VECTOR

EXAM QUESTIONS

Part B

Question 1 (**)

The vectors $\mathbf{a}$ and $\mathbf{b}$, are not parallel.

Simplify fully the following expression

$$(2\mathbf{a}+\mathbf{b})\wedge(\mathbf{a}-2\mathbf{b}).$$

$\boxed{}$, $\boxed{5\mathbf{b}\wedge\mathbf{a}=-5\mathbf{a}\wedge\mathbf{b}}$

USING THE FACT THAT THE "CROSS PRODUCT" IS DISTRIBUTIVE OVER ADDITION & SUBTRACTION, WE OBTAIN

$$(2\underline{a}+\underline{b})\wedge(\underline{a}-2\underline{b}) = 2\underline{a}\wedge\underline{a} - 4\underline{a}\wedge\underline{b} + \underline{b}\wedge\underline{a} - 2\underline{b}\wedge\underline{b}$$

NEXT WE USE THE PROPERTIES

- $\underline{u}\wedge\underline{u}=\underline{0}$ FOR ALL $\underline{u}$
- $\underline{u}\wedge\underline{v} = -\underline{v}\wedge\underline{u}$ FOR ALL $\underline{u}$ & $\underline{v}$

$$\dots = \underline{0} + 4\underline{b}\wedge\underline{a} + \underline{b}\wedge\underline{a} - \underline{0}$$

$$= 5\underline{b}\wedge\underline{a}$$

[OR INDEED $-5\underline{a}\wedge\underline{b}$]

Question 2 (**)

The vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are not parallel.

Simplify fully

$$\mathbf{a}\cdot\left[\mathbf{b}\wedge(\mathbf{c}+\mathbf{a})\right].$$

$\boxed{}$, $\boxed{\mathbf{a}\cdot(\mathbf{b}\wedge\mathbf{c})}$

APPLY THE CROSS PRODUCT FIRST

$$\Longrightarrow \underline{a}\cdot[\underline{b}\wedge(\underline{c}+\underline{a})] = \underline{a}\cdot[\underline{b}\wedge\underline{c} + \underline{b}\wedge\underline{a}]$$

$$= \underline{a}\cdot\underline{b}\wedge\underline{c} + \underline{a}\cdot\underline{b}\wedge\underline{a}$$

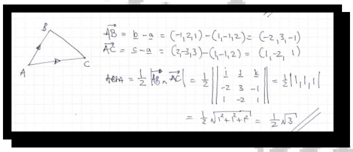
NOW $\underline{b}\wedge\underline{a}$ IS PERPENDICULAR TO $\underline{a}$, SO $\underline{a}\cdot(\underline{b}\wedge\underline{a})=0$

$$\therefore \underline{a}\cdot[\underline{b}\wedge(\underline{c}+\underline{a})] = \underline{a}\cdot\underline{b}\wedge\underline{c}$$

Question 3 (**)

Find the area of the triangle with vertices at $A(1,-1,2)$, $B(-1,2,1)$ and $C(2,-3,3)$.

$\boxed{\frac{1}{2}\sqrt{3}}$



$\overrightarrow{AB} = \underline{b} - \underline{a} = (-1,2,1)-(1,-1,2) = (-2,3,-1)$

$\overrightarrow{AC} = \underline{c} - \underline{a} = (2,-3,3)-(1,-1,2) = (1,-2,1)$

AREA $= \frac{1}{2}\left|\overrightarrow{AB} \wedge \overrightarrow{AC}\right| = \frac{1}{2}\left\| \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -2 & 3 & -1 \\ 1 & -2 & 1 \end{vmatrix} \right\| = \frac{1}{2}|1,1,1|$

$= \frac{1}{2}\sqrt{1^2+1^2+1^2} = \frac{1}{2}\sqrt{3}$

Question 4 (**)

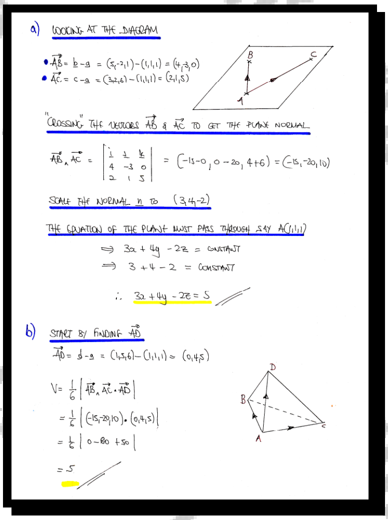
Referred to a fixed origin the coordinates of the following points are given

$$A(1,1,1),\ B(5,-2,1),\ C(3,2,6) \text{ and } D(1,5,6).$$

a) Find a Cartesian equation for the plane containing the points A, B and C.

b) Determine the volume of the tetrahedron $ABCD$.

$\boxed{3x+4y-2z=5}$, $\boxed{\text{volume}=5}$



a) LOOKING AT THE DIAGRAM

- $\overrightarrow{AB} = \underline{b} - \underline{a} = (5,-2,1)-(1,1,1) = (4,-3,0)$
- $\overrightarrow{AC} = \underline{c} - \underline{a} = (3,2,6)-(1,1,1) = (2,1,5)$

"CROSSING" THE VECTORS $\overrightarrow{AB}$ & $\overrightarrow{AC}$ TO GET THE PLANE NORMAL

$\overrightarrow{AB} \wedge \overrightarrow{AC} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 4 & -3 & 0 \\ 2 & 1 & 5 \end{vmatrix} = (-15-0,\ 0-20,\ 4+6) = (-15,-20,10)$

SCALE THE NORMAL $\underline{n}$ TO $(3,4,-2)$

THE EQUATION OF THE PLANE MUST PASS THROUGH SAY $A(1,1,1)$

$\Rightarrow 3x+4y-2z = \text{CONSTANT}$

$\Rightarrow 3+4-2 = \text{CONSTANT}$

$\therefore\ 3x+4y-2z=5$

b) START BY FINDING $\overrightarrow{AD}$

$\overrightarrow{AD} = \underline{d} - \underline{a} = (1,5,6)-(1,1,1) = (0,4,5)$

$V = \frac{1}{6}\left|\overrightarrow{AB} \wedge \overrightarrow{AC} \cdot \overrightarrow{AD}\right|$

$= \frac{1}{6}\left|(-15,-20,10)\cdot(0,4,5)\right|$

$= \frac{1}{6}\left|0-80+50\right|$

$= 5$

Question 5 (**)

The position vectors of the points A, B and C are given below

$$\overrightarrow{OA} = -\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}, \quad \overrightarrow{OB} = 3\mathbf{i} + 4\mathbf{j} - \mathbf{k} \quad \text{and} \quad \overrightarrow{OC} = \mathbf{i} + 4\mathbf{j} + \mathbf{k}.$$

a) Show that $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ are linearly dependent.

b) Find the area of the triangle ABC.

$\boxed{}$, $\boxed{\text{area} = 3}$

a) LINEARLY DEPENDENT $\Rightarrow$ "THEY DO NOT SPAN 3D SPACE"

$\Rightarrow$ "VOLUME OF THE PARALLELEPIPED THEY DEFINE MUST BE ZERO

HENCE WE FORM $\overrightarrow{OA} \wedge \overrightarrow{OB} \cdot \overrightarrow{OC}$

$$\begin{vmatrix} -1 & 2 & 2 \\ 3 & 4 & -1 \\ 1 & 4 & 1 \end{vmatrix} = -1\begin{vmatrix} 4 & -1 \\ 4 & 1 \end{vmatrix} - 2\begin{vmatrix} 3 & -1 \\ 1 & 1 \end{vmatrix} + 2\begin{vmatrix} 3 & 4 \\ 1 & 4 \end{vmatrix}$$

$$= -1(4+4) - 2(3+1) + 2(12-4)$$

$$= -8 - 8 + 16$$

$$= 0$$

INDEED LINEARLY DEPENDENT

b) WORK OUT ANY TWO SIDES OF ABC

- $\overrightarrow{AB} = \underline{b} - \underline{a} = (3,4,-1) - (-1,2,2) = (4,2,-3)$
- $\overrightarrow{AC} = \underline{c} - \underline{a} = (1,4,1) - (-1,2,2) = (2,2,-1)$

- $\text{AREA} = \frac{1}{2}\left|\overrightarrow{AB} \wedge \overrightarrow{AC}\right| = \frac{1}{2}\left\|\begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 2 & -3 \\ 2 & 2 & -1 \end{matrix}\right\| = \frac{1}{2}\left|-2+6, -6+4, 8-4\right|$

$$= \frac{1}{2}\left|4,-2,4\right| = \frac{1}{2}\sqrt{16+4+16}$$

$$= 3$$

Question 6 (**)

Find the equation of the straight line which is common to the planes

$$x-2y+4z=9 \quad \text{and} \quad 2x-3y+z=4 .$$

$\boxed{\mathbf{r}=(\mathbf{i}+2\mathbf{k})+\lambda(10\mathbf{i}+7\mathbf{j}+\mathbf{k}) \text{ or } [\mathbf{r}-(\mathbf{i}+2\mathbf{k})]\wedge(10\mathbf{i}+7\mathbf{j}+\mathbf{k})=\mathbf{0}}$

FIRST APPROACH (BY ROW REDUCING)

$$\left[\begin{array}{ccc|c} 1 & -2 & 4 & 9 \\ 2 & -3 & 1 & 4 \end{array}\right] \quad r_{12}(-2) \quad \left[\begin{array}{cccc} 1 & -2 & 4 & 9 \\ 0 & 1 & -7 & -14 \end{array}\right] \quad r_{21}(2)$$

$$\left[\begin{array}{cccc} 1 & 0 & -10 & -19 \\ 0 & 1 & -7 & -14 \end{array}\right] \quad \text{ie} \quad \begin{array}{l} x-10z=-19 \\ y-7z=-14 \end{array}$$

$$\begin{array}{l} x=-19+10z \\ y=-14+7z \\ z=0+z \end{array} \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} -19 \\ -14 \\ 0 \end{pmatrix}+t\begin{pmatrix} 10 \\ 7 \\ 1 \end{pmatrix}$$

TIDY A BIT TO MAKE THE NUMBERS SMALLER A BIT

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} -19 \\ -14 \\ 0 \end{pmatrix}+(\lambda+2)\begin{pmatrix} 10 \\ 7 \\ 1 \end{pmatrix}=\begin{pmatrix} -19 \\ -14 \\ 0 \end{pmatrix}+\lambda\begin{pmatrix} 10 \\ 7 \\ 1 \end{pmatrix}+\begin{pmatrix} 20 \\ 14 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}+\lambda\begin{pmatrix} 10 \\ 7 \\ 1 \end{pmatrix}$$

SECOND APPROACH – FIND THE DIRECTION BY CROSSING THE NORMALS

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 4 \\ 2 & -3 & 1 \end{vmatrix}=(-2+12,\ 8-1,\ -3+4)=(10,7,1)$$

NOW LET SAY $x=1$ IN THE EQUATIONS

$$\left.\begin{array}{l} 1-2y+4z=9 \\ 2-3y+z=4 \end{array}\right\} \Longrightarrow \left.\begin{array}{l} -2y+4z=8 \\ -3y+z=2 \end{array}\right\} \Rightarrow z=3y+2$$

SUB INTO THE OTHER

$\Rightarrow -2y+4(3y+2)=8$

$\Rightarrow -2y+12y+8=8$

$\Rightarrow 8y=0$

$\Rightarrow y=0$

AND SINCE $z=3y+2$, $z=2$

USING THE COMMON POINT $(1,0,2)$ AND DIRECTION $(10,7,1)$

$$\underline{r}=\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}+\lambda\begin{pmatrix} 10 \\ 7 \\ 1 \end{pmatrix}$$

AS BEFORE

Question 7 (+)**

The following vectors are given.

$$\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$$
$$\mathbf{b} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$$
$$\mathbf{c} = \mathbf{j} + 3\mathbf{k}$$

a) Show the three vectors are coplanar.

b) Express $\mathbf{a}$ in terms of $\mathbf{b}$ and $\mathbf{c}$.

$$\boxed{\mathbf{a} = 2\mathbf{b} - \mathbf{c}}$$

(a) $\underline{a} = (2,3,-1)$ If COPLANAR $\underline{a} \wedge \underline{b} \cdot \underline{c} = 0$

$\underline{b} = (1,2,1)$

$\underline{c} = (0,1,3)$

$$\begin{vmatrix} 0 & 1 & 3 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{vmatrix} = -\begin{vmatrix} 1 & 1 \\ 0 & 3 \end{vmatrix} + 3\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = -3 + 3 = 0$$

(b) $\lambda \underline{b} + \mu \underline{c} = \underline{a}$

$\lambda(1,2,1) + \mu(0,1,3) = (2,3,-1)$

$(\lambda, 2\lambda + \mu, \lambda + 3\mu) = (2,3,-1)$

$\therefore \lambda = 2$

$2\lambda + \mu = 3$

$4 + \mu = 3$

$\mu = -1$

$\therefore \underline{a} = 2\underline{b} - \underline{c}$

Question 8 (**+)

The vectors $\mathbf{a}$ and $\mathbf{b}$ are such so that

$$|\mathbf{a}| = \sqrt{10}, \quad |\mathbf{b}| = 10 \quad \text{and} \quad \mathbf{a} \cdot \mathbf{b} = 30.$$

Find the value of $|\mathbf{a} \wedge \mathbf{b}|$.

$\boxed{}$, $\boxed{|\mathbf{a} \wedge \mathbf{b}| = 10}$

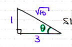
FROM THE DEFINITION OF THE "DOT PRODUCT"

$\Rightarrow \underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}|\cos\theta$

$\Rightarrow 30 = \sqrt{10} \times 10 \times \cos\theta$

$\Rightarrow \cos\theta = \frac{3}{\sqrt{10}}$

$\Rightarrow \theta \simeq 18.43^{\circ}\ldots$ OR



$\sin\theta = \frac{1}{\sqrt{10}}$

HENCE WE OBTAIN BY THE DEFINITION OF THE "CROSS" PRODUCT

$\Rightarrow \underline{a} \wedge \underline{b} = |\underline{a}||\underline{b}|\sin\theta\,\hat{\underline{n}}$

$\Rightarrow |\underline{a} \wedge \underline{b}| = \big|\,|\underline{a}||\underline{b}|\sin\theta\,\hat{\underline{n}}\,\big|$

$\Rightarrow |\underline{a} \wedge \underline{b}| = |\underline{a}||\underline{b}||\sin\theta||\hat{\underline{n}}|$

$\Rightarrow |\underline{a} \wedge \underline{b}| = \sqrt{10} \times 10 \times \frac{1}{\sqrt{10}} \times 1$ $\quad |\hat{\underline{n}}| = 1$

(or $\sin 18.43^{\circ}$)

$\therefore |\underline{a} \wedge \underline{b}| = 10$

Question 9 (**+)

With respect to a fixed origin O, the points A and B have position vectors given by

$$\mathbf{a} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k} \quad \text{and} \quad \mathbf{b} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}.$$

a) Find a Cartesian equation of the plane that passes through O, A and B.

A straight line has a vector equation

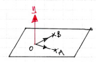
$$\left[\mathbf{r} - (4\mathbf{i} + \mathbf{j} + 6\mathbf{k})\right] \wedge (\mathbf{i} + \mathbf{j} + \mathbf{k}) = \mathbf{0}.$$

b) Determine the coordinates of the point C, where C is the intersection between the straight line and the plane.

$\boxed{}$, $\boxed{x - 7y - 5z = 0}$, $\boxed{C(1,-2,3)}$

a) OBTAIN A NORMAL

$\vec{OA} \wedge \vec{OB} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & -1 & 2 \\ 2 & 1 & -1 \end{vmatrix} = (-1,7,5)$



USING THE ORIGIN (0,0,0)

$-x + 7y + 5z = 0$

$x - 7y - 5z = 0$

b) REWRITE THE EQUATION OF THE LINE IN PARAMETRIC

$\underline{r} = (4,1,6) + \lambda(1,1,1)$

$\underline{r} = (\lambda+4, \lambda+1, \lambda+6)$

SOLVING SIMULTANEOUSLY

- $x = \lambda + 4$
- $y = \lambda + 1$
- $z = \lambda + 6$

& $x - 7y - 5z = 0$

$\Rightarrow (\lambda+4) - 7(\lambda+1) - 5(\lambda+6) = 0$

$\Rightarrow \lambda + 4 - 7\lambda - 7 - 5\lambda - 30 = 0$

$\Rightarrow -11\lambda = 33$

$\Rightarrow \lambda = -3$

$\therefore C(1,-2,3)$

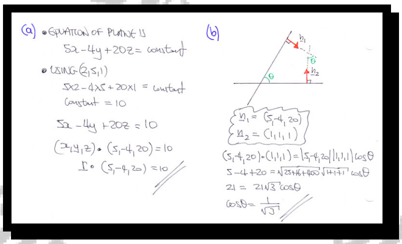
Question 10 (+)**

The plane Π_1 passes through the point with coordinates $(2,5,1)$ and is perpendicular to the vector $5\mathbf{i}-4\mathbf{j}+20\mathbf{k}$.

a) Find a vector equation of Π_1, in the form $\mathbf{r}\cdot\mathbf{n}=d$.

b) Calculate the exact value of the cosine of the acute angle between Π_1 and the plane Π_2 with equation $x+y+z=10$.

$\boxed{\mathbf{r}\cdot(5\mathbf{i}-4\mathbf{j}+20\mathbf{k})=10}$, $\boxed{\cos\theta=\frac{1}{\sqrt{3}}}$



Question 11 **(**+)**

The following three vectors are given

$$\mathbf{a} = \mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$$
$$\mathbf{b} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$$
$$\mathbf{c} = \mathbf{i} + 2\mathbf{j} + \lambda\mathbf{k}$$

where λ is a scalar constant.

a) If the three vectors given above are coplanar, find the value of λ.

b) Express $\mathbf{a}$ in terms of $\mathbf{b}$ and $\mathbf{c}$.

$\boxed{}$, $\boxed{\lambda = 1}$, $\boxed{\mathbf{a} = 3\mathbf{c} - \mathbf{b}}$

a) IF THE VECTORS ARE COPLANAR, THE CROSS PRODUCT OF ANY TWO WILL BE PERPENDICULAR TO THE THIRD

$$\Rightarrow (\underline{a} \wedge \underline{b}) \cdot \underline{c} = 0$$

$$\Rightarrow \begin{vmatrix} 1 & 2 & \lambda \\ 1 & 3 & 2 \\ 2 & 3 & 1 \end{vmatrix} = 0$$

$$\Rightarrow 1\begin{vmatrix} 3 & 2 \\ 3 & 1 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} + \lambda\begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix} = 0$$

$$\Rightarrow (3-6) - 2(1-4) + \lambda(3-6) = 0$$

$$\Rightarrow -3 + 6 - 3\lambda = 0$$

$$\Rightarrow 3 = 3\lambda$$

$$\Rightarrow \lambda = 1$$

b) SETTING UP AN EQUATION

$$\underline{a} = p\underline{b} + q\underline{c}$$

$$\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = p\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + q\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

EQUATE SAY $\underline{i}$ & $\underline{k}$ (THE $\underline{j}$ SHOULD BALANCE)

$$\left.\begin{array}{l} 2p + q = 1 \\ p + q = 2 \end{array}\right\} \Rightarrow p = -1 \quad \& \quad q = 3$$

$$\therefore \underline{a} = 3\underline{c} - \underline{b}$$

Question 12 (***)

The vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are such so that

$$\mathbf{c} \wedge \mathbf{a} = \mathbf{i} \quad \text{and} \quad \mathbf{b} \wedge \mathbf{c} = 2\mathbf{k}.$$

Express $(\mathbf{a}+\mathbf{b}) \wedge (\mathbf{a}+\mathbf{b}+2\mathbf{c})$ in terms of $\mathbf{i}$ and $\mathbf{k}$.

$\boxed{-2\mathbf{i}+4\mathbf{k}}$

$$(\underline{a}+\underline{b})_\wedge(\underline{a}+\underline{b}+2\underline{c}) = \cancel{(\underline{a}+\underline{b})_\wedge(\underline{a}+\underline{b})} + (\underline{a}+\underline{b})_\wedge 2\underline{c}$$
$$= 2\underline{a}_\wedge\underline{c} + 2\underline{b}_\wedge\underline{c}$$
$$= -2\underline{i} + 2(2\underline{k})$$
$$= -2\underline{i} + 4\underline{k}$$

Question 13 (***)

Relative to a fixed origin O, the position vectors of the points A, B and C are

$$\overrightarrow{OA} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \quad \overrightarrow{OB} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \quad \text{and} \quad \overrightarrow{OC} = \begin{pmatrix} 4 \\ -1 \\ 5 \end{pmatrix}.$$

a) Show that $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ are linearly independent.

b) Evaluate $\overrightarrow{OA} \cdot \overrightarrow{OB}$.

c) Show that $\overrightarrow{OB} \wedge \overrightarrow{OC} = k\overrightarrow{OA}$, where k is a constant.

The points O, A, B and C are vertices of a solid.

d) Describe the solid geometrically and state its volume.

