VECTOR EXAM QUESTIONS

Part A

Question 1 (**)

The straight line l_1 passes through the points with coordinates $(5,1,6)$ and $(2,2,1)$.

a) Find a vector equation of l_1.

A different straight line l_2 passes though the point $C(6,6,-4)$ and is parallel to the vector $4\mathbf{i}-2\mathbf{j}+3\mathbf{k}$.

b) Show clearly that l_1 and l_2 are skew.

$\boxed{}$, $\boxed{\mathbf{r} = 5\mathbf{i}+\mathbf{j}+6\mathbf{k}+\lambda(3\mathbf{i}-\mathbf{j}+5\mathbf{k})}$

a) OBTAIN DIRECTION VECTOR FROM $A(5,1,6)$ & $B(2,2,1)$

$\overrightarrow{AB} = \underline{b} - \underline{a} = (2,2,1) - (5,1,6) = (-3,1,-5)$

"SCALE IT" TO HAVE LESS NEGATIVES

$\overrightarrow{AB} = (3,-1,5)$

$\Rightarrow \underline{r} = (5,1,6) + \lambda(3,-1,5)$

$\Rightarrow \underline{r} = (3\lambda+5, 1-\lambda, 5\lambda+6)$

b) REWRITE THE TWO LINES INCLUDING $\underline{r}_2 = (6,6,-4) + \mu(4,-2,3)$

- $\underline{r}_1 = (3\lambda+5, 1-\lambda, 5\lambda+6)$
- $\underline{r}_2 = (4\mu+6, 6-2\mu, 3\mu-4)$

$$\left.\begin{array}{ll} \underline{i} & 3\lambda+5 = 4\mu+6 \\ \underline{j} & 1-\lambda = 6-2\mu \end{array}\right\} \Rightarrow \lambda = 2\mu-5$$

$\therefore 3(2\mu-5)+5 = 4\mu+6$
$6\mu - 15 + 5 = 4\mu + 6$
$2\mu = 16$
$\mu = 8$

$\lambda = 2\times8-5$
$\lambda = 11$

CHECKING $\underline{k}$ FOR THESE VALUES

$5\lambda+6 = 5\times11+6 = 61$
$3\mu+4 = 3\times8-4 = 20$

$61 \neq 20$

LINES NOT PARALLEL & NOT INTERSECTING, SO SKEW

Question 2 (**)

Relative to a fixed origin O, the respective position vectors of three points A, B and C are

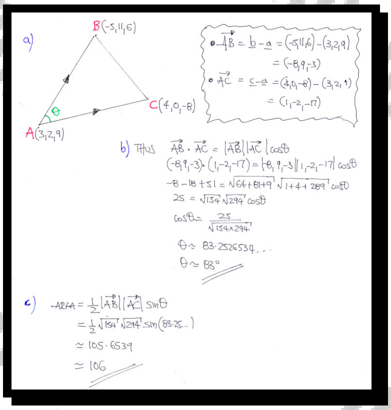
$$\begin{pmatrix} 3 \\ 2 \\ 9 \end{pmatrix}, \quad \begin{pmatrix} -5 \\ 11 \\ 6 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 4 \\ 0 \\ -8 \end{pmatrix}.$$

a) Determine, in component form, the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

b) Hence find, to the nearest degree, the angle BAC.

c) Calculate the area of the triangle BAC.

$\boxed{\overrightarrow{AB} = -8\mathbf{i} + 9\mathbf{j} - 3\mathbf{k}}$, $\boxed{\overrightarrow{AC} = \mathbf{i} - 2\mathbf{j} - 17\mathbf{k}}$, $\boxed{\theta \approx 83°}$, $\boxed{\text{area} \approx 106}$



Question 3 **(**)**

The straight line l_1 passes through the points $A(2,5,9)$ and $B(6,0,10)$.

a) Find a vector equation for l_1.

The straight line l_2 has vector equation

$$\mathbf{r} = \begin{pmatrix} 8 \\ 8 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix},$$

where μ is a scalar parameter.

b) Show that the point A is the intersection of l_1 and l_2.

c) Show further that l_1 and l_2 are perpendicular to each other.

$$\boxed{\mathbf{r} = 2\mathbf{i} + 5\mathbf{j} + 9\mathbf{k} + \lambda(4\mathbf{i} - 5\mathbf{j} + \mathbf{k})}$$

(a) $\overrightarrow{AB} = \underline{b} - \underline{a} = (6,0,10) - (2,5,9) = (4,-5,1)$

$\therefore \underline{r} = (2,5,9) + \lambda(4,-5,1)$

$\underline{r} = (4\lambda+2,\ 5-5\lambda,\ \lambda+9)$

(b) $\underline{r} = (8,8,0) + \mu(2,1,-3) = (2\mu+8,\ \mu+8,\ -3\mu)$

- POINT $A(2,5,9)$ LIES ON l_1 (WE USED IT TO FIND l_1)
- BY INSPECTION (LOOKING AT $\underline{k}$), IF $\mu = -3$, $(2\mu+8, \mu+8, -3\mu)$ GIVES $(2,5,9)$

$\therefore$ A IS THE INTERSECTION OF l_1 & l_2

(c) DOTTING DIRECTION VECTORS:

$(4,-5,1)\cdot(2,1,-3) = (4\times2) - 5\times1 + 1(-3) = 8-5-3 = 0$

$\therefore l_1 \perp l_2$

Question 4 (**)

Relative to a fixed origin O, the points A and B have respective position vectors

$$\mathbf{i}+7\mathbf{j}+5\mathbf{k} \text{ and } 5\mathbf{i}+\mathbf{j}-5\mathbf{k}.$$

a) Find a vector equation of the straight line l_1 which passes through A and B.

The straight line l_2 has vector equation

$$\mathbf{r}_2 = 5\mathbf{i}-4\mathbf{j}+4\mathbf{k}+\mu(\mathbf{i}-4\mathbf{j}+2\mathbf{k}),$$

where μ is a scalar parameter.

The point C is the point of intersection between l_1 and l_2.

b) Find the position vector of C.

c) Show that C is the midpoint of AB.

$\boxed{\mathbf{r}=\mathbf{i}+7\mathbf{j}+5\mathbf{k}+\lambda(2\mathbf{i}-3\mathbf{j}-5\mathbf{k})}$, $\boxed{\overrightarrow{OC}=3\mathbf{i}+4\mathbf{j}}$

(a) $\overrightarrow{AB} = \underline{b}-\underline{a} = (5,1,-5)-(1,7,5) = (4,-6,-10)$

SCALE DIRECTION $(2,-3,-5)$

$\therefore\ \underline{r}_1 = (1,7,5)+\lambda(2,-3,-5)$

$\underline{r}_1 = (2\lambda+1,\ 7-3\lambda,\ 5-5\lambda)$

(b) $\underline{r}_2 = (5,-4,4)+\mu(1,-4,2)$

$\underline{r}_2 = (\mu+5,\ -4\mu-4,\ 2\mu+4)$

EQUATE $\underline{i}$ & $\underline{j}$

$\left.\begin{array}{l}(\underline{i})\ \ 2\lambda+1=\mu+5\\ (\underline{j})\ \ 7-3\lambda=-4\mu-4\end{array}\right\}\Rightarrow \mu = 2\lambda-4$ SUBSTITUTE

$\Rightarrow 7-3\lambda = -4(2\lambda-4)-4$

$\Rightarrow 7-3\lambda = -8\lambda+16-4$

$\Rightarrow 5\lambda = 5$

$\Rightarrow \boxed{\lambda = 1}$

USING $\lambda=1$ WE OBTAIN $(2\lambda+1,\ 7-3\lambda,\ 5-5\lambda) = (3,4,0)$

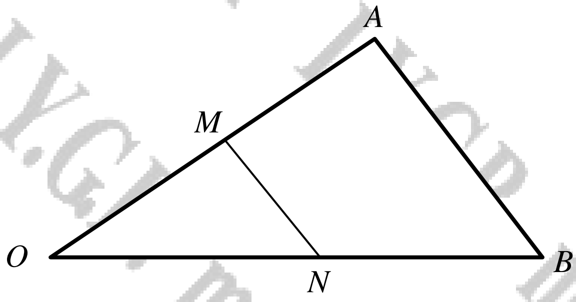
$\therefore\ C(3,4,0)$

(c) MIDPOINT OF $AB = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right) = \left(\frac{1+5}{2}, \frac{7+1}{2}, \frac{5-5}{2}\right)$

$= (3,4,0)$

Question 5 (**)

The figure below shows the triangle OAB.

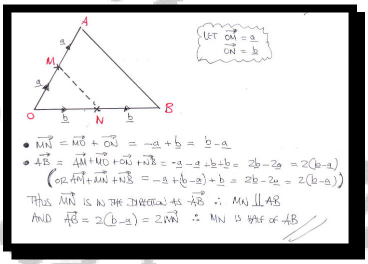


The point M is the midpoint of OA and the point N is the midpoint of OB.

Let $\overrightarrow{OM} = \mathbf{a}$ and $\overrightarrow{ON} = \mathbf{b}$.

By finding simplified expressions for $\overrightarrow{MN}$ and $\overrightarrow{AB}$, in terms of $\mathbf{a}$ and $\mathbf{b}$, show that MN is parallel to AB, and half its length.

$\boxed{\text{proof}}$



LET $\overrightarrow{OM} = \underline{a}$
$\overrightarrow{ON} = \underline{b}$

- $\overrightarrow{MN} = \overrightarrow{MO} + \overrightarrow{ON} = -\underline{a} + \underline{b} = \underline{b} - \underline{a}$
- $\overrightarrow{AB} = \overrightarrow{AM} + \overrightarrow{MO} + \overrightarrow{ON} + \overrightarrow{NB} = -\underline{a} - \underline{a} + \underline{b} + \underline{b} = 2\underline{b} - 2\underline{a} = 2(\underline{b} - \underline{a})$
  $\left(\text{OR } \overrightarrow{AM} + \overrightarrow{MN} + \overrightarrow{NB} = -\underline{a} + (\underline{b} - \underline{a}) + \underline{b} = 2\underline{b} - 2\underline{a} = 2(\underline{b} - \underline{a})\right)$

THUS $\overrightarrow{MN}$ IS IN THE DIRECTION AS $\overrightarrow{AB}$ $\therefore$ $MN \parallel AB$

AND $\overrightarrow{AB} = 2(\underline{b} - \underline{a}) = 2\overrightarrow{MN}$ $\therefore$ MN IS HALF OF AB //

Question 6 (+)**

The points $A(2,4,4)$, $B(6,8,4)$, $C(6,4,0)$, $D(2,0,0)$ and $M(4,4,2)$ are given.

The straight line l_1 has equation

$$\mathbf{r}_1 = 6\mathbf{i} + 4\mathbf{j} + \lambda(\mathbf{i} + \mathbf{j}),$$

where λ is a scalar parameter.

The straight line l_2 passes through the points C and M.

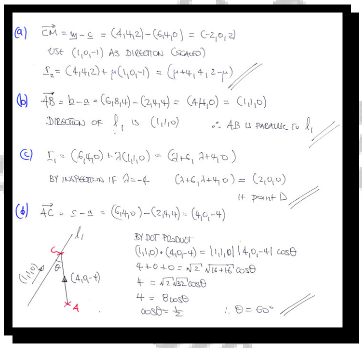
a) Find a vector equation of l_2.

b) Show that $\overrightarrow{AB}$ is parallel to l_1.

c) Verify that D lies on l_1.

d) Find the acute angle between $\overrightarrow{AC}$ and l_1.

$\boxed{\mathbf{r}_2 = 4\mathbf{i} + 4\mathbf{j} + 2\mathbf{k} + \mu(\mathbf{i} - \mathbf{k})}$, $\boxed{60°}$



(a) $\overrightarrow{CM} = \underline{m} - \underline{c} = (4,4,2) - (6,4,0) = (-2,0,2)$

USE $(1,0,-1)$ AS DIRECTION (SCALED)

$\underline{r}_2 = (4,4,2) + \mu(1,0,-1) = (\mu+4, 4, 2-\mu)$

(b) $\overrightarrow{AB} = \underline{b} - \underline{a} = (6,8,4) - (2,4,4) = (4,4,0) = (1,1,0)$

DIRECTION OF l_1 IS $(1,1,0)$ $\therefore$ AB IS PARALLEL TO l_1

(c) $\underline{r}_1 = (6,4,0) + \lambda(1,1,0) = (\lambda+6, \lambda+4, 0)$

BY INSPECTION IF $\lambda = -4$ $(\lambda+6, \lambda+4, 0) = (2,0,0)$

i.e. point D

(d) $\overrightarrow{AC} = \underline{c} - \underline{a} = (6,4,0) - (2,4,4) = (4,0,-4)$

BY DOT PRODUCT

$(1,1,0) \cdot (4,0,-4) = |1,1,0|\,|4,0,-4| \cos\theta$

$4 + 0 + 0 = \sqrt{2}\sqrt{16+16}\cos\theta$

$4 = \sqrt{2}\sqrt{32}\cos\theta$

$4 = 8\cos\theta$

$\cos\theta = \frac{1}{2}$ $\therefore \theta = 60°$

Question 7 (**+)

The straight lines l_1 and l_2 have the following vector equations

$$\mathbf{r}_1 = 4\mathbf{i} + 3\mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} + 4\mathbf{j} + 3\mathbf{k})$$

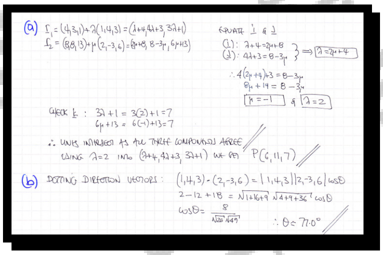
$$\mathbf{r}_2 = 8\mathbf{i} + 8\mathbf{j} + 13\mathbf{k} + \mu(2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}),$$

where λ and μ are scalar parameters.

a) Show that l_1 and l_2 intersect at some point P and find its coordinates.

b) Calculate the acute angle between l_1 and l_2.

$\boxed{P(6,11,7)}$, $\boxed{77.0°}$



(a) $\underline{r}_1 = (4,3,1) + \lambda(1,4,3) = (\lambda+4, 4\lambda+3, 3\lambda+1)$
$\underline{r}_2 = (8,8,13) + \mu(2,-3,6) = (2\mu+8, 8-3\mu, 6\mu+13)$

EQUATE $\underline{i}$ & $\underline{j}$

(i): $\lambda + 4 = 2\mu + 8$
(j): $4\lambda + 3 = 8 - 3\mu$ $\Rightarrow \boxed{\lambda = 2\mu + 4}$

$\therefore 4(2\mu+4) + 3 = 8 - 3\mu$
$8\mu + 14 = 8 - 3\mu$
$\boxed{\mu = -1}$ & $\boxed{\lambda = 2}$

CHECK $\underline{k}$: $3\lambda + 1 = 3(2) + 1 = 7$
$6\mu + 13 = 6(-1) + 13 = 7$

$\therefore$ LINES INTERSECT AS ALL THREE COMPONENTS AGREE
USING $\lambda = 2$ INTO $(\lambda+4, 4\lambda+3, 3\lambda+1)$ WE GET $P(6,11,7)$

(b) DOTTING DIRECTION VECTORS: $(1,4,3)\cdot(2,-3,6) = |1,4,3||2,-3,6|\cos\theta$
$2 - 12 + 18 = \sqrt{1+16+9}\sqrt{4+9+36}\cos\theta$
$\cos\theta = \frac{8}{\sqrt{26}\sqrt{49}}$
$\therefore \theta \approx 77.0°$

Question 8 (**+)

Relative to a fixed origin O, the points P and Q have respective position vectors

$$(-7\mathbf{j}+4\mathbf{k}) \text{ and } (3\mathbf{i}-8\mathbf{j}+2\mathbf{k}).$$

The straight line l_1 passes through the points P and Q.

a) Determine a vector equation for l_1.

The straight line l_2 has vector equation

$$\mathbf{r}=(7\mathbf{i}+a\mathbf{j}+b\mathbf{k})+\mu(\mathbf{i}+4\mathbf{j}-\mathbf{k}),$$

where a and b are scalar constants, and μ is a scalar parameter.

b) Given that l_1 and l_2 intersect at Q, find the value of a and the value of b.

c) Calculate the acute angle between l_1 and l_2.

$\boxed{}$, $\boxed{\mathbf{r}=-7\mathbf{j}+4\mathbf{k}+\lambda(3\mathbf{i}-\mathbf{j}-2\mathbf{k})}$, $\boxed{a=8}$, $\boxed{b=-2}$, $\boxed{86.4°}$

(a) $\vec{PQ} = q - p = (3,-8,2)-(0,-7,4) = (3,-1,-2)$

THUS $r = (0,-7,4)+\lambda(3,-1,-2)$

$r = (3\lambda, -\lambda-7, -2\lambda+4)$

(b) l_2 PASSES THROUGH $Q(3,-8,2)$ $\quad r_2 = (7+\mu, a+4\mu, b-\mu)$

(i): $7+\mu=3$, $\mu=-4$ $\quad$ (j): $a+4\mu=-8$, $a-16=-8$, $a=8$ $\quad$ (k): $b-\mu=2$, $b+4=2$, $b=-2$

(c) DOTTING THE DIRECTION VECTORS OF THE TWO LINES

$(3,-1,-2)\cdot(1,4,-1) = |3,-1,-2||1,4,-1|\cos\theta$

$3-4+2 = \sqrt{9+1+4}\sqrt{1+16+1}\cos\theta$

$1 = \sqrt{14\times18}\cos\theta$

$\cos\theta = \frac{1}{42}\sqrt{7}$

$\theta \approx 86.4°$

Question 9 (+)**

The straight lines l_1 and l_2 have the following vector equations

$$\mathbf{r}_1 = 2\mathbf{i} + 2\mathbf{j} + \lambda(\mathbf{i} + \mathbf{j})$$

$$\mathbf{r}_2 = 2\mathbf{i} + 5\mathbf{j} + 7\mathbf{k} + \mu(2\mathbf{i} + \mathbf{j} - \mathbf{k}),$$

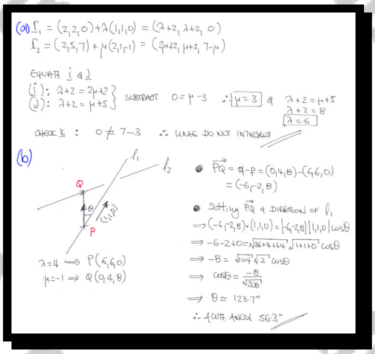
where λ and μ are scalar parameters.

a) Show that l_1 and l_2 **do not** intersect.

The point P lies on l_1 where $\lambda = 4$ and the point Q lies on l_2 where $\mu = -1$.

b) Find the acute angle between PQ and l_1.

$\boxed{}$, $\boxed{56.3°}$



Question 10 (+)**

Relative to a fixed origin O, the points A, B and C have respective position vectors

$$\mathbf{a} = 8\mathbf{i} + \mathbf{j}, \quad \mathbf{b} = 5\mathbf{j} + 8\mathbf{k} \quad \text{and} \quad \mathbf{c} = 14\mathbf{i} + \mathbf{j} + 15\mathbf{k}.$$

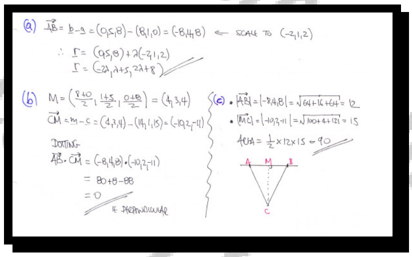
a) Find a vector equation of the straight line which passes through A and B.

The point M is the midpoint of AB.

b) Show that CM is perpendicular to AB.

c) Determine the area of the triangle ABC.

$\boxed{\mathbf{r} = 5\mathbf{j} + 8\mathbf{k} + \lambda(-2\mathbf{i} + \mathbf{j} + 2\mathbf{k})}$, $\boxed{\text{area} = 90}$


(a) $\overrightarrow{AB} = \underline{b} - \underline{a} = (0,5,8) - (8,1,0) = (-8,4,8) \leftarrow$ SCALE TO $(-2,1,2)$

$\therefore \underline{r} = (0,5,8) + \lambda(-2,1,2)$

$\underline{r} = (-2\lambda, \lambda+5, 2\lambda+8)$

(b) $M = \left(\frac{8+0}{2}, \frac{1+5}{2}, \frac{0+8}{2}\right) = (4,3,4)$

$\overrightarrow{CM} = \underline{m} - \underline{c} = (4,3,4) - (14,1,15) = (-10,2,-11)$

DOTTING

$\overrightarrow{AB} \cdot \overrightarrow{CM} = (-8,4,8) \cdot (-10,2,-11)$

$= 80 + 8 - 88$

$= 0$

// PERPENDICULAR

(c) • $|\overrightarrow{AB}| = |-8,4,8| = \sqrt{64+16+64} = 12$

• $|\overrightarrow{MC}| = |-10,2,-11| = \sqrt{100+4+121} = 15$

AREA $= \frac{1}{2} \times 12 \times 15 = 90$

Question 11 (**+)

With respect to a fixed origin O, the respective position vectors of the points A, B and C are

$$\begin{pmatrix} 2 \\ 9 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} 12 \\ 4 \\ 7 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 10 \\ -3 \\ 7 \end{pmatrix}.$$

a) Find the position vector of the midpoint of AC.

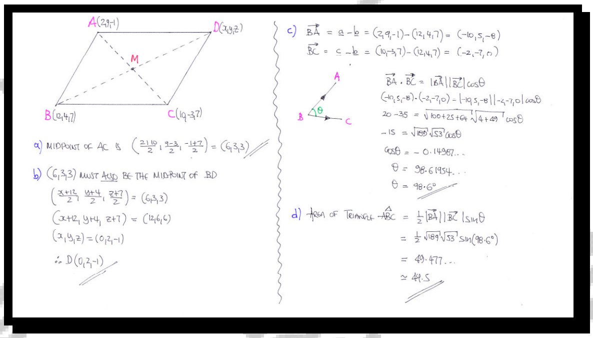
The point D is such so that $ABCD$ is a parallelogram.

b) Determine the position vector of D.

c) Calculate, correct to one decimal place, the angle ABC.

d) Hence, calculate the area of the triangle ABC.

$\boxed{6\mathbf{i}+3\mathbf{j}+3\mathbf{k}}$, $\boxed{2\mathbf{j}-\mathbf{k}}$, $\boxed{\theta \approx 98.6°}$, $\boxed{\text{area} \approx 49.5}$



Question 12 (**+)

$OABC$ is a square.

The point M is the midpoint of AB and the point N is the midpoint of MC.

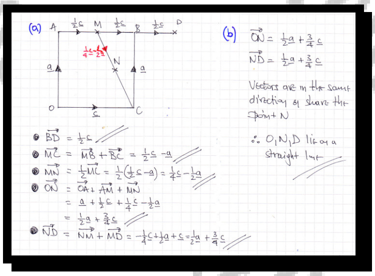
The point D is such so that $\overrightarrow{AD} = \frac{3}{2}\overrightarrow{AB}$.

Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OC} = \mathbf{c}$.

a) Find simplified expressions, in terms of $\mathbf{a}$ and $\mathbf{c}$, for each of the vectors $\overrightarrow{BD}$, $\overrightarrow{MC}$, $\overrightarrow{MN}$, $\overrightarrow{ON}$ and $\overrightarrow{ND}$.

b) Deduce, showing your reasoning, that O, N and D are collinear.

$\boxed{\overrightarrow{BD} = \frac{1}{2}\mathbf{c}}$, $\boxed{\overrightarrow{MC} = \frac{1}{2}\mathbf{c} - \mathbf{a}}$, $\boxed{\overrightarrow{MN} = \frac{1}{4}\mathbf{c} - \frac{1}{2}\mathbf{a}}$, $\boxed{\overrightarrow{ON} = \frac{1}{2}\mathbf{a} + \frac{3}{4}\mathbf{c}}$, $\boxed{\overrightarrow{ND} = \frac{1}{2}\mathbf{a} + \frac{3}{4}\mathbf{c}}$



Question 13 (***)

Relative to a fixed origin O, the points P and Q have respective position vectors

$$5\mathbf{i}+2\mathbf{k} \text{ and } 3\mathbf{i}+3\mathbf{j}.$$

a) Determine a vector equation of the straight line l which passes through the points P and Q.

The straight line m has a vector equation

$$\mathbf{r} = 4\mathbf{i}+8\mathbf{j}-\mathbf{k}+\mu(5\mathbf{i}-\mathbf{j}+3\mathbf{k}),$$

where μ is a scalar parameter.

b) Show that l and m intersect at some point A and find its position vector.

c) Find the size of the acute angle θ, formed by l and m.

$\boxed{\mathbf{r} = 5\mathbf{i}+2\mathbf{k}+\lambda(2\mathbf{i}-3\mathbf{j}+2\mathbf{k})}$, $\boxed{\overrightarrow{OA} = -\mathbf{i}+9\mathbf{j}-4\mathbf{k}}$, $\boxed{\theta \approx 38.8°}$

(a) $PQ = \overrightarrow{PQ} = \underline{q}-\underline{p} = (3,3,0)-(5,0,2) = (-2,3,-2)$ "SCALE" IT BY -1

$\therefore \underline{r}_1 = (5,0,2)+\lambda(2,-3,2)$

$\underline{r}_1 = (2\lambda+5, -3\lambda, 2\lambda+2)$

(b) $\underline{r}_1 = (2\lambda+5, -3\lambda, 2\lambda+2)$, $\underline{r}_2 = (5\mu+4, 8-\mu, 3\mu-1)$ EQUATE $\underline{i}$ & $\underline{k}$

$(\underline{i})$: $2\lambda+5 = 5\mu+4$, $(\underline{k})$: $2\lambda+2 = 3\mu-1$ SUBTRACT $3 = 2\mu+5$

$\boxed{\mu=-1}$ $\boxed{\lambda=-3}$

CHECK $\underline{j}$: $-3\lambda = -3(-3) = 9$, $8-\mu = 8-(-1) = 9$ • AS ALL THREE COMPONENTS AGREE THE LINES INTERSECT

• USING $\mu=-1$ INTO $(5\mu+4, 8-\mu, 3\mu-1)$ WE OBTAIN $A(-1,9,-4)$

IE $\underline{a} = -\underline{i}+9\underline{j}-4\underline{k}$

(c) DOTTING DIRECTIONS $(2,-3,2)\cdot(5,-1,3) = |2,-3,2||5,-1,3|\cos\theta$

$10+3+6 = \sqrt{4+9+4}\sqrt{25+1+9}\cos\theta$

$19 = \sqrt{17}\sqrt{35}\cos\theta$

$\cos\theta = \frac{19}{\sqrt{17\times 35}}$ $\therefore \theta \approx 38.8°$

Question 14 (***)

The points $A(2,10,7)$ and $B(0,15,12)$ are given.

a) Determine a vector equation of the straight line l_1 that passes through the points A and B.

The vector equation of the straight line l_2 is

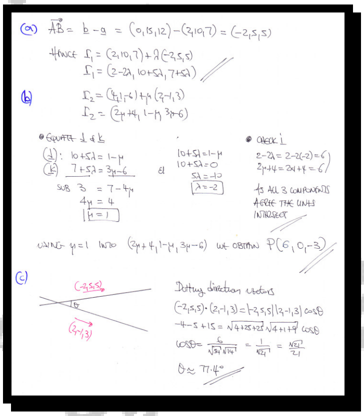
$$\mathbf{r}_2 = 4\mathbf{i} + \mathbf{j} - 6\mathbf{k} + \mu(2\mathbf{i} - \mathbf{j} + 3\mathbf{k}),$$

where μ is a scalar parameter.

b) Show that l_1 and l_2 intersect at some point P and find its coordinates.

c) Calculate the acute angle between l_1 and l_2.

$\boxed{}$, $\boxed{\mathbf{r}_1 = 2\mathbf{i} + 10\mathbf{j} + 7\mathbf{k} + \lambda(-2\mathbf{i} + 5\mathbf{j} + 5\mathbf{k})}$, $\boxed{P(6,0,-3)}$, $\boxed{77.4°}$



(a) $\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = (0,15,12) - (2,10,7) = (-2,5,5)$

HENCE $\mathbf{r}_1 = (2,10,7) + \lambda(-2,5,5)$

$\mathbf{r}_1 = (2-2\lambda, 10+5\lambda, 7+5\lambda)$

(b) $\mathbf{r}_2 = (4,1,-6) + \mu(2,-1,3)$

$\mathbf{r}_2 = (2\mu+4, 1-\mu, 3\mu-6)$

• EQUATE $\mathbf{j}$ & $\mathbf{k}$

($\mathbf{j}$): $10+5\lambda = 1-\mu$

($\mathbf{k}$): $7+5\lambda = 3\mu-6$

SUB $3 = 7-4\mu$

$4\mu = 4$

$\boxed{\mu = 1}$

& $10+5\lambda = 1-\mu$

$10+5\lambda = 0$

$5\lambda = -10$

$\boxed{\lambda = -2}$

• CHECK $\mathbf{i}$

$2-2\lambda = 2-2(-2) = 6$

$2\mu+4 = 2\times1+4 = 6$

AS ALL 3 COMPONENTS AGREE THE LINES INTERSECT

USING $\mu = 1$ INTO $(2\mu+4, 1-\mu, 3\mu-6)$ WE OBTAIN $P(6,0,-3)$

(c) $(-2,5,5)$ $(2,-1,3)$ θ

Dotting direction vectors

$(-2,5,5)\cdot(2,-1,3) = |-2,5,5||2,-1,3|\cos\theta$

$-4-5+15 = \sqrt{4+25+25}\sqrt{4+1+9}\cos\theta$

$\cos\theta = \frac{6}{\sqrt{54}\sqrt{14}} = \frac{1}{\sqrt{21}} = \frac{\sqrt{21}}{21}$

$\theta \approx 77.4°$

Question 15 (***)

Relative to a fixed origin O the following position vectors are given.

$$\overrightarrow{OA} = \begin{pmatrix} 1 \\ 6 \\ 11 \end{pmatrix}, \ \overrightarrow{OB} = \begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix}, \ \overrightarrow{OC} = \begin{pmatrix} 6 \\ 1 \\ 1 \end{pmatrix} \text{ and } \overrightarrow{OD} = \begin{pmatrix} 2 \\ 8 \\ 9 \end{pmatrix}.$$

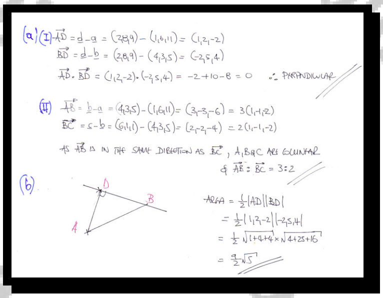
a) Show clearly that …

i. … $\overrightarrow{AD}$ is perpendicular to $\overrightarrow{BD}$.

ii. … the points A, B and C are collinear and state the ratio $AB : BC$.

b) Determine the **exact** area of the triangle ABD.

$\boxed{AB : BC = 3 : 2}$, $\boxed{\text{area} = \frac{9}{2}\sqrt{5}}$



(a)(I) $\overrightarrow{AD} = \underline{d} - \underline{a} = (2,8,9) - (1,6,11) = (1,2,-2)$

$\overrightarrow{BD} = \underline{d} - \underline{b} = (2,8,9) - (4,3,5) = (-2,5,4)$

$\overrightarrow{AD} \cdot \overrightarrow{BD} = (1,2,-2)\cdot(-2,5,4) = -2+10-8 = 0$ ∴ PERPENDICULAR

(II) $\overrightarrow{AB} = \underline{b} - \underline{a} = (4,3,5) - (1,6,11) = (3,-3,-6) = 3(1,-1,-2)$

$\overrightarrow{BC} = \underline{c} - \underline{b} = (6,1,1) - (4,3,5) = (2,-2,-4) = 2(1,-1,-2)$

AS $\overrightarrow{AB}$ IS IN THE SAME DIRECTION AS $\overrightarrow{BC}$, A, B & C ARE COLLINEAR

& $\overrightarrow{AB} : \overrightarrow{BC} = 3:2$

(b)

$\text{AREA} = \frac{1}{2}|AD||BD|$

$= \frac{1}{2}|1,2,-2||-2,5,4|$

$= \frac{1}{2}\sqrt{1+4+4} \times \sqrt{4+25+16}$

$= \frac{9}{2}\sqrt{5}$

Question 16 (***)

Relative to a fixed origin O, the points A and B have respective position vectors

$$4\mathbf{i}+3\mathbf{j}-3\mathbf{k} \quad \text{and} \quad 2\mathbf{i}+3\mathbf{k}\,.$$

a) Determine a vector equation of the straight line l_1 which passes through the points A and B.

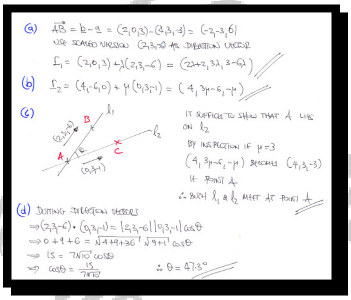
The straight line l_2 passes through the point C with position vector $4\mathbf{i}-6\mathbf{j}$ and is parallel to the vector $3\mathbf{j}-\mathbf{k}$.

b) Write down a vector equation of l_2.

c) Show that l_1 and l_2 intersect at the point A.

d) Find the acute angle between l_1 and l_2.

$\boxed{\mathbf{r}_1 = 2\mathbf{i}+3\mathbf{k}+\lambda(2\mathbf{i}+3\mathbf{j}-6\mathbf{k})}$, $\boxed{\mathbf{r}_2 = 4\mathbf{i}-6\mathbf{k}+\mu(3\mathbf{j}-\mathbf{k})}$, $\boxed{47.3°}$


(a) $\overrightarrow{AB} = \underline{b}-\underline{a} = (2,0,3)-(4,3,-3) = (-2,-3,6)$

USE SCALED VERSION $(2,3,-6)$ AS DIRECTION VECTOR

$\underline{r}_1 = (2,0,3)+\lambda(2,3,-6) = (2\lambda+2, 3\lambda, 3-6\lambda)$

(b) $\underline{r}_2 = (4,-6,0)+\mu(0,3,-1) = (4, 3\mu-6, -\mu)$

(c) IT SUFFICES TO SHOW THAT A LIES ON l_2

BY INSPECTION IF $\mu=3$

$(4, 3\mu-6, -\mu)$ BECOMES $(4,3,-3)$

IE POINT A

$\therefore$ BOTH l_1 & l_2 MEET AT POINT A

(d) DOTTING DIRECTION VECTORS

$\Rightarrow (2,3,-6)\cdot(0,3,-1) = |2,3,-6||0,3,-1|\cos\theta$

$\Rightarrow 0+9+6 = \sqrt{4+9+36}\sqrt{9+1}\cos\theta$

$\Rightarrow 15 = 7\sqrt{10}\cos\theta$

$\Rightarrow \cos\theta = \frac{15}{7\sqrt{10}}$ $\quad \therefore \theta = 47.3°$

Question 17 (***)

Relative to a fixed origin O, the straight lines L and M have vector equations

$$\mathbf{r}_1 = \begin{pmatrix} 4 \\ 5 \\ 0 \end{pmatrix} + t\begin{pmatrix} -2 \\ 4 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{r}_2 = \begin{pmatrix} -4 \\ -1 \\ 3 \end{pmatrix} + s\begin{pmatrix} 5 \\ 1 \\ -2 \end{pmatrix},$$

where t and s are scalar parameters.

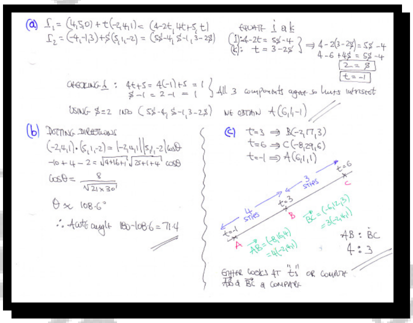
a) Show that L and M intersect at some point A and find its coordinates.

b) Find the size of the acute angle θ, formed by L and M.

The points B and C lie on the L where $t = 3$ and $t = 6$, respectively.

c) Find the ratio $AB : BC$.

$\boxed{}$, $\boxed{A(6,1,-1)}$, $\boxed{\theta \approx 71.4}$, $\boxed{AB : BC = 4:3}$



Question 18 (***)

The straight lines l_1 and l_2 have the following vector equations

$$\mathbf{r}_1 = 2\mathbf{i}+\mathbf{j}+\lambda(-\mathbf{i}+2\mathbf{j}+3\mathbf{k})$$

$$\mathbf{r}_2 = \mathbf{i}+5\mathbf{j}+4\mathbf{k}+\mu(\mathbf{i}-2\mathbf{k})$$

where λ and μ are scalar parameters.

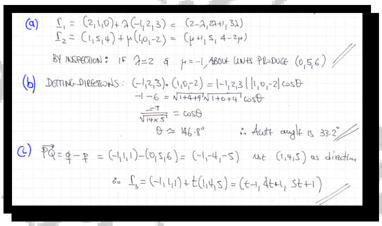
a) Verify that both l_1 and l_2 pass through the point P, whose position vector is $5\mathbf{j}+6\mathbf{k}$.

b) Find the acute angle between l_1 and l_2 .

The point Q has position vector $-\mathbf{i}+\mathbf{j}+\mathbf{k}$.

c) Find a vector equation of the straight line l_3 that passes through the point Q, so that all three straight lines intersect.

$\boxed{33.2°}$, $\boxed{\mathbf{r}_3 = -\mathbf{i}+\mathbf{j}+\mathbf{k}+t(\mathbf{i}+4\mathbf{j}+5\mathbf{k})}$


(a) $\underline{r}_1 = (2,1,0)+\lambda(-1,2,3) = (2-\lambda, 2\lambda+1, 3\lambda)$
$\underline{r}_2 = (1,5,4)+\mu(1,0,-2) = (\mu+1, 5, 4-2\mu)$

BY INSPECTION: IF $\lambda=2$ & $\mu=-1$, ABOVE LINES PRODUCE $(0,5,6)$

(b) DOTTING DIRECTIONS: $(-1,2,3)\cdot(1,0,-2) = |-1,2,3||1,0,-2|\cos\theta$
$-1-6 = \sqrt{1+4+9}\sqrt{1+0+4}\cos\theta$
$\frac{-7}{\sqrt{14\times5}} = \cos\theta$
$\theta \simeq 146.8°$ $\therefore$ Acute angle is $33.2°$

(c) $\overrightarrow{PQ} = \underline{q}-\underline{p} = (-1,1,1)-(0,5,6) = (-1,-4,-5)$ use $(1,4,5)$ as direction

$\therefore \underline{r}_3 = (-1,1,1)+t(1,4,5) = (t-1, 4t+1, 5t+1)$

Question 19 (***)

With respect to a fixed origin O, the point A has position vector $8\mathbf{i}-6\mathbf{j}+5\mathbf{k}$ and the point B has position vector $t\mathbf{i}+t\mathbf{j}+2t\mathbf{k}$.

a) Show clearly that

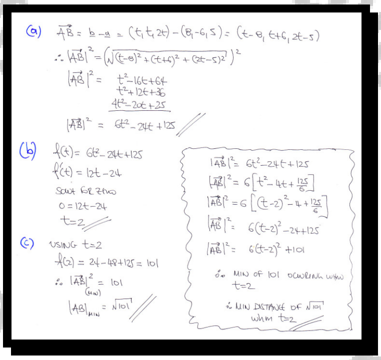
$$|AB|^2 = 6t^2 - 24t + 125 .$$

Let $f(t) = 6t^2 - 24t + 125$.

b) Find the value of t for which $f(t)$ takes a minimum value.

c) Hence determine the closest distance between A and B.

$\boxed{t=2}$, $\boxed{\sqrt{101}}$


(a) $\overrightarrow{AB} = \underline{b} - \underline{a} = (t, t, 2t) - (8, -6, 5) = (t-8, t+6, 2t-5)$

$\therefore |\overrightarrow{AB}|^2 = \left(\sqrt{(t-8)^2 + (t+6)^2 + (2t-5)^2}\right)^2$

$|\overrightarrow{AB}|^2 = t^2-16t+64$
$t^2+12t+36$
$4t^2-20t+25$

$|\overrightarrow{AB}|^2 = 6t^2-24t+125$

(b) $f(t) = 6t^2-24t+125$

$f'(t) = 12t-24$

SOLVE FOR ZERO

$0 = 12t-24$

$t=2$

(c) USING $t=2$

$f(2) = 24-48+125 = 101$

$\therefore |\overrightarrow{AB}|^2_{(MIN)} = 101$

$|AB|_{MIN} = \sqrt{101}$

$|\overrightarrow{AB}|^2 = 6t^2-24t+125$

$|\overrightarrow{AB}|^2 = 6\left[t^2-4t+\frac{125}{6}\right]$

$|\overrightarrow{AB}|^2 = 6\left[(t-2)^2-4+\frac{125}{6}\right]$

$|\overrightarrow{AB}|^2 = 6(t-2)^2-24+125$

$|\overrightarrow{AB}|^2 = 6(t-2)^2+101$

$\therefore$ MIN OF 101 OCCURING WHEN $t=2$

$\therefore$ MIN DISTANCE OF $\sqrt{101}$ WHEN $t=2$

Question 20 (***)

Relative to a fixed origin O, the points A and B have respective coordinates

$$(2,-3,3) \quad \text{and} \quad (5,1,b),$$

where b is a constant.

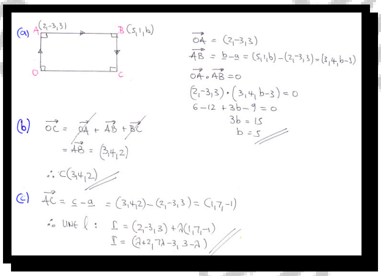
The point C is such so that $OABC$ is a rectangle, where O is the origin.

a) Show clearly that $b = 5$.

b) Determine the position vector of C.

c) Find the vector equation of the straight line l that passes through A and C.

$\boxed{C(3,4,2)}$, $\boxed{\mathbf{r} = 2\mathbf{i} - 3\mathbf{j} + 3\mathbf{k} + \lambda(\mathbf{i} + 7\mathbf{j} - \mathbf{k})}$



Question 21 (***)

The following vectors are given

$$\mathbf{a} = 6\mathbf{i} - 3\mathbf{j} + 2\mathbf{k} \quad \text{and} \quad \mathbf{b} = (4p+1)\mathbf{i} + (p-2)\mathbf{j} + \mathbf{k},$$

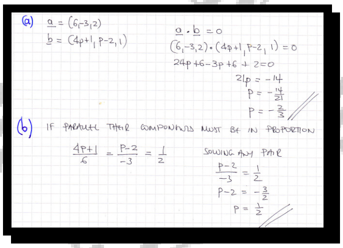
where p is a scalar constant.

Find the value of p if …

a) … $\mathbf{a}$ and $\mathbf{b}$ are perpendicular.

b) … $\mathbf{a}$ and $\mathbf{b}$ are parallel.

$\boxed{p = -\frac{2}{3}}$, $\boxed{p = \frac{1}{2}}$


(a) $\underline{a} = (6,-3,2)$

$\underline{b} = (4p+1, p-2, 1)$

$\underline{a} \cdot \underline{b} = 0$

$(6,-3,2)\cdot(4p+1, p-2, 1) = 0$

$24p + 6 - 3p + 6 + 2 = 0$

$21p = -14$

$p = -\frac{14}{21}$

$p = -\frac{2}{3}$

(b) IF PARALLEL THEIR COMPONENTS MUST BE IN PROPORTION

$\frac{4p+1}{6} = \frac{p-2}{-3} = \frac{1}{2}$

SOLVING ANY PAIR

$\frac{p-2}{-3} = \frac{1}{2}$

$p - 2 = -\frac{3}{2}$

$p = \frac{1}{2}$

Question 22 (***)

$OABC$ is a parallelogram and the point M is the midpoint of AB.

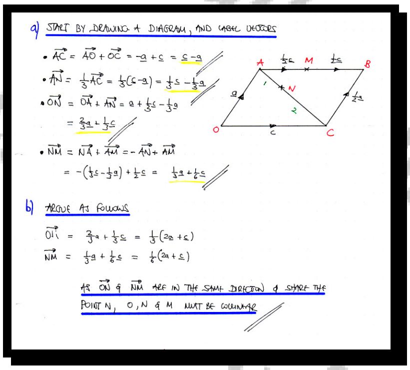
The point N lies on the diagonal AC so that $AN : NC = 1 : 2$.

Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OC} = \mathbf{c}$.

a) Find simplified expressions, in terms of $\mathbf{a}$ and $\mathbf{c}$, for each of the vectors $\overrightarrow{AC}$, $\overrightarrow{AN}$, $\overrightarrow{ON}$ and $\overrightarrow{NM}$.

b) Deduce, showing your reasoning, that O, N and M are collinear.

$\boxed{}$, $\boxed{\overrightarrow{AC} = \mathbf{c} - \mathbf{a}}$, $\boxed{\overrightarrow{AN} = \frac{1}{3}\mathbf{c} - \frac{1}{3}\mathbf{a}}$, $\boxed{\overrightarrow{ON} = \frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{c}}$, $\boxed{\overrightarrow{NM} = \frac{1}{3}\mathbf{a} + \frac{1}{6}\mathbf{c}}$



a) START BY DRAWING A DIAGRAM, AND LABEL VECTORS

- $\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC} = -\mathbf{a} + \mathbf{c} = \mathbf{c} - \mathbf{a}$
- $\overrightarrow{AN} = \frac{1}{3}\overrightarrow{AC} = \frac{1}{3}(\mathbf{c} - \mathbf{a}) = \frac{1}{3}\mathbf{c} - \frac{1}{3}\mathbf{a}$
- $\overrightarrow{ON} = \overrightarrow{OA} + \overrightarrow{AN} = \mathbf{a} + \frac{1}{3}\mathbf{c} - \frac{1}{3}\mathbf{a} = \frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{c}$
- $\overrightarrow{NM} = \overrightarrow{NA} + \overrightarrow{AM} = -\overrightarrow{AN} + \overrightarrow{AM} = -\left(\frac{1}{3}\mathbf{c} - \frac{1}{3}\mathbf{a}\right) + \frac{1}{2}\mathbf{c} = \frac{1}{3}\mathbf{a} + \frac{1}{6}\mathbf{c}$

b) ARGUE AS FOLLOWS

$\overrightarrow{ON} = \frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{c} = \frac{1}{3}(2\mathbf{a} + \mathbf{c})$

$\overrightarrow{NM} = \frac{1}{3}\mathbf{a} + \frac{1}{6}\mathbf{c} = \frac{1}{6}(2\mathbf{a} + \mathbf{c})$

AS $\overrightarrow{ON}$ & $\overrightarrow{NM}$ ARE IN THE SAME DIRECTION & SHARE THE POINT N, O, N & M MUST BE COLLINEAR

Question 23 (***)

The points with coordinates $A(1,4,3)$, $B(2,2,1)$ and $C(5,4,0)$ are given.

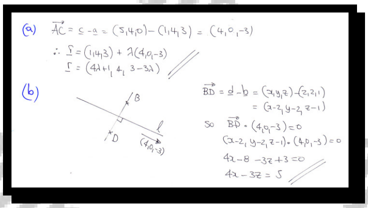
a) Find a vector equation of the straight line l, that passes through A and C.

The point $D(x,y,z)$ is such so that BD is perpendicular to l.

b) Show clearly that

$$4x - 3z = 5.$$

$$\boxed{\mathbf{r} = \mathbf{i} + 4\mathbf{j} + 3\mathbf{k} + \lambda(4\mathbf{i} - 3\mathbf{k})}$$



(a) $\overrightarrow{AC} = \underline{c} - \underline{a} = (5,4,0) - (1,4,3) = (4,0,-3)$

$\therefore \underline{r} = (1,4,3) + \lambda(4,0,-3)$

$\underline{r} = (4\lambda+1, 4, 3-3\lambda)$

(b) $\overrightarrow{BD} = \underline{d} - \underline{b} = (x,y,z) - (2,2,1)$

$= (x-2, y-2, z-1)$

so $\overrightarrow{BD} \cdot (4,0,-3) = 0$

$(x-2, y-2, z-1) \cdot (4,0,-3) = 0$

$4x - 8 - 3z + 3 = 0$

$4x - 3z = 5$

Question 24 (***)

The straight line L_1 passes through the points $A(3,0,3)$ and $B(5,5,2)$.

The straight line L_2 has a vector equation given by

$$\mathbf{r} = \begin{pmatrix} 5 \\ 5 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix},$$

where μ is a scalar parameter.

a) Write down the coordinates of the point of intersection of L_1 and L_2.

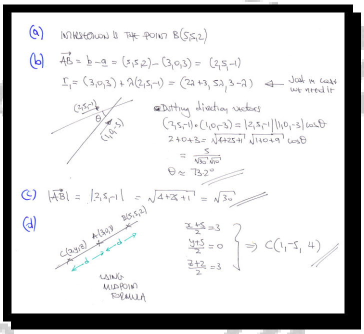
b) Find the size of the acute angle θ, between L_1 and L_2.

c) Calculate the distance AB.

The point C lies on L_1 so that the distance AB is equal to the distance AC.

d) Determine the coordinates of C.

$\boxed{P(5,5,2)}$, $\boxed{\theta \approx 73.2°}$, $\boxed{|AB| = \sqrt{30}}$, $\boxed{C(1,-5,4)}$


(a) INTERSECTION IS THE POINT B(5,5,2)

(b) $\vec{AB} = \underline{b} - \underline{a} = (5,5,2) - (3,0,3) = (2,5,-1)$

$\underline{r}_1 = (3,0,3) + \lambda(2,5,-1) = (2\lambda+3, 5\lambda, 3-\lambda)$ ← Just in case we need it

Dotting direction vectors

$(2,5,-1)\cdot(1,0,-3) = |2,5,-1||1,0,-3|\cos\theta$

$2+0+3 = \sqrt{4+25+1}\sqrt{1+0+9}\cos\theta$

$= \frac{5}{\sqrt{30}\sqrt{10}}$

$\theta \simeq 73.2°$

(c) $|\vec{AB}| = |2,5,-1| = \sqrt{4+25+1} = \sqrt{30}$

(d) USING MIDPOINT FORMULA

$\frac{x+5}{2} = 3$, $\frac{y+5}{2} = 0$, $\frac{z+2}{2} = 3$ $\Rightarrow C(1,-5,4)$

Question 25 (***)

A tunnel is to be dug through a mountain in order to link two cities.

Digging at one end of the tunnel begins at the point with coordinates $(-3,-3,9)$ and continues in the direction $2\mathbf{i}+2\mathbf{j}-\mathbf{k}$.

The digging at the other end of the tunnel starts at the point with coordinates $(-19,-7,-3)$ and continues in the direction $6\mathbf{i}+3\mathbf{j}+2\mathbf{k}$.

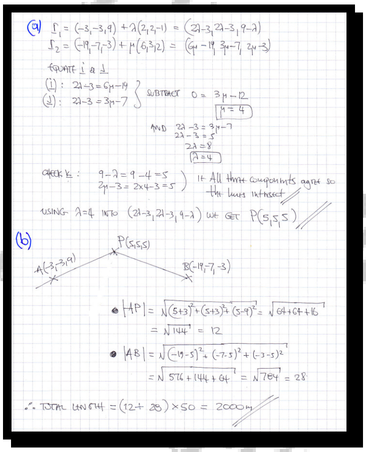
Both sections are assumed to be straight lines.

The coordinates are measured relative to a fixed origin O, where one unit is 50 metres.

a) Show that the two sections of the tunnel will eventually meet at a point P, and find the coordinates of this point.

b) Find the total length of the tunnel.

$\boxed{P(5,5,5)}$, $\boxed{\text{length} = 2000 \text{ metres}}$



(a) $\mathbf{r}_1 = (-3,-3,9) + \lambda(2,2,-1) = (2\lambda-3, 2\lambda-3, 9-\lambda)$

$\mathbf{r}_2 = (-19,-7,-3) + \mu(6,3,2) = (6\mu-19, 3\mu-7, 2\mu-3)$

EQUATE $\underline{i}$ & $\underline{j}$

$(\underline{i})$: $2\lambda-3 = 6\mu-19$
$(\underline{j})$: $2\lambda-3 = 3\mu-7$ } SUBTRACT $0 = 3\mu-12$ $\boxed{\mu=4}$

AND $2\lambda-3 = 3\mu-7$
$2\lambda-3 = 5$
$2\lambda = 8$
$\boxed{\lambda=4}$

CHECK $\underline{k}$: $9-\lambda = 9-4 = 5$, $2\mu-3 = 2\times4-3 = 5$) ∴ All three components agree so the lines intersect

USING $\lambda=4$ INTO $(2\lambda-3, 2\lambda-3, 9-\lambda)$ WE GET $P(5,5,5)$

(b)

P(5,5,5)

A(-3,-3,9)

B(-19,-7,-3)

• $|AP| = \sqrt{(5+3)^2+(5+3)^2+(5-9)^2} = \sqrt{64+64+16} = \sqrt{144} = 12$

• $|AB| = \sqrt{(-19-5)^2+(-7-5)^2+(-3-5)^2} = \sqrt{576+144+64} = \sqrt{784} = 28$

∴ TOTAL LENGTH $= (12+28)\times 50 = 2000$ m

Question 26 (***)

Relative to a fixed origin O, the straight lines l_1 and l_2 have respective vector equations given by

$$\mathbf{r}_1 = \begin{pmatrix} 7 \\ 2 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 8 \\ -1 \\ 2 \end{pmatrix} \quad \text{and} \quad \mathbf{r}_2 = \begin{pmatrix} 1 \\ p \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 9 \\ -2 \\ 5 \end{pmatrix},$$

where λ and μ are scalar parameters and p is a scalar constant.

The point T is the point of intersection between l_1 and l_2.

Find in any order …

a) … the size of acute angle between l_1 and l_2.

b) … the value of p.

c) … the coordinates of T.

$\boxed{\theta \approx 15.4°}$, $\boxed{p = 1}$, $\boxed{T(-17, 5, -9)}$

(a) $(8,-1,2)\cdot(9,-2,5) = |8,-1,2||9,-2,5|\cos\theta$

$72+2+10 = \sqrt{64+1+4}\sqrt{81+4+25}\cos\theta$

$84 = \sqrt{69}\sqrt{110}\cos\theta$

$\cos\theta = \frac{84}{\sqrt{69\times110}}$

$\theta \approx 15.4°$

(b) $\mathbf{r}_1 = (8\lambda+7, 2-\lambda, 2\lambda-3)$

$\mathbf{r}_2 = (9\mu+1, p-2\mu, 5\mu+1)$

EQUATE $\underline{i}$ & $\underline{k}$

$(\underline{i})$: $8\lambda+7 = 9\mu+1$

$(\underline{k})$: $2\lambda-3 = 5\mu+1$ $(\times -4)$

$8\lambda+7 = 9\mu+1$

$-8\lambda+12 = -20\mu-4$

$19 = -11\mu-3$

$11\mu = -22$

$\boxed{\mu = -2}$

$\therefore$ $2\lambda-3 = 5\mu+1$

$2\lambda-3 = -10+1$

$2\lambda = -6$

$\boxed{\lambda = -3}$

LOOKING AT $\underline{j}$ $\Rightarrow$ $2-\lambda = p-2\mu$

$2+3 = p+4$

$1 = p$

(c) USING $\lambda = -3$

$(8\lambda+7, 2-\lambda, 2\lambda-3)$ becomes $(-17, 5, -9)$

Question 27 (***)

With respect to a fixed origin O, the points $A(2,6,5)$ and $B(5,0,-4)$ are given.

a) Find a vector equation of the straight line L_1, which passes through A and B.

The straight line L_2 has a vector equation

$$\mathbf{r}_2 = \begin{pmatrix} -4 \\ 4 \\ -5 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

where μ is a scalar parameter.

b) Show clearly that L_1 and L_2 intersect, and find the coordinates of their point of intersection.

