MATRICES

EXAM QUESTIONS

(Part Two)

Question 1 (**)

Find the eigenvalues and the corresponding eigenvectors of the following 2×2 matrix.

$$\mathbf{A}=\begin{pmatrix}7 & 6\\ 6 & 2\end{pmatrix}.$$

$\boxed{\lambda=-2,\mathbf{u}=\alpha\begin{pmatrix}2\\-3\end{pmatrix}}$, $\boxed{\lambda=11,\mathbf{u}=\beta\begin{pmatrix}3\\2\end{pmatrix}}$

$A=\begin{pmatrix}7 & 6\\ 6 & 2\end{pmatrix}$

CHARACTERISTIC EQUATION

$\begin{vmatrix}7-\lambda & 6\\ 6 & 2-\lambda\end{vmatrix}=0$

$\Rightarrow (7-\lambda)(2-\lambda)-36=0$

$\Rightarrow 14-7\lambda-2\lambda+\lambda^2-36=0$

$\Rightarrow \lambda^2-9\lambda-22=0$

$\Rightarrow (\lambda-11)(\lambda+2)=0$

$\lambda = -2,\ 11$

• IF $\lambda=-2$ $\left.\begin{array}{l}7x+6y=-2x\\6x+2y=-2y\end{array}\right\}\Rightarrow \begin{array}{l}9x+6y=0\\6x+4y=0\end{array}$

$\therefore y=-\frac{3}{2}x \quad \therefore \begin{pmatrix}1\\-\frac{3}{2}\end{pmatrix}\sim\begin{pmatrix}2\\-3\end{pmatrix}$

• IF $\lambda=11$ $\left.\begin{array}{l}7x+6y=11x\\6x+2y=11y\end{array}\right\}\Rightarrow \begin{array}{l}6y=4x\\6x=9y\end{array}$

$\therefore y=\frac{2}{3}x \quad \therefore \begin{pmatrix}1\\\frac{2}{3}\end{pmatrix}\sim\begin{pmatrix}3\\2\end{pmatrix}$

Question 2 (**)

A transformation in three dimensional space is defined by the following 3×3 matrix, where x is a scalar constant.

$$\mathbf{C}=\begin{pmatrix}2 & -2 & 4\\ 5 & x-2 & 2\\ -1 & 3 & x\end{pmatrix}.$$

Show that $\mathbf{C}$ is non singular for all values of x.

$\boxed{}$, $\boxed{\text{proof}}$

EVALUATING THE DETERMINANT OF THE MATRIX AFTER SIMPLIFICATION WITH ELEMENTARY OPERATIONS

$|\underline{C}| = \begin{vmatrix}2 & -2 & 4\\ 5 & x-2 & 2\\ -1 & 3 & x\end{vmatrix} \begin{array}{c}C_{12}(1)\\C_{13}(-2)\end{array} = \begin{vmatrix}2 & 0 & 0\\ 5 & x+3 & -8\\ -1 & 2 & x+2\end{vmatrix}$

EXPANDING BY THE FIRST ROW

$\dots = 2\begin{vmatrix}x+3 & -8\\ 2 & x+2\end{vmatrix} = 2\left[(x+2)(x+3)+16\right]$

$= 2\left[x^2+5x+6+16\right]$

$= 2\left[x^2+5x+22\right]$

$= 2\left[\left(x+\frac{5}{2}\right)^2-\frac{25}{4}+22\right]$

$= 2\left(x+\frac{5}{2}\right)^2+\frac{63}{2} > 0$ FOR ALL x

THEREFORE $\underline{C}$ IS NON SINGULAR FOR ALL x

Question 3 (**)

The 2×2 matrix $\mathbf{A}$ is given below.

$$\mathbf{A} = \begin{pmatrix} 1 & 8 \\ 8 & -11 \end{pmatrix}.$$

a) Find the eigenvalues of $\mathbf{A}$.

b) Determine an eigenvector for each of the corresponding eigenvalues of $\mathbf{A}$.

c) Find a 2×2 matrix $\mathbf{P}$, so that

$$\mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

where λ_1 and λ_2 are the eigenvalues of $\mathbf{A}$, with $\lambda_1 < \lambda_2$.