$\boxed{\overrightarrow{OA} \cdot \overrightarrow{OB} = 0}$, $\boxed{k = 14}$, $\boxed{\text{cuboid, volume} = 42}$

(a) $$\begin{vmatrix} 1 & -1 & -1 \\ 2 & 3 & -1 \\ 4 & -1 & 5 \end{vmatrix} \underset{C_{13}(1)}{\overset{C_{12}(1)}{=}} \begin{vmatrix} 1 & 0 & 0 \\ 2 & 5 & 1 \\ 4 & 3 & 9 \end{vmatrix} = \begin{vmatrix} 5 & 1 \\ 3 & 9 \end{vmatrix} = 42 \neq 0$$

∴ LINEARLY INDEPENDENT

(b) $\overrightarrow{OA} \cdot \overrightarrow{OB} = (1,-1,-1)\cdot(2,3,-1) = 2-3+1 = 0$

(c) $$\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 3 & -1 \\ 4 & -1 & 5 \end{vmatrix} = (14,-14,-14) = 14(1,-1,-1) = k\,\overrightarrow{OA}$$

$k = 14$

(d) IT IS A CUBOID AS $\overrightarrow{OA} \perp \overrightarrow{OB} \perp \overrightarrow{OC}$ & ITS VOLUME IS 42

part (a)

Question 14 (***)

Relative to a fixed origin O, the plane Π_1 passes through the points A, B and C with position vectors $\mathbf{i}-\mathbf{j}+2\mathbf{k}$, $6\mathbf{i}-\mathbf{j}+\mathbf{k}$ and $3\mathbf{i}-2\mathbf{j}+2\mathbf{k}$, respectively.

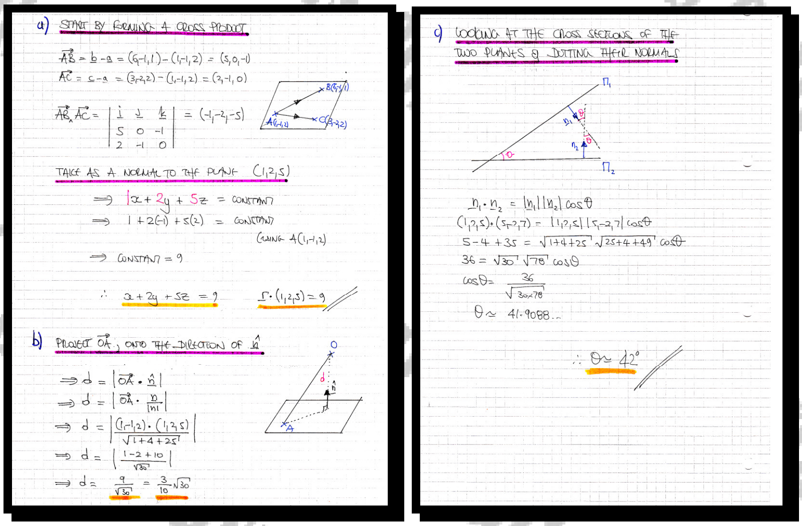
a) Determine an equation of Π_1 in the form $\mathbf{r}\cdot\mathbf{n}=c$, where $\mathbf{n}$ is the normal to Π_1 and c is a scalar constant.

b) Find, in exact surd form, the shortest distance of Π_1 from the origin O.

The plane Π_2 passes through the point A and has normal $5\mathbf{i}-2\mathbf{j}+7\mathbf{k}$.

c) Calculate, to the nearest degree, the acute angle between Π_1 and Π_2.

$\boxed{\text{FP3-Q}}$, $\boxed{\mathbf{r}\cdot(\mathbf{i}+2\mathbf{j}+5\mathbf{k})=9}$, $\boxed{\frac{3}{10}\sqrt{10}}$, $\boxed{42°}$



Question 15 (***)

Relative to a fixed origin O, the points A, B and C have position vectors

$$\mathbf{a} = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix} \quad \text{and} \quad \mathbf{c} = \begin{pmatrix} 3-2\lambda \\ \lambda+5 \\ \lambda+17 \end{pmatrix},$$

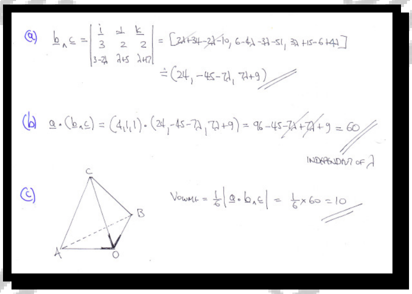
where λ is a scalar parameter.

a) Find the $\mathbf{b} \wedge \mathbf{c}$ in terms of λ.

b) Show that $\mathbf{a} \cdot (\mathbf{b} \wedge \mathbf{c})$ is independent of λ.

c) Find the volume of the tetrahedron and $OABC$.

$\boxed{24\mathbf{i} - (7\lambda+45)\mathbf{j} + (7\lambda+9)\mathbf{k}}$, $\boxed{\text{area} = 10}$



Question 16 (***)

With respect to a fixed origin O, the points $A(0,1,2)$, $B(2,3,1)$ and $C(1,1,3)$ are all contained by the plane Π.

a) Calculate the area of the triangle ABC.

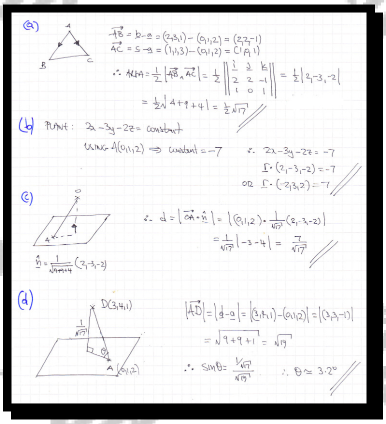
b) Determine an equation of Π, giving the answer in the form $\mathbf{r}\cdot\mathbf{n} = c$, where $\mathbf{n}$ is a normal to Π and c is a scalar constant.

c) Find the distance of Π from the origin O.

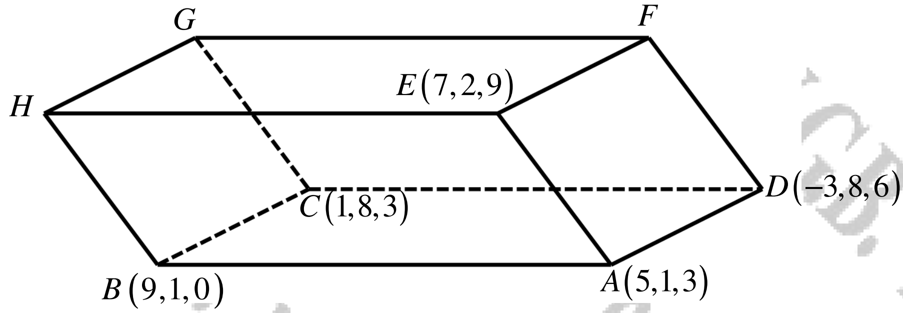
The distance of the point $D(3,4,1)$ from the plane Π is $\dfrac{1}{\sqrt{17}}$.

d) Calculate, correct to one decimal place, the acute angle between AD and Π.

$\boxed{\text{area} = \tfrac{1}{2}\sqrt{17}}$, $\boxed{\mathbf{r}\cdot(2\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}) = -7}$, $\boxed{\text{distance} = \dfrac{7}{\sqrt{17}}}$, $\boxed{3.2°}$



Question 17 (***)



The figure above shows a parallelepiped.

Relative to a fixed origin O, the vertices of the parallelepiped at A, B, C, D and E have respective position vectors

$$\begin{aligned}\mathbf{a} &= 5\mathbf{i}+\mathbf{j}+3\mathbf{k}\,,\\ \mathbf{b} &= 9\mathbf{i}+\mathbf{j}\,,\\ \mathbf{c} &= \mathbf{i}+8\mathbf{j}+3\mathbf{k}\,,\\ \mathbf{d} &= -3\mathbf{i}+8\mathbf{j}+6\mathbf{k}\\ \mathbf{e} &= 7\mathbf{i}+2\mathbf{j}+9\mathbf{k}\,.\end{aligned}$$

a) Calculate the area of the face $ABCD$.

b) Show that the volume of parallelepiped is 222 cubic units.

c) Hence, find the distance between the faces $ABCD$ and $EFGH$

$\boxed{}$, $\boxed{\text{area}=37}$, $\boxed{\text{distance}=6}$

a) CALCULATE THE RELEVANT VECTORS FOR A CROSS PRODUCT

$\overrightarrow{AB} = \underline{b}-\underline{a} = (9,1,0)-(5,1,3) = (4,0,-3)$

$\overrightarrow{AD} = \underline{d}-\underline{a} = (-3,8,6)-(5,1,3) = (-8,7,3)$

$\text{AREA} = |\overrightarrow{AB} \wedge \overrightarrow{AD}| = \left\| \begin{matrix} \underline{i} & \underline{j} & \underline{k} \\ 4 & 0 & -3 \\ -8 & 7 & 3 \end{matrix} \right\| = |21,12,28|$

$= \sqrt{21^2+12^2+28^2} = \sqrt{1369} = 37$

b) VOLUME IS $|\overrightarrow{AE}\cdot(\overrightarrow{AB}\wedge\overrightarrow{AD})|$, SO WE OBTAIN

$\Rightarrow V = |\overrightarrow{AE}\cdot(21,12,28)|$

$\Rightarrow V = |(\underline{e}-\underline{a})\cdot(21,12,28)|$

$\Rightarrow V = |[(7,2,9)-(5,1,3)]\cdot(21,12,28)|$

$\Rightarrow V = |(2,1,6)\cdot(21,12,28)|$

$\Rightarrow V = |42+12+168|$

$\Rightarrow V = 222$

AS REQUIRED

c) WE SHOULD OBTAIN THE VOLUME AS

$\Rightarrow V = \text{BASE AREA} \times \text{HEIGHT}$

$\Rightarrow 222 = 37 \times h$

$\Rightarrow h = 6$

∴ THE REQUIRED DISTANCE IS 6

Question 18 (***)

Two non zero vectors $\mathbf{a}$ and $\mathbf{b}$ have respective magnitudes a and b, respectively.

Given that $c = |\mathbf{a} \wedge \mathbf{b}|$ and $d = |\mathbf{a} \cdot \mathbf{b}|$, show that

$$c^2 + d^2 = a^2 b^2 .$$

proof

$c = |\underline{a} \wedge \underline{b}| = \left| |a||b| \sin\theta \, \hat{\underline{n}} \right| = \left||a|\right| \left||b|\right| |\sin\theta| |\hat{\underline{n}}| = a\,b \sin\theta \times 1 = ab\sin\theta$

$d = |\underline{a} \cdot \underline{b}| = \left| |\underline{a}||\underline{b}| \cos\theta \right| = \left||\underline{a}|\right| \left||\underline{b}|\right| |\cos\theta| = a\,b\,|\cos\theta|$

HENCE $c^2 + d^2 = (ab\sin\theta)^2 + (ab|\cos\theta|)^2 = a^2b^2\sin^2\theta + a^2b^2\cos^2\theta$

$= a^2b^2(\sin^2\theta + \cos^2\theta) = a^2b^2$

AS REQUIRED

Question 19 (***)

Relative to a fixed origin O, the points $A(-2,3,5)$, $B(1,-3,1)$ and $C(4,-6,-7)$ lie on the plane Π.

a) Find a Cartesian equation for Π.

The perpendicular from the point $P(26,2,7)$ meets the Π at the point Q.

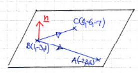
b) Determine the coordinates of Q.

$\boxed{}$, $\boxed{12x + 4y + 3z = 3}$, $\boxed{Q(2,-6,1)}$

a) START BY FINDING A NORMAL TO THE PLANE

$\overrightarrow{BC} = \underline{c} - \underline{b} = (4,-6,-7) - (1,-3,1) = (3,-3,-8)$

$\overrightarrow{BA} = \underline{a} - \underline{b} = (-2,3,5) - (1,-3,1) = (-3,6,4)$



$\overrightarrow{BC} \wedge \overrightarrow{BA} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & -3 & -8 \\ -3 & 6 & 4 \end{vmatrix} = (36,12,9)$

SCALING THE NORMAL VECTOR

$\underline{n} = (12,4,3)$

EQUATION OF PLANE USING $B(1,-3,1)$

$\Rightarrow 12x + 4y + 3z = \text{CONSTANT}$

$\Rightarrow (12\times1) + 4(-3) + (3\times1) = \text{CONSTANT}$

$\Rightarrow \text{CONSTANT} = 3$

$\therefore 12x + 4y + 3z = 3$

b) EQUATION OF A LINE PASSING THROUGH $P(26,2,7)$, PERPENDICULAR TO THE PLANE

$\underline{r} = (26,2,7) + \lambda(12,4,3)$

$(x,y,z) = (12\lambda+26,\ 4\lambda+2,\ 3\lambda+7)$

SOLVING SIMULTANEOUSLY WITH THE EQUATION OF THE PLANE

$\Rightarrow 12(12\lambda+26) + 4(4\lambda+2) + 3(3\lambda+7) = 3$

$\Rightarrow 144\lambda + 312 + 16\lambda + 8 + 9\lambda + 21 = 3$

$\Rightarrow 169\lambda = -338$

$\Rightarrow \lambda = -2$

$\therefore Q(2,-6,1)$

Question 20 (***)

The points $A(3,1,0)$, $B(0,2,2)$ and $C(3,3,1)$ form a plane Π.

a) Show that $\mathbf{i}-\mathbf{j}+2\mathbf{k}$ is a normal to Π.

b) Find a Cartesian equation for Π.

The straight line L passes through the point $P(3,1,3)$ and meets Π at right angles at the point Q.

c) Determine the distance PQ.

$\boxed{}$, $\boxed{x-y+2z=2}$, $\boxed{|PQ|=\sqrt{6}}$

a) BY VERIFICATION

$\vec{AB} = \underline{b}-\underline{a} = (0,2,2)-(3,1,0) = (-3,1,2)$

$\vec{AC} = \underline{c}-\underline{a} = (3,3,1)-(3,1,0) = (0,2,1)$

DOTTING EACH OF THESE VECTORS WITH THE NORMAL GIVEN

$(-3,1,2)\cdot(1,-1,2) = -3-1+4 = 0$

$(0,2,1)\cdot(1,-1,2) = 0-2+2 = 0$

INDEED THE NORMAL TO Π

b) THE EQUATION OF THE PLANE WILL BE

$x-y+2z = \text{CONSTANT}$

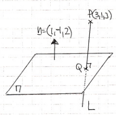
USING ANY OF THE 3 POINTS, SAY $B(0,2,2)$

$0-2+2\times 2 = \text{CONSTANT}$

$\text{CONSTANT} = 2$

$\therefore\ x-y+2z=2$

c) STANDARD APPROACH



$L:\ \underline{r} = (3,1,3)+\lambda(1,-1,2)$

$\underline{r} = (\lambda+3,\ 1-\lambda,\ 2\lambda+3)$

$\uparrow\ x\quad \uparrow\ y\quad \uparrow\ z$

SOLVING SIMULTANEOUSLY WITH Π

$\Rightarrow x-y+2z=2$

$\Rightarrow (\lambda+3)-(1-\lambda)+2(2\lambda+3)=2$

$\Rightarrow \lambda+3-1+\lambda+4\lambda+6=2$

$\Rightarrow 6\lambda+8=2$

$\Rightarrow 6\lambda=-6$

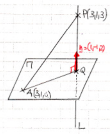
$\Rightarrow \lambda=-1$

$\therefore Q(\underset{\lambda+3}{2},\ \underset{1-\lambda}{2},\ \underset{2\lambda+3}{1})$

$|PQ| = |\underline{q}-\underline{p}| = |(2,2,1)-(3,1,3)| = |-1,1,-2| = \sqrt{1+1+4}$

$\therefore |PQ| = \sqrt{6}$

ALTERNATIVE FOR PART (c)



- $\vec{PA} = \underline{a}-\underline{p} = (3,1,0)-(3,1,3) = (0,0,-3)$
- $\hat{\underline{n}} = \frac{1}{\sqrt{1+1+4}}(1,-1,2) = \frac{1}{\sqrt{6}}(1,-1,2)$
- $|PQ| = |\vec{PA}\cdot\hat{\underline{n}}|$

$= \left|(0,0,-3)\cdot\frac{1}{\sqrt{6}}(1,-1,2)\right|$

$= \frac{1}{\sqrt{6}}\left|(0,0,-3)\cdot(1,-1,2)\right|$

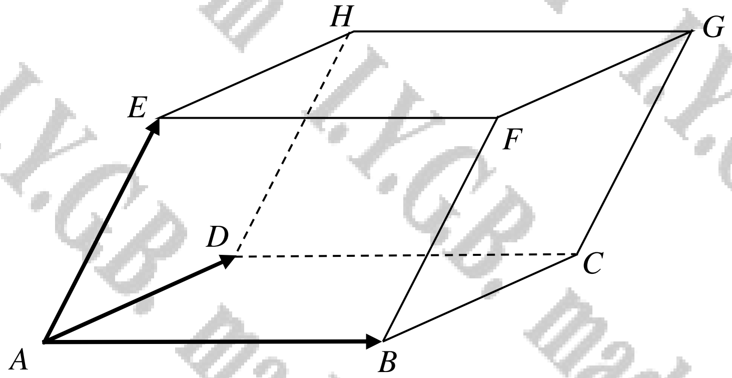
$= \frac{1}{\sqrt{6}}\left|0+0-6\right|$

$= \frac{6}{\sqrt{6}}$

$= \sqrt{6}$

AS BEFORE

Question 21 (***)



The figure above shows a parallelepiped, whose vertices are located at the points $A(2,1,t)$, $B(3,3,2)$, $D(4,0,5)$ and $E(1,-2,7)$, where t is a constant.

a) Calculate $\overrightarrow{AB} \wedge \overrightarrow{AD}$, in terms of t.

b) Find the value of $\overrightarrow{AB} \wedge \overrightarrow{AD} \cdot \overrightarrow{AE}$

The volume of the parallelepiped is 22 cubic units.

c) Determine the possible values of t.

$\boxed{(12-3t)\mathbf{i}+(-t-1)\mathbf{j}-5\mathbf{k}}$, $\boxed{11t-44}$, $\boxed{t=2,6}$

(a) $\overrightarrow{AB} = \underline{b}-\underline{a} = (3,3,2)-(2,1,t) = (1,2,2-t)$

$\overrightarrow{AD} = \underline{d}-\underline{a} = (4,0,5)-(2,1,t) = (2,-1,5-t)$

$\overrightarrow{AB} \wedge \overrightarrow{AD} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 2 & 2-t \\ 2 & -1 & 5-t \end{vmatrix} = (10-2t+2-t,\ 4-2t-5+t,\ -1-4) = (12-3t,\ -1-t,\ -5)$

(b) $\overrightarrow{AE} = \underline{e}-\underline{a} = (1,-2,7)-(2,1,t) = (-1,-3,7-t)$

$\therefore \overrightarrow{AB} \wedge \overrightarrow{AD} \cdot \overrightarrow{AE} = (12-3t,-1-t,-5)\cdot(-1,-3,7-t) = 3t-12+3+3t-35+5t$

$= 11t-44$

(c) $\left.\begin{array}{l} V = |11t-44| \\ 22 = |11t-44| \end{array}\right\} \Rightarrow \left.\begin{array}{l} 11t-44 = 22 \\ 11t-44 = -22 \end{array}\right\} \Rightarrow \left.\begin{array}{l} 11t = 66 \\ 11t = 22 \end{array}\right\} \Rightarrow t = \begin{cases} 6 \\ 2 \end{cases}$

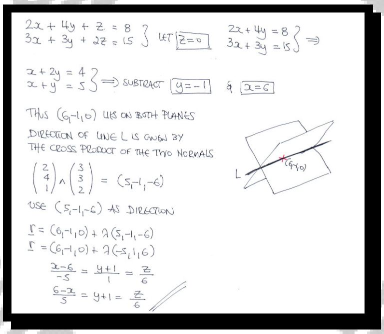
Question 22 (***)

Find in Cartesian form the equation of the intersection between the planes with the following equations

$$2x + 4y + z = 0$$

$$3x + 3y + 2z = 15.$$

$$\boxed{\frac{6-x}{5} = y+1 = \frac{z}{6}}$$



Question 23 (***)

Two planes have Cartesian equations

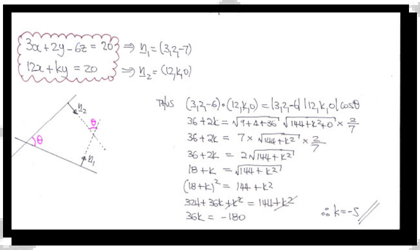
$$3x + 2y - 6z = 20 \quad \text{and} \quad 12x + ky = 20,$$

where k is a non zero constant.

The acute angle between the two planes is θ.

Given that $\cos\theta = \frac{2}{7}$, determine the value of k.

$$\boxed{k = -5}$$



Question 24 (***)

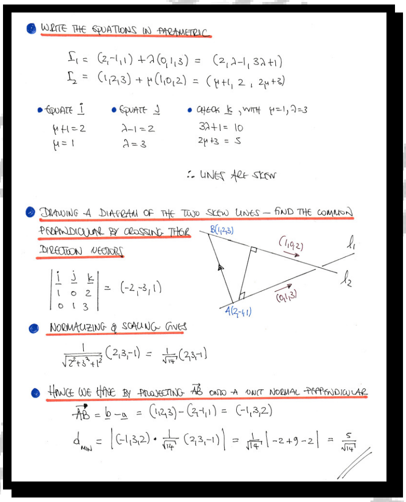
The straight lines l_1 and l_2 have respective vector equations

$$\mathbf{r}_1 = 2\mathbf{i} - \mathbf{j} + \mathbf{k} + \lambda(\mathbf{j} + 3\mathbf{k})$$
$$\mathbf{r}_2 = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + \mu(\mathbf{i} + 2\mathbf{k})$$

where λ and μ are scalar parameters.