The straight line L_3 is in the direction $\begin{pmatrix} 1 \\ k \\ 1 \end{pmatrix}$.

c) Given the acute angle between L_2 and L_3 is $60°$, show clearly that $k = \pm\sqrt{6}$

$\boxed{\mathbf{r}_1 = 2\mathbf{i} + 6\mathbf{j} + 5\mathbf{k} + \lambda(-\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})}$, $\boxed{(3,4,2)}$

(a) $\overrightarrow{AB} = \underline{b} - \underline{a} = (5,0,-4) - (2,6,5) = (3,-6,-9)$ SCALE DIRECTION TO $(-1,2,3)$

$\underline{r}_1 = (2,6,5) + \lambda(-1,2,3) = (2-\lambda, 2\lambda+6, 3\lambda+5)$

(b) $\underline{r}_2 = (-4,4,-5) + \mu(1,0,1) = (\mu-4, 4, \mu-5)$

• EQUATE $\underline{i}$ & $\underline{j}$

$\left.\begin{array}{l} 2-\lambda = \mu-4 \\ 2\lambda+6 = 4 \end{array}\right\} \Rightarrow \boxed{\lambda=-1}, \boxed{\mu=7}$

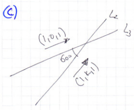
• CHECK $\underline{k}$: $3\lambda+5 = 3(-1)+5 = 2$

$\mu-5 = 7-5 = 2$

All 3 components agree so lines intersect

USING $\mu=7$ INTO $(\mu-4, 4, \mu-5) = (3,4,2)$

(c)



BY DOT PRODUCT

$(1,0,1)\cdot(1,k,1) = |1,0,1||1,k,1|\cos 60°$

$1+0+1 = \sqrt{1+0+1}\sqrt{1+k^2+1} \times \frac{1}{2}$

$2 = \sqrt{2}\sqrt{k^2+2} \times \frac{1}{2}$

$\frac{4}{\sqrt{2}} = \sqrt{k^2+2}$

$8 = k^2+2$

$6 = k^2$

$k = \pm\sqrt{6}$

AS REQUIRED

Question 28 (***)

The straight lines l_1 and l_2 have the following vector equations

$$\mathbf{r}_1 = 2\mathbf{i} + 3\mathbf{j} + \lambda(2\mathbf{i} + \mathbf{j} + 4\mathbf{k})$$

$$\mathbf{r}_2 = 5\mathbf{i} + 3\mathbf{j} + 9\mathbf{k} + \mu(\mathbf{i} + 2\mathbf{j} - \mathbf{k}),$$

where λ and μ are scalar parameters.

a) Show that l_1 and l_2 intersect at some point A, and find its coordinates.

b) Show further that l_1 and l_2 intersect at right angles.

The point B lies on l_1 where $\lambda = -1$ and the point C lies on l_2 where $\mu = 3$.

c) Find the exact area of the triangle BAC.

$\boxed{A(6,5,8)}$, $\boxed{\text{area} = 9\sqrt{14}}$

(a) $\mathbf{r}_1 = (2,3,0) + \lambda(2,1,4) = (2\lambda+2, \lambda+3, 4\lambda)$
$\mathbf{r}_2 = (5,3,9) + \mu(1,2,-1) = (\mu+5, 2\mu+3, 9-\mu)$

EQUATE $\underline{i}$ & $\underline{k}$

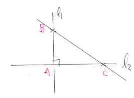
$(\underline{i})$: $2\lambda+2 = \mu+5$
$(\underline{k})$: $4\lambda = 9-\mu$ } Add: $6\lambda+2 = 14$, $6\lambda = 12$, $\boxed{\lambda = 2}$ & $4\lambda = 9-\mu$, $8 = 9-\mu$, $\boxed{\mu = 1}$

CHECK $\underline{j}$: $\lambda+3 = 2+3 = 5$
$2\mu+3 = 2\times1+3 = 5$ All 3 components agree so lines intersect

USING $\mu=1$ INTO $(\mu+5, 2\mu+3, 9-\mu)$ GIVES $A(6,5,8)$

(b) DOTTING DIRECTION VECTORS $(2,1,4)\cdot(1,2,-1)$
$= 2+2-4$
$= 0$
$\therefore$ PERPENDICULAR

(c) If $\lambda=-1$ $B(0,2,-4)$
If $\mu=3$ $C(8,9,6)$



• $|AB| = |\underline{b}-\underline{a}| = |(0,2,-4)-(6,5,8)|$
$= |-6,-3,-12|$
$= \sqrt{36+9+144} = \sqrt{189}$

• $|AC| = |\underline{c}-\underline{a}| = |(8,9,6)-(6,5,8)| = |2,4,-2| = \sqrt{4+16+4} = \sqrt{24}$

$\therefore$ AREA $= \frac{1}{2}\times$ BASE $\times$ HEIGHT $= \frac{1}{2}\times\sqrt{24}\times\sqrt{189} = \frac{1}{2}\times 2\sqrt{6}\times 3\sqrt{21}$
$= 3\sqrt{6}\sqrt{21} = 3\sqrt{2}\sqrt{3}\sqrt{3}\sqrt{7} = 9\sqrt{14}$

Question 29 (***)

The straight lines l_1 and l_2 have the following vector equations

$$\mathbf{r}_1 = 4\mathbf{i} + 7\mathbf{j} + 4\mathbf{k} + \lambda(\mathbf{i} - \mathbf{j})$$

$$\mathbf{r}_2 = 8\mathbf{i} + 5\mathbf{j} + 2\mathbf{k} + \mu(\mathbf{i} - \mathbf{k}),$$

where λ and μ are scalar parameters.

a) Show that l_1 and l_2 intersect at some point A and find its coordinates.

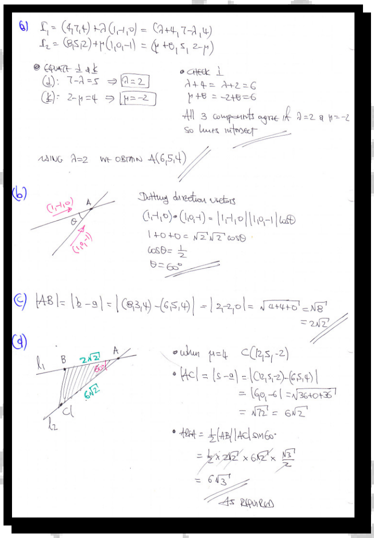
b) Calculate the acute angle between l_1 and l_2.

The point $B(8,3,4)$ lies on l_1 and the point C lies on l_2 where $\mu = 4$.

c) Find the distance AB.

d) Show that the area of the triangle ABC is $6\sqrt{3}$ square units.

$\boxed{A(6,5,4)}$, $\boxed{60°}$, $\boxed{|AB| = 2\sqrt{2}}$


(a) $\underline{r}_1 = (4,7,4) + \lambda(1,-1,0) = (\lambda+4, 7-\lambda, 4)$
$\underline{r}_2 = (8,5,2) + \mu(1,0,-1) = (\mu+8, 5, 2-\mu)$

• EQUATE $\underline{j}$ & $\underline{k}$
($\underline{j}$): $7-\lambda = 5 \Rightarrow \boxed{\lambda = 2}$
($\underline{k}$): $2-\mu = 4 \Rightarrow \boxed{\mu = -2}$

• CHECK $\underline{i}$
$\lambda + 4 = 2 + 2 = 6$
$\mu + 8 = -2 + 8 = 6$
All 3 components agree if $\lambda = 2$ & $\mu = -2$
so lines intersect

USING $\lambda = 2$ WE OBTAIN $A(6,5,4)$

(b) Dotting direction vectors
$(1,-1,0)\cdot(1,0,-1) = |1,-1,0||1,0,-1|\cos\theta$
$1+0+0 = \sqrt{2}\sqrt{2}\cos\theta$
$\cos\theta = \frac{1}{2}$
$\theta = 60°$

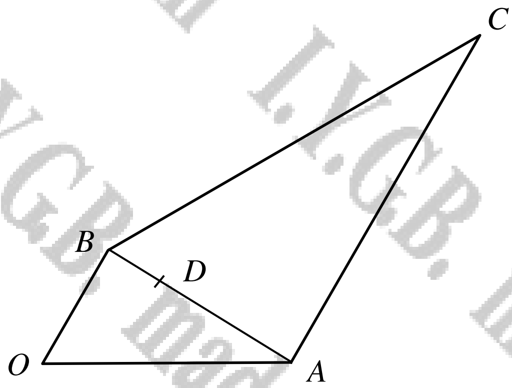
(c) $|AB| = |\underline{b} - \underline{a}| = |(8,3,4) - (6,5,4)| = |2,-2,0| = \sqrt{4+4+0} = \sqrt{8}$
$= 2\sqrt{2}$

(d) • when $\mu = 4$ $C(12,5,-2)$
• $|AC| = |\underline{c} - \underline{a}| = |(12,5,-2) - (6,5,4)|$
$= |6,0,-6| = \sqrt{36+0+36}$
$= \sqrt{72} = 6\sqrt{2}$

• AREA $= \frac{1}{2}|AB||AC|\sin 60°$
$= \frac{1}{2} \times 2\sqrt{2} \times 6\sqrt{2} \times \frac{\sqrt{3}}{2}$
$= 6\sqrt{3}$
AS REQUIRED

Question 30 (***)

The figure below shows a trapezium $OBCA$ where OB is parallel to AC.



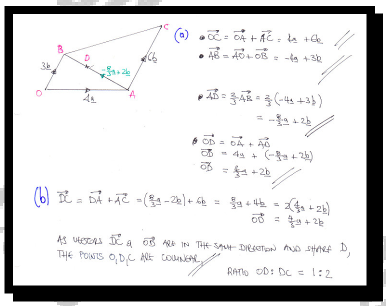
The point D lies on BA so that $BD : DA = 1:2$.

Let $\overrightarrow{OA} = 4\mathbf{a}$, $\overrightarrow{OB} = 3\mathbf{b}$ and $\overrightarrow{AC} = 6\mathbf{b}$.

a) Find simplified expressions, in terms of $\mathbf{a}$ and $\mathbf{b}$, for each of the vectors $\overrightarrow{OC}$, $\overrightarrow{AB}$, $\overrightarrow{AD}$ and $\overrightarrow{OD}$.

b) Deduce, showing your reasoning, that O, D and C are collinear and state the ratio of $OD : DC$.

$\boxed{\overrightarrow{OC} = 4\mathbf{a} + 6\mathbf{b}}$, $\boxed{\overrightarrow{AB} = -4\mathbf{a} + 3\mathbf{b}}$, $\boxed{\overrightarrow{AD} = -\frac{8}{3}\mathbf{a} + 2\mathbf{b}}$, $\boxed{\overrightarrow{OD} = \frac{4}{3}\mathbf{a} + 2\mathbf{b}}$, $\boxed{OD : DC = 1:2}$



Question 31 (***)

The points $A(5,1,3)$, $B(3,1,5)$, $C(5,3,5)$ and $D(4,0,3)$ are given.

a) Show that the triangle ABC is equilateral and find its area.

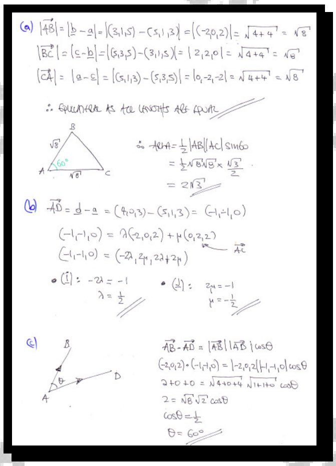
b) Show further that

$$\overrightarrow{AD} = \lambda\overrightarrow{AB} + \mu\overrightarrow{AC},$$

stating the exact values of the scalar constants λ and μ.

c) Find the size of the angle BAD.

$\boxed{\text{area} = 2\sqrt{3}}$, $\boxed{\lambda = \frac{1}{2}, \mu = -\frac{1}{2}}$, $\boxed{\measuredangle BAD = 60°}$



(a) $|\overrightarrow{AB}| = |\underline{b} - \underline{a}| = |(3,1,5) - (5,1,3)| = |(-2,0,2)| = \sqrt{4+4} = \sqrt{8}$

$|\overrightarrow{BC}| = |\underline{c} - \underline{b}| = |(5,3,5) - (3,1,5)| = |2,2,0| = \sqrt{4+4} = \sqrt{8}$

$|\overrightarrow{CA}| = |\underline{a} - \underline{c}| = |(5,1,3) - (5,3,5)| = |0,-2,-2| = \sqrt{4+4} = \sqrt{8}$

$\therefore$ EQUILATERAL AS ALL LENGTHS ARE EQUAL

$\therefore$ AREA $= \frac{1}{2}|AB||AC|\sin 60$

$= \frac{1}{2}\sqrt{8}\sqrt{8} \times \frac{\sqrt{3}}{2}$

$= 2\sqrt{3}$

(b) $\overrightarrow{AD} = \underline{d} - \underline{a} = (4,0,3) - (5,1,3) = (-1,-1,0)$

$(-1,-1,0) = \lambda(-2,0,2) + \mu(0,2,2) \leftarrow \overrightarrow{AC}$

$(-1,-1,0) = (-2\lambda, 2\mu, 2\lambda + 2\mu)$

• (i): $-2\lambda = -1$, $\lambda = \frac{1}{2}$

• (j): $2\mu = -1$, $\mu = -\frac{1}{2}$

(c)

$\overrightarrow{AB} \cdot \overrightarrow{AD} = |\overrightarrow{AB}||\overrightarrow{AB}|\cos\theta$

$(-2,0,2) \cdot (-1,-1,0) = |-2,0,2||-1,-1,0|\cos\theta$

$2 + 0 + 0 = \sqrt{4+0+4}\sqrt{1+1+0}\cos\theta$

$2 = \sqrt{8}\sqrt{2}\cos\theta$

$\cos\theta = \frac{1}{2}$

$\theta = 60°$

Question 32 (***)

With respect to a fixed origin O, the straight lines l_1 and l_2 have respective vector equations given by

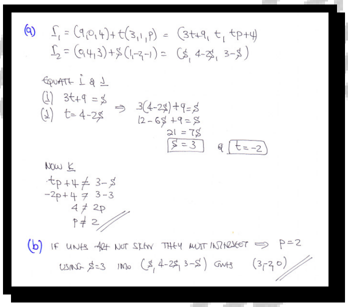
$$\mathbf{r}_1 = \begin{pmatrix} 9 \\ 0 \\ 4 \end{pmatrix} + t\begin{pmatrix} 3 \\ 1 \\ p \end{pmatrix} \quad \text{and} \quad \mathbf{r}_2 = \begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix} + s\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix},$$

where t and s are scalar parameters.

a) If l_1 and l_2 are skew find the value that p **cannot** take.

b) If l_1 and l_2 are not skew find the coordinates of their point of intersection.

$\boxed{p \neq 2}$, $\boxed{(3,-2,0)}$


(a) $\mathbf{r}_1 = (9,0,4) + t(3,1,p) = (3t+9, t, tp+4)$
$\mathbf{r}_2 = (0,4,3) + s(1,-2,-1) = (s, 4-2s, 3-s)$

EQUATE i & j

(i) $3t+9 = s$
(j) $t = 4-2s$ $\Rightarrow$ $3(4-2s)+9 = s$
$12-6s+9 = s$
$21 = 7s$
$\boxed{s=3}$ & $\boxed{t=-2}$

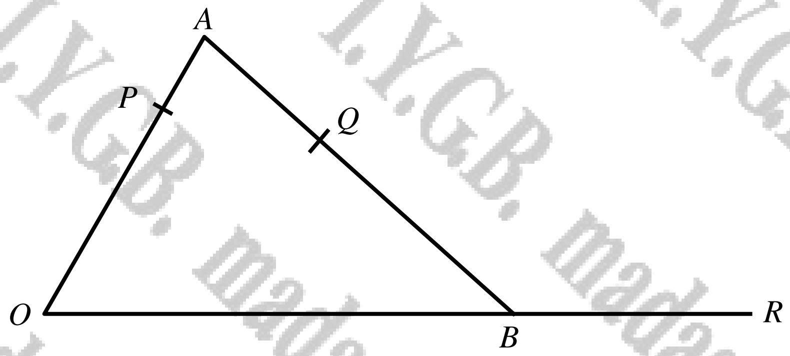
NOW k

$tp+4 \neq 3-s$
$-2p+4 \neq 3-3$
$4 \neq 2p$
$p \neq 2$

(b) IF LINES ARE NOT SKEW THEY MUST INTERSECT $\Rightarrow$ $p=2$
USING $s=3$ INTO $(s, 4-2s, 3-s)$ GIVES $(3,-2,0)$

Question 33 (***)

The figure below shows a triangle OAB.



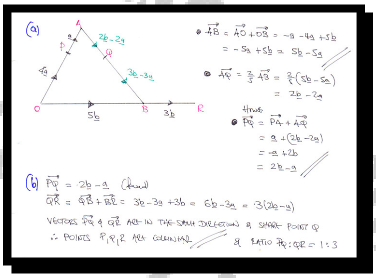
- The point P lies on OA so that $OP : PA = 4 : 1$.
- The point Q lies on AB so that $AQ : QB = 2 : 3$
- The side OB is extended to the point R so that $OB : BR = 5 : 3$.

Let $\overrightarrow{PA} = \mathbf{a}$ and $\overrightarrow{OB} = 5\mathbf{b}$.

a) Find simplified expressions, in terms of $\mathbf{a}$ and $\mathbf{b}$, for each of the vectors $\overrightarrow{AB}$, $\overrightarrow{AQ}$ and $\overrightarrow{PQ}$.

b) Deduce, showing your reasoning, that P, Q and R are collinear and state the ratio of $PQ : QR$.

$\boxed{\overrightarrow{AB} = 5\mathbf{b} - 5\mathbf{a}}$, $\boxed{\overrightarrow{AQ} = 2\mathbf{b} - 2\mathbf{a}}$, $\boxed{\overrightarrow{PQ} = 2\mathbf{b} - \mathbf{a}}$, $\boxed{PQ : QR = 1 : 3}$



Question 34 (***)

The straight lines l_1 and l_2 have respective vector equations

$$\mathbf{r}_1 = \begin{pmatrix} 6 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 4 \\ a \end{pmatrix} \quad \text{and} \quad \mathbf{r}_2 = \begin{pmatrix} 5 \\ 8 \\ b \end{pmatrix} + \mu \begin{pmatrix} 7 \\ 1 \\ 0 \end{pmatrix},$$

where λ and μ are scalar parameters while a and b are positive constants.

Given that l_1 and l_2 intersect at some point P, forming an angle of $60°$, determine in any order …

a) … the value of a.

b) … the value of b.

c) … the coordinates of P.

$\boxed{a = 5}$, $\boxed{b = 10}$, $\boxed{P(12,9,10)}$

(a) $\mathbf{r}_1 = (6,1,0) + \lambda(3,4,a) = (3\lambda+6, 4\lambda+1, a\lambda)$

$\mathbf{r}_2 = (5,8,b) + \mu(7,1,0) = (7\mu+5, \mu+8, b)$

ANGLES MEET AT $60°$ $\quad (3,4,a)\cdot(7,1,0) = \sqrt{9+16+a^2}\sqrt{49+1}\cos 60$

$21 + 4 + 0 = \sqrt{a^2+25}\sqrt{50} \times \frac{1}{2}$

$50 = \sqrt{a^2+25}\sqrt{50}$

$\sqrt{50} = \sqrt{a^2+25}$

$50 = a^2 + 25$

$a^2 = 25$

$a = 5 \quad (a>0)$

(c) LINES INTERSECT (GIVEN)

$\underline{i}: 3\lambda+6 = 7\mu+5$

$\underline{j}: 4\lambda+1 = \mu+8$

$\Rightarrow \boxed{\mu = 4\lambda-7} \Rightarrow 3\lambda+6 = 7(4\lambda-7)+5$

$3\lambda+6 = 28\lambda-49+5$

$50 = 25\lambda$

$\boxed{\lambda = 2}$

$\boxed{\mu = 1}$

$\therefore$ INTERSECTION POINT IS $(3\lambda+6, 4\lambda+1, a\lambda) = (12,9,10)$

(b) $\therefore b = 10$

Question 35 (***)

The points with coordinates $A(8,0,12)$ and $B(9,-2,14)$ are given.

a) Find the vector equation of the straight line l_1 that passes through A and B.

The straight line l_2 has equation

$$\mathbf{r} = \mathbf{i} + 9\mathbf{j} + 2\mathbf{k} + \mu(2\mathbf{i} + \mathbf{j}),$$

where μ is a scalar parameter.

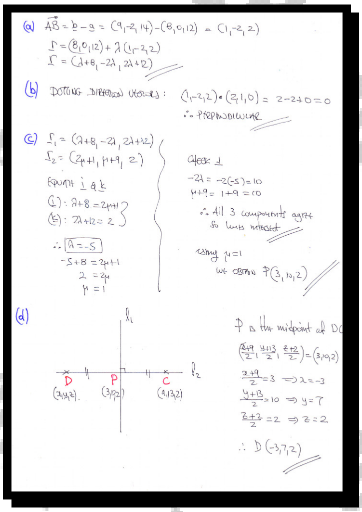
b) Show that l_1 and l_2 are perpendicular.

c) Show further that l_1 and l_2 intersect at some point P and state the coordinates of P.

The point $C(9,13,2)$ lies on l_2 and the point D is the reflection of C about l_1.

d) Determine the coordinates of D.

$\boxed{\mathbf{r}_1 = 8\mathbf{i} + 12\mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})}$, $\boxed{P(3,10,2)}$, $\boxed{D(-3,7,2)}$



Question 36 (***)

Relative to a fixed origin O, the point A has position vector $7\mathbf{i}+4\mathbf{j}$ and the point B has position vector $-3\mathbf{j}+7\mathbf{k}$. The straight line L_1 passes through the points A and B.

a) Find a vector equation for L_1.

The straight line L_2 has a vector equation

$$\mathbf{r}=3\mathbf{i}-2\mathbf{j}-4\mathbf{k}+\mu(\mathbf{i}+2\mathbf{j}+3\mathbf{k}),$$

where μ is a scalar parameter.

b) Show that L_1 and L_2 intersect at some point C, and find its position vector.

c) Show further that L_1 and L_2 are perpendicular.

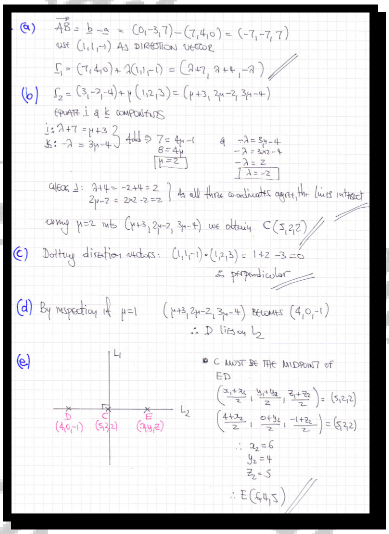
The point D has position vector $4\mathbf{i}-\mathbf{k}$.

d) Verify that D lies on L_2.

The point E is the image of D after reflection about L_1.

e) Find the position vector of E.

$\boxed{}$, $\boxed{\mathbf{r}=7\mathbf{i}+4\mathbf{j}+\lambda(\mathbf{i}+\mathbf{j}-\mathbf{k})}$, $\boxed{\overrightarrow{OC}=5\mathbf{i}+2\mathbf{j}+2\mathbf{k}}$, $\boxed{\overrightarrow{OE}=6\mathbf{i}+4\mathbf{j}+5\mathbf{k}}$



(a) $\overrightarrow{AB}=\underline{b}-\underline{a}=(0,-3,7)-(7,4,0)=(-7,-7,7)$

USE $(1,1,-1)$ AS DIRECTION VECTOR

$\underline{r}_1=(7,4,0)+\lambda(1,1,-1)=(\lambda+7,\lambda+4,-\lambda)$

(b) $\underline{r}_2=(3,-2,-4)+\mu(1,2,3)=(\mu+3,2\mu-2,3\mu-4)$

EQUATE $\underline{i}$ & $\underline{k}$ COMPONENTS

$\underline{i}: \lambda+7=\mu+3$
$\underline{k}: -\lambda=3\mu-4$ } Add $\Rightarrow 7=4\mu-1$, $8=4\mu$, $\boxed{\mu=2}$ & $-\lambda=3\mu-4$, $-\lambda=3\times2-4$, $-\lambda=2$, $\boxed{\lambda=-2}$

CHECK $\underline{j}$: $\lambda+4=-2+4=2$, $2\mu-2=2\times2-2=2$ } As all three co-ordinates agree, the lines intersect

using $\mu=2$ into $(\mu+3,2\mu-2,3\mu-4)$ we obtain $C(5,2,2)$

(c) Dotting direction vectors: $(1,1,-1)\cdot(1,2,3)=1+2-3=0$

$\therefore$ perpendicular

(d) By inspection if $\mu=1$ $(\mu+3,2\mu-2,3\mu-4)$ BECOMES $(4,0,-1)$

$\therefore D$ lies on L_2

(e) L_1, L_2, D $(4,0,-1)$, C $(5,2,2)$, E (x,y,z)

- C MUST BE THE MIDPOINT OF ED

$\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2},\frac{z_1+z_2}{2}\right)=(5,2,2)$

$\left(\frac{4+x_2}{2},\frac{0+y_2}{2},\frac{-1+z_2}{2}\right)=(5,2,2)$

$\therefore x_2=6$, $y_2=4$, $z_2=5$

$\therefore E(6,4,5)$

Question 37 (***+)

Relative to a fixed origin O, the points A and C have respective coordinates

$$(7,2,3) \quad \text{and} \quad (3,-2,1).$$

a) Find the vector $\overrightarrow{AC}$.

b) State the coordinates of the midpoint of AC.

The straight line l has vector equation

$$\mathbf{r} = \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -4 \end{pmatrix},$$

where λ is a scalar parameter.

c) Show that $\overrightarrow{AC}$ is perpendicular to l.

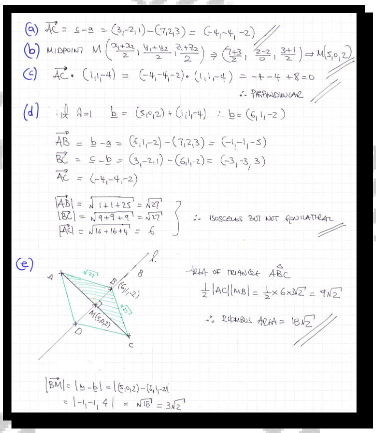
The point B lies on l, where $\lambda = 1$.

d) Show further that the triangle ABC is isosceles but not equilateral.

The point D is such, so that $ABCD$ is a rhombus.

e) Show that the area of this rhombus is $18\sqrt{2}$ square units.

$\boxed{}$, $\boxed{\overrightarrow{AC} = -4\mathbf{i} - 4\mathbf{j} - 2\mathbf{k}}$, $\boxed{(5,0,2)}$



(a) $\overrightarrow{AC} = \underline{c} - \underline{a} = (3,-2,1) - (7,2,3) = (-4,-4,-2)$

(b) MIDPOINT $M\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right) \Rightarrow \left(\frac{7+3}{2}, \frac{2-2}{0}, \frac{3+1}{2}\right) \Rightarrow M(5,0,2)$

(c) $\overrightarrow{AC} \cdot (1,1,-4) = (-4,-4,-2) \cdot (1,1,-4) = -4-4+8=0$

$\therefore$ PERPENDICULAR

(d) if $\lambda=1$ $\underline{b} = (5,0,2) + (1,1,-4)$ $\therefore \underline{b} = (6,1,-2)$

$\overrightarrow{AB} = \underline{b} - \underline{a} = (6,1,-2) - (7,2,3) = (-1,-1,-5)$

$\overrightarrow{BC} = \underline{c} - \underline{b} = (3,-2,1) - (6,1,-2) = (-3,-3,3)$

$\overrightarrow{AC} = (-4,-4,-2)$

$\left.\begin{array}{l} |\overrightarrow{AB}| = \sqrt{1+1+25} = \sqrt{27} \\ |\overrightarrow{BC}| = \sqrt{9+9+9} = \sqrt{27} \\ |\overrightarrow{AC}| = \sqrt{16+16+4} = 6 \end{array}\right\}$ $\therefore$ ISOSCELES BUT NOT EQUILATERAL

(e)

AREA OF TRIANGLE $\hat{ABC}$

$\frac{1}{2}|AC||MB| = \frac{1}{2} \times 6 \times 3\sqrt{2} = 9\sqrt{2}$

$\therefore$ RHOMBUS AREA $= 18\sqrt{2}$

$|\overrightarrow{BM}| = |\underline{m} - \underline{b}| = |(5,0,2) - (6,1,-2)|$

$= |-1,-1,4| = \sqrt{18} = 3\sqrt{2}$

Question 38 **(***+)**

Relative to a fixed origin O, the points A and B have respective position vectors

$$3\mathbf{i}-\mathbf{j}+2\mathbf{k} \quad \text{and} \quad -\mathbf{i}+\mathbf{j}+9\mathbf{k}.$$

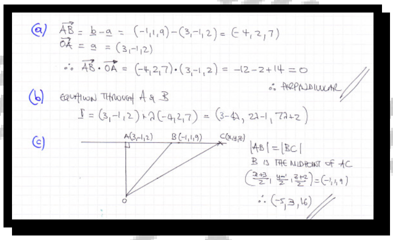
a) Show that $\overrightarrow{OA}$ and $\overrightarrow{AB}$ are perpendicular.

b) Find a vector equation of the straight line l, that passes through A and B.

The point C lies on l, so that the areas of the triangles OAB and OBC are equal.

c) Determine the position vector of C.

$\boxed{\quad}$, $\boxed{\mathbf{r}=3\mathbf{i}-\mathbf{j}+2\mathbf{k}+\lambda(-4\mathbf{i}+2\mathbf{j}+7\mathbf{k})}$, $\boxed{\overrightarrow{OC}=-5\mathbf{i}+3\mathbf{j}+16\mathbf{k}}$


(a) $\overrightarrow{AB} = \underline{b}-\underline{a} = (-1,1,9)-(3,-1,2) = (-4,2,7)$
$\overrightarrow{OA} = \underline{a} = (3,-1,2)$
$\therefore \overrightarrow{AB}\cdot\overrightarrow{OA} = (-4,2,7)\cdot(3,-1,2) = -12-2+14 = 0$
$\therefore$ PERPENDICULAR

(b) EQUATION THROUGH A & B
$\underline{r} = (3,-1,2)+\lambda(-4,2,7) = (3-4\lambda, 2\lambda-1, 7\lambda+2)$

(c)

$|AB| = |BC|$
B IS THE MIDPOINT OF AC
$\left(\frac{x+3}{2}, \frac{y-1}{2}, \frac{z+2}{2}\right) = (-1,1,9)$
$\therefore (-5,3,16)$

Question 39 (***+)

Relative to a fixed origin O, the points A, B and C have respective position vectors

$$\mathbf{i}+10\mathbf{k}\,, \quad 4\mathbf{i}+3\mathbf{j}+7\mathbf{k} \quad \text{and} \quad 8\mathbf{i}+7\mathbf{j}+3\mathbf{k}\,.$$

a) Show that A, B and C are collinear, and find the ratio $AB:BC$.

b) Find a vector equation for the straight line l that passes through A, B and C.

c) Show that OB is perpendicular to l.

d) Calculate the area of the triangle OAC.

$$\boxed{AB:BC=3:4}\,, \quad \boxed{\mathbf{r}=\mathbf{i}+10\mathbf{k}+\lambda(\mathbf{i}+\mathbf{j}-\mathbf{k})}\,, \quad \boxed{\text{area}=\frac{7}{2}\sqrt{222}\approx 52.15}$$

(a). $\overrightarrow{AB} = \underline{b}-\underline{a} = (4,3,7)-(1,0,10) = (3,3,-3) = 3(1,1,-1)$

$\overrightarrow{BC} = \underline{c}-\underline{b} = (8,7,3)-(4,3,7) = (4,4,-4) = 4(1,1,-1)$

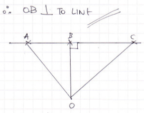
$\overrightarrow{AB}$ IS IN THE SAME DIRECTION AS $\overrightarrow{BC}$, SO A,B,C COLLINEAR AND RATIO IS $3:4$

(b) $\underline{r} = (1,0,10)+\lambda(1,1,-1) = (\lambda+1,\lambda,10-\lambda)$

(c) $\overrightarrow{OB}\cdot(\text{DIRECTION OF LINE}) = (4,3,7)\cdot(1,1,-1) = 4+3-7=0$

$\therefore$ $OB \perp$ TO LINE

(d)



$|\overrightarrow{OB}| = |4,3,7| = \sqrt{16+9+49} = \sqrt{74}$

$|\overrightarrow{AC}| = |\underline{c}-\underline{a}| = |(8,7,3)-(1,0,10)| = |7,7,-7| = \sqrt{49+49+49} = 7\sqrt{3}$

$\therefore$ AREA $= \frac{1}{2}|OB||AC| = \frac{1}{2}\times\sqrt{74}\times 7\sqrt{3} = \frac{7}{2}\sqrt{222}$

Question 40 (***+)

The points A and B have coordinates $(11,15,4)$ and $(13,23,7)$, respectively.

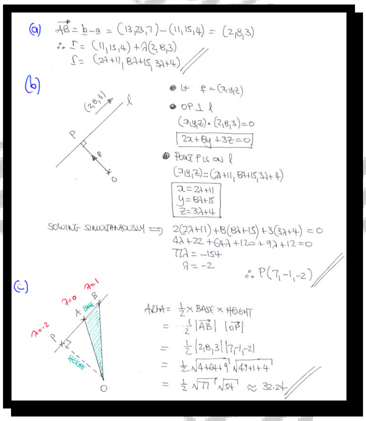
a) Find a vector equation for the straight line l that passes through A and B.

The point P lies on l, so that OP is perpendicular to l, where O is the origin.

b) Show, without verification, that the coordinates of P are $(7,-1,-2)$.

c) Calculate the area of the triangle OAB.

$\boxed{}$, $\boxed{\mathbf{r} = 11\mathbf{i} + 15\mathbf{j} + 4\mathbf{k} + \lambda(2\mathbf{i} + 8\mathbf{j} + 3\mathbf{k})}$, $\boxed{\text{area} = \frac{3}{2}\sqrt{462} \approx 32.24}$



Question 41 (***+)

Relative to a fixed origin O, the position vectors of the points A, B, C and D are

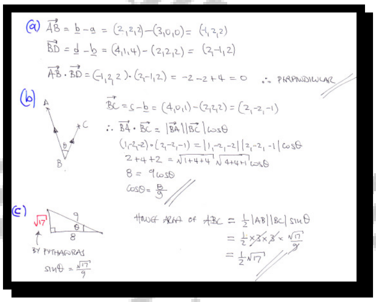
$$3\mathbf{i}, \quad 2\mathbf{i}+2\mathbf{j}+2\mathbf{k}, \quad 4\mathbf{i}+\mathbf{k} \quad \text{and} \quad 4\mathbf{i}+\mathbf{j}+4\mathbf{k}, \text{ respectively.}$$

a) Show that $\overrightarrow{AB}$ and $\overrightarrow{BD}$ are perpendicular.

b) Find the exact value of the cosine of the angle ABC.

c) Determine the exact value of the area of triangle ABC.

$\boxed{\cos(\measuredangle ABC) = \frac{8}{9}}$, $\boxed{\text{area} = \frac{1}{2}\sqrt{17}}$



(a) $\overrightarrow{AB} = \underline{b} - \underline{a} = (2,2,2) - (3,0,0) = (-1,2,2)$

$\overrightarrow{BD} = \underline{d} - \underline{b} = (4,1,4) - (2,2,2) = (2,-1,2)$

$\overrightarrow{AB} \cdot \overrightarrow{BD} = (-1,2,2) \cdot (2,-1,2) = -2 - 2 + 4 = 0$ $\therefore$ PERPENDICULAR

(b) $\overrightarrow{BC} = \underline{c} - \underline{b} = (4,0,1) - (2,2,2) = (2,-2,-1)$

$\therefore \overrightarrow{BA} \cdot \overrightarrow{BC} = |\overrightarrow{BA}||\overrightarrow{BC}|\cos\theta$

$(1,-2,-2) \cdot (2,-2,-1) = |1,-2,-2||2,-2,-1|\cos\theta$

$2 + 4 + 2 = \sqrt{1+4+4}\sqrt{4+4+1}\cos\theta$

$8 = 9\cos\theta$

$\cos\theta = \frac{8}{9}$

(c) BY PYTHAGORAS $\sin\theta = \frac{\sqrt{17}}{9}$

HENCE AREA OF $ABC = \frac{1}{2}|AB||BC|\sin\theta$

$= \frac{1}{2} \times 3 \times 3 \times \frac{\sqrt{17}}{9}$

$= \frac{1}{2}\sqrt{17}$

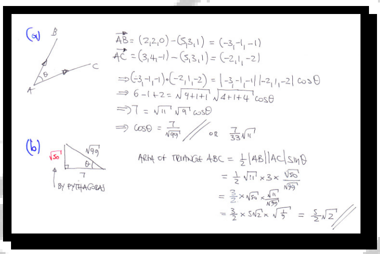
Question 42 (***+)

Relative to a fixed origin O, the points A, B and C have respective coordinates $(5,3,1)$, $(2,2,0)$ and $(3,4,-1)$.

a) Find the exact value of the cosine of the angle BAC.

b) Show that the exact area of the triangle ABC is $\frac{5}{2}\sqrt{2}$.

$\boxed{\cos(\measuredangle BAC) = \frac{7}{33}\sqrt{11}}$



(a) $\overrightarrow{AB} = (2,2,0) - (5,3,1) = (-3,-1,-1)$

$\overrightarrow{AC} = (3,4,-1) - (5,3,1) = (-2,1,-2)$

$\Rightarrow (-3,-1,-1) \cdot (-2,1,-2) = |-3,-1,-1||-2,1,-2|\cos\theta$

$\Rightarrow 6 - 1 + 2 = \sqrt{9+1+1}\sqrt{4+1+4}\cos\theta$

$\Rightarrow 7 = \sqrt{11}\sqrt{9}\cos\theta$

$\Rightarrow \cos\theta = \frac{7}{\sqrt{99}}$ OR $\frac{7}{33}\sqrt{11}$

(b) BY PYTHAGORAS

AREA OF TRIANGLE $ABC = \frac{1}{2}|AB||AC|\sin\theta$

$= \frac{1}{2}\sqrt{11} \times 3 \times \frac{\sqrt{50}}{\sqrt{99}}$

$= \frac{3}{2} \times \sqrt{50} \times \frac{\sqrt{11}}{\sqrt{99}}$

$= \frac{3}{2} \times 5\sqrt{2} \times \sqrt{\frac{1}{9}} = \frac{5}{2}\sqrt{2}$

Question 43 (***+)

The straight line l passes through the points $P(-5,9,-9)$ and $Q(a,b,11)$, where a and b are scalar constants.

The vector equation of l is given by

$$\mathbf{r} = \begin{pmatrix} 1 \\ 7 \\ c \end{pmatrix} + \lambda \begin{pmatrix} d \\ -1 \\ 2 \end{pmatrix},$$

where c and d are scalar constants and λ is a scalar parameter.

a) Determine in any order the value of each the constants a, b, c and d.

The point T with x coordinate 4 lies on l.

b) Show clearly that …

i. … OT is perpendicular to l, where O is the origin.

ii. … $PT : TQ = 3:7$.

$\boxed{a=25}$, $\boxed{b=-1}$, $\boxed{c=-5}$, $\boxed{d=3}$

(a) $\underline{r} = (1,7,c) + \lambda(d,-1,2) = (1+\lambda d, 7-\lambda, c+2\lambda)$

• NOW $P(-5,9,-9)$ LIES ON THE LINE l

$\underline{j}$: $7-\lambda = 9$, $-2=\lambda$, $\boxed{\lambda=-2}$

$\underline{i}$: $1+\lambda d = -5$, $1-2d=-5$, $6=2d$, $d=3$

$\underline{k}$: $c+2\lambda=-9$, $c-4=-9$, $c=-5$

AND $Q(a,b,11)$ ALSO LIES ON THE LINE $\underline{r} = (1+3\lambda, 7-\lambda, 2\lambda-5)$

$\underline{k}$: $2\lambda-5=11$, $2\lambda=16$, $\boxed{\lambda=8}$

$\underline{i}$: $1+3\lambda=a$, $1+3\times8=a$, $a=25$

$\underline{j}$: $7-\lambda=b$, $b=7-8$, $b=-1$

(b)(I) $T(4,y,z) = (1+3\lambda, 7-\lambda, 2\lambda-5)$

$\underline{i}$: $1+3\lambda=4$, $3\lambda=3$, $\boxed{\lambda=1}$

$\underline{j}$: $7-\lambda$, $7-1=6$

$\underline{k}$: $2\lambda-5$, $2-5$, -3

IE $T(4,6,-3)$

$\vec{OT} = (4,6,-3)$ DIRECTION OF LINE IS $(d,-1,-2) = (3,-1,-2)$

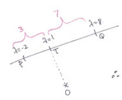
HENCE $(4,6,-3)\cdot(3,-1,2) = 12-6-6=0$

$\therefore$ PERPENDICULAR INDEED

(II) AT P $\lambda=-2$

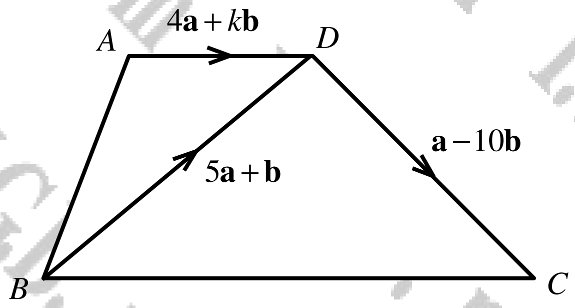
AT Q $\lambda=8$

AT T $\lambda=1$



$\therefore PT:TQ = 3:7$

Question 44 (*+)**



The figure above shows a trapezium $ABCD$ where AD is parallel to BC.

The following information is given for this trapezium.

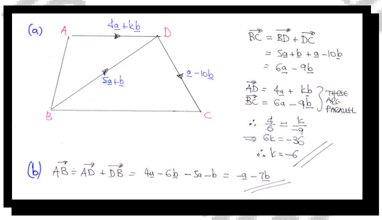
$$\overrightarrow{BD} = 5\mathbf{a}+\mathbf{b}, \qquad \overrightarrow{DC} = \mathbf{a}-10\mathbf{b} \qquad \text{and} \qquad \overrightarrow{AD} = 4\mathbf{a}+k\mathbf{b},$$

where k is an integer.

a) Find the value of k.

b) Find a simplified expression for $\overrightarrow{AB}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

$\boxed{k=-6}$, $\boxed{\overrightarrow{AB} = -\mathbf{a}-7\mathbf{b}}$



Question 45 (***+)

The straight lines L_1 and L_2 have respective vector equations

$$\mathbf{r}_1 = \begin{pmatrix} 4 \\ -3 \\ 3 \end{pmatrix} + t\begin{pmatrix} 5 \\ -1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{r}_2 = \begin{pmatrix} 9 \\ 8 \\ -2 \end{pmatrix} + s\begin{pmatrix} -5 \\ -5 \\ 2 \end{pmatrix},$$

where t and s are scalar parameters.

a) Show that L_1 and L_2 intersect at some point P, and find its coordinates.

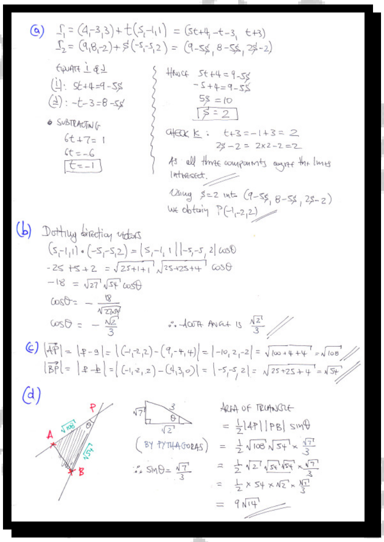
b) Find the exact value of the cosine of the acute angle θ, between L_1 and L_2.

The point $A(9,-4,4)$ lies on L_1 and the point $B(4,3,0)$ lies on L_2.

c) Find the distance of AP and the distance of BP.

d) Show the area of the triangle APB is $9\sqrt{14}$.

$\boxed{P(-1,-2,2)}$, $\boxed{\cos\theta = \frac{\sqrt{2}}{3}}$, $\boxed{|AP| = 6\sqrt{3}}$, $\boxed{|BP| = 3\sqrt{6}}$



Question 46 (***+)

With respect to a fixed origin O, the straight line l has vector equation

$$\mathbf{r} = \begin{pmatrix} a \\ b \\ 10 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix},$$

where a and b are scalar constants and λ is a scalar parameter.

a) If l passes though the point $P(7,3,6)$, find the value of a and the value of b.

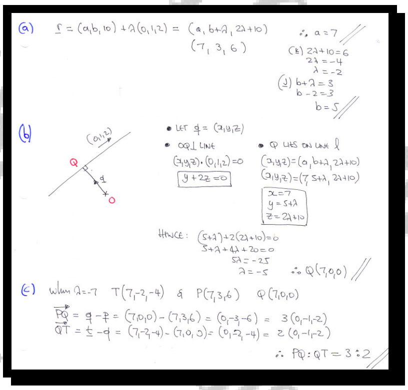
The point Q lies on l so that OQ is perpendicular to l.

b) Find the coordinates of Q.

The point T lies on l where $\lambda = -7$.

c) Find the ratio $PQ : QT$.

$\boxed{\text{C4}}$, $\boxed{a = 7}$, $\boxed{b = 5}$, $\boxed{Q(7,0,0)}$, $\boxed{PQ : QT = 3:2}$


(a) $\underline{r} = (a,b,10) + \lambda(0,1,2) = (a, b+\lambda, 2\lambda+10)$

$(7, 3, 6)$

$\therefore a = 7$

(k) $2\lambda + 10 = 6$
$2\lambda = -4$
$\lambda = -2$

(j) $b + \lambda = 3$
$b - 2 = 3$
$b = 5$

(b) $(0,1,2)$, Q, $\underline{q}$, O

• LET $\underline{q} = (x,y,z)$

• $OQ \perp$ LINE

$(x,y,z)\cdot(0,1,2) = 0$

$\boxed{y + 2z = 0}$

• Q LIES ON LINE l

$(x,y,z) = (a, b+\lambda, 2\lambda+10)$

$(x,y,z) = (7, 5+\lambda, 2\lambda+10)$

$\boxed{\begin{matrix} x = 7 \\ y = 5+\lambda \\ z = 2\lambda+10 \end{matrix}}$

HENCE: $(5+\lambda) + 2(2\lambda+10) = 0$
$5 + \lambda + 4\lambda + 20 = 0$
$5\lambda = -25$
$\lambda = -5$

$\therefore Q(7,0,0)$

(c) when $\lambda = -7$ $T(7,-2,-4)$ & $P(7,3,6)$ $Q(7,0,0)$

$\overrightarrow{PQ} = \underline{q} - \underline{p} = (7,0,0) - (7,3,6) = (0,-3,-6) = 3(0,-1,-2)$

$\overrightarrow{QT} = \underline{t} - \underline{q} = (7,-2,-4) - (7,0,0) = (0,-2,-4) = 2(0,-1,-2)$

$\therefore PQ : QT = 3:2$

Question 47 (***+)

Relative to a fixed origin O, the straight lines l and m have respective vector equations

$$\mathbf{r}_1 = \begin{pmatrix} 6 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ a \end{pmatrix} \quad \text{and} \quad \mathbf{r}_2 = \begin{pmatrix} 4 \\ -3 \\ 7 \end{pmatrix} + s \begin{pmatrix} 2 \\ 2 \\ -5 \end{pmatrix},$$

where t and s are scalar parameters, and a is a constant.

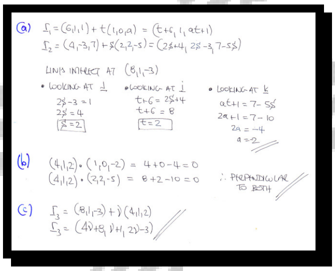
l and m intersect at the point $P(8,1,-3)$.

a) Find the value of a.

b) Show that the vector $\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$ is perpendicular to both l and m.

c) Determine a vector equation of the straight line n, such that all three straight lines intersect, with the n being perpendicular to both l and m.

$\boxed{a = -2}$, $\boxed{\mathbf{r} = 8\mathbf{i} + \mathbf{j} - 3\mathbf{k} + \nu(4\mathbf{i} + \mathbf{j} + 2\mathbf{k})}$


(a) $r_1 = (6,1,1) + t(1,0,a) = (t+6, 1, at+1)$
$r_2 = (4,-3,7) + s(2,2,-5) = (2s+4, 2s-3, 7-5s)$

LINES INTERSECT AT $(8,1,-3)$

• LOOKING AT $\underline{j}$: $2s-3=1$, $2s=4$, $\boxed{s=2}$

• LOOKING AT $\underline{i}$: $t+6=2s+4$, $t+6=8$, $\boxed{t=2}$

• LOOKING AT $\underline{k}$: $at+1=7-5s$, $2a+1=7-10$, $2a=-4$, $a=-2$

(b) $(4,1,2)\cdot(1,0,-2) = 4+0-4=0$
$(4,1,2)\cdot(2,2,-5) = 8+2-10=0$
$\therefore$ PERPENDICULAR TO BOTH

(c) $r_3 = (8,1,-3) + \nu(4,1,2)$
$r_3 = (4\nu+8, \nu+1, 2\nu-3)$

Question 48 (***+)

With respect to a fixed origin O, the straight line l has vector equation

$$\mathbf{r} = \begin{pmatrix} 24 \\ 6 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix},$$

where λ is a scalar parameter.

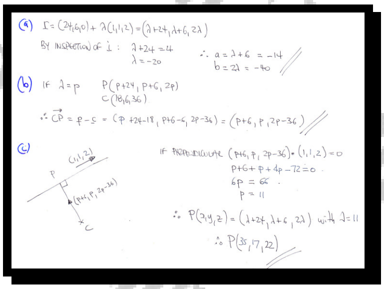
a) If the point $D(4, a, b)$ lies on l, find the value of a and the value of b.

The point P lies on l where $\lambda = p$, and the point C has coordinates $(18, 6, 36)$.

b) Show that $\overrightarrow{CP} = \begin{pmatrix} p+6 \\ p \\ 2p-36 \end{pmatrix}$.

c) Given further that $\overrightarrow{CP}$ is perpendicular to l, find the coordinates of P.

$\boxed{a = -14}$, $\boxed{b = -40}$, $\boxed{P(35, 17, 22)}$


(a) $\underline{r} = (24,6,0) + \lambda(1,1,2) = (\lambda+24, \lambda+6, 2\lambda)$

BY INSPECTION OF $\underline{i}$: $\lambda + 24 = 4$

$\lambda = -20$

$\therefore a = \lambda + 6 = -14$

$b = 2\lambda = -40$

(b) IF $\lambda = p$ $P(p+24, p+6, 2p)$

$C(18, 6, 36)$

$\therefore \overrightarrow{CP} = \underline{p} - \underline{c} = (p+24-18, p+6-6, 2p-36) = (p+6, p, 2p-36)$

(c)

IF PERPENDICULAR $(p+6, p, 2p-36) \cdot (1,1,2) = 0$

$p + 6 + p + 4p - 72 = 0$

$6p = 66$

$p = 11$

$\therefore P(x,y,z) = (\lambda+24, \lambda+6, 2\lambda)$ with $\lambda = 11$

$\therefore P(35, 17, 22)$

Question 49 (***+)

The straight line l has the following vector equation

$$\mathbf{r} = -2\mathbf{i} - 12\mathbf{j} - 9\mathbf{k} + \lambda(\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}),$$

where λ is a scalar parameter.

The point $P(a,b,3)$ lies on l.

a) Find the value of each of the scalar constants a and b.

The point O represents a fixed origin.

The point Q lies on l, so that OQ is perpendicular to l.

b) Show that the coordinates of Q are $(2,0,-1)$.

You may not verify this fact by using the coordinates of Q.

c) Find the exact area of the triangle OPQ.

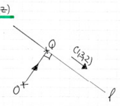
$\boxed{}$, $\boxed{a=4}$, $\boxed{b=6}$, $\boxed{\text{area} = \sqrt{70}}$

a) WRITE THE LINE IN FULL PARAMETRIC FORM

$$\underline{r} = \begin{pmatrix} -2 \\ -12 \\ -9 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} \lambda-2 \\ 3\lambda-12 \\ 2\lambda-9 \end{pmatrix}$$

- $\lambda - 2 = a$
- $3\lambda - 12 = b$
- $2\lambda - 9 = 3$, $2\lambda = 12$, $\lambda = 6$
- $3\times6 - 12 = b$, $b = 6$
- $6 - 2 = a$, $a = 4$

b) LOOKING AT THE DIAGRAM, LET $\underline{q} = (x,y,z)$



$\vec{OQ} \perp l \Rightarrow (x,y,z)\cdot(1,3,2) = 0$

$\Rightarrow \boxed{x+3y+2z=0}$

BUT $Q(x,y,z)$ LIES ON l

$\Rightarrow \boxed{\begin{array}{l} x = \lambda-2 \\ y = 3\lambda-12 \\ z = 2\lambda-9 \end{array}}$

$\Rightarrow (\lambda-2) + 3(3\lambda-12) + 2(2\lambda-9) = 0$

$\Rightarrow \lambda - 2 + 9\lambda - 36 + 4\lambda - 18 = 0$

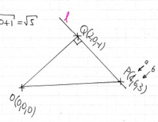
$\Rightarrow 14\lambda = 56$

$\Rightarrow \lambda = 4$

$\therefore Q(4-2, 3\times4-12, 2\times4-9)$

$Q(2,0,-1)$

c) LOOKING AT THE DIAGRAM



$|\vec{OQ}| = |\underline{q}| = |2,0,-1| = \sqrt{4+0+1} = \sqrt{5}$

$|\vec{PQ}| = |\underline{q} - \underline{p}| = |(2,0,-1)-(4,6,3)|$

$= |-2,-6,-4|$

$= \sqrt{4+36+16}$

$= \sqrt{56}$

$\therefore \text{AREA} = \frac{1}{2}|OQ||PQ| = \frac{1}{2}\sqrt{5}\times\sqrt{56} = \frac{1}{2}\sqrt{280}$

$= \sqrt{70}$

Question 50 (***+)

The straight line L has the vector equation

$$\mathbf{r} = \begin{pmatrix} 3 \\ -4 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix},$$

where λ is a scalar parameter.

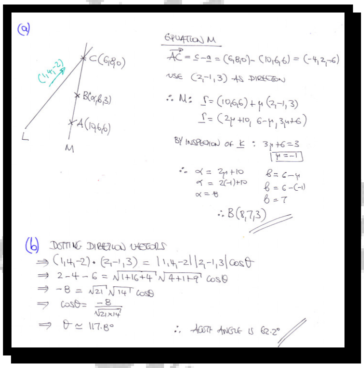
The straight line M passes through the points with coordinates $A(10,6,6)$ and $B(\alpha,\beta,3)$, where α and β are scalar constants.

L and M intersect at the point $C(6,8,0)$.

a) Find the coordinates of B.

b) Calculate the acute angle between L and M.

$\boxed{P(8,7,3)}$, $\boxed{62.2°}$



Question 51 (***+)

The straight line l_1 passes through the points $A(2,8,1)$ and $B(2,4,3)$.

a) Find a vector equation for l_1, in terms of a scalar parameter λ.

The straight line l_2 has a vector equation

$$\mathbf{r} = \begin{pmatrix} 5 \\ 2 \\ a \end{pmatrix} + \mu \begin{pmatrix} 1 \\ b \\ -1 \end{pmatrix},$$

where a and b are scalar constants, and μ is a scalar parameter.

The point $C(2,-4,c)$, where c is a scalar constant, is the point of intersection between l_1 and l_2.

b) Find the value of each of the scalar constants a, b and c.

c) Determine the ratio $AB : BC$.