$$\boxed{\lambda_1 = -15,\ \lambda_2 = 5},\ \boxed{\mathbf{u} = \begin{pmatrix} 1 \\ -2 \end{pmatrix},\ \mathbf{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}},\ \boxed{\mathbf{P} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}}$$

(a) $A = \begin{pmatrix} 1 & 8 \\ 8 & -11 \end{pmatrix}$

CHARACTERISTIC EQUATION

$\begin{vmatrix} 1-\lambda & 8 \\ 8 & -11-\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)(-11-\lambda) - 64 = 0$

$\Rightarrow (\lambda-1)(\lambda+11) - 64 = 0$

$\Rightarrow \lambda^2 + 10\lambda - 75 = 0$

$\Rightarrow (\lambda+15)(\lambda-5) = 0$

$\Rightarrow \lambda = \begin{cases} 5 \\ -15 \end{cases}$

(b) $\lambda = 5$ $\left.\begin{array}{l} x + 8y = 5x \\ 8x - 11y = 5y \end{array}\right\} \Rightarrow x = 2y \therefore \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\lambda = -15$ $\left.\begin{array}{l} x + 8y = -15x \\ 8x - 11y = -15y \end{array}\right\} \Rightarrow y = -2x \therefore \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

(c) NORMALIZE EIGENVECTORS, BOTH HAVE LENGTH $\sqrt{5}$

$P = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}$

Question 4 (**)

Describe fully the transformation given by the following 3×3 matrix.

$$\begin{pmatrix} 0.28 & -0.96 & 0 \\ 0.96 & 0.28 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

rotation in the z axis, anticlockwise, by $\arcsin(0.96)$

$$\begin{pmatrix} 0.28 & -0.96 & 0 \\ 0.96 & 0.28 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\underline{k} \longmapsto \underline{k}$ $\therefore$ Z AXIS IS INVARIANT

ROTATION BY $\arcsin(0.96)$ OR $\arccos(0.28)$ ABOUT THE Z AXIS

Question 5 (**)

A transformation in three dimensional space is defined by the following 3×3 matrix, where k is a scalar constant.

$$\mathbf{A} = \begin{pmatrix} 1 & -2 & k \\ k & 2 & 0 \\ 2 & 3 & 1 \end{pmatrix}.$$

Show that the transformation defined by $\mathbf{A}$ can be inverted for all values of k.

proof

- FOR THE TRANSFORMATION TO BE INVERTIBLE, $|\underline{A}| \neq 0$, SO EXPANDING THE DETERMINANT BY THE THIRD COLUMN YIELDS

$$\det \underline{A} = \begin{vmatrix} 1^{+} & -2^{-} & k^{+} \\ k^{-} & 2^{+} & 0^{-} \\ 2^{+} & 3^{-} & 1^{+} \end{vmatrix} = k\begin{vmatrix} k & 2 \\ 2 & 3 \end{vmatrix} - 0\cancel{\begin{vmatrix} 1 & -2 \\ 2 & 3 \end{vmatrix}} + 1\begin{vmatrix} 1 & -2 \\ k & 2 \end{vmatrix}$$

$$= k(3k-4) + 2 + 2k$$

$$= 3k^2 - 4k + 2k + 2$$

$$= 3k^2 - 2k + 2 > 0$$

SINCE $b^2 - 4ac = (-2)^2 - 4\times3\times2$
$= 4 - 24$
$= -20 < 0$

- AS THE DETERMINANT IS ALWAYS POSITIVE, I.E $|\underline{A}| \neq 0$, THE TRANSFORMATION DESCRIBED BY $\underline{A}$ WILL BE INVERTIBLE FOR ALL k

Question 6 (**)

The 3×3 matrix $\mathbf{A}$ is given below.