Show that l_1 and l_2 are skew and hence find the shortest distance between them.

$\boxed{}$, $\boxed{\dfrac{5}{\sqrt{14}}}$



WRITE THE EQUATIONS IN PARAMETRIC

$\underline{r}_1 = (2,-1,1) + \lambda(0,1,3) = (2, \lambda-1, 3\lambda+1)$

$\underline{r}_2 = (1,2,3) + \mu(1,0,2) = (\mu+1, 2, 2\mu+3)$

• EQUATE $\underline{i}$: $\mu+1 = 2$, $\mu = 1$

• EQUATE $\underline{j}$: $\lambda-1 = 2$, $\lambda = 3$

• CHECK $\underline{k}$, WITH $\mu=1, \lambda=3$: $3\lambda+1 = 10$, $2\mu+3 = 5$

∴ LINES ARE SKEW

DRAWING A DIAGRAM OF THE TWO SKEW LINES – FIND THE COMMON PERPENDICULAR BY CROSSING THEIR DIRECTION VECTORS

$$\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{vmatrix} = (-2,-3,1)$$

NORMALIZING & SCALING GIVES

$$\frac{1}{\sqrt{2^2+3^2+1^2}}(2,3,-1) = \frac{1}{\sqrt{14}}(2,3,-1)$$

HENCE WE HAVE BY PROJECTING $\overrightarrow{AB}$ ONTO A UNIT NORMAL PERPENDICULAR

$$\overrightarrow{AB} = \underline{b} - \underline{a} = (1,2,3) - (2,-1,1) = (-1,3,2)$$

$$d_{\text{MIN}} = \left|(-1,3,2)\cdot\frac{1}{\sqrt{14}}(2,3,-1)\right| = \frac{1}{\sqrt{14}}\left|-2+9-2\right| = \frac{5}{\sqrt{14}}$$

Question 25 (***)

The points $A(1,-3,1)$, $B(-1,-2,0)$ and $C(0,-1,-4)$ define a plane Π.

a) Show that $\mathbf{i}+3\mathbf{j}+\mathbf{k}$ is a normal to Π.

b) Determine a Cartesian equation for Π.

The straight line L has equation

$$\mathbf{r} = 2\mathbf{i}+\mathbf{k}+\lambda(5\mathbf{i}+\mathbf{j}+2\mathbf{k}),$$

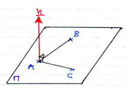
where λ is a scalar parameter.

c) Find the coordinates of the point of intersection between Π and L.

d) Calculate the size of the acute angle between Π and L.

$\boxed{}$, $\boxed{x+3y+z+7=0}$, $\boxed{(-3,-1,-1)}$, $\boxed{33.4°}$

a) SHOWING THAT $\vec{AB}$ & $\vec{AC}$ ARE BOTH PERPENDICULAR TO THE GIVEN VECTOR



- $\vec{AB} = \underline{b} - \underline{a} = (-1,-2,0)-(1,-3,1)$
  $\vec{AB} = (-2,1,-1)$
- $\vec{AC} = \underline{c} - \underline{a} = (0,-1,-4)-(1,-3,1)$
  $\vec{AC} = (-1,2,-5)$

$$\left(\begin{array}{l}\vec{AB}\cdot(1,3,1) = (-2,1,-1)\cdot(1,3,1) = -2+3-1=0\\ \vec{AC}\cdot(1,3,1) = (-1,2,-5)\cdot(1,3,1) = -1+6-5=0\end{array}\right)$$

INDEED A NORMAL TO Π

b) THE CARTESIAN EQUATION OF Π MUST BE

$1x+3y+1z = \text{CONSTANT}$
$x+3y+z = \text{CONSTANT}$

USING $A(1,-3,1)$ GIVES

$1+3(-3)+1 = \text{CONSTANT}$
$-7 = \text{CONSTANT}$

$\therefore x+3y+z=-7$
$x+3y+z+7=0$

c) SOLVING SIMULTANEOUSLY

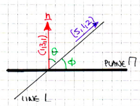
$\Pi: x+3y+z+7=0$

$\underline{r} = (2,0,1)+\lambda(5,1,2)$
$\underline{r} = (5\lambda+2, \lambda, 2\lambda+1)$

$(5\lambda+2)+3\lambda+(2\lambda+1)+7=0$
$10\lambda+10=0$
$\lambda=-1$

$\therefore [5(-1)+2, -1, 2(-1)+1]$ YIELDS $(-3,-1,-1)$

d)



LOOKING AT THE DIAGRAM

$\Rightarrow (1,3,1)\cdot(5,1,2) = |1,3,1|\,|5,1,2|\cos\theta$
$\Rightarrow 5+3+2 = \sqrt{1+9+1}\sqrt{25+1+4}\cos\theta$
$\Rightarrow 10 = \sqrt{11}\sqrt{30}\cos\theta$
$\Rightarrow \cos\theta = \frac{10}{\sqrt{11\times30}}$
$\Rightarrow \theta \approx 56.599°...$

$\therefore$ REQUIRED ANGLE IS ϕ

$\Rightarrow \phi = 90-\theta$
$\Rightarrow \phi \approx 33.4°$

Question 26 (***+)

A tetrahedron has its four vertices at the points $A(-3,6,4)$, $B(0,11,0)$, $C(4,1,28)$ and $D(7,k,24)$, where k is a constant.

a) Calculate the area of the triangle ABC.

b) Find the volume of the tetrahedron $ABCD$, in terms of k.

The volume of the tetrahedron is 150 cubic units.

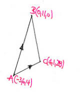
c) Determine the possible values of k.

$\boxed{}$, $\boxed{\text{area} = 75}$, $\boxed{\text{volume} = \frac{50}{3}|k-6|}$, $\boxed{k=-3, \quad k=15}$

a) START BY WORKING OUT THE RELEVANT VECTORS

$\vec{AB} = \underline{b} - \underline{a} = (0,11,0) - (-3,6,4) = (3,5,-4)$

$\vec{AC} = \underline{c} - \underline{a} = (4,1,28) - (-3,6,4) = (7,-5,24)$



HENCE USING THE STANDARD FORMULA

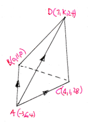
$$\text{AREA} = \frac{1}{2}\left|\vec{AB} \wedge \vec{AC}\right| = \frac{1}{2}\left\|\begin{matrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & 5 & -4 \\ 7 & -5 & 24 \end{matrix}\right\| = \frac{1}{2}\left|120-20, -28-72, -15-35\right|$$

$$= \frac{1}{2}\left|100,-100,-50\right| = \frac{1}{2} \times 50 \left|2,-2,1\right| = 25\sqrt{4+4+1} = \underline{75}$$

b) WORK OUT THE VECTOR $\vec{AD}$ IN TERMS OF k

$\vec{AD} = \underline{d} - \underline{a} = (7,k,24) - (-3,6,4) = (10, k-6, 20)$

USING THE "STANDARD FORMULA" FOR TETRAHEDRON



$$\text{VOLUME} = \frac{1}{6}\left|\vec{AB} \wedge \vec{AC} \cdot \vec{AD}\right|$$

$$= \frac{1}{6}\left\|\begin{matrix} 10 & k-6 & 20 \\ 3 & 5 & -4 \\ 7 & -5 & 24 \end{matrix}\right\|$$

$$= \frac{1}{6}\left|(100,-100,-50)\cdot(10,k-6,20)\right|$$

$$= \frac{1}{6}\left|\cancel{1000} - 100(k-6) - \cancel{1000}\right|$$

$$= \frac{1}{6}\left|-100\right|\left|k-6\right|$$

$$= \underline{\frac{50}{3}\left|k-6\right|}$$

c) USING PART (b)

$$\frac{50}{3}|k-6| = 150$$

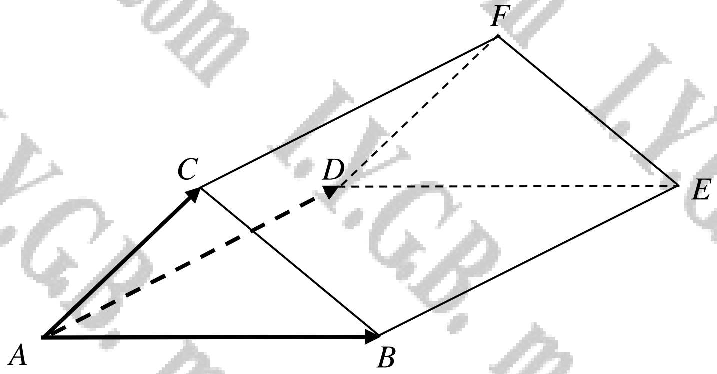
$$50|k-6| = 450$$

$$|k-6| = 9$$

$$k-6 = \begin{cases} 9 \\ -9 \end{cases}$$

$$k = \begin{cases} \underline{15} \\ \underline{-3} \end{cases}$$

Question 27 (***+)



A triangular prism has vertices at $A(3,3,3)$, $B(1,3,t)$, $C(5,1,5)$ and $F(8,0,10)$, where t is a constant.

The face ABC is parallel to the face DEF and the lines AD, BE and CF are parallel to each other.

a) Calculate $\overrightarrow{AB} \wedge \overrightarrow{AC}$, in terms of t.

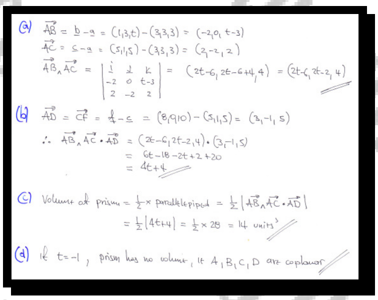
b) Find the value of $\overrightarrow{AB} \wedge \overrightarrow{AC} \cdot \overrightarrow{AD}$, in terms of t.

The value of t is taken to be 6.

c) Determine the volume of the prism for this value of t.

d) Explain the geometrical significance if $t = -1$.

$\boxed{(2t-6)\mathbf{i}+(2t-2)\mathbf{j}+4\mathbf{k}}$, $\boxed{4t+4}$, $\boxed{V = 14 \text{ cubic units}}$, $\boxed{A, B, C, D \text{ are coplanar, so no volume}}$



(a) $\overrightarrow{AB} = \underline{b} - \underline{a} = (1,3,t)-(3,3,3) = (-2,0,t-3)$

$\overrightarrow{AC} = \underline{c} - \underline{a} = (5,1,5)-(3,3,3) = (2,-2,2)$

$\overrightarrow{AB} \wedge \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 0 & t-3 \\ 2 & -2 & 2 \end{vmatrix} = (2t-6, 2t-6+4, 4) = (2t-6, 2t-2, 4)$

(b) $\overrightarrow{AD} = \overrightarrow{CF} = \underline{f} - \underline{c} = (8,0,10)-(5,1,5) = (3,-1,5)$

$\therefore \overrightarrow{AB} \wedge \overrightarrow{AC} \cdot \overrightarrow{AD} = (2t-6, 2t-2, 4)\cdot(3,-1,5)$
$= 6t-18-2t+2+20$
$= 4t+4$

(c) Volume of prism $= \frac{1}{2} \times$ parallelepiped $= \frac{1}{2}\left|\overrightarrow{AB} \wedge \overrightarrow{AC} \cdot \overrightarrow{AD}\right|$
$= \frac{1}{2}|4t+4| = \frac{1}{2} \times 28 = 14$ units^3

(d) if $t=-1$, prism has no volume, ie A, B, C, D are coplanar

Question 28 (***+)

Relative to a fixed origin O the point P has coordinates $(1,2,1)$.

A plane Π has Cartesian equation

$$2x + y + 3z = 21.$$

The straight line L passes through the point P and it is perpendicular to Π.

a) Find the coordinates of the point M, where M is the intersection of Π and L.

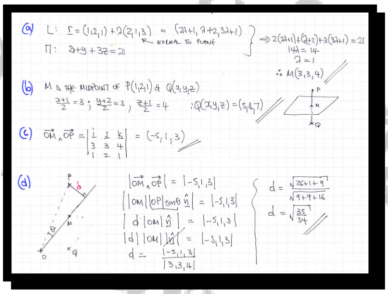
The point Q is the reflection of P about Π.

b) Find the coordinates of Q.

c) Find $\overrightarrow{OM} \wedge \overrightarrow{OP}$.

d) Hence, or otherwise, find the shortest distance from the point P to the straight line OM, giving the answer in exact form.

$\boxed{M(3,3,4)}$, $\boxed{Q(5,4,7)}$, $\boxed{5\mathbf{i} - \mathbf{j} - 3\mathbf{k}}$, $\boxed{\text{distance} = \sqrt{\frac{35}{34}}}$



Question 29 **(***+)**

The plane Π has an equation given by

$$\mathbf{r} = 4\mathbf{i} + \mathbf{k} + \lambda(2\mathbf{j} - \mathbf{k}) + \mu(3\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}),$$

where λ and μ are scalar parameters.

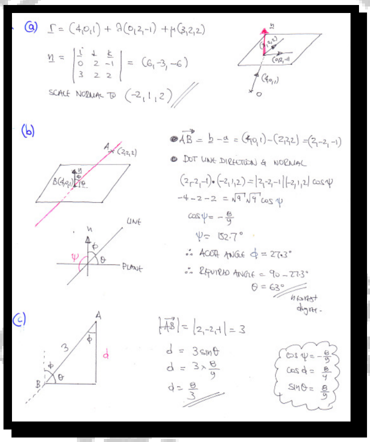
a) Find a normal vector to this plane.

The straight line L passes through the point $A(2,2,2)$ and meets Π at the point $B(4,0,1)$.

b) Calculate, to the nearest degree, the acute angle between L and Π.

c) Hence, or otherwise, find the shortest distance from A to Π.

$\boxed{\mathbf{n} = -2\mathbf{i} + \mathbf{j} + 2\mathbf{k}}$, $\boxed{63°}$, $\boxed{\text{distance} = \frac{8}{3}}$



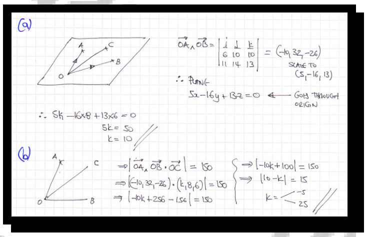
Question 30 (***+)

With respect to a fixed origin O the points A, B and C, have respective coordinates $(6,10,10)$, $(11,14,13)$ and $(k,8,6)$, where k is a constant.

a) Given that all the three points lie on a plane which contains the origin, find the value of k.

b) Given instead that OA, OB, OC are edges of a parallelepiped of volume 150 cubic units determine the possible values of k.

$\boxed{k=10}$, $\boxed{k=-5, \quad k=25}$



Question 31 (***+)

The straight lines L_1 and L_2 have respective Cartesian equations

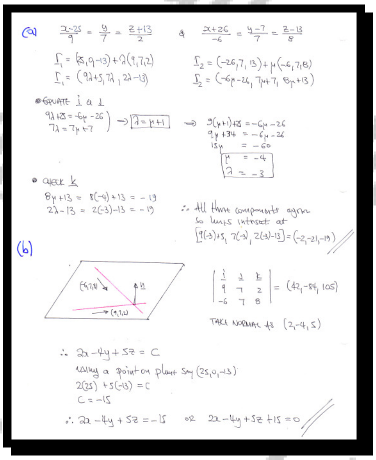
$$\frac{x-25}{9}=\frac{y}{7}=\frac{z+13}{2} \qquad \text{and} \qquad \frac{x+26}{-6}=\frac{y-7}{7}=\frac{z-13}{8}.$$

a) Show that L_1 and L_2 intersect at some point and find its coordinates.

The plane Π contains both L_1 and L_2.

b) Find a Cartesian equation for Π.

$\boxed{(-2,-21,-19)}$, $\boxed{2x-4y+5z+15=0}$



(a) $\frac{x-25}{9}=\frac{y}{7}=\frac{z+13}{2}$ & $\frac{x+26}{-6}=\frac{y-7}{7}=\frac{z-13}{8}$

$\underline{r}_1=(25,0,-13)+\lambda(9,7,2)$ $\quad$ $\underline{r}_2=(-26,7,13)+\mu(-6,7,8)$

$\underline{r}_1=(9\lambda+25,7\lambda,2\lambda-13)$ $\quad$ $\underline{r}_2=(-6\mu-26,7\mu+7,8\mu+13)$

• EQUATE $\underline{i}$ & $\underline{j}$

$9\lambda+25=-6\mu-26$

$7\lambda=7\mu+7$ $\Rightarrow \boxed{\lambda=\mu+1}$ $\Rightarrow$ $9(\mu+1)+25=-6\mu-26$

$9\mu+34=-6\mu-26$

$15\mu=-60$

$\boxed{\mu=-4 \quad \lambda=-3}$

• CHECK $\underline{k}$

$8\mu+13=8(-4)+13=-19$

$2\lambda-13=2(-3)-13=-19$

$\therefore$ All three components agree so lines intersect at

$[9(-3)+25,7(-3),2(-3)-13]=(-2,-21,-19)$

(b)

$(-6,7,8)$ $\quad$ $\underline{n}$ $\quad$ $(9,7,2)$

$\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ 9 & 7 & 2\\ -6 & 7 & 8\end{vmatrix}=(42,-84,105)$

TAKE NORMAL AS $(2,-4,5)$

$\therefore 2x-4y+5z=C$

using a point on plane say $(25,0,-13)$

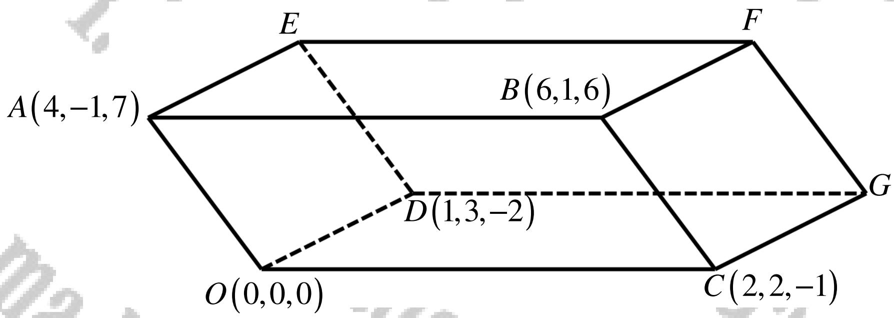
$2(25)+5(-13)=C$

$C=-15$

$\therefore 2x-4y+5z=-15$ or $2x-4y+5z+15=0$

Question 32 (***+)

The figure below shows a parallelepiped.



Relative to an origin O the points A, B, C and D have respective position vectors

$$\mathbf{a} = 4\mathbf{i} - \mathbf{j} + 7\mathbf{k}, \quad \mathbf{b} = 6\mathbf{i} + \mathbf{j} + 6\mathbf{k}, \quad \mathbf{c} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k} \quad \text{and} \quad \mathbf{d} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}.$$

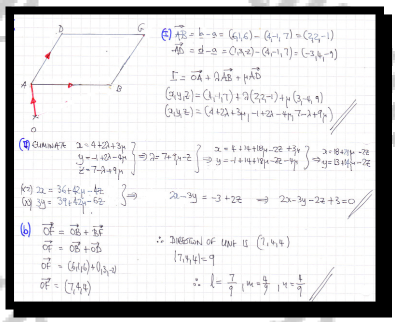
a) Find an equation of the plane $ABDG$ in the form …

i. … $\mathbf{r} = \mathbf{u} + \lambda\mathbf{v} + \mu\mathbf{w}$.

ii. … $ax + by + cz + d = 0$.

b) Hence determine the direction cosines of the straight line through O and F.

$\boxed{\mathbf{r} = 4\mathbf{i} - \mathbf{j} + 7\mathbf{k} + \lambda(2\mathbf{i} + 2\mathbf{j} - \mathbf{k}) + \mu(3\mathbf{i} - 4\mathbf{j} + 9\mathbf{k})}$, $\boxed{2x - 3y - 2z + 3 = 0}$, $\boxed{l = \frac{7}{9}, m = \frac{4}{9}, n = \frac{4}{9}}$



Question 33 (***+)

The planes Π_1 and Π_2 have the following Cartesian equations.

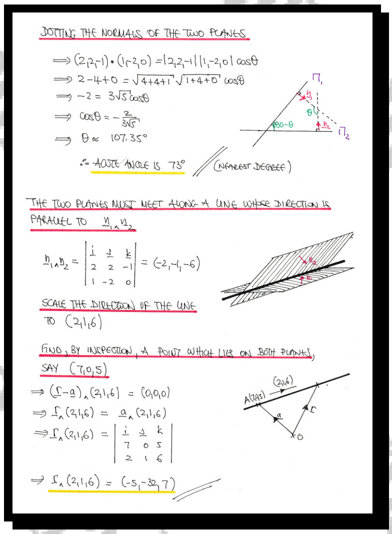
$$2x + 2y - z = 9$$
$$x - 2y = 7$$

a) Find, to the nearest degree, the acute angle between Π_1 and Π_2.

The two planes intersect along the straight line L.

b) Determine an equation of L in the form $\mathbf{r} \wedge \mathbf{a} = \mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are vectors with integer components.