$\boxed{\mathbf{r}_1 = 2\mathbf{i}+8\mathbf{j}+\mathbf{k}+\lambda(2\mathbf{j}-\mathbf{k})}$, $\boxed{a=4}$, $\boxed{b=2}$, $\boxed{c=7}$, $\boxed{AB:BC=1:2}$

(a) $\overrightarrow{AB} = \underline{b} - \underline{a} = (2,4,3)-(2,8,1) = (0,-4,2)$

USE $(0,2,-1)$ AS A DIRECTION

$\underline{r}_1 = (2,8,1) + \lambda(0,2,-1)$

$\underline{r}_1 = (2, 2\lambda+8, 1-\lambda)$

(b) $\underline{r}_2 = (5,2,a) + \mu(1,b,-1)$

$\underline{r}_2 = (\mu+5, \mu b+2, a-\mu)$

EQUATE ALL COMPONENTS

$$\left.\begin{aligned} (i)&: 2 = \mu+5 = 2 \\ (j)&: 2\lambda+8 = \mu b+2 = -4 \\ (k)&: 1-\lambda = a-\mu = c \end{aligned}\right\} \Rightarrow \boxed{\mu=-3}$$

$2\lambda+8 = -3b+2 = -4$

$1-\lambda = a+3 = c$

$\therefore$ $2\lambda+8=-4$, $2\lambda=-12$, $\boxed{\lambda=-6}$

$-3b+2=-4$, $-3b=-6$, $\boxed{b=2}$

$1-\lambda = c$, $1+6=c$, $\boxed{c=7}$

$a+3=7$, $\boxed{a=4}$

$\therefore$ $a=4$, $b=2$, $c=7$

(c) $\overrightarrow{BC} = \underline{c}-\underline{b} = (2,-4,7)-(2,4,3) = (0,-8,4) = 4(0,-2,1)$

$\overrightarrow{AB} = (0,-4,2) = 2(0,-2,1)$

$\therefore$ RATIO IS $2:4$ IE $1:2$

Question 52 (***+)

The straight lines L_1 and L_2 have vector equations

$$\mathbf{r}_1 = \begin{pmatrix} 12 \\ 7 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -3 \\ -1 \end{pmatrix} \quad \text{and} \quad \mathbf{r}_2 = \begin{pmatrix} 0 \\ 1 \\ 21 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix},$$

where λ and μ are scalar parameters.

a) Show that L_1 and L_2 intersect at the point P, and find its coordinates.

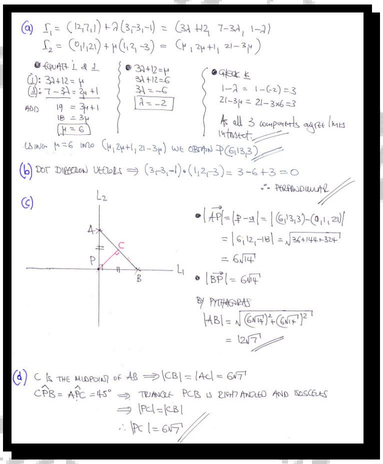
b) Show further that L_1 and L_2 are perpendicular to each other.

The point $A(0,1,21)$ lies on L_2 and the point B lies on L_1 so that $|\overrightarrow{AP}| = |\overrightarrow{PB}|$.

c) Find the distance AB.

d) Hence state the shortest distance of P from the line through A and B.

$\boxed{P(6,13,3)}$, $\boxed{|AB| = 12\sqrt{7}}$, $\boxed{6\sqrt{7}}$



Question 53 (***+)

The straight lines l_1 and l_2 have the following Cartesian equations

$$l_1: \quad x - a = \frac{y+4}{-4} = \frac{z}{-2}$$

$$l_2: \quad \frac{x-a}{2} = \frac{y+1}{-5} = \frac{z-1}{-3}$$

where a is a scalar constant.

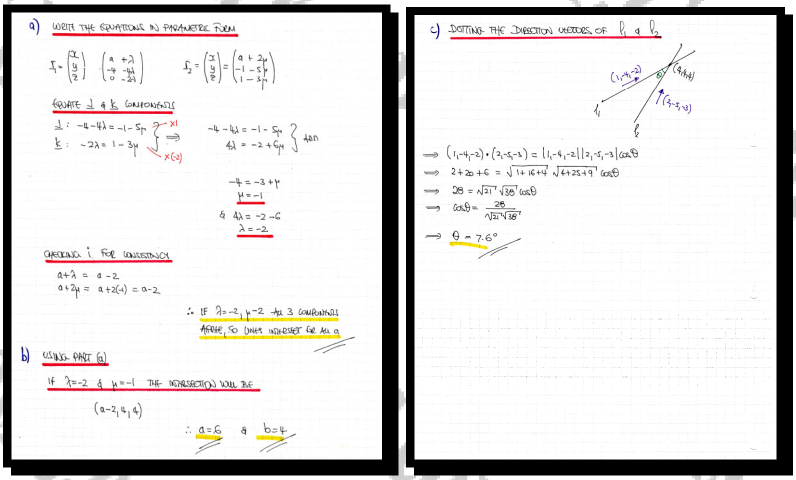
a) Show that l_1 and l_2 intersect at for all values of a.

The intersection point of l_1 and l_2 has coordinates (b,b,b), where b is a scalar constant.

b) Find the value of a and the value of b.

c) Calculate the acute angle formed by l_1 and l_2.

$\boxed{}$, $\boxed{a=6}$, $\boxed{b=4}$, $\boxed{\theta \approx 7.6°}$



Question 54 (***+)

OAB is a triangle with the point P being the midpoint of OB and the point Q being the midpoint of AB.

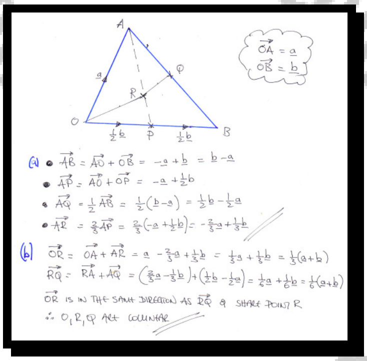
The point R is such so that $\overrightarrow{AR} = \frac{2}{3}\overrightarrow{AP}$.

Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

a) Find simplified expressions, in terms of $\mathbf{a}$ and $\mathbf{b}$, for each of the vectors $\overrightarrow{AB}$, $\overrightarrow{AP}$, $\overrightarrow{AQ}$ and $\overrightarrow{AR}$.

b) By finding simplified expressions, in terms $\mathbf{a}$ and $\mathbf{b}$, for two more suitable vectors, show that the points O, R and Q are collinear.

$\boxed{\overrightarrow{AB} = \mathbf{b} - \mathbf{a}}$, $\boxed{\overrightarrow{AP} = \frac{1}{2}\mathbf{b} - \mathbf{a}}$, $\boxed{\overrightarrow{AQ} = \frac{1}{2}\mathbf{b} - \frac{1}{2}\mathbf{a}}$, $\boxed{\overrightarrow{AR} = \frac{1}{3}\mathbf{b} - \frac{2}{3}\mathbf{a}}$



Question 55 (***+)

The points $A(1,1,2)$, $B(2,1,5)$, $C(4,0,1)$ and D form the parallelogram $ABCD$, where the above coordinates are measured relative to a fixed origin.

a) Find the coordinates of D.

The points E, B and D are collinear, so that B is the midpoint of ED.

b) Determine the coordinates of E.

The point F is such so that $ABEF$ is also a parallelogram.

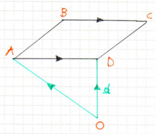
c) Find the coordinates of F.

d) Show that B is the midpoint of FC.

e) Prove that $ADBF$ is another parallelogram.

$\boxed{}$, $\boxed{D(3,0,-2)}$, $\boxed{E(1,2,12)}$, $\boxed{F(0,2,9)}$

a) LOOKING AT THE DIAGRAM



$\Rightarrow \overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD}$) PARALLEL
$\Rightarrow \overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{BC}$

$\Rightarrow \underline{d} = \underline{a} + (\underline{c} - \underline{b})$

$\Rightarrow \underline{d} = \underline{a} + \underline{c} - \underline{b}$

$\Rightarrow \underline{d} = (1,1,2) + (4,0,1) - (2,1,5)$

$\Rightarrow \underline{d} = (3,0,-2)$

A(1,1,2)
B(2,1,5)
C(4,0,1)

$\therefore D(3,0,-2)$

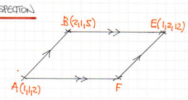
b) E(1,2,12)   B(2,1,5)   D(3,0,-2)

BY SEQUENCING / INSPECTION / MIDPOINT FORMULA

$x: \ 1 \xleftarrow{-1} 2 \xleftarrow{-1} 3$
$y: \ 2 \xleftarrow{+1} 1 \xleftarrow{+1} 0$
$z: \ 12 \xleftarrow{+7} 5 \xleftarrow{+7} -2$

E   B   D

c) AS IN PART (a) OR INSPECTION



(B) to (E)
$2 \xrightarrow{-1} 1$
$1 \xrightarrow{+1} 2$
$5 \xrightarrow{+7} 12$

(A) to (F)
$1 \xrightarrow{-1} 0$
$1 \xrightarrow{+1} 2$
$2 \xrightarrow{+7} 9$

$\therefore F(0,2,9)$

d) F(0,2,9)   C(4,0,1)   B(2,1,5)

MIDPOINT OF FC $= \left(\frac{0+4}{2}, \frac{2+0}{2}, \frac{9+1}{2}\right) = (2,1,5)$ WHICH IS B

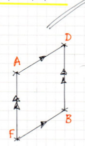
e)

$\overrightarrow{AD} = \underline{d} - \underline{a} = (3,0,-2) - (1,1,2) = (2,-1,-4)$

$\overrightarrow{FB} = \underline{b} - \underline{f} = (2,1,5) - (0,2,9) = (2,-1,-4)$

$\overrightarrow{FA} = \underline{a} - \underline{f} = (1,1,2) - (0,2,9) = (1,-1,-7)$

$\overrightarrow{BD} = \underline{d} - \underline{b} = (3,0,-2) - (2,1,5) = (1,-1,-7)$



REFERRING TO THE CALCULATIONS & THE DIAGRAM

OPPOSITE SIDES ARE PARALLEL (AND EQUAL)

INDEED ADBF IS A PARALLELOGRAM

Question 56 (***+)

The straight lines l_1 and l_2 have the following vector equations

$$\mathbf{r}_1 = 2\mathbf{i} + \mathbf{j} + 8\mathbf{k} + \lambda(\mathbf{j} - 2\mathbf{k})$$

$$\mathbf{r}_2 = \mathbf{i} + \mu(a\mathbf{i} + b\mathbf{j} + 2\mathbf{k}),$$

where λ and μ are scalar parameters, and a and b are scalar constants.

l_1 and l_2 intersect at right angles at the point P.

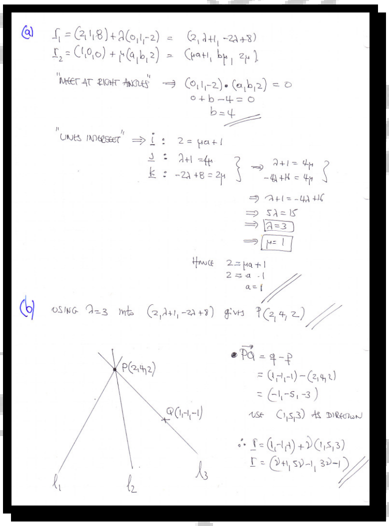
a) Find the value of a and the value of b.

b) Determine the coordinates of P.

The straight line l_3 passes through the point $Q(1,-1,-1)$.

c) Find a vector equation for l_3, given that all three lines intersect at the same point.

$\boxed{a = 1,\ b = 4}$, $\boxed{P(2,4,2)}$, $\boxed{\mathbf{r} = \mathbf{i} - \mathbf{j} - \mathbf{k} + \nu(\mathbf{i} + 5\mathbf{j} + 3\mathbf{k})}$


(a) $\underline{r}_1 = (2,1,8) + \lambda(0,1,-2) = (2, \lambda+1, -2\lambda+8)$

$\underline{r}_2 = (1,0,0) + \mu(a,b,2) = (\mu a+1, b\mu, 2\mu)$

"MEET AT RIGHT ANGLES" $\Rightarrow (0,1,-2)\cdot(a,b,2) = 0$

$0 + b - 4 = 0$

$b = 4$

"LINES INTERSECT" $\Rightarrow$ $\underline{i}$: $2 = \mu a + 1$

$\underline{j}$: $\lambda + 1 = 4\mu$

$\underline{k}$: $-2\lambda + 8 = 2\mu$

$\Rightarrow \lambda + 1 = 4\mu$, $-4\lambda + 16 = 4\mu$

$\Rightarrow \lambda + 1 = -4\lambda + 16$

$\Rightarrow 5\lambda = 15$

$\Rightarrow \boxed{\lambda = 3}$

$\Rightarrow \boxed{\mu = 1}$

HENCE $2 = \mu a + 1$

$2 = a + 1$

$a = 1$

(b) USING $\lambda = 3$ into $(2, \lambda+1, -2\lambda+8)$ gives $P(2,4,2)$

P(2,4,2)

Q(1,−1,−1)

l_1 l_2 l_3

• $\vec{PQ} = \underline{q} - \underline{p}$

$= (1,-1,-1) - (2,4,2)$

$= (-1,-5,-3)$

USE $(1,5,3)$ AS DIRECTION

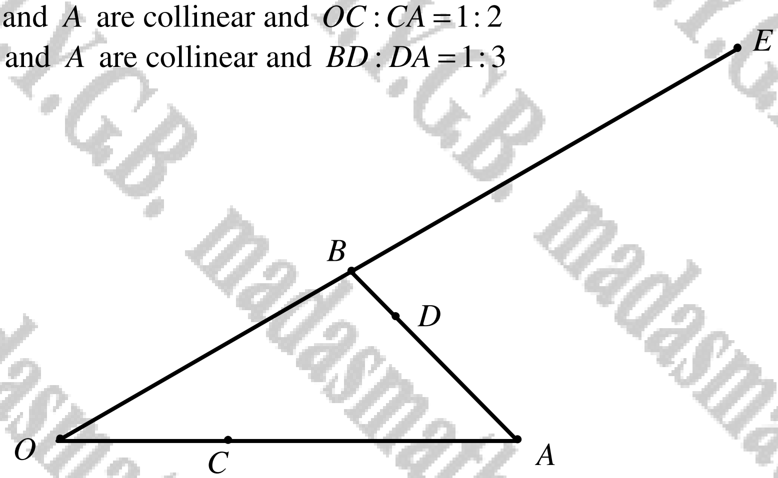
$\therefore \underline{r} = (1,-1,-1) + \nu(1,5,3)$

$\underline{r} = (\nu+1, 5\nu-1, 3\nu-1)$

Question 57 (***+)

The figure below shows the points O, C, A, D, B and E, which are related as follows.

- O, B and E are collinear and $OB : BE = 1:2$
- O, C and A are collinear and $OC : CA = 1:2$
- B, D and A are collinear and $BD : DA = 1:3$



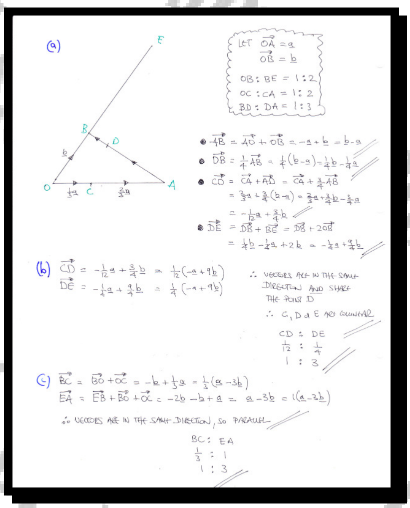
Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

a) Find simplified expressions, in terms of $\mathbf{a}$ and $\mathbf{b}$, for each of the vectors $\overrightarrow{AB}$, $\overrightarrow{DB}$, $\overrightarrow{CD}$ and $\overrightarrow{DE}$.

b) Show that the points C, D and E are collinear, and find the ratio $CD : DE$.

c) Show further that BC is parallel to EA, and find the ratio $BC : EA$.

$\boxed{\overrightarrow{AB} = \mathbf{b} - \mathbf{a}}$, $\boxed{\overrightarrow{DB} = \frac{1}{4}\mathbf{b} - \frac{1}{4}\mathbf{a}}$, $\boxed{\overrightarrow{CD} = -\frac{1}{12}\mathbf{a} + \frac{3}{4}\mathbf{b}}$, $\boxed{\overrightarrow{DE} = -\frac{1}{4}\mathbf{a} + \frac{9}{4}\mathbf{b}}$, $\boxed{CD : DE = 1:3}$, $\boxed{BC : EA = 1:3}$



Question 58 (***+)

With respect to a fixed origin O, the following points are given

$$A(2,2,5),\ B(12,7,0),\ C(0,0,1) \text{ and } D(9,k,4),$$

where k is a scalar constant.

a) Find the vector equation of the straight line l_1 that passes through A and B.

The straight line l_2 passes through C and D, and intersects l_1 at the point P.

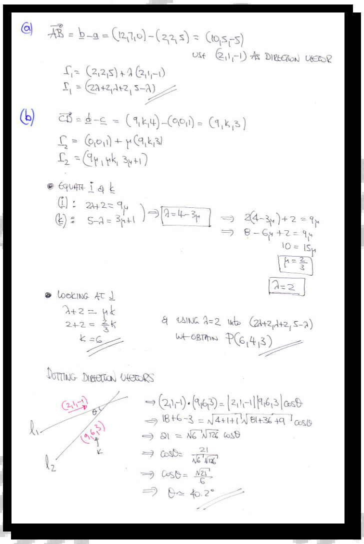
b) Determine in any order …

i. … the coordinates of P.

ii. … the value of k.

iii. … the acute angle between l_1 and l_2.

$\boxed{}$, $\boxed{\mathbf{r} = 2\mathbf{i} + 2\mathbf{j} + 5\mathbf{k} + \lambda(2\mathbf{i} + \mathbf{j} - \mathbf{k})}$, $\boxed{P(6,4,3)}$, $\boxed{k=6}$, $\boxed{40.2°}$



(a) $\vec{AB} = \underline{b} - \underline{a} = (12,7,0) - (2,2,5) = (10,5,-5)$

USE $(2,1,-1)$ AS DIRECTION VECTOR

$\underline{r}_1 = (2,2,5) + \lambda(2,1,-1)$

$\underline{r}_1 = (2\lambda+2, \lambda+2, 5-\lambda)$

(b) $\vec{CD} = \underline{d} - \underline{c} = (9,k,4) - (0,0,1) = (9,k,3)$

$\underline{r}_2 = (0,0,1) + \mu(9,k,3)$

$\underline{r}_2 = (9\mu, \mu k, 3\mu+1)$

• EQUATE $\underline{i}$ & $\underline{k}$

$(\underline{i})$: $2\lambda + 2 = 9\mu$

$(\underline{k})$: $5 - \lambda = 3\mu + 1$

$\Rightarrow \boxed{\lambda = 4 - 3\mu} \Rightarrow 2(4-3\mu) + 2 = 9\mu$

$\Rightarrow 8 - 6\mu + 2 = 9\mu$

$10 = 15\mu$

$\boxed{\mu = \frac{2}{3}}$

$\boxed{\lambda = 2}$

• LOOKING AT $\underline{j}$

$\lambda + 2 = \mu k$

$2 + 2 = \frac{2}{3}k$

$k = 6$

& USING $\lambda = 2$ into $(2\lambda+2, \lambda+2, 5-\lambda)$

WE OBTAIN $P(6,4,3)$

DOTTING DIRECTION VECTORS

l_1, l_2, $(2,1,-1)$, $(9,6,3)$, θ, k

$\Rightarrow (2,1,-1)\cdot(9,6,3) = |2,1,-1||9,6,3|\cos\theta$

$\Rightarrow 18 + 6 - 3 = \sqrt{4+1+1}\sqrt{81+36+9}\cos\theta$

$\Rightarrow 21 = \sqrt{6}\sqrt{126}\cos\theta$

$\Rightarrow \cos\theta = \frac{21}{\sqrt{6}\sqrt{126}}$

$\Rightarrow \cos\theta = \frac{\sqrt{21}}{6}$

$\Rightarrow \theta \simeq 40.2°$

Question 59 (****)

Two submarines S_1 and S_2, are travelling through the ocean.

They both appeared on the radar screen of a tracking station at the same time. The distances are measured in hundreds of metres and the time t, in seconds, is measured from the instant they were both observed on the radar screen of the tracking station.

The coordinates of S_1 and S_2, relative to a fixed origin O, are given by

$$S_1: \ \mathbf{r}_1 = (2t-4)\mathbf{i} + (t-15)\mathbf{j} + (t+5)\mathbf{k}$$

$$S_2: \ \mathbf{r}_2 = 10\mathbf{i} + (-2t+6)\mathbf{j} + (2t-2)\mathbf{k}$$

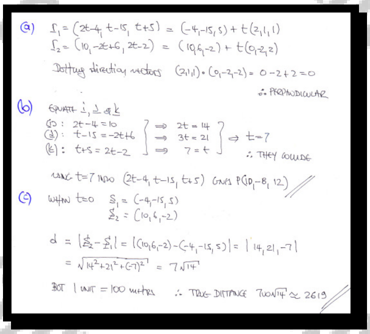
a) Show that S_1 and S_2 are travelling in perpendicular directions to each other.

Suppose that S_1 and S_2 continue to travel according to the above vector equations.

b) Show further that S_1 and S_2, will eventually collide at some point P, and further determine the coordinates of P.

c) Calculate, to the nearest metre, the distance between S_1 and S_2, when they were first observed by the tracking station.

$\boxed{P(10,-8,12)}$, $\boxed{\text{distance} = 700\sqrt{14} \approx 2619\,\text{m}}$



(a) $S_1 = (2t-4, t-15, t+5) = (-4,-15,5) + t(2,1,1)$
$S_2 = (10, -2t+6, 2t-2) = (10,6,-2) + t(0,-2,2)$
Dotting direction vectors $(2,1,1)\cdot(0,-2,-2) = 0-2+2=0$
$\therefore$ PERPENDICULAR

(b) EQUATE $\underline{i}, \underline{j}$ & $\underline{k}$
(i): $2t-4=10 \Rightarrow 2t=14$
(j): $t-15=-2t+6 \Rightarrow 3t=21$
(k): $t+5=2t-2 \Rightarrow 7=t$
$\Rightarrow t=7$ $\therefore$ THEY COLLIDE

USING $t=7$ INTO $(2t-4, t-15, t+5)$ GIVES $P(10,-8,12)$

(c) WHEN $t=0$ $\underline{S}_1 = (-4,-15,5)$
$\underline{S}_2 = (10,6,-2)$

$d = |\underline{S}_2 - \underline{S}_1| = |(10,6,-2)-(-4,-15,5)| = |14,21,-7|$
$= \sqrt{14^2+21^2+(-7)^2} = 7\sqrt{14}$

BUT 1 UNIT = 100 metres $\therefore$ TRUE DISTANCE $700\sqrt{14} \simeq 2619$

Question 60 (***+)

The straight line l passes through the points with coordinates

$$A(-1,-4,8) \quad \text{and} \quad B(1,-2,5).$$

a) Find a vector equation of l.

The origin is denoted by O.

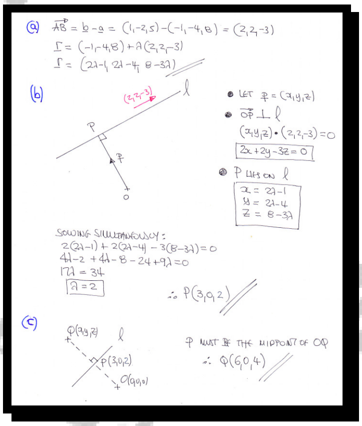
The point P lies on l, so that OP is perpendicular to l.

b) Determine the coordinates of P.

The point Q is the reflection of O, about l.

c) State the coordinates of Q.

$\boxed{\mathbf{r} = -\mathbf{i} - 4\mathbf{j} + 8\mathbf{k} + \lambda(2\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})}$, $\boxed{P(3,0,2)}$, $\boxed{Q(6,0,4)}$



(a) $\overrightarrow{AB} = \underline{b} - \underline{a} = (1,-2,5)-(-1,-4,8) = (2,2,-3)$

$\underline{r} = (-1,-4,8) + \lambda(2,2,-3)$

$\underline{r} = (2\lambda-1, 2\lambda-4, 8-3\lambda)$

(b) $(2,2,-3)$ l P $\underline{p}$ O

- LET $\underline{p} = (x,y,z)$
- $\overrightarrow{OP} \perp l$

$(x,y,z)\cdot(2,2,-3)=0$

$\boxed{2x+2y-3z=0}$

- P LIES ON l

$\boxed{\begin{aligned} x &= 2\lambda-1 \\ y &= 2\lambda-4 \\ z &= 8-3\lambda \end{aligned}}$

SOLVING SIMULTANEOUSLY:

$2(2\lambda-1)+2(2\lambda-4)-3(8-3\lambda)=0$

$4\lambda-2+4\lambda-8-24+9\lambda=0$

$17\lambda = 34$

$\boxed{\lambda=2}$

$\therefore P(3,0,2)$

(c) $Q(x,y,z)$ l $P(3,0,2)$ $O(0,0,0)$

P MUST BE THE MIDPOINT OF OQ

$\therefore Q(6,0,4)$

Question 61 (***+)

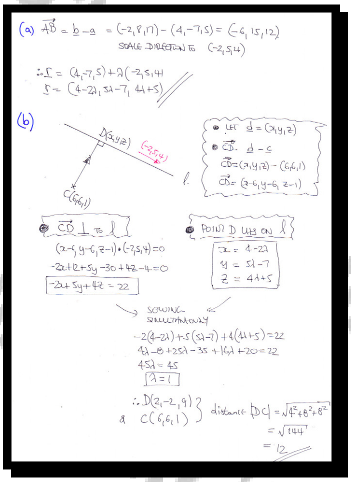
The points A and B have coordinates $(4,-7,5)$ and $(-2,8,17)$, respectively.

a) Find the equation of the straight line l, which passes through A and B.

The point C has coordinates $(6,6,1)$.

b) Find the shortest distance from C to l.

$\boxed{\mathbf{r} = 4\mathbf{i} - 7\mathbf{j} + 5\mathbf{k} + \lambda(-2\mathbf{i} + 5\mathbf{j} + 4\mathbf{k})}$, $\boxed{\text{shortest distance} = 12}$


(a) $\overrightarrow{AB} = \underline{b} - \underline{a} = (-2,8,17) - (4,-7,5) = (-6,15,12)$

SCALE DIRECTION TO $(-2,5,4)$

$\therefore \underline{r} = (4,-7,5) + \lambda(-2,5,4)$

$\underline{r} = (4-2\lambda, 5\lambda-7, 4\lambda+5)$

(b)

- LET $\underline{d} = (x,y,z)$
- $\overrightarrow{CD} = \underline{d} - \underline{c}$

$\overrightarrow{CD} = (x,y,z) - (6,6,1)$

$\overrightarrow{CD} = (x-6, y-6, z-1)$

- $\overrightarrow{CD} \perp$ TO l

$(x-6, y-6, z-1) \cdot (-2,5,4) = 0$

$-2x + 12 + 5y - 30 + 4z - 4 = 0$

$\boxed{-2x + 5y + 4z = 22}$

- POINT D LIES ON l

$\boxed{\begin{aligned} x &= 4-2\lambda \\ y &= 5\lambda-7 \\ z &= 4\lambda+5 \end{aligned}}$

SOLVING SIMULTANEOUSLY

$-2(4-2\lambda) + 5(5\lambda-7) + 4(4\lambda+5) = 22$

$4\lambda - 8 + 25\lambda - 35 + 16\lambda + 20 = 22$

$45\lambda = 45$

$\boxed{\lambda = 1}$

$\therefore D(2,-2,9)$ & $C(6,6,1)$ distance $|DC| = \sqrt{4^2+8^2+8^2}$

$= \sqrt{144}$

$= 12$

Question 62 (***+)

Relative to a fixed origin O, the points A, B and C have respective coordinates $(-2,5,13)$, $(1,1,1)$ and $(3,5,5)$.

a) Determine the size of the angle ABC.

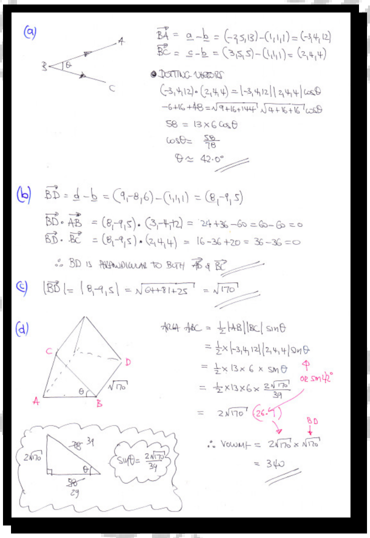
The point D has coordinates $(9,-8,6)$.

b) Show that BD is perpendicular to both AB and BC.

c) Find the distance BD.

d) Calculate the volume of the right triangular prism with base the triangle ABC and height BD.

$\boxed{42.0°}$, $\boxed{|BD| = \sqrt{170}}$, $\boxed{340 \text{ cubic units}}$



Question 63 (***+)

The points with coordinates $A(3,0,3)$ and $B(4,-1,5)$ are given.

a) Find a vector equation of the straight line l_1 that passes through A and B.

The straight line l_2 has equation

$$\mathbf{r} = 5\mathbf{i} + 10\mathbf{j} + 4\mathbf{k} + \mu(\mathbf{i} + 3\mathbf{j} + \mathbf{k}),$$

where μ is a scalar parameter.

b) Show that l_1 and l_2 are perpendicular.

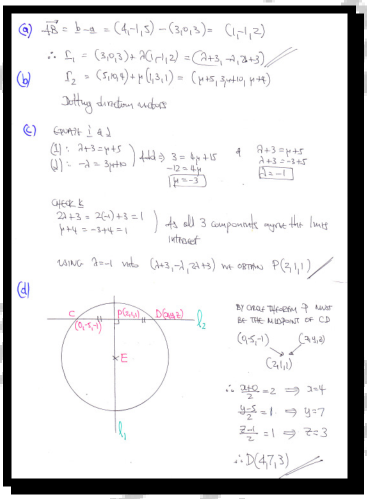
c) Show further that l_1 and l_2 intersect at some point P and find its coordinates.

The point E is on the l_1.

A circle with centre at E is drawn so that it cuts l_2 at the points C and D.

d) Given that the coordinates of C are $(0,-5,-1)$, find the coordinates of D.

$\boxed{\quad}$, $\boxed{\mathbf{r}_1 = 3\mathbf{i} + 3\mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} + 2\mathbf{k})}$, $\boxed{P(2,1,1)}$, $\boxed{D(4,7,3)}$



(a) $\vec{AB} = \underline{b} - \underline{a} = (4,-1,5) - (3,0,3) = (1,-1,2)$

$\therefore \ r_1 = (3,0,3) + \lambda(1,-1,2) = (\lambda+3, -\lambda, 2\lambda+3)$

(b) $r_2 = (5,10,4) + \mu(1,3,1) = (\mu+5, 3\mu+10, \mu+4)$

Dotting direction vectors

(c) EQUATE i & j

(1): $\lambda+3 = \mu+5$
(2): $-\lambda = 3\mu+10$
Add $\Rightarrow 3 = 4\mu+15$, $-12 = 4\mu$, $\boxed{\mu = -3}$ & $\lambda+3 = \mu+5$, $\lambda+3 = -3+5$, $\boxed{\lambda=-1}$

CHECK k
$2\lambda+3 = 2(-1)+3 = 1$
$\mu+4 = -3+4 = 1$
As all 3 components agree the lines intersect

USING $\lambda=-1$ into $(\lambda+3,-\lambda,2\lambda+3)$ we OBTAIN $P(2,1,1)$

(d)

BY CIRCLE THEOREM P MUST BE THE MIDPOINT OF CD

$(0,-5,-1)$ (x,y,z) $\to$ $(2,1,1)$

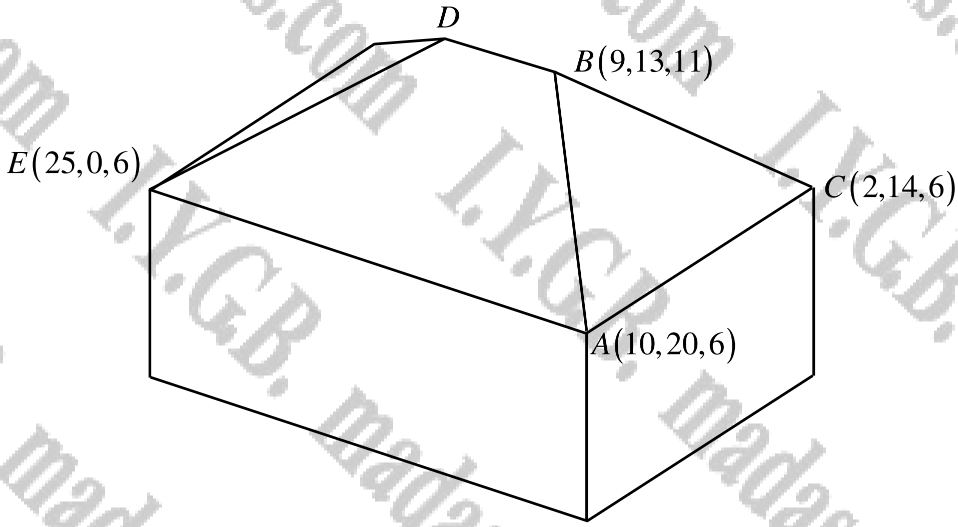
$\therefore \frac{x+0}{2} = 2 \Rightarrow x=4$

$\frac{y-5}{2} = 1 \Rightarrow y=7$

$\frac{z-1}{2} = 1 \Rightarrow z=3$

$\therefore D(4,7,3)$

Question 64 (****)



The figure above shows a solid, modelling a house with a standard slanted roof, where all the distances are measured in metres. With respect to a fixed origin, the coordinates of some of the vertices of the solid are marked in the diagram.

a) Find a vector equation of AE.

b) Show that AE is perpendicular to AC.

c) Find the cosine of the angle ABC.

The straight line BD is parallel to AE. The length of BD is 10 metres.

d) Determine the coordinates of D.

$\boxed{}$, $\boxed{\mathbf{r} = 10\mathbf{i} + 20\mathbf{j} + 6\mathbf{k} + \lambda(3\mathbf{i} - 4\mathbf{j})}$, $\boxed{\cos(\measuredangle ABC) = \frac{1}{3}}$, $\boxed{D(15,5,11)}$

(a) $\vec{AE} = \underline{e} - \underline{a} = (25,0,6) - (10,20,6) = (15,-20,0)$ • SCALE IT TO $(3,-4,0)$

$\therefore \underline{r} = (10,20,6) + \lambda(3,-4,0)$

$\underline{r} = (3\lambda + 10, 20 - 4\lambda, 6)$

(b) $\vec{AC} = \underline{c} - \underline{a} = (2,14,6) - (10,20,6) = (-8,-6,0)$

$\vec{AE} \cdot \vec{AC} = (15,-20,20) \cdot (-8,-6,0) = -120 + 120 = 0$ INDEED PERPENDICULAR

(c) $\vec{AB} = \underline{b} - \underline{a} = (9,13,11) - (10,20,6) = (-1,-7,5) \Rightarrow |\vec{AB}| = \sqrt{1+49+25} = \sqrt{75}$

$\vec{CB} = \underline{b} - \underline{c} = (9,13,11) - (2,14,6) = (7,-1,5) \Rightarrow |\vec{CB}| = \sqrt{49+1+25} = \sqrt{75}$

BY DOT PRODUCT $\vec{AB} \cdot \vec{CB} = |\vec{AB}||\vec{CB}|\cos\theta$

$(-1,-7,5) \cdot (7,-1,5) = \sqrt{75}\sqrt{75}\cos\theta$

$25 = 75\cos\theta$

$\cos\theta = \frac{1}{3}$

(d) BD IS PARALLEL TO AE, SO IN DIRECTION $(3,-4,0)$

$|3,-4,0| = \sqrt{9+16+0} = 5$

BUT $|BD| = 10$

$\therefore \vec{BD} = 2(3,-4,0) = (6,-8,0)$

$\underline{d} - \underline{b} = (6,-8,0)$

$\underline{d} - (9,13,11) = (6,-8,0)$

$\therefore \underline{d} = (15,5,11)$

$D(15,5,11)$

Question 65 (****)

Relative to a fixed origin O the following position vectors are given.

$$\overrightarrow{OA} = \begin{pmatrix} 0 \\ 8 \\ 3 \end{pmatrix} \quad \text{and} \quad \overrightarrow{OB} = \begin{pmatrix} 1 \\ 13 \\ 1 \end{pmatrix}.$$

a) Find a vector equation for the line straight l_1 which passes through A and B.

The straight line l_2 has vector equation

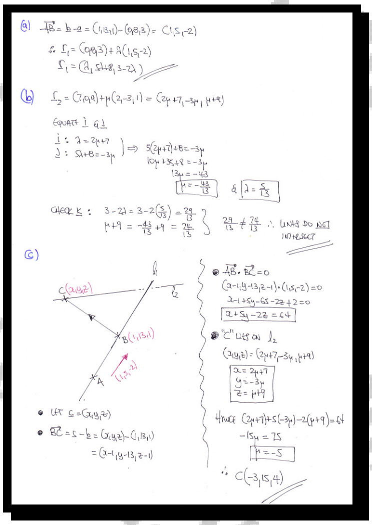
$$\mathbf{r} = \begin{pmatrix} 7 \\ 0 \\ 9 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix},$$

where μ is a scalar parameter.

b) Show that l_1 and l_2 do not intersect.

c) Find the position vector of C, given it lies on l_2 and $\measuredangle ABC = 90°$.

$\boxed{}$, $\boxed{\mathbf{r} = 8\mathbf{j} + 3\mathbf{k} + \lambda(\mathbf{i} + 5\mathbf{j} - 2\mathbf{k})}$, $\boxed{C(-3,15,4)}$



Question 66 (****)

The straight line L_1 passes through the points A and B, whose respective position vectors relative to a fixed origin O are

$$\begin{pmatrix}1\\2\\5\end{pmatrix} \text{ and } \begin{pmatrix}2\\1\\6\end{pmatrix}.$$

a) Find a vector equation for L_1.

The angle ABC is denoted by θ, where C is the point with position vector $\begin{pmatrix}3\\0\\1\end{pmatrix}$.

b) Show clearly that $\cos\theta = \frac{1}{3}$.

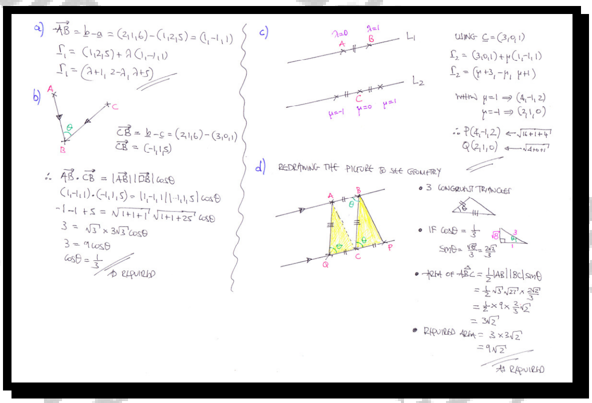
The straight line L_2 passes through C and is parallel to L_1.

The points P and Q both lie on L_2 so that $|AB| = |CP| = |CQ|$.

c) Determine the position vector of P and the position vector of Q, given that P is furthest away from O.

d) Show further that the area of the quadrilateral $ABPQ$ is $9\sqrt{2}$.

$\boxed{\mathbf{r}_1 = \mathbf{i} + 2\mathbf{j} + 5\mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} + \mathbf{k})}$, $\boxed{\overrightarrow{OP} = 4\mathbf{i} - \mathbf{j} + 2\mathbf{k}}$, $\boxed{\overrightarrow{OQ} = 2\mathbf{i} + \mathbf{j}}$



Question 67 (****)

The points A and B have position vectors $9\mathbf{i}+3\mathbf{j}+5\mathbf{k}$ and $9\mathbf{i}+4\mathbf{j}+\mathbf{k}$, respectively.

a) Find a vector equation of the straight line l_1 that passes through A and B.

The straight line l_2 has the vector equation

$$\mathbf{r}_2 = 6\mathbf{i}+3\mathbf{j}-4\mathbf{k}+\mu(\mathbf{i}+\mathbf{j}-\mathbf{k}),$$

where μ is a scalar parameter.

b) Show that l_1 and l_2 intersect and find the position vector of their point of intersection.

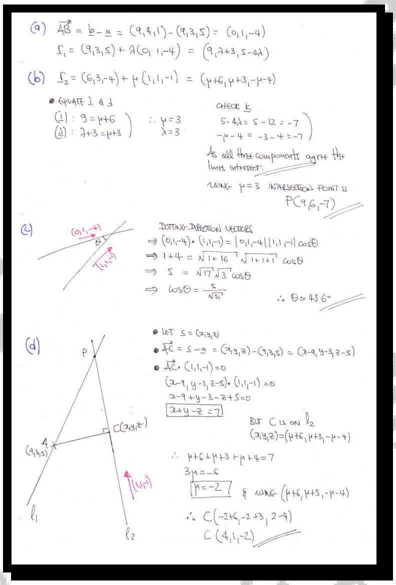
c) Find the acute angle between l_1 and l_2.

The point C lies on l_2 in such a position so that is closest to A.

d) Show that the position vector of C is given by

$$\mathbf{c} = 4\mathbf{i}+\mathbf{j}-2\mathbf{k}.$$

$\boxed{\mathbf{r}_1 = 9\mathbf{i}+3\mathbf{j}+5\mathbf{k}+\lambda(\mathbf{j}-4\mathbf{k})}$, $\boxed{9\mathbf{i}+6\mathbf{j}-7\mathbf{k}}$, $\boxed{45.6°}$



Question 68 (****)

With respect to a fixed origin O, the points with coordinates $A(2,3,5)$, $B(6,-1,5)$, $C(9,2,2)$ and $D(5,6,2)$ are given.

Prove that $ABCD$ is a rectangle and show that its area is $12\sqrt{6}$ square units.

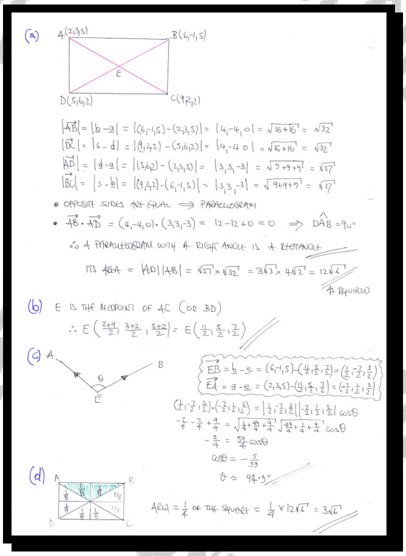
The diagonals of the rectangle intersect at the point E.

a) Find the coordinates of E.

b) Find the size of the angle BEA.

c) State the exact area of the triangle BEA.

$\boxed{E\left(\frac{11}{2},\frac{5}{2},\frac{7}{2}\right)}$, $\boxed{\measuredangle BEA = 94.9°}$, $\boxed{\text{area}(\triangle BEA) = 3\sqrt{6}}$

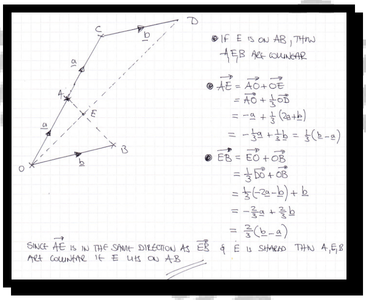


Question 69 (****)

Let $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\overrightarrow{OC} = 2\mathbf{a}$ and $\overrightarrow{OD} = 2\mathbf{a} + \mathbf{b}$.

If $\overrightarrow{OE} = \dfrac{1}{3}\overrightarrow{OD}$ prove that the point E lies on the straight line AB.

$\boxed{\text{proof}}$



Question 70 (****)

The point A has position vector $-5\mathbf{j}+7\mathbf{k}$.

a) Find a vector equation of the straight line l that passes through A and is parallel to the vector $\mathbf{i}+3\mathbf{j}-\mathbf{k}$.

The point P lies on l so that OP is perpendicular to l, where O is the origin.

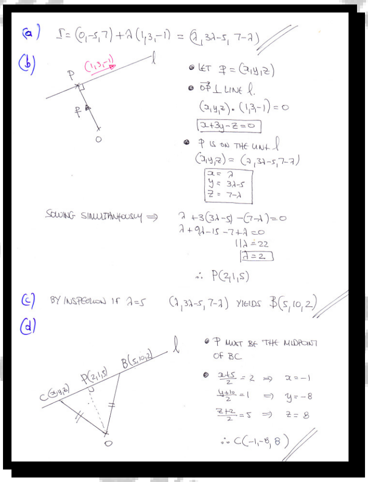
b) Determine the coordinates of P.

c) Show that the point $B(5,10,2)$ lies on l.

The point C is on l so that $|OB|=|OC|$.

d) Find the coordinates of C.

$\boxed{\mathbf{r}=-5\mathbf{j}+7\mathbf{k}+\lambda(\mathbf{i}+3\mathbf{j}-\mathbf{k})}$, $\boxed{P(2,1,5)}$, $\boxed{C(-1,-8,8)}$



(a) $\mathbf{r}=(0,-5,7)+\lambda(1,3,-1)=(\lambda,3\lambda-5,7-\lambda)$

(b) • LET $\mathbf{p}=(x,y,z)$

• $\vec{OP}\perp$ LINE l.

$(x,y,z)\cdot(1,3,-1)=0$

$\boxed{x+3y-z=0}$

• P IS ON THE LINE l

$(x,y,z)=(\lambda,3\lambda-5,7-\lambda)$

$\boxed{\begin{array}{l}x=\lambda\\ y=3\lambda-5\\ z=7-\lambda\end{array}}$

SOLVING SIMULTANEOUSLY $\Rightarrow$ $\lambda+3(3\lambda-5)-(7-\lambda)=0$

$\lambda+9\lambda-15-7+\lambda=0$

$11\lambda=22$

$\boxed{\lambda=2}$

$\therefore P(2,1,5)$

(c) BY INSPECTION IF $\lambda=5$ $(\lambda,3\lambda-5,7-\lambda)$ YIELDS $B(5,10,2)$

(d) • P MUST BE THE MIDPOINT OF BC

• $\frac{x+5}{2}=2 \Rightarrow x=-1$

$\frac{y+10}{2}=1 \Rightarrow y=-8$

$\frac{z+2}{2}=5 \Rightarrow z=8$

$\therefore C(-1,-8,8)$

Question 71 **(****)**

Relative to a fixed origin O, the straight lines l_1 and l_2 have vector equations

$$\mathbf{r}_1 = \begin{pmatrix} 2 \\ a \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -2 \\ -2 \end{pmatrix} \quad \text{and} \quad \mathbf{r}_2 = \begin{pmatrix} b \\ 2 \\ 14 \end{pmatrix} + \mu \begin{pmatrix} 7 \\ -4 \\ 6 \end{pmatrix},$$

where λ and μ are scalar parameters, and a and b are constants.

l_1 and l_2 intersect at the point P, whose z coordinate is 8.

a) Find the coordinates of the point P.

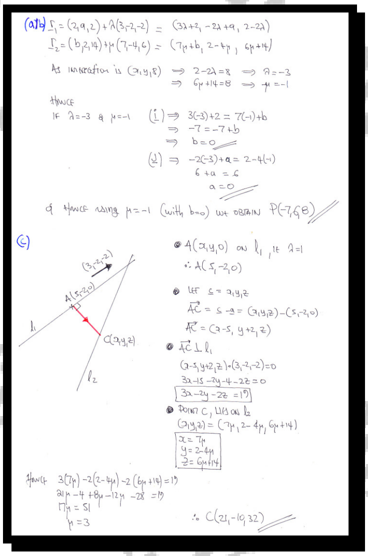
b) Show that the value of both a and b, is zero.

The point A, whose z coordinate is zero, lies on l_1.

The point C lies on l_2, so that AC is perpendicular to l_1.

c) Determine the coordinates of C.

$\boxed{P(-7,6,8)}$, $\boxed{C(21,-10,32)}$


(a)(b) $\underline{r}_1 = (2,a,2) + \lambda(3,-2,-2) = (3\lambda+2, -2\lambda+a, 2-2\lambda)$
$\underline{r}_2 = (b,2,14) + \mu(7,-4,6) = (7\mu+b, 2-4\mu, 6\mu+14)$

As intersection is $(x,y,8) \Rightarrow 2-2\lambda = 8 \Rightarrow \lambda = -3$
$\Rightarrow 6\mu+14 = 8 \Rightarrow \mu = -1$

Hence
If $\lambda = -3$ & $\mu = -1$ (i) $\Rightarrow 3(-3)+2 = 7(-1)+b$
$\Rightarrow -7 = -7+b$
$\Rightarrow b = 0$
(j) $\Rightarrow -2(-3)+a = 2-4(-1)$
$6+a = 6$
$a = 0$

& hence using $\mu = -1$ (with $b=0$) we obtain $P(-7,6,8)$

(c)

• $A(x,y,0)$ on l_1, if $\lambda = 1$
$\therefore A(5,-2,0)$

• Let $\underline{c} = x,y,z$
$\vec{AC} = \underline{c} - \underline{a} = (x,y,z) - (5,-2,0)$
$\vec{AC} = (x-5, y+2, z)$

• $\vec{AC} \perp l_1$
$(x-5, y+2, z)\cdot(3,-2,-2) = 0$
$3x-15-2y-4-2z = 0$
$\boxed{3x-2y-2z = 19}$

• Point C, lies on l_2
$(x,y,z) = (7\mu, 2-4\mu, 6\mu+14)$
$\boxed{x = 7\mu,\ y = 2-4\mu,\ z = 6\mu+14}$

Hence $3(7\mu) - 2(2-4\mu) - 2(6\mu+14) = 19$
$21\mu - 4 + 8\mu - 12\mu - 28 = 19$
$17\mu = 51$
$\mu = 3$

$\therefore C(21,-10,32)$

Question 72 (****)

The straight line l passes through the points A and C whose respective coordinates are $(-2,7,9)$ and $(8,-3,-1)$.

a) Find a vector equation for l.

The point $E(2,p,q)$ lies on l and the point B has coordinates $(-4,1,1)$.

b) Determine the value of p and the value of q.

c) Show that BE is perpendicular to l.

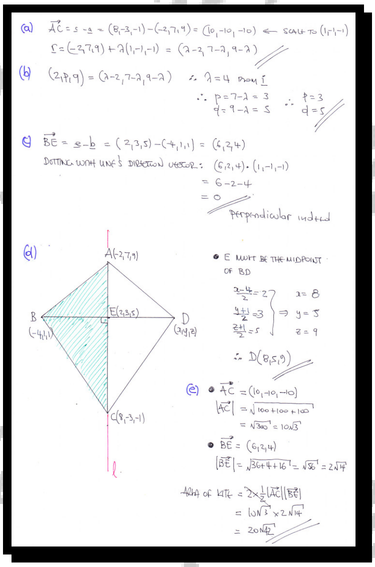
The point D is such, so that $ABCD$ is a kite with $\measuredangle ABC = \measuredangle ADC$.

Determine …

d) … the coordinates of D.

e) … the area of the kite $ABCD$.

$\boxed{}$, $\boxed{\mathbf{r} = -2\mathbf{i} + 7\mathbf{j} + 9\mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} - \mathbf{k})}$, $\boxed{p = 3 \text{ and } q = 5}$, $\boxed{D(8,5,9)}$, $\boxed{20\sqrt{42}}$



Question 73 (****)

The straight line l passes through the points with coordinates $(4,-1,1)$ and $(-1,4,6)$.

a) Determine a vector equation of l.

The points C and D have coordinates $(4,-2,-3)$ and $(p,q,-1)$, respectively.

The midpoint of CD is the point M, where M lies on l.

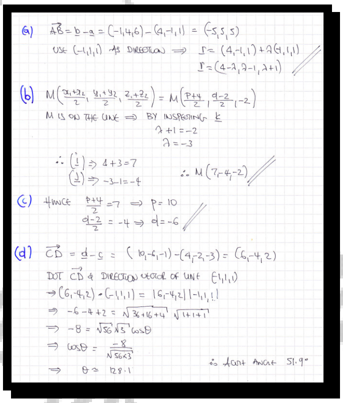
Find in any order …

b) … the coordinates of M.

c) … the value of p and the value of q.

d) … the size of the acute angle θ, between CD and l.

$\boxed{\text{C4M}}$, $\boxed{\mathbf{r}=4\mathbf{i}-\mathbf{j}+\mathbf{k}+\lambda(-\mathbf{i}+\mathbf{j}+\mathbf{k})}$, $\boxed{M(7,-4,-2)}$, $\boxed{p=10,\ q=-6}$, $\boxed{\theta\approx 51.9°}$



Question 74 (****)

Relative to a fixed origin O, the straight lines L and M have vector equations

$$\mathbf{r}_1 = \begin{pmatrix} 4 \\ 10 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} \quad \text{and} \quad \mathbf{r}_2 = \begin{pmatrix} 0 \\ 14 \\ -7 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -2 \\ 4 \end{pmatrix},$$

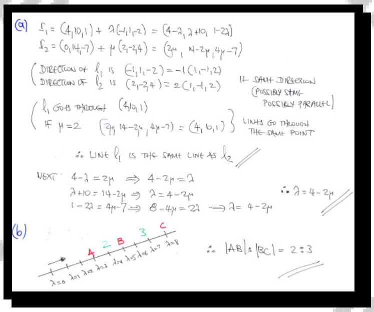
where λ and μ are scalar parameters.

a) Show that L and M represent the same straight line and find a linear relationship between λ and μ, giving the answer in the form $\lambda = f(\mu)$.

The points A, B and C lie on L, where $\lambda = 3$, $\lambda = 5$ and $\lambda = 8$ respectively.

b) State the ratio $AB : BC$.

$\boxed{}$, $\boxed{\lambda = 4 - 2\mu}$, $\boxed{AB : BC = 2 : 3}$


(a) $r_1 = (4,10,1) + \lambda(-1,1,-2) = (4-\lambda, \lambda+10, 1-2\lambda)$
$r_2 = (0,14,-7) + \mu(2,-2,4) = (2\mu, 14-2\mu, 4\mu-7)$

DIRECTION OF l_1 IS $(-1,1,-2) = -1(1,-1,2)$
DIRECTION OF l_2 IS $(2,-2,4) = 2(1,-1,2)$
IE SAME DIRECTION (POSSIBLY SAME POSSIBLY PARALLEL)

l_1 GOES THROUGH $(4,10,1)$
IF $\mu = 2$ $(2\mu, 14-2\mu, 4\mu-7) = (4,10,1)$
LINES GO THROUGH THE SAME POINT

$\therefore$ LINE l_1 IS THE SAME LINE AS l_2

NEXT: $4-\lambda = 2\mu \Rightarrow 4-2\mu = \lambda$
$\lambda+10 = 14-2\mu \Rightarrow \lambda = 4-2\mu$
$1-2\lambda = 4\mu-7 \Rightarrow 8-4\mu = 2\lambda \Longrightarrow \lambda = 4-2\mu$
$\therefore \lambda = 4-2\mu$

(b)

$\therefore |AB| : |BC| = 2 : 3$

Question 75 (****)

Relative to a fixed origin O, the points A and B have respective position vectors

$$\mathbf{a} = 2\mathbf{i} + 4\mathbf{j} + 6\mathbf{k} \quad \text{and} \quad \mathbf{b} = 4\mathbf{i} + 4\mathbf{j} - 4\mathbf{k}.$$

a) Find the position vector of the point C, given that $\mathbf{c} = \mathbf{a} + \mathbf{b}$.

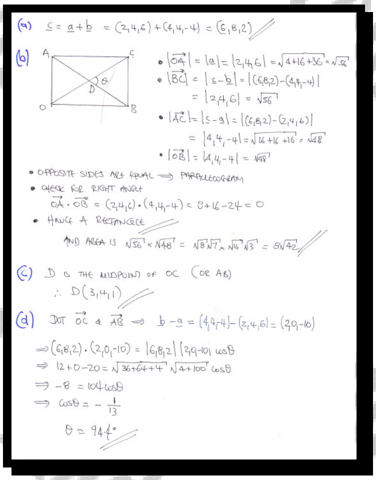
b) Show that $OACB$ is a rectangle, and calculate its area.

The diagonals of the rectangle $OACB$, OC and AB, meet at the point D.

c) State the position vector of D.

d) Calculate the size of the angle BDC.