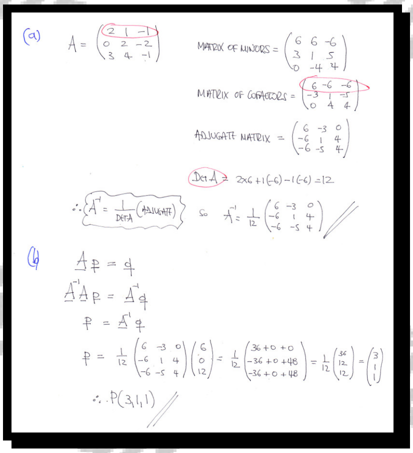
$$\mathbf{A} = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 2 & -2 \\ 3 & 4 & -1 \end{pmatrix}.$$

a) Find the inverse of $\mathbf{A}$.

The point P has been mapped by $\mathbf{A}$ onto the point $Q(6,0,12)$.

b) Determine the coordinates of P.

$$\boxed{\mathbf{A}^{-1} = \frac{1}{12}\begin{pmatrix} 6 & -3 & 0 \\ -6 & 1 & 4 \\ -6 & -5 & 4 \end{pmatrix}}, \quad \boxed{P(3,1,1)}$$



Question 7 (**)

The 3×3 matrix $\mathbf{A}$ is given below.

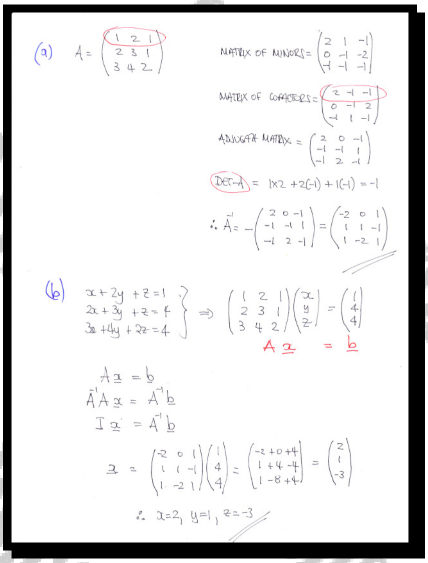
$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 3 & 4 & 2 \end{pmatrix}.$$

a) Find the inverse of $\mathbf{A}$.

b) Hence, or otherwise, solve the system of equations

$$\begin{aligned} x+2y+z &= 1 \\ 2x+3y+z &= 4 \\ 3x+4y+2z &= 4 \end{aligned}$$

$$\boxed{\mathbf{A}^{-1} = \begin{pmatrix} -2 & 0 & 1 \\ 1 & 1 & -1 \\ 1 & -2 & 1 \end{pmatrix}}, \quad \boxed{x=2, \quad y=1, \quad z=-3}$$



(a) $A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 3 & 4 & 2 \end{pmatrix}$

MATRIX OF MINORS $= \begin{pmatrix} 2 & 1 & -1 \\ 0 & -1 & -2 \\ -1 & -1 & -1 \end{pmatrix}$

MATRIX OF COFACTORS $= \begin{pmatrix} 2 & -1 & -1 \\ 0 & -1 & 2 \\ -1 & 1 & -1 \end{pmatrix}$

ADJUGATE MATRIX $= \begin{pmatrix} 2 & 0 & -1 \\ -1 & -1 & 1 \\ -1 & 2 & -1 \end{pmatrix}$

DET-A $= 1\times2 + 2(-1) + 1(-1) = -1$

$\therefore A^{-1} = -\begin{pmatrix} 2 & 0 & -1 \\ -1 & -1 & 1 \\ -1 & 2 & -1 \end{pmatrix} = \begin{pmatrix} -2 & 0 & 1 \\ 1 & 1 & -1 \\ 1 & -2 & 1 \end{pmatrix}$

(b) $\left.\begin{aligned} x+2y+z&=1 \\ 2x+3y+z&=4 \\ 3x+4y+2z&=4 \end{aligned}\right\} \Rightarrow \underbrace{\begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 3 & 4 & 2 \end{pmatrix}}_{A}\underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\underline{x}} = \underbrace{\begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix}}_{\underline{b}}$

$A\underline{x} = \underline{b}$

$A^{-1}A\underline{x} = A^{-1}\underline{b}$

$I\underline{x} = A^{-1}\underline{b}$

$\underline{x} = \begin{pmatrix} -2 & 0 & 1 \\ 1 & 1 & -1 \\ 1 & -2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix} = \begin{pmatrix} -2+0+4 \\ 1+4-4 \\ 1-8+4 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$