$\boxed{73°}$, $\boxed{\mathbf{r} \wedge (2\mathbf{i} + \mathbf{j} + 6\mathbf{k}) = -5\mathbf{i} - 32\mathbf{j} + 7\mathbf{k}}$



DOTTING THE NORMALS OF THE TWO PLANES

$\Rightarrow (2,2,-1) \cdot (1,-2,0) = |2,2,-1||1,-2,0| \cos\theta$

$\Rightarrow 2 - 4 + 0 = \sqrt{4+4+1}\sqrt{1+4+0} \cos\theta$

$\Rightarrow -2 = 3\sqrt{5} \cos\theta$

$\Rightarrow \cos\theta = -\frac{2}{3\sqrt{5}}$

$\Rightarrow \theta \approx 107.35°$

$\therefore$ ACUTE ANGLE IS $73°$ // (NEAREST DEGREE)

THE TWO PLANES MUST MEET ALONG A LINE WHOSE DIRECTION IS PARALLEL TO $\underline{n}_1 \wedge \underline{n}_2$

$\underline{n}_1 \wedge \underline{n}_2 = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 2 & -1 \\ 1 & -2 & 0 \end{vmatrix} = (-2,-1,-6)$

SCALE THE DIRECTION OF THE LINE TO $(2,1,6)$

FIND, BY INSPECTION, A POINT WHICH LIES ON BOTH PLANES, SAY $(7,0,5)$

$\Rightarrow (\underline{r} - \underline{a}) \wedge (2,1,6) = (0,0,0)$

$\Rightarrow \underline{r} \wedge (2,1,6) = \underline{a} \wedge (2,1,6)$

$\Rightarrow \underline{r} \wedge (2,1,6) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 7 & 0 & 5 \\ 2 & 1 & 6 \end{vmatrix}$

$\Rightarrow \underline{r} \wedge (2,1,6) = (-5,-32,7)$ //

Question 34 (***+)

The straight line l has Cartesian equation

$$\frac{x-2}{2}=\frac{y-3}{3}=\frac{z-4}{2}.$$

a) Show that the point P with coordinates $(16,24,18)$ lies on l.

The point A has coordinates $(8,19,6)$ and the direction vector of l is denoted by $\mathbf{d}$.

b) Calculate $\dfrac{\overrightarrow{AP}\wedge\mathbf{d}}{|\mathbf{d}|}$.

c) Hence show that the shortest distance of A from l is exactly 6 units.

$$\boxed{\frac{(20\mathbf{i}-4\mathbf{j}-14\mathbf{k})}{\sqrt{17}}}$$

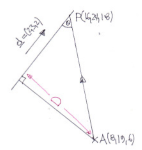
(a) $P(16,24,18)$ THUS $\left.\begin{array}{l}\frac{16-2}{2}=7\\ \frac{24-3}{3}=7\\ \frac{18-4}{2}=7\end{array}\right\}$ $\therefore$ P LIES ON l.

(b) $\dfrac{AP\wedge\underline{d}}{|\underline{d}|}=\dfrac{(\underline{p}-\underline{a})\wedge\underline{d}}{|\underline{d}|}=\dfrac{[(16,24,18)-(8,5,12)]\wedge(2,3,2)}{|2,3,2|}$

$=\dfrac{(8,19,6)\wedge(2,3,2)}{\sqrt{17}}=\dfrac{1}{\sqrt{17}}\begin{vmatrix}\underline{i}&\underline{j}&\underline{k}\\8&19&6\\2&3&2\end{vmatrix}$

$=\dfrac{1}{\sqrt{17}}(38-18,12-16,24-38)=\dfrac{1}{\sqrt{17}}(20,-4,-14)$

(c)



$D=|\overrightarrow{AP}|\sin\theta$

$D=|1||\overrightarrow{AP}|\sin\theta$

$D=|\hat{\underline{d}}||\overrightarrow{AP}|\sin\theta$

$\hat{\underline{n}}D=|\hat{\underline{d}}||\overrightarrow{AP}|\sin\theta\,\hat{\underline{n}}$

$\hat{\underline{n}}D=\hat{\underline{d}}\wedge\overrightarrow{AP}$

$\hat{\underline{n}}D=\dfrac{\underline{d}}{|\underline{d}|}\wedge\overrightarrow{AP}$

$\hat{\underline{n}}D=\dfrac{1}{\sqrt{17}}(20,-4,-14)$

$|\hat{\underline{n}}D|=\left|\dfrac{1}{\sqrt{17}}(20,-4,-14)\right|$

$D=\dfrac{1}{\sqrt{17}}\times\sqrt{400+16+196}$

$D=6$

Question 35 (***+)

The three vertices of the parallelogram $ABCD$ have coordinates

$$A(7,1,-6), \quad B(4,0,7) \quad \text{and} \quad D(-2,6,1).$$

The diagonals of the parallelogram meet at the point M.

a) Determine in any order the coordinates of M and the coordinates of C.

b) Calculate in exact simplified surd form, the area of $ABCD$.

The straight line l passes through C and is perpendicular to $ABCD$.

c) Find an equation of l, giving the answer in the form $(\mathbf{r}-\mathbf{a})\wedge\mathbf{b}=\mathbf{0}$, where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors to be found.

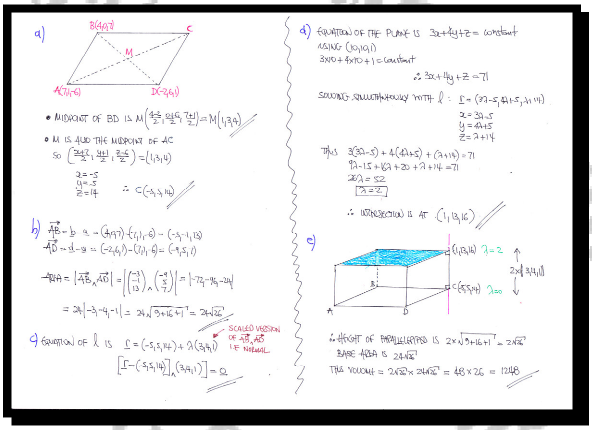
The plane Π is parallel to $ABCD$ and passes through the point with coordinates $(10,10,1)$.

d) Determine the coordinates of the point of intersection between Π and l.

The parallelogram $ABCD$ is one of the six faces of a parallelepiped whose opposite face lies in Π.

e) Calculate the volume of this parallelepiped.

$\boxed{M(1,3,4)}$, $\boxed{C(-5,5,14)}$, $\boxed{\text{area}=24\sqrt{26}}$, $\boxed{\mathbf{a}=-5\mathbf{i}+5\mathbf{j}+14\mathbf{k}}$, $\boxed{\mathbf{b}=3\mathbf{i}+4\mathbf{j}+\mathbf{k}}$, $\boxed{(1,13,6)}$, $\boxed{\text{volume}=1248}$



Question 36 (***+)

Three planes have the following Cartesian equations.

$$x-3y-2z=2$$
$$2x-2y+3z=1$$
$$5x-7y+4z=k$$

where k is a constant.

Determine the intersection of the three planes, stating any restrictions in the value of k.

$\boxed{}$, $\boxed{\mathbf{r}=3\mathbf{i}+\mathbf{j}-\mathbf{k}+t(13\mathbf{i}+7\mathbf{j}-4\mathbf{k})}$

- WRITE THE EQUATIONS OF THE PLANES IN MATRIX FORM

$$\left.\begin{array}{l} x-3y-2z=2 \\ 2x-2y+3z=1 \\ 5x-7y+4z=k \end{array}\right\} \Rightarrow \begin{bmatrix} 1 & -3 & -2 \\ 2 & -2 & 3 \\ 5 & -7 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 2 \\ 1 \\ k \end{bmatrix}$$

- CHECK IF A UNIQUE SOLUTION EXISTS (IN TERMS OF k)

$$\begin{vmatrix} 1 & -3 & -2 \\ 2 & -2 & 3 \\ 5 & -7 & 4 \end{vmatrix} = \begin{vmatrix} -2 & 3 \\ -7 & 4 \end{vmatrix} +3\begin{vmatrix} 2 & 3 \\ 5 & 4 \end{vmatrix} -2\begin{vmatrix} 2 & -2 \\ 5 & -7 \end{vmatrix}$$
$$=(-8+21)+3(8-15)-2(-14+10)$$
$$=13-21+8$$
$$=0$$

NO UNIQUE SOLUTION EXISTS

- WRITE THE SYSTEM AS AN AUGMENTED MATRIX & ROW REDUCE

$$\left[\begin{array}{ccc|c} 1 & -3 & -2 & 2 \\ 2 & -2 & 3 & 1 \\ 5 & -7 & 4 & k \end{array}\right] \xrightarrow[r_{13}(-5)]{r_{12}(-2)} \left[\begin{array}{ccc|c} 1 & -3 & -2 & 2 \\ 0 & 4 & 7 & -3 \\ 0 & 8 & 14 & k-10 \end{array}\right]$$

$$\xrightarrow{r_2(\frac{1}{4})} \left[\begin{array}{ccc|c} 1 & -3 & -2 & 2 \\ 0 & 1 & \frac{7}{4} & -\frac{3}{4} \\ 0 & 8 & 14 & k-10 \end{array}\right] \xrightarrow{r_{23}(-8)} \left[\begin{array}{ccc|c} 1 & -3 & -2 & 2 \\ 0 & 1 & \frac{7}{4} & -\frac{3}{4} \\ 0 & 0 & 0 & k-4 \end{array}\right]$$

- FOR A SOLUTION $k=4$

$$\xrightarrow{r_{21}(3)} \left[\begin{array}{ccc|c} 1 & 0 & \frac{13}{4} & -\frac{1}{4} \\ 0 & 1 & \frac{7}{4} & -\frac{3}{4} \end{array}\right]$$

- EXTRACTING A SOLUTION OUT

$$\left.\begin{array}{l} x+\frac{13}{4}z=-\frac{1}{4} \\ y+\frac{7}{4}z=-\frac{3}{4} \end{array}\right\} \Rightarrow \begin{array}{l} x=-\frac{1}{4}-\frac{13}{4}z \\ y=-\frac{3}{4}-\frac{7}{4}z \end{array}$$

- LET $z=-4t-1$

$$\left.\begin{array}{l} x=-\frac{1}{4}-\frac{13}{4}(-4t-1) \\ y=-\frac{3}{4}-\frac{7}{4}(-4t-1) \\ z=-4t-1 \end{array}\right\} \Rightarrow \begin{array}{l} x=-\frac{1}{4}+13t+\frac{13}{4} \\ y=-\frac{3}{4}+7t+\frac{7}{4} \\ z=-4t-1 \end{array}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3+13t \\ 1+7t \\ -1-4t \end{pmatrix}$$

Question 37 (***+)

The planes Π_1 and Π_2 have respective Cartesian equations

$$x + 2y - z = 1 \qquad \text{and} \qquad x + 3y + z = 6.$$

a) Find the acute angle between Π_1 and Π_2.

b) Show that Π_1 and Π_2 intersect along the straight line with equation

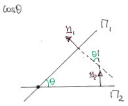
$$\mathbf{r} = (5\lambda - 9)\mathbf{i} + (5 - 2\lambda)\mathbf{j} + \lambda\mathbf{k},$$

where λ is a scalar parameter.

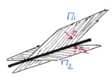
$\boxed{42.4°}$

a) DOTTING THE NORMALS OF THE PLANES

$$(1,2,-1)\cdot(1,3,1) = |1,2,-1||1,3,1|\cos\theta$$
$$1+6-1 = \sqrt{1+4+1}\sqrt{1+9+1}\cos\theta$$
$$6 = \sqrt{6\times 11}\cos\theta$$
$$\cos\theta = \frac{6}{\sqrt{66}}$$
$$\theta \approx 42.4°$$



b) THE INTERSECTION OF THE TWO PLANES WILL BE A LINE IN THE DIRECTION PARALLEL TO $n_1 \wedge n_2$ – HENCE WE HAVE:



$$n_1 \wedge n_2 = \begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ 1 & 3 & 1 \end{vmatrix} = (5,-2,1)$$

NEXT DETERMINE A POINT WHICH LIES ON BOTH PLANES

$$\left.\begin{array}{l} x+2y-z=1 \\ x+3y+z=6 \end{array}\right\} \Rightarrow \text{LET } y=0 \ldots \quad \begin{array}{l} x-z=1 \\ x+z=6 \end{array}$$

$$\Rightarrow x = \tfrac{7}{2},\ z = \tfrac{5}{2}$$

THE EQUATION OF LINE CAN BE WRITTEN AS

$$\underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{7}{2} \\ 0 \\ \frac{5}{2} \end{pmatrix} + t\begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 5t+\frac{7}{2} \\ -2t \\ t+\frac{5}{2} \end{pmatrix}$$

LOOKING AT THE REQUIRED RESULT, $\lambda = t + \frac{5}{2}$ ($\underline{k}$ COMPONENT)

$$\underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{7}{2} \\ 0 \\ \frac{5}{2} \end{pmatrix} + \left(\lambda - \frac{5}{2}\right)\begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}$$

$$\underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{7}{2} \\ 0 \\ \frac{5}{2} \end{pmatrix} + \lambda\begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix} + \begin{pmatrix} -\frac{25}{2} \\ 5 \\ -\frac{5}{2} \end{pmatrix}$$

$$\underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -9 \\ 5 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}$$

$$\underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5\lambda-9 \\ 5-2\lambda \\ \lambda \end{pmatrix}$$

AS REQUIRED

Question 38 (*+)**

It is given that the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ satisfy

$$\mathbf{b} \wedge \mathbf{c} = 2\mathbf{i} \qquad \text{and} \qquad \mathbf{a} \wedge \mathbf{c} = \mu \mathbf{j},$$

where μ is a scalar constant.

It is further given that the vector expression defined as

$$(\mathbf{a} + 2\mathbf{b} - 3\mathbf{c}) \wedge (\mathbf{a} + 2\mathbf{b} + k\mathbf{c}),$$

where k is a scalar constant, is parallel to the vector $\mathbf{i} - \mathbf{j}$.

Determine the condition that μ and k must satisfy.

$\boxed{}$, $\boxed{k \neq 3}$, $\boxed{\mu = 4}$

PROCEED AS FOLLOWS

$(\mathbf{a} + 2\mathbf{b} - 3\mathbf{c}) \wedge (\mathbf{a} + 2\mathbf{b} + k\mathbf{c}) = \lambda(\mathbf{i} - \mathbf{j})$

AS THE "CROSS PRODUCT" IS DISTRIBUTIVE OVER ADDITION/SUBTRACTION

$\Rightarrow [[\mathbf{a} + 2\mathbf{b}] - 3\mathbf{c}] \wedge [[\mathbf{a} + 2\mathbf{b}] + k\mathbf{c}] = \lambda(\mathbf{i} - \mathbf{j})$

$\Rightarrow \cancel{(\mathbf{a} + 2\mathbf{b}) \wedge (\mathbf{a} + 2\mathbf{b})} + (\mathbf{a} + 2\mathbf{b}) \wedge k\mathbf{c} - \cancel{3k\mathbf{c} \wedge \mathbf{c}} = \lambda(\mathbf{i} - \mathbf{j})$
$\qquad - 3\mathbf{c} \wedge (\mathbf{a} + 2\mathbf{b})$

$\Rightarrow (\mathbf{a} + 2\mathbf{b}) \wedge k\mathbf{c} + 3(\mathbf{a} + 2\mathbf{b}) \wedge \mathbf{c} = \lambda(\mathbf{i} - \mathbf{j})$

$\Rightarrow k\mathbf{a} \wedge \mathbf{c} + 2k\mathbf{b} \wedge \mathbf{c} + 3\mathbf{a} \wedge \mathbf{c} + 6\mathbf{b} \wedge \mathbf{c} = \lambda(\mathbf{i} - \mathbf{j})$

$\Rightarrow (k+3)(\mathbf{a} \wedge \mathbf{c}) + (2k+6)(\mathbf{b} \wedge \mathbf{c}) = \lambda(\mathbf{i} - \mathbf{j})$

BUT $\mathbf{b} \wedge \mathbf{c} = 2\mathbf{i}$ & $\mathbf{a} \wedge \mathbf{c} = \mu\mathbf{j}$

$\Rightarrow (k+3)(\mu\mathbf{j}) + (2k+6)(2\mathbf{i}) = \lambda\mathbf{i} - \lambda\mathbf{j}$

COMPARING COMPONENTS

$\left\{ \begin{array}{l} 4k+12 = \lambda \\ (k+3)\mu = -\lambda \end{array} \right\}$ ADDING GIVES

$4k+12 + \mu(k+3) = 0$
$4(k+3) + \mu(k+3) = 0$
$(k+3)(\mu+4) = 0$
$k = 3$ OR $\mu = -4$

FINALLY WE HAVE

$k \neq 3$ AND $\mu = -4$

(YIELDS NO DIRECTION)

Question 39 (***+)

The position vector $\mathbf{r}$ of a variable point traces the plane Π with equation

$$\mathbf{r} = (4+\lambda+5\mu)\mathbf{i} + (8+2\lambda-4\mu)\mathbf{j} + (-5+\lambda+7\mu)\mathbf{k},$$

where λ and μ are parameters.

a) Express the equation of Π in the form

$$\mathbf{r}\cdot\mathbf{n} = c,$$

where $\mathbf{n}$ and c is a vector and scalar constant, respectively.

The point $P(12,-1,44)$ is reflected about Π onto the point P'.

b) Determine the coordinates of P'.

$\boxed{}$, $\boxed{\mathbf{r}\cdot(9\mathbf{i}-\mathbf{j}-7\mathbf{k}) = 63}$, $\boxed{P'(48,-5,16)}$

a) ELIMINATE TO CARTESIAN FIRST

$$\left.\begin{array}{l} x = 4+\lambda+5\mu \\ y = 8+2\lambda-4\mu \\ z = -5+\lambda+7\mu \end{array}\right\} \Rightarrow \lambda = z+5-7\mu$$

SUBSTITUTE INTO THE FIRST TWO EQUATIONS

$$\left.\begin{array}{l} x = 4+(z+5-7\mu)+5\mu \\ y = 8+2(z+5-7\mu)-4\mu \end{array}\right\} \Rightarrow$$

$$\begin{array}{ll} x = 9+z-2\mu & \times 9 \\ y = 18+2z-18\mu & \times(-1) \end{array}$$

$$\left.\begin{array}{l} 9x = 81+9z-18\mu \\ -y = -18-2z+18\mu \end{array}\right\} \text{ADDING}$$

$$9x-y = 63+7z$$

$$9x-y-7z = 63$$

FINALLY THE EQUATION OF THE PLANE CAN BE WRITTEN AS

$$(9,-1,-7)\cdot(x,y,z) = 63$$

$$\underline{r}\cdot\begin{pmatrix}9\\-1\\-7\end{pmatrix} = 63$$

b) DETERMINE THE EQUATION OF A LINE THROUGH $P(12,-1,44)$ & IN THE DIRECTION OF THE NORMAL

$$r = (12,-1,44) + t(9,-1,-7)$$

$$(x,y,z) = (9t+12, -t-1, -7t+44)$$

SOLVE SIMULTANEOUSLY WITH THE EQUATION OF THE PLANE

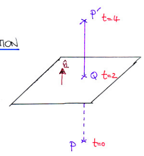
$$\begin{array}{l} x = 9t+12 \\ y = -t-1 \\ z = 44-7t \end{array} \quad \& \quad 9x-y-7z = 63$$

$$\Rightarrow 9(9t+12)-(-t-1)-7(44-7t) = 63$$

$$\Rightarrow 81t+t+49t+108+1-308 = 63$$

$$\Rightarrow 131t = 262$$

$$\Rightarrow t = 2$$



USING $t=4$ WE OBTAIN THE REFLECTION

$P'(48,-5,16)$

ALTERNATIVE/VARIATION

- USE $t=2$ TO FIND $Q(30,-3,30)$

  THIS LIES ON $9x-y-7z=63$

- THEN USE "MIDPOINT PATTERNS"

$$12 \xrightarrow{+18} 30 \xrightarrow{+18} 48$$

$$-1 \xrightarrow{-2} -3 \xrightarrow{-2} -5$$

$$44 \xrightarrow{-14} 30 \xrightarrow{-14} 16$$

P   Q   P'

Question 40 (****)

The plane Π has a vector equation

$$\mathbf{r} = (1 + 4\lambda + 3\mu)\mathbf{i} + (3 + \lambda + 2\mu)\mathbf{j} + (4 + 2\lambda - \mu)\mathbf{k},$$

where λ and μ are scalar parameters.

The straight line L has a vector equation

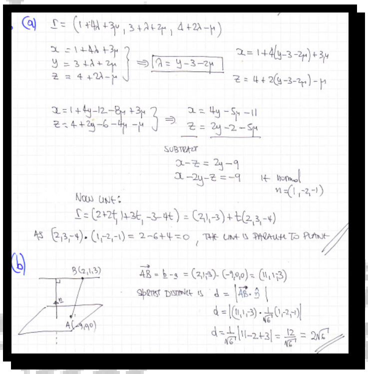
$$\mathbf{r} = (2 + 2t)\mathbf{i} + (1 + 3t)\mathbf{j} + (-3 - 4t)\mathbf{k},$$

where t is a scalar parameter.

a) Show that L is parallel to Π.

b) Find the shortest distance between L and Π.

$\boxed{2\sqrt{6}}$



(a) $\underline{r} = (1+4\lambda+3\mu,\ 3+\lambda+2\mu,\ 4+2\lambda-\mu)$

$\left.\begin{array}{l} x = 1+4\lambda+3\mu \\ y = 3+\lambda+2\mu \\ z = 4+2\lambda-\mu \end{array}\right\} \Rightarrow \boxed{\lambda = y-3-2\mu}$

$x = 1+4(y-3-2\mu)+3\mu$

$z = 4+2(y-3-2\mu)-\mu$

$\left.\begin{array}{l} x = 1+4y-12-8\mu+3\mu \\ z = 4+2y-6-4\mu-\mu \end{array}\right\} \Rightarrow \begin{array}{l} x = 4y-5\mu-11 \\ z = 2y-2-5\mu \end{array}$

SUBTRACT

$x - z = 2y - 9$

$x - 2y - z = -9$ it normal $n = (1,-2,-1)$

NOW LINE:

$\underline{r} = (2+2t,\ 1+3t,\ -3-4t) = (2,1,-3) + t(2,3,-4)$

AS $(2,3,-4)\cdot(1,-2,-1) = 2-6+4 = 0$, THE LINE IS PARALLEL TO PLANE

(b) $B(2,1,3)$ $A(-9,0,0)$ n

$\vec{AB} = \underline{b} - \underline{a} = (2,1,3) - (-9,0,0) = (11,1,3)$

SHORTEST DISTANCE IS $d = |\vec{AB}\cdot\hat{n}|$

$d = \left|(11,1,-3)\cdot\frac{1}{\sqrt{6}}(1,-2,-1)\right|$

$d = \frac{1}{\sqrt{6}}|11-2+3| = \frac{12}{\sqrt{6}} = 2\sqrt{6}$

Question 41 (****)

Relative to a fixed origin O, the following points are given.

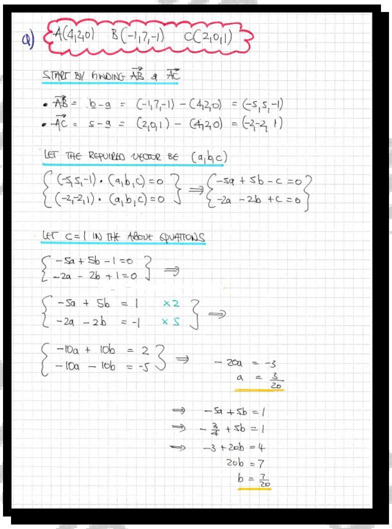
$$A(4,2,0), \quad B(-1,7,-1) \quad \text{and} \quad C(2,0,1).$$

a) Determine a vector, with integer components, which is perpendicular to both $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

*You may **NOT** use the vector (cross) product for this part.*

b) Deduce a Cartesian equation of the plane, which passes through A, B and C.

$\boxed{}$, $\boxed{3\mathbf{i}+7\mathbf{j}+20\mathbf{k}}$, $\boxed{3x+7y+20z=26}$



a) $A(4,2,0) \quad B(-1,7,-1) \quad C(2,0,1)$

START BY FINDING $\overrightarrow{AB}$ & $\overrightarrow{AC}$

- $\overrightarrow{AB} = \underline{b} - \underline{a} = (-1,7,-1) - (4,2,0) = (-5,5,-1)$
- $\overrightarrow{AC} = \underline{c} - \underline{a} = (2,0,1) - (4,2,0) = (-2,-2,1)$

LET THE REQUIRED VECTOR BE (a,b,c)

$$\left\{\begin{array}{l}(-5,5,-1)\cdot(a,b,c)=0\\(-2,-2,1)\cdot(a,b,c)=0\end{array}\right\} \Longrightarrow \left\{\begin{array}{l}-5a+5b-c=0\\-2a-2b+c=0\end{array}\right\}$$

LET $c=1$ IN THE ABOVE EQUATIONS

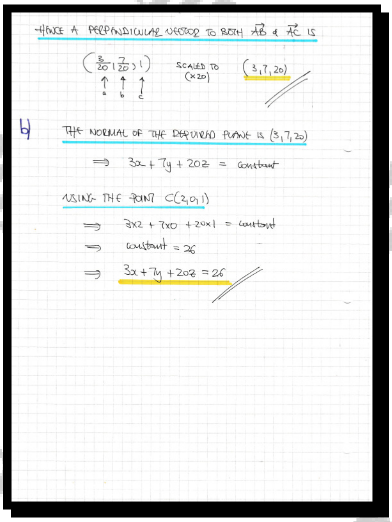
$$\left\{\begin{array}{l}-5a+5b-1=0\\-2a-2b+1=0\end{array}\right\} \Longrightarrow$$

$$\left\{\begin{array}{ll}-5a+5b=1 & \times 2\\-2a-2b=-1 & \times 5\end{array}\right\} \Longrightarrow$$

$$\left\{\begin{array}{l}-10a+10b=2\\-10a-10b=-5\end{array}\right\} \Longrightarrow \quad -20a=-3$$

$$a = \frac{3}{20}$$

$$\Longrightarrow -5a+5b=1$$
$$\Longrightarrow -\frac{3}{4}+5b=1$$
$$\Longrightarrow -3+20b=4$$
$$20b=7$$
$$b=\frac{7}{20}$$



HENCE A PERPENDICULAR VECTOR TO BOTH $\overrightarrow{AB}$ & $\overrightarrow{AC}$ IS

$$\left(\frac{3}{20}, \frac{7}{20}, 1\right) \quad \underset{(\times 20)}{\text{SCALED TO}} \quad (3,7,20)$$

b) THE NORMAL OF THE REQUIRED PLANE IS $(3,7,20)$

$$\Longrightarrow 3x+7y+20z = \text{constant}$$

USING THE POINT $C(2,0,1)$

$$\Longrightarrow 3\times2+7\times0+20\times1 = \text{constant}$$
$$\Longrightarrow \text{constant} = 26$$
$$\Longrightarrow 3x+7y+20z=26$$

Question 42 (****)

The straight lines L_1 and L_2 have respective Cartesian equations

$$\frac{x-2}{2}=\frac{y-3}{4}=z \quad \text{and} \quad \frac{x+2}{2}=\frac{4y}{11}=\frac{z+10}{3}.$$

a) Show that L_1 and L_2 intersect at some point P and find its coordinates.

b) Show further that the Cartesian vector $37\mathbf{i}-16\mathbf{j}-10\mathbf{k}$ is perpendicular to both L_1 and L_2.

The plane Π is defined by L_1 and L_2.

The point $Q(2,5,-2)$ does not lie on Π.

The straight line L_3 passes through Q and P.

c) Calculate the acute angle formed between L_3 and Π.

$\boxed{}$, $\boxed{P(6,11,2)}$, $\boxed{\theta \approx 2.00°}$

a) START BY WRITING THE EQUATIONS IN PARAMETRIC

$$L_1: \frac{x-2}{2}=\frac{y-3}{4}=\frac{z-0}{1}$$
$$L_2: \frac{x+2}{2}=\frac{y-0}{11/4}=\frac{z+10}{3} \Rightarrow$$

$$\mathbf{r}_1 = (2,3,0)+\lambda(2,4,1) = (2\lambda+2, 4\lambda+3, \lambda)$$
$$\mathbf{r}_2 = (-2,0,-10)+\mu'(2,\tfrac{11}{4},3) + \mu(8,11,12) = (8\mu-2, 11\mu, 12\mu-10)$$

EQUATE $\underline{j}$ & $\underline{k}$

$\underline{j}$: $11\mu = 4\lambda+3$
$\underline{k}$: $\lambda = 12\mu-10$

$\Rightarrow 11\mu = 4(12\mu-10)+3$
$\Rightarrow 11\mu = 48\mu-37$
$37 = 37\mu$
$\mu = 1$
&
$\lambda = 2$

CHECK $\underline{i}$: $2\lambda+2 = 2\times2+2 = 6$
$8\mu-2 = 8\times1-2 = 6$

AS ALL 3 COMPONENTS AGREE (IF $\lambda=2, \mu=1$), THE LINES MEET AT THE POINT $P(6,11,2)$

b) DOTTING THE GIVEN VECTOR WITH THE DIRECTION VECTORS

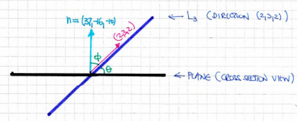
$(37,-16,-10)\cdot(2,4,1) = 74-64-10 = 0$
$(37,-16,-10)\cdot(8,11,12) = 296-176-120 = 0$ INDEED PERPENDICULAR

c) THE EQUATION OF THE PLANE IS NOT ACTUALLY NEEDED

- THE PLANE NORMAL IS $\underline{n} = (37,-16,-10)$
- THE LINE L_3 PASSES THROUGH THE INTERSECTION $(6,11,2)$ AND THE GIVEN POINT $(2,5,-2)$
- DIRECTION OF L_3 IS GIVEN BY

$$(6,11,2)-(2,5,-2) = (4,6,4) \sim (2,3,2)$$

HENCE WE HAVE LOOKING AT A DIAGRAM



$\Rightarrow (37,-16,-10)\cdot(2,3,2) = |37,-16,-10||2,3,2|\cos\phi$
$\Rightarrow 74-48-20 = \sqrt{1369+256+100}\sqrt{4+9+4}\cos\phi$
$\Rightarrow 6 = \sqrt{1725\times17}\cos\phi$
$\Rightarrow \phi = 87.992\ldots$

$\therefore$ REQUIRED ANGLE $\theta \simeq 2.00°$

Question 43 (****)

Relative to a fixed origin O, the following points are given.

$$A(7,2,6), \quad B(9,10,4) \quad \text{and} \quad C(-3,-2,-2).$$

a) Determine a vector, with integer components, which is perpendicular to both $\overrightarrow{AB}$ and $\overrightarrow{AC}$, and hence deduce a Cartesian equation of the plane Π, which passes through A, B and C.

*You may **NOT** use the vector (cross) product for this part.*

The straight line l is perpendicular to Π and passes through the point $P(11,3,-4)$.

The point Q is the intersection of l and Π.

b) Find the coordinates of Q.

c) Calculate the distance PQ.

$\boxed{}$, $\boxed{2\mathbf{i}-\mathbf{j}+2\mathbf{k}}$, $\boxed{2x-y-2z=0}$, $\boxed{Q(5,6,2)}$, $\boxed{|PQ|=3}$

$A(7,2,6) \quad B(9,10,4) \quad C(-3,-2,-2)$

a) START BY FINDING $\overrightarrow{AB}$ & $\overrightarrow{AC}$

$\overrightarrow{AB} = (9,10,4)-(7,2,6) = (2,8,-2) \sim (1,4,-1)$

$\overrightarrow{AC} = (-3,-2,-2)-(7,2,6) = (-10,-4,-8) \sim (5,2,4)$

LET THE REQUIRED VECTOR BE (p,q,r)

$\left\{\begin{array}{l}(p,q,r)\cdot(1,4,-1)=0\\(p,q,r)\cdot(5,2,4)=0\end{array}\right\} \Rightarrow \left\{\begin{array}{l}p+4q-r=0\\5p+2q+4r=0\end{array}\right\}$

LET $r=1$ IN THE ABOVE EQUATIONS

$\left\{\begin{array}{ll}p+4q-1=0 & \times -5\\5p+2q+4=0 & \times 1\end{array}\right\} \Rightarrow$

$\left\{\begin{array}{l}-5p-20q+5=0\\5p+2q+4=0\end{array}\right\} \Rightarrow -18q+9=0$

$\Rightarrow 18q=9$

$\Rightarrow q=\frac{1}{2}$

$\Rightarrow p+4q-1=0$

$\Rightarrow p+2-1=0$

$\Rightarrow p=-1$

HENCE A PERPENDICULAR VECTOR TO BOTH $\overrightarrow{AB}$ & $\overrightarrow{AC}$ IS

$(p,q,r) = (-1,\frac{1}{2},1) \underset{\times -2}{\sim} (2,-1,-2)$

HENCE AN EQUATION OF THE REQUIRED PLANE IS

$2x-y-2z = \text{constant}$

USING THE POINT $A(7,2,6)$

$(2\times7)-2-(2\times6) = \text{constant}$

$\text{constant} = 0$

$\therefore 2x-y-2z=0$

b) THE REQUIRED LINE HAS DIRECTION VECTOR $(2,-1,2)$

$\Rightarrow \underline{r} = (\text{FIXED POINT}) + \lambda(\text{DIRECTION VECTOR})$

$\Rightarrow \underline{r} = (11,3,-4) + \lambda(2,-1,-2)$

$\Rightarrow (x,y,z) = (2\lambda+11, -\lambda+3, -2\lambda-4)$

SOLVING SIMULTANEOUSLY WITH $2x-y-2z=0$

$\Rightarrow 2(2\lambda+11)-(-\lambda+3)-2(-2\lambda-4)=0$

$\Rightarrow 4\lambda+22+\lambda-3+4\lambda+8=0$

$\Rightarrow 9\lambda=-27$

$\Rightarrow \lambda=-3$

$2\lambda+11 \quad -\lambda-3 \quad -2\lambda-4$

$\therefore Q(5,6,2)$

c) FINALLY TO FIND THE DISTANCE

$\left.\begin{array}{l}P(11,3,-4)\\Q(5,6,2)\end{array}\right\} \Rightarrow |\overrightarrow{PQ}| = |\underline{q}-\underline{p}|$

$= |(5,6,2)-(11,3,-4)|$

$= |(-6,3,6)|$

$= \sqrt{36+9+36}$

$= \sqrt{81}$

$= 9$

ALTERNATIVE

SINCE $|\text{DIRECTION VECTOR}| = |2,-1,-2| = 3$
AND $\lambda=-3$, THE REQUIRED DISTANCE
WILL BE $3\times|\lambda| = 9$

Question 44 (****)

The straight line L and the plane Π have equations

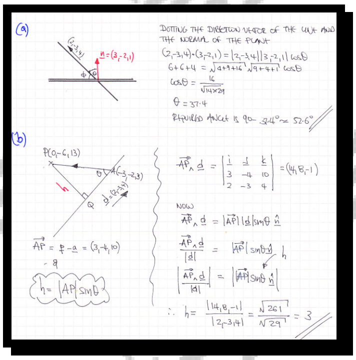
$$L : \quad \mathbf{r} = -3\mathbf{i} - 2\mathbf{j} + 3\mathbf{k} + \lambda(2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k})$$

$$\Pi : \quad 3x - 2y + z = 5$$

a) Find the size of the acute angle between L and Π.

b) Use a method involving the cross product to show that the shortest distance of the point $(0,-6,13)$ from L is 3 units.

$\boxed{52.6°}$



Question 45 (****)

The equations of two planes are given below

$$\mathbf{r}\cdot(6\mathbf{i}-3\mathbf{j}+2\mathbf{k})=42 \quad \text{and} \quad \mathbf{r}\cdot(17\mathbf{i}+2\mathbf{j}+\mathbf{k})=-7.$$

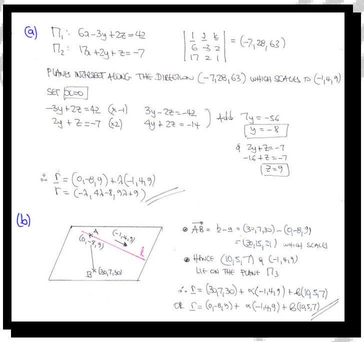
The straight line l is the intersection of the two planes.

a) Find an equation for l, in the form $\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors and λ is a scalar parameter.

A third plane Π_3 contains l and the point with position vector $30\mathbf{i}+7\mathbf{j}+30\mathbf{k}$.

b) Find an equation for Π_3, in the form $\mathbf{r}=\mathbf{u}+\alpha\mathbf{v}+\beta\mathbf{w}$, where $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are constant vectors and α and β are scalar parameters.

$\boxed{\mathbf{r}=-8\mathbf{j}+9\mathbf{k}+\lambda(-\mathbf{i}+4\mathbf{j}+9\mathbf{k})}$, $\boxed{\mathbf{r}=(-8\mathbf{j}+9\mathbf{k})+\alpha(-\mathbf{i}+4\mathbf{j}+9\mathbf{k})+\beta(10\mathbf{i}+5\mathbf{j}+7\mathbf{k})}$



(a) $\Pi_1: 6x-3y+2z=42$

$\Pi_2: 17x+2y+z=-7$

$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & -3 & 2 \\ 17 & 2 & 1 \end{vmatrix} = (-7,28,63)$

PLANES INTERSECT ALONG THE DIRECTION $(-7,28,63)$ WHICH SCALES TO $(-1,4,9)$

SET $\boxed{x=0}$

$-3y+2z=42 \quad (\times -1) \qquad 3y-2z=-42$

$2y+z=-7 \quad (\times 2) \qquad 4y+2z=-14$

ADD $7y=-56$, $\boxed{y=-8}$

& $2y+z=-7$

$-16+z=-7$

$\boxed{z=9}$

$\therefore \underline{r}=(0,-8,9)+\lambda(-1,4,9)$

$r=(-\lambda, 4\lambda-8, 9\lambda+9)$

(b)

A $(0,-8,9)$, $(-1,4,9)$, l, B $(30,7,30)$

• $\overrightarrow{AB}=\underline{b}-\underline{a}=(30,7,30)-(0,-8,9)$

$=(30,15,21)$ WHICH SCALES

• HENCE $(10,5,7)$ & $(-1,4,9)$ LIE ON THE PLANE Π_3

$\therefore \underline{r}=(30,7,30)+\alpha(-1,4,9)+\beta(10,5,7)$

OR $\underline{r}=(0,-8,9)+\alpha(-1,4,9)+\beta(10,5,7)$

Question 46 (****)

A triangle has vertices at $A(-2,-2,0)$, $B(6,8,6)$ and $C(-6,8,12)$.

a) Find the area of the triangle ABC.

The plane Π_1 contains the point B and is perpendicular to AB.

b) Show that an equation of Π_1 is

$$4x+5y+3z=82.$$

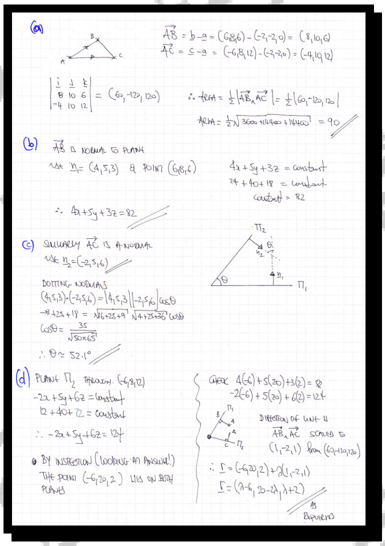
The plane Π_2 contains the point C and is perpendicular to AC.

c) Find the size of the acute angle between Π_1 and Π_2.

d) Show that the intersection of Π_1 and Π_2 is

$$(\lambda-6)\mathbf{i}+(20-2\lambda)\mathbf{j}+(2\lambda+2)\mathbf{k}.$$

$\boxed{\text{area}=90}$, $\boxed{52.1°}$



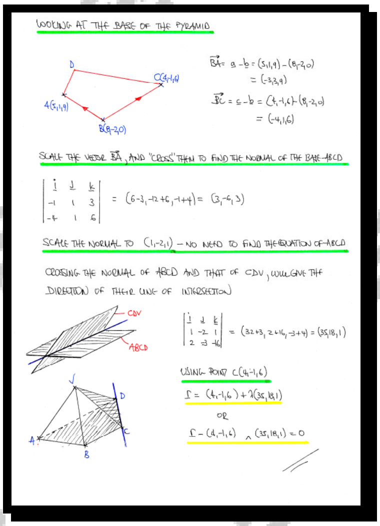
Question 47 (****)

The plane quadrilateral $ABCD$ is the base of a pyramid with vertex V.

The coordinates of the points A, B and C are $(5,1,9)$, $(8,-2,0)$ and $(4,-1,6)$, respectively.

If the equation of the face CDV is $2x-3y-16z+85=0$ determine the vector equation of the line CD.

$$\boxed{\mathbf{r}=(4\mathbf{i}-\mathbf{j}+6\mathbf{k})+\lambda(35\mathbf{i}+18\mathbf{j}+\mathbf{k}) \text{ or } [\mathbf{r}-(4\mathbf{i}-\mathbf{j}+6\mathbf{k})]\wedge(35\mathbf{i}+18\mathbf{j}+\mathbf{k})=\mathbf{0}}$$



Question 48 (****)

A straight line L and a plane Π have respective cartesian equations

$$L: \ x-3=2-y=\tfrac{1}{4}(2z-5) \quad \text{and} \quad \Pi: \ 2x+ky+z=13,$$

where k is a constant.

Given that the acute angle between L and Π is $30°$, find the possible values of k.

$\boxed{}$, $\boxed{k=1 \ \cup \ k=-17}$

EXTRACT DIRECTIONS FROM THE GIVEN EQUATIONS

- $x-3 = 2-y = \frac{2z-5}{4}$

  $\frac{x-3}{1} = \frac{y-2}{-1} = \frac{z-5/2}{2}$

  $\therefore$ DIRECTION OF L IS $(1,-1,2)$

- $2x+ky+z=13$

  NORMAL OF Π IS $(2,k,1)$

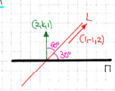
BY THE DOT PRODUCT LOOKING AT THE DIAGRAM

$|(1,-1,2)\cdot(2,k,1)| = |1,-1,2|\,|2,k,1|\cos 60°$

(mod because of acute)

$|2-k+2| = \sqrt{1+1+4} \times \sqrt{4+k^2+1} \times \frac{1}{2}$

$|4-k| = \frac{1}{2}\sqrt{6}\sqrt{k^2+5}$



SQUARING BOTH SIDES

$\Rightarrow |4-k|^2 = \left[\frac{1}{2}\sqrt{6}\sqrt{k^2+5}\right]^2$

$\Rightarrow (4-k)^2 = \frac{1}{4}\times 6 \times (k^2+5)$

$\Rightarrow 16-8k+k^2 = \frac{3}{2}(k^2+5)$

$\Rightarrow 32-16k+2k^2 = 3k^2+15$

$\Rightarrow k^2+16k-17=0$

$\Rightarrow (k-1)(k+17)=0$

$\therefore k = 1, \ -17$

Question 49 (****)

With respect to a fixed origin O the point A has position vector $\overrightarrow{OA} = -4\mathbf{i} + \mathbf{j} - 2\mathbf{k}$.

The straight line L has vector equation

$$\mathbf{r} \wedge \overrightarrow{OA} = 5\mathbf{i} - 10\mathbf{k}.$$

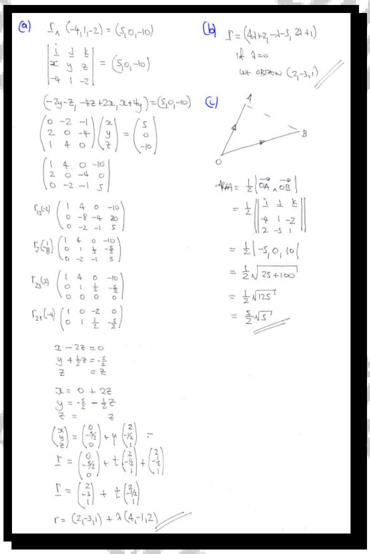
a) Find, in terms of a scalar parameter λ, a vector equation of L.

Give the answer in the form $\mathbf{r} = \mathbf{p} + \lambda\mathbf{q}$, where $\mathbf{p}$ and $\mathbf{q}$ are constant vectors.

b) Verify that the point B, with position vector $\overrightarrow{OB} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$, lies on L.

c) Find the exact area of the triangle OAB.

$\boxed{\mathbf{r} = -\frac{5}{2}\mathbf{j} + \lambda(4\mathbf{i} - \mathbf{j} + 2\mathbf{k})}$, $\boxed{\text{area} = \frac{5}{2}\sqrt{5}}$



Question 50 (****)

The planes Π_1 and Π_2 have respective Cartesian equations

$$6x+2y+9z=5 \quad \text{and} \quad 10x-y-11z=4.$$

a) Find the acute angle between Π_1 and Π_2.

b) Show that Π_1 and Π_2 intersect along the straight line with equation

$$\mathbf{r}=\mathbf{i}-5\mathbf{j}+\mathbf{k}+t(\mathbf{i}-12\mathbf{j}+2\mathbf{k}),$$

where t is a scalar parameter.

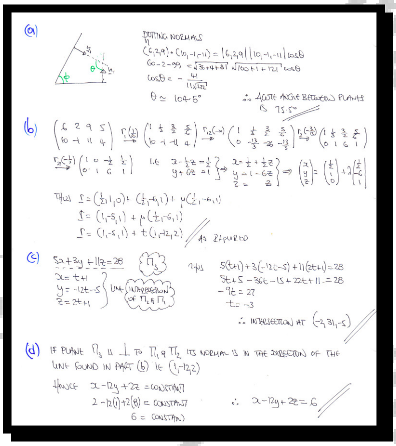
The plane Π_3 has Cartesian equation

$$5x+3y+11z=28.$$

c) Find the coordinates of the point of intersection of all three planes.

d) Determine an equation of the plane that passes through the point $(2,1,8)$ and is perpendicular to both Π_1 and Π_2.

$\boxed{75.5°}$, $\boxed{(-2,31,-5)}$, $\boxed{x-12y+2z=6}$


(a) DOTTING NORMALS

$(6,2,9)\cdot(10,-1,-11)=|6,2,9||10,-1,-11|\cos\theta$

$60-2-99=\sqrt{36+4+81}\sqrt{100+1+121}\cos\theta$

$\cos\theta=-\frac{41}{11\sqrt{222}}$

$\theta\simeq 104.5°$ ∴ ACUTE ANGLE BETWEEN PLANES IS $75.5°$

(b) $\begin{pmatrix}6&2&9&5\\10&-1&11&4\end{pmatrix} \xrightarrow{r_1(\frac{1}{6})} \begin{pmatrix}1&\frac{1}{3}&\frac{3}{2}&\frac{5}{6}\\10&-1&-11&4\end{pmatrix} \xrightarrow{r_{12}(-10)} \begin{pmatrix}1&\frac{1}{3}&\frac{3}{2}&\frac{5}{6}\\0&-\frac{13}{3}&-26&-\frac{13}{3}\end{pmatrix} \xrightarrow{r_2(-\frac{3}{13})} \begin{pmatrix}1&\frac{1}{3}&\frac{3}{2}&\frac{5}{6}\\0&1&6&1\end{pmatrix}$

$\xrightarrow{r_{21}(-\frac{1}{3})} \begin{pmatrix}1&0&-\frac{1}{2}&\frac{1}{2}\\0&1&6&1\end{pmatrix}$ i.e. $\left.\begin{array}{l}x-\frac{1}{2}z=\frac{1}{2}\\ y+6z=1\end{array}\right\} \Rightarrow \left.\begin{array}{l}x=\frac{1}{2}+\frac{1}{2}z\\ y=1-6z\\ z=z\end{array}\right\} \Rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}\frac{1}{2}\\1\\0\end{pmatrix}+\lambda\begin{pmatrix}\frac{1}{2}\\-6\\1\end{pmatrix}$

Thus $\mathbf{r}=(\frac{1}{2},1,0)+(\frac{1}{2},-6,1)+\mu(\frac{1}{2},-6,1)$

$\mathbf{r}=(1,-5,1)+\mu(\frac{1}{2},-6,1)$

$\mathbf{r}=(1,-5,1)+t(1,-12,2)$ AS REQUIRED

(c) $5x+3y+11z=28$ Π_3

$\left.\begin{array}{l}x=t+1\\ y=-12t-5\\ z=2t+1\end{array}\right\}$ LINE (INTERSECTION OF Π_2 & Π_1)

Thus $5(t+1)+3(-12t-5)+11(2t+1)=28$

$5t+5-36t-15+22t+11=28$

$-9t=27$

$t=-3$

∴ INTERSECTION AT $(-2,31,-5)$

(d) IF PLANE Π_3 IS $\perp$ TO Π_1 & Π_2 ITS NORMAL IS IN THE DIRECTION OF THE LINE FOUND IN PART (b) I.E. $(1,-12,2)$

HENCE $x-12y+2z=$ CONSTANT

$2-12(1)+2(8)=$ CONSTANT

$6=$ CONSTANT

∴ $x-12y+2z=6$

Question 51 **(****)**

The points $P(2,2,1)$ and $Q(6,-7,-1)$ lie on the plane Π with Cartesian equation

$$cx + 4y - 12z = k \, ,$$

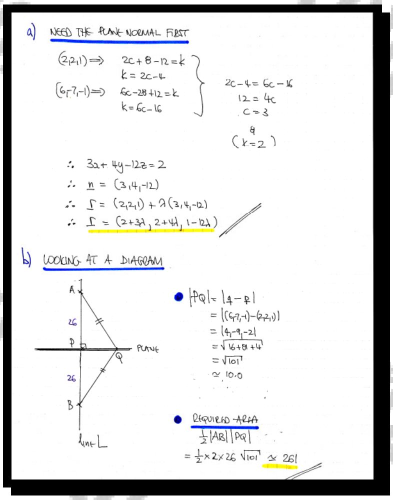
where c and k are constants.

a) Determine an equation of the straight line L, which is perpendicular to Π and passing through P.

The points A and B are both located on L and each of these points is at a distance of 26 units from Π.

b) Show that the area of the triangle ABQ is approximately 261 square units.

$\boxed{\mathbf{r} = (3\lambda + 2)\mathbf{i} + (4\lambda + 2)\mathbf{j} + (1 - 12\lambda)\mathbf{k}}$


a) NEED THE PLANE NORMAL FIRST

$(2,2,1) \Longrightarrow 2c + 8 - 12 = k$, $k = 2c - 4$

$(6,-7,-1) \Longrightarrow 6c - 28 + 12 = k$, $k = 6c - 16$

$2c - 4 = 6c - 16$
$12 = 4c$
$c = 3$
& $(k = 2)$

$\therefore 3x + 4y - 12z = 2$
$\therefore \underline{n} = (3,4,-12)$
$\therefore \underline{r} = (2,2,1) + \lambda(3,4,-12)$
$\therefore \underline{r} = (2+3\lambda, 2+4\lambda, 1-12\lambda)$

b) LOOKING AT A DIAGRAM

- $|PQ| = |\underline{q} - \underline{p}|$
$= |(6,-7,-1) - (2,2,1)|$
$= |4,-9,-2|$
$= \sqrt{16+81+4}$
$= \sqrt{101}$
$\simeq 10.0$

- REQUIRED AREA
$\frac{1}{2}|AB||PQ|$
$= \frac{1}{2} \times 2 \times 26\sqrt{101} \simeq 261$

Question 52 (****)

The plane Π_1 contains the origin O and the points $A(2,0,-1)$ and $B(4,3,1)$.

a) Find a Cartesian equation of Π_1.

The plane Π_2 contains the point B and has normal vector $\mathbf{n} = 3\mathbf{i} + \mathbf{j} - \mathbf{k}$.

b) Determine an equation of the plane in the form $\mathbf{r} \cdot \mathbf{n} = d$.

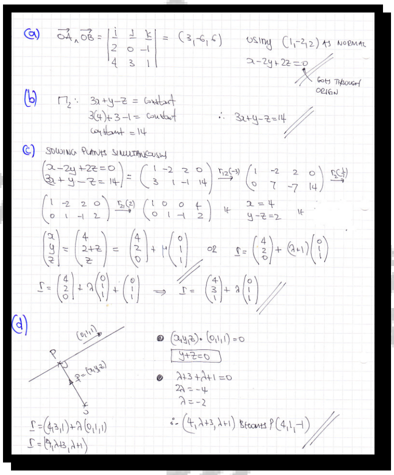
The straight line L is the intersection of Π_1 and Π_2.

The point P lies on L so that OP is perpendicular to L.

c) Find a vector equation of L.

d) Determine the coordinates of P.

$\boxed{x - 2y + 2z = 0}$, $\boxed{\mathbf{r} \cdot (3\mathbf{i} + \mathbf{j} - \mathbf{k}) = 14}$, $\boxed{\mathbf{r} = 4\mathbf{i} + 3\mathbf{j} + \mathbf{k} + \lambda(\mathbf{j} + \mathbf{k})}$, $\boxed{P(4,1,-1)}$



Question 53 (****)

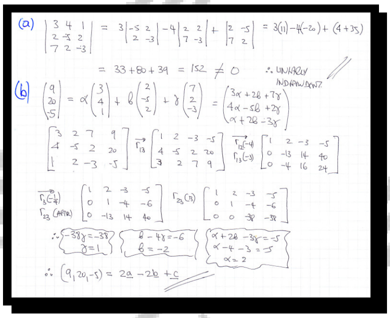
The following vectors are given

$$\mathbf{a} = 3\mathbf{i} + 4\mathbf{j} + \mathbf{k}$$
$$\mathbf{b} = 2\mathbf{i} - 5\mathbf{j} + 2\mathbf{k}$$
$$\mathbf{c} = 7\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$$

a) Show that the vectors are linearly independent.

b) Express the vector $9\mathbf{i} + 20\mathbf{j} - 5\mathbf{k}$ in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

$$\boxed{9\mathbf{i} + 20\mathbf{j} - 5\mathbf{k} = 2\mathbf{a} - 2\mathbf{b} + \mathbf{c}}$$



(a) $\begin{vmatrix} 3 & 4 & 1 \\ 2 & -5 & 2 \\ 7 & 2 & -3 \end{vmatrix} = 3\begin{vmatrix} -5 & 2 \\ 2 & -3 \end{vmatrix} - 4\begin{vmatrix} 2 & 2 \\ 7 & -3 \end{vmatrix} + \begin{vmatrix} 2 & -5 \\ 7 & 2 \end{vmatrix} = 3(11) - 4(-20) + (4+35)$

$= 33 + 80 + 39 = 152 \neq 0$ ∴ LINEARLY INDEPENDENT

(b) $\begin{pmatrix} 9 \\ 20 \\ -5 \end{pmatrix} = \alpha\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} + \beta\begin{pmatrix} 2 \\ -5 \\ 2 \end{pmatrix} + \gamma\begin{pmatrix} 7 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 3\alpha + 2\beta + 7\gamma \\ 4\alpha - 5\beta + 2\gamma \\ \alpha + 2\beta - 3\gamma \end{pmatrix}$

$\begin{bmatrix} 3 & 2 & 7 & 9 \\ 4 & -5 & 2 & 20 \\ 1 & 2 & -3 & -5 \end{bmatrix} \xrightarrow{r_{13}} \begin{bmatrix} 1 & 2 & -3 & -5 \\ 4 & -5 & 2 & 20 \\ 3 & 2 & 7 & 9 \end{bmatrix} \xrightarrow[r_{13}(-3)]{r_{12}(-4)} \begin{bmatrix} 1 & 2 & -3 & -5 \\ 0 & -13 & 14 & 40 \\ 0 & -4 & 16 & 24 \end{bmatrix}$

$\xrightarrow[r_{23}\ (\text{AFTER})]{r_3(-\frac{1}{4})} \begin{bmatrix} 1 & 2 & -3 & -5 \\ 0 & 1 & -4 & -6 \\ 0 & -13 & 14 & 40 \end{bmatrix} \xrightarrow{r_{23}(13)} \begin{bmatrix} 1 & 2 & -3 & -5 \\ 0 & 1 & -4 & -6 \\ 0 & 0 & -38 & -38 \end{bmatrix}$

∴ $-38\gamma = -38$, $\gamma = 1$; $\beta - 4\gamma = -6$, $\beta = -2$; $\alpha + 2\beta - 3\gamma = -5$, $\alpha - 4 - 3 = -5$, $\alpha = 2$

∴ $(9, 20, -5) = 2\underline{a} - 2\underline{b} + \underline{c}$

Question 54 (****)

The points $A(0,2,1)$, $B(8,6,0)$ and $C(-4,1,1)$ form a plane Π_1.

a) Find a Cartesian equation for Π_1.

The point $T(1,2,t)$ lies outside Π_1.

b) Show that the shortest distance of T from Π_1 is

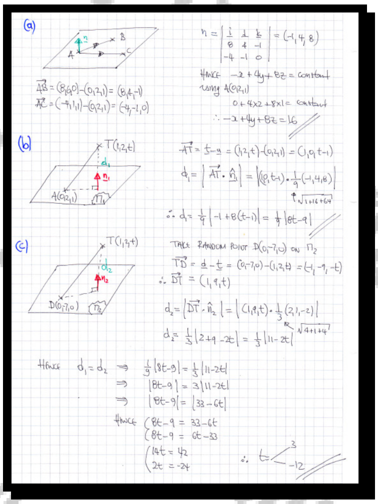
$$\left|\frac{1}{9}(8t-9)\right|.$$

The plane Π_2 has Cartesian equation

$$2x+y-2z+7=0.$$

c) Given that the T is equidistant from Π_1 and Π_2 find the possible values of t.

$\boxed{-x+4y+8z=16}$, $\boxed{t=-12,3}$



(a)

$\vec{AB} = (8,6,0)-(0,2,1) = (8,4,-1)$

$\vec{AC} = (-4,1,1)-(0,2,1) = (-4,-1,0)$

$n = \begin{vmatrix} i & j & k \\ 8 & 4 & -1 \\ -4 & -1 & 0 \end{vmatrix} = (-1,4,8)$

Hence $-x+4y+8z = \text{constant}$

using $A(0,2,1)$

$0+4\times 2+8\times 1 = \text{constant}$

$\therefore -x+4y+8z=16$

(b)

$\vec{AT} = \underline{t}-\underline{a} = (1,2,t)-(0,2,1) = (1,0,t-1)$

$d_1 = \left|\vec{AT}\cdot\hat{\underline{n}}_1\right| = \left|(1,0,t-1)\cdot\frac{1}{9}(-1,4,8)\right|$

$\sqrt{1+16+64}$

$\therefore d_1 = \frac{1}{9}\left|-1+8(t-1)\right| = \frac{1}{9}\left|8t-9\right|$

(c)

TAKE RANDOM POINT $D(0,-7,0)$ ON Π_2

$\vec{TD} = \underline{d}-\underline{t} = (0,-7,0)-(1,2,t) = (-1,-9,-t)$

$\therefore \vec{DT} = (1,9,t)$

$d_2 = \left|\vec{DT}\cdot\hat{n}_2\right| = \left|(1,9,t)\cdot\frac{1}{3}(2,1,-2)\right|$

$\sqrt{4+1+4}$

$d_2 = \frac{1}{3}\left|2+9-2t\right| = \frac{1}{3}\left|11-2t\right|$

Hence $d_1 = d_2 \Rightarrow \frac{1}{9}|8t-9| = \frac{1}{3}|11-2t|$

$\Rightarrow |8t-9| = 3|11-2t|$

$\Rightarrow |8t-9| = |33-6t|$

Hence $\begin{cases} 8t-9 = 33-6t \\ 8t-9 = 6t-33 \end{cases}$

$\begin{cases} 14t = 42 \\ 2t = -24 \end{cases}$

$\therefore t = 3, -12$

Question 55 (****)

With respect to a fixed origin O, the points $A(3,0,0)$, $B(0,2,-1)$ and $C(2,0,1)$ have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, respectively.

a) Calculate $\overrightarrow{AC} \wedge \overrightarrow{OB}$.

The plane Π contains the point C and the straight line L with vector equation

$$(\mathbf{r}-\mathbf{a}) \wedge \mathbf{b} = \mathbf{0},$$

where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors to be found.

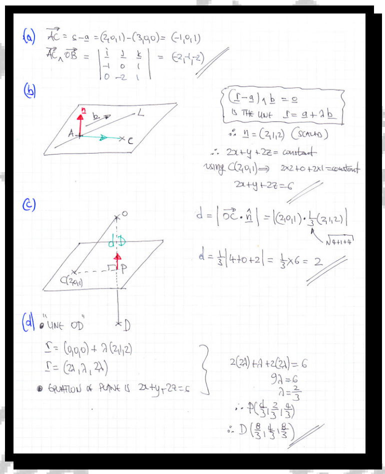
b) Find a Cartesian equation of Π.

c) Calculate the shortest distance of Π from O.

The point D is the reflection of O about Π.

d) Determine the coordinates of D.

$\boxed{-2\mathbf{i}-\mathbf{i}-2\mathbf{k}}$, $\boxed{2x+y+2z=6}$, $\boxed{\text{distance}=2}$, $\boxed{D\left(\frac{8}{3},\frac{4}{3},\frac{8}{3}\right)}$



Question 56 (****)

Relative to a fixed origin O, the point A has position vector $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$.

The plane Π has vector equation

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c},$$

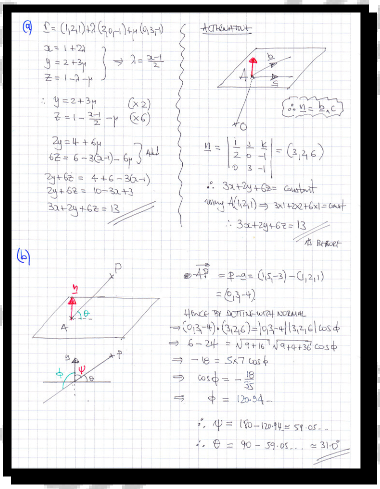
where $\mathbf{b} = 2\mathbf{i} - \mathbf{k}$ and $\mathbf{c} = 3\mathbf{j} - \mathbf{k}$.

a) Find a Cartesian equation of Π.

The point P has position vector $\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$.

b) Calculate, to the nearest degree, the acute angle between AP and Π.

$\boxed{3x + 2y + 6z = 13}$, $\boxed{31°}$



Question 57 (****)

The system of equations below has a unique solution.

$$\begin{aligned} 5x + y + 6z &= 9 \\ 3x + 6y + 2z &= 8 \\ 4x + 2y - 9z &= 75 \end{aligned}$$

a) Show that $z = -5$ and find the values of x and y.

The straight line L and the plane Π have respective vector equations

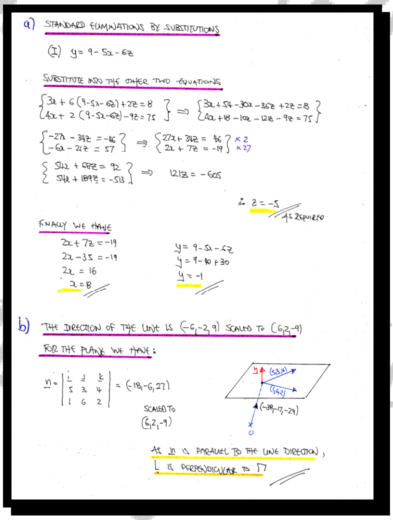
$$\mathbf{r}_1 = \begin{pmatrix} -29 \\ -9 \\ 46 \end{pmatrix} + t \begin{pmatrix} -6 \\ -2 \\ 9 \end{pmatrix} \quad \text{and} \quad \mathbf{r}_2 = \begin{pmatrix} -38 \\ -17 \\ -29 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 3 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 6 \\ 2 \end{pmatrix},$$

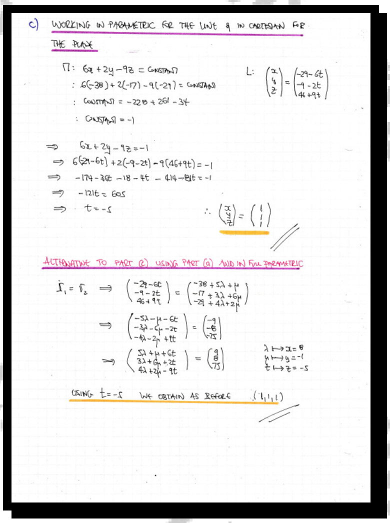
where t, λ and μ are scalar parameters.

b) Show that L is perpendicular to Π.

c) Show further that L meets Π at the point with coordinates $(1,1,1)$.

V, $\boxed{}$, $\boxed{x = 8,\ y = -1}$





Question 58 (****)

The straight line L has vector equation

$$\mathbf{r} = \begin{pmatrix} 3 \\ 7 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 2 \\ -3 \end{pmatrix},$$

where λ is a scalar parameter.

The plane Π passes through the points $A(11,13,5)$ and $B(15,12,5)$.

It is further given that Π is parallel to L.

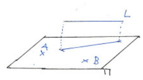
a) Find a Cartesian equation for Π and hence calculate the distance between L and Π.

The straight line M is the reflection of L about Π.

b) Determine a vector equation for M.

$\boxed{\quad}$, $\boxed{x+4y+2z=73}$, $\boxed{\text{distance} = 2\sqrt{21}}$, $\boxed{\mathbf{r} = 7\mathbf{i}+23\mathbf{j}+8\mathbf{k}+\mu(2\mathbf{i}-2\mathbf{j}+3\mathbf{k})}$

a) START BY OBTAINING A NORMAL BY "CROSSING" $\vec{AB}$ & THE DIRECTION OF L



$\vec{AB} = \underline{b} - \underline{a} = (15,12,5) - (11,13,5)$
$= (4,-1,0)$

$\underline{n} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 4 & -1 & 0 \\ -2 & 2 & -3 \end{vmatrix} = (3,12,6)$

SCALING THE NORMAL TO $(1,4,2)$ & USING POINT $A(11,13,5)$

$x + 4y + 2z = \text{CONSTANT}$
$11 + 4\times13 + 2\times5 = \text{CONSTANT}$
$\text{CONSTANT} = 11 + 52 + 10$
$\text{CONSTANT} = 73$

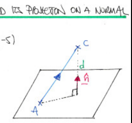
$\therefore \; x + 4y + 2z = 73$

NOW TO FIND THE SHORTEST DISTANCE, TAKE ANY POINT ON THE LINE SAY $C(3,7,0)$, FIND $\vec{AC}$ AND FIND ITS PROJECTION ON A NORMAL

$\vec{AC} = \underline{c} - \underline{a} = (3,7,0) - (15,12,5) = (-12,-5,-5)$

ALSO $\hat{\underline{n}} = \frac{1}{\sqrt{1^2+4^2+2^2}}(1,4,2)$

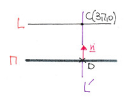
$\hat{\underline{n}} = \frac{(1,4,2)}{\sqrt{21}}$



$d = |\vec{AC} \cdot \hat{\underline{n}}| = \left| (-12,-5,-5) \cdot \frac{(1,4,2)}{\sqrt{21}} \right| = \left| \frac{-12-20-10}{\sqrt{21}} \right|$

$= \left| \frac{-42}{\sqrt{21}} \right| = \frac{42\sqrt{21}}{21} = 2\sqrt{21}$

b) FIND AN EQUATION OF L'



$\underline{r} = (x,y,z) = (3,7,0) + \lambda(1,4,2)$
$(x,y,z) = (\lambda+3, 4\lambda+7, 2\lambda)$

SOLVING SIMULTANEOUSLY WITH THE EQUATION OF THE PLANE

$\Longrightarrow x + 4y + 2z = 73$
$\Longrightarrow (\lambda+3) + 4(4\lambda+7) + 2(2\lambda) = 73$
$\Longrightarrow \lambda + 3 + 16\lambda + 28 + 4\lambda = 73$
$\Longrightarrow 21\lambda = 42$
$\Longrightarrow \lambda = 2$

THUS $D(5,15,4)$

HENCE WE HAVE THE REFLECTION OF $C(3,7,0)$ ABOUT Π AS THE POINT $E(7,23,8)$ (BY INSPECTION AS D IS THE MIDPOINT OF CE)

$\therefore$ REQUIRED LINE WILL BE

$\underline{r} = (7,23,8) + \mu(2,-2,3)$

Question 59 (****)

The point $P(1,3,8)$ lies on the plane Π_1.

The straight line L, whose Cartesian equation is given below also lies on Π_1.

$$x-4=\frac{y-3}{3}=\frac{2-z}{4}.$$

a) Find a Cartesian equation of Π_1.

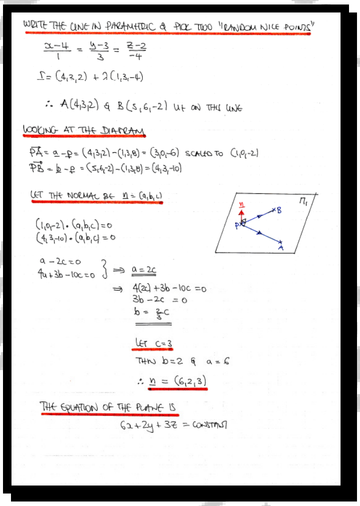
*You may not use the vector product (cross product) in part **(a)**.*

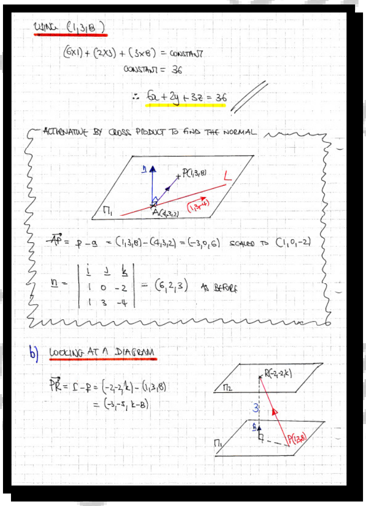
The point $R(-2,-2,k)$, where k is a constant, lies on another plane Π_2, which is parallel to Π_1.

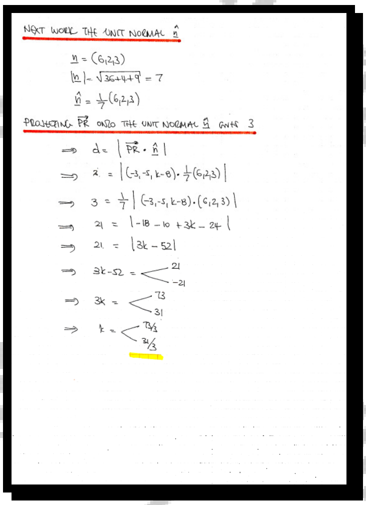
b) Given that the distance between Π_1 and Π_2 is 3 units determine, in exact fractional form, the possible values of k.

*You may not use the standard formula which finds the distance between two parallel planes in part **(b)**.*

$\boxed{6x+2y+3z=36}$, $\boxed{k=\frac{73}{3},\ k=\frac{31}{3}}$







Question 60 (****)

With respect to a fixed origin O, four points have the following coordinates

$$A(-1,3,-1),\ B(1,2,-2),\ C(1,2,2) \text{ and } D(k,k,k),$$

where k is a constant.

a) Determine the shortest distance between the straight lines AB and CD.

b) Find, in terms of k, the volume of the tetrahedron $ABCD$.

$\boxed{}$, $\boxed{d_{\text{min}} = 2\sqrt{2}}$, $\boxed{\text{volume} = \frac{2}{3}|3k-5|}$

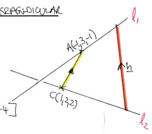
a) START BY OBTAINING THE DIRECTION VECTORS OF THE TWO LINES

$\overrightarrow{AB} = \underline{b} - \underline{a} = (1,2,-2) - (-1,3,-1) = (2,-1,-1)$

$\overrightarrow{CD} = \underline{d} - \underline{c} = (k,k,k) - (1,2,2) = (k-1, k-2, k-2)$

FIND THE DIRECTION OF THE COMMON PERPENDICULAR

$$\underline{n} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ k-1 & k-2 & k-2 \\ 2 & -1 & -1 \end{vmatrix}$$



$\underline{n} = [-k+2+k-2,\ 2k-4+k-1,\ -k+. -2k+4]$

$\underline{n} = (0, 3k-5, -3k+5)$

$\underline{n} = (3k-5)[0,1,-1]$

SCALE n AND MAKING IT UNIT YIELDS $\frac{1}{\sqrt{2}}(0,1,-1)$

FINALLY OBTAIN THE VECTOR $\overrightarrow{CA}$ & PROJECT IT ONTO THE UNIT PERPENDICULAR BETWEEN THE TWO LINES

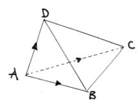
$\overrightarrow{CA} = \underline{a} - \underline{c} = (-1,3,-1) - (1,2,2) = (-2,1,-3)$

$d = |\overrightarrow{CA} \cdot \hat{\underline{n}}| = |(-2,1,-3) \cdot \frac{1}{\sqrt{2}}(0,1,-1)| = \frac{1}{\sqrt{2}}|0+1+3|$

$= \frac{4}{\sqrt{2}}$

$= 2\sqrt{2}$

b)



$\overrightarrow{AB} = (2,-1,-1)$ (FROM EARLIER)

$\overrightarrow{AC} = \underline{c} - \underline{a} = (1,2,2) - (-1,3,-1) = (2,-1,3)$

$\overrightarrow{AD} = \underline{d} - \underline{a} = (k,k,k) - (-1,3,-1) = (k+1, k-3, k+1)$

THE REQUIRED VOLUME IS GIVEN BY

$$V = \frac{1}{6}|\overrightarrow{AB} \wedge \overrightarrow{AC} \cdot \overrightarrow{AD}| = \frac{1}{6}\left\|\begin{matrix} k+1 & k-3 & k+1 \\ 2 & -1 & -1 \\ 2 & -1 & 3 \end{matrix}\right\| \quad r_{23}(-1)$$

$$= \frac{1}{6}\left\|\begin{matrix} k+1 & k-3 & k+1 \\ 2 & -1 & -1 \\ 0 & 0 & 4 \end{matrix}\right\|$$

$= \frac{1}{6} \times |4(-k-1-2k+6)|$

$= \frac{2}{3}|5-3k|$

or $\frac{2}{3}|3k-5|$

Question 61 (****+)

The straight line L has Cartesian equation

$$x-9=\frac{y-a}{2}=\frac{z-1}{b},$$

where a and b are non zero constants.

The plane Π has Cartesian equation

$$x+y-2z=12.$$

a) If L is contained by Π, determine the value of a and the value of b.

b) Given instead that L meets Π at the point where $x=0$, and is inclined at an angle $\arcsin\frac{5}{6}$ to Π, determine the value of a.

$\boxed{}$, $\boxed{a=5,\ b=\frac{2}{3}}$, $\boxed{a=50}$

a) WRITE THE LINE IN PARAMETRIC FORM

$$\frac{x-9}{1}=\frac{y-a}{2}=\frac{z-1}{b} \implies \underline{r}=(9,a,1)+\lambda(1,2,b)$$
$$\implies (x,y,z)=(\lambda+9,\ 2\lambda+a,\ \lambda b+1)$$

IF THE LINE IS CONTAINED BY THE PLANE ITS DIRECTION VECTOR MUST BE PERPENDICULAR TO THE NORMAL OF THE PLANE

$\implies$ (PLANE NORMAL) $\cdot$ (LINE DIRECTION VECTOR) $=0$

$\implies (1,1,-2)\cdot(1,2,b)=0$

$\implies 1+2-2b=0$

$\implies 2b=3$

$\implies b=\frac{3}{2}$

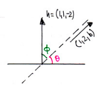
ALSO THE POINT ON THE LINE $(9,a,1)$ MUST ALSO LIE ON THE PLANE

$\implies x+y-2z=12$

$\implies 9+a-2=12$

$\implies a=5$

b) IF THE LINE MEETS THE PLANE AT $\theta=\arcsin\frac{5}{6}$, THEN IT MUST MEET THE NORMAL TO THE PLANE AT $\phi=\arccos\frac{5}{6}$, AS θ & ϕ ARE SUPPLEMENTARY



$\implies (1,1,-2)\cdot(1,2,b)=|1,1,-2||1,2,b|\cos\phi$

$\implies 1+2-2b=\sqrt{1+1+4}\sqrt{1+4+b^2}\cos\phi$

$\implies 3-2b=\sqrt{6}\sqrt{5+b^2}\times\frac{5}{6}$

$\implies 6(3-2b)=5\sqrt{6}\sqrt{5+b^2}$

SQUARING BOTH SIDES

$\implies 36(3-2b)^2=25\times6\times(5+b^2)$

$\implies 6(3-2b)^2=25(5+b^2)$

$\implies 6(9-12b+4b^2)=125+25b^2$

$\implies 54-72b+24b^2=125+25b^2$

$\implies 0=b^2+72b+71$

$\implies 0=(b+1)(b+71)$

$\implies b=\begin{cases}-1\\-71\end{cases}$ (DO BOTH WORK DUE TO SQUARING?)

FINALLY TO FIND a

- IF $b=-1$

$(x,y,z)=(\lambda+9,\ 2\lambda+a,\ \lambda b+1)$
$(x,y,z)=(\lambda+9,\ 2\lambda+a,\ -\lambda+1)$
$(0,y,z)=(\lambda+9,\ 2\lambda+a,\ -\lambda+1)$

$\implies \lambda=-9$
$\implies z=10$
$\implies y=-18+a$

HENCE
$x+y-2z=12$
$0+(-18+a)-2\times10=12$
$-18+a-20=12$
$a=50$

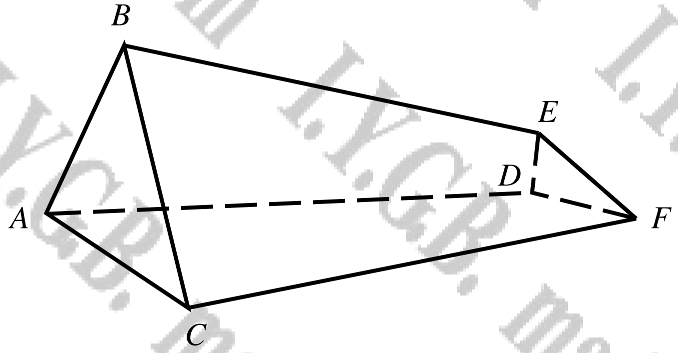
- IF $b=-71$

$(x,y,z)=(\lambda+9,\ 2\lambda+a,\ \lambda b+1)$
$(x,y,z)=(\lambda+9,\ 2\lambda+a,\ -71\lambda+1)$
$(0,y,z)=(\lambda+9,\ 2\lambda+a,\ -71\lambda+1)$

$\implies \lambda=-9$
$\implies z=640$ $\left\{\begin{array}{r}71\\ \times 9\\ \hline 639\end{array}\right\}$
$\implies y=-18+a$

HENCE
$x+y-2z=12$
$0+(-18+a)-2\times640=12$
$-18+a-1280=12$
$a=1310$

Question 62 (****+)



The figure above shows an irregular hollow shape, consisting of two non-congruent, non-parallel triangular faces ABC and DEF, and two non-congruent quadrilateral faces $ABED$ and $BCFE$.

The respective equations of the straight lines AD and DE are

$$\mathbf{r}_1 = -5\mathbf{i}+6\mathbf{j}+\mathbf{k}+\lambda(2\mathbf{i}+3\mathbf{j}) \quad \text{and} \quad \mathbf{r}_2 = -\mathbf{i}+12\mathbf{j}+\mathbf{k}+\mu(-2\mathbf{i}+7\mathbf{j}-7\mathbf{k}),$$

where λ and μ are scalar parameters.

a) If the plane face $BCFE$ has equation $21x-14y+20z=111$, determine an equation of the straight line BE.

The straight line BC has equation

$$\mathbf{r}_3 = -\mathbf{i}-8\mathbf{j}+\mathbf{k}+\nu(-2\mathbf{i}+7\mathbf{j}+7\mathbf{k}),$$

where ν is a scalar parameter.

b) Given further that the point G has position vector $5\mathbf{i}+7\mathbf{j}$, determine the acute angle between the plane face $BCFE$ and the straight line BG.

$\boxed{}$, $\boxed{\mathbf{r}=\mathbf{i}+5\mathbf{j}+8\mathbf{k}+t(2\mathbf{i}+3\mathbf{j})}$, $\boxed{\theta \approx 13.5°}$

[solutions overleaf]

a) START BY CROSSING THE DIRECTIONS AD & DE IN ORDER TO FIND THE NORMAL TO THE PLANE ABED

$$\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 3 & 0 \\ -2 & 7 & -7 \end{vmatrix} = (-21-0, 0+14, 14+6) = (-21, 14, 20)$$

EQUATION OF ABED

$$-21x + 14y + 20z = \text{CONSTANT}$$
$$-21(-5)+14(6)+20(1) = \text{CONSTANT} \qquad \text{USING } (-5,6,1) \text{ OR } (-1,12,1)$$
$$105 + 84 + 20 = \text{CONSTANT}$$
$$\text{CONSTANT} = 209$$

$$\therefore \ -21x + 14y + 20z = 209$$

SOLVING THE EQUATIONS OF THE TWO PLANES

$$\left.\begin{array}{ll} \text{ABED}: & -21x + 14y + 20z = 209 \\ \text{BCFE}: & 21x - 14y + 20z = 111 \end{array}\right\} \text{ ADDING } \quad 40z = 320$$
$$z = 8 \quad \text{(ALWAYS)}$$

BOTH EQUATIONS REDUCE TO

$$-21x + 14y = 49$$
$$-3x + 2y = 7$$
$$2y = 3x + 7$$
$$y = \tfrac{3}{2}x + \tfrac{7}{2}$$

THUS WE HAVE

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ \frac{3}{2}x + \frac{7}{2} \\ 8 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{7}{2} \\ 8 \end{pmatrix} + \xi \begin{pmatrix} 1 \\ 3/2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 7/2 \\ 8 \end{pmatrix} + (\eta + 1)\begin{pmatrix} 1 \\ 3/2 \\ 0 \end{pmatrix}$$

$$\Rightarrow \underline{r} = \begin{pmatrix} 1 \\ 5 \\ 8 \end{pmatrix} + \eta \begin{pmatrix} 1 \\ 3/2 \\ 0 \end{pmatrix}$$

$$\Rightarrow \underline{r} = \begin{pmatrix} 1 \\ 5 \\ 8 \end{pmatrix} + 2t \begin{pmatrix} 1 \\ 3/2 \\ 0 \end{pmatrix}$$

$$\Rightarrow \underline{r} = (1,5,8) + t(2,3,0)$$

b) NOW SOLVE THE LINES BC & BE TO FIND B

$$\text{BC}: \ \underline{r}_3 = (-1-2v, -8+7v, 1+7v)$$
$$\text{BE}: \ \underline{r} = (1+2t, 5+3t, 8)$$

LINES DO INTERSECT SO NO VERIFICATION NEEDED

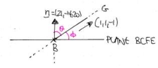
$$7v + 1 = 8$$
$$7v = 7$$
$$v = 1$$
$$(t = -2) \qquad \therefore B(-3, -1, 8)$$

NOW $B(-3,-1,8)$ & $G(5,7,0)$

$$\Rightarrow \overrightarrow{BG} = \underline{g} - \underline{b} = (5,7,0) - (-3,-1,8) = (8,8,-8)$$

SCALE IT TO DIRECTION $(1,1,-1)$

DRAWING A DIAGRAM



DOTTING NORMAL TO BCFE AND DIRECTION BG

$$(21,-14,20)\cdot(1,1,-1) = |21,-14,20||1,1,-1| \cos\theta$$
$$21 - 14 - 20 = \sqrt{441+196+400}\,\sqrt{3}\,\cos\theta$$
$$-13 = \sqrt{1137 \times 3}\,\cos\theta$$
$$\cos\theta = \frac{-13}{\sqrt{3411}}$$
$$\theta = 103.478\ldots$$

WE REQUIRE ACUTE ANGLE (SEE DIAGRAM)

$$\theta = 180 - 103.478$$
$$\theta = 76.52^\circ\ldots$$

FINALLY THE REQUIRED ANGLE IS ϕ

$$\phi = 90 - \theta$$
$$\phi \simeq 13.5^\circ$$

Question 63 (****+)

The skew straight lines L_1 and L_2 have vector equations

$$\mathbf{r}_1 = (-13\mathbf{j} + \mathbf{k}) + \lambda(-3\mathbf{i} + 4\mathbf{j} - 7\mathbf{k}),$$

$$\mathbf{r}_2 = (5\mathbf{i} + 25\mathbf{j}) + \mu(2\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}),$$

where λ and μ are scalar parameters.

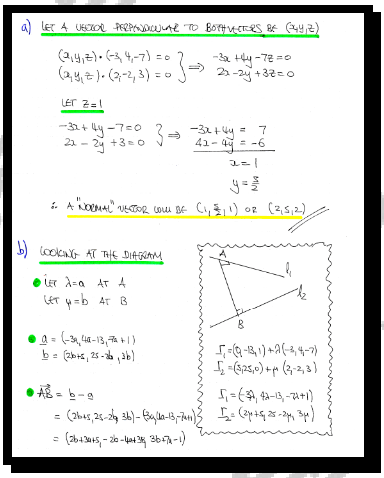
a) Find a vector which mutually perpendicular to L_1 and L_2.

You may not use the vector (cross) product in answering part ***(a)***.

The point A lies on L_1 and the point B lies on L_2.

b) Given that the distance AB is least, determine the coordinates of A and B.

$\boxed{}$, $\boxed{A(-3,-9-6)}$, $\boxed{B(9,21,6)}$



a) LET A VECTOR PERPENDICULAR TO BOTH VECTORS BE (x,y,z)

$$\left.\begin{array}{l}(x,y,z)\cdot(-3,4,-7)=0\\(x,y,z)\cdot(2,-2,3)=0\end{array}\right\}\Rightarrow\begin{array}{l}-3x+4y-7z=0\\2x-2y+3z=0\end{array}$$

LET $z=1$

$$\left.\begin{array}{l}-3x+4y-7=0\\2x-2y+3=0\end{array}\right\}\Rightarrow\begin{array}{l}-3x+4y=7\\4x-4y=-6\\\hline x=1\\y=\frac{5}{2}\end{array}$$

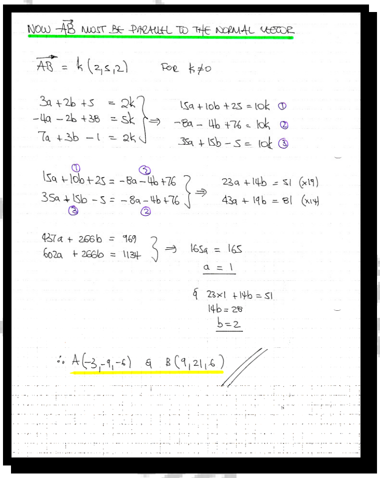
$\therefore$ A "NORMAL" VECTOR WILL BE $(1,\frac{5}{2},1)$ OR $(2,5,2)$

b) LOOKING AT THE DIAGRAM

- LET $\lambda=a$ AT A
  LET $\mu=b$ AT B

- $\underline{a}=(-3a,4a-13,-7a+1)$
  $\underline{b}=(2b+5,25-2b,3b)$

- $\overrightarrow{AB}=\underline{b}-\underline{a}$
  $=(2b+5,25-2b,3b)-(-3a,4a-13,-7a+1)$
  $=(2b+3a+5,-2b-4a+38,3b+7a-1)$



NOW $\overrightarrow{AB}$ MUST BE PARALLEL TO THE NORMAL VECTOR

$\overrightarrow{AB}=k(2,5,2)$ FOR $k\neq0$

$$\left.\begin{array}{l}3a+2b+5=2k\\-4a-2b+38=5k\\7a+3b-1=2k\end{array}\right\}\Rightarrow\begin{array}{ll}15a+10b+25=10k & ①\\-8a-4b+76=10k & ②\\35a+15b-5=10k & ③\end{array}$$

$$\left.\begin{array}{l}15a+10b+25=-8a-4b+76\\35a+15b-5=-8a-4b+76\end{array}\right\}\Rightarrow\begin{array}{ll}23a+14b=51 & (\times19)\\43a+19b=81 & (\times14)\end{array}$$

$$\left.\begin{array}{l}437a+266b=969\\602a+266b=1134\end{array}\right\}\Rightarrow\begin{array}{l}165a=165\\a=1\end{array}$$

& $23\times1+14b=51$
$14b=28$
$b=2$

$\therefore A(-3,-9,-6)$ & $B(9,21,6)$

Question 64 (*****)

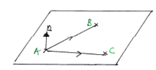
The points A, B and C have respective position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, relative to a fixed origin O.

Show that the equation of the plane through A, B and C can be written as

$$(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\cdot(\mathbf{a}\wedge\mathbf{b}+\mathbf{b}\wedge\mathbf{c}+\mathbf{c}\wedge\mathbf{a})=\mathbf{a}\cdot\mathbf{b}\wedge\mathbf{c}$$

$\boxed{}$, $\boxed{\text{proof}}$

LOOKING AT THE DIAGRAM



$\Longrightarrow \underline{n} = \overrightarrow{AB}\wedge\overrightarrow{AC}$

$\Longrightarrow \underline{n} = (\underline{b}-\underline{a})\wedge(\underline{c}-\underline{a})$

$\Longrightarrow \underline{n} = \underline{b}\wedge\underline{c} - \underline{b}\wedge\underline{a} - \underline{a}\wedge\underline{c} + \cancel{\underline{a}\wedge\underline{a}}$

$\Longrightarrow \underline{n} = \underline{b}\wedge\underline{c} + \underline{a}\wedge\underline{b} + \underline{c}\wedge\underline{a}$

$\Longrightarrow \underline{n} = \underline{a}\wedge\underline{b} + \underline{b}\wedge\underline{c} + \underline{c}\wedge\underline{a}$

USING THE POINT A, AND LETTING $\underline{r} = (x,y,z)$

$\Longrightarrow (\underline{r}-\underline{a})\cdot\underline{n} = 0$

$\Longrightarrow \underline{r}\cdot\underline{n} - \underline{a}\cdot\underline{n} = 0$

$\Longrightarrow \underline{r}\cdot\underline{n} = \underline{a}\cdot\underline{n}$

$\Longrightarrow \begin{pmatrix}x\\y\\z\end{pmatrix}\cdot(\underline{a}\wedge\underline{b}+\underline{b}\wedge\underline{c}+\underline{c}\wedge\underline{a}) = \underline{a}\cdot[\underline{a}\wedge\underline{b}+\underline{b}\wedge\underline{c}+\underline{c}\wedge\underline{a}]$

$= \underbrace{\cancel{\underline{a}\cdot\underline{a}\wedge\underline{b}}}_{\text{PERPENDICULAR TO } \underline{a}} + \underline{a}\cdot\underline{b}\wedge\underline{c} + \underbrace{\cancel{\underline{a}\cdot\underline{c}\wedge\underline{a}}}_{\text{PERPENDICULAR TO } \underline{a}}$

$\therefore (x,y,z)\cdot(\underline{a}\wedge\underline{b}+\underline{b}\wedge\underline{c}+\underline{c}\wedge\underline{a}) = \underline{a}\cdot\underline{b}\wedge\underline{c}$ //

Question 65 (*****)

An irregular pyramid with a triangular base ABC has vertex at the point V.

The equation of the straight line VC is

$$\mathbf{r} = 2\mathbf{i} + 4\mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} + 4\mathbf{k})$$

where λ is a scalar parameter.

The plane face ABV has equation $2x - 3y - z = 1$.

If the point D lies on the plane face VBC and has position vector $\frac{10}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + 5\mathbf{k}$, show that the equation of the line VB can be written as

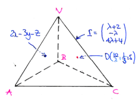
$$\mathbf{r} = 3\mathbf{i} - \mathbf{j} + 8\mathbf{k} + \mu(2\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}),$$

where μ is a scalar parameter.

V, $\boxed{}$, $\boxed{\text{proof}}$

STARTING WITH A DIAGRAM

THE LINE VB IS THE INTERSECTION OF THE PLANES VBA (GIVEN) AND THE PLANE VBC (TO BE FOUND)



TAKE 3 POINTS ON VBC

$\lambda = 0 \quad P(2,0,4)$
$\lambda = -1 \quad Q(1,1,0)$
$D(\frac{10}{3}, \frac{1}{3}, 5)$

$\vec{PQ} = \underline{q} - \underline{p} = (1,1,0) - (2,0,4) = (-1,1,-4)$ SCALE IT TO $(1,-1,4)$
$\vec{PD} = \underline{d} - \underline{p} = (\frac{10}{3}, \frac{1}{3}, 5) - (2,0,4) = (\frac{4}{3}, \frac{1}{3}, 1)$ SCALE IT TO $(4,1,3)$

CROSSING THESE DIRECTIONS TO GET THE NORMAL OF VBC

$$\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -1 & 4 \\ 4 & 1 & 3 \end{vmatrix} = (-7, 13, 5) \leftarrow \text{NORMAL OF VBC}$$

NEXT CROSSING THE NORMALS OF ABV & VBC TO GET THE DIRECTION OF VB

$$\begin{array}{l} \\ \text{VBC} \rightarrow \\ \text{ABV} \rightarrow \end{array} \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -7 & 13 & 5 \\ 2 & -3 & -1 \end{vmatrix} = (2, 3, -5) \leftarrow \text{DIRECTION VECTOR OF VB}$$

NOW INTERSECTING THE PLANE ABV & VC TO FIND V

$$2x - 3y - z = 1 \quad \& \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \lambda+2 \\ -\lambda \\ 4\lambda+4 \end{pmatrix}$$

$\Rightarrow 2(\lambda+2) - 3(-\lambda) - (4\lambda+4) = 1$
$\Rightarrow 2\lambda + \cancel{4} + 3\lambda - 4\lambda - \cancel{4} = 1$
$\Rightarrow \lambda = 1$

$\therefore V(3,-1,8)$

FINALLY THE LINE VB, USING $V(3,-1,8)$ & DIRECTION $(2,3,-5)$

$\underline{r} = (3,-1,8) + \mu(2,3,-5)$ // AS REQUIRED

Question 66 (*****)

The straight line L_1 has vector equation

$$\mathbf{r} = 4\mathbf{i} - 3\mathbf{j} + 7\mathbf{k} + \lambda(3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}),$$

where λ is a scalar parameter.

The plane Π has vector equation

$$\mathbf{r} \cdot (4\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}) = 17.$$

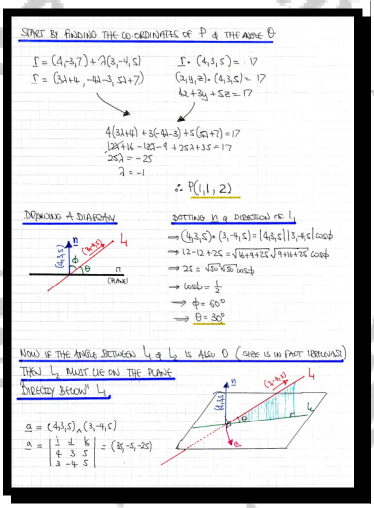
The point P is the intersection of L_1 and Π.

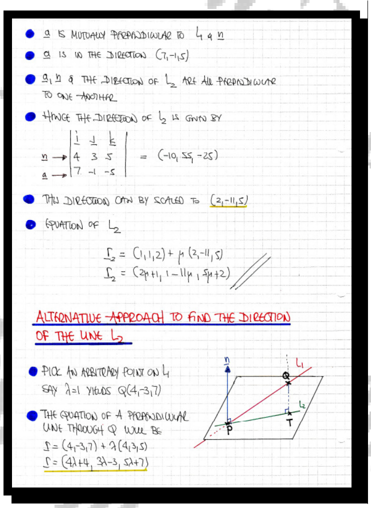
The acute angle θ is formed between L_1 and Π.

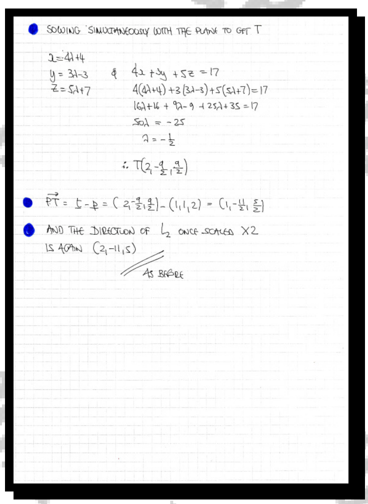
The straight line L_2 lies on Π, passes through P so that the acute angle between L_1 and L_2 is also θ.

Determine the value of θ and find a vector equation for L_2.

$\boxed{}$, $\boxed{\theta = 30°}$, $\boxed{\mathbf{r}_2 = \mathbf{i} + \mathbf{j} + 2\mathbf{k} + \mu(2\mathbf{i} - 11\mathbf{j} + 5\mathbf{k})}$







Question 67 (*****)

With respect to a fixed origin O, the points A, B and C have respective position vectors

$$\mathbf{a} = 3\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}, \quad \mathbf{b} = 6\mathbf{i} + 2\mathbf{k} \quad \text{and} \quad \mathbf{c} = 3\mathbf{j} + 5\mathbf{k},$$

so that the plane Π contains A, B and C.

The straight line L is **parallel** to Π and has vector equation

$$\mathbf{r} = (13\mathbf{i} - 9\mathbf{j}) + \lambda(-7\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}),$$

where λ is a scalar parameter.

The point P lies outside the plane so that PC is perpendicular to Π.

The point Q lies on L so that PQ is perpendicular to L.

Given further that P is equidistant from Π and L, find the position vector of P and the position vector of Q.

$\boxed{}$, $\boxed{\mathbf{p} = -6\mathbf{i} - 4\mathbf{k}}$, $\boxed{\mathbf{q} = -\mathbf{i} + \mathbf{j} + 6\mathbf{k}}$

START BY OBTAINING THE EQUATION OF THE PLANE

$\overrightarrow{CA} = \underline{a} - \underline{c} = (3,3,3) - (0,3,5) = (3,0,-2)$

$\overrightarrow{CB} = \underline{b} - \underline{c} = (6,0,2) - (0,3,5) = (6,-3,-3)$ ← SCALE TO $(-2,1,1)$

$\overrightarrow{CA} \wedge \overrightarrow{CB} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & 0 & -2 \\ -2 & 1 & 1 \end{vmatrix} = (2,1,3)$ ← NORMAL

$\Rightarrow 2x + y + 3z = \text{CONSTANT}$

$\Rightarrow 2\times0 + 3 + 3\times5 = \text{CONSTANT}$ ← USING $(0,3,5)$

$\Rightarrow \text{CONSTANT} = 18$

$\Rightarrow 2x + y + 3z = 18$

NEXT WE OBTAIN THE EQUATION OF THE LINE THROUGH P & C

$\Rightarrow \underline{r} = (0,3,5) + \mu(2,1,3)$

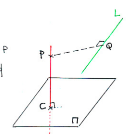
$\Rightarrow \underline{r} = (x,y,z) = (2\mu, \mu+3, 3\mu+5)$

THUS WE NOW HAVE

$\underline{p} = (2p, p+3, 3p+5)$ FOR SOME $\mu = p$

$\underline{q} = (13-7q, 5q-9, 3q)$ FOR SOME $\lambda = q$

$\underline{c} = (0,3,5)$



- $\overrightarrow{CP} = \underline{p} - \underline{c} = (2p, p+3, 3p+5) - (0,3,5) = (2p, p, 3p)$
- $\overrightarrow{QP} = \underline{p} - \underline{q} = (2p, p+3, 3p+5) - (13-7q, 5q-9, 3q) = (2p+7q-13, p-5q+12, 3p-3q+5)$

NEXT WE USE THE FACT THAT $\overrightarrow{QP} \perp L$

$\Rightarrow (2p+7q-13, p-5q+12, 3p-3q+5)\cdot(-7,5,3) = 0$

$\Rightarrow \begin{matrix} \cancel{-14p} - 49q + 91 \\ \cancel{5p} - 25q + 60 \\ \cancel{9p} - 9q + 15 \end{matrix} = 0$

$\Rightarrow -83q + 166 = 0$

$\Rightarrow q = 2$

REWRITE THE VECTORS $\overrightarrow{CP}$ & $\overrightarrow{QP}$ WITH q KNOWN

- $\overrightarrow{CP} = (2p, p, 3p)$
- $\overrightarrow{QP} = (2p+1, p+2, 3p-1)$

$|CP| = |QP| \Rightarrow |2p, p, 3p| = |2p+1, p+2, 3p-1|$

$\Rightarrow \sqrt{(2p)^2 + p^2 + (3p)^2} = \sqrt{(2p+1)^2 + (p+2)^2 + (3p-1)^2}$

$\Rightarrow \cancel{4p^2} + \cancel{p^2} + \cancel{9p^2} = \cancel{4p^2} + 4p + 1 + \cancel{p^2} + 4p + 4 + \cancel{9p^2} - 6p + 1$

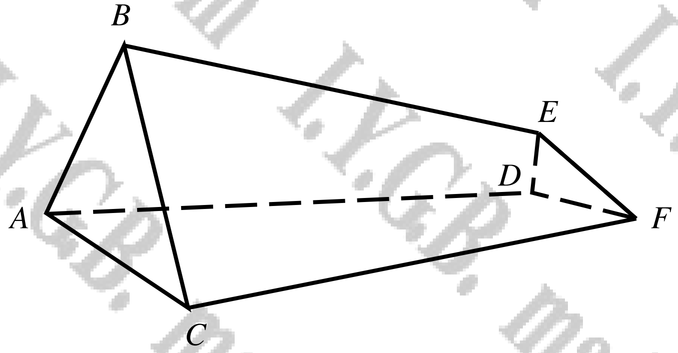
$\Rightarrow -6 = 2p$

$\Rightarrow p = -3$

FINALLY WE HAVE

$P(-6, 0, -4)$ & $Q(-1, 1, 6)$

Question 68 (*****)



The figure above shows an irregular hollow shape, consisting of two non-congruent, non-parallel triangular faces ABC and DEF, and two non-congruent quadrilateral faces $ABED$ and $BCFE$.

The respective equations of the straight lines AD, DE and BC are

$$\mathbf{r}_1 = -5\mathbf{i} + 6\mathbf{j} + \mathbf{k} + \lambda(2\mathbf{i} + 3\mathbf{j})$$
$$\mathbf{r}_2 = -\mathbf{i} + 12\mathbf{j} + \mathbf{k} + \mu(-2\mathbf{i} + 7\mathbf{j} - 7\mathbf{k})$$
$$\mathbf{r}_3 = -\mathbf{i} - 8\mathbf{j} + \mathbf{k} + \nu(-2\mathbf{i} + 7\mathbf{j} + 7\mathbf{k})$$

where λ, μ and ν are scalar parameters.

If the plane face $BCFE$ has equation $21x - 14y + 20z = 111$ and the point G has position vector $5\mathbf{i} + 7\mathbf{j}$, show that the acute angle between the plane face $BCFE$ and the straight line BG is

$$\frac{\pi}{2} - \arccos\left[\frac{13}{\sqrt{3111}}\right].$$

$\boxed{}$, $\boxed{\text{proof}}$

[solutions overleaf]

PLANNING

- FIND NORMAL OF ABED
- FIND EQUATION OF ABED
- FIND LINE "BE" BY INTERSECTING ABED & BCFE
- FIND POINT B BY INTERSECTING LINE "BE" AND "BC"
- FIND DIRECTION OF BG
- FIND ANGLE BETWEEN BG AND BCFE

CROSSING DIRECTIONS OF AD & DE

$$\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 3 & 0 \\ -2 & 7 & -7 \end{vmatrix} = (-21, 14, 20) \leftarrow \text{NORMAL OF ABED}$$

EQUATION OF ABED, USING EITHER $(-5,6,1)$ OR $(-1,12,1)$

$$-21x + 14y + 20z = \text{CONSTANT}$$
$$-21(-5) + 14(6) + 20(1) = \text{CONSTANT}$$
$$105 + 84 + 20 = \text{CONSTANT}$$
$$\text{CONSTANT} = 209$$

$\therefore$ ABED $-21x + 14y + 20z = 209$

INTERSECTING THE PLANES TO FIND THE LINE "BE"

- ABED: $-21x + 14y + 20z = 209$
- BCFE: $21x - 14y + 20z = 111$

ADDING $40z = 320$
$z = 8$ (ALWAYS)

BOTH PLANES NOW REDUCE TO

$$-21x + 14y = 49$$
$$-3x + 2y = 7$$
$$2y = 3x + 7$$
$$y = \frac{3}{2}x + \frac{7}{2}$$

THUS THE LINE "BE" MUST BE

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ \frac{3}{2}x + \frac{7}{2} \\ 8 \end{pmatrix} = \begin{pmatrix} 0 \\ 7/2 \\ 8 \end{pmatrix} + x\begin{pmatrix} 1 \\ 3/2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 7/2 \\ 8 \end{pmatrix} + (\eta + 1)\begin{pmatrix} 1 \\ 3/2 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 5 \\ 8 \end{pmatrix} + \eta\begin{pmatrix} 1 \\ 3/2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 8 \end{pmatrix} + 2t\begin{pmatrix} 1 \\ 3/2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 8 \end{pmatrix} + t\begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$$

$$\Gamma_{BE} = (1,5,8) + t(2,3,0)$$

NEXT FIND THE POINT B BY INTERSECTING THE LINES

- BE: $(x,y,z) = (1+2t, 5+3t, 8)$
- BC: $(x,y,z) = (-1-2\lambda, -8+7\lambda, 1+7\lambda)$

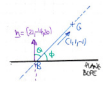
LINES MUST INTERSECT, SO
BY INSPECTING $\underline{k}$ COMPONENT
$\lambda = 1$ $(t = -2)$

$\therefore B(-3,-1,8)$

FIND THE DIRECTION OF BG, $B(-3,-1,8)$ & $G(5,7,0)$

$$\overrightarrow{BG} = \underline{g} - \underline{b} = (5,7,0) - (-3,-1,8) = (8,8,-8) \quad \text{SCALE TO} \quad (1,1,-1)$$

FINALLY LOOKING AT THE DIAGRAM



$$(21,-14,20)\cdot(1,1,-1) = |21,-14,20||1,1,-1|\cos\theta$$
$$21 - 14 - 20 = \sqrt{441+196+400}\sqrt{3}\cos\theta$$
$$-13 = \sqrt{3111}\cos\theta$$
$$\cos\theta = \frac{-13}{\sqrt{3111}}$$

$$\begin{array}{r} 441 \\ 196 \\ +400 \\ \hline 1037 \end{array} \qquad \begin{array}{r} 1037 \\ \times\ 3 \\ \hline 3111 \end{array}$$

BUT θ MUST BE ACUTE SO

$$\cos\theta = \frac{13}{\sqrt{3111}}$$
$$\theta = \arccos\left(\frac{13}{\sqrt{3111}}\right)$$

HENCE THE REQUIRED ANGLE ϕ IS

$$\frac{\pi}{2} - \arccos\left(\frac{13}{\sqrt{3111}}\right)$$

// AS REQUIRED