$\boxed{\mathbf{c} = 6\mathbf{i} + 8\mathbf{j} + 2\mathbf{k}}$, $\boxed{\text{area} = 8\sqrt{42}}$, $\boxed{\mathbf{d} = 3\mathbf{i} + 4\mathbf{j} + \mathbf{k}}$, $\boxed{94.4°}$



(a) $\underline{c} = \underline{a} + \underline{b} = (2,4,6) + (4,4,-4) = (6,8,2)$

(b)

- $|\overrightarrow{OA}| = |\underline{a}| = |2,4,6| = \sqrt{4+16+36} = \sqrt{56}$
- $|\overrightarrow{BC}| = |\underline{c} - \underline{b}| = |(6,8,2)-(4,4,-4)| = |2,4,6| = \sqrt{56}$
- $|\overrightarrow{AC}| = |\underline{c} - \underline{a}| = |(6,8,2)-(2,4,6)| = |4,4,-4| = \sqrt{16+16+16} = \sqrt{48}$
- $|\overrightarrow{OB}| = |4,4,-4| = \sqrt{48}$

- OPPOSITE SIDES ARE EQUAL $\Rightarrow$ PARALLELOGRAM
- CHECK FOR RIGHT ANGLE
  $\overrightarrow{OA} \cdot \overrightarrow{OB} = (2,4,6) \cdot (4,4,-4) = 8 + 16 - 24 = 0$
- HENCE A RECTANGLE

AND AREA IS $\sqrt{56} \times \sqrt{48} = \sqrt{8}\sqrt{7} \times \sqrt{16}\sqrt{3} = 8\sqrt{42}$

(c) D IS THE MIDPOINT OF OC (OR AB)

$\therefore D(3,4,1)$

(d) DOT $\overrightarrow{OC}$ & $\overrightarrow{AB}$ $\Rightarrow$ $\underline{b} - \underline{a} = (4,4,-4) - (2,4,6) = (2,0,-10)$

$\Rightarrow (6,8,2) \cdot (2,0,-10) = |6,8,2|\,|2,0,-10| \cos\theta$

$\Rightarrow 12 + 0 - 20 = \sqrt{36+64+4}\sqrt{4+100} \cos\theta$

$\Rightarrow -8 = 104 \cos\theta$

$\Rightarrow \cos\theta = -\frac{1}{13}$

$\theta = 94.4°$

Question 76 (****)

With respect to a fixed origin O, the points A and B have coordinates $(5,-1,-1)$ and $(1,-5,7)$, respectively.

a) Find a vector equation of the straight line l which passes through A and B.

The point C has coordinates $(4,-2,1)$.

b) Show that C lies on l.

c) Show further that $\overrightarrow{OC}$ is perpendicular to l.

The point D lie on l so that $\left|\overrightarrow{CD}\right| = 2\left|\overrightarrow{CA}\right|$.

d) Find the **two** possible sets for the coordinates of D.

$\boxed{}$, $\boxed{\mathbf{r} = 5\mathbf{i} - \mathbf{j} - \mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} - 2\mathbf{k})}$, $\boxed{D(2,-4,5) \text{ or } D(6,0,-3)}$

(a) $\underline{a} = (5,-1,-1)$ $\quad \overrightarrow{AB} = \underline{b} - \underline{a} = (1,-5,7) - (5,-1,-1)$

$\underline{b} = (1,-5,7)$ $\quad = (-4,-4,8)$

SCALE DIRECTION TO $(1,1,-2)$

HENCE $\underline{r} = (5,-1,-1) + \lambda(1,1,-2)$

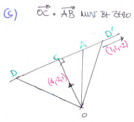
$\underline{r} = (\lambda+5, \lambda-1, -2\lambda-1)$

(b) $\underline{c} = (4,-2,1)$ IF $\lambda = -1$ $\underline{r}$ GIVES $(-1+5, -1-1, -2(-1)-1) = (4,-2,1)$

$\therefore \underline{c}$ LIES ON l

(c) $\overrightarrow{OC} \cdot \overrightarrow{AB}$ MUST BE ZERO, THUS: $(4,-2,1)\cdot(1,1,-2) = 4-2-2 = 0$

$\therefore$ INDEED PERPENDICULAR



(d) $|CD| = 2|CA|$

AT A $\quad \lambda = 0$ (BY INSPECTION)

AT C $\quad \lambda = -1$ (FOUND EARLIER)

$\therefore$ AT D $\quad \lambda = -3$ OR $\lambda = 1$

$\therefore D(2,-4,5)$

$D(6,0,-3)$

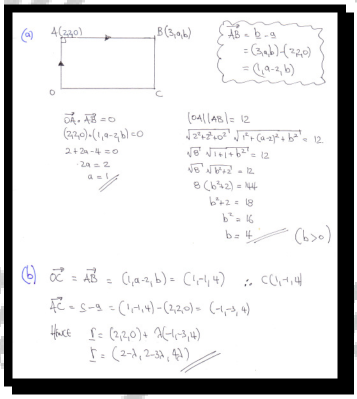
Question 77 (****)

$OABC$ is a rectangle, with $A(2,2,0)$, $B(3,a,b)$, where a and b are positive constants and O is a fixed origin.

a) Given that the area of $OABC$ is 12 square units determine the value of a and the value of b.

b) Find a vector equation of the straight line l that passes through A and C.

$\boxed{a=1}$, $\boxed{b=4}$, $\boxed{\mathbf{r}=2\mathbf{i}+2\mathbf{j}+\lambda(-\mathbf{i}-3\mathbf{j}+4\mathbf{k})}$



(a) $A(2,2,0)$ $B(3,a,b)$ O C

$\overrightarrow{AB} = \underline{b} - \underline{a}$
$= (3,a,b)-(2,2,0)$
$= (1,a-2,b)$

$\overrightarrow{OA}\cdot\overrightarrow{AB}=0$
$(2,2,0)\cdot(1,a-2,b)=0$
$2+2a-4=0$
$2a=2$
$a=1$

$|OA||AB|=12$
$\sqrt{2^2+2^2+0^2}\,\sqrt{1^2+(a-2)^2+b^2}=12.$
$\sqrt{8}\,\sqrt{1+1+b^2}=12$
$\sqrt{8}\,\sqrt{b^2+2}=12$
$8(b^2+2)=144$
$b^2+2=18$
$b^2=16$
$b=4$ $(b>0)$

(b) $\overrightarrow{OC}=\overrightarrow{AB}=(1,a-2,b)=(1,-1,4)$ $\therefore C(1,-1,4)$
$\overrightarrow{AC}=\underline{c}-\underline{a}=(1,-1,4)-(2,2,0)=(-1,-3,4)$
Hence $\underline{r}=(2,2,0)+\lambda(-1,-3,4)$
$\underline{r}=(2-\lambda,2-3\lambda,4\lambda)$

Question 78 (****)

With respect to a fixed origin O, the points A and B have coordinates $(1,5,4)$ and $(3,4,5)$, respectively.

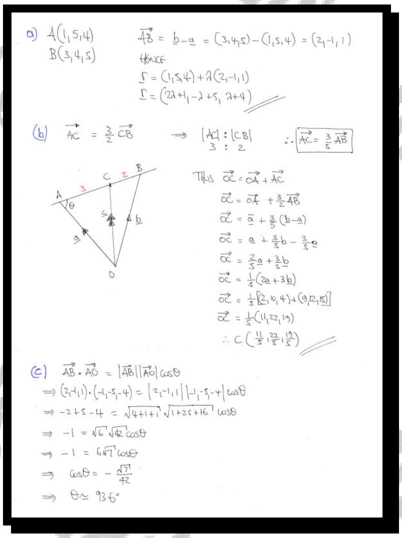
a) Find a vector equation of the straight line l that passes through A and B.

The point C lie on l so that $\overrightarrow{AC} = \frac{1}{2}\overrightarrow{CB}$.

b) Determine the coordinates of C.

c) Calculate the size of the angle OAC.

$\boxed{\mathbf{r} = \mathbf{i} + 5\mathbf{j} + 4\mathbf{k} + \lambda(2\mathbf{i} - \mathbf{j} + \mathbf{k})}$, $\boxed{C\left(\frac{11}{5}, \frac{22}{5}, \frac{19}{5}\right)}$, $\boxed{93.6°}$



a) $A(1,5,4)$ $B(3,4,5)$ $\quad \overrightarrow{AB} = \underline{b} - \underline{a} = (3,4,5) - (1,5,4) = (2,-1,1)$

HENCE

$\underline{r} = (1,5,4) + \lambda(2,-1,1)$

$\underline{r} = (2\lambda+1, -\lambda+5, \lambda+4)$

(b) $\overrightarrow{AC} = \frac{3}{2}\overrightarrow{CB} \Rightarrow |AC| : |CB|$ $\quad 3 : 2$ $\quad \therefore \boxed{\overrightarrow{AC} = \frac{3}{5}\overrightarrow{AB}}$

Thus $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC}$

$\overrightarrow{OC} = \overrightarrow{OA} + \frac{3}{2}\overrightarrow{AB}$

$\overrightarrow{OC} = \underline{a} + \frac{3}{5}(\underline{b} - \underline{a})$

$\overrightarrow{OC} = \underline{a} + \frac{3}{5}\underline{b} - \frac{3}{5}\underline{a}$

$\overrightarrow{OC} = \frac{2}{5}\underline{a} + \frac{3}{5}\underline{b}$

$\overrightarrow{OC} = \frac{1}{5}(2\underline{a} + 3\underline{b})$

$\overrightarrow{OC} = \frac{1}{5}[(2,10,4) + (9,12,15)]$

$\overrightarrow{OC} = \frac{1}{5}(11,22,19)$

$\therefore C\left(\frac{11}{5}, \frac{22}{5}, \frac{19}{5}\right)$

(c) $\overrightarrow{AB} \cdot \overrightarrow{AO} = |\overrightarrow{AB}||\overrightarrow{AO}|\cos\theta$

$\Rightarrow (2,-1,1)\cdot(-1,-5,-4) = |2,-1,1||-1,-5,-4|\cos\theta$

$\Rightarrow -2+5-4 = \sqrt{4+1+1}\sqrt{1+25+16}\cos\theta$

$\Rightarrow -1 = \sqrt{6}\sqrt{42}\cos\theta$

$\Rightarrow -1 = 6\sqrt{7}\cos\theta$

$\Rightarrow \cos\theta = -\frac{\sqrt{7}}{42}$

$\Rightarrow \theta \simeq 93.6°$

Question 79 (****)

Relative to a fixed origin O, the points A and B have position vectors $8\mathbf{i}+5\mathbf{j}+7\mathbf{k}$ and $8\mathbf{i}+6\mathbf{j}+3\mathbf{k}$, respectively.

a) Find a vector equation of the straight line l_1 which passes through A and B.

The straight line l_2 has the vector equation

$$\mathbf{r}_2 = 5\mathbf{i}+5\mathbf{j}-2\mathbf{k}+\mu(\mathbf{i}+\mathbf{j}-\mathbf{k}),$$

where μ is a scalar parameter.

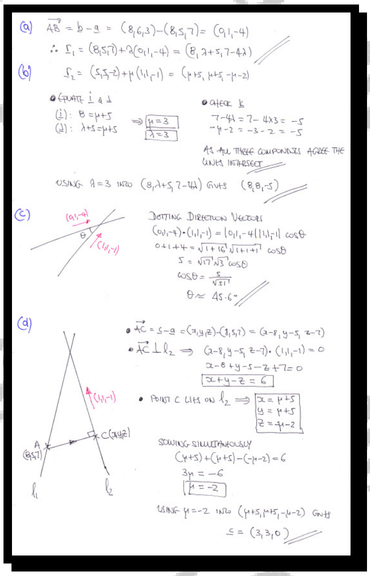
b) Show that l_1 and l_2 intersect and find the position vector of their point of intersection.

c) Calculate the acute angle between l_1 and l_2.

The point C lies on l_2 so that C is as close as possible to A.

d) Find the position vector of C.

$\boxed{\mathbf{r}_1 = 8\mathbf{i}+5\mathbf{j}+7\mathbf{k}+\lambda(\mathbf{j}-4\mathbf{k})}$, $\boxed{8\mathbf{i}+8\mathbf{j}-5\mathbf{k}}$, $\boxed{45.6°}$, $\boxed{\overrightarrow{OC} = 3\mathbf{i}+3\mathbf{j}}$



(a) $\overrightarrow{AB} = \underline{b}-\underline{a} = (8,6,3)-(8,5,7) = (0,1,-4)$

$\therefore\ \underline{r}_1 = (8,5,7)+\lambda(0,1,-4) = (8,\lambda+5,7-4\lambda)$

(b) $\underline{r}_2 = (5,5,-2)+\mu(1,1,-1) = (\mu+5,\mu+5,-\mu-2)$

• EQUATE $\underline{i}$ & $\underline{j}$

($\underline{i}$): $8 = \mu+5$

($\underline{j}$): $\lambda+5 = \mu+5$

$\Rightarrow \boxed{\mu=3}$ $\boxed{\lambda=3}$

• CHECK $\underline{k}$

$7-4\lambda = 7-4\times3 = -5$

$-\mu-2 = -3-2 = -5$

AS ALL THREE COMPONENTS AGREE THE LINES INTERSECT

USING $\lambda=3$ INTO $(8,\lambda+5,7-4\lambda)$ GIVES $(8,8,-5)$

(c) $(0,1,-4)$ θ $(1,1,-1)$

DOTTING DIRECTION VECTORS

$(0,1,-4)\cdot(1,1,-1) = |0,1,-4||1,1,-1|\cos\theta$

$0+1+4 = \sqrt{1+16}\sqrt{1+1+1}\cos\theta$

$5 = \sqrt{17}\sqrt{3}\cos\theta$

$\cos\theta = \frac{5}{\sqrt{51}}$

$\theta \approx 45.6°$

(d) $(1,1,-1)$ $A\ (8,5,7)$ $C(x,y,z)$ l_1 l_2

• $\overrightarrow{AC} = \underline{c}-\underline{a} = (x,y,z)-(8,5,7) = (x-8,y-5,z-7)$

• $\overrightarrow{AC}\perp l_2 \Rightarrow (x-8,y-5,z-7)\cdot(1,1,-1) = 0$

$x-8+y-5-z+7=0$

$\boxed{x+y-z=6}$

• POINT C LIES ON $l_2 \Rightarrow$ $\boxed{\begin{array}{l} x=\mu+5 \\ y=\mu+5 \\ z=-\mu-2 \end{array}}$

SOLVING SIMULTANEOUSLY

$(\mu+5)+(\mu+5)-(-\mu-2)=6$

$3\mu=-6$

$\boxed{\mu=-2}$

USING $\mu=-2$ INTO $(\mu+5,\mu+5,-\mu-2)$ GIVES

$\underline{c} = (3,3,0)$

Question 80 **(****)**

Relative to a fixed origin O, the position vectors of three points A, B and C are

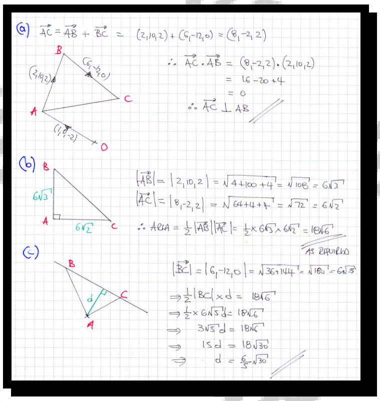
$$\overrightarrow{OA} = \mathbf{i} - 2\mathbf{k}, \quad \overrightarrow{AB} = 2\mathbf{i} + 10\mathbf{j} + 2\mathbf{k} \quad \text{and} \quad \overrightarrow{BC} = 6\mathbf{i} - 12\mathbf{j}.$$

a) Show that $\overrightarrow{AC}$ is perpendicular to $\overrightarrow{AB}$.

b) Show further that the area of the triangle ABC is $18\sqrt{6}$.

c) Hence, or otherwise, determine the shortest distance of A from the straight line through B and C.

$\boxed{}$, $\boxed{\text{distance} = \frac{6}{5}\sqrt{30}}$



(a) $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = (2,10,2) + (6,-12,0) = (8,-2,2)$

$\therefore \overrightarrow{AC} \cdot \overrightarrow{AB} = (8,-2,2) \cdot (2,10,2)$
$= 16 - 20 + 4$
$= 0$
$\therefore \overrightarrow{AC} \perp AB$

(b) $|\overrightarrow{AB}| = |2,10,2| = \sqrt{4+100+4} = \sqrt{108} = 6\sqrt{3}$
$|\overrightarrow{AC}| = |8,-2,2| = \sqrt{64+4+4} = \sqrt{72} = 6\sqrt{2}$
$\therefore \text{AREA} = \frac{1}{2}|\overrightarrow{AB}||\overrightarrow{AC}| = \frac{1}{2} \times 6\sqrt{3} \times 6\sqrt{2} = 18\sqrt{6}$
AS REQUIRED

(c) $|\overrightarrow{BC}| = |6,-12,0| = \sqrt{36+144} = \sqrt{180} = 6\sqrt{5}$
$\Rightarrow \frac{1}{2}|BC| \times d = 18\sqrt{6}$
$\Rightarrow \frac{1}{2} \times 6\sqrt{5}\, d = 18\sqrt{6}$
$\Rightarrow 3\sqrt{5}\, d = 18\sqrt{6}$
$\Rightarrow 15d = 18\sqrt{30}$
$\Rightarrow d = \frac{6}{5}\sqrt{30}$

Question 81 (****)

The straight line l_1 passes through the points $A(6,2,0)$ and $B(5,0,5)$.

a) Find a vector equation of l_1.

The straight line l_2 has vector equation

$$\mathbf{r}_2 = \begin{pmatrix} -7 \\ 6 \\ -4 \end{pmatrix} + \mu \begin{pmatrix} -5 \\ 0 \\ 2 \end{pmatrix},$$

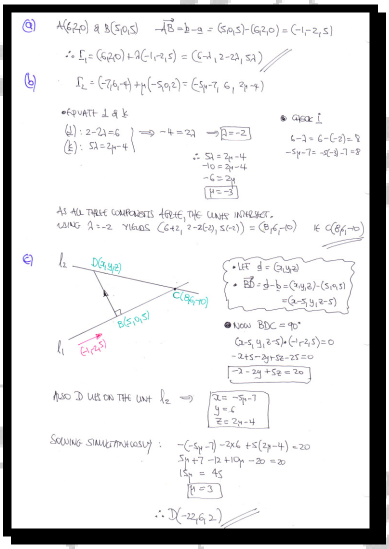
where μ is a scalar parameter.

b) Show that l_1 and l_2 intersect at some point C, and find its coordinates.

The point D lies on l_2 so that $DBC = 90°$.

c) Determine the coordinates of D.

$\boxed{}$, $\boxed{\mathbf{r} = 6\mathbf{i} + 2\mathbf{j} + \lambda(-5\mathbf{i} + 2\mathbf{k})}$, $\boxed{C(8,6,-10)}$, $\boxed{D(-22,6,2)}$



Question 82 (****)

The straight lines L_1 and L_2 have vector equations

$$\mathbf{r}_1 = \begin{pmatrix} 2 \\ 10 \\ 14 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \quad \text{and} \quad \mathbf{r}_2 = \begin{pmatrix} a \\ 8 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ b \\ 1 \end{pmatrix},$$

where λ and μ are scalar parameters, and a and b are scalar constants.

L_1 and L_2 intersect at the point P whose z coordinate is 6, and the acute angle between L_1 and L_2, is θ.

a) Determine the coordinates of P.

b) Find the value of a and the value of b.

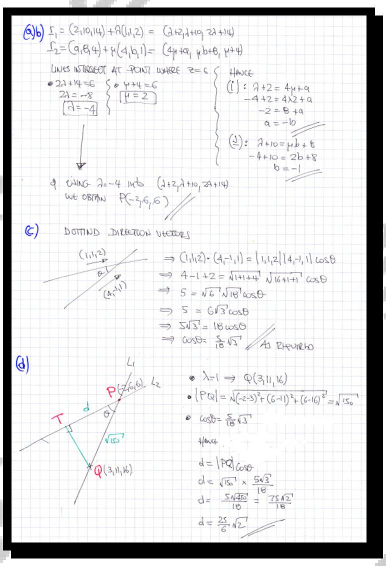
c) Show that $\cos\theta = \frac{5}{18}\sqrt{3}$.

The point Q lies on L_1 where $\lambda = 1$.

The point T lies on L_2 so that $\overrightarrow{QT}$ is perpendicular to L_2.

d) Determine the exact distance PT.

$\boxed{\quad}$, $\boxed{P(-2,6,6)}$, $\boxed{a=-10, \quad b=-1}$, $\boxed{|PT| = \frac{25}{6}\sqrt{2}}$



Question 83 (****)

The points A and B have coordinates $(3,-1,2)$ and $(2,0,2)$, respectively.

a) Find a vector equation of the straight line l_1 that passes through A and B.

The straight line l_2 has equation

$$\mathbf{r} = 2\mathbf{i} + \mathbf{j} + \mathbf{k} + \mu(\mathbf{i} - \mathbf{k}),$$

where μ is a scalar parameter.

b) Show that l_1 and l_2 intersect at some point P and find its coordinates.

c) Verify that the point $C(9,1,-6)$ lies on l_2.

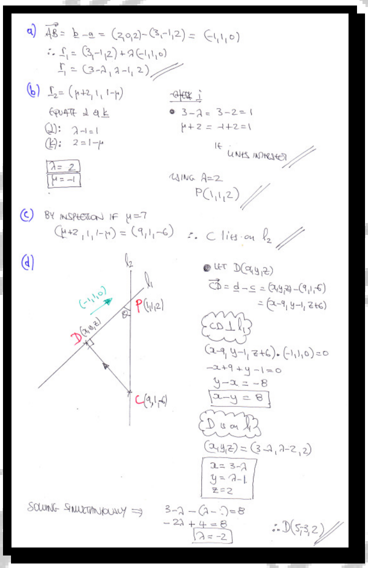
The point D lies on l_1 so that CD is perpendicular to l_1.

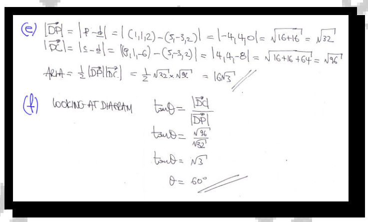
d) Determine the coordinates of D.

e) Calculate the area of the triangle PDC.

f) Deduce the acute angle between l_1 and l_2.

$\boxed{\mathbf{r}_1 = 2\mathbf{i} + 2\mathbf{k} + \lambda(-\mathbf{i} + \mathbf{j})}$, $\boxed{P(1,1,2)}$, $\boxed{D(5,-3,2)}$, $\boxed{\text{area} = 16\sqrt{3}}$, $\boxed{60°}$





Question 84 (****)

Relative to a fixed origin O, the points A and B have respective position vectors

$$2\mathbf{i}+10\mathbf{j}+2\mathbf{k} \quad \text{and} \quad 2\mathbf{i}+\mathbf{j}+2\mathbf{k}.$$

The angle AOB is θ.

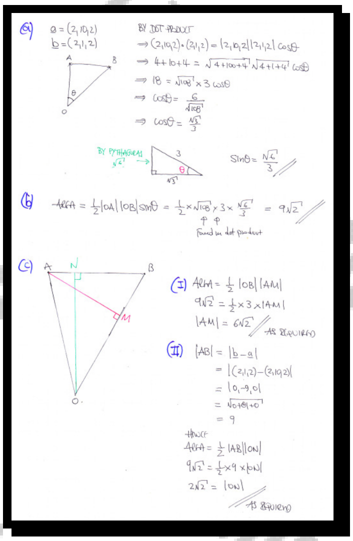
a) Show that $\sin\theta = \dfrac{\sqrt{6}}{3}$.

b) Calculate the exact area of the triangle AOB.

c) Show further that the shortest distance of …

- **i.** … A from the straight line OB is $6\sqrt{2}$,
- **ii.** … the straight line AB from O is $2\sqrt{2}$.

$\boxed{}$, $\boxed{\text{area} = 9\sqrt{2}}$



Question 85 (****)

The straight lines l_1 and l_2 have the following vector equations

$$\mathbf{r}_1 = 2\mathbf{i}+\mathbf{j}+5\mathbf{k}+\lambda(\mathbf{i}-\mathbf{k})$$

$$\mathbf{r}_2 = 2\mathbf{i}+\mathbf{j}+5\mathbf{k}+\mu(\mathbf{i}+4\mathbf{j}-\mathbf{k})$$

where λ and μ are scalar parameters.

The point A is the intersection of l_1 and l_2.

The point $B(b,1,-1)$, where b is a scalar constant, lies on l_1.

The point $D(4,d,3)$, where d is a scalar constant, lies on l_2.

a) Find the value of b and the value of d.

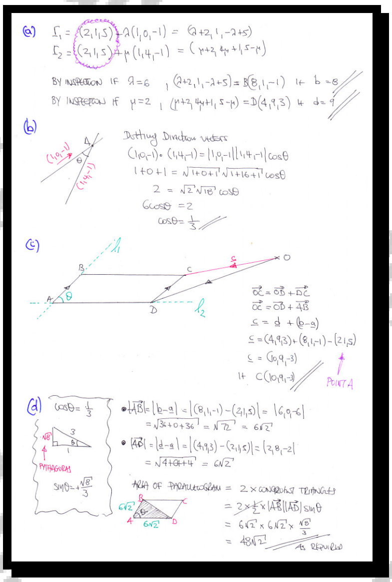
b) Calculate the cosine of θ, where θ is the acute angle formed by l_1 and l_2.

The point C is such so that $ABCD$ is a parallelogram.

c) Determine the coordinates of C.

d) Show that the area of the parallelogram $ABCD$ is $48\sqrt{2}$ square units.

$\boxed{}$, $\boxed{b=8,\ \ d=9}$, $\boxed{\cos\theta=\frac{1}{3}}$, $\boxed{C(10,9,-3)}$



Question 86 (****)

The straight line L_1 passes through the point $A(5,-2,1)$ and is parallel to the vector

$$\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}.$$

a) Find a vector equation for L_1, in terms of a scalar parameter λ.

The straight line L_2 has a vector equation

$$\mathbf{r} = \begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix},$$

where μ is scalar parameter.

b) Show that the lines intersect at some point P, and find its coordinates.

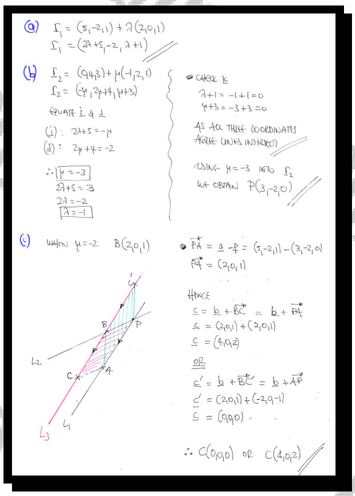
The point B lies on L_2 where $\mu = -2$.

The point C lies on a straight line which is parallel to L_1 and passes through B.

The points A, B, C and P are vertices of a parallelogram.

c) Show that one of the possible positions for C is the origin O and find the coordinates of the other possible position for C.

$\boxed{}$, $\boxed{\mathbf{r} = 5\mathbf{i} - 2\mathbf{j} + \mathbf{k} + \lambda(2\mathbf{i} + \mathbf{k})}$, $\boxed{P(3,-2,0)}$, $\boxed{C(4,0,2)}$



Question 87 (****)

The straight line L has vector equation

$$\mathbf{r} = \mathbf{i} - 2\mathbf{j} + 5\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} - \mathbf{k}),$$

where λ is a scalar parameter.

The point A has position vector $\mathbf{i} + \mathbf{j} - \mathbf{k}$.

The point P lies on L so that AP is perpendicular to L.

a) Find the position vector of P.

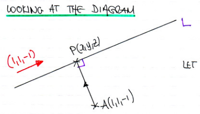
The point B is the reflection of A about L.

b) Determine the position vector of B.

$\boxed{}$, $\boxed{\overrightarrow{OP} = 4\mathbf{i} + \mathbf{j} + 2\mathbf{k}}$, $\boxed{\overrightarrow{OB} = 7\mathbf{i} + \mathbf{j} + 5\mathbf{k}}$

a) $\underline{r} = (1,-2,5) + \lambda(1,1,-1) = (\lambda+1, \lambda-2, 5-\lambda)$

LOOKING AT THE DIAGRAM



LET $P(x,y,z)$

I.E $\underline{p} = (x,y,z)$

$\overrightarrow{AP}$ IS PERPENDICULAR TO L

$\Rightarrow \overrightarrow{AP} \cdot (1,1,-1) = 0$

$\Rightarrow (\underline{p} - \underline{a}) \cdot (1,1,-1) = 0$

$\Rightarrow [(x,y,z) - (1,1,-1)] \cdot (1,1,-1) = 0$

$\Rightarrow (x-1, y-1, z+1) \cdot (1,1,-1) = 0$

$\Rightarrow x - 1 + y - 1 - z - 1 = 0$

$\Rightarrow \boxed{x + y - z = 3}$

P LIES ON L (LINE)

$\Rightarrow (x,y,z) = (\lambda+1, \lambda-2, 5-\lambda)$

$\Rightarrow \begin{pmatrix} x = \lambda+1 \\ y = \lambda-2 \\ z = 5-\lambda \end{pmatrix}$

SOLVING SIMULTANEOUSLY WE OBTAIN

$\Rightarrow (\lambda+1) + (\lambda-2) - (5-\lambda) = 3$

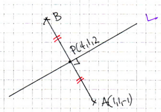
$\Rightarrow 3\lambda - 6 = 3$

$\Rightarrow 3\lambda = 9$

$\Rightarrow \underline{\underline{\lambda = 3}}$

$\therefore P(4,1,2)$

b) LOOKING AT THE DIAGRAM



P MUST BE THE MIDPOINT OF AB

BY INSPECTION

$$\begin{array}{ccccc} A & & P & & B \\ \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} & \begin{array}{c} \xrightarrow{+3} \\ \xrightarrow{+0} \\ \xrightarrow{+3} \end{array} & \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} & \begin{array}{c} \xrightarrow{+3} \\ \xrightarrow{+0} \\ \xrightarrow{+3} \end{array} & \begin{pmatrix} 7 \\ 1 \\ 5 \end{pmatrix} \end{array}$$

$\therefore B(7,1,5)$

Question 88 (****)

All the position vectors and coordinates in this question are measured from a fixed origin O.

The point P lies on the straight line l with vector equation

$$\mathbf{r} = \mathbf{i} - 3\mathbf{k} + \lambda(2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}),$$

where λ is a scalar parameter.

The point Q has position vector $3\mathbf{i} + 9\mathbf{j} + 6\mathbf{k}$.

a) Determine, in terms of λ, an expression for the vector $\overrightarrow{QP}$.

b) By considering $\left|\overrightarrow{QP}\right|^2$, find the value of λ which makes $\left|\overrightarrow{QP}\right|$ minimum.

c) Hence, or otherwise, find the shortest of distance of Q from l.

$\boxed{}$, $\boxed{\overrightarrow{QP} = (2\lambda - 2)\mathbf{i} + (3\lambda - 9)\mathbf{j} + (5\lambda - 9)\mathbf{k}}$, $\boxed{\lambda = 2}$, $\boxed{\sqrt{14}}$

$\underline{r} = (1,0,-3) + \lambda(2,3,5) = (2\lambda+1, 3\lambda, 5\lambda-3)$

a) AS P IS ON THIS LINE

$\underline{p} = (2\lambda+1, 3\lambda, 5\lambda-3)$, FOR SOME λ

$\overrightarrow{QP} = \underline{p} - \underline{q} = (2\lambda+1, 3\lambda, 5\lambda-3) - (3,9,6)$

$\overrightarrow{QP} = (2\lambda-2, 3\lambda-9, 5\lambda-9)$ //

b) LOOKING AT THE MODULUS

$\Rightarrow |\overrightarrow{QP}| = |2\lambda-2, 3\lambda-9, 5\lambda-9|$

$\Rightarrow |\overrightarrow{QP}| = \sqrt{(2\lambda-2)^2 + (3\lambda-9)^2 + (5\lambda-9)^2}$

$\Rightarrow |\overrightarrow{QP}|^2 = \begin{matrix} 4\lambda^2 - 8\lambda + 4 \\ 9\lambda^2 - 54\lambda + 81 \\ 25\lambda^2 - 90\lambda + 81 \end{matrix}$

$\Rightarrow |\overrightarrow{QP}|^2 = 38\lambda^2 - 152\lambda + 166$

BY COMPLETING THE SQUARE (OR CALCULUS)

$\Rightarrow |\overrightarrow{QP}|^2 = 38\left[\lambda^2 - 4\lambda + \frac{83}{19}\right]$

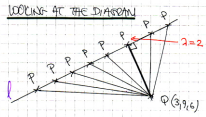
$\Rightarrow |\overrightarrow{QP}|^2 = 38\left[(\lambda-2)^2 - 4 + \frac{83}{19}\right]$

$\Rightarrow |\overrightarrow{QP}|^2 = 38(\lambda-2)^2 - 152 + 166$

$\Rightarrow |\overrightarrow{QP}|^2 = 38(\lambda-2)^2 + 14$

$\therefore \lambda = 2$

c) LOOKING AT THE DIAGRAM



THE SHORTEST DISTANCE OF Q FROM l IS $\sqrt{14}$ //

[BY LOOKING AT $|\overrightarrow{QP}| = \sqrt{38(\lambda-2)^2 + 14}$]

ALTERNATIVE BY CALCULUS

LOOKING AT $|QP|^2 = 38\lambda^2 - 152\lambda + 166$

$\Rightarrow f(\lambda) = 38\lambda^2 - 152\lambda + 166$ ← MINIMUM

$\Rightarrow f'(\lambda) = 76\lambda - 152$

$\Rightarrow 0 = 76\lambda - 152$ ← look for STATIONARY POINTS

$\Rightarrow 152 = 76\lambda$

$\Rightarrow \lambda = 2$

$\Rightarrow f(2) = 38 \times 2^2 - 152 \times 2 + 166$

$\Rightarrow f(2) = 14$

$\Rightarrow |QP|^2_{MIN} = 14$

$\Rightarrow |QP|_{MIN} = \sqrt{14}$

Question 89 (****)

The points $A(-1,4)$, $B(2,3)$ and $C(8,1)$ lie on the x-y plane, where O is the origin.

a) Show that A, B and C are collinear.

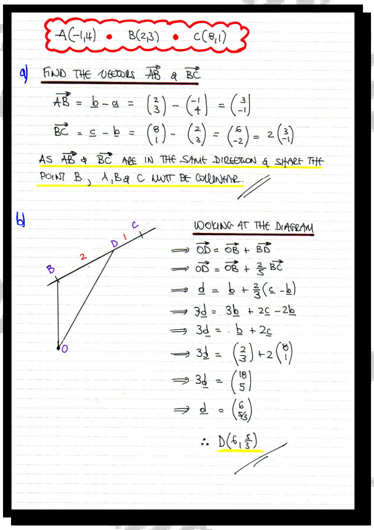
The point D lies on BC so that $\overrightarrow{BD}:\overrightarrow{BC} = 2:3$.

b) Find the coordinates of D.

The straight line OB is extended to the point P, so that $\overrightarrow{AP}$ is parallel to $\overrightarrow{OC}$.

c) Determine the coordinates of P.

$\boxed{}$, $\boxed{D\left(6,\frac{5}{3}\right)}$, $\boxed{P\left(3,\frac{9}{2}\right)}$



$A(-1,4)$ • $B(2,3)$ • $C(8,1)$

a) FIND THE VECTORS $\overrightarrow{AB}$ & $\overrightarrow{BC}$

$\overrightarrow{AB} = \underline{b} - \underline{a} = \begin{pmatrix}2\\3\end{pmatrix} - \begin{pmatrix}-1\\4\end{pmatrix} = \begin{pmatrix}3\\-1\end{pmatrix}$

$\overrightarrow{BC} = \underline{c} - \underline{b} = \begin{pmatrix}8\\1\end{pmatrix} - \begin{pmatrix}2\\3\end{pmatrix} = \begin{pmatrix}6\\-2\end{pmatrix} = 2\begin{pmatrix}3\\-1\end{pmatrix}$

AS $\overrightarrow{AB}$ & $\overrightarrow{BC}$ ARE IN THE SAME DIRECTION & SHARE THE POINT B, A, B & C MUST BE COLLINEAR.

b)

LOOKING AT THE DIAGRAM

$\Rightarrow \overrightarrow{OD} = \overrightarrow{OB} + \overrightarrow{BD}$

$\Rightarrow \overrightarrow{OD} = \overrightarrow{OB} + \frac{2}{3}\overrightarrow{BC}$

$\Rightarrow \underline{d} = \underline{b} + \frac{2}{3}(\underline{c} - \underline{b})$

$\Rightarrow 3\underline{d} = 3\underline{b} + 2\underline{c} - 2\underline{b}$

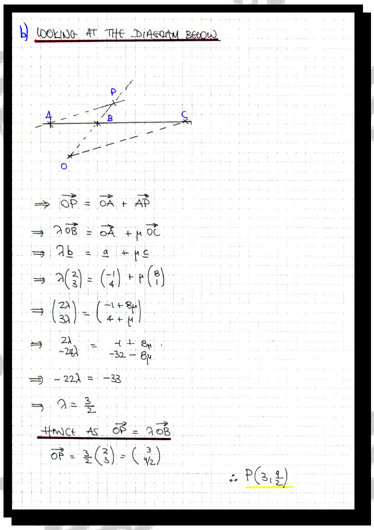
$\Rightarrow 3\underline{d} = \underline{b} + 2\underline{c}$

$\Rightarrow 3\underline{d} = \begin{pmatrix}2\\3\end{pmatrix} + 2\begin{pmatrix}8\\1\end{pmatrix}$

$\Rightarrow 3\underline{d} = \begin{pmatrix}18\\5\end{pmatrix}$

$\Rightarrow \underline{d} = \begin{pmatrix}6\\5/3\end{pmatrix}$

$\therefore D\left(6, \frac{5}{3}\right)$



b) LOOKING AT THE DIAGRAM BELOW

$\Rightarrow \overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$

$\Rightarrow \lambda\overrightarrow{OB} = \overrightarrow{OA} + \mu\overrightarrow{OC}$

$\Rightarrow \lambda\underline{b} = \underline{a} + \mu\underline{c}$

$\Rightarrow \lambda\begin{pmatrix}2\\3\end{pmatrix} = \begin{pmatrix}-1\\4\end{pmatrix} + \mu\begin{pmatrix}8\\1\end{pmatrix}$

$\Rightarrow \begin{pmatrix}2\lambda\\3\lambda\end{pmatrix} = \begin{pmatrix}-1+8\mu\\4+\mu\end{pmatrix}$

$\Rightarrow \begin{matrix}2\lambda\\-24\lambda\end{matrix} = \begin{matrix}-1+8\mu\\-32-8\mu\end{matrix}$

$\Rightarrow -22\lambda = -33$

$\Rightarrow \lambda = \frac{3}{2}$

HENCE AS $\overrightarrow{OP} = \lambda\overrightarrow{OB}$

$\overrightarrow{OP} = \frac{3}{2}\begin{pmatrix}2\\3\end{pmatrix} = \begin{pmatrix}3\\9/2\end{pmatrix}$

$\therefore P\left(3, \frac{9}{2}\right)$

Question 90 (****)

Relative to a fixed origin O, the points A, B and C have respective position vectors

$$-6\mathbf{i}-5\mathbf{j}-21\mathbf{k}, \quad 8\mathbf{i}+9\mathbf{j} \quad \text{and} \quad u\mathbf{i}-3\mathbf{j}+v\mathbf{k},$$

where u and v are scalar constants.

A, B and C lie on the straight line l.

a) Find a vector equation of l.

b) Determine the value of u and the value of v.

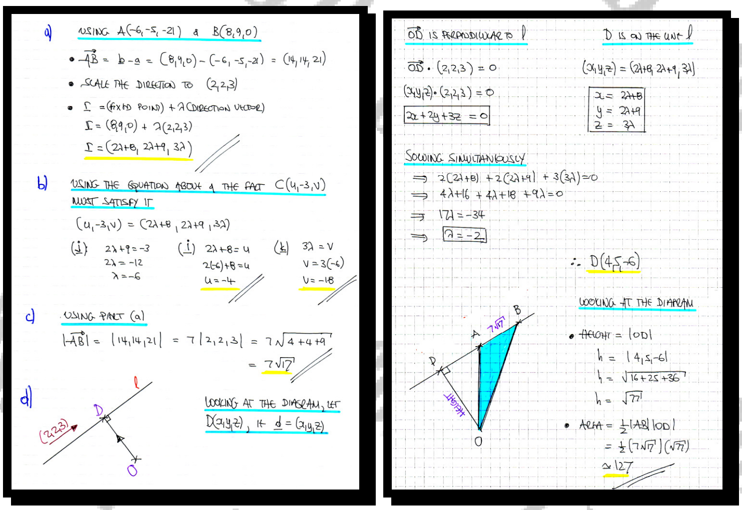
c) Calculate the distance AB.

The point D lies on l so that $\overrightarrow{OD}$ is perpendicular to l.

d) Determine the position vector of D.

e) Calculate, correct to three significant figures, the area of the triangle OAB.

$\boxed{\mathbf{r}=8\mathbf{i}+9\mathbf{j}+\lambda(2\mathbf{i}+2\mathbf{j}+3\mathbf{k})}$, $\boxed{u=-4}$, $\boxed{v=-18}$, $\boxed{|AB|=7\sqrt{17}}$, $\boxed{\mathbf{d}=4\mathbf{i}+5\mathbf{j}-6\mathbf{k}}$, $\boxed{\text{area}\approx 127}$



Question 91 (****)

The position vectors and coordinates in this question are relative to a fixed origin O.

The straight lines l_1, l_2 and l_3 have the following vector equations

$$\mathbf{r}_1 = 10\mathbf{i} + 6\mathbf{j} + 9\mathbf{k} + \lambda(3\mathbf{i} + \mathbf{j} + 4\mathbf{k})$$

$$\mathbf{r}_2 = -4\mathbf{j} + 13\mathbf{k} + \mu(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})$$

$$\mathbf{r}_3 = -3\mathbf{i} - 4\mathbf{k} + \nu(4\mathbf{i} + 3\mathbf{j} + \mathbf{k}),$$

where λ, μ and ν are scalar parameters.

a) Show that l_1 and l_2 intersect at some point A, and find its coordinates.

b) Verify that $B(5,6,-2)$ lies on both l_2 and l_3.

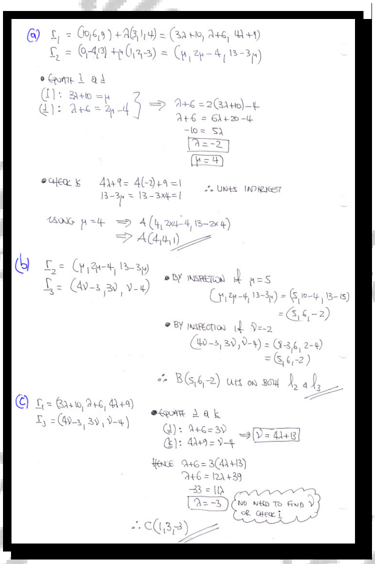
The point C is the intersection of l_1 and l_3.

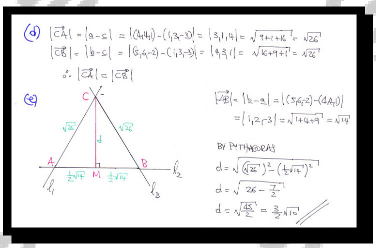
c) Find the coordinates of C.

d) Show that $|CA| = |CB|$.

e) Hence calculate the shortest distance of C from l_2.

$\boxed{A(4,4,1)}$, $\boxed{C(1,3,-3)}$, $\boxed{\text{distance} = \frac{3}{2}\sqrt{10}}$





Question 92 (****)

The position vectors and coordinates in this question are relative to a fixed origin O.

The straight lines l_1 and l_2 have the following vector equations

$$\mathbf{r}_1 = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k} + \lambda(4\mathbf{i} + 4\mathbf{j} + 3\mathbf{k})$$

$$\mathbf{r}_2 = 9\mathbf{i} + \mu(\mathbf{i} - 3\mathbf{j} + a\mathbf{k})$$

where λ and μ are scalar parameters, and a is a scalar constant.

The point A is the intersection between l_1 and l_2, and the acute angle between them is denoted by θ.

a) Find in any order …

i. … the value of a.

ii. … the coordinates of A.

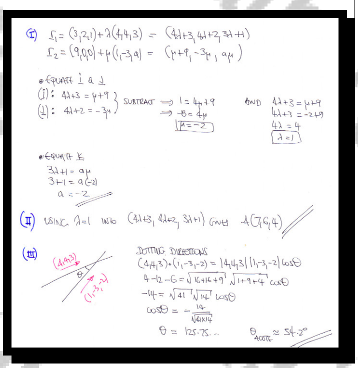
iii. … the value of θ.

The point B has coordinates $(5,13,11)$.

The point P lies on l_1 so that the angle APB is $90°$.

b) Calculate the distance BP.

$\boxed{a=-2}$, $\boxed{A(7,6,4)}$, $\boxed{\theta = 54.2°}$, $\boxed{|BP| = \sqrt{61}}$

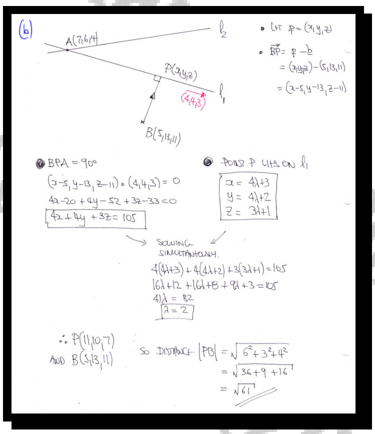

(I) $r_1 = (3,2,1) + \lambda(4,4,3) = (4\lambda+3, 4\lambda+2, 3\lambda+1)$
$r_2 = (9,0,0) + \mu(1,-3,a) = (\mu+9, -3\mu, a\mu)$

• EQUATE i & j
(i): $4\lambda+3 = \mu+9$
(j): $4\lambda+2 = -3\mu$
SUBTRACT $\Rightarrow 1 = 4\mu+9 \Rightarrow -8 = 4\mu$, $\boxed{\mu=-2}$
AND $4\lambda+3 = \mu+9$, $4\lambda+3 = -2+9$, $4\lambda = 4$, $\boxed{\lambda=1}$

• EQUATE k
$3\lambda+1 = a\mu$
$3+1 = a(-2)$
$a = -2$

(II) USING $\lambda=1$ INTO $(4\lambda+3, 4\lambda+2, 3\lambda+1)$ GIVES $A(7,6,4)$

(III) $(4,4,3)$ θ $(1,-3,-2)$
DOTTING DIRECTIONS
$(4,4,3)\cdot(1,-3,-2) = |4,4,3||1,-3,-2|\cos\theta$
$4-12-6 = \sqrt{16+16+9}\sqrt{1+9+4}\cos\theta$
$-14 = \sqrt{41}\sqrt{14}\cos\theta$
$\cos\theta = -\frac{14}{\sqrt{41\times14}}$
$\theta = 125.75\ldots$ $\quad \theta_{ACUTE} \approx 54.2°$


(b) $A(7,6,4)$ l_2 $P(x,y,z)$ l_1 $(4,4,3)$ $B(5,13,11)$
• LET $p = (x,y,z)$
• $\vec{BP} = p - b = (x,y,z) - (5,13,11) = (x-5, y-13, z-11)$

• BPA = 90°
$(x-5, y-13, z-11)\cdot(4,4,3) = 0$
$4x-20+4y-52+3z-33=0$
$\boxed{4x+4y+3z=105}$

• POINT P LIES ON l_1
$x = 4\lambda+3$
$y = 4\lambda+2$
$z = 3\lambda+1$

SOLVING SIMULTANEOUSLY.
$4(4\lambda+3)+4(4\lambda+2)+3(3\lambda+1) = 105$
$16\lambda+12+16\lambda+8+9\lambda+3 = 105$
$41\lambda = 82$
$\boxed{\lambda=2}$

$\therefore P(11,10,7)$ AND $B(5,13,11)$
SO DISTANCE $|PB| = \sqrt{6^2+3^2+4^2} = \sqrt{36+9+16} = \sqrt{61}$

Question 93 (****)

The position vectors and coordinates in this question are relative to a fixed origin O.

The points A and B have respective position vectors

$$2\mathbf{i}+3\mathbf{j} \quad \text{and} \quad 6\mathbf{i}-2\mathbf{j}+3\mathbf{k}.$$

a) Find a vector equation of the straight line l_1 that passes through A and B.

The straight line l_2 has vector equation

$$\mathbf{r}=5\mathbf{i}-3\mathbf{j}+6\mathbf{k}+\mu(\mathbf{i}-2\mathbf{j}+2\mathbf{k}),$$

where μ is a scalar parameter.

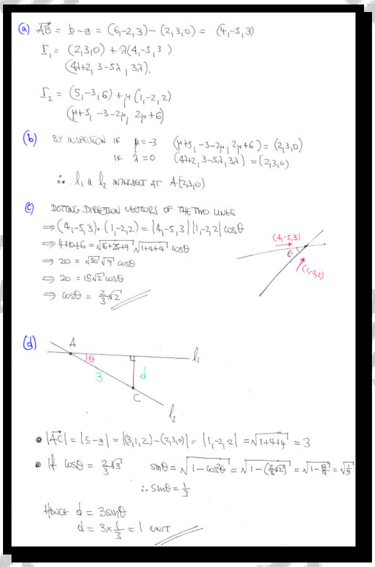
b) Show that l_1 and l_2 intersect at the point A.

c) Find the exact value of $\cos\theta$, where θ is the acute angle between l_1 and l_2.

The point C with position vector $3\mathbf{i}+\mathbf{j}+2\mathbf{k}$ lies on l_2.

d) Show that the shortest distance from C to l_1 is exactly one unit.

$\boxed{\mathbf{r}=2\mathbf{i}+3\mathbf{j}+\lambda(4\mathbf{i}-5\mathbf{j}+3\mathbf{k})}$, $\boxed{\cos\theta=\frac{2}{3}\sqrt{2}}$



(a) $\overrightarrow{AB} = \underline{b}-\underline{a} = (6,-2,3)-(2,3,0) = (4,-5,3)$

$\underline{r}_1 = (2,3,0)+\lambda(4,-5,3)$

$(4\lambda+2,\ 3-5\lambda,\ 3\lambda)$.

$\underline{r}_2 = (5,-3,6)+\mu(1,-2,2)$

$(\mu+5,\ -3-2\mu,\ 2\mu+6)$

(b) BY INSPECTION IF $\mu=-3$ $(\mu+5, -3-2\mu, 2\mu+6)=(2,3,0)$

IF $\lambda=0$ $(4\lambda+2, 3-5\lambda, 3\lambda)=(2,3,0)$

$\therefore$ l_1 & l_2 INTERSECT AT $A(2,3,0)$

(c) DOTTING DIRECTION VECTORS OF THE TWO LINES

$\Rightarrow (4,-5,3)\cdot(1,-2,2)=|4,-5,3||1,-2,2|\cos\theta$

$\Rightarrow 4+10+6=\sqrt{16+25+9}\sqrt{1+4+4}\cos\theta$

$\Rightarrow 20=\sqrt{50}\sqrt{9}\cos\theta$

$\Rightarrow 20=15\sqrt{2}\cos\theta$

$\Rightarrow \cos\theta=\frac{2}{3}\sqrt{2}$

(d)

- $|\overrightarrow{AC}|=|\underline{c}-\underline{a}|=|(3,1,2)-(2,3,0)|=|1,-2,2|=\sqrt{1+4+4}=3$
- If $\cos\theta=\frac{2}{3}\sqrt{3}$ $\quad\sin\theta=\sqrt{1-\cos^2\theta}=\sqrt{1-\left(\frac{2}{3}\sqrt{2}\right)^2}=\sqrt{1-\frac{8}{9}}=\sqrt{\frac{1}{9}}$

$\therefore \sin\theta=\frac{1}{3}$

HENCE $d=3\sin\theta$

$d=3\times\frac{1}{3}=1$ UNIT

Question 94 (****)

The position vectors and coordinates in this question are relative to a fixed origin O.

The straight line l has vector equation

$$\mathbf{r} = \begin{pmatrix} 10 \\ 7 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix},$$

where λ is a scalar parameter.

The point $P(14,15,15)$ lies on l and the point A has coordinates $(5,1,2)$.

a) Calculate the size of the acute angle that AP makes with l.

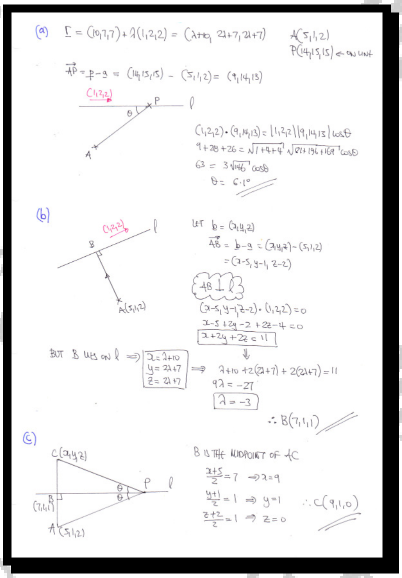
The point B lies on l so that $ABP = 90°$.

b) Determine the coordinates of B.

The point C is such so that l is the angle bisector of APC.

c) Find a set of the possible coordinates of C.

$\boxed{\theta \approx 6.1°}$, $\boxed{B(7,1,1)}$, $\boxed{C(9,1,0)}$


(a) $\underline{r} = (10,7,7) + \lambda(1,2,2) = (\lambda+10, 2\lambda+7, 2\lambda+7)$ $A(5,1,2)$

$P(14,15,15)$ ← ON LINE

$\overrightarrow{AP} = \underline{p} - \underline{a} = (14,15,15) - (5,1,2) = (9,14,13)$

$(1,2,2) \cdot (9,14,13) = |1,2,2||9,14,13| \cos\theta$

$9 + 28 + 26 = \sqrt{1+4+4}\sqrt{81+196+169}\cos\theta$

$63 = 3\sqrt{446}\cos\theta$

$\theta = 6.1°$

(b) LET $\underline{b} = (x,y,z)$

$\overrightarrow{AB} = \underline{b} - \underline{a} = (x,y,z) - (5,1,2)$

$= (x-5, y-1, z-2)$

$AB \perp l$

$(x-5, y-1, z-2) \cdot (1,2,2) = 0$

$x - 5 + 2y - 2 + 2z - 4 = 0$

$\boxed{x + 2y + 2z = 11}$

BUT B LIES ON l $\Rightarrow$ $\boxed{\begin{array}{l} x = \lambda+10 \\ y = 2\lambda+7 \\ z = 2\lambda+7 \end{array}}$ $\Rightarrow$ $\lambda + 10 + 2(2\lambda+7) + 2(2\lambda+7) = 11$

$9\lambda = -27$

$\boxed{\lambda = -3}$

$\therefore B(7,1,1)$

(c) B IS THE MIDPOINT OF AC

$\frac{x+5}{2} = 7 \Rightarrow x = 9$

$\frac{y+1}{2} = 1 \Rightarrow y = 1$

$\frac{z+2}{2} = 1 \Rightarrow z = 0$

$\therefore C(9,1,0)$

Question 95 (****)

The position vectors and coordinates in this question are relative to a fixed origin O.

The points A, B and C have coordinates $(0,0,8)$, $(2,6,4)$ and $(8,8,0)$, respectively.

The point D is such so that $ABCD$ is a parallelogram.

The angle ABC is θ.

a) Determine the coordinates of D.

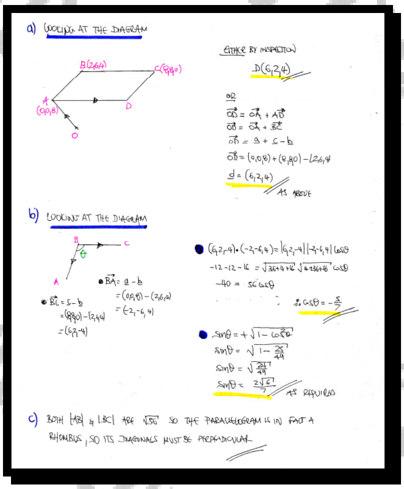
b) Use the scalar product to find an exact value for $\cos\theta$ and hence show

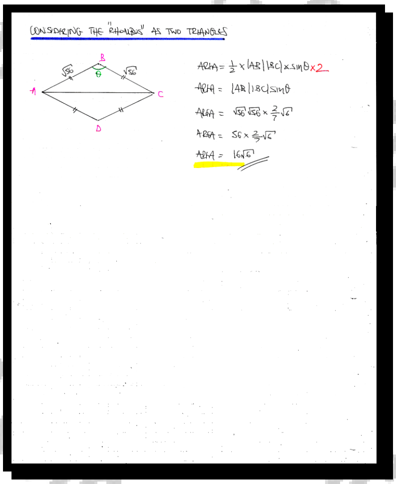
$$\sin\theta = \frac{2}{7}\sqrt{6}.$$

c) Explain, with reference to the calculations of part **(b)**, why AC must be perpendicular to BD.

d) Show that the area of the parallelogram is $16\sqrt{6}$.