$\therefore x=2, \ y=1, \ z=-3$

Question 8 (**)

The 3×3 matrix $\mathbf{A}$ is given below.

$$\mathbf{A} = \begin{pmatrix} -4 & -4 & 4 \\ -1 & 0 & 1 \\ -7 & -6 & 7 \end{pmatrix}.$$

Given that $\mathbf{I}$ is the 3×3 identity matrix, determine the values of the constant λ, so that $\mathbf{A}+\lambda\mathbf{I}$ is singular.

$$\boxed{\lambda = 0, -1, -2}$$

$$A+\lambda I = \begin{pmatrix} -4+\lambda & -4 & 4 \\ -1 & \lambda & 1 \\ -7 & -6 & 7+\lambda \end{pmatrix}$$

$$\begin{vmatrix} \lambda-4 & -4 & 4 \\ -1 & \lambda & 1 \\ -7 & -6 & \lambda+7 \end{vmatrix} = 0$$

$$C_{32}(1)\begin{vmatrix} \lambda-4 & 0 & 4 \\ -1 & \lambda+1 & 1 \\ -7 & \lambda+1 & \lambda+7 \end{vmatrix} = 0$$

$$r_{32}(-1)\begin{vmatrix} \lambda-4 & 0 & 4 \\ 6 & 0 & -\lambda-6 \\ -7 & \lambda+1 & \lambda+7 \end{vmatrix} = 0$$

EXPAND BY MIDDLE COLUMN

$$-(\lambda+1)\begin{vmatrix} \lambda-4 & 4 \\ 6 & -\lambda-6 \end{vmatrix} = 0$$

$$-(\lambda+1)\left[(\lambda-4)(-\lambda-6)-24\right] = 0$$

$$(\lambda+1)\left[(\lambda-4)(\lambda+6)+24\right] = 0$$

$$(\lambda+1)(\lambda^2+2\lambda\cancel{-24}\cancel{+24}) = 0$$

$$(\lambda+1)\times\lambda(\lambda+2) = 0$$

$$\therefore \lambda = -1,\ 0,\ -2$$

Question 9 (**)

The 3×3 matrix $\mathbf{A}$ is defined in terms of the scalar constant k by

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 3 \\ k & 2 & 4 \\ k-2 & 3 & k+7 \end{pmatrix}.$$

Given that $|\mathbf{A}| = 8$, find the possible values of k.

$\boxed{}$, $\boxed{k=-2,\quad k=-8}$

EXPANDING THE DETERMINANT OF THE MATRIX BY THE FIRST ROW YIELDS

$$\Rightarrow \begin{vmatrix} 2^{+} & -1^{-} & 3^{+} \\ k & 2 & 4 \\ k-2 & 3 & k+7 \end{vmatrix} = 8$$

$$\Rightarrow 2\begin{vmatrix} 2 & 4 \\ 3 & k+7 \end{vmatrix} - (-1)\begin{vmatrix} k & 4 \\ k-2 & k+7 \end{vmatrix} + 3\begin{vmatrix} k & 2 \\ k-2 & 3 \end{vmatrix} = 8$$

$$\Rightarrow 2\left[2(k+7)-12\right] + k(k+7)-4(k-2) + 3\left[3k-2(k-2)\right] = 8$$

$$\Rightarrow 2\left[2k+2\right] + k^2+7k-4k+8 + 3\left[k+4\right] = 8$$

$$\Rightarrow 4k+4+k^2+3k\cancel{+8}+3k+12=\cancel{8}$$

$$\Rightarrow k^2+10k+16=0$$

$$\Rightarrow (k+2)(k+8)=0$$

$$\Rightarrow k = \begin{cases} -2 \\ -8 \end{cases}$$

ALTERNATIVE BY ELEMENTARY OPERATIONS FIRST

$$\Rightarrow \begin{vmatrix} 2 & -1 & 3 \\ k & 2 & 4 \\ k-2 & 3 & k+7 \end{vmatrix} = 8$$

$$\begin{array}{l} \\ r_{12}(2) \\ r_{13}(3) \end{array} \begin{vmatrix} 2 & -1 & 3 \\ k+4 & 0 & 10 \\ k+4 & 0 & k+16 \end{vmatrix} = 8$$

& NOW EXPANDING BY THE MIDDLE COLUMN

$$\Rightarrow (k+4)(k+16)-10(k+4)=8$$

$$\Rightarrow k^2+20k+64-10k-40=8$$

$$\Rightarrow k^2+10k+16=0$$

ETC ETC AS BEFORE

Question 10 (**)

Find the eigenvalues and the corresponding equations of invariant lines of the following 2×2 matrix

$$\mathbf{B}=\begin{pmatrix}4 & -5\\ 6 & -9\end{pmatrix}.$$

$\boxed{\lambda=1,\ y=\frac{3}{5}x}$, $\boxed{\lambda=-6,\ y=2x}$

$B=\begin{pmatrix}4 & -5\\ 6 & -9\end{pmatrix}$

CHARACTERISTIC EQUATION

$\begin{vmatrix}4-\lambda & -5\\ 6 & -9-\lambda\end{vmatrix}=0$

$\Rightarrow (4-\lambda)(-9-\lambda)+30=0$

$\Rightarrow (\lambda-4)(\lambda+9)+30=0$

$\Rightarrow \lambda^2+5\lambda-6=0$

$\Rightarrow (\lambda-1)(\lambda+6)$

$\Rightarrow \lambda = 1, -6$

• IF $\lambda=1$ $\left.\begin{array}{l}4x-5y=x\\ 6x-9y=y\end{array}\right\}\Rightarrow 5y=3x$

$y=\frac{3}{5}x$

• IF $\lambda=-6$ $\left.\begin{array}{l}4x-5y=-6x\\ 6x-9y=-6y\end{array}\right\}\Rightarrow 5y=10x$

$y=2x$

Question 11 (**)

A transformation in three dimensional space is defined by the following 3×3 matrix, where y is a scalar constant.

$$\mathbf{M}=\begin{pmatrix}y-3 & -2 & 0\\ 1 & y & -2\\ -1 & y-1 & y-1\end{pmatrix}$$

If $|\mathbf{M}|=0$, find the possible values of y.

$\boxed{}$, $\boxed{y=-1,\quad y=0,\quad y=3}$

• EVALUATING THE DETERMINANT IN TERMS OF y – EXPAND BY 1ST ROW

$|\underline{M}| = \begin{vmatrix}\overset{+}{y-3} & \overset{-}{-2} & \overset{+}{0}\\ 1 & y & -2\\ -1 & y-1 & y-1\end{vmatrix}$

$= (y-3)\begin{vmatrix}y & -2\\ y-1 & y-1\end{vmatrix} + 2\begin{vmatrix}1 & -2\\ -1 & y-1\end{vmatrix} + 0\cancel{\begin{vmatrix}1 & y\\ -1 & y-1\end{vmatrix}}$

$= (y-3)\left[y(y-1)+2(y-1)\right] + 2\left[y-1-2\right]$

$= (y-3)(y^2-y+2y-2) + 2(y-3)$

$= (y-3)\left[y^2-y+2y-2+2\right]$

$= (y-3)(y^2+y)$

$= y(y+1)(y-3)$

• HENCE IF $|\underline{M}|=0$

$y(y+1)(y-3)=0$

$y = -1, 0, 3$

Question 12 (**)

The 3×3 matrix $\mathbf{M}$ is given below.

$$\mathbf{M} = \begin{pmatrix} 5 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 3 & 1 \end{pmatrix}.$$

a) Find the inverse of $\mathbf{M}$.

The point A has been transformed by $\mathbf{M}$ into the point $B(5,2,-1)$.

b) Determine the coordinates of A.

$$\boxed{\mathbf{M}^{-1} = \frac{1}{9}\begin{pmatrix} 2 & -1 & -1 \\ -1 & -4 & 5 \\ 1 & 13 & -5 \end{pmatrix}}, \quad \boxed{A(1,-2,4)}$$

(a) $M = \begin{pmatrix} 5 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 3 & 1 \end{pmatrix}$

MATRIX OF MINORS $= \begin{pmatrix} -2 & -1 & -1 \\ -1 & 4 & 13 \\ 1 & 5 & 5 \end{pmatrix}$

MATRIX OF COFACTORS $= \begin{pmatrix} -2 & 1 & -1 \\ 1 & 4 & -13 \\ 1 & -5 & 5 \end{pmatrix}$

ADJUGATE MATRIX $= \begin{pmatrix} -2 & 1 & 1 \\ 1 & 4 & -5 \\ -1 & -13 & 5 \end{pmatrix}$

DET M $= 5(-2) + 2\times 1 + 1(-1) = -9$

$\therefore M^{-1} = -\frac{1}{9}\begin{pmatrix} -2 & 1 & 1 \\ 1 & 4 & -5 \\ -1 & -13 & 5 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 2 & -1 & -1 \\ -1 & -4 & 5 \\ 1 & 13 & -5 \end{pmatrix}$

(b) LET $\underline{b} = \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix}$, $\underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$M\underline{x} = \underline{b}$

$M^{-1}M\underline{x} = M^{-1}\underline{b}$

$I\underline{x} = M^{-1}\underline{b}$

$\underline{x} = \frac{1}{9}\begin{pmatrix} 2 & -1 & -1 \\ -1 & -4 & 5 \\ 1 & 13 & -5 \end{pmatrix}\begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 10-2+1 \\ -5-8-5 \\ 5+26+5 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$

$\therefore A(1,-2,4)$

Question 13 (**)

A non invertible transformation in three dimensional space is defined by the following 3×3 matrix, where a is a scalar constant.

$$\mathbf{A}=\begin{pmatrix} a & 1 & 2 \\ 2 & -1 & a \\ 3 & a & 4 \end{pmatrix}$$

Determine the possible values of a.

FP1-K , $\boxed{a=1,\ \ a=-2}$

- IF THE TRANSFORMATION IS NOT INVERTIBLE, THE MATRIX $\underline{A}$ IS NOT INVERTIBLE, SO $\det \underline{A}=0$ — EXPAND BY THE FIRST ROW

$$\Rightarrow |\underline{A}|=0$$

$$\Rightarrow \begin{vmatrix} a & 1 & 2 \\ 2 & -1 & a \\ 3 & a & 4 \end{vmatrix}=0$$

$$\Rightarrow a\begin{vmatrix} -1 & a \\ a & 4 \end{vmatrix}-1\begin{vmatrix} 2 & a \\ 3 & 4 \end{vmatrix}+2\begin{vmatrix} 2 & -1 \\ 3 & a \end{vmatrix}=0$$

$$\Rightarrow a(-4-a^2)-(8-3a)+2(2a+3)=0$$

$$\Rightarrow \cancel{-4a}-a^3-8+3a+\cancel{4a}+6=0$$

$$\Rightarrow 0=a^3-3a+2$$

- BY INSPECTION $a=1$ IS A SOLUTION — BY LONG DIVISION (OR MANIPULATION)

$$\Rightarrow a^2(a-1)+a(a-1)-2(a-1)=0$$

$$\Rightarrow (a-1)(a^2+a-2)=0$$

$$\Rightarrow (a-1)(a-1)(a+2)=0$$

$$\Rightarrow a=\begin{cases} 1 \\ -2 \end{cases} \text{ (REPEATED)}$$

Question 14 (**)

The 3×3 matrix $\mathbf{M}$ is given below.

$$\mathbf{M} = \begin{pmatrix} 3 & 2 & 1 \\ 1 & -2 & -1 \\ 1 & 0 & 3 \end{pmatrix}.$$

a) Find the inverse of $\mathbf{M}$.

b) Hence, or otherwise, solve the following system of equations.

$$\begin{aligned} 3x + 2y + z &= 7 \\ x - 2y - z &= 1 \\ x \quad\quad + 3z &= 11 \end{aligned}$$

$\boxed{\mathbf{M}^{-1} = \frac{1}{12}\begin{pmatrix} 3 & 3 & 0 \\ 2 & -4 & -2 \\ -1 & -1 & 4 \end{pmatrix}}$, $\boxed{x = 2, \quad y = -1, \quad z = 3}$

a)

- $\underline{M} = \begin{bmatrix} 3 & 2 & 1 \\ 1 & -2 & -1 \\ 1 & 0 & 3 \end{bmatrix}$

MATRIX OF MINORS $= \begin{bmatrix} -6 & 4 & 2 \\ 6 & 8 & -2 \\ 0 & -4 & -8 \end{bmatrix}$

MATRIX OF COFACTORS $= \begin{bmatrix} -6 & -4 & 2 \\ -6 & 8 & +2 \\ 0 & +4 & -8 \end{bmatrix}$

ADJUGATE MATRIX $= \begin{bmatrix} -6 & -6 & 0 \\ -4 & 8 & 4 \\ 2 & 2 & -8 \end{bmatrix}$

- $\det \underline{M} = 3(-6) + 2(-4) + 2\times 1 = -24$

- $\underline{M}^{-1} = \frac{1}{|M|} \times (\text{ADJUGATE}) = \frac{1}{-24}\begin{vmatrix} -6 & -6 & 0 \\ -4 & 8 & 4 \\ 2 & 2 & -8 \end{vmatrix} = \frac{1}{12}\begin{bmatrix} 3 & 3 & 0 \\ 2 & -4 & -2 \\ -1 & -1 & 4 \end{bmatrix}$

b) REWRITING THE SYSTEM IN MATRIX FORM AND MANIPULATING

$\Rightarrow \underline{M}\underline{x} = \underline{b}$ WHERE $\underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ & $\underline{b} = \begin{pmatrix} 7 \\ 1 \\ 11 \end{pmatrix}$

$\Rightarrow \underline{M}^{-1}\underline{M}\underline{x} = \underline{M}^{-1}\underline{b}$

$\Rightarrow \underline{I}\,\underline{x} = \frac{1}{12}\begin{bmatrix} 3 & 3 & 0 \\ 2 & -4 & -2 \\ -1 & -1 & 4 \end{bmatrix}\begin{bmatrix} 7 \\ 1 \\ 11 \end{bmatrix}$

$\Rightarrow \underline{x} = \frac{1}{12}\begin{bmatrix} 3\times 7 + 3\times 1 + 0 \\ 2\times 7 - 4\times 1 - 2\times 11 \\ -1\times 7 - 1\times 1 + 4\times 11 \end{bmatrix} = \frac{1}{12}\begin{pmatrix} 24 \\ -12 \\ 36 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$

(IE $x=2$, $y=-1$, $z=3$)

Question 15 (**)

A 3×3 matrix $\mathbf{T}$ represents the linear transformation

$$\mathbf{T}\begin{pmatrix} x \\ y \\ z \end{pmatrix} :\mapsto \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

so that

$$\mathbf{T}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} :\mapsto \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}, \quad \mathbf{T}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} :\mapsto \begin{pmatrix} 6 \\ 1 \\ 5 \end{pmatrix}, \quad \mathbf{T}\begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix} :\mapsto \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

Find the elements of $\mathbf{T}$.

$$\boxed{}, \quad \boxed{\mathbf{T} = \begin{pmatrix} 3 & 3 & 2 \\ 4 & -3 & 1 \\ 2 & 3 & 2 \end{pmatrix}}$$

- ONE OF THE MAPPED VECTORS WE ARE GIVEN IS $\underline{i}$ SO WE KNOW THE FIRST COLUMN OF THE MATRIX

$$\underline{T} = \begin{bmatrix} 3 & a & b \\ 4 & c & d \\ 2 & e & f \end{bmatrix}$$

- NOW WE MAP THE OTHER TWO VECTORS, OBTAINING SIMULTANEOUS EQUATIONS

$$\begin{bmatrix} 3 & a & b \\ 4 & c & d \\ 2 & e & f \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 0 & -4 \end{bmatrix} = \begin{bmatrix} 6 & 1 \\ 1 & 1 \\ 5 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 3+a & 6+a-4b \\ 4+c & 8+c-4d \\ 2+e & 4+e-4f \end{bmatrix} \begin{bmatrix} 6 & 1 \\ 1 & 1 \\ 5 & -1 \end{bmatrix}$$

- THESE EQUATION YIELD

• $a+3=6$, $\underline{a=3}$ • $c+4=1$, $\underline{c=-3}$ • $e+2=5$, $\underline{e=3}$

• $6+a-4b=1$, $9-4b=1$, $8=4b$, $\underline{b=2}$ • $8+c-4d=1$, $8-3-4d=1$, $4=4d$, $\underline{d=1}$ • $4+e-4f=-1$, $4+3-4f=-1$, $8=4f$, $\underline{f=2}$

- HENCE THE REQUIRED MATRIX IS

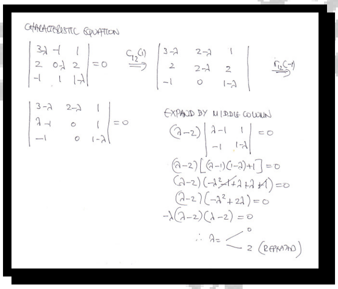
$$\underline{T} = \begin{bmatrix} 3 & 3 & 2 \\ 4 & -3 & 1 \\ 2 & 3 & 2 \end{bmatrix}$$

Question 16 (+)**

Find the eigenvalues of the following 3×3 matrix.

$$\mathbf{A}=\begin{pmatrix}3 & -1 & 1\\ 2 & 0 & 2\\ -1 & 1 & 1\end{pmatrix}.$$

$\boxed{\lambda=0,\ 2}$



CHARACTERISTIC EQUATION

$$\begin{vmatrix}3-\lambda & -1 & 1\\ 2 & 0-\lambda & 2\\ -1 & 1 & 1-\lambda\end{vmatrix}=0 \xRightarrow{C_{12}(1)} \begin{vmatrix}3-\lambda & 2-\lambda & 1\\ 2 & 2-\lambda & 2\\ -1 & 0 & 1-\lambda\end{vmatrix} \xRightarrow{r_{12}(-1)}$$

$$\begin{vmatrix}3-\lambda & 2-\lambda & 1\\ \lambda-1 & 0 & 1\\ -1 & 0 & 1-\lambda\end{vmatrix}=0$$

EXPAND BY MIDDLE COLUMN

$$(\lambda-2)\begin{vmatrix}\lambda-1 & 1\\ -1 & 1-\lambda\end{vmatrix}=0$$

$$(\lambda-2)\left[(\lambda-1)(1-\lambda)+1\right]=0$$

$$(\lambda-2)(-\lambda^2\cancel{-1}+\lambda+\lambda\cancel{+1})=0$$

$$(\lambda-2)(-\lambda^2+2\lambda)=0$$

$$-\lambda(\lambda-2)(\lambda-2)=0$$

$$\therefore \lambda = 0,\ 2 \text{ (REPEATED)}$$

Question 17 (+)**

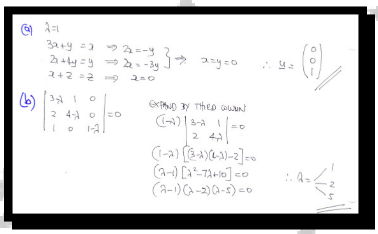
The 3×3 matrix $\mathbf{A}$ is given below.

$$\mathbf{A}=\begin{pmatrix}3 & 1 & 0\\ 2 & 4 & 0\\ 1 & 0 & 1\end{pmatrix}.$$

a) Given that $\lambda=1$ is an eigenvalue of $\mathbf{A}$ find the corresponding eigenvector.

b) Find the other two eigenvalues of $\mathbf{A}$.

$\boxed{\mathbf{u}=\begin{pmatrix}0\\0\\1\end{pmatrix}}$, $\boxed{\lambda=2,\ 5}$



(a) $\lambda=1$

$3x+y=x \Rightarrow 2x=-y$

$2x+4y=y \Rightarrow 2x=-3y$ $\Bigg\} \Rightarrow x=y=0$ $\quad \therefore \underline{u}=\begin{pmatrