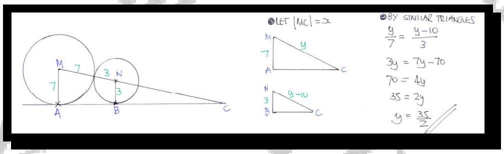
Question 2 (**+)



The figure above shows two circles, with centres at M and N, with respective radii 7 cm and 3 cm touch each other. The points M, N and C lie in a straight line and the straight line ABC is a common tangent to the circles.

Determine the length of MC.

$$\boxed{|MC| = \frac{35}{2}}$$



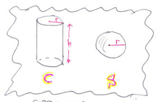
Question 3 (*+)**

A circular cylinder and a sphere both have radius r cm.

The total surface area of the cylinder is twice as large as the surface area of the sphere.

Determine the ratio of the volume of the cylinder to the volume of the sphere.

$\boxed{9:4}$



$$\frac{\text{SURFACE OF C}}{\text{SURFACE OF S}} = 2$$

$$\Rightarrow \frac{(\pi r^2 \times 2) + (2\pi r \times h)}{4\pi r^2} = 2$$

$$\Rightarrow \frac{2\pi r^2 + 2\pi r h}{4\pi r^2} = 2$$

$$\Rightarrow 2\pi r^2 + 2\pi r h = 8\pi r^2$$

DIVIDE BY $2\pi r$

$$\Rightarrow r + h = 4r$$

$$\Rightarrow \boxed{h = 3r}$$

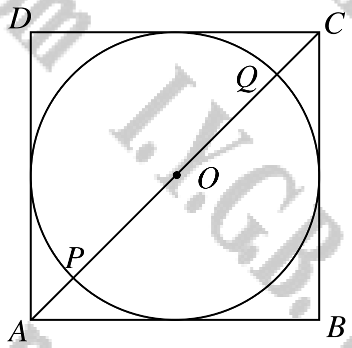
$$\text{VOLUME OF C} = \pi r^2 h = \pi r^2(3r) = 3\pi r^3$$

$$\text{VOLUME OF S} = \frac{4}{3}\pi r^3$$

$$\therefore \frac{\text{VOLUME OF C}}{\text{VOLUME OF S}} = \frac{3\cancel{\pi r^3}}{\frac{4}{3}\cancel{\pi r^3}} = \frac{3}{4/3} = \frac{9}{4}$$

$$\therefore 9:4$$

Question 4 (****)

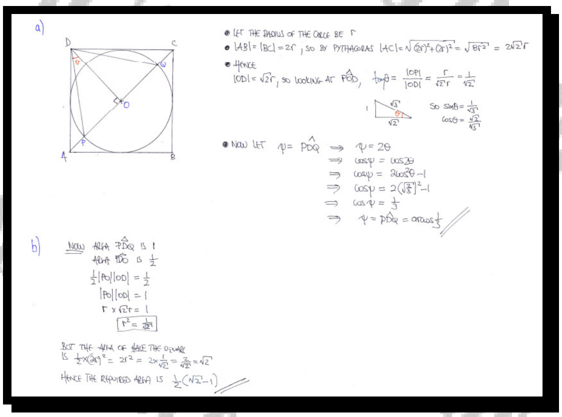


The figure above shows a circle with centre at O, inscribed in a square $ABCD$. The diagonal AC meets the circle at the points P and Q.

a) Show clearly that $\measuredangle PDQ = \arccos\frac{1}{3}$.

b) Given that the triangle PDQ is of unit area, determine the exact area of the triangle APD.

$$\boxed{\text{area} = \frac{1}{2}\left(\sqrt{2}-1\right)}$$


a)

- LET THE RADIUS OF THE CIRCLE BE r
- $|AB| = |BC| = 2r$, SO BY PYTHAGORAS $|AC| = \sqrt{(2r)^2+(2r)^2} = \sqrt{8r^2} = 2\sqrt{2}r$
- HENCE
$|OD| = \sqrt{2}r$, SO LOOKING AT $\hat{POD}$, $\tan\theta = \frac{|OP|}{|OD|} = \frac{r}{\sqrt{2}r} = \frac{1}{\sqrt{2}}$

SO $\sin\theta = \frac{1}{\sqrt{3}}$
$\cos\theta = \frac{\sqrt{2}}{\sqrt{3}}$

- NOW LET $\psi = P\hat{D}Q \Rightarrow \psi = 2\theta$
$\Rightarrow \cos\psi = \cos 2\theta$
$\Rightarrow \cos\psi = 2\cos^2\theta - 1$
$\Rightarrow \cos\psi = 2\left(\sqrt{\frac{2}{3}}\right)^2 - 1$
$\Rightarrow \cos\psi = \frac{1}{3}$
$\Rightarrow \psi = P\hat{D}Q = \arccos\frac{1}{3}$

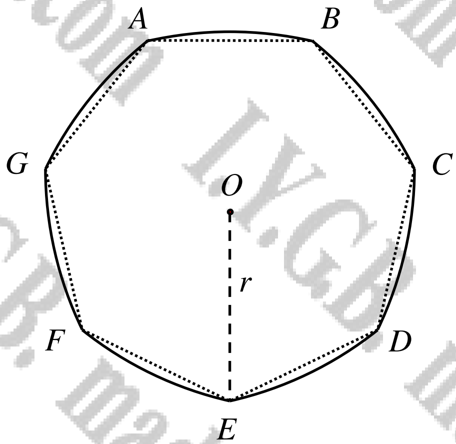
b)

NOW AREA $P\hat{D}Q$ IS 1
AREA $P\hat{D}O$ IS $\frac{1}{2}$
$\frac{1}{2}|PO||OD| = \frac{1}{2}$
$|PO||OD| = 1$
$r \times \sqrt{2}r = 1$
$\boxed{r^2 = \frac{1}{\sqrt{2}}}$

BUT THE AREA OF HALF THE SQUARE IS $\frac{1}{2}\times(2r)^2 = 2r^2 = 2\times\frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$

HENCE THE REQUIRED AREA IS $\frac{1}{2}(\sqrt{2}-1)$

Question 5 (*****)



The figure above shows a Reuleaux heptagon, $ABCDEFG$, constructed as follows.

Firstly a regular heptagon $ABCDEFG$ with centre at O and radius r is constructed. This is shown dotted in the figure.

A circular arc $\overset{\frown}{AB}$ is drawn with centre at E and radius EA. A second circular arc $\overset{\frown}{BC}$ is drawn with centre at F and radius FB.

A third circular arc $\overset{\frown}{CD}$ is drawn with centre at G and radius GC and the process is repeated , forming a curved heptagon known as a Reuleaux heptagon.

Show that the area of this Reuleaux heptagon is

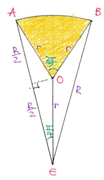
$$r^2\left[2\pi\cos^2\left(\frac{\pi}{14}\right)-\sin\left(\frac{\pi}{7}\right)\right].$$

$\boxed{}$, $\boxed{\text{proof}}$

- LOOKING AT ONE SECTOR OF THE HEPTAGON MARK SOME OF THE ANGLES IN THE DIAGRAM

$\hat{AOB} = \frac{2\pi}{7}$

$\hat{AEB} = \frac{1}{2}\times\frac{2\pi}{7} = \frac{\pi}{7}$ (CIRCLE THEOREM)



- NOW WE HAVE

$|AE| = R = 2\times\frac{R}{2} = 2\times r\cos\frac{\pi}{14}$

- AREA OF SECTOR ABC

$\frac{1}{2}R^2\left(\frac{\pi}{7}\right) = \frac{1}{2}\left(2r\cos\frac{\pi}{14}\right)^2\times\frac{\pi}{7}$

$= \frac{2}{7}\pi r^2\cos^2\frac{\pi}{14}$

- AREA OF TRIANGLE $\hat{AOE}$

$\frac{1}{2}|AE||OE|\sin\frac{\pi}{14} = \frac{1}{2}Rr\sin\frac{\pi}{14}$

$= \cancel{\tfrac{1}{2}}\times \cancel{2}r\cos\frac{\pi}{14}\times r\sin\frac{\pi}{14}$

$= r^2\cos\frac{\pi}{14}\sin\frac{\pi}{14}$

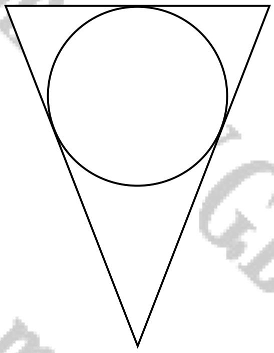
- "YELLOW" REGION $= \frac{2}{7}\pi r^2\cos^2\frac{\pi}{14} - 2\times\left[r^2\cos\frac{\pi}{14}\sin\frac{\pi}{14}\right]$

$= \frac{2\pi}{7}r^2\cos^2\frac{\pi}{14} - r^2\sin\frac{\pi}{7}$

- REQUIRED AREA $= 7\left(\frac{2\pi}{7}r^2\cos^2\frac{\pi}{14} - r^2\sin\frac{\pi}{7}\right)$

$= r^2\left[2\pi\cos^2\frac{\pi}{14} - \sin\frac{\pi}{7}\right]$

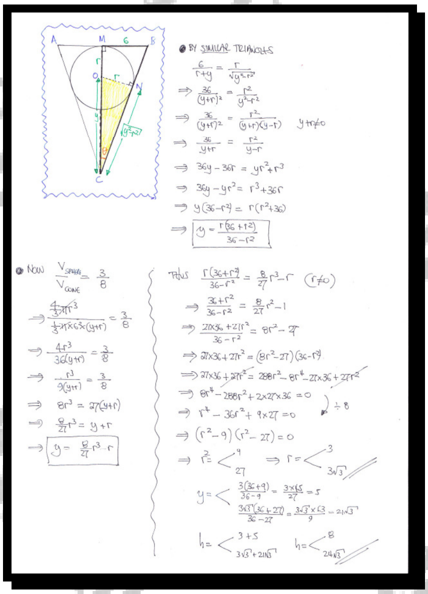
Question 6 **(*****)**



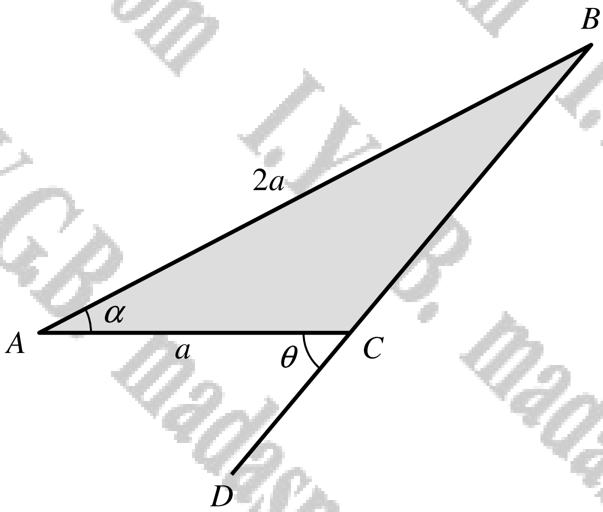
The figure above shows the cross sectional view of a solid sphere that **just fits** inside a right circular conical shell of radius 6 cm and height $h \text{ cm}$.

If the sphere occupies $\frac{3}{8}$ of the volume of the conical shell determine the two possible values of h.

$$\boxed{h = 8,\ 24\sqrt{3}}$$



Question 7 (*****)



The figure above shows a triangle ABC, where $|AB| = a$ and $|AC| = 2a$.

The angle BAC is α, where $\tan\alpha = \frac{3}{4}$.

The side BC is extended to the point D so that the angle ACD is denoted by θ.

Show clearly that $\theta = \arctan 2$

$\boxed{\text{proof}}$