$\boxed{}$, $\boxed{D(6,2,4)}$, $\boxed{\cos\theta = -\frac{5}{7}}$





Question 96 (****)

The position vectors and coordinates in this question are relative to a fixed origin O.

The straight lines l_1 and l_2 have the following vector equations

$$\mathbf{r}_1 = 5\mathbf{i} + 3\mathbf{j} + 6\mathbf{k} + \lambda(2\mathbf{i} + \mathbf{j} + 2\mathbf{k})$$

$$\mathbf{r}_2 = -\mathbf{i} + 5\mathbf{j} + a\mathbf{k} + \mu(\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}),$$

where λ and μ are scalar parameters, and a is a scalar constant.

The point A lies on both l_1 and l_2.

a) Find the value of a and the coordinates of A.

The point $P(11, p, 12)$, where p is a scalar constant, lies on l_1.

The point $Q(q, -9, -8)$, where q is a scalar constant, lies on l_2.

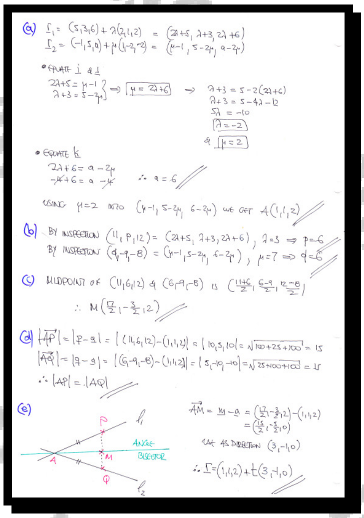
b) Find the value of p and the value of q.

c) Determine the coordinates of the midpoint of PQ.

d) Show that $|AP| = |AQ|$.

e) Hence, or otherwise, find a vector equation of the angle bisector of $\measuredangle PAQ$.

$\boxed{}$, $\boxed{a = 6}$, $\boxed{A(1,1,2)}$, $\boxed{p = 6,\ q = 6}$, $\boxed{M\left(\frac{17}{2}, -\frac{3}{2}, 2\right)}$, $\boxed{\mathbf{r} = \mathbf{i} + \mathbf{j} + 2\mathbf{k} + t(3\mathbf{i} - \mathbf{j})}$


(a) $\underline{r}_1 = (5,3,6) + \lambda(2,1,2) = (2\lambda+5, \lambda+3, 2\lambda+6)$
$\underline{r}_2 = (-1,5,a) + \mu(1,-2,-2) = (\mu-1, 5-2\mu, a-2\mu)$

• EQUATE $\underline{i}$ & $\underline{j}$

$\left.\begin{array}{l} 2\lambda+5 = \mu-1 \\ \lambda+3 = 5-2\mu \end{array}\right\} \Rightarrow \boxed{\mu = 2\lambda+6} \Rightarrow \lambda+3 = 5-2(2\lambda+6)$

$\lambda+3 = 5-4\lambda-12$

$5\lambda = -10$

$\boxed{\lambda = -2}$

& $\boxed{\mu = 2}$

• EQUATE $\underline{k}$

$2\lambda+6 = a-2\mu$

$\cancel{-4}+6 = a\cancel{-4}$ $\quad \therefore a = 6$

USING $\mu=2$ INTO $(\mu-1, 5-2\mu, 6-2\mu)$ WE GET $A(1,1,2)$

(b) BY INSPECTION $(11,p,12) = (2\lambda+5, \lambda+3, 2\lambda+6)$, $\lambda=3 \Rightarrow p=6$
BY INSPECTION $(q,-9,-8) = (\mu-1, 5-2\mu, 6-2\mu)$, $\mu=7 \Rightarrow q=6$

(c) MIDPOINT OF $(11,6,12)$ & $(6,-9,-8)$ IS $\left(\frac{11+6}{2}, \frac{6-9}{2}, \frac{12-8}{2}\right)$

$\therefore M\left(\frac{17}{2}, -\frac{3}{2}, 2\right)$

(d) $|\overrightarrow{AP}| = |\underline{p}-\underline{a}| = |(11,6,12)-(1,1,2)| = |10,5,10| = \sqrt{100+25+100} = 15$
$|\overrightarrow{AQ}| = |\underline{q}-\underline{a}| = |(6,-9,-8)-(1,1,2)| = |5,-10,-10| = \sqrt{25+100+100} = 15$
$\therefore |AP| = |AQ|$

(e)

$\overrightarrow{AM} = \underline{m}-\underline{a} = \left(\frac{17}{2}, -\frac{3}{2}, 2\right)-(1,1,2)$
$= \left(\frac{15}{2}, -\frac{5}{2}, 0\right)$

USE AS DIRECTION $(3,-1,0)$

$\therefore \underline{r} = (1,1,2) + t(3,-1,0)$

Question 97 (****)

The points with coordinates $A(4,0,-4)$ and $B(5,-1,-6)$ lie on the line L, where the point O is a fixed origin.

a) Find a vector equation of the line L.

b) Find the distance between the points A and B.

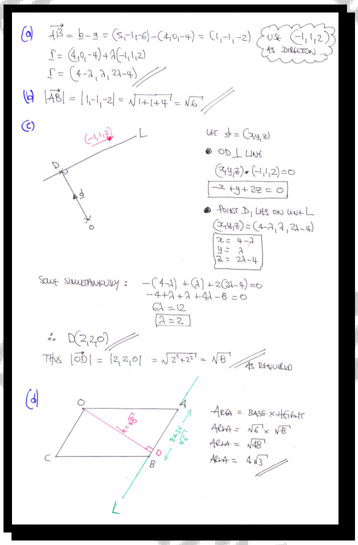
The point D lies on the line L, so that OD is perpendicular to L.

c) Find the coordinates of the point D, and hence show that $\left|\overrightarrow{OD}\right| = \sqrt{8}$

The point C is such so that $OABC$ is parallelogram.

d) Find the exact area of the parallelogram $OABC$.

$\boxed{\mathbf{r} = 4\mathbf{i} - 4\mathbf{k} + \lambda(-\mathbf{i} + \mathbf{j} + 2\mathbf{k})}$, $\boxed{|AB| = \sqrt{6}}$, $\boxed{D(2,2,0)}$, $\boxed{\text{area} = 4\sqrt{3}}$


(a) $\overrightarrow{AB} = \underline{b} - \underline{a} = (5,-1,-6) - (4,0,-4) = (1,-1,-2)$ USE $(-1,1,2)$ AS DIRECTION

$\underline{r} = (4,0,-4) + \lambda(-1,1,2)$

$\underline{r} = (4-\lambda, \lambda, 2\lambda-4)$

(b) $|\overrightarrow{AB}| = |1,-1,-2| = \sqrt{1+1+4} = \sqrt{6}$

(c) LET $\underline{d} = (x,y,z)$

- $OD \perp$ LINE

$(x,y,z) \cdot (-1,1,2) = 0$

$\boxed{-x + y + 2z = 0}$

- POINT D, LIES ON LINE L

$(x,y,z) = (4-\lambda, \lambda, 2\lambda-4)$

$x = 4-\lambda$, $y = \lambda$, $z = 2\lambda - 4$

SOLVE SIMULTANEOUSLY: $-(4-\lambda) + (\lambda) + 2(2\lambda-4) = 0$

$-4 + \lambda + \lambda + 4\lambda - 8 = 0$

$6\lambda = 12$

$\boxed{\lambda = 2}$

$\therefore D(2,2,0)$

THUS $|\overrightarrow{OD}| = |2,2,0| = \sqrt{2^2+2^2} = \sqrt{8}$ AS REQUIRED

(d) AREA = BASE × HEIGHT

AREA = $\sqrt{6} \times \sqrt{8}$

AREA = $\sqrt{48}$

AREA = $4\sqrt{3}$

Question 98 **(****)**

The lines l_1 passes through the point $A(2,2,-2)$ and has direction vector $\mathbf{i}+\mathbf{j}+2\mathbf{k}$.

a) Find a vector equation for l_1.

The line l_2 has the vector equation

$$\mathbf{r}_2 = 7\mathbf{i}+\mathbf{j}-2\mathbf{k}+\mu(2\mathbf{i}-\mathbf{j}-\mathbf{k}),$$

where μ is a scalar parameter.

The lines intersect at the point B and the acute angle between the two lines is θ.

b) Find the coordinates of B.

c) Show that $\cos\theta = \frac{1}{6}$.

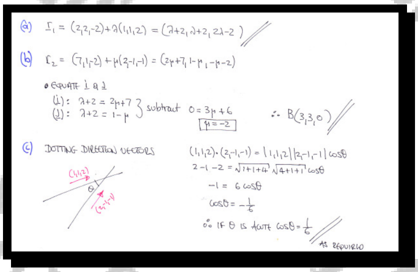
The point $P(15,-3,-6)$ lies on l_2 and the point Q lies on l_1 so that $\measuredangle PQB = 90°$.

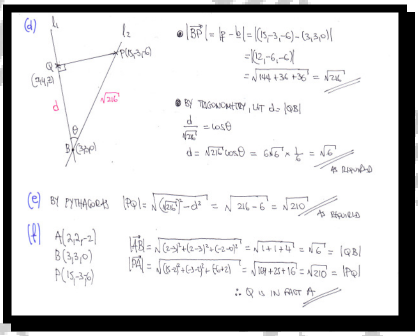
d) Find $|\overrightarrow{BP}|$ and show that $|\overrightarrow{BQ}| = \sqrt{6}$.

e) Show that $|\overrightarrow{PQ}| = \sqrt{210}$.

f) Verify that the point Q is in fact the same the point as A.

$\boxed{\mathbf{r}_1 = 2\mathbf{i}+2\mathbf{j}-2\mathbf{k}+\lambda(\mathbf{i}+\mathbf{j}+2\mathbf{k})}$, $\boxed{B(3,3,0)}$ $\boxed{|\overrightarrow{BP}| = \sqrt{216} = 6\sqrt{6}}$





Question 99 (****)

With respect to a fixed origin, the points A, B and C have coordinates $(-2,-4,6)$, $(-16,1,4)$ and $(4,8,-6)$, respectively.

a) Find a vector equation for the line L_1, through the points A and B.

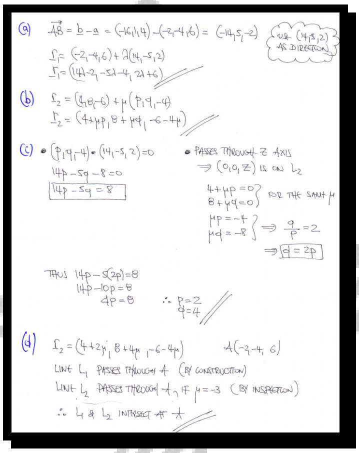
b) Find a vector equation for the line L_2, that passes through the point C and is parallel to the vector $p\mathbf{i}+q\mathbf{j}-4\mathbf{k}$, where p and q are scalar constants.

The line L_2 passes through the z axis, and is perpendicular to L_1.

c) Find the values of p and q.

d) Verify that L_1 and L_2 lines intersect at the point A.

$\boxed{\mathbf{r}_1 = -2\mathbf{i}-4\mathbf{j}+6\mathbf{k}+\lambda(14\mathbf{i}-5\mathbf{j}+2\mathbf{k})}$, $\boxed{\mathbf{r}_2 = 4\mathbf{i}+8\mathbf{j}-6\mathbf{k}+\lambda(p\mathbf{i}+q\mathbf{j}-4\mathbf{k})}$, $\boxed{p=2,\ \ q=4}$



(a) $\vec{AB} = \underline{b}-\underline{a} = (-16,1,4)-(-2,-4,6) = (-14,5,-2)$

USE $(14,-5,2)$ AS DIRECTION

$\underline{r}_1 = (-2,-4,6)+\lambda(14,-5,2)$

$\underline{r}_1 = (14\lambda-2, -5\lambda-4, 2\lambda+6)$

(b) $\underline{r}_2 = (4,8,-6)+\mu(p,q,-4)$

$\underline{r}_2 = (4+\mu p, 8+\mu q, -6-4\mu)$

(c) • $(p,q,-4)\cdot(14,-5,2)=0$

$14p-5q-8=0$

$\boxed{14p-5q=8}$

• PASSES THROUGH z AXIS

$\Rightarrow (0,0,z)$ IS ON L_2

$\left.\begin{array}{l}4+\mu p=0\\8+\mu q=0\end{array}\right\}$ FOR THE SAME μ

$\left.\begin{array}{l}\mu p=-4\\\mu q=-8\end{array}\right\} \Rightarrow \frac{q}{p}=2$

$\Rightarrow \boxed{q=2p}$

THUS $14p-5(2p)=8$

$14p-10p=8$

$4p=8$

$\therefore p=2$, $q=4$

(d) $\underline{r}_2 = (4+2\mu, 8+4\mu, -6-4\mu)$ $\quad A(-2,-4,6)$

LINE L_1 PASSES THROUGH A (BY CONSTRUCTION)

LINE L_2 PASSES THROUGH A, IF $\mu=-3$ (BY INSPECTION)

$\therefore L_1$ & L_2 INTERSECT AT A

Question 100 (****)

The straight lines L and M have the following vector equations

$$L: \mathbf{r}_1 = (3\lambda + 3)\mathbf{i} + (4 - 4\lambda)\mathbf{j} + 2\lambda\mathbf{k}$$

$$M: \mathbf{r}_2 = (3\mu + 12)\mathbf{i} + (20 - 4\mu)\mathbf{j} + (2\mu + 4)\mathbf{k},$$

where λ and μ are scalar parameters.

a) Show that L and M are parallel.

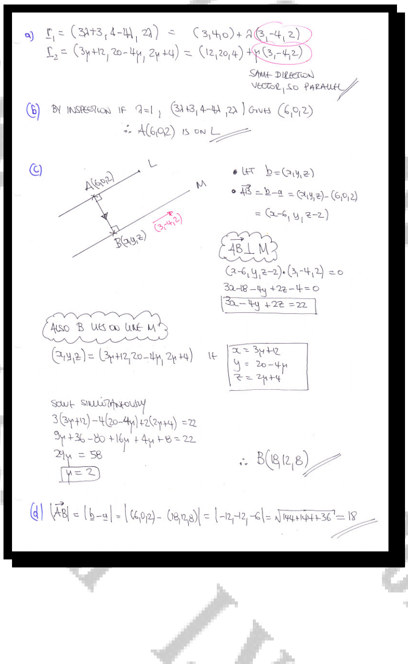
b) Show further that $A(6,0,2)$ lies on L.

The point B lies on M so that AB is perpendicular to M.

c) Find the coordinates of B.

d) Hence determine the shortest distance between L and M.

$\boxed{B(18,12,8)}$, $\boxed{\text{distance} = 18 \text{ units}}$



a) $\underline{r}_1 = (3\lambda+3, 4-4\lambda, 2\lambda) = (3,4,0) + \lambda(3,-4,2)$
$\underline{r}_2 = (3\mu+12, 20-4\mu, 2\mu+4) = (12,20,4) + \mu(3,-4,2)$

SAME DIRECTION VECTOR, SO PARALLEL

(b) BY INSPECTION IF $\lambda = 1$, $(3\lambda+3, 4-4\lambda, 2\lambda)$ GIVES $(6,0,2)$

$\therefore A(6,0,2)$ IS ON L

(c)

• LET $\underline{b} = (x,y,z)$

• $\vec{AB} = \underline{b} - \underline{a} = (x,y,z) - (6,0,2)$
$= (x-6, y, z-2)$

$\vec{AB} \perp M$

$(x-6, y, z-2) \cdot (3,-4,2) = 0$
$3x - 18 - 4y + 2z - 4 = 0$
$\boxed{3x - 4y + 2z = 22}$

ALSO B LIES ON LINE M

$(x,y,z) = (3\mu+12, 20-4\mu, 2\mu+4)$ i.e. $\boxed{\begin{aligned} x &= 3\mu+12 \\ y &= 20-4\mu \\ z &= 2\mu+4 \end{aligned}}$

SOLVE SIMULTANEOUSLY

$3(3\mu+12) - 4(20-4\mu) + 2(2\mu+4) = 22$
$9\mu + 36 - 80 + 16\mu + 4\mu + 8 = 22$
$29\mu = 58$
$\boxed{\mu = 2}$

$\therefore B(18,12,8)$

(d) $|\vec{AB}| = |\underline{b} - \underline{a}| = |(6,0,2) - (18,12,8)| = |-12,-12,-6| = \sqrt{144+144+36} = 18$

Question 101 (****)

The straight lines l_1 and l_2 have the following vector equations

$$\mathbf{r}_1 = -2\mathbf{i} - \mathbf{j} - 2\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j})$$

$$\mathbf{r}_2 = 8\mathbf{i} - 5\mathbf{j} + 26\mathbf{k} + \mu(\mathbf{i} - \mathbf{j} + 4\mathbf{k})$$

where λ and μ are scalar parameters.

a) Given l_1 and l_2 intersect at the point P, find the coordinates of P.

b) Show l_1 and l_2 are perpendicular.

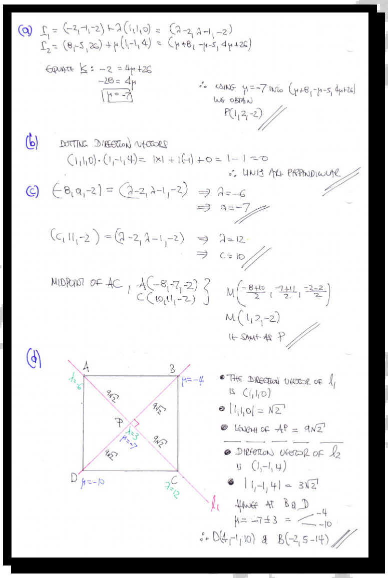
The points $A(-8,a,-2)$ and $C(c,11,-2)$ lie on l_1.

c) Find the value of each of the constants a and c, and show further that P is the midpoint of AC.

The quadrilateral $ABCD$ is a square.

d) Determine the coordinates of the points B and D.

$\boxed{P(1,2,-2)}$, $\boxed{a=-7}$, $\boxed{c=10}$, $\boxed{B(4,-1,10) \ \& \ D(-2,5,-14) \text{ in any order}}$



(a) $\underline{r}_1 = (-2,-1,-2) + \lambda(1,1,0) = (\lambda-2, \lambda-1, -2)$
$\underline{r}_2 = (8,-5,26) + \mu(1,-1,4) = (\mu+8, -\mu-5, 4\mu+26)$

EQUATE $\underline{k}$: $-2 = 4\mu + 26$
$-28 = 4\mu$
$\boxed{\mu = -7}$

$\therefore$ USING $\mu = -7$ INTO $(\mu+8, -\mu-5, 4\mu+26)$ WE OBTAIN $P(1,2,-2)$

(b) DOTTING DIRECTION VECTORS
$(1,1,0)\cdot(1,-1,4) = 1\times1 + 1(-1) + 0 = 1 - 1 = 0$
$\therefore$ LINES ARE PERPENDICULAR

(c) $(-8,a,-2) = (\lambda-2, \lambda-1, -2) \Rightarrow \lambda = -6$
$\Rightarrow a = -7$

$(c,11,-2) = (\lambda-2, \lambda-1, -2) \Rightarrow \lambda = 12$
$\Rightarrow c = 10$

MIDPOINT OF AC, $\left.\begin{matrix} A(-8,-7,-2) \\ C(10,11,-2) \end{matrix}\right\}$ $M\left(\frac{-8+10}{2}, \frac{-7+11}{2}, \frac{-2-2}{2}\right)$
$M(1,2,-2)$
IE SAME AS P

(d)

- THE DIRECTION VECTOR OF l_1 IS $(1,1,0)$
- $|1,1,0| = \sqrt{2}$
- LENGTH OF AP $= 9\sqrt{2}$
- DIRECTION VECTOR OF l_2 IS $(1,-1,4)$
- $|1,-1,4| = 3\sqrt{2}$

HENCE AT B & D
$\mu = -7 \pm 3 = \begin{matrix} -4 \\ -10 \end{matrix}$

$\therefore D(4,-1,10)$ & $B(-2,5-14)$

Question 102 (****)

Relative to a fixed origin, the points P and Q have position vectors $9\mathbf{j}-2\mathbf{k}$ and $7\mathbf{i}-8\mathbf{j}+11\mathbf{k}$, respectively.

a) Determine the distance between the points P and Q.

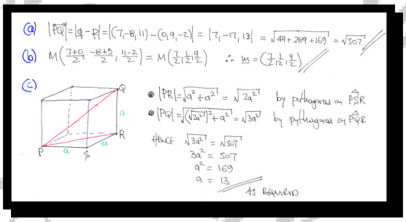
b) Find the position vector of the point M, where M is the midpoint of PQ.

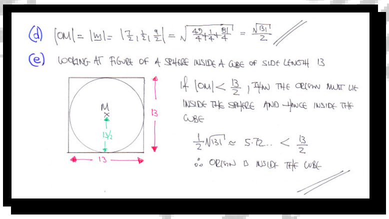
The points P and Q are vertices of a cube, so that PQ is one of the longest diagonals of the cube.

c) Show that the length of one of the sides of the cube is 13 units.

d) Show that the origin O lies inside the cube.

$\boxed{}$, $\boxed{|PQ| = \sqrt{507}}$, $\boxed{\overrightarrow{OM} = \frac{7}{2}\mathbf{i}+\frac{1}{2}\mathbf{j}+\frac{9}{2}\mathbf{k}}$





Question 103 (****)

The points $A(7,a,5)$ and $B(b,1,12)$ lie on the straight line L, with vector equation

$$\mathbf{r} = 19\mathbf{i} - 2\mathbf{j} - 9\mathbf{k} + \lambda(6\mathbf{i} - \mathbf{j} - 7\mathbf{k}),$$

where λ is a scalar parameter.

a) Find the value of a and the value of b.

The point C has coordinates $(-3,19,10)$ and M is the midpoint of BC.

b) Determine the coordinates of M.

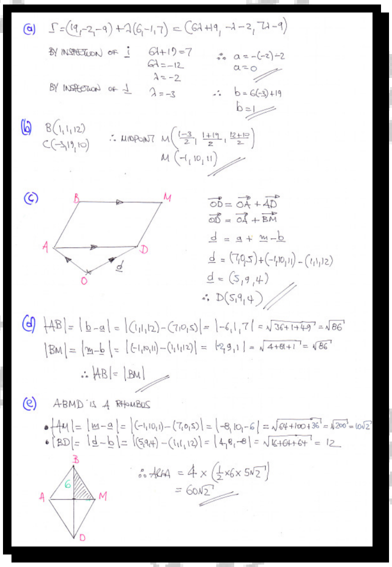
The point D is such so that $ABMD$ is a parallelogram.

c) Find the coordinates of D.

d) Show that $|AB| = |BM|$.

e) Find the exact area of the parallelogram $ABMD$.

$\boxed{a=0,\ b=1}$, $\boxed{A(-1,10,11)}$, $\boxed{D(5,9,4)}$, $\boxed{\text{area} = 60\sqrt{2}}$



Question 104 (****)

The straight line L_1 passes through the points $A(1,-2,5)$ and $B(4,-3,3)$.

a) Find a vector equation for L_1.

The straight line L_2 has a vector equation

$$\mathbf{r} = \begin{pmatrix} 8 \\ p \\ q \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix},$$

where μ is a scalar parameter, and p and q are scalar constants.

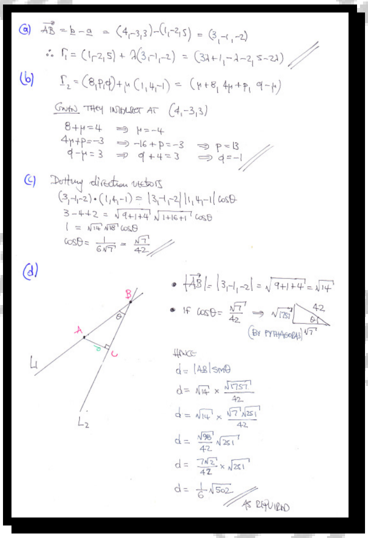
b) Given that L_1 and L_2 intersect at B, find the value of p and the value of q.

c) Find the cosine of the acute angle θ, between L_1 and L_2.

The point C is on L_2, so that AC is perpendicular to L_2.

d) Show that the length of AC is $\dfrac{\sqrt{502}}{6}$.

$\boxed{\mathbf{r} = \mathbf{i} - 2\mathbf{j} + 5\mathbf{k} + \lambda(3\mathbf{i} - \mathbf{j} - 2\mathbf{k})}$, $\boxed{p = 13,\ q = -1}$, $\boxed{\cos\theta = \frac{1}{42}\sqrt{7}}$


(a) $\overrightarrow{AB} = \underline{b} - \underline{a} = (4,-3,3) - (1,-2,5) = (3,-1,-2)$

$\therefore\ r_1 = (1,-2,5) + \lambda(3,-1,-2) = (3\lambda+1, -\lambda-2, 5-2\lambda)$

(b) $r_2 = (8,p,q) + \mu(1,4,-1) = (\mu+8, 4\mu+p, q-\mu)$

GIVEN THEY INTERSECT AT $(4,-3,3)$

$8+\mu = 4 \Rightarrow \mu = -4$

$4\mu + p = -3 \Rightarrow -16 + p = -3 \Rightarrow p = 13$

$q - \mu = 3 \Rightarrow q + 4 = 3 \Rightarrow q = -1$

(c) Dotting direction vectors

$(3,-1,-2)\cdot(1,4,-1) = |3,-1,-2||1,4,-1|\cos\theta$

$3 - 4 + 2 = \sqrt{9+1+4}\sqrt{1+16+1}\cos\theta$

$1 = \sqrt{14}\sqrt{18}\cos\theta$

$\cos\theta = \frac{1}{6\sqrt{7}} = \frac{\sqrt{7}}{42}$

(d)

- $|\overrightarrow{AB}| = |3,-1,-2| = \sqrt{9+1+4} = \sqrt{14}$
- IF $\cos\theta = \frac{\sqrt{7}}{42} \Rightarrow$ $\sqrt{1757}$, 42, $\sqrt{7}$ (BY PYTHAGORAS)

HENCE

$d = |AB|\sin\theta$

$d = \sqrt{14} \times \frac{\sqrt{1757}}{42}$

$d = \sqrt{14} \times \frac{\sqrt{7}\sqrt{251}}{42}$

$d = \frac{\sqrt{98}}{42}\sqrt{251}$

$d = \frac{7\sqrt{2}}{42} \times \sqrt{251}$

$d = \frac{1}{6}\sqrt{502}$

AS REQUIRED

Question 105 (****)

With respect to a fixed origin O, the variable points A and B have the following position vectors

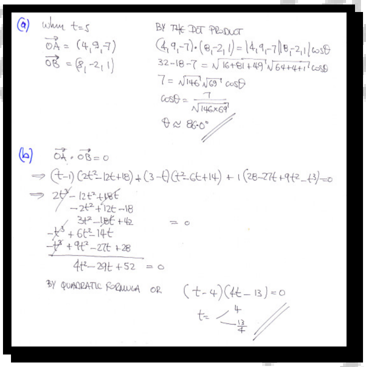
$$\overrightarrow{OA} = \begin{pmatrix} t-1 \\ t^2-6t+14 \\ 28-27t+9t^2-t^3 \end{pmatrix} \quad \text{and} \quad \overrightarrow{OB} = \begin{pmatrix} 2t^2-12t+18 \\ 3-t \\ 1 \end{pmatrix},$$

where t is a scalar parameter.

a) Calculate the angle AOB when $t=5$

b) Find the values of t for which the angle AOB is a right angle.

$\boxed{\theta \approx 86.0}$, $\boxed{t=4 \text{ or } t=\frac{13}{4}}$



(a) when $t=5$

$\overrightarrow{OA} = (4,9,-7)$

$\overrightarrow{OB} = (8,-2,1)$

BY THE DOT PRODUCT

$(4,9,-7)\cdot(8,-2,1) = |4,9,-7||8,-2,1|\cos\theta$

$32-18-7 = \sqrt{16+81+49}\sqrt{64+4+1}\cos\theta$

$7 = \sqrt{146}\sqrt{69}\cos\theta$

$\cos\theta = \frac{7}{\sqrt{146\times 69}}$

$\theta \approx 86.0°$

(b) $\overrightarrow{OA}\cdot\overrightarrow{OB} = 0$

$\Rightarrow (t-1)(2t^2-12t+18)+(3-t)(t^2-6t+14)+1(28-27t+9t^2-t^3)=0$

$\Rightarrow \cancel{2t^3}-12t^2+\cancel{18t}$

$-2t^2+12t-18$

$3t^2-\cancel{18t}+42 \quad = 0$

$-\cancel{t^3}+6t^2-14t$

$-\cancel{t^3}+9t^2-27t+28$

$4t^2-29t+52 = 0$

BY QUADRATIC FORMULA OR $(t-4)(4t-13)=0$

$t = 4, \frac{13}{4}$

Question 106 (****)

The points A and B have coordinates $(7,13,14)$ and $(15,19,15)$, respectively.

a) Find a vector equation of the straight line l_1 which passes through A and B.

The line l_2 has vector equation

$$\mathbf{r} = 5\mathbf{i} - 8\mathbf{j} - 9\mathbf{k} + \mu(-2\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}),$$

where μ is a scalar parameter.

b) Show that l_1 and l_2 do not intersect.

c) Find a vector with integer components in their simplest proportions, which is perpendicular to both lines.

[*you may not use the cross product for this part*]

$\boxed{\mathbf{r} = 7\mathbf{i} + 13\mathbf{j} + 14\mathbf{k} + \lambda(8\mathbf{i} + 6\mathbf{j} + \mathbf{k})}$, $\boxed{\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}}$

(a) $\overrightarrow{AB} = \underline{b} - \underline{a} = (15,19,15) - (7,13,14) = (8,6,1)$

$\underline{r}_1 = (7,13,14) + \lambda(8,6,1)$

$\underline{r}_1 = (8\lambda+7, 6\lambda+13, \lambda+14)$

(b) $\underline{r}_2 = (5,-8,-9) + \mu(-2,5,3) = (5-2\mu, 5\mu-8, 3\mu-9)$

EQUATE $\underline{i}$ & $\underline{k}$

$(\underline{i})$: $8\lambda+7 = 5-2\mu$
$(\underline{k})$: $\lambda+14 = 3\mu-9$ $\Rightarrow \boxed{\lambda = 3\mu-23}$

HENCE $8(3\mu-23)+7 = 5-2\mu$
$24\mu - 184 + 7 = 5 - 2\mu$
$26\mu = 182$
$\boxed{\mu = 7}$
$\lambda = 3(7) - 23$
$\boxed{\lambda = -2}$

CHECK $\underline{j}$

$6\lambda+13 = 6(-2)+13 = 1$
$5\mu-8 = 5(7)-8 = 27$

y co-ords do not agree, so lines do not intersect

(c) DOTTING DIRECTION VECTORS TO ZERO WITH A THIRD VECTOR (x,y,z)

$(8,6,1)\cdot(x,y,z) = 0$
$(-2,5,3)\cdot(x,y,z) = 0$ $\Rightarrow$ $8x+6y+z=0$, $-2x+5y+3z=0$

• LET $y=1$

$8x+6+z=0$
$-2x+5+3z=0$ $\Rightarrow$ $8x+6+z=0$, $-8x+20+12z=0$

$26+13z=0$
$13z=-26$
$z=-2$

IF $z=-2$: $8x+6-2=0$, $8x=-4$, $x=-\frac{1}{2}$

$\therefore \left(-\frac{1}{2}, 1, -2\right)$ SCALED TO $(1,-2,4)$ ($\times(-2)$)

Question 107 (****)

The straight lines l_1 and l_2 have respective vector equations

$$\mathbf{r}_1 = 5\mathbf{i}+3\mathbf{j}+\mathbf{k}+\lambda(\mathbf{i}+\mathbf{j}+\mathbf{k}) \quad \text{and} \quad \mathbf{r}_2 = -3\mathbf{i}+4\mathbf{j}+8\mathbf{k}+\mu(2\mathbf{i}-\mathbf{j}-3\mathbf{k}),$$

where λ and μ are scalar parameters.

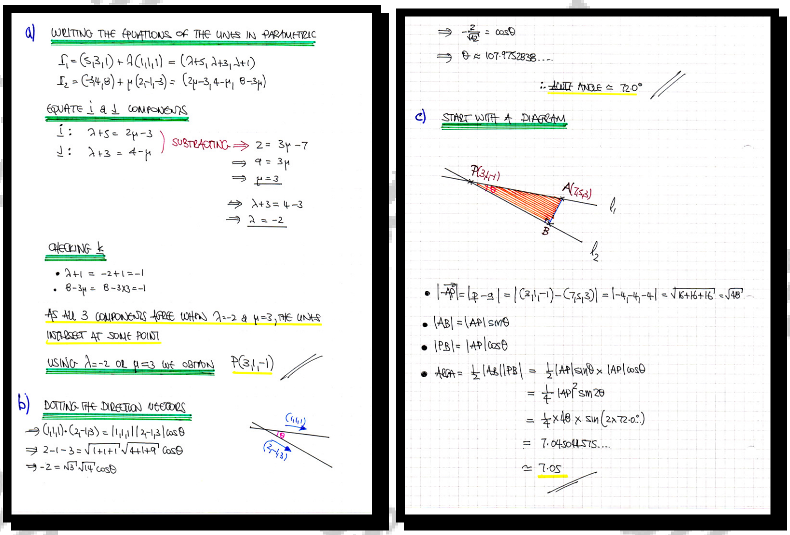
a) Show that l_1 and l_2 intersect at some point P, further finding its coordinates.

b) Calculate the acute angle between l_1 and l_2.

The point $A(7,5,3)$ lies on l_1 and the point B lies on l_2, such that the straight line AB is perpendicular to l_2.

c) Determine the area of the triangle ABP.

$\boxed{}$, $\boxed{P(3,1,-1)}$, $\boxed{72.0°}$, $\boxed{\text{area}(ABC) = \frac{8}{7}\sqrt{38} \approx 7.05}$



Question 108 (****)

The straight line L passes through the points $B(1,4,0)$ and $D(2,2,6)$.

a) Find a vector equation of L.

The point $A(1,0,p)$, where p is a scalar constant, is such so that $\measuredangle BAD = 90°$.

b) Find the possible values of p.

The rectangle $ABCD$ has an area of $12\sqrt{2}$ square units.

c) Find the coordinates of C.

$\boxed{}$, $\boxed{\mathbf{r} = \mathbf{i} + 4\mathbf{j} + \lambda(\mathbf{i} - 2\mathbf{j} + 6\mathbf{k})}$, $\boxed{p = 2, 4}$, $\boxed{C(2,6,2)}$

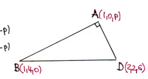
a) FIND A DIRECTION VECTOR FOR L

$\overrightarrow{BD} = \underline{d} - \underline{b} = (2,2,6) - (1,4,0) = (1,-2,6)$

$\underline{r} = (1,4,0) + \lambda(1,-2,6)$

$(x,y,z) = (\lambda+1, 4-2\lambda, 6\lambda)$

b) LOOKING AT THE DIAGRAM



- $\overrightarrow{AB} = \underline{b} - \underline{a} = (1,4,0) - (1,0,p) = (0,4,-p)$
- $\overrightarrow{AD} = \underline{d} - \underline{a} = (2,2,6) - (1,0,p) = (1,2,6-p)$

$\Rightarrow \overrightarrow{AB} \cdot \overrightarrow{AD} = 0$

$\Rightarrow (0,4,-p) \cdot (1,2,6-p) = 0$

$\Rightarrow 0 + 8 - p(6-p) = 0$

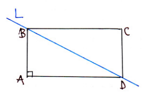
$\Rightarrow 8 - 6p + p^2 = 0$

$\Rightarrow p^2 - 6p + 8 = 0$

$\Rightarrow (p-4)(p-2) = 0$

$p = \begin{cases} 2 \\ 4 \end{cases}$

c) LOOKING AT THE RECTANGLE



- AREA $\overset{\square}{ABCD} = 12\sqrt{2}$

  (AREA $\overset{\triangle}{BAD} = 6\sqrt{2}$)

- IF $p=2$

  $|AB||AD| = |0,4,-2||1,2,4| = \sqrt{16+4}\sqrt{1+4+16} \neq 12\sqrt{2}$

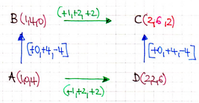
- IF $p=4$

  $|AB||BD| = |0,4,-4||1,2,2| = \sqrt{16+16}\sqrt{1+4+4} = \sqrt{32} \times 3$

  $= 12\sqrt{2}$

$\therefore A(1,0,4)$

HENCE BY INSPECTION



$\therefore C(2,6,2)$

Question 109 (****)

The straight lines l_1 and l_2 have the following vector equations

$$\mathbf{r}_1 = 12\mathbf{i} + 7\mathbf{j} + 3\mathbf{k} + \lambda(2\mathbf{i} + \mathbf{j} + \mathbf{k})$$

$$\mathbf{r}_2 = \mathbf{i} + 3\mathbf{j} + \mu(3\mathbf{i} - \mathbf{k}),$$

where λ and μ are scalar parameters.

a) Show that l_1 and l_2 intersect at some point A, further finding its coordinates.

b) Calculate the acute angle between l_1 and l_2.

The point $B(16,9,5)$ lies on l_1 and the point D lies on l_2.

c) If BD is perpendicular to l_2 find the coordinates of D.

d) Find the coordinates of a point C so that the triangle ABC is isosceles.

$\boxed{}$, $\boxed{A(4,3,-1)}$, $\boxed{49.8°}$, $\boxed{D(13,3,-4)}$, $\boxed{C(22,3,-7)}$

a) WRITE THE EQUATIONS IN PARAMETRIC FORM

$\underline{r}_1 = (12,7,3) + \lambda(2,1,1) = (2\lambda+12, \lambda+7, \lambda+3)$
$\underline{r}_2 = (1,3,0) + \mu(3,0,-1) = (3\mu+1, 3, -\mu)$

EQUATE $\underline{j}$ & $\underline{k}$

$\left.\begin{array}{l}\underline{j}: \lambda+7=3 \\ \underline{k}: \lambda+3=-\mu\end{array}\right\} \Rightarrow \lambda=-4 \text{ \& } \mu=1$

CHECK $\underline{i}$

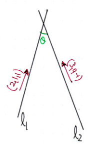
$2\lambda+12 = 2(-4)+12 = 4$
$3\mu+1 = 3\times1+1 = 4$

AS ALL 3 COMPONENTS AGREE WHEN $\lambda=-4$ & $\mu=1$ THE LINES INTERSECT AT SOME POINT

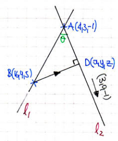
USING $\lambda=-4$ OR $\mu=1$ YIELDS $A(4,3,-1)$

b) DOTTING THE DIRECTION VECTORS

$(2,1,1)\cdot(3,0,-1) = |2,1,1|\times|3,0,-1|\cos\theta$
$6+0-1 = \sqrt{4+1+1}\sqrt{9+0+1}\cos\theta$
$5 = \sqrt{6}\sqrt{10}\cos\theta$
$\cos\theta = \frac{5}{\sqrt{60}}$
$\theta \simeq 49.8°$



c) LOOKING AT THE DIAGRAM



- $\overrightarrow{BD} = \underline{d} - \underline{b} = (x,y,z)-(16,9,5)$
  $= (x-16, y-9, z-5)$
- $(x-16, y-9, z-5)\cdot(3,0,-1) = 0$
  (AS $B\hat{D}A = 90°$)

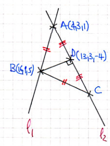
$3x-48-z+5=0$
$3x-z=43$

- BUT $(x,y,z) = (3\mu+1, 3, -\mu)$

$\Rightarrow 3(3\mu+1)-(-\mu)=43$
$\Rightarrow 10\mu+3=43$
$\Rightarrow 10\mu=40$
$\Rightarrow \mu=4$

$\therefore D(13,3,-4)$

d) THE POINT C IS NOT UNIQUE, HOWEVER C COULD BE SUCH SO THAT D IS THE MIDPOINT OF AC



A		D		C
4	$\xrightarrow{+9}$	13	$\xrightarrow{+9}$	22
3	$\xrightarrow{+0}$	3	$\xrightarrow{+0}$	3
−1	$\xrightarrow{-3}$	−4	$\xrightarrow{-3}$	−7

$\therefore C(22,3,-7)$

Question 110 (****)

The vectors $\mathbf{p}$ and $\mathbf{q}$ are defined as

$$\mathbf{p} = \mathbf{a} + 2\mathbf{b} \qquad \text{and} \qquad \mathbf{q} = 5\mathbf{a} - 4\mathbf{b}$$

where $\mathbf{a}$ and $\mathbf{b}$ are unit vectors.

Given that $\mathbf{p}$ and $\mathbf{q}$ are perpendicular, determine the acute angle between $\mathbf{a}$ and $\mathbf{b}$.

$\boxed{}$, $\boxed{\theta = 60°}$

WE ARE GIVEN THAT $|\underline{a}| = |\underline{b}| = 1$

THEN WE HAVE $\underline{p} \perp \underline{q}$, IF $\underline{p} \cdot \underline{q} = 0$

$\Rightarrow (\underline{a} + 2\underline{b}) \cdot (5\underline{a} - 4\underline{b}) = 0$

$\Rightarrow 5\underline{a} \cdot \underline{a} - 4\underline{a} \cdot \underline{b} + 10\underline{a} \cdot \underline{b} - 8\underline{b} \cdot \underline{b} = 0$

$\Rightarrow 5|\underline{a}||\underline{a}|\cos 0 + 6\underline{a} \cdot \underline{b} - 8|\underline{b}||\underline{b}|\cos 0 = 0$

$\Rightarrow 5 \times 1 \times 1 \times 1 + 6\underline{a} \cdot \underline{b} - 8 \times 1 \times 1 \times 1 = 0$

$\Rightarrow 6\underline{a} \cdot \underline{b} = 3$

$\Rightarrow \underline{a} \cdot \underline{b} = \frac{1}{2}$

$\Rightarrow |\underline{a}||\underline{b}|\cos\theta = \frac{1}{2}$

$\Rightarrow 1 \times 1 \times \cos\theta = \frac{1}{2}$

$\Rightarrow \cos\theta = \frac{1}{2}$

$\Rightarrow \theta = 60°$ //

Question 111 (****)

The straight lines l_1 and l_2 have the following Cartesian equations

$$l_1: \quad \frac{x-8}{1} = \frac{y+1}{-1} = \frac{z-2}{1}.$$

$$l_2: \quad \frac{x-3}{-1} = \frac{y-4}{1} = \frac{z-1}{1}.$$

a) Show that l_1 and l_2 intersect at some point P, and find its coordinates.

b) Find the exact value of $\cos\theta$, where θ is the acute angle formed by l_1 and l_2.

The point $A(6,1,0)$ lies on l_1 and the point $B(4,3,0)$ lies on l_2.

c) By considering $|AP|$ and $|BP|$ show further that the angle bisector of $\measuredangle APB$ is parallel to the vector $\mathbf{k}$.

$\boxed{}$, $\boxed{P(5,2,-1)}$, $\boxed{\cos\theta = \frac{1}{3}}$

a) WRITE THE EQUATIONS IN PARAMETRIC

$\frac{x-8}{1} = \frac{y+1}{-1} = \frac{z-2}{1} = \lambda \Rightarrow \underline{r}_1 = (8,-1,2) + \lambda(1,-1,1)$

$\frac{x-3}{-1} = \frac{y-4}{1} = \frac{z-1}{1} = \mu \Rightarrow \underline{r}_2 = (3,4,1) + \mu(-1,1,1)$

$\underline{r}_1 = (\lambda+8, -\lambda-1, \lambda+2)$

$\underline{r}_2 = (-\mu+3, \mu+4, \mu+1)$

EQUATE $\underline{j}$ & $\underline{k}$ (EQUATING $\underline{i}$ & $\underline{j}$ YIELD NOTHING)

$\left.\begin{array}{l} \underline{j}: -\lambda-1 = \mu+4 \\ \underline{k}: \lambda+2 = \mu+1 \end{array}\right\}$ ADDING $\Rightarrow 1 = 2\mu + 5$

$\Rightarrow -4 = 2\mu$

$\underline{\mu = -2}$ & $\underline{\lambda = -3}$

CHECK $\underline{i}$

$\lambda + 8 = -3 + 8 = 5$

$-\mu + 3 = -(-2) + 3 = 5$

INDEED THE LINES INTERSECT AT $P(5,2,-1)$

b) DOTTING THE DIRECTION VECTORS

$(1,-1,1)\cdot(-1,1,1) = |1,-1,1||-1,1,1|\cos\theta$

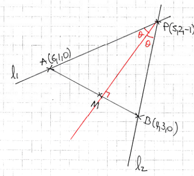
$-1-1+1 = \sqrt{1+1+1}\sqrt{1+1+1}\cos\theta$

$-1 = \sqrt{3}\sqrt{3}\cos\theta$

$\cos\theta = -\frac{1}{3}$

$\therefore$ ACUTE ANGLE WILL HAVE $\cos\theta = \frac{1}{3}$

c) LOOKING AT THE DIAGRAM BELOW



$|\overrightarrow{AP}| = |\underline{p} - \underline{a}| = |(5,2,-1)-(6,1,0)| = |-1,1,-1| = \sqrt{3}$

$|\overrightarrow{BP}| = |\underline{p} - \underline{b}| = |(5,2,-1)-(4,3,0)| = |1,-1,-1| = \sqrt{3}$

HENCE $\hat{APB}$ IS ISOSCELES SO THE PERPENDICULAR BISECTOR OF $\hat{APB}$ MUST GO THROUGH THE MIDPOINT OF AB

$M\left(\frac{6+4}{2}, \frac{3+1}{2}, \frac{0+0}{2}\right) \Rightarrow M(5,2,0)$

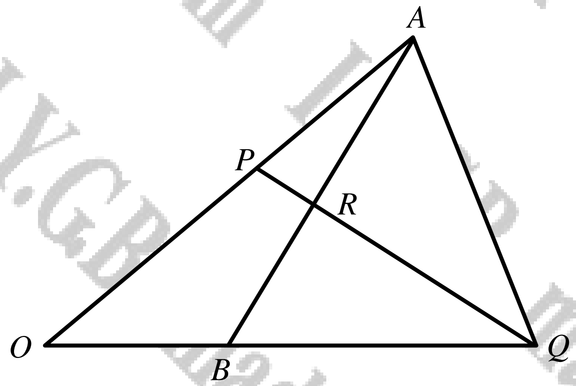
$\overrightarrow{MP} = \underline{p} - \underline{m} = (5,2,-1) - (5,2,0) = (0,0,-1)$

IE MULTIPLE OF $\underline{k}$

$\therefore$ PARALLEL TO $\underline{k}$

Question 112 (****+)

The figure below shows a triangle OAQ.



- The point P lies on OA so that $OP:OA = 3:5$.
- The point B lies on OQ so that $OB:BQ = 1:2$.

Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

a) Given that $\overrightarrow{AR} = h\overrightarrow{AB}$, where h is a scalar parameter with $0 < h < 1$, show that

$$\overrightarrow{OR} = (1-h)\mathbf{a} + h\mathbf{b}.$$

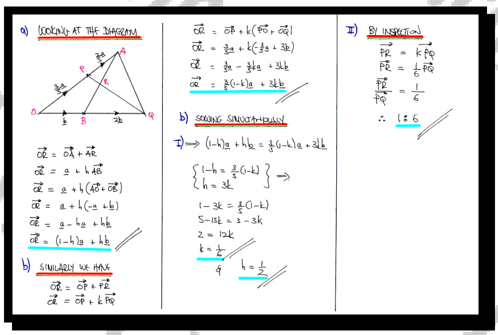
b) Given further that $\overrightarrow{PR} = k\overrightarrow{PQ}$, where k is a scalar parameter with $0 < k < 1$, find a similar expression for $\overrightarrow{OR}$ in terms of k, $\mathbf{a}$, $\mathbf{b}$.

c) Determine …

  i. … the value of k and the value of h.

  ii. … the ratio of $\overrightarrow{PR}:\overrightarrow{PQ}$.

[illegible], $\boxed{\overrightarrow{OR} = \frac{3}{5}(1-k)\mathbf{a} + k\mathbf{b}}$, $\boxed{k = \frac{1}{6}}$, $\boxed{h = \frac{1}{2}}$, $\boxed{PR:PQ = 1:6}$



Question 113 (****+)

The points A and B have coordinates $(4,0,2)$ and $(7,0,-1)$, respectively.

a) Find the vector $\overrightarrow{AB}$.

The straight line l has vector equation

$$\mathbf{r} = -3\mathbf{i} - 4\mathbf{j} + \mathbf{k} + \lambda(7\mathbf{i} + 4\mathbf{j} + \mathbf{k}),$$

where λ is a scalar parameter.

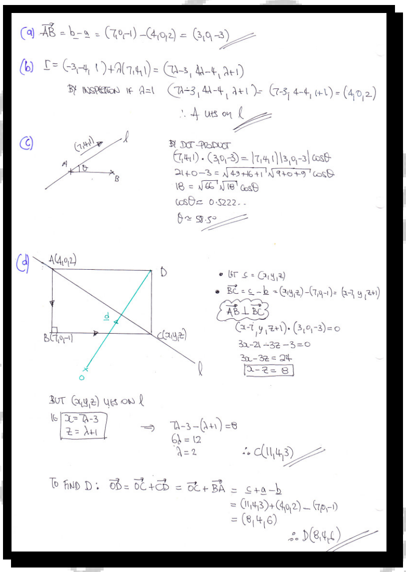
b) Show that A lies on l.

c) Calculate the acute angle between $\overrightarrow{AB}$ and l.

The point C lies on l so that $ABCD$ is a rectangle.

d) Find the coordinates of D.

$\boxed{\overrightarrow{AB} = 3\mathbf{i} - 3\mathbf{k}}$, $\boxed{\theta \approx 58.5°}$, $\boxed{D(8,4,6)}$



Question 114 (****+)

The straight lines l_1 and l_2 have the following vector equations

$$\mathbf{r}_1 = 3\mathbf{j} + \mathbf{k} + \lambda(2\mathbf{i} - \mathbf{j} + 2\mathbf{k})$$

$$\mathbf{r}_2 = 2\mathbf{i} + 7\mathbf{k} + \mu(\mathbf{i} - \mathbf{j} + 2\mathbf{k}),$$

where λ and μ are scalar parameters.

a) Show that l_1 and l_2 intersect at some point P and find its position vector.

The points A and C lie on l_1 and the points B and D lie on l_2.

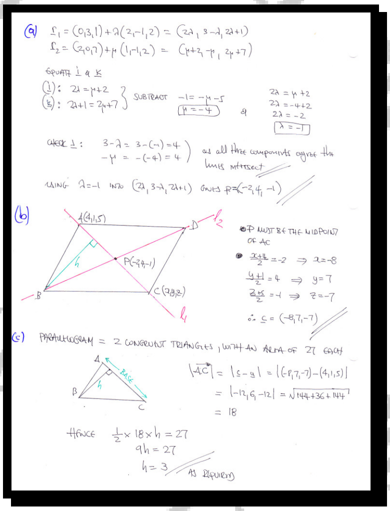
The point A has position vector $4\mathbf{i} + \mathbf{j} + 5\mathbf{k}$.

The quadrilateral $ABCD$ is a parallelogram with an area of 54 square units.

b) State the position vector of the point C.

c) Show that the distance of the point B from l_1 is 3 units.

$\boxed{\mathbf{p} = -2\mathbf{i} + 4\mathbf{j} - \mathbf{k}}$, $\boxed{\mathbf{c} = -8\mathbf{i} + 7\mathbf{j} - 7\mathbf{k}}$



Question 115 (****+)

With respect to a fixed origin O, the points A and B have respective position vectors

$$\mathbf{i}+11\mathbf{j}+3\mathbf{k} \quad \text{and} \quad 11\mathbf{i}-4\mathbf{j}+8\mathbf{k}.$$

a) Find a vector equation for the straight line l, which passes through A and B.

The point C is the point on l closest to the origin O.

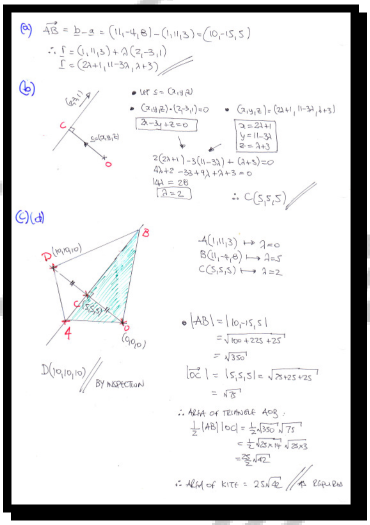
b) Determine the position vector of C.

The point D is the reflection of O about l.

c) State the position vector of D.

d) Show that the area of the kite $OADB$ is $25\sqrt{42}$ square units.

$\boxed{\mathbf{r}=\mathbf{i}+11\mathbf{j}+3\mathbf{k}+\lambda(2\mathbf{i}-3\mathbf{j}+\mathbf{k})}$, $\boxed{\overrightarrow{OC}=5\mathbf{i}+5\mathbf{j}+5\mathbf{k}}$, $\boxed{\overrightarrow{OD}=10\mathbf{i}+10\mathbf{j}+10\mathbf{k}}$



Question 116 (****+)

Relative to a fixed origin O, the points $A(3t-19, 2t-14, 28-t)$, $B(t+1, t-2, 5t)$ and $C(2t-11, 10-t, 2t+4)$, represent the coordinates of the paths of three helicopters, where t represents the time in minutes after a certain instant.

All distances are in kilometres with the coordinates axes Ox, Oy, Oz oriented due east, due north and vertically upwards, respectively.

a) Show that all three helicopters pass through a point P and find its coordinates.

b) Explain why only two of the helicopters will collide at the point P if they maintain their courses as described in this problem.

c) Show that the paths of A and B are perpendicular.

$\boxed{P(5,2,20)}$

(a) $\underline{r}_A = (3t-19, 2t-14, 28-t)$ $\qquad$ $\underline{r}_A = (3T-19, 2T-14, 28-T)$

$\underline{r}_B = (t+1, t-2, 5t)$

$\underline{r}_C = (2t-11, 10-t, 2t+4)$

EQUATE $\underline{j}$ & $\underline{k}$ OF A & B

$(\underline{j})$: $2T-14 = t-2$

$(\underline{k})$: $28-T = 5t$ $\Rightarrow$ $\boxed{t = 2T-12}$

$28-T = 5(2T-12)$

$28-T = 10T-60$

$88 = 11T$

$\boxed{T=8}$ & $\boxed{t=4}$

CHECK $\underline{i}$: $3T-19 = 3\times8-19 = 5$

$t+1 = 4+1 = 5$

$\therefore$ HELICOPTERS' A & B PATHS WILL CROSS

USING $t=4$ $(t+1, t-2, 5t)$ WE OBTAIN $P(5,2,20)$

NOW LOOKING AT $\underline{r}_C = (2t-11, 10-t, 2t+4)$ & $P(5,2,20)$

BY INSPECTION IF $t=8$ HELICOPTER C ALSO PASSES THROUGH $P(5,2,20)$

$\therefore$ THE PATHS OF ALL THREE PASS THROUGH $P(5,2,20)$

(b) HELICOPTER A, PASSES THROUGH P when $t=8$

B, PASSES THROUGH P when $t=4$

C, PASSES THROUGH P when $t=8$

HENCE ONLY A & C WILL COLLIDE

(c) DIRECTION VECTOR OF "A" IS $(3,2,-1)$

DIRECTION VECTOR OF "B" IS $(1,1,5)$

$(3,2,-1)\cdot(1,1,5) = 3+2-5 = 0$ $\therefore$ PATHS ARE PERPENDICULAR

Question 117 (****+)

The straight lines l_1 and l_2, where λ and μ are scalar parameters, have the following vector equations

$$\mathbf{r}_1 = (\lambda+2)\mathbf{i} + (2\lambda+6)\mathbf{j} + (-\lambda-1)\mathbf{k}$$

$$\mathbf{r}_2 = (2\mu-4)\mathbf{i} + (4-\mu)\mathbf{j} + (3-\mu)\mathbf{k}.$$

l_1 and l_2 intersect at the point A and the acute angle between l_1 and l_2 is θ.

a) Find in any order…

i. … the coordinates of A.

ii. … the exact value of $\cos\theta$.

The point B lies on l_1 and the point C lies on l_2. The triangle ABC is isosceles with $|AB| = |AC| = 6\sqrt{6}$.

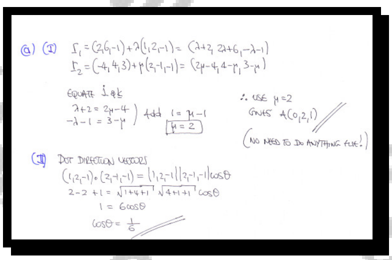
b) Find the two possible sets of coordinates for the points B and C.

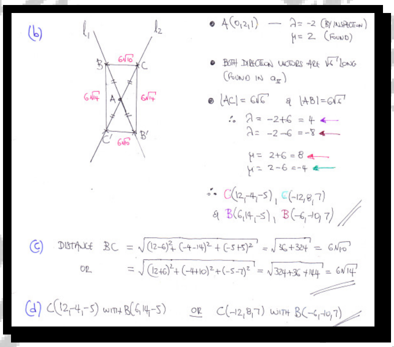
c) Show that either $|BC| = 6\sqrt{14}$ or $|BC| = 6\sqrt{10}$

In the triangle ABC the angle BAC is acute.

a) In the triangle ABC, determine the two possible pairings for the coordinates of the point B and the corresponding coordinates of the point C.

$$\boxed{A(0,2,1)}, \boxed{\cos\theta = \frac{1}{6}}, \boxed{B_1(-6,-10,7) \text{ or } B_2(6,14,-5)},$$
$$\boxed{C_1(-12,8,7) \text{ or } C_2(12,-4,-5)}, \boxed{B_1 \ \& \ C_1 \text{ or } B_2 \ \& \ C_2}$$





Question 118 (****+)

Relative to a fixed origin O, the points A, B, C and D have coordinates $(7,6,2)$, $(12,10,5)$, $(1,-4,-8)$ and $(11,4,-2)$, respectively.

a) Find the vector equation of the straight line l_1 which passes through the point A and B and the vector equation of the straight line l_2 which passes through the point C and D.

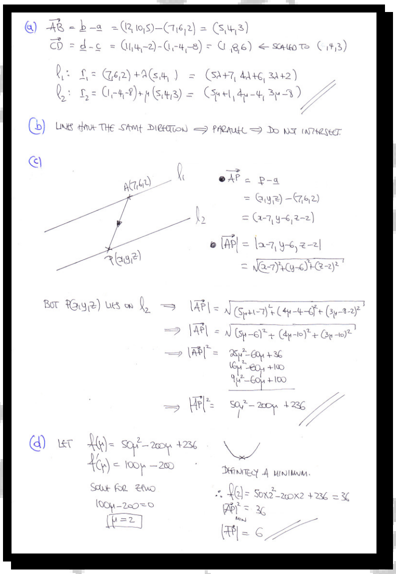
b) Explain why l_1 and l_2 do not intersect.

The point P lies on l_2.

c) Find an expression for $\left|\overrightarrow{AP}\right|^2$, in terms of μ.

d) Calculate the distance between l_1 and l_2.

$\boxed{\mathbf{r}_1 = 7\mathbf{i}+6\mathbf{j}+2\mathbf{k}+\lambda(5\mathbf{i}+4\mathbf{j}+3\mathbf{k})}$, $\boxed{\mathbf{r}_2 = \mathbf{i}-4\mathbf{j}-8\mathbf{k}+\mu(5\mathbf{i}+4\mathbf{j}+3\mathbf{k})}$, $\boxed{\left|\overrightarrow{PA}\right|^2 = 50\mu^2-200\mu+236}$, $\boxed{6 \text{ units}}$



(a) $\overrightarrow{AB} = \underline{b}-\underline{a} = (12,10,5)-(7,6,2) = (5,4,3)$

$\overrightarrow{CD} = \underline{d}-\underline{c} = (11,4,-2)-(1,-4,-8) = (10,8,6) \leftarrow$ SCALED TO $(5,4,3)$

$l_1: \underline{r}_1 = (7,6,2)+\lambda(5,4,3) = (5\lambda+7, 4\lambda+6, 3\lambda+2)$

$l_2: \underline{r}_2 = (1,-4,-8)+\mu(5,4,3) = (5\mu+1, 4\mu-4, 3\mu-8)$

(b) LINES HAVE THE SAME DIRECTION $\Rightarrow$ PARALLEL $\Rightarrow$ DO NOT INTERSECT.

(c)

• $\overrightarrow{AP} = \underline{p}-\underline{a} = (x,y,z)-(7,6,2) = (x-7, y-6, z-2)$

• $|\overrightarrow{AP}| = |x-7, y-6, z-2| = \sqrt{(x-7)^2+(y-6)^2+(z-2)^2}$

BUT $P(x,y,z)$ LIES ON l_2 $\Rightarrow |\overrightarrow{AP}| = \sqrt{(5\mu+1-7)^2+(4\mu-4-6)^2+(3\mu-8-2)^2}$

$\Rightarrow |\overrightarrow{AP}| = \sqrt{(5\mu-6)^2+(4\mu-10)^2+(3\mu-10)^2}$

$\Rightarrow |\overrightarrow{AP}|^2 = \begin{array}{l} 25\mu^2-60\mu+36 \\ 16\mu^2-80\mu+100 \\ 9\mu^2-60\mu+100 \end{array}$

$\Rightarrow |\overrightarrow{AP}|^2 = 50\mu^2-200\mu+236$

(d) LET $f(\mu) = 50\mu^2-200\mu+236$

$f'(\mu) = 100\mu-200$

SOLVE FOR ZERO

$100\mu-200=0$

$\boxed{\mu=2}$

DEFINITELY A MINIMUM.

$\therefore f(2) = 50\times2^2-200\times2+236 = 36$

$|\overrightarrow{AP}|^2_{\text{MIN}} = 36$

$|\overrightarrow{AP}| = 6$

Question 119 (**+)**

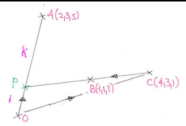
Relative to a fixed origin O, the points A, B and C have coordinates $(2,3,5)$, $(1,1,1)$ and $(4,3,1)$, respectively.

The line segment CB is extended to the point P.

It is further given that P lies on the line segment OA so that $|OP|:|PA| = 1:k$.

Determine the value of k.

$\boxed{k=4}$



FIRSTLY $\overrightarrow{CB} = \underline{c} - \underline{b} = (4,3,1)-(1,1,1) = (3,2,0)$

$$\left.\begin{array}{l}\overrightarrow{OP} = \lambda\overrightarrow{OA} \\ \overrightarrow{OP} = \overrightarrow{OC} + \mu\overrightarrow{CB}\end{array}\right\} \quad \left(\lambda = \frac{1}{k+1}\right)$$

$\Rightarrow \lambda\overrightarrow{OA} = \overrightarrow{OC} + \mu\overrightarrow{CB}$

$\Rightarrow \lambda(2,3,5) = (4,3,1) + \mu(3,2,0)$

$\Rightarrow (2\lambda, 3\lambda, 5\lambda) = (3\mu+4, 2\mu+3, 1)$

FROM THE $\underline{k}$ COMPONENT $5\lambda = 1$

$\lambda = 1/5$

$\therefore \frac{1}{5} = \frac{1}{k+1}$

$k+1 = 5$

$k = 4$

CHECK THE REST OF THE CO-ORDINATES

- $\frac{2}{5} = 3\mu + 4$
  $-\frac{18}{5} = 3\mu$
  $\mu = -\frac{6}{5}$
- $\frac{3}{5} = 2\mu + 3$
  $\frac{3}{5} = 2\mu + 3$
  $2\mu = -\frac{12}{5}$
  $\mu = -\frac{6}{5}$

Question 120 (****+)

The straight line l_1 passes through the points with coordinates $A(-2,3,4)$ and $B(8,-1,14)$.

a) Find a vector equation for l_1.

The straight line l_2 has vector equation

$$\mathbf{r} = \mathbf{i} + 5\mathbf{j} - 5\mathbf{k} + \mu(\mathbf{i} - 2\mathbf{j} + 7\mathbf{k}),$$

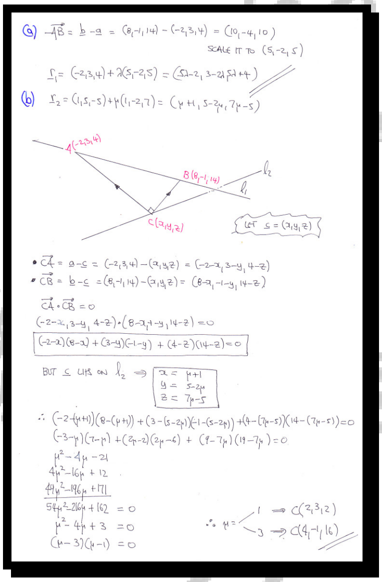
where μ is a scalar parameter.

The point C lies on l_2 so that AC is perpendicular to BC.

b) Show that one possible position for the point C has coordinates $(2,3,2)$ and find the other.

c) Assuming further that C has coordinates $(2,3,2)$, show that the area of the triangle ABC is $14\sqrt{5}$ square units.

$\boxed{\mathbf{r}_1 = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k} + \lambda(5\mathbf{i} - 2\mathbf{j} + 5\mathbf{k})}$, $\boxed{(4,-1,16)}$



(a) $\vec{AB} = \underline{b} - \underline{a} = (8,-1,14) - (-2,3,4) = (10,-4,10)$

SCALE IT TO $(5,-2,5)$

$\underline{r}_1 = (-2,3,4) + \lambda(5,-2,5) = (5\lambda-2, 3-2\lambda, 5\lambda+4)$

(b) $\underline{r}_2 = (1,5,-5) + \mu(1,-2,7) = (\mu+1, 5-2\mu, 7\mu-5)$

• $\vec{CA} = \underline{a} - \underline{c} = (-2,3,4) - (x,y,z) = (-2-x, 3-y, 4-z)$

• $\vec{CB} = \underline{b} - \underline{c} = (8,-1,14) - (x,y,z) = (8-x, -1-y, 14-z)$

$\vec{CA} \cdot \vec{CB} = 0$

$(-2-x, 3-y, 4-z) \cdot (8-x, -1-y, 14-z) = 0$

$\boxed{(-2-x)(8-x) + (3-y)(-1-y) + (4-z)(14-z) = 0}$

BUT $\underline{c}$ LIES ON l_2 $\Rightarrow$ $\boxed{\begin{aligned} x &= \mu+1 \\ y &= 5-2\mu \\ z &= 7\mu-5 \end{aligned}}$

$\therefore (-2-(\mu+1))(8-(\mu+1)) + (3-(5-2\mu))(-1-(5-2\mu)) + (4-(7\mu-5))(14-(7\mu-5)) = 0$

$(-3-\mu)(7-\mu) + (2\mu-2)(2\mu-6) + (9-7\mu)(19-7\mu) = 0$

$\mu^2 - 4\mu - 21$

$4\mu^2 - 16\mu + 12$

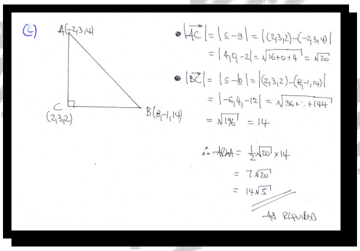
$49\mu^2 - 196\mu + 171$

$54\mu^2 - 216\mu + 162 = 0$

$\mu^2 - 4\mu + 3 = 0$

$(\mu-3)(\mu-1) = 0$

$\therefore \mu = 1 \Rightarrow C(2,3,2)$ or $\mu = 3 \Rightarrow C(4,-1,16)$



(c)

• $|\vec{AC}| = |\underline{c} - \underline{a}| = |(2,3,2) - (-2,3,4)|$
$= |4,0,-2| = \sqrt{16+0+4} = \sqrt{20}$

• $|\vec{BC}| = |\underline{c} - \underline{b}| = |(2,3,2) - (8,-1,14)|$
$= |-6,4,-12| = \sqrt{36+16+144}$
$= \sqrt{196} = 14$

$\therefore$ AREA $= \frac{1}{2}\sqrt{20} \times 14$
$= 7\sqrt{20}$
$= 14\sqrt{5}$

AS REQUIRED

Question 121 (****+)

The straight lines l_1 and l_2 have the following vector equations

$$\mathbf{r}_1 = 13\mathbf{i} - 5\mathbf{j} + 8\mathbf{k} + \lambda(6\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})$$

$$\mathbf{r}_2 = -5\mathbf{i} - 4\mathbf{j} + 8\mathbf{k} + \mu(2\mathbf{i} + \mathbf{j} - 2\mathbf{k}),$$

where λ and μ are scalar parameters.

a) Show that l_1 and l_2 intersect at some point C and find its coordinates.

b) Find the cosine of the acute angle between l_1 and l_2.

The point A lies on l_1 where $\lambda = -1$ and the point B lies on l_2 where $\mu = 4$.

c) Determine a vector equation of the angle bisector of $\measuredangle ACB$.

$\boxed{}$, $\boxed{C(1,-1,2)}$, $\boxed{\cos\theta = \frac{4}{21}}$, $\boxed{\mathbf{r} = \mathbf{i} - \mathbf{j} + 2\mathbf{k} + t(32\mathbf{i} + \mathbf{j} - 5\mathbf{k})}$

a) $\underline{r}_1 = (13,-5,8) + \lambda(6,-2,3) = (6\lambda+13, -2\lambda-5, 3\lambda+8)$
$\underline{r}_2 = (-5,-4,8) + \mu(2,1,-2) = (2\mu-5, \mu-4, 8-2\mu)$

- EQUATE $\underline{i}$ & $\underline{k}$

$\underline{i}: \ 6\lambda+13 = 2\mu-5$
$\underline{k}: \ 3\lambda+8 = 8-2\mu$
ADDING $\Rightarrow 9\lambda + 21 = 3 \Rightarrow 9\lambda = -18 \Rightarrow \boxed{\lambda = -2}$

AND $6\lambda+13 = 2\mu-5$
$1 = 2\mu - 5$
$6 = 2\mu$
$\boxed{\mu = 3}$

- CHECK $\underline{j}$

$3\lambda+8 = 3(-2)+8 = 2$
$8-2\mu = 8-2\times3 = 2$

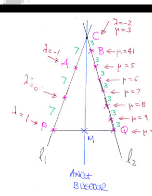
- AS ALL THREE COMPONENTS AGREE WHEN $\lambda=-2$ & $\mu=3$, THE LINES INTERSECT
- USING $\mu=3$ WE OBTAIN $C(1,-1,2)$

b) DOTTING THE DIRECTION VECTORS

$(6,-2,3)\cdot(2,1,-2) = |6,-2,3||2,1,-2|\cos\theta$
$12-2-6 = \sqrt{36+4+9}\sqrt{4+1+4}\cos\theta$
$4 = 7\times3\cos\theta$
$\cos\theta = \frac{4}{21}$

c)
- THE DIRECTION VECTOR OF l_1 IS 7 UNITS LONG & THE DIRECTION VECTOR OF l_2 IS 3 UNITS LONG

EXTEND BOTH TO 21 UNITS



- HENCE $|CP| = |CQ| = 21$

$P(19,-7,11) \leftarrow \lambda=1$
$Q(15,6,-12) \leftarrow \mu=10$

- BY INSPECTION

$M(17,-\frac{1}{2},-\frac{1}{2})$

$\overrightarrow{MC} = \underline{c} - \underline{m} = (1,-1,2) - (17,-\frac{1}{2},-\frac{1}{2}) = (-16,-\frac{1}{2},\frac{5}{2})$

SCALE IT TO A DIRECTION VECTOR $(-32,-1,5)$
OR BETTER $(32,1,-5)$

- USING $C(1,-1,2)$ AS A FIXED POINT

$\underline{r} = (1,-1,2) + t(32,1,-5)$
$\underline{r} = (32t+1, t-1, 2-5t)$

Question 122 (****+)

The straight lines l_1 and l_2 have the following vector equations

$$\mathbf{r}_1 = 9\mathbf{i} + 7\mathbf{j} + 11\mathbf{k} + \lambda(4\mathbf{i} + 3\mathbf{j} + 5\mathbf{k})$$

$$\mathbf{r}_2 = -2\mathbf{i} + 5\mathbf{j} - 4\mathbf{k} + \mu(3\mathbf{i} - 4\mathbf{j} + a\mathbf{k}),$$

where λ and μ are scalar parameters and a is a scalar constant.

The point A is the intersection of l_1 and l_2.

b) Find in any order …

i. … the value of a.

ii. … the coordinates of A.

The acute angle between l_1 and l_2 is θ.

c) Show that $\theta = 60°$.

The point B lies on l_1 and the point C lies on l_2.

The triangle ABC is equilateral with sides of length $15\sqrt{2}$.

d) Find the **two** possible pairings for the coordinates of B and C.

$\boxed{a=5}$, $\boxed{A(1,1,1)}$,

$\boxed{B(13,10,16)\ \&\ C(10,-11,16) \text{ or } B(-11,-8,-14)\ \&\ C(-8,13,-14)}$

a) $\boxed{\begin{aligned} \mathbf{r}_1 &= (9,7,11) + \lambda(4,3,5) = (4\lambda+9,\ 3\lambda+7,\ 5\lambda+11) \\ \mathbf{r}_2 &= (-2,5,-4) + \mu(3,-4,a) = (3\mu-2,\ 5-4\mu,\ a\mu-4) \end{aligned}}$

I) • EQUATE $\mathbf{i}$ & $\mathbf{j}$

$\mathbf{i}$: $4\lambda+9 = 3\mu-2$ $(\times 4)$ $\Rightarrow$ $16\lambda+36 = 12\mu-8$

$\mathbf{j}$: $3\lambda+7 = 5-4\mu$ $(\times 3)$ $\Rightarrow$ $9\lambda+21 = 15-12\mu$

• ADDING GIVES $25\lambda + 57 = 7$

$25\lambda = -50$

$\boxed{\lambda = -2}$

• AND $4\lambda+9 = 3\mu-2$

$-8+9 = 3\mu-2$

$3 = 3\mu$

$\boxed{\mu = 1}$

• EQUATE $\mathbf{k}$ WITH $\lambda=-2$ & $\mu=1$

$5\lambda+11 = a\mu-4$

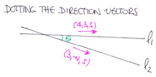
$-10+11 = a-4$

$a = 5$

II) USING $\lambda=-2$ INTO $(4\lambda+9,\ 3\lambda+7,\ 5\lambda+11)$ GIVES

$A(1,1,1)$

b) DOTTING THE DIRECTION VECTORS



$\Rightarrow (4,3,5)\cdot(3,-4,5) = |4,3,5||3,-4,5|\cos\theta$

$\Rightarrow 12-12+25 = \sqrt{16+9+25}\sqrt{9+16+25}\cos\theta$

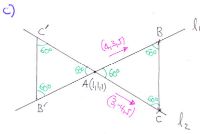
$\Rightarrow 25 = \sqrt{50}\sqrt{50}\cos\theta$

$\Rightarrow 25 = 50\cos\theta$

$\Rightarrow \cos\theta = \frac{1}{2}$

$\Rightarrow \theta = \frac{\pi}{3}$ OR $60°$

c)



WE FOUND EARLIER $|4,3,5| = \sqrt{50} = 5\sqrt{2}$

$|3,-4,5| = \sqrt{50} = 5\sqrt{2}$

THIS MEANS THAT INCREASING λ OF μ BY 1 UNIT, ADVANCES UP BY A DISTANCE OF $5\sqrt{2}$ ON EITHER LINE.

SINCE THE SIDE LENGTH IS $15\sqrt{2}$ THIS CORRESPONDS TO 3 UNITS IN λ OR μ

AT THE INTERSECTION $A(1,1,1)$ WE FOUND $\lambda=-2$ & $\mu=1$

$\therefore$ B MUST BE AT $\lambda=-2\pm3$ & C MUST BE AT $\mu=1\pm3$

$\lambda=1$: $B(13,10,16)$ $\qquad$ $\mu=4$: $C(10,-11,16)$

$\lambda=-5$: $B(-11,-8,-14)$ $\qquad$ $\mu=-2$: $C(-8,13,-14)$

FINALLY WE NEED TO FIND THE DISTANCE $|BC|$ FOR ANY PAIR TO SEE WHICH WAY THEY PAIR UP (OTHERWISE WE MAY PICK THE "OBTUSE" TRIANGLES BETWEEN THE 2 EQUILATERAL

SAY FOR THE "FIRST PAIR" $|BC| = \sqrt{(13-10)^2+(10+11)^2+\cancel{(16-16)^2}}$

$= \sqrt{9+441} = \sqrt{450} = \sqrt{225\times2}$

$= 15\sqrt{2}$

SO THIS IS ONE OF THE CORRECT PAIRS

$\therefore$ $B(13,10,16)$, $C(10,-11,16)$ OR $B(-11,-8,-14)$, $C(-8,13,-14)$

Question 123 (****+)

The points $A(3,2,14)$, $B(0,1,13)$ and $C(5,6,8)$ are defined with respect to a fixed origin O.

a) Show that the cosine of the angle ABC is $\dfrac{3}{\sqrt{33}}$.

The straight line L passes through A and it is parallel to the vector $\overrightarrow{BC}$.

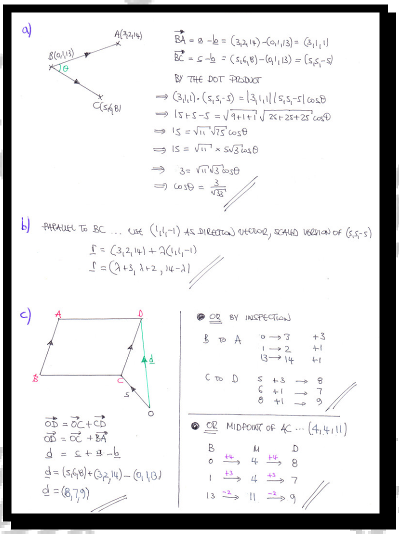
b) Find a vector equation of L.

The point D lies on L so that $ABCD$ is a parallelogram.

c) Find the coordinates of D.

d) If instead $ABCD$ is an isosceles trapezium and the point D still lies on L, determine the new coordinates of D.

$\boxed{}$, $\boxed{\mathbf{r} = 3\mathbf{i}+2\mathbf{j}+14\mathbf{k}+\lambda(\mathbf{i}+\mathbf{j}-\mathbf{k})}$, $\boxed{D(8,7,9)}$, $\boxed{D(6,5,11)}$



a)

$\overrightarrow{BA} = \mathbf{a} - \mathbf{b} = (3,2,14)-(0,1,13) = (3,1,1)$

$\overrightarrow{BC} = \mathbf{c} - \mathbf{b} = (5,6,8)-(0,1,13) = (5,5,-5)$

BY THE DOT PRODUCT

$\Rightarrow (3,1,1)\cdot(5,5,-5) = |3,1,1||5,5,-5|\cos\theta$

$\Rightarrow 15+5-5 = \sqrt{9+1+1}\sqrt{25+25+25}\cos\theta$

$\Rightarrow 15 = \sqrt{11}\sqrt{75}\cos\theta$

$\Rightarrow 15 = \sqrt{11}\times 5\sqrt{3}\cos\theta$

$\Rightarrow 3 = \sqrt{11}\sqrt{3}\cos\theta$

$\Rightarrow \cos\theta = \frac{3}{\sqrt{33}}$

b) PARALLEL TO BC ... USE $(1,1,-1)$ AS DIRECTION VECTOR, SCALED VERSION OF $(5,5,-5)$

$\mathbf{r} = (3,2,14) + \lambda(1,1,-1)$

$\mathbf{r} = (\lambda+3, \lambda+2, 14-\lambda)$

c)

$\overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{CD}$

$\overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{BA}$

$\mathbf{d} = \mathbf{c} + \mathbf{a} - \mathbf{b}$

$\mathbf{d} = (5,6,8)+(3,2,14)-(0,1,13)$

$\mathbf{d} = (8,7,9)$

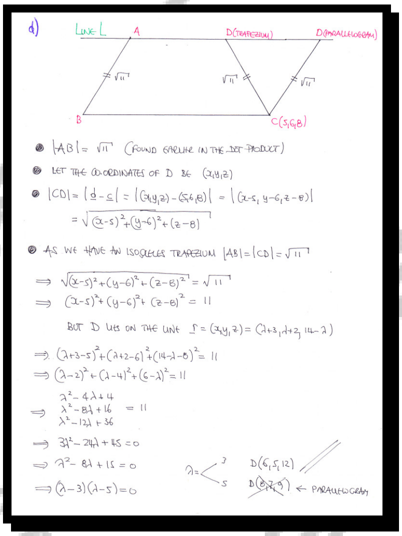
OR BY INSPECTION

B TO A: $0 \to 3$ $+3$; $1 \to 2$ $+1$; $13 \to 14$ $+1$

C TO D: 5 $+3$ $\to 8$; 6 $+1$ $\to 7$; 8 $+1$ $\to 9$

OR MIDPOINT OF AC ... $(4,4,11)$

B		M		D
0	$\xrightarrow{+4}$	4	$\xrightarrow{+4}$	8
1	$\xrightarrow{+3}$	4	$\xrightarrow{+3}$	7
13	$\xrightarrow{-2}$	11	$\xrightarrow{-2}$	9



- $|AB| = \sqrt{11}$ (FOUND EARLIER IN THE DOT PRODUCT)
- LET THE CO-ORDINATES OF D BE (x,y,z)
- $|CD| = |\mathbf{d}-\mathbf{c}| = |(x,y,z)-(5,6,8)| = |(x-5, y-6, z-8)|$ $= \sqrt{(x-5)^2+(y-6)^2+(z-8)}$
- AS WE HAVE AN ISOSCELES TRAPEZIUM $|AB| = |CD| = \sqrt{11}$

$\Rightarrow \sqrt{(x-5)^2+(y-6)^2+(z-8)^2} = \sqrt{11}$

$\Rightarrow (x-5)^2+(y-6)^2+(z-8)^2 = 11$

BUT D LIES ON THE LINE $\mathbf{r} = (x,y,z) = (\lambda+3, \lambda+2, 14-\lambda)$

$\Rightarrow (\lambda+3-5)^2+(\lambda+2-6)^2+(14-\lambda-8)^2 = 11$

$\Rightarrow (\lambda-2)^2+(\lambda-4)^2+(6-\lambda)^2 = 11$

$\Rightarrow \begin{matrix}\lambda^2-4\lambda+4\\ \lambda^2-8\lambda+16\\ \lambda^2-12\lambda+36\end{matrix} = 11$

$\Rightarrow 3\lambda^2-24\lambda+45=0$

$\Rightarrow \lambda^2-8\lambda+15=0$

$\Rightarrow (\lambda-3)(\lambda-5)=0$

$\lambda = \begin{cases} 3 & D(6,5,12) \\ 5 & \cancel{D(8,7,9)} \leftarrow \text{PARALLELOGRAM} \end{cases}$

Question 124 (****+)

With respect to a fixed origin O, the point A and the point B have position vectors $\mathbf{i} - 7\mathbf{j} + 5\mathbf{k}$ and $-9\mathbf{j} + 6\mathbf{k}$, respectively.

a) Find a vector equation of the straight line l which passes through A and B.

A variable vector is defined as

$$\mathbf{p} = (p+6)\mathbf{i} + (2p+3)\mathbf{j} - p\mathbf{k},$$

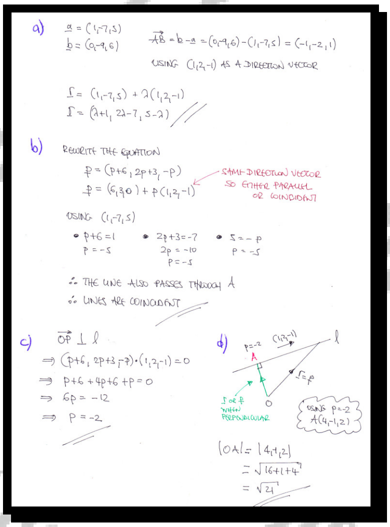
where p is a scalar parameter.

b) Show that for all values of p, the point P with position vector $\mathbf{p}$, lies on l.

c) Determine the value of p for which $\overrightarrow{OP}$ is perpendicular to l.

d) Hence, or otherwise, find the shortest distance of l from the origin O.

$\boxed{}$, $\boxed{\mathbf{r} = \mathbf{i} - 7\mathbf{j} + 5\mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} - \mathbf{k})}$, $\boxed{p = -2}$, $\boxed{\text{shortest distance} = \sqrt{21}}$



a) $\underline{a} = (1,-7,5)$
$\underline{b} = (0,-9,6)$
$\overrightarrow{AB} = \underline{b} - \underline{a} = (0,-9,6) - (1,-7,5) = (-1,-2,1)$
USING $(1,2,-1)$ AS A DIRECTION VECTOR
$\underline{r} = (1,-7,5) + \lambda(1,2,-1)$
$\underline{r} = (\lambda+1, 2\lambda-7, 5-\lambda)$

b) REWRITE THE EQUATION
$\underline{p} = (p+6, 2p+3, -p)$
$\underline{p} = (6,3,0) + p(1,2,-1)$
SAME DIRECTION VECTOR SO EITHER PARALLEL OR COINCIDENT
USING $(1,-7,5)$
• $p+6=1$, $p=-5$
• $2p+3=-7$, $2p=-10$, $p=-5$
• $5=-p$, $p=-5$
∴ THE LINE ALSO PASSES THROUGH A
∴ LINES ARE COINCIDENT

c) $\overrightarrow{OP} \perp l$
$\Rightarrow (p+6, 2p+3, -p)\cdot(1,2,-1) = 0$
$\Rightarrow p+6+4p+6+p=0$
$\Rightarrow 6p=-12$
$\Rightarrow p=-2$

d)

$|OA| = |4,-1,2|$
$= \sqrt{16+1+4}$
$= \sqrt{21}$

Question 125 (****+)

The points with coordinates $A(7,6,10)$, $B(6,5,6)$ and $C(1,0,4)$ are the vertices of the parallelogram $ABCD$.

a) Find …

i. … the coordinates of D.

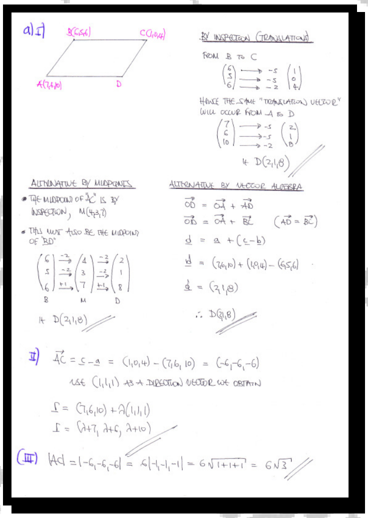
ii. … a vector equation of the straight line l which passes through the points A and C.

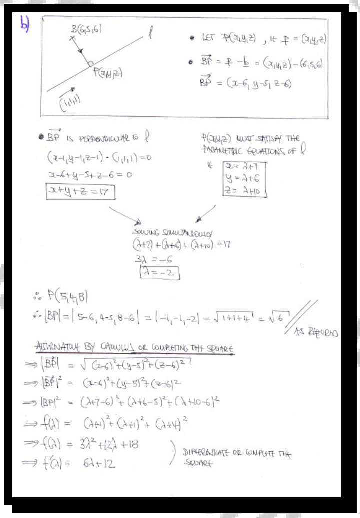
iii. … the distance AC.

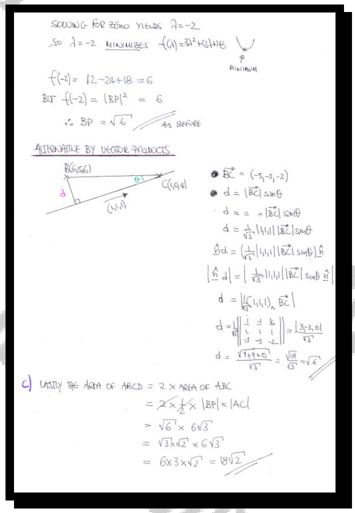
b) Show that the shortest distance of l from B is $\sqrt{6}$ units.

c) Hence find the exact area of the parallelogram $ABCD$.

$\boxed{}$, $\boxed{D(2,1,8)}$, $\boxed{\mathbf{r} = 7\mathbf{i}+6\mathbf{j}+10\mathbf{k}+\lambda(\mathbf{i}+\mathbf{j}+\mathbf{k})}$, $\boxed{|AC| = 6\sqrt{3}}$, $\boxed{18\sqrt{2}}$







Question 126 (****+)

The straight lines l_1 and l_2 have the following vector equations

$$\mathbf{r}_1 = \mathbf{i} - 5\mathbf{j} + \lambda(4\mathbf{j} - \mathbf{k})$$

$$\mathbf{r}_2 = 4\mathbf{i} - 3\mathbf{j} + \mathbf{k} + \mu(3\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$$

where λ and μ are scalar parameters.

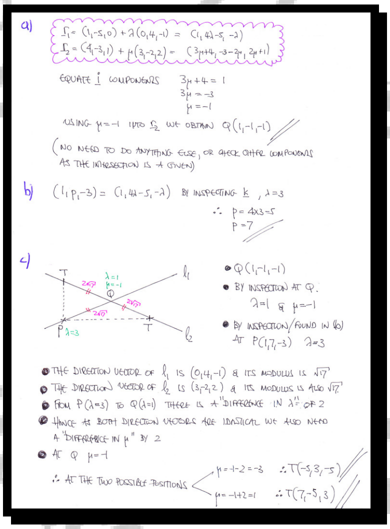
a) Given that l_1 and l_2 intersect at some point Q, find the position vector of Q.

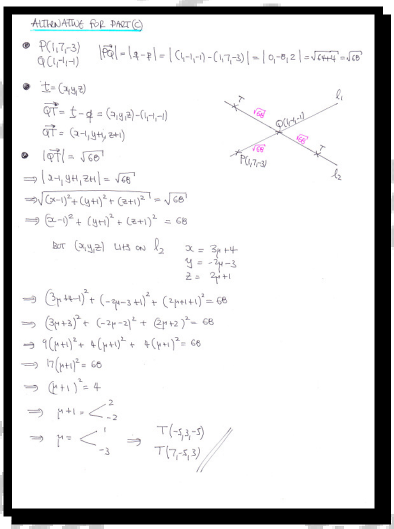
b) Given further that the point P lies on l_1 and has position vector $\mathbf{i} + p\mathbf{j} - 3\mathbf{k}$, find the value of p.

The point T lies on l_2 so that $|\overrightarrow{PQ}| = |\overrightarrow{QT}|$.

c) Determine the two possible position vectors for T.

$\boxed{}$, $\boxed{\mathbf{q} = \mathbf{i} - \mathbf{j} - \mathbf{k}}$, $\boxed{p = 7}$, $\boxed{\mathbf{t} = -5\mathbf{i} + 3\mathbf{j} - 5\mathbf{k} \text{ or } \mathbf{t} = 7\mathbf{i} - 5\mathbf{j} + 3\mathbf{k}}$





Question 127 (****+)

The point P has position vector $2\mathbf{i}+2\mathbf{j}+21\mathbf{k}$.

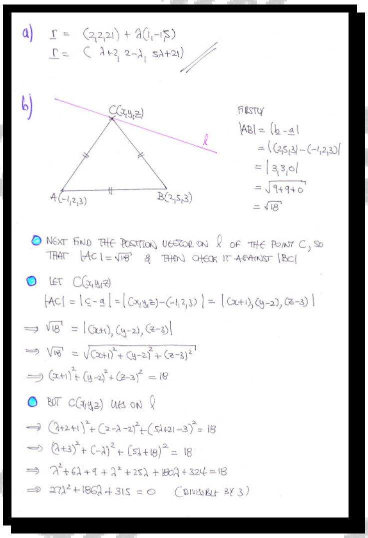
a) Find the vector equation of the straight line l which passes through P and is parallel to the vector $\mathbf{i}-\mathbf{j}+5\mathbf{k}$.

The points A and B have coordinates $(-1,2,3)$ and $(2,5,3)$, respectively.

The point C lies on l so that the triangle ABC is equilateral.

b) Find the two possible position vectors for C.

$\boxed{}$, $\boxed{\mathbf{r}=2\mathbf{i}+2\mathbf{j}+21\mathbf{k}+\lambda(\mathbf{i}-\mathbf{j}+5\mathbf{k})}$, $\boxed{\mathbf{c}=-\mathbf{i}+5\mathbf{j}+6\mathbf{k} \quad \text{or} \quad \mathbf{c}=-\frac{17}{9}\mathbf{i}+\frac{53}{9}\mathbf{j}+\frac{14}{9}\mathbf{k}}$



a) $\underline{r} = (2,2,21) + \lambda(1,-1,5)$

$\underline{r} = (\lambda+2,\ 2-\lambda,\ 5\lambda+21)$

b)

FIRSTLY

$|AB| = |\underline{b}-\underline{a}|$
$= |(2,5,3)-(-1,2,3)|$
$= |3,3,0|$
$= \sqrt{9+9+0}$
$= \sqrt{18}$

NEXT FIND THE POSITION VECTOR ON l OF THE POINT C, SO THAT $|AC|=\sqrt{18}$ & THEN CHECK IT AGAINST $|BC|$

LET $C(x,y,z)$

$|AC| = |\underline{c}-\underline{a}| = |(x,y,z)-(-1,2,3)| = |(x+1),(y-2),(z-3)|$

$\Rightarrow \sqrt{18} = |(x+1),(y-2),(z-3)|$

$\Rightarrow \sqrt{18} = \sqrt{(x+1)^2+(y-2)^2+(z-3)^2}$

$\Rightarrow (x+1)^2+(y-2)^2+(z-3)^2 = 18$

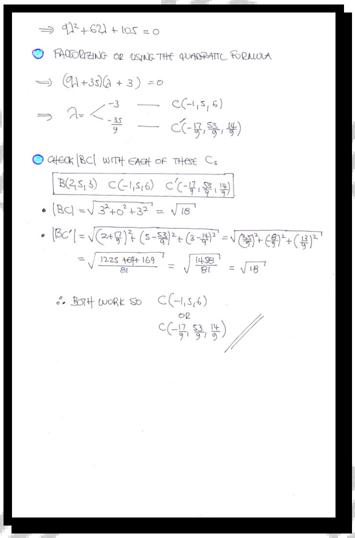
BUT $C(x,y,z)$ LIES ON l

$\Rightarrow (\lambda+2+1)^2+(2-\lambda-2)^2+(5\lambda+21-3)^2 = 18$

$\Rightarrow (\lambda+3)^2+(-\lambda)^2+(5\lambda+18)^2 = 18$

$\Rightarrow \lambda^2+6\lambda+9+\lambda^2+25\lambda^2+180\lambda+324 = 18$

$\Rightarrow 27\lambda^2+186\lambda+315 = 0$ (DIVISIBLE BY 3)



$\Rightarrow 9\lambda^2+62\lambda+105=0$

FACTORIZING OR USING THE QUADRATIC FORMULA

$\Rightarrow (9\lambda+35)(\lambda+3)=0$

$\Rightarrow \lambda = -3$ — $C(-1,5,6)$; $\lambda = -\frac{35}{9}$ — $C'(-\frac{17}{9},\frac{53}{9},\frac{14}{9})$

CHECK $|BC|$ WITH EACH OF THESE Cs

$\boxed{B(2,5,3) \quad C(-1,5,6) \quad C'(-\frac{17}{9},\frac{53}{9},\frac{14}{9})}$

- $|BC| = \sqrt{3^2+0^2+3^2} = \sqrt{18}$
- $|BC'| = \sqrt{(2+\frac{17}{9})^2+(5-\frac{53}{9})^2+(3-\frac{14}{9})^2} = \sqrt{(\frac{35}{9})^2+(-\frac{8}{9})^2+(\frac{13}{9})^2}$
  $= \sqrt{\frac{1225+64+169}{81}} = \sqrt{\frac{1458}{81}} = \sqrt{18}$

$\therefore$ BOTH WORK SO $C(-1,5,6)$ OR $C(-\frac{17}{9},\frac{53}{9},\frac{14}{9})$

Question 128 (****+)

The quadrilateral $ABCD$ is a rectangle with the vertex A having coordinates $(2,1,2)$.

The diagonals of the rectangle intersect at the point with coordinates $(7,0,4)$.

a) Find the coordinates of the point C.

The points B and D both lie on the straight line with vector equation

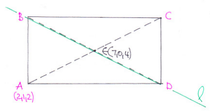
$$\mathbf{r} = 4\mathbf{i} + 15\mathbf{j} + 10\mathbf{k} + \lambda(\mathbf{i} - 5\mathbf{j} - 2\mathbf{k}),$$

where λ is a scalar parameter.

b) Determine the coordinates of B and D.

$\boxed{}$, $\boxed{C(12,-1,6)}$, $\boxed{B(6,5,6),\ D(8,-5,2) \text{ in any order}}$

a)



BY INSPECTION (SEQUENTIAL COUNTING) AS E IS THE MIDPOINT OF AC

$$\begin{pmatrix}2\\1\\2\end{pmatrix} \xrightarrow[+2]{+5,\ -1} \begin{pmatrix}7\\0\\4\end{pmatrix} \xrightarrow[+2]{+5,\ -1} \begin{pmatrix}12\\-1\\6\end{pmatrix} \qquad \therefore C(12,-1,6)$$

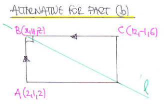
b)
- $|AE| = |\underline{e} - \underline{a}| = |(7,0,4)-(2,1,2)| = |5,-1,2| = \sqrt{25+1+4} = \sqrt{30}$
- THE DIRECTION VECTOR OF ℓ IS $(1,-5,-2)$ & $|1,-5,-2| = \sqrt{30}$
- THE EQUATION OF ℓ IS
$$\underline{r} = (4,15,10) + \lambda(1,-5,-2)$$
$$\underline{r} = (\lambda+4,\ 15-5\lambda,\ 10-2\lambda)$$

BY INSPECTION THE VALUE OF λ AT $E(7,0,4)$ IS 3

- AS $|AE|$ = LENGTH OF THE DIRECTION VECTOR OF ℓ THEN AT THE POINTS B & D $\lambda = 3 \pm 1$
$$\lambda = \begin{cases}4\\2\end{cases}$$
$$\therefore B(8,-5,2) \text{ \& } D(6,5,6)$$

IN EITHER ORDER

ALTERNATIVE FOR PART (b)



- LET $B(x,y,z) \implies \underline{b} = (x,y,z)$
- $\overrightarrow{AB} = \underline{b} - \underline{a} = (x-2,\ y-1,\ z-2)$
- $\overrightarrow{CB} = \underline{b} - \underline{c} = (x-12,\ y+1,\ z-6)$
- $\hat{ABC} = 90$, SO $\overrightarrow{AB} \cdot \overrightarrow{CB} = 0$
$$(x-2,\ y-1,\ z-2)\cdot(x-12,\ y+1,\ z-6) = 0$$
$$\boxed{(x-2)(x-12) + (y-1)(y+1) + (z-2)(z-6) = 0}$$
- BUT POINT B LIES ON ℓ
$$\underline{r} = (4,15,10) + \lambda(1,-5,-2)$$
$$\underline{r} = (\lambda+4,\ 15-5\lambda,\ 10-2\lambda)$$
$$\boxed{\begin{aligned}x &= \lambda+4\\ y &= 15-5\lambda\\ z &= 10-2\lambda\end{aligned}}$$
- SOLVING SIMULTANEOUSLY THE TWO "BOXED" EXPRESSIONS
$$(\lambda+2)(\lambda-8) + (14-5\lambda)(16-5\lambda) + (8-2\lambda)(4-2\lambda) = 0$$
$$(\lambda^2-6\lambda-16) + (25\lambda^2-150\lambda+224) + (4\lambda^2-24\lambda+32) = 0$$
$$30\lambda^2 - 180\lambda + 240 = 0$$
$$\lambda^2 - 6\lambda + 8 = 0$$
$$(\lambda-2)(\lambda-4) = 0$$
$$\lambda = \begin{cases}2\\4\end{cases} \implies \begin{matrix}B(6,5,6)\\B(8,-5,2)\end{matrix}$$

AS BEFORE

Question 129 **(****+)**

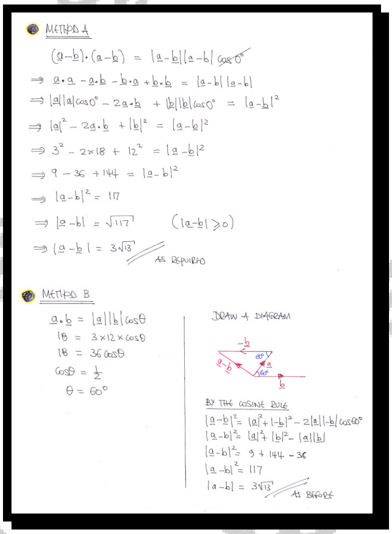
The vectors $\mathbf{a}$ and $\mathbf{b}$ are such so that

$$|\mathbf{a}| = 3, \ |\mathbf{b}| = 12 \quad \text{and} \quad \mathbf{a}\cdot\mathbf{b} = 18 .$$

Show clearly that

$$|\mathbf{a} - \mathbf{b}| = 3\sqrt{13} .$$

$\boxed{}$, $\boxed{\text{proof}}$



METHOD A

$$(\underline{a}-\underline{b})\cdot(\underline{a}-\underline{b}) = |\underline{a}-\underline{b}||\underline{a}-\underline{b}|\cancel{\cos 0^\circ}$$

$$\Rightarrow \underline{a}\cdot\underline{a} - \underline{a}\cdot\underline{b} - \underline{b}\cdot\underline{a} + \underline{b}\cdot\underline{b} = |\underline{a}-\underline{b}||\underline{a}-\underline{b}|$$

$$\Rightarrow |\underline{a}||\underline{a}|\cos 0^\circ - 2\underline{a}\cdot\underline{b} + |\underline{b}||\underline{b}|\cos 0^\circ = |\underline{a}-\underline{b}|^2$$

$$\Rightarrow |\underline{a}|^2 - 2\underline{a}\cdot\underline{b} + |\underline{b}|^2 = |\underline{a}-\underline{b}|^2$$

$$\Rightarrow 3^2 - 2\times 18 + 12^2 = |\underline{a}-\underline{b}|^2$$

$$\Rightarrow 9 - 36 + 144 = |\underline{a}-\underline{b}|^2$$

$$\Rightarrow |\underline{a}-\underline{b}|^2 = 117$$

$$\Rightarrow |\underline{a}-\underline{b}| = \sqrt{117} \qquad (|\underline{a}-\underline{b}| \geqslant 0)$$

$$\Rightarrow |\underline{a}-\underline{b}| = 3\sqrt{13}$$

AS REQUIRED

METHOD B

$$\underline{a}\cdot\underline{b} = |\underline{a}||\underline{b}|\cos\theta$$
$$18 = 3\times 12\times\cos\theta$$
$$18 = 36\cos\theta$$
$$\cos\theta = \tfrac{1}{2}$$
$$\theta = 60^\circ$$

DRAW A DIAGRAM

BY THE COSINE RULE

$$|\underline{a}-\underline{b}|^2 = |\underline{a}|^2 + |-\underline{b}|^2 - 2|\underline{a}||-\underline{b}|\cos 60^\circ$$
$$|\underline{a}-\underline{b}|^2 = |\underline{a}|^2 + |\underline{b}|^2 - |\underline{a}||\underline{b}|$$
$$|\underline{a}-\underline{b}|^2 = 9 + 144 - 36$$
$$|\underline{a}-\underline{b}|^2 = 117$$
$$|\underline{a}-\underline{b}| = 3\sqrt{13}$$

AS BEFORE

Question 130 (****+)

It is given that

$$\mathbf{w} = \mathbf{u} + \mathbf{v},$$

where $\mathbf{w} = 2\mathbf{i} + 8\mathbf{j} - \mathbf{k}$.

Given further that $\mathbf{u}$ is in the direction $\mathbf{i} + \mathbf{j} + \mathbf{k}$, and the vectors $\mathbf{u}$ and $\mathbf{v}$ are perpendicular to one another, determine $\mathbf{u}$ and $\mathbf{v}$ in component form.

$\boxed{}$, $\boxed{\mathbf{u} = 3\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}}$, $\boxed{\mathbf{v} = -\mathbf{i} + 5\mathbf{j} - 4\mathbf{k}}$

METHOD A

$\underline{w} = \underline{u} + \underline{v}$

DOTTING THE EQUATION BY $\underline{u}$

$\underline{w} \cdot \underline{u} = \underline{u} \cdot \underline{u} + \cancel{\underline{v} \cdot \underline{u}}$ PERPENDICULAR

LET $\underline{u} = \lambda(1,1,1)$, $\lambda \neq 0$

$\Rightarrow \begin{pmatrix} 2 \\ 8 \\ -1 \end{pmatrix} \cdot \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\Rightarrow \lambda(2+8-1) = \lambda^2(1+1+1)$

$\Rightarrow 9\lambda = 3\lambda^2$

$\Rightarrow 3 = \lambda \quad (\lambda \neq 0)$

FINALLY AS $\underline{w} = \underline{u} + \underline{v}$

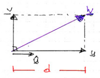
$\begin{pmatrix} 2 \\ 8 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} + \underline{v}$

$\therefore \underline{v} = \begin{pmatrix} -1 \\ 5 \\ -4 \end{pmatrix}$ & $\underline{u} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$

METHOD B

START WITH A DIAGRAM WITH $\underline{u}$ & $\underline{v}$ PERPENDICULAR, AND $\underline{w}$ THEIR "RESULTANT

PROJECT $\underline{w}$ ONTO THE DIRECTION OF $\underline{u}$



$d = \underline{w} \cdot \hat{\underline{u}}$

THUS WE HAVE

$\Rightarrow \underline{u} = d\hat{\underline{u}}$

$\Rightarrow \underline{u} = (\underline{w} \cdot \hat{\underline{u}})\hat{\underline{u}}$

$\Rightarrow \underline{u} = \left[(2,8,-1) \cdot \frac{(1,1,1)}{\sqrt{3}}\right] \frac{(1,1,1)}{\sqrt{3}}$

$\Rightarrow \underline{u} = \frac{1}{3}\left[(2,8,-1) \cdot (1,1,1)\right](1,1,1)$

$\Rightarrow \underline{u} = \frac{1}{3}(2+8-1)(1,1,1)$

$\Rightarrow \underline{u} = 3(1,1,1)$

$\Rightarrow \underline{u} = (3,3,3)$

& FINALLY SINCE $\underline{w} = \underline{u} + \underline{v}$

$(2,8,-1) = (3,3,3) + \underline{v}$

$\underline{v} = (-1,5,-4)$

METHOD C

LET $\underline{v} = (x,y,z)$, $\underline{u} = \lambda(1,1,1)$, $\lambda \neq 0$

$\underline{v} \perp \underline{u} \Rightarrow (x,y,z) \cdot \lambda(1,1,1) = 0$

$\Rightarrow \lambda(x+y+z) = 0$

$\Rightarrow x+y+z = 0$

NOW WE HAVE $\underline{w} = \underline{u} + \underline{v}$

$\Rightarrow (2,8,-1) = (\lambda,\lambda,\lambda) + (x,y,z)$

$\Rightarrow \begin{cases} x+\lambda = 2 \\ y+\lambda = 8 \\ z+\lambda = -1 \end{cases}$

ADDING THESE EQUATIONS

$x+y+z+3\lambda = 9$

$0 + 3\lambda = 9$

$\lambda = 3$

$\therefore \underline{u} = (3,3,3)$ & $\underline{v} = (x,y,z) = (-1,5,-4)$

Question 131 (****+)

The points $A(3,3,2)$, $B(6,4,3)$ and $C(5,1,4)$ are referred with respect to a fixed origin O. The point M is the midpoint of AC.

a) Show that $\overrightarrow{BM}$ is perpendicular to $\overrightarrow{AC}$.

The point D is such so that $ABCD$ is a kite with an area of $6\sqrt{6}$.

The straight line BD is a line of symmetry for the kite $ABCD$.

b) Find the coordinates of D.

$\boxed{\text{SPC}}$, $\boxed{D(0,-2,3)}$

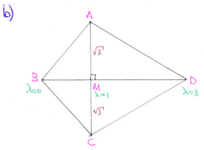
a) $A(3,3,2)$ $B(6,4,3)$ $C(5,1,4)$

- $M(4,2,3)$ BY INSPECTION
- $\overrightarrow{BM} = \underline{m} - \underline{b} = (4,2,3) - (6,4,3) = (-2,-2,0)$
- $\overrightarrow{AC} = \underline{c} - \underline{a} = (5,1,4) - (3,3,2) = (2,-2,2)$

$\overrightarrow{BM} \cdot \overrightarrow{AC} = (-2,-2,0) \cdot (2,-2,2) = -4+4+0 = 0$

INDEED PERPENDICULAR

b)



- FIRSTLY $|AC| = |2,-2,2| = \sqrt{4+4+4} = \sqrt{12} = 2\sqrt{3}$

  $|AM| = |MC| = \sqrt{3}$

- AREA OF KITE = AREA OF 2 TRIANGLES

$$6\sqrt{6} = 2\times\left[\tfrac{1}{2}|BD||AM|\right]$$
$$6\sqrt{6} = |BD||AM|$$
$$6\sqrt{6} = |BD|\sqrt{3}$$
$$6\sqrt{2}\cancel{\sqrt{3}} = |BD|\cancel{\sqrt{3}}$$
$$|BD| = 6\sqrt{2}$$

- SET AN EQUATION OF LINE STARTING AT B ($\lambda=0$) AND "HEADING" IN THE DIRECTION BM (IE AT M $\lambda=1$)

$$\underline{r} = \underline{b} + \lambda\overrightarrow{BM}$$
$$\underline{r} = (6,4,3) + \lambda(-2,-2,0)$$
$$r = (6-2\lambda, 4-2\lambda, 3)$$

BUT $|\overrightarrow{BM}| = |-2,-2,0| = \sqrt{4+4+0} = \sqrt{8} = 2\sqrt{2}$

$|\overrightarrow{BD}| = 6\sqrt{2}$

$\therefore$ BY INSPECTION $\lambda=3$ AT D

$\therefore D(0,-2,3)$

Question 132 (****+)

Relative to a fixed origin O, the straight lines l and m have vector equations

$$\mathbf{r}_1 = \begin{pmatrix} p \\ 4 \\ 5 \end{pmatrix} + t\begin{pmatrix} q \\ -1 \\ 2 \end{pmatrix} \quad \text{and} \quad \mathbf{r}_2 = \begin{pmatrix} 9 \\ 0 \\ 16 \end{pmatrix} + s\begin{pmatrix} 1 \\ -2 \\ 7 \end{pmatrix},$$

where t and s are scalar parameters, and p and q are scalar constants.

The point A is the intersection of l and m, and the cosine of acute angle θ between l and m is $\frac{1}{3}\sqrt{6}$.

a) Find the value of p and the value of q, given that q is a positive integer.

b) Determine the coordinates of A.

The point B has coordinates $B(12,5,9)$.

c) Find the cosine of the acute angle φ between AB and l.

d) Hence show, without the use of any calculating aid, that

$$\varphi = 2\theta.$$

$\boxed{p=0}$, $\boxed{q=2}$, $\boxed{A(8,2,9)}$, $\boxed{\cos\varphi = \frac{1}{3}}$

(a) $\underline{r}_1 = (p,4,5) + t(q,-1,2) = (p+qt,\ 4-t,\ 2t+5)$

$\underline{r}_2 = (9,0,16) + s(1,-2,7) = (s+9,\ -2s,\ 7s+16)$

BY DOT PRODUCT: $(q,-1,2)\cdot(1,-2,7) = |q,-1,2||1,-2,7|\cos\theta$

$\Rightarrow q+2+14 = \sqrt{q^2+1+4}\sqrt{1+4+49}\,\frac{\sqrt{6}}{3}$

$\Rightarrow q+16 = \sqrt{q^2+5}\sqrt{54}\,\frac{\sqrt{6}}{3}$

$\Rightarrow q+16 = 6\sqrt{q^2+5}$

$\Rightarrow (q+16)^2 = 36(q^2+5)$

$\Rightarrow q^2+32q+256 = 36q^2+180$

$\Rightarrow 0 = 35q^2-32q-76$

BY QUADRATIC FORMULA OR OTHERWISE

$\Rightarrow 0 = (q-2)(35q+38)$

$\therefore q = 2$ or $\cancel{-\frac{38}{35}}$ $(q>0)$

EQUATE COMPONENTS IN THE LINES, GIVEN THEY INTERSECT

(i): $p+2t = s+9$ ①

(j): $4-t = -2s$ ② $\Rightarrow \boxed{t = 2s+4}$ SUB INTO ① & ③

(k): $2t+5 = 7s+16$ ③

$\left.\begin{array}{l} p+2(2s+4) = s+9 \\ 2(2s+4)+5 = 7s+16 \end{array}\right\} \Rightarrow \left.\begin{array}{l} p+4s+8 = s+9 \\ 4s+8+5 = 7s+16 \end{array}\right\} \Rightarrow \left.\begin{array}{l} p = -3s+1 \\ -3 = 3s \end{array}\right\}$

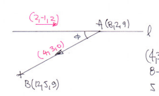
$\therefore s = -1$

$p = 4$

$(t=2)$

USING $t=2$ (OR $s=-1$) WE OBTAIN $A(8,2,9)$

(c) $\overrightarrow{AB} = \underline{b} - \underline{a} = (12,5,9)-(8,2,9) = (4,3,0)$



$(4,3,0)\cdot(2,-1,2) = |4,3,0||2,-1,2|\cos\phi$

$8-3+0 = \sqrt{16+9+0}\sqrt{4+1+4}\cos\phi$

$5 = 5\times 3\cos\phi$

$\cos\phi = \frac{1}{3}$

(d) $\cos 2\theta = 2\cos^2\theta - 1 = 2\left(\frac{\sqrt{6}}{3}\right)^2 - 1 = 2\times\frac{6}{9} - 1 = \frac{1}{3} = \cos\phi$

Question 133 (****+)

The straight lines l_1 and l_2 have the vector equations given below

$$\mathbf{r}_1 = 3\mathbf{i} + \mathbf{j} + 7\mathbf{k} + \lambda(2\mathbf{i} - \mathbf{j} + 3\mathbf{k}),$$

$$\mathbf{r}_2 = 3\mathbf{i} + 3\mathbf{k} + \mu(2\mathbf{i} - 2\mathbf{j} - \mathbf{k}),$$

where λ and μ are scalar parameters.

a) Show that l_1 and l_2 intersect at some point P and find its coordinates.

The points A and C lie on l_1 and the points B and D lie on l_2, such that $ABCD$ forms a parallelogram.

The point A has coordinates $(7,-1,13)$.

b) Find …

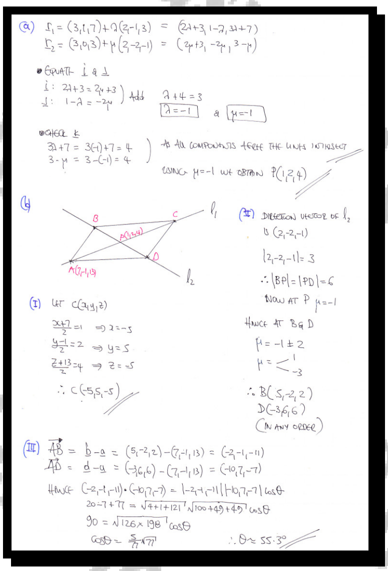
i. … the coordinates of C.

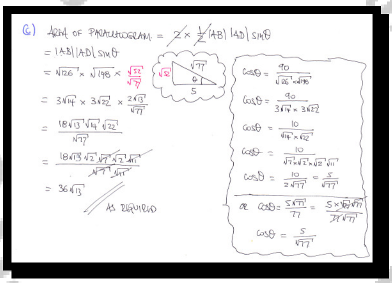
ii. … the coordinates of B and D, given further that $|BD| = 12$.

iii. … the angle $\measuredangle BAD$.

c) Show that the exact area of the parallelogram $ABCD$ is $36\sqrt{13}$.

$\boxed{P(1,2,4)}$, $\boxed{C(-5,5,-5)}$, $\boxed{B(5,-2,2),\ D(-3,6,6) \text{ in any order}}$, $\boxed{\measuredangle BAD \approx 55.3°}$





Question 134 (****+)

The straight lines l_1 and l_2 have the following vector equations

$$\mathbf{r}_1 = 7\mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} + a\mathbf{k})$$

$$\mathbf{r}_2 = 3\mathbf{i} + b\mathbf{j} + 5\mathbf{k} + \mu(\mathbf{i} - \mathbf{k}),$$

where λ and μ are scalar parameters, and a and b are scalar constants.

It is further given that the point A is the intersection of l_1 and l_2, and the acute angle between l_1 and l_2 is $60°$.

Find in any order …

… the two possible pairings for the value of a and the value b.

… the possible coordinates of A for each possible pair of a and b.

$\boxed{}$, $\boxed{a = 0 \text{ with } b = 4 \text{ and } A(5,4,3)}$, $\boxed{a = 4 \text{ with } b = \frac{12}{5} \text{ and } A\left(\frac{33}{5}, \frac{12}{5}, \frac{7}{5}\right)}$

$$\boxed{\begin{aligned} r_1 &= (7,2,3) + \lambda(1,-1,a) = (\lambda+7,\ 2-\lambda,\ a\lambda+3) \\ r_2 &= (3,b,5) + \mu(1,0,-1) = (\mu+3,\ b,\ 5-\mu) \end{aligned}}$$

- BY THE DOT PRODUCT USING THE FACT THAT THE LINES INTERSECT AT 60°

$\Rightarrow (1,-1,a)\cdot(1,0,-1) = |1,-1,a||1,0,-1|\cos 60°$

$\Rightarrow 1+0-a = \sqrt{1+1+a^2}\sqrt{1+0+1} \times \frac{1}{2}$

$\Rightarrow 1-a = \sqrt{a^2+2}\sqrt{2} \times \frac{1}{2}$

$\Rightarrow 2(1-a) = \sqrt{a^2+2}\sqrt{2}$

$\Rightarrow 4(1-a)^2 = (a^2+2)\times 2$

$\Rightarrow 4(1-2a+a^2) = 2(a^2+2)$

$\Rightarrow 2(a^2-2a+1) = a^2+2$

$\Rightarrow 2a^2-4a+\cancel{2} = a^2+\cancel{2}$

$\Rightarrow 2a^2-4a=0$

$\Rightarrow a(a-4)=0$

$\Rightarrow a = 0 \text{ or } 4$

- NEXT WE NEED TO FIND THE VALUE OF b & THE CORRESPONDING POINT OF INTERSECTION, FOR EACH VALUE OF a

IF $a=0$

$r_1 = (\lambda+7, 2-\lambda, 3)$

$r_2 = (\mu+3, b, 5-\mu)$

EQUATE COMPONENTS

$\underline{k}$: $3 = 5-\mu$

$\boxed{\mu=2}$

$\underline{i}$: $\lambda+7=\mu+3$

$\boxed{\lambda=-2}$

$\underline{j}$: $2-\lambda = b$

$\boxed{b=4}$

& INTERSECTION IS AT $A(5,4,3)$

IF $a=4$

$r_1 = (\lambda+7, 2-\lambda, 4\lambda+3)$

$r_2 = (\mu+3, b, 5-\mu)$

EQUATING COMPONENTS

$\underline{i}$: $\lambda+7=\mu+3$

$\underline{k}$: $4\lambda+3 = 5-\mu$

Adding the equations yields

$5\lambda+10=8$

$5\lambda=-2$

$\boxed{\lambda=-\frac{2}{5}}$

HENCE $\lambda+7=\mu+3$

$-\frac{2}{5}+7=\mu+3$

$4-\frac{2}{5}=\mu$

$\boxed{\mu=\frac{18}{5}}$ ← NOT ACTUALLY NEEDED HERE

& $b = 2-\lambda$

$b = 2+\frac{2}{5}$

$\boxed{b=\frac{12}{5}}$

& INTERSECTION AT

$A\left(-\frac{2}{5}+7,\ 2+\frac{2}{5},\ 4\left(-\frac{2}{5}\right)+3\right)$

$\therefore A\left(\frac{33}{5}, \frac{12}{5}, \frac{7}{5}\right)$

- THUS WE OBTAIN

$a = 0$ or 4, $b = 4$ or $\frac{12}{5}$, INTERSECTION $A(5,4,3)$ or $A\left(\frac{33}{5},\frac{12}{5},\frac{7}{5}\right)$

Question 135 (****+)

The point A has position vector $-\mathbf{i}+7\mathbf{j}-\mathbf{k}$.

a) Find the vector equation of the straight line l_1 which passes through A and is parallel to the vector $3\mathbf{i}-2\mathbf{j}+2\mathbf{k}$.

The straight line l_2 has equation

$$\mathbf{r}_2 = 9\mathbf{i}-9\mathbf{j}+8\mathbf{k}+\mu(3\mathbf{i}-3\mathbf{j}+4\mathbf{k}),$$

where μ is a scalar parameter.

b) Show that …

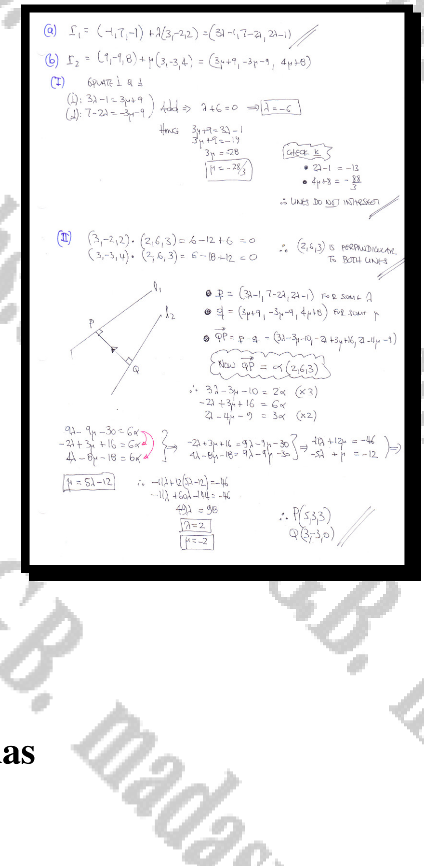
i. … l_1 and l_2 do not intersect.

ii. … the vector $2\mathbf{i}+6\mathbf{j}+3\mathbf{k}$ is perpendicular to both l_1 and l_2.

The point P lies on l_1 and the point Q lies on l_2 so that the distance PQ is least.

c) Find the coordinates of P and Q.

$\boxed{\mathbf{r}_1 = -\mathbf{i}+7\mathbf{j}-\mathbf{k}+\lambda(3\mathbf{i}-2\mathbf{j}+2\mathbf{k})}$, $\boxed{P(5,3,3) \ \ \& \ \ Q(3,-3,0)}$



Question 136 (****+)

With respect to a fixed origin O, the straight lines L_1 and L_2 have respective vector equations

$$\mathbf{r}_1 = \begin{pmatrix} -1 \\ 5 \\ -1 \end{pmatrix} + t\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{r}_2 = \begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix} + s\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},$$

where t and s are scalar parameters.

The points A and C lie on L_1 and L_2, where $t = 0$ and $s = 0$, respectively.

a) Find $\left|\overrightarrow{AC}\right|$, in exact surd form.

b) Show that L_1 and L_2 intersect at some point B and find its coordinates.

c) Find the size of the angle θ, between L_1 and L_2.

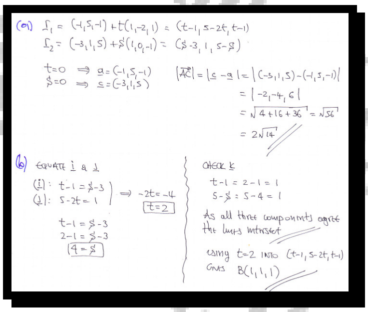
The point D is such so that $ABCD$ is a kite.

d) Show further that …

i. … the area of the kite is $16\sqrt{3}$ square units.

ii. … the length of BD is $\frac{8}{7}\sqrt{42}$.

$\boxed{\left|\overrightarrow{AC}\right| = 2\sqrt{14}}$, $\boxed{B(1,1,1)}$, $\boxed{\theta = 90°}$



(a) $r_1 = (-1,5,-1) + t(1,-2,1) = (t-1, 5-2t, t-1)$
$r_2 = (-3,1,5) + s(1,0,-1) = (s-3, 1, 5-s)$

$t=0 \Rightarrow a = (-1,5,-1)$
$s=0 \Rightarrow c = (-3,1,5)$

$|\overrightarrow{AC}| = |c - a| = |(-3,1,5)-(-1,5,-1)|$
$= |-2,-4,6|$
$= \sqrt{4+16+36} = \sqrt{56}$
$= 2\sqrt{14}$

(b) EQUATE i & j

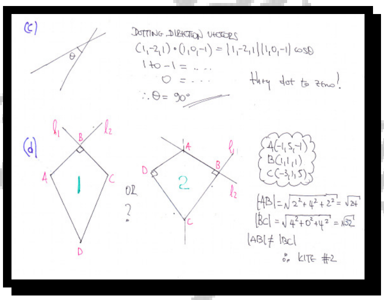
(i): $t-1 = s-3$
(j): $5-2t = 1$
$\Rightarrow -2t = -4$
$\boxed{t=2}$

$t-1 = s-3$
$2-1 = s-3$
$\boxed{4 = s}$

CHECK k
$t-1 = 2-1 = 1$
$5-s = 5-4 = 1$

As all three components agree the lines intersect

using $t=2$ into $(t-1, 5-2t, t-1)$
gives $B(1,1,1)$

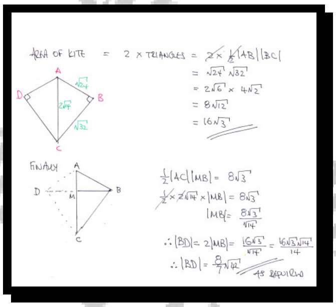


(c) DOTTING DIRECTION VECTORS
$(1,-2,1)\cdot(1,0,-1) = |1,-2,1||1,0,-1|\cos\theta$
$1+0-1 = \ldots$
$0 = \ldots$
they dot to zero!
$\therefore \theta = 90°$

(d)

$A(-1,5,-1)$
$B(1,1,1)$
$C(-3,1,5)$

$|AB| = \sqrt{2^2+4^2+2^2} = \sqrt{24}$
$|BC| = \sqrt{4^2+0^2+4^2} = \sqrt{32}$
$|AB| \neq |BC|$
$\therefore$ KITE #2



AREA OF KITE $= 2 \times$ TRIANGLES $= \cancel{2} \times \cancel{\frac{1}{2}}|AB||BC|$
$= \sqrt{24}\sqrt{32}$
$= 2\sqrt{6} \times 4\sqrt{2}$
$= 8\sqrt{12}$
$= 16\sqrt{3}$

$\frac{1}{2}|AC||MB| = 8\sqrt{3}$
$\cancel{\frac{1}{2}} \times \cancel{2}\sqrt{14} \times |MB| = 8\sqrt{3}$
$|MB| = \frac{8\sqrt{3}}{\sqrt{14}}$
$\therefore |BD| = 2|MB| = \frac{16\sqrt{3}}{\sqrt{14}} = \frac{16\sqrt{3}\sqrt{14}}{14}$
$\therefore |BD| = \frac{8}{7}\sqrt{42}$ AS REQUIRED

Question 137 (****+)

A person standing at a fixed origin O observes an insect taking off from a point A on level horizontal ground. The position vector of the insect $\mathbf{r}$ metres, t seconds after taking off, is given by

$$\mathbf{r} = (t+1)\mathbf{i} + \left(2t+\tfrac{1}{2}\right)\mathbf{j} + 2t\mathbf{k} .$$

All distances are in metres and the coordinates axes Ox, Oy, Oz are oriented due east, due north and vertically upwards, respectively.

a) Find …

- **i.** … the bearing of the insect's flight path.
- **ii.** … the angle between the flight path and the horizontal ground.

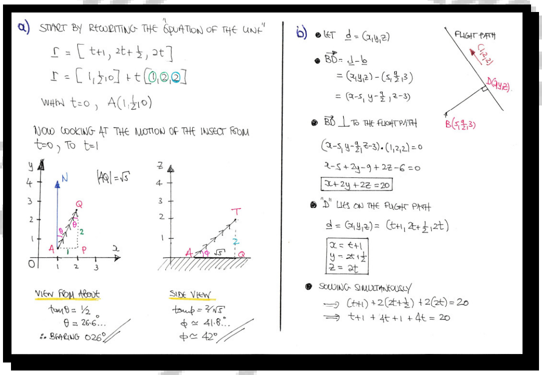
The roof top of a garden shed is located at $B\left(5, \frac{9}{2}, 3\right)$.

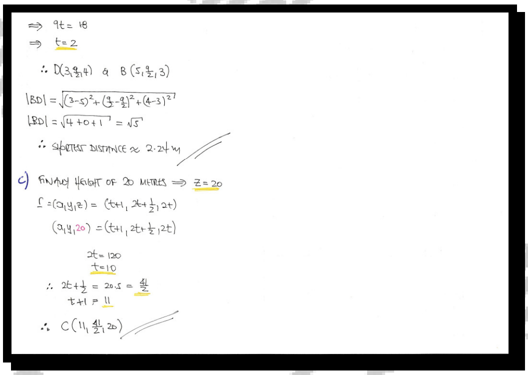
b) Calculate the shortest distance between the insect's path and the point B.

When the insect reaches a height of 20 metres above the ground, at the point C, the insect gets eaten by a bird.

c) Determine the coordinates of C.

$\boxed{\text{bearing} \approx 027°}$, $\boxed{\theta \approx 42°}$, $\boxed{\sqrt{5}}$, $\boxed{C\left(11, \frac{41}{2}, 20\right)}$





Question 138 (****+)

With respect to a fixed origin O, the points with coordinates $A(4,3,-1)$, $B(5,1,2)$, $C(2,0,3)$ and $D(4,2,-1)$ are given.

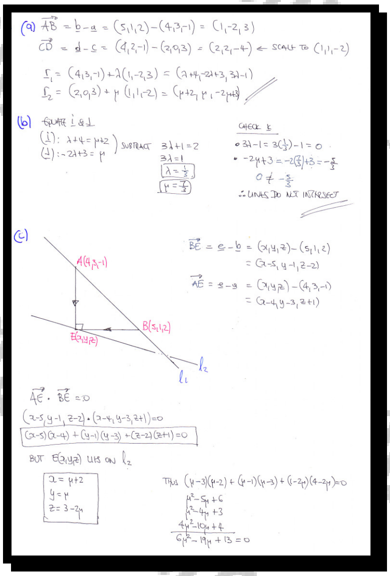
a) Find the vector equation of the line l_1 which passes through A and B, and the vector equation of the line l_2 which passes through C and D.

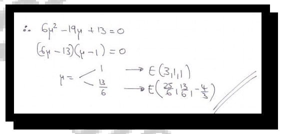
b) Show that l_1 and l_2 do not intersect.

The point E is on l_2 so that $\measuredangle AEB = 90°$.

c) Show that one possible position for E has coordinates $\left(\frac{25}{6}, \frac{13}{6}, -\frac{4}{3}\right)$ and find the coordinates of the other possible position.

$\boxed{\mathbf{r}_1 = 4\mathbf{i} + 3\mathbf{j} - \mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})}$, $\boxed{\mathbf{r}_2 = 2\mathbf{i} + 3\mathbf{k} + \mu(\mathbf{i} + \mathbf{j} - 2\mathbf{k})}$, $\boxed{E(3,1,1)}$





Question 139 (****+)

Relative to a fixed origin O, the points A and B have respectively position vectors $\mathbf{i}-3\mathbf{j}-9\mathbf{k}$ and $-4\mathbf{j}-10\mathbf{k}$.

a) Find the vector equation of the straight line l_1 which passes through A and B.

The straight line l_2 has the vector equation

$$\mathbf{r}_2 = 6\mathbf{i}+\mathbf{k}+\mu(-\mathbf{i}+p\mathbf{j}+q\mathbf{k}),$$

where μ is a scalar parameter, and p and q are scalar constants.

b) Given that l_1 and l_2 are perpendicular, write an equation in terms of p and q.

c) Given further that l_1 and l_2 intersect, find the value of p and the value of q.

d) Determine the position vector of the point of intersection of l_1 and l_2.

$\boxed{\mathbf{r}_1 = \mathbf{i}-3\mathbf{j}-9\mathbf{k}+\lambda(\mathbf{i}+\mathbf{j}+\mathbf{k})}$, $\boxed{p+q=1}$, $\boxed{p=-3,\ q=4}$, $\boxed{7\mathbf{i}+3\mathbf{j}-3\mathbf{k}}$

(a) $\vec{AB} = \underline{b}-\underline{a} = (0,-4,-10)-(1,-3,-9) = (-1,-1,-1)$

use $(1,1,1)$ as direction

$\underline{r}_1 = (1,-3,-9)+\lambda(1,1,1) = (\lambda+1, \lambda-3, \lambda-9)$

(b) $\underline{r}_2 = (6,0,1)+\mu(-1,p,q) = (6-\mu, \mu p, \mu q+1)$

$(1,1,1)\cdot(-1,p,q)=0$

$-1+p+q=0$

$p+q=1$

(c) • EQUATE $\underline{i}, \underline{j}$ & $\underline{k}$

$$\left.\begin{aligned} (\underline{i})\ & \lambda+1 = 6-\mu \\ (\underline{j})\ & \lambda-3 = \mu p \\ (\underline{k})\ & \lambda-9 = \mu q+1 \end{aligned}\right\} \Longrightarrow \boxed{p=1-q} \qquad \begin{aligned} \lambda+1 &= 6-\mu \\ \lambda-3 &= \mu(1-q) \\ \lambda-9 &= \mu q+1 \end{aligned}$$

$$\left.\begin{aligned} \lambda+1 &= 6-\mu \\ \lambda-3 &= \mu-\mu q \\ \lambda-9 &= \mu q+1 \end{aligned}\right\} \text{ADD THE LAST TWO} \qquad \left.\begin{aligned} \lambda+1 &= 6-\mu \\ 2\lambda-12 &= \mu+1 \end{aligned}\right\} \text{Add again}$$

$3\lambda-11=7$

$3\lambda=18$

$\boxed{\lambda=6}$

$\therefore\ \lambda+1=6-\mu$

$7=6-\mu$

$\boxed{\mu=-1}$

$\therefore\ \lambda-9=\mu q+1$

$6-9=-q+1$

$\boxed{q=4}$

$\therefore\ p=1-q$

$\boxed{p=-3}$

(d) USING $\lambda=6$ in $(\lambda+1, \lambda-3, \lambda-9)$ we get $(7,3,-3)$

Question 140 (****+)

The straight lines l_1 and l_2 have respective vector equations

$$\mathbf{r}_1 = 3\lambda\mathbf{i} + (6-2\lambda)\mathbf{j} + (2\lambda+1)\mathbf{k}$$

$$\mathbf{r}_2 = (3\mu+10)\mathbf{i} + (-3\mu-10)\mathbf{j} + (4\mu+10)\mathbf{k}$$

where λ and μ are scalar parameters.

a) Show that l_1 and l_2 do not intersect.

The point P lies on l_1 and the point Q lies on l_2 so that the distance PQ is least.

b) Find the coordinates of P and the coordinates of Q.

$\boxed{P(6,2,5) \quad \& \quad Q(4,-4,2)}$

(a) $\mathbf{r}_1 = (3\lambda, 6-2\lambda, 2\lambda+1) = (0,6,1) + \lambda(3,-2,2)$

$\mathbf{r}_2 = (3\mu+10, -3\mu-10, 4\mu+10) = (10,-10,10) + \mu(3,-3,4)$

- EQUATE $\mathbf{i}$ & $\mathbf{j}$

$(\mathbf{i})$: $3\lambda = 3\mu + 10$

$(\mathbf{j})$: $6 - 2\lambda = -3\mu - 10$

ADD: $\lambda + 6 = 0$, $\boxed{\lambda = -6}$

THUS $3\lambda = 3\mu + 10$

$-18 = 3\mu + 10$

$\boxed{\mu = -\frac{28}{3}}$

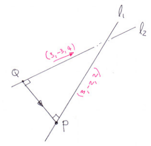
- CHECK $\mathbf{k}$:

$2\lambda + 1 = 2(-6) + 1 = -11$

$4\mu + 10 = 4\left(-\frac{28}{3}\right) + 10 = -\frac{88}{3}$

$\therefore$ LINES DO NOT INTERSECT.

(b)



- LET $\mathbf{p} = (3\lambda, 6-2\lambda, 2\lambda+1)$

$\mathbf{q} = (3\mu+10, -3\mu-10, 4\mu+10)$

FOR SOME λ & μ

- $\overrightarrow{QP} = \mathbf{p} - \mathbf{q}$

$\overrightarrow{QP} = (3\lambda - 3\mu - 10, 3\mu - 2\lambda + 16, 2\lambda - 4\mu - 9)$

- $\overrightarrow{QP} \cdot (3,-3,4) = 0$

$\overrightarrow{QP} \cdot (3,-2,2) = 0$

HENCE

$(3\lambda - 3\mu - 10, 3\mu - 2\lambda + 16, 2\lambda - 4\mu - 9) \cdot (3,-3,4) = 0$

$9\lambda - 9\mu - 30 - 9\mu + 6\lambda - 48 + 8\lambda - 16\mu - 36 = 0$

$\boxed{23\lambda - 34\mu = 114}$

$(3\lambda - 3\mu - 10, 3\mu - 2\lambda + 16, 2\lambda - 4\mu - 9) \cdot (3,-2,2) = 0$

$9\lambda - 9\mu - 30 - 6\mu + 4\lambda - 32 + 4\lambda - 8\mu - 18 = 0$

$\boxed{17\lambda - 23\mu = 80}$

$23\lambda - 34\mu = 114 \quad (\times 17)$

$17\lambda - 23\mu = 80 \quad (\times 23)$

$391\lambda - 578\mu = 1938$

$391\lambda - 529\mu = 1840$

SUBTRACT $-49\mu = 98$

$\boxed{\mu = -2}$

AND $17\lambda - 23(-2) = 80$

$17\lambda = 34$

$\boxed{\lambda = 2}$

HENCE $P(6,2,5)$ & $Q(4,-4,2)$

Question 141 (****+)

Relative to a fixed origin O, the straight lines l_1 and l_2 have the following respective vector equations

$$\mathbf{r}_1 = \begin{pmatrix} 8 \\ q \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ p \end{pmatrix} \quad \text{and} \quad \mathbf{r}_2 = \begin{pmatrix} 3 \\ -4 \\ -5 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

where λ and μ are scalar parameters, and p and q are scalar constants.

a) Given that l_1 and l_2 are perpendicular, determine the value of p.

The point D is the intersection of l_1 and l_2

b) Find the value of q and the coordinates of D.

Another straight line l_3 intersects **both** l_1 and l_2, and is also perpendicular to **both** l_1 and l_2.

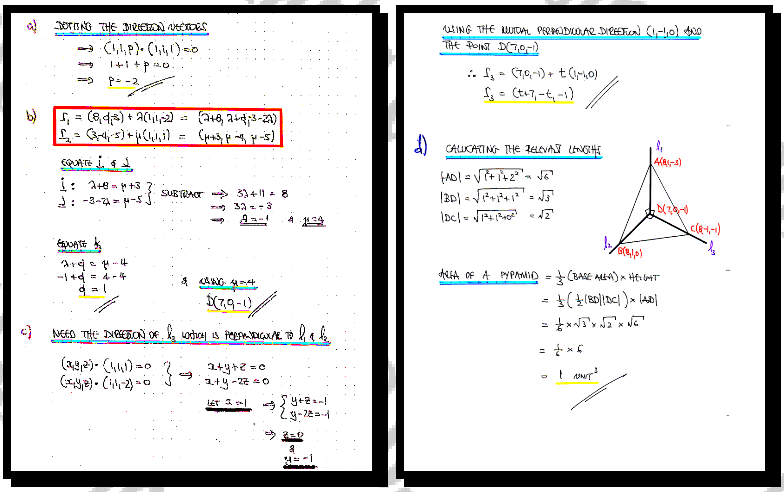
c) Find a vector equation for l_3.

You may not use the vector (cross) product in this part

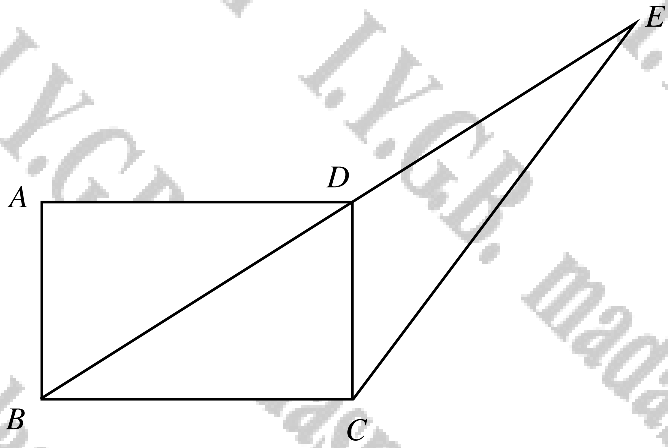
The points $A(8,1,-3)$, $B(8,1,0)$ and $C(8,-1,-1)$ lie on l_1, l_2 and l_3, respectively.

d) Show that the volume of the triangle based pyramid with vertices at A, B, C and D is 1 cubic unit.

$\boxed{}$, $\boxed{p=-2}$, $\boxed{q=1}$, $\boxed{D(7,0,-1)}$, $\boxed{\mathbf{r} = 7\mathbf{i} - \mathbf{k} + \nu(\mathbf{i} - \mathbf{j})}$



Question 142 (*****)

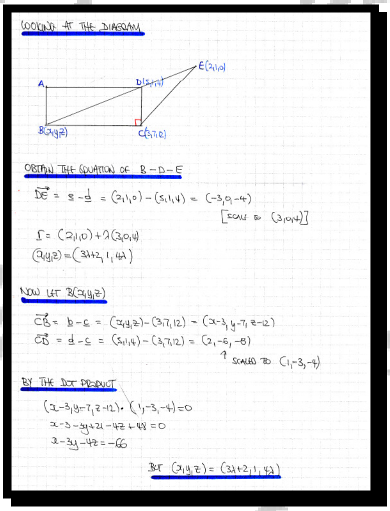


The figure above shows the rectangle $ABCD$, where $C(3,7,12)$ and $D(5,1,4)$.

The point $E(2,1,0)$ is such so that BDE and EC are straight lines.

Use vector methods to determine the coordinates of A.

$\boxed{}$, $\boxed{A(19,-5,12)}$



LOOKING AT THE DIAGRAM

OBTAIN THE EQUATION OF B – D – E

$\vec{DE} = \underline{e} - \underline{d} = (2,1,0) - (5,1,4) = (-3,0,-4)$

[SCALE TO $(3,0,4)$]

$\underline{r} = (2,1,0) + \lambda(3,0,4)$

$(x,y,z) = (3\lambda+2, 1, 4\lambda)$

NOW LET $B(x,y,z)$

$\vec{CB} = \underline{b} - \underline{c} = (x,y,z) - (3,7,12) = (x-3, y-7, z-12)$

$\vec{CD} = \underline{d} - \underline{c} = (5,1,4) - (3,7,12) = (2,-6,-8)$

SCALED TO $(1,-3,-4)$

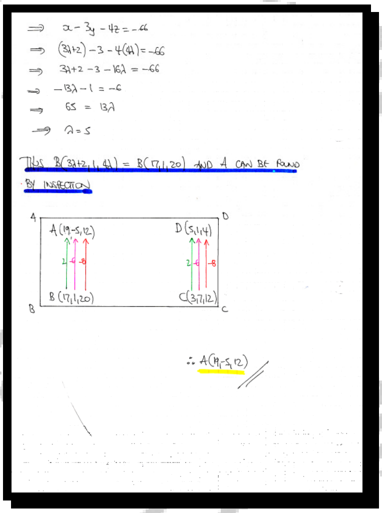
BY THE DOT PRODUCT

$(x-3, y-7, z-12)\cdot(1,-3,-4) = 0$

$x - 3 - 3y + 21 - 4z + 48 = 0$

$x - 3y - 4z = -66$

BUT $(x,y,z) = (3\lambda+2, 1, 4\lambda)$



$\Rightarrow x - 3y - 4z = -66$

$\Rightarrow (3\lambda+2) - 3 - 4(4\lambda) = -66$

$\Rightarrow 3\lambda + 2 - 3 - 16\lambda = -66$

$\Rightarrow -13\lambda - 1 = -66$

$\Rightarrow 65 = 13\lambda$

$\Rightarrow \lambda = 5$

THUS $B(3\lambda+2, 1, 4\lambda) = B(17,1,20)$ AND A CAN BE FOUND BY INSPECTION

$\therefore A(19,-5,12)$

Question 143 (*****)

The coordinates in this question are relative to a fixed origin O at $(0,0,0)$.

The straight line l_1 has vector equation

$$\mathbf{i}+3\mathbf{j}+5\mathbf{k}+\lambda(-\mathbf{i}+3\mathbf{j}+\mathbf{k}),$$

where λ is a scalar parameter.

The straight line l_2 passes thought the point with coordinates $(6,0,6)$ and is in the direction $2\mathbf{i}-3\mathbf{j}+\mathbf{k}$.

a) Verify that $A(4,3,5)$ is the intersection of l_1 and l_2, and show further that $B(12,-9,9)$ lies on l_2.

The point $C(6,-3,3)$ lies on l_1.

The straight line l_3 passes through B and C.

The straight line l_4 is parallel to l_2 and passes through C.

The straight line l_5 is perpendicular to l_3 and passes through A.

b) Given that l_4 and l_5 intersect at the point D, find the coordinates of D.

$\boxed{}$, $\boxed{D(4,0,2)}$

a) l_1: $\mathbf{r}_1=(3,6,6)+\lambda(-1,3,1)=(3-\lambda,3\lambda+6,\lambda+6)$
l_2: $\mathbf{r}_2=(6,0,6)+\mu(2,-3,1)=(2\mu+6,-3\mu,\mu+6)$

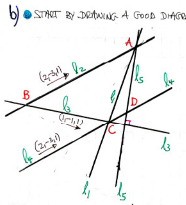
- CHECKING THE POINT $(4,3,5)$, BY INSPECTION
  - IF $\lambda=-1$, $[3-(-1),3(-1)+6,-1+6]=(4,3,5)$
  - IF $\mu=-1$ $[2(-1)+6,-3(-1),-1+6]=(4,3,5)$

$\therefore A(4,3,5)$ IS THE INTERSECTION OF l_1 & l_2

- CHECKING $B(12,-9,9)$
  - $2\mu+6=12$, $2\mu=6$, $\mu=3$
  - $-3\mu=-9$, $\mu=3$
  - $\mu+6=9$, $\mu=3$

$\therefore B(12,-9,9)$ LIES ON l_2

b)
- START BY DRAWING A GOOD DIAGRAM



$A(4,3,5)$
$B(12,-9,9)$
$C(6,-3,3)$

- FIND AN EQUATION OF l_4 (PARALLEL TO l_2 & PASSING THROUGH C)

$\mathbf{r}_4=(6,-3,3)+t(2,-3,1)=(2t+6,-3t-3,t+3)$

- NEXT FIND THE DIRECTION VECTOR OF l_3, IE FIND $\overrightarrow{BC}$

$\overrightarrow{BC}=\underline{c}-\underline{b}=(6,-3,3)-(12,-9,9)=(-6,6,-6)$

$\downarrow$ SCALE IT TO $(1,-1,1)$

- NEXT LET THE POINT $D(x,y,z)$ HAVE POSITION VECTOR $\underline{d}=(x,y,z)$

$\overrightarrow{AD}=\underline{d}-\underline{a}=(x,y,z)-(4,3,5)=(x-4,y-3,z-5)$

- BUT $\overrightarrow{AD}$ IS PERPENDICULAR TO l_3

$\Rightarrow \overrightarrow{AD}\cdot(\text{DIRECTION VECTOR OF } l_3)=0$
$\Rightarrow (x-4,y-3,z-5)\cdot(1,-1,1)=0$
$\Rightarrow x-4-y+3+z-5=0$
$\Rightarrow x-y+z=6$

- BUT $D(x,y,z)$ LIES ON l_4, SO ITS CO-ORDINATES MUST SATISFY THE PARAMETRIC EQUATIONS OF l_4

$$\boxed{\begin{aligned}x&=2t+6\\y&=-3t-3\\z&=t+3\end{aligned}}$$

- SOLVING SIMULTANEOUSLY THE LAST TWO SETS OF EQUATIONS

$\Rightarrow (2t+6)-(-3t-3)+(t+3)=6$
$\Rightarrow 6t+6+3+3=6$
$\Rightarrow 6t=-6$
$\Rightarrow t=-1$

$\therefore D(4,0,2)$

Question 144 (*****)

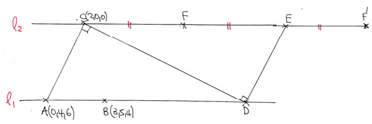
Relative to a fixed origin O at $(0,0,0)$ the points A, B and C have coordinates $(0,4,6)$, $(3,5,4)$ and $(2,0,0)$, respectively.

- The straight line l_1 passes through A and B.
- The straight line l_2 passes through C and is parallel to l_1.
- The point D lies on l_1 so that $\measuredangle ACD = 90°$.
- The point E lies on l_2 so that $\measuredangle CDE = 90°$.
- The point F lies on l_2 so that $|EC| = 2|EF|$.

Determine the coordinates of the possible positions of F.

$\boxed{}$, $\boxed{F(8,2,-4) \cup F(20,6,-12)}$

- START WITH A GOOD DIAGRAM – NOTE THAT "ACED" IS A PARALLELOGRAM



- FIND EQUATION OF l_1
  - $\vec{AB} = \underline{b} - \underline{a} = (3,5,4) - (0,4,6) = (3,1,-2)$
  - $\underline{r}_1 = (0,4,6) + \lambda(3,1,-2)$
    $\underline{r}_1 = (3\lambda, \lambda+4, 6-2\lambda)$
- FIND VECTOR/DIRECTION OF AC
  $\vec{AC} = \underline{c} - \underline{a} = (2,0,0) - (0,4,6) = (2,-4,-6)$
  SCALE IT TO $(-1,2,3)$
- NEXT FIND THE CO.ORDS OF $D(x,y,z)$
  - $\vec{CD} = \underline{d} - \underline{c} = (x,y,z) - (2,0,0) = (x-2, y, z)$
  - $\vec{CD} \perp \vec{AC} \Rightarrow (x-2,y,z)\cdot(-1,2,3) = 0$
    $-x+2+2y+3z = 0$
    $-x+2y+3z = -2$
    $x-2y-3z = 2$
  - BUT (x,y,z) MUST SATISFY $\underline{r}_1$ IE $x = 3\lambda$, $y = \lambda+4$, $z = 6-2\lambda$

$$\Rightarrow 3\lambda - 2(\lambda+4) - 3(6-2\lambda) = 2$$
$$\Rightarrow 3\lambda - 2\lambda - 8 - 18 + 6\lambda = 2$$
$$\Rightarrow 7\lambda = 28$$
$$\Rightarrow \lambda = 4$$

$\therefore D(12,8,-2)$

- NEXT FIND THE CO-ORDINATES OF E

$$\begin{array}{ccc} A & & C \\ 0 & \xrightarrow{+2} & 2 \\ 4 & \xrightarrow{-4} & 0 \\ 6 & \xrightarrow{-6} & 0 \end{array} \quad \Rightarrow \quad \begin{array}{ccc} D & & E \\ 12 & \xrightarrow{+2} & 14 \\ 8 & \xrightarrow{-4} & 4 \\ -2 & \xrightarrow{-6} & -8 \end{array}$$

$\therefore E(14,4,-8)$

- FIND THE MIDPOINT OF $C(2,0,0)$ & $E(14,4,-8)$, BY INSPECTION $(8,2,-4)$
- FINALLY

$$\begin{array}{ccccccc} C & & F & & E & & F' \\ 2 & \xrightarrow{+6} & 8 & \xrightarrow{+6} & 14 & \xrightarrow{+6} & 20 \\ 0 & \xrightarrow{+2} & 2 & \xrightarrow{+2} & 4 & \xrightarrow{+2} & 6 \\ 0 & \xrightarrow{-4} & -4 & \xrightarrow{-4} & -8 & \xrightarrow{-4} & -12 \end{array}$$

$\therefore$ EITHER $(8,2,-4)$ OR $(20,6,-12)$

Question 145 (*****)

With respect to a fixed origin O, the points A, B and C have position vectors

$$\mathbf{a} = \begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix}, \ \mathbf{b} = \begin{pmatrix} 8 \\ 2 \\ 7 \end{pmatrix} \text{ and } \mathbf{c} = \begin{pmatrix} 11 \\ 0 \\ 1 \end{pmatrix}.$$

a) Determine the volume of the cube, with vertices the points A, B and C.

The points P, Q and R are vertices of a different cube, so that

$$\overrightarrow{PQ} = \begin{pmatrix} 0 \\ 1 \\ 7 \end{pmatrix} \text{ and } \overrightarrow{PR} = \begin{pmatrix} k \\ 4 \\ 3 \end{pmatrix},$$

where k is a positive constant.

b) Given that $\measuredangle QPR = 60°$, determine …

i. … the value of k.

ii. … the length of the diagonal of the second cube.

$\boxed{}$, $\boxed{\text{volume} = 343}$, $\boxed{k = 5}$, $\boxed{\sqrt{75} = 5\sqrt{3}}$

a) AS WE DO NOT THE LOCATION OF THE VERTICES WE NEED TO FIND ALL THE POSSIBLE LENGTHS BETWEEN A, B & C

$|\overrightarrow{AB}| = |\underline{b} - \underline{a}| = |(8,2,7)-(0,5,2)| = |8,-3,5| = \sqrt{64+9+25} = \sqrt{98}$

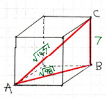
$|\overrightarrow{AC}| = |\underline{c} - \underline{a}| = |(11,0,1)-(0,5,2)| = |11,-5,-1| = \sqrt{121+25+1} = \sqrt{147}$

$|\overrightarrow{BC}| = |\underline{c} - \underline{b}| = |(11,0,1)-(8,2,7)| = |3,-2,-6| = \sqrt{9+4+36} = \sqrt{49} = 7$

THUS THE CONFIGURATION IS AS OPPOSITE

$\therefore$ SIDE LENGTH IS 7 UNITS

$\therefore$ VOLUME $= 7 \times 7 \times 7$
$= 7 \times 49$
$= 280 + 63$
$= 343$



b)I) DRAWING THE SECOND CUBE

BY THE DOT-PRODUCT

$\Rightarrow \overrightarrow{PQ} \cdot \overrightarrow{PR} = |\overrightarrow{PQ}||\overrightarrow{PR}|\cos\theta$

$\Rightarrow (0,1,7)\cdot(k,4,3) = |0,1,7||k,4,3|\cos 60°$

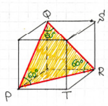
$\Rightarrow 0+4+21 = \sqrt{0+1+49}\sqrt{k^2+16+9} \times \frac{1}{2}$

$\Rightarrow 25 = \sqrt{50}\sqrt{k^2+25} \times \frac{1}{2}$

$\Rightarrow 50 = \sqrt{50}\sqrt{k^2+25}$

$\Rightarrow \frac{50}{\sqrt{50}} = \sqrt{k^2+25}$

$\Rightarrow \frac{2500}{50} = k^2+25$



$\Rightarrow k^2 + 25 = 50$

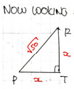
$\Rightarrow k^2 = 25$

$\Rightarrow k = 5$ $(k>0)$

II) LOOKING AT THE PREVIOUS DIAGRAM

$|PQ| = |QR| = |PR| = \sqrt{50}$

NOW LOOKING AT $P\hat{R}T$, ON THE "FLOOR"



$x^2 + x^2 = (\sqrt{50})^2$
$2x^2 = 50$
$x^2 = 25$
$x = 5$
SIDE LENGTH

$\therefore$ LENGTH OF THE LONGEST DIAGONAL IS $\sqrt{5^2+5^2+5^2}$
$= \sqrt{75}$
$= 5\sqrt{3}$

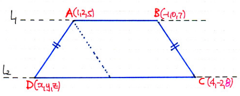
Question 146 (*****)

Relative to a fixed origin O at $(0,0,0)$ the points A, B and C have coordinates $(1,2,5)$, $(-1,0,7)$ and $(4,-2,8)$, respectively.

The point D is such so that $ABCD$ is an isosceles trapezium with $|BC| = |AD|$.

Determine the coordinates of D.

$\boxed{}$, $\boxed{D\left(\frac{14}{3}, -\frac{4}{3}, \frac{22}{3}\right)}$



- $\vec{AB} = \underline{b} - \underline{a} = (-1,0,7) - (1,2,5) = (-2,-2,2)$
- USING $(1,1,-1)$ AS A DIRECTION, WE CAN FIND AN EQUATION OF L_2

$$\underline{r}_2 = (4,-2,8) + \lambda(1,1,-1)$$
$$\underline{r}_2 = (\lambda+4, \lambda-2, 8-\lambda)$$

- FIND THE LENGTH OF BC

$$|\vec{BC}| = |\underline{c} - \underline{b}| = |(4,-2,8)-(-1,0,7)| = |5,-2,1|$$
$$= \sqrt{25+4+1} = \sqrt{30}$$

- NOW $|\vec{AD}| = \sqrt{30}$ WITH $D(x,y,z)$ LYING ON L_2

$$\Rightarrow |\vec{AD}| = \sqrt{30}$$
$$\Rightarrow |\underline{d} - \underline{a}| = \sqrt{30}$$
$$\Rightarrow |(x,y,z) - (1,2,5)| = \sqrt{30}$$
$$\Rightarrow |x-1, y-2, z-5| = \sqrt{30}$$
$$\Rightarrow \sqrt{(x-1)^2 + (y-2)^2 + (z-5)^2} = \sqrt{30}$$
$$\Rightarrow (x-1)^2 + (y-2)^2 + (z-5)^2 = 30$$
$$\Rightarrow [(\lambda+4)-1]^2 + [(\lambda-2)-2]^2 + [(8-\lambda)-5]^2 = 30$$
$$\Rightarrow (\lambda+3)^2 + (\lambda-4)^2 + (3-\lambda)^2 = 30$$
$$\Rightarrow \begin{pmatrix} \lambda^2+6\lambda+9 \\ \lambda^2-8\lambda+16 \\ \lambda^2-6\lambda+9 \end{pmatrix} = 30$$
$$\Rightarrow 3\lambda^2 - 8\lambda + 4 = 0$$
$$\Rightarrow (3\lambda-2)(\lambda-2) = 0$$
$$\Rightarrow \lambda = \begin{cases} 2 \\ 2/3 \end{cases}$$

- NOW $D(6,0,6)$ OR $D\left(\frac{14}{3}, -\frac{4}{3}, \frac{22}{3}\right)$

$|\vec{AB}| = |\underline{b} - \underline{a}| = |-2,-2,2| = \sqrt{4+4+4} = \sqrt{12}$

$|CD| = |\underline{d} - \underline{c}| = |(6,0,6) - (4,-2,8)| = |2,2,-2| = \sqrt{12}$

$\therefore$ D CANNOT BE $(6,0,6)$ AS THIS YIELDS A PARALLELOGRAM

$\therefore D\left(\frac{14}{3}, -\frac{4}{3}, \frac{22}{3}\right)$

Question 147 (*****)

Relative to a fixed origin O, the points A and B have position vectors $4\mathbf{i}+5\mathbf{j}+8\mathbf{k}$ and $6\mathbf{i}+6\mathbf{j}+7\mathbf{k}$, respectively. The straight line l_1 passes through A and , B and crosses the $y-z$ plane at the point C. The straight line l_2 passes through the point D with position vector $p\mathbf{j}+(2p+2)\mathbf{k}$, where p is a scalar constant.

Given that l_1 and l_2 are perpendicular, and intersect at C, find the value of p.

$\boxed{}$, $\boxed{p=5}$

- FIND THE VECTOR EQUATION OF l_1

$$\vec{AB} = \underline{b}-\underline{a} = (6,6,7)-(4,5,8) = (2,1,-1)$$
$$\underline{r}_1 = (4,5,8)+\lambda(2,1,-1)$$
$$\underline{r}_1 = (2\lambda+4, \lambda+5, 8-\lambda)$$

- NEXT FIND THE CO-ORDINATES OF THE POINT C

"CROSSES THE $y-z$ PLANE" $\Rightarrow x=0$
$\Rightarrow 2\lambda+4=0$
$\Rightarrow \lambda=-2$
$\Rightarrow C(0,3,10)$

- NEXT WRITE AN EXPRESSION FOR THE EQUATION OF l_2

$$\underline{r}_2 = (0, p, 2p+2) + \mu(a,b,c)$$
$$\underline{r}_2 = (\mu a, \mu b+p, \mu c+2p+2)$$

- NOW l_1 & l_2 INTERSECT AT $C(0,3,10)$

$$\left.\begin{array}{l} \underline{i}: \mu a = 0 \\ \underline{j}: \mu b + p = 3 \\ \underline{k}: \mu c + 2p + 2 = 10 \end{array}\right\} \text{ LOOKING AT } \underline{i}$$

$\mu a = 0$, $\mu \neq 0$
OTHERWISE $p=3$
AND $\underline{k}$ COMPONENT
YIELDS $8 \neq 10$

$\therefore a=0$

- LINES ARE ALSO PERPENDICULAR, SO THEIR DIRECTION VECTORS MUST DOT TO ZERO

$$(a,b,c)\cdot(2,1,-1)=0$$
$$\cancel{2a}+b-c=0$$
$$b=c$$

- HENCE THE $\underline{j}$ & $\underline{k}$ COMPONENT EQUATIONS BECOME

$$\left.\begin{array}{l} \mu b + p = 3 \\ \mu c + 2p + 2 = 10 \end{array}\right\} \Rightarrow \begin{array}{l} \mu b + p = 3 \\ \mu b + 2p + 2 = 10 \\ \hline p+2 = 7 \end{array}$$

$p=5$

Question 148 (*****)

The points A and B, have respective position vectors $\mathbf{a}$ and $\mathbf{b}$, relative to a fixed origin O.

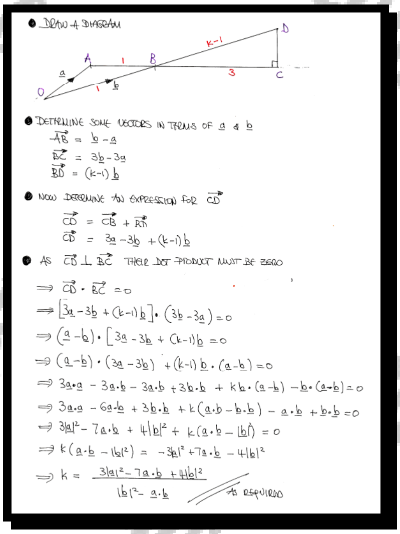
The point C lies on AB produced such that $|AB| : |AC| = 1 : 4$

The point D lies on OB produced such that $|OB| : |OD| = 1 : k$, where $|OB| : |OD| = 1 : k$ is a scalar constant.

Given that AB is perpendicular to CD show that

$$k = \frac{3|\mathbf{a}|^2 - 7\mathbf{a}\cdot\mathbf{b} + 4|\mathbf{b}|^2}{|\mathbf{b}|^2 - \mathbf{a}\cdot\mathbf{b}}.$$

proof



- DRAW A DIAGRAM

- DETERMINE SOME VECTORS IN TERMS OF $\underline{a}$ & $\underline{b}$

$\overrightarrow{AB} = \underline{b} - \underline{a}$

$\overrightarrow{BC} = 3\underline{b} - 3\underline{a}$

$\overrightarrow{BD} = (k-1)\underline{b}$

- NOW DETERMINE AN EXPRESSION FOR $\overrightarrow{CD}$

$\overrightarrow{CD} = \overrightarrow{CB} + \overrightarrow{BD}$

$\overrightarrow{CD} = 3\underline{a} - 3\underline{b} + (k-1)\underline{b}$

- AS $\overrightarrow{CD} \perp \overrightarrow{BC}$ THEIR DOT PRODUCT MUST BE ZERO

$\Rightarrow \overrightarrow{CD} \cdot \overrightarrow{BC} = 0$

$\Rightarrow [3\underline{a} - 3\underline{b} + (k-1)\underline{b}] \cdot (3\underline{b} - 3\underline{a}) = 0$

$\Rightarrow (\underline{a} - \underline{b}) \cdot [3\underline{a} - 3\underline{b} + (k-1)\underline{b} = 0$

$\Rightarrow (\underline{a} - \underline{b}) \cdot (3\underline{a} - 3\underline{b}) + (k-1)\underline{b} \cdot (\underline{a} - \underline{b}) = 0$

$\Rightarrow 3\underline{a}\cdot\underline{a} - 3\underline{a}\cdot\underline{b} - 3\underline{a}\cdot\underline{b} + 3\underline{b}\cdot\underline{b} + k\underline{b}\cdot(\underline{a} - \underline{b}) - \underline{b}\cdot(\underline{a} - \underline{b}) = 0$

$\Rightarrow 3\underline{a}\cdot\underline{a} - 6\underline{a}\cdot\underline{b} + 3\underline{b}\cdot\underline{b} + k(\underline{a}\cdot\underline{b} - \underline{b}\cdot\underline{b}) - \underline{a}\cdot\underline{b} + \underline{b}\cdot\underline{b} = 0$

$\Rightarrow 3|\underline{a}|^2 - 7\underline{a}\cdot\underline{b} + 4|\underline{b}|^2 + k(\underline{a}\cdot\underline{b} - |\underline{b}|^2) = 0$

$\Rightarrow k(\underline{a}\cdot\underline{b} - |\underline{b}|^2) = -3|\underline{a}|^2 + 7\underline{a}\cdot\underline{b} - 4|\underline{b}|^2$

$\Rightarrow k = \dfrac{3|\underline{a}|^2 - 7\underline{a}\cdot\underline{b} + 4|\underline{b}|^2}{|\underline{b}|^2 - \underline{a}\cdot\underline{b}}$ AS REQUIRED

Question 149 (*****)

OAB is a triangle and $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

- The point C lies on OB so that $OC : CB = 3:1$.
- The point P lies on AC so that $AP : PC = 2:1$.
- The point Q lies on AB so that O, P and Q are collinear.

Determine the ratio $AQ : QB$.

$\boxed{\quad}$, $\boxed{AQ : QB = 3:2}$

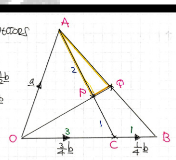
- START BY DETERMINING SOME MORE VECTORS

$\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC} = -\underline{a} + \frac{3}{4}\underline{b}$

$\overrightarrow{AP} = \frac{2}{3}\overrightarrow{AC} = \frac{2}{3}\left(-\underline{a} + \frac{3}{4}\underline{b}\right) = -\frac{2}{3}\underline{a} + \frac{1}{2}\underline{b}$

$\overrightarrow{PC} = \frac{1}{2}\overrightarrow{AP} = \frac{1}{2}\left(-\frac{2}{3}\underline{a} + \frac{1}{2}\underline{b}\right) = -\frac{1}{3}\underline{a} + \frac{1}{4}\underline{b}$

$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\underline{a} + \underline{b} = \underline{b} - \underline{a}$



- LET $\overrightarrow{AQ} = \lambda\overrightarrow{AB}$, $0 < \lambda < 1$

$\overrightarrow{PQ} = \mu\overrightarrow{OP}$, $0 < \mu < 1$

- NOW BY LOOKING AT THE TRIANGLE $\overset{\triangle}{APQ}$

$\Rightarrow \overrightarrow{AQ} = \overrightarrow{AP} + \overrightarrow{PQ}$

$\Rightarrow \lambda\overrightarrow{AB} = \overrightarrow{AP} + \mu\overrightarrow{OP}$

$\Rightarrow \lambda\overrightarrow{AB} = \overrightarrow{AP} + \mu\left(\overrightarrow{OA} + \overrightarrow{AP}\right)$

$\Rightarrow \lambda\overrightarrow{AB} = \overrightarrow{AP} + \mu\overrightarrow{OA} + \mu\overrightarrow{AP}$

$\Rightarrow \lambda\overrightarrow{AB} = (\mu+1)\overrightarrow{AP} + \mu\overrightarrow{OA}$

$\Rightarrow \lambda(\underline{b} - \underline{a}) = (\mu+1)\left(-\frac{2}{3}\underline{a} + \frac{1}{2}\underline{b}\right) + \mu\underline{a}$

$\Rightarrow -\lambda\underline{a} + \lambda\underline{b} = \left[(\mu+1)\left(-\frac{2}{3}\underline{a}\right) + \mu\underline{a}\right] + \left[(\mu+1)\left(\frac{1}{2}\underline{b}\right)\right]$

$\Rightarrow -\lambda\underline{a} + \lambda\underline{b} = \left[\mu - \frac{2}{3}\mu - \frac{2}{3}\right]\underline{a} + \left[\frac{1}{2}(\mu+1)\right]\underline{b}$

$\Rightarrow -\lambda\underline{a} + \lambda\underline{b} = \left(\frac{1}{3}\mu - \frac{2}{3}\right)\underline{a} + \left(\frac{1}{2}\mu + \frac{1}{2}\right)\underline{b}$

- EQUATING VECTORS $\underline{a}$ & $\underline{b}$ IN BOTH SIDES

$-\lambda = \frac{1}{3}\mu - \frac{2}{3}$ $\qquad$ $\lambda = \frac{1}{2}\mu + \frac{1}{2}$

$-3\lambda = \mu - 2$ $\qquad$ $2\lambda = \mu + 1$

$\mu = 2 - 3\lambda$ $\qquad$ $\mu = 2\lambda - 1$

$2\lambda - 1 = 2 - 3\lambda$

$5\lambda = 3$

$\lambda = \frac{3}{5}$

- THUS LOOKING IN THE DIAGRAM IF $\lambda = \frac{3}{5}$, $\overrightarrow{AQ} = \frac{3}{5}\overrightarrow{AB}$

$\overrightarrow{AQ} : \overrightarrow{QB}$

$3 : 2$

Question 150 (*****)

Relative to a fixed origin O, the straight lines l_1 and l_2 have vector equations

$$\mathbf{r}_1 = \begin{pmatrix}7\\1\\2\end{pmatrix} + \lambda\begin{pmatrix}2\\0\\-1\end{pmatrix} \quad \text{and} \quad \mathbf{r}_2 = \begin{pmatrix}14\\19\\3\end{pmatrix} + \mu\begin{pmatrix}-2\\4\\-3\end{pmatrix},$$

where λ and μ are scalar parameters.

The point A lies on l_1 and the point B lies on l_2, so that the distance AB is least.

Find the coordinates of A and the coordinates of B.

$\boxed{}$, $\boxed{A(13,1,-1)}$, $\boxed{B(18,11,9)}$

WITHOUT USING THE CROSS PRODUCT

$$\left.\begin{array}{l}(a,b,c)\cdot(2,0,-1)=0\\(a,b,c)\cdot(-2,4,-3)=0\end{array}\right\}\Rightarrow \begin{array}{l}2a-c=0\\-2a+4b-3c=0\end{array}$$

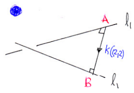
LET ONE OF THE VARIABLES TAKE A NON ZERO VALUE SAY $c=2$

THEN $\left.\begin{array}{l}2a-2=0\\-2a+4b-6=0\end{array}\right\}\Rightarrow \left.\begin{array}{l}a=1\\-2+4b-6=0\end{array}\right\}\Rightarrow \begin{array}{l}a=1\\b=2\\c=2\end{array}$

I.E $(1,2,2)$

BY THE CROSS PRODUCT

$$\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ 2 & 0 & -1\\ -2 & 4 & -3\end{vmatrix} = (0+4,\ 2+6,\ 8-0) = (4,8,8)$$ WHICH SCALES TO $(1,2,2)$ AS ABOVE



$\underline{r}_1 = (7,1,2)+\lambda(2,0,-1) = (2\lambda+7,\ 1,\ 2-\lambda)$ A

$\underline{r}_2 = (14,19,3)+\mu(-2,4,-3) = (14-2\mu,\ 4\mu+19,\ 3-3\mu)$ B

$\vec{AB} = \underline{b}-\underline{a} = (14-2\mu-2\lambda-7,\ 4\mu+19-1,\ 3-3\mu-2+\lambda)$

$\vec{AB} = (7-2\mu-2\lambda,\ 4\mu+18,\ \lambda-3\mu+1)$

NOW $(7-2\mu-2\lambda,\ 4\mu+18,\ \lambda-3\mu+1) = k(1,2,2)$

$$\left.\begin{array}{l}7-2\mu-2\lambda = k\\4\mu+18 = 2k\\\lambda-3\mu+1 = 2k\end{array}\right\}\Rightarrow \boxed{k=2\mu+9} \quad \therefore \left.\begin{array}{l}7-2\mu-2\lambda = 2\mu+9\\\lambda-3\mu+1 = 4\mu+18\end{array}\right\}\Rightarrow \left\{\begin{array}{l}4\mu+2\lambda=-2\\\lambda-7\mu=17\end{array}\right\}$$

$$\Rightarrow \left\{\begin{array}{l}2\mu+\lambda=-1\\-7\mu+\lambda=17\end{array}\right\}\Rightarrow 9\mu=-18,\ \boxed{\mu=-2},\ \boxed{\lambda=3}$$

$\therefore A(13,1,-1)$
$B(18,11,9)$

Question 151 (*****)

The point P lies on the straight line L_1, which is parallel to the vector $2\mathbf{i}+\mathbf{j}+2\mathbf{k}$ and passes through the point with coordinates $(10,3,7)$, relative to an origin at $(0,0,0)$.

The point Q lies on another straight line L_2, which is in the direction of the vector $4\mathbf{i}-\mathbf{j}+\mathbf{k}$ and passes through the point with coordinates $(9,1,0)$.

The straight line L_3 is perpendicular to both L_1 and L_2, and meets L_1 and L_2 at the points P and Q, respectively.

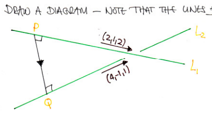
Find the coordinates of P and Q.

$\boxed{}$, $\boxed{P(4,0,1),\ Q(5,2,-1)}$

- WRITE DOWN THE EQUATIONS OF THE TWO LINES L_1 & L_2

$$L_1: \quad \mathbf{r}_1 = (10,3,7) + \lambda(2,1,2) = (2\lambda+10, \lambda+3, 2\lambda+7)$$
$$L_2: \quad \mathbf{r}_2 = (9,1,0) + \mu(4,-1,1) = (4\mu+9, 1-\mu, \mu)$$

- DRAW A DIAGRAM – NOTE THAT THE LINES ~~HAVE~~ TO BE SKEW



- LET $\mathbf{p} = (2a+10, a+3, 2a+7)$ FOR $\lambda = a$

$\quad\ \ \mathbf{q} = (4b+9, 1-b, b)$ FOR $\mu = b$

$$\Rightarrow \overrightarrow{PQ} = \mathbf{q} - \mathbf{p} = (4b+9, 1-b, b) - (2a+10, a+3, 2a+7)$$
$$\Rightarrow \overrightarrow{PQ} = [4b - 2a - 1, -b-a-2, b-2a-7]$$

- NOW $\overrightarrow{PQ}$ IS PERPENDICULAR TO BOTH LINES

$$\left.\begin{array}{l}(4b-2a-1, -b-a-2, b-2a-7)\cdot(2,1,2) = 0\\ (4b-2a-1, -b-a-2, b-2a-7)\cdot(4,-1,1) = 0\end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{l}\left.\begin{array}{l}8b-4a-2\\ -b-a-2\\ 2b-4a-14\end{array}\right\} = 0\\ \\ \left.\begin{array}{l}16b-8a-4\\ b+a+2\\ b-2a-7\end{array}\right\} = 0\end{array}\right\} \Rightarrow \left.\begin{array}{l}9b-9a-18 = 0\\ 18b-9a-9 = 0\end{array}\right\} \Rightarrow$$

$$\Rightarrow \begin{array}{l}b - a = 2\\ 2b - a = 1\end{array}$$

$$\Rightarrow -b = 1$$
$$\Rightarrow b = -1$$

THUS $a = -3$

- HENCE $P(4,0,1)$ $Q(5,2,-1)$

Question 152 (*****)

The straight line l_1, where λ is a scalar parameter, has vector equation

$$\mathbf{r} = 10\mathbf{i} + 8\mathbf{j} + 5\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} + \mathbf{k}).$$

The points $A(4,1,3)$ and $B(6,5,-3)$ lie on the straight line l_2.

a) Given that l_1 and l_2 lie on the same plane, show that l_1 is perpendicular to l_2.

The points C and D lie on l_1 so that the resulting quadrilateral $ACBD$ is a kite, whose line of symmetry is l_2.

b) Given further that the area of the kite is $8\sqrt{42}$ square units, determine the possible coordinates of the points C and D.

$\boxed{}$, $\boxed{\mathbf{r} = 4\mathbf{i} + \mathbf{j} + 3\mathbf{k} + \mu(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})}$, $\boxed{|CD| = 8\sqrt{3}}$, $\boxed{C(9,7,4) \text{ \& } D(1,-1,-4) \text{ in any order}}$

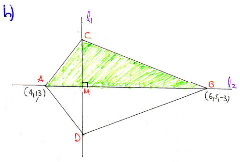
a) $\vec{AB} = \underline{b} - \underline{a} = (6,5,-3) - (4,1,3) = (2,4,-6)$

DIRECTION OF l_2 IS $(1,2,-3)$

$(1,1,1) \cdot (1,2,-3) = 1 + 2 - 3 = 0$

INDEED PERPENDICULAR

b)



START BY A GOOD DIAGRAM — AREA OF THE KITE IS $8\sqrt{42}$

$$2\left[\tfrac{1}{2}|AB||MC|\right] = 8\sqrt{42}$$
$$|AB||MC| = 8\sqrt{42}$$
$$|2,4,-6||MC| = 8\sqrt{42}$$
$$2|1,2,-3||MC| = 8\sqrt{42}$$
$$\sqrt{1+4+9}\,|MC| = 4\sqrt{42}$$
$$\sqrt{14}\,|MC| = 4\sqrt{42}$$
$$\cancel{\sqrt{14}}\,|MC| = 4\cancel{\sqrt{14}}\sqrt{3}$$
$$\underline{\underline{|MC| = 4\sqrt{3}}}$$

NEXT WE NEED THE INTERSECTION OF l_1 & l_2

l_1: $\underline{r}_1 = (10,8,5) + \lambda(1,1,1) = (\lambda+10, \lambda+8, \lambda+5)$

l_2: $\underline{r}_2 = (4,1,3) + \mu(1,2,-3) = (\mu+4, 2\mu+1, 3-3\mu)$

EQUATE $\underline{i}$ & $\underline{j}$

$\left.\begin{aligned} \underline{i}&: \lambda+10 = \mu+4 \\ \underline{j}&: \lambda+8 = 2\mu+1 \end{aligned}\right\}$ SUBTRACT "UPWARDS" $\Rightarrow -2 = \mu - 3$

$\mu = 1$ & $\lambda = -5$

$\therefore \underline{\underline{M(5,3,0)}}$

NOW THE VALUE OF λ AT M IS -5

THE DIRECTION VECTOR OF l_1 IS $(1,1,1)$, WHOSE MODULUS IS $\sqrt{3}$, WHICH IMPLIES THAT EVERY "STEP" IN λ, ADVANCES BY $\sqrt{3}$

BUT $|MC| = 4\sqrt{3}$ & SIMILARLY $|MD| = 4\sqrt{3}$

THUS THE REQUIRED POINTS WILL BE PRODUCED FROM $\lambda = -5 \pm 4$

$\therefore$ IF $\lambda = -1 \Rightarrow (9,7,4)$

IF $\lambda = -9 \Rightarrow (1,-1,-4)$

(IN ANY ORDER)

Question 153 (*****)

The straight line L_1 passes through the points A and B, whose respective position vectors relative to a fixed origin O are

$$\begin{pmatrix}1\\2\\5\end{pmatrix} \text{ and } \begin{pmatrix}2\\1\\6\end{pmatrix}.$$

The point C has position vector $\begin{pmatrix}3\\0\\1\end{pmatrix}$.

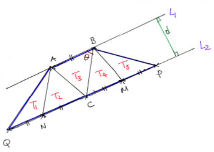
The straight line L_2 passes through C and is parallel to L_1.

The points P and Q both lie on L_2 so that $|CP| = |CQ| = 2|AB|$.

Find the area of the quadrilateral with vertices at A, B, P and Q.

$\boxed{}$, $\boxed{15\sqrt{2}}$

● START WITH A GOOD DIAGRAM – LET M & N BE THE MIDPOINTS OF CP & CQ



● AS ALL THE TRIANGLES IN THE DIAGRAM HAVE EQUAL BASES & THEY ALL HAVE THE SAME HEIGHT d (PARALLEL LINES), THEIR AREAS ARE EQUAL

● IT SUFFICES TO FIND THE AREA OF T_3 – LET $A\hat{B}C = \theta$

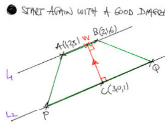
$$\left.\begin{matrix}A(1,2,5)\\B(2,1,6)\\C(3,0,1)\end{matrix}\right\} \Rightarrow \begin{matrix}\vec{AB} = \underline{b}-\underline{a} = (2,1,6)-(1,2,5) = (1,-1,1)\\ \vec{CB} = \underline{b}-c = (2,1,6)-(3,0,1) = (-1,1,5)\end{matrix}$$

● BY THE DOT PRODUCT
$$\vec{AB}\cdot\vec{CB} = |\vec{AB}||\vec{CB}|\cos\theta$$
$$(1,-1,1)\cdot(-1,1,5) = |1,-1,1||-1,1,5|\cos\theta$$
$$-1-1+5 = \sqrt{1+1+1}\sqrt{1+1+25}\cos\theta$$
$$3 = \sqrt{3}\sqrt{27}\cos\theta$$
$$3 = 9\cos\theta$$
$$\cos\theta = \frac{1}{3}$$
$$\therefore \sin\theta = \frac{\sqrt{8}}{3}$$

● AREA OF ABPQ = 5 (AREA OF T_3) $= 5\times\left[\frac{1}{2}|AB||CB|\sin\theta\right]$
$$= 5\times\frac{1}{2}\sqrt{3}\times\sqrt{27}\times\frac{\sqrt{8}}{3} = \frac{5}{2}\times9\times\frac{2\sqrt{2}}{3} = 15\sqrt{2}$$

ALTERNATIVE APPROACH (FINDING THE DISTANCE BETWEEN THE LINES)

● START AGAIN WITH A GOOD DIAGRAM – FIND SOME AUXILIARIES



$$\vec{AB} = \underline{b}-\underline{a} = (2,1,6)-(1,2,5) = (1,-1,1)$$
$$|\vec{AB}| = |1,-1,1| = \sqrt{3}$$
$$\therefore |PQ| = 2|CP| = 4\sqrt{3}$$

● EQUATION OF L_1
$$\boxed{\underline{r} = (1,2,5)+\lambda(1,-1,1) = (\lambda+1, 2-\lambda, \lambda+5)}$$

● LET POINT W LIE ON L_1 SO THAT $\vec{CW} \perp L_1$
- SUPPOSE $\underline{w} = (x,y,z)$
- $\vec{CW} = \underline{w}-\underline{c} = (x,y,z)-(3,0,1) = (x-3, y, z-1)$

BUT $(x,y,z) = (\lambda+1, 2-\lambda, \lambda+5)$
$$\therefore \vec{CW} = (\lambda+1-3, 2-\lambda, \lambda+5-1)$$
$$\boxed{\vec{CW} = (\lambda-2, 2-\lambda, \lambda+4)}$$

● BUT $\vec{CW}$ IS PERPENDICULAR TO L_1
$$(\lambda-2, 2-\lambda, \lambda+4)\cdot(1,-1,1) = 0$$
$$\lambda-2+\lambda-2+\lambda+4 = 0$$
$$\lambda = 0$$
$$\therefore \underline{w} = (1,2,5) \text{ IE } \underline{w} = \underline{a}$$

● HENCE THE HEIGHT OF THE TRAPEZIUM IS $|\vec{CW}| = |-2,2,4|$
$$= \sqrt{4+4+16}$$
$$= \sqrt{24}$$

● FINALLY THE AREA IS
$$\frac{|AB|+|PQ|}{2}\times|CW| = \frac{\sqrt{3}+4\sqrt{3}}{2}\times\sqrt{24}$$
$$= \frac{5\sqrt{3}}{2}\times2\sqrt{6}$$
$$= 5\sqrt{3}\times\sqrt{6}$$
$$= 5\sqrt{3}\times\sqrt{3}\times\sqrt{2}$$
$$= 15\sqrt{2}$$

AS BEFORE.

Question 154 (*****)

Relative to a fixed origin O, the straight line l passes through the points $A(a,-3,6)$, $B(2,b,2)$ and $C(3,3,0)$, where a and b are constants.

a) Find the value of a and the value of b, and hence find a vector equation of l.

The points P and Q lie on the l so that $|OP| = |OQ|$ and $\measuredangle POQ = 90°$.

b) Find the coordinates of P and the coordinates of Q.

$\boxed{}$, $\boxed{a=0,\ b=1}$, $\boxed{\mathbf{r} = -3\mathbf{j} + 6\mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})}$, $\boxed{(1,-1,4)\ \&\ (3,3,0)}$

(a) $A(a,-3,6)$, $B(2,b,2)$, $C(3,3,0)$

$\overrightarrow{AB} = t \times \overrightarrow{BC}$, FOR SOME SCALAR t

$(\underline{b}-\underline{a}) = t(\underline{c}-\underline{b})$

$[(2,b,2)-(a,-3,6)] = t[(3,3,0)-(2,b,2)]$

$(2-a,\ b+3,\ -4) = t(1,\ 3-b,\ -2)$

HENCE (i): $2-a = t$
(j): $b+3 = (3-b)t$
(k): $-4 = -2t$

$\therefore t=2$ (from k)

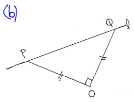
$\therefore$ i: $2-a=2$
$\boxed{a=0}$

j: $b+3=(3-b)\times 2$
$b+3 = 6-2b$
$3b=3$
$\boxed{b=1}$

$\therefore A(0,-3,6)$ & $\overrightarrow{AB} = (2,4,-4)$

$\therefore \underline{r} = (0,-3,6) + \lambda(1,2,-2) = (\lambda,\ 2\lambda-3,\ 6-2\lambda)$

(b)



- LET $\underline{p} = (p,\ 2p-3,\ 6-2p)$
$\underline{q} = (q,\ 2q-3,\ 6-2q)$

FOR SOME VALUES $\lambda=p$ & $\lambda=q$
$p \neq q$

- $\overrightarrow{OP} \cdot \overrightarrow{OQ} = 0$

$(p,2p-3,6-2p)\cdot(q,2q-3,6-2q)=0$

$pq+4pq-6p-6q+9+36-12q-12p+4pq=0$

$9pq-18p-18q+45=0$

$\boxed{pq-2p-2q+5=0}$

- $|\overrightarrow{OP}| = |p,\ 2p-3,\ 6-2p|$
$= \sqrt{p^2+(2p-3)^2+(6-2p)^2}$
$= \sqrt{p^2+4p^2-12p+9+36-24p+4p^2}$
$= \sqrt{9p^2-36p+45}$

- $|\overrightarrow{OQ}| = \sqrt{9q^2-36q+45}$

NOW $|\overrightarrow{OP}| = |\overrightarrow{OQ}|$

$\Rightarrow \sqrt{9p^2-36p+45} = \sqrt{9q^2-36q+45}$

$\Rightarrow 9p^2-36p+\cancel{45} = 9q^2-36q+\cancel{45}$

$\Rightarrow p^2-4p = q^2-4q$

$\Rightarrow p^2-q^2 = 4p-4q$

$\Rightarrow (p-q)(p+q) = 4(p-q)$

BUT $p \neq q$

$\Rightarrow \boxed{p+q=4}$

HENCE $\left\{\begin{array}{l} pq-2p-2q+5=0 \\ p+q=4 \end{array}\right\}$

$p(4-p)-2p-2(4-p)+5=0$

$4p-p^2-\cancel{2p}-8+\cancel{2p}+5=0$

$0=p^2-4p-3$

$(p-1)(p-3)=0$

$p = \begin{cases}1\\3\end{cases} \Rightarrow q = \begin{cases}3\\1\end{cases}$

$\therefore P(1,-1,4)$ & $Q(3,3,0)$

IN EITHER ORDER

Question 155 (*****)

The straight line l has vector equation

$$\mathbf{r} = 3\mathbf{i} + 3\mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}),$$

where λ is a scalar parameter.

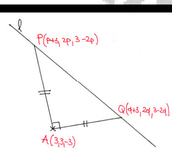
The point A has coordinates $(3,3,-3)$, relative to a fixed origin O.

The points P and Q lie on the l so that $|AP| = |AQ|$.

Given further that $\measuredangle PAQ = 90°$, find the coordinates of P and the coordinates of Q.

$\boxed{}$, $\boxed{(4,2,1) \ \& \ (6,6,-3)}$

$$\boxed{\begin{aligned} \underline{r} &= (3,0,3) + \lambda(1,2,-2) \\ \underline{r} &= (\lambda+3, 2\lambda, 3-2\lambda) \end{aligned}}$$



- LET THE CO-ORDINATES OF P & Q BE FOR $\lambda = p$ & $\lambda = q$

$P(p+3, 2p, 3-2p)$

$Q(q+3, 2q, 3-2q)$

- $\vec{AP} = \underline{p} - \underline{a} = (p+3, 2p, 3-2p) - (3,3,-3) = (p, 2p-3, 6-2p)$

$\vec{AQ} = \underline{q} - \underline{a} = (q+3, 2q, 3-2q) - (3,3,-3) = (q, 2q-3, 6-2q)$

- $\vec{AP} \perp \vec{AQ}$ SO THEIR DOT PRODUCT MUST BE ZERO

$(p, 2p-3, 6-2p) \cdot (q, 2q-3, 6-2q) = 0$

$pq + (2p-3)(2q-3) + (6-2p)(6-2q) = 0$

$pq + 4pq - 6p - 6q + 9 + 36 - 12q - 12p + 4pq = 0$

$9pq - 18p - 18q + 45 = 0$

$\boxed{pq - 2p - 2q + 5 = 0}$

- NEXT WE NEED THE LENGTHS TO BE EQUAL

$|\vec{AP}| = |p, 2p-3, 6-2p| = \sqrt{p^2 + (2p-3)^2 + (6-2p)^2}$

$= \sqrt{p^2 + 4p^2 - 12p + 9 + 36 - 24p + 4p^2}$

$= \sqrt{9p^2 - 36p + 45}$

$= 3\sqrt{p^2 - 4p + 5}$

$|\vec{AQ}| = 3\sqrt{q^2 - 4q + 5}$ (BY ANALOGY)

- FINALLY ANOTHER EQUATION FROM $|AP| = |AQ|$

$\cancel{3}\sqrt{p^2 - 4p + 5} = \cancel{3}\sqrt{q^2 - 4q + 5}$

$p^2 - 4p \cancel{+5} = q^2 - 4q \cancel{+5}$

$p^2 - q^2 = 4p - 4q$

$(p-q)(p+q) = 4(p-q)$

$\boxed{p + q = 4}$

(AS $p \neq q$, AS THE POINTS ARE DISTINCT)

- SOLVING SIMULTANEOUSLY

$\boxed{p + q = 4}$ & $pq - 2p - 2q + 5 = 0$

$pq - 2(p+q) + 5 = 0$

$pq - 2 \times 4 + 5 = 0$

$\boxed{pq = 3}$

BY INSPECTION

$p = 1 \quad q = 3$ OR $q = 1 \quad p = 3$

(OR BY SOLVING)

$\therefore$ THE TWO POINTS ARE

$(p+3, 2p, 3-2p) \Rightarrow P(4,2,1)$

$(q+3, 2q, 3-2q) \Rightarrow Q(6,6,-3)$

OR THE OTHER WAY ROUND

Question 156 (*****)

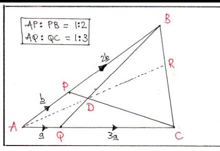
In the acute triangle ABC the following information is given.

- The point P lies on AB so that $AP:PB = 1:2$.
- The point Q lies on AC so that $AQ:QC = 1:3$.
- The point D is the intersection of CP and BQ.

The straight line through A and D is extended so that it meets BC at the point R.

Determine the ratio $BR:RC$.

$\boxed{\text{SPEN}}$, $\boxed{BR:RC = 2:5}$



- LET $\vec{AP} = \underline{b}$, $\vec{AQ} = \underline{a}$
- CONSEQUENTLY $\vec{PB} = 2\underline{b}$, $\vec{QC} = 3\underline{a}$
- MARK THESE ONTO THE DIAGRAM OPPOSITE

- FIRSTLY WE CALCULATE SOME "OBVIOUS PATHS" SUCH AS

$$\vec{BC} = \vec{BA} + \vec{AC} = -3\underline{b} + 4\underline{a} = 4\underline{a} - 3\underline{b}$$
$$\vec{BQ} = \vec{BA} + \vec{AQ} = -3\underline{b} + \underline{a} = \underline{a} - 3\underline{b}$$
$$\vec{CP} = \vec{CA} + \vec{AP} = -4\underline{a} + \underline{b}$$

- NEXT WE LOOK AT THE VECTOR $\vec{AD}$

$$\Rightarrow \begin{cases} \vec{AD} = \vec{AB} + \lambda \vec{BQ} & (\text{FOR SOME SCALAR } \lambda,\ 0<\lambda<1) \\ \vec{AD} = \vec{AC} + \mu \vec{CP} & (\text{FOR SOME SCALAR } \mu,\ 0<\mu<1) \end{cases}$$

$$\Rightarrow \vec{AB} + \lambda \vec{BQ} = \vec{AC} + \mu \vec{CP}$$
$$\Rightarrow 3\underline{b} + \lambda(\underline{a} - 3\underline{b}) = 4\underline{a} + \mu(-4\underline{a} + \underline{b})$$
$$\Rightarrow (3-3\lambda)\underline{b} + \lambda\underline{a} = (4-4\mu)\underline{a} + \mu\underline{b}$$

$$\left.\begin{array}{l} \lambda = 4 - 4\mu \\ \mu = 3 - 3\lambda \end{array}\right\} \Rightarrow \mu = 3 - 3(4-4\mu)$$
$$\Rightarrow \mu = 3 - 12 + 12\mu$$
$$\Rightarrow 9 = 11\mu$$
$$\Rightarrow \boxed{\mu = \tfrac{9}{11}}$$
$$\Rightarrow \lambda = 4 - 4 \times \tfrac{9}{11} = \tfrac{44-36}{11} = \tfrac{8}{11}$$
$$\Rightarrow \boxed{\lambda = \tfrac{8}{11}}$$

- NEXT WE CONSIDER THE VECTOR $\vec{AR}$ FROM TWO DIFFERENT PATHS (LOOPS)

  - FIRSTLY $\vec{AD} = \vec{AB} + \lambda \vec{BQ}$

$$\vec{AD} = 3\underline{b} + \tfrac{8}{11}(\underline{a} - 3\underline{b}) = \tfrac{8}{11}\underline{a} + \left(3 - \tfrac{24}{11}\right)\underline{b}$$
$$\vec{AD} = \tfrac{8}{11}\underline{a} + \tfrac{9}{11}\underline{b}$$

  - $\vec{AR} = \vec{AB} + \vec{BR}$
    $= 3\underline{b} + k\vec{BC}$ $(k \text{ a SCALAR},\ 0<k<1)$
  - $\vec{AR} = m\vec{AD}$ $(m \text{ a SCALAR},\ m>1)$

- FINALLY EQUATING EXPRESSIONS FOR $\vec{AR}$

$$\Rightarrow 3\underline{b} + k\vec{BC} = m\vec{AD}$$
$$\Rightarrow 3\underline{b} + k(4\underline{a} - 3\underline{b}) = m\left(\tfrac{8}{11}\underline{a} + \tfrac{9}{11}\underline{b}\right)$$
$$\Rightarrow 4\underline{a}k + (3-3k)\underline{b} = \tfrac{8}{11}m\underline{a} + \tfrac{9}{11}m\underline{b}$$

$$\left.\begin{array}{l} \tfrac{8}{11}m = 4k \\ \tfrac{9}{11}m = 3-3k \end{array}\right\} \rightarrow \text{DIVIDE} \quad \frac{8}{9} = \frac{4k}{3-3k}$$
$$\frac{2}{3} = \frac{k}{1-k} \quad \div 4,\ \times 3$$
$$2 - 2k = 3k$$
$$2 = 5k$$
$$\boxed{k = \tfrac{2}{5}}$$

(m IS NOT NEEDED)

- REQUIRED RATIO $BR:RC = 2:5$

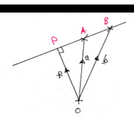
Question 157 (*****)

Relative to a fixed origin O, the position vectors of two points A and B are denoted by $\mathbf{a}$ and $\mathbf{b}$. The point P is the foot of the perpendicular from O to the straight line through A and B.

Show that if $\mathbf{p}$ denotes the position vector of P, then

$$\mathbf{p} = \mathbf{a} - \frac{\mathbf{a}\cdot(\mathbf{a}-\mathbf{b})(\mathbf{a}-\mathbf{b})}{|\mathbf{a}-\mathbf{b}|^2}.$$

$\boxed{}$, $\boxed{\text{proof}}$



STARTING BY THE POSITION VECTOR OF THE POINT P

$$\underline{p} = \underline{a} + \lambda(\underline{b}-\underline{a})$$

ALSO $\vec{OP} \cdot \vec{AB} = 0$

$$\underline{p}\cdot(\underline{b}-\underline{a}) = 0$$
$$[\underline{a} + \lambda(\underline{b}-\underline{a})]\cdot(\underline{b}-\underline{a}) = 0$$
$$\underline{a}\cdot(\underline{b}-\underline{a}) + \lambda(\underline{b}-\underline{a})\cdot(\underline{b}-\underline{a}) = 0$$
$$\lambda|\underline{b}-\underline{a}|^2 = \underline{a}\cdot(\underline{a}-\underline{b})$$
$$\lambda|\underline{a}-\underline{b}|^2 = \underline{a}\cdot(\underline{a}-\underline{b})$$
$$\lambda = \frac{\underline{a}\cdot(\underline{a}-\underline{b})}{|\underline{a}-\underline{b}|^2}$$

FINALLY WE OBTAIN

$$\underline{p} = \underline{a} + \frac{\underline{a}\cdot(\underline{a}-\underline{b})(\underline{b}-\underline{a})}{|a-b|^2}$$
$$\underline{p} = \underline{a} - \frac{a\cdot(\underline{a}-\underline{b})(\underline{a}-\underline{b})}{|\underline{a}-\underline{b}|^2}$$

Question 158 (*****)

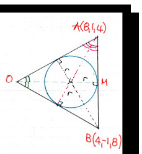
Relative to a fixed origin O located at the point with coordinates $(0,0,0)$, the points $A(8,1,4)$ and $B(4,-1,8)$ are given.

A circle, with centre at the point P and radius r, is drawn so that the three sides of the triangle OAB are tangents to this circle.

Determine the coordinates of P and the exact value of r.

$\boxed{\text{SP-J}}$, $\boxed{P\left(\frac{9}{2},0,\frac{9}{2}\right)}$, $\boxed{r=\frac{3}{2}\sqrt{2}}$

- START BY DRAWING THE TRIANGLE

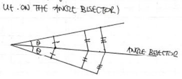


- BY INSPECTION THE TRIANGLE IS ISOSCELES

$$|OA| = |8,1,4| = \sqrt{64+1+16} = 9$$
$$|OB| = |4,-1,8| = \sqrt{16+1+64} = 9$$

- THE CENTRE OF THE CIRCLE MUST LIE ON THE POINT OF INTERSECTION BETWEEN TWO OF ITS ANGLE BISECTORS

(IF A POINT IS EQUIDISTANT FROM TWO INTERSECTING LINES THEN IT MUST LIE ON THE ANGLE BISECTOR)



- THE MIDPOINT OF AB IS $M(6,0,6)$
- EQUATION OF LINE THROUGH O & M IS $\underline{r}_1 = (0,0,0) + \lambda(1,0,1)$ ← SCALED

$$\underline{r}_1 = (\lambda,0,\lambda)$$

THIS IS ALSO AN ANGLE BISECTOR AS THE TRIANGLE IS ISOSCELES

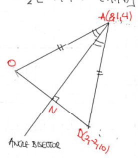
- NEXT FIND THE LENGTH OF AB

$$|AB| = |\underline{b}-\underline{a}| = |(4,-1,8)-(8,1,4)| = |-4,-2,4| = \sqrt{16+4+16}$$
$$= \sqrt{36} = 6$$

- EXTEND AB FROM 6 TO 9, SO WE CREATE ANOTHER ISOSCELES TRIANGLE

$$\overrightarrow{OD} = \overrightarrow{OA} + \tfrac{3}{2}(\overrightarrow{AB}) = \underline{a} + \tfrac{3}{2}(\underline{b}-\underline{a}) = \tfrac{3}{2}\underline{b} - \tfrac{1}{2}\underline{a} = \tfrac{1}{2}(3\underline{b}-\underline{a})$$

$$= \tfrac{1}{2}\left[3(4,-1,8) - (8,1,4)\right] = \tfrac{1}{2}(4,-4,20) = (2,-2,10)$$



N IS THE MIDPOINT OF OD

$N(1,-1,5)$

$$\overrightarrow{AN} = \underline{n} - \underline{a} = (1,-1,5)-(8,1,4) = (-7,-2,1)$$

SCALE THE DIRECTION TO $(7,2,-1)$

EQUATION OF THE ANGLE BISECTOR WILL BE

$$\underline{r}_2 = (1,-1,5) + \mu(7,2,-1)$$
$$\underline{r}_2 = (7\mu+1,\ 2\mu-1,\ 5-\mu)$$

- SOLVING SIMULTANEOUSLY THE TWO LINES (ANGLE BISECTORS) TO FIND THE CENTRE OF THE CIRCLE

$$\left.\begin{array}{l}\underline{r}_1 = (\lambda,0,\lambda)\\ \underline{r}_2 = (7\mu+1,2\mu-1,5-\mu)\end{array}\right\}$$

$\underline{j}$: $2\mu - 1 = 0$
$\mu = \frac{1}{2}$

$\underline{k}$: $\lambda = 5-\mu$
$\lambda = \frac{9}{2}$

$$\underline{i}:\ \left.\begin{array}{l}\lambda = 7\mu+\frac{1}{2}\\ \lambda = 7\times\frac{1}{2}+\frac{1}{2}\\ \lambda = \frac{9}{2}\end{array}\right\}\text{ EXPECT TO BALANCE}$$

$\therefore$ CENTRE IS AT $(\lambda,0,\lambda)$ IE $\left(\frac{9}{2},0,\frac{9}{2}\right)$

- FINALLY TO FIND THE RADIUS OF THE CIRCLE WE CONSIDER THE SHORTEST DISTANCE OF $\left(\frac{9}{2},0,\frac{9}{2}\right)$ FROM THE LINE OA

– LINE OA: $(x,y,z) = (0,0,0) + t(8,1,4)$
$(x,y,z) = (8t,t,4t)$

– DISTANCE SQUARED OF THE GENERAL POINT $(8t,t,4t)$ FROM $\left(\frac{9}{2},0,\frac{9}{2}\right)$ IS GIVEN BY

$$d^2 = \left(8t-\tfrac{9}{2}\right)^2 + t^2 + \left(4t-\tfrac{9}{2}\right)^2$$
$$\frac{d}{dt}(d^2) = 16\left(8t-\tfrac{9}{2}\right) + 2t + 8\left(4t-\tfrac{9}{2}\right)$$

– SOLVE FOR ZERO

$$0 = 16\left(8t-\tfrac{9}{2}\right) + 2t + 8\left(4t-\tfrac{9}{2}\right)$$
$$0 = 8\left(8t-\tfrac{9}{2}\right) + t + 4\left(4t-\tfrac{9}{2}\right)$$
$$0 = 64t - 36 + t + 16t - 18$$
$$54 = 81t$$
$$t = \tfrac{54}{81} = \tfrac{6}{9} = \tfrac{2}{3}$$

$$\therefore\ d^2 = \left(8\times\tfrac{2}{3}-\tfrac{9}{2}\right)^2 + \left(\tfrac{2}{3}\right)^2 + \left(4\times\tfrac{2}{3}-\tfrac{9}{2}\right)^2$$
$$d^2 = \left(\tfrac{16}{3}-\tfrac{9}{2}\right)^2 + \tfrac{4}{9} + \left(\tfrac{8}{3}-\tfrac{9}{2}\right)^2$$
$$d^2 = \left(\tfrac{32-27}{6}\right)^2 + \tfrac{4}{9} + \left(\tfrac{16-27}{6}\right)^2 = \left(\tfrac{5}{6}\right)^2 + \tfrac{4}{9} + \left(\tfrac{11}{6}\right)^2$$
$$d^2 = \tfrac{25}{36} + \tfrac{16}{36} + \tfrac{121}{36} = \tfrac{162}{36} = \tfrac{81}{18} = \tfrac{9}{2}$$
$$d^2 = \tfrac{18}{4} = \tfrac{9}{4}\times 2$$
$$d = \tfrac{3}{2}\sqrt{2}$$

Question 159 (*****)

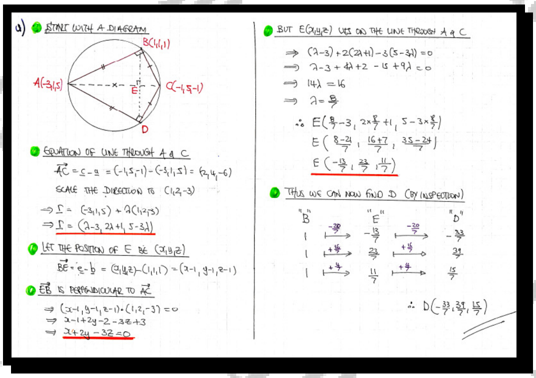
The points $A(-3,1,5)$, $B(1,1,1)$ and $C(-1,5,-1)$ are three of the vertices of the kite $ABCD$, which is circumscribed by a circle.

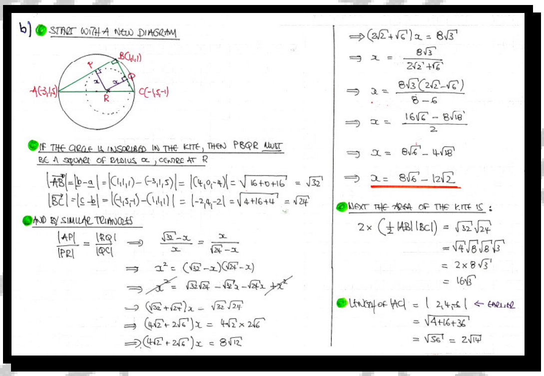
a) Given that $|AB| = |AD|$ and $|BC| = |DC|$, find the exact coordinates of D.

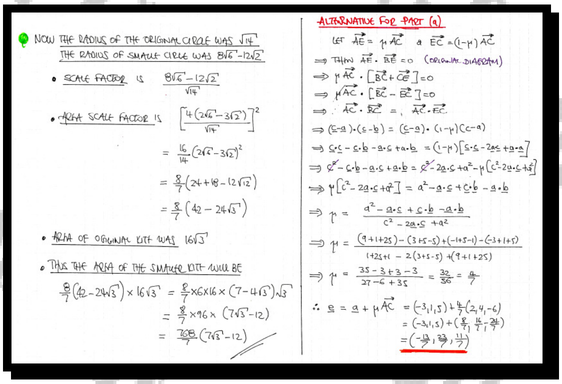
A smaller circle is circumscribed by the kite, and a smaller kite similar to $ABCD$ is circumscribed by the smaller circle.

b) Determine in exact form the area of the smaller kite.

$D\left(-\frac{33}{7}, \frac{39}{7}, \frac{15}{7}\right)$, area $= \dfrac{768\left(7\sqrt{3}-12\right)}{7}$





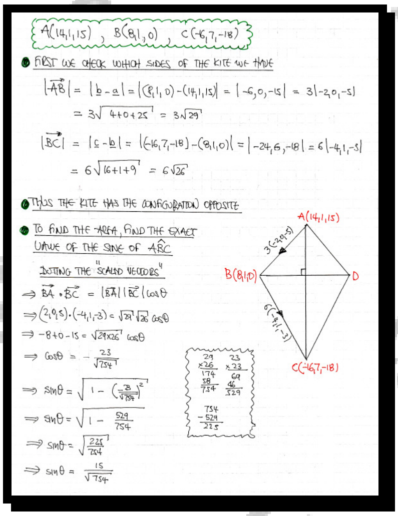


Question 160 (*****)

The points $A(14,1,15)$, $B(8,1,0)$ and $C(-16,7,-18)$ are three of the vertices of the kite $ABCD$. A circle of radius r is circumscribed by the kite.

Find the area of the kite and hence or otherwise determine, in exact simplified surd form, the value of r.

$\boxed{\text{area} = 270}$, $\boxed{r = \frac{6}{5}\left(2\sqrt{26} - \sqrt{29}\right)}$



$A(14,1,15)$, $B(8,1,0)$, $C(-16,7,-18)$

- FIRST WE CHECK WHICH SIDES OF THE KITE WE HAVE

$|\overrightarrow{AB}| = |\underline{b} - \underline{a}| = |(8,1,0)-(14,1,15)| = |-6,0,-15| = 3|-2,0,-5|$
$= 3\sqrt{4+0+25} = 3\sqrt{29}$

$|\overrightarrow{BC}| = |\underline{c} - \underline{b}| = |(-16,7,-18)-(8,1,0)| = |-24,6,-18| = 6|-4,1,-3|$
$= 6\sqrt{16+1+9} = 6\sqrt{26}$

- THUS THE KITE HAS THE CONFIGURATION OPPOSITE

- TO FIND THE AREA, FIND THE EXACT VALUE OF THE SINE OF $A\hat{B}C$

DOTTING THE "SCALED VECTORS"

$\Rightarrow \overrightarrow{BA} \cdot \overrightarrow{BC} = |\overrightarrow{BA}||\overrightarrow{BC}|\cos\theta$

$\Rightarrow (2,0,5)\cdot(-4,1,-3) = \sqrt{29}\sqrt{26}\cos\theta$

$\Rightarrow -8+0-15 = \sqrt{29\times 26}\cos\theta$

$\Rightarrow \cos\theta = -\frac{23}{\sqrt{754}}$

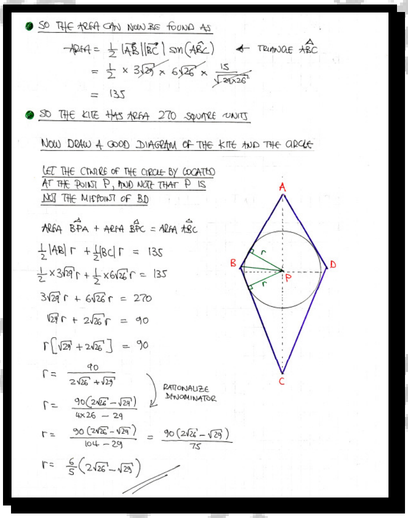
$\Rightarrow \sin\theta = \sqrt{1-\left(\frac{-23}{\sqrt{754}}\right)^2}$

$\Rightarrow \sin\theta = \sqrt{1-\frac{529}{754}}$

$\Rightarrow \sin\theta = \sqrt{\frac{225}{754}}$

$\Rightarrow \sin\theta = \frac{15}{\sqrt{754}}$

$29 \times 26 = 174 + 580 = 754$, $23 \times 23 = 69 + 460 = 529$, $754 - 529 = 225$



- SO THE AREA CAN NOW BE FOUND AS

AREA $= \frac{1}{2}|\overrightarrow{AB}||\overrightarrow{BC}|\sin(A\hat{B}C)$ ← TRIANGLE $A\hat{B}C$

$= \frac{1}{2} \times 3\sqrt{29} \times 6\sqrt{26} \times \frac{15}{\sqrt{29\times 26}}$

$= 135$

- SO THE KITE HAS AREA 270 SQUARE UNITS

NOW DRAW A GOOD DIAGRAM OF THE KITE AND THE CIRCLE

LET THE CENTRE OF THE CIRCLE BE LOCATED AT THE POINT P, AND NOTE THAT P IS NOT THE MIDPOINT OF BD

AREA $B\hat{P}A$ + AREA $B\hat{P}C$ = AREA $A\hat{B}C$

$\frac{1}{2}|AB|r + \frac{1}{2}|BC|r = 135$

$\frac{1}{2}\times 3\sqrt{29}r + \frac{1}{2}\times 6\sqrt{26}r = 135$

$3\sqrt{29}r + 6\sqrt{26}r = 270$

$\sqrt{29}r + 2\sqrt{26}r = 90$

$r\left[\sqrt{29} + 2\sqrt{26}\right] = 90$

$r = \frac{90}{2\sqrt{26}+\sqrt{29}}$

RATIONALIZE DENOMINATOR

$r = \frac{90(2\sqrt{26}-\sqrt{29})}{4\times 26 - 29}$

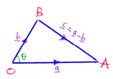
$r = \frac{90(2\sqrt{26}-\sqrt{29})}{104-29} = \frac{90(2\sqrt{26}-\sqrt{29})}{75}$

$r = \frac{6}{5}\left(2\sqrt{26}-\sqrt{29}\right)$

Question 161 (*****)

Use a vector method involving the scalar product to prove the validity of the Cosine Rule.

proof



$\overrightarrow{BA} = \underline{a} - \underline{b}$

Let $\underline{c} = \underline{a} - \underline{b}$

$\Rightarrow \underline{c} = \underline{a} - \underline{b}$

$\Rightarrow \underline{c}\cdot\underline{c} = (\underline{a} - \underline{b})\cdot\underline{c}$

$\Rightarrow \underline{c}\cdot\underline{c} = (\underline{a} - \underline{b})\cdot(\underline{a} - \underline{b})$

$\Rightarrow \underline{c}\cdot\underline{c} = \underline{a}\cdot\underline{a} - \underline{a}\cdot\underline{b} - \underline{b}\cdot\underline{a} + \underline{b}\cdot\underline{b}$

$\Rightarrow \underline{c}\cdot\underline{c} = \underline{a}\cdot\underline{a} - 2\underline{a}\cdot\underline{b} + \underline{b}\cdot\underline{b}$

$\Rightarrow |\underline{c}||\underline{c}|\cancel{\cos 0} = |\underline{a}||\underline{a}|\cancel{\cos 0} - 2|\underline{a}||\underline{b}|\cos\theta + |\underline{b}||\underline{b}|\cancel{\cos 0}$

$\Rightarrow |\underline{c}|^2 = |\underline{a}|^2 - 2|\underline{a}||\underline{b}|\cos\theta + |\underline{b}|^2$

$\Rightarrow c^2 = a^2 + b^2 - 2ab\cos\theta$

Question 162 (*****)

Three points in space A, B and K are such so that $\overrightarrow{KB} = 2\overrightarrow{AK}$.

Prove that if M is a fourth distinct arbitrary point in space, then

$$2\left|\overrightarrow{MA}\right|^2 + \left|\overrightarrow{MB}\right|^2 - 3\left|\overrightarrow{MK}\right|^2 = \text{constant}.$$

proof

MANIPULATE AS FOLLOWS

$2|\overrightarrow{MA}|^2 + |\overrightarrow{MB}|^2 - 3|MK|^2$

$= 2|\overrightarrow{MK} + \overrightarrow{KA}|^2 + |\overrightarrow{MK} + \overrightarrow{KB}|^2 - 3|\overrightarrow{MK}|^2$

$= 2(\overrightarrow{MK} + \overrightarrow{KA})\cdot(\overrightarrow{MK} + \overrightarrow{KA}) + (\overrightarrow{MK} + \overrightarrow{KB})\cdot(\overrightarrow{MK} + \overrightarrow{KB}) - \ldots$

$= 2[\overrightarrow{MK}\cdot\overrightarrow{MK} + 2\overrightarrow{KA}\cdot\overrightarrow{MK} + \overrightarrow{KA}\cdot\overrightarrow{KA}] + [\overrightarrow{MK}\cdot\overrightarrow{MK} + 2\overrightarrow{MK}\cdot\overrightarrow{KB} + \overrightarrow{KB}\cdot\overrightarrow{KB}] - 3|\overrightarrow{MK}|^2$

NOTE $\underline{a}\cdot\underline{a} = |\underline{a}||\underline{a}|\cos 0 = |\underline{a}||\underline{a}| = |\underline{a}|^2$

$= 2[|\overrightarrow{MK}|^2 + 2\overrightarrow{KA}\cdot\overrightarrow{MK} + |\overrightarrow{KA}|^2] + [|\overrightarrow{MK}|^2 - 2\overrightarrow{MK}\cdot\overrightarrow{KB} + |\overrightarrow{KB}|^2] - 3|\overrightarrow{MK}|^2$

$= \cancel{2|MK|^2} + 4\overrightarrow{KA}\cdot\overrightarrow{MK} + 2|\overrightarrow{KA}|^2 + \cancel{|\overrightarrow{MK}|^2} - 2\overrightarrow{MK}\cdot\overrightarrow{KB} + |\overrightarrow{KB}|^2 - \cancel{3|\overrightarrow{MK}|^2}$

$= 4\overrightarrow{KA}\cdot\overrightarrow{MK} - 2\overrightarrow{MK}\cdot\overrightarrow{KB} + 2|\overrightarrow{KA}|^2 + |\overrightarrow{KB}|^2$

$= 2\overrightarrow{MK}\cdot[2\overrightarrow{KA} - \overrightarrow{KB}] + 2|\overrightarrow{KA}|^2 + |\overrightarrow{KB}|^2$

WE WERE GIVEN THAT $\overrightarrow{KB} = 2\overrightarrow{AK}$

$= \cancel{2\overrightarrow{MK}\cdot[2\overrightarrow{KA} - 2\overrightarrow{AK}]} + 2|\overrightarrow{KA}|^2 + |2\overrightarrow{AK}|^2$

$= 2|\overrightarrow{KA}|^2 + |-2\overrightarrow{KA}|^2$

$= 2|\overrightarrow{KA}| + 4|\overrightarrow{KA}|^2$

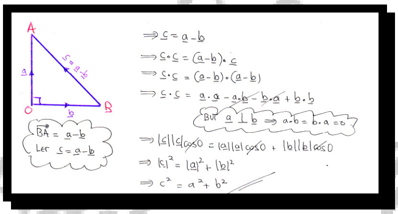
$= 6|\overrightarrow{KA}|^2$

IS CONSTANT

Question 163 (*****)

Use a vector method involving the scalar product to prove the validity of Pythagoras' Theorem.

$\boxed{}$, $\boxed{\text{proof}}$



$\Rightarrow \underline{c} = \underline{a} - \underline{b}$

$\Rightarrow \underline{c} \cdot \underline{c} = (\underline{a} - \underline{b}) \cdot \underline{c}$

$\Rightarrow \underline{c} \cdot \underline{c} = (\underline{a} - \underline{b}) \cdot (\underline{a} - \underline{b})$

$\Rightarrow \underline{c} \cdot \underline{c} = \underline{a} \cdot \underline{a} - \cancel{\underline{a} \cdot \underline{b}} - \cancel{\underline{b} \cdot \underline{a}} + \underline{b} \cdot \underline{b}$

BUT $\underline{a} \perp \underline{b} \Rightarrow a \cdot b = b \cdot a = 0$

$\Rightarrow |\underline{c}||\underline{c}|\cancel{\cos 0} = |\underline{a}||\underline{a}|\cancel{\cos 0} + |\underline{b}||\underline{b}|\cancel{\cos 0}$

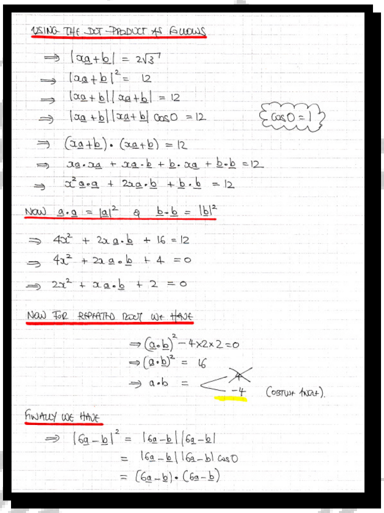
$\Rightarrow |\underline{c}|^2 = |\underline{a}|^2 + |\underline{b}|^2$

$\Rightarrow c^2 = a^2 + b^2$

Question 164 (*****)

Find the modulus of $6\mathbf{a} - \mathbf{b}$, given that the equation $|x\mathbf{a} + \mathbf{b}| = 2\sqrt{3}$ has repeated roots in x, where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors.

$\boxed{}$, $\boxed{4\sqrt{13}}$



USING THE DOT PRODUCT AS FOLLOWS

$\Rightarrow |x\underline{a} + \underline{b}| = 2\sqrt{3}$

$\Rightarrow |x\underline{a} + \underline{b}|^2 = 12$

$\Rightarrow |x\underline{a} + \underline{b}||x\underline{a} + \underline{b}| = 12$

$\Rightarrow |x\underline{a} + \underline{b}||x\underline{a} + \underline{b}| \cos 0 = 12$ $\quad$ $\cos 0 = 1$

$\Rightarrow (x\underline{a} + \underline{b}) \cdot (x\underline{a} + \underline{b}) = 12$

$\Rightarrow x\underline{a} \cdot x\underline{a} + x\underline{a} \cdot \underline{b} + \underline{b} \cdot x\underline{a} + \underline{b} \cdot \underline{b} = 12$

$\Rightarrow x^2 \underline{a} \cdot \underline{a} + 2x\underline{a} \cdot \underline{b} + \underline{b} \cdot \underline{b} = 12$

NOW $\underline{a} \cdot \underline{a} = |\underline{a}|^2$ & $\underline{b} \cdot \underline{b} = |\underline{b}|^2$

$\Rightarrow 4x^2 + 2x\underline{a} \cdot \underline{b} + 16 = 12$

$\Rightarrow 4x^2 + 2x\underline{a} \cdot \underline{b} + 4 = 0$

$\Rightarrow 2x^2 + x\underline{a} \cdot \underline{b} + 2 = 0$

NOW FOR REPEATED ROOT WE HAVE

$\Rightarrow (\underline{a} \cdot \underline{b})^2 - 4 \times 2 \times 2 = 0$

$\Rightarrow (\underline{a} \cdot \underline{b})^2 = 16$

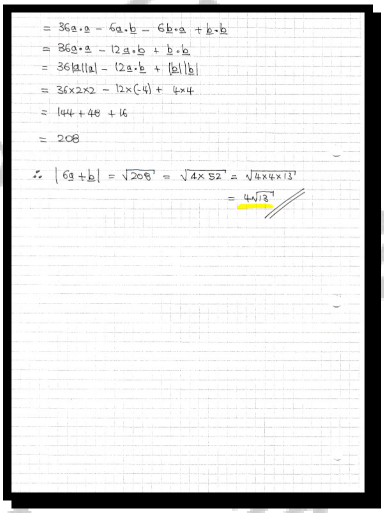
$\Rightarrow a \cdot b = \cancel{4}$ or -4 (OBTUSE ANGLE).

FINALLY WE HAVE

$\Rightarrow |6\underline{a} - \underline{b}|^2 = |6\underline{a} - \underline{b}||6\underline{a} - \underline{b}|$

$= |6\underline{a} - \underline{b}||6\underline{a} - \underline{b}| \cos 0$

$= (6\underline{a} - \underline{b}) \cdot (6\underline{a} - \underline{b})$



$= 36\underline{a} \cdot \underline{a} - 6\underline{a} \cdot \underline{b} - 6\underline{b} \cdot \underline{a} + \underline{b} \cdot \underline{b}$

$= 36\underline{a} \cdot \underline{a} - 12\underline{a} \cdot \underline{b} + \underline{b} \cdot \underline{b}$

$= 36|\underline{a}||\underline{a}| - 12\underline{a} \cdot \underline{b} + |\underline{b}||\underline{b}|$

$= 36 \times 2 \times 2 - 12 \times (-4) + 4 \times 4$

$= 144 + 48 + 16$

$= 208$

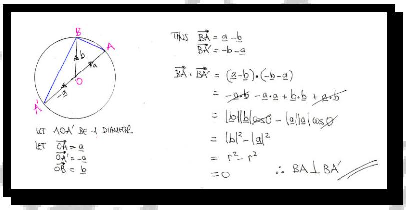
$\therefore |6\underline{a} + \underline{b}| = \sqrt{208} = \sqrt{4 \times 52} = \sqrt{4 \times 4 \times 13}$

$= 4\sqrt{13}$

Question 165 (*****)

Use a vector method involving the scalar product to prove that an inscribed angle in a circle which corresponds to a diameter is always a right angle.

[ ], [proof]



Question 166 (*****)

The vertices of the triangle OAB have coordinates $A(6,-18,-6)$, $B(7,-1,3)$, where O is a fixed origin.

The point N lies on OA so that $ON : NA = 1:2$.

The point M is the midpoint of OB.

The point P is the intersection of AM and BN.

By using vector methods, or otherwise, determine the coordinates of P.

[ ], $\boxed{P(4,-4,0)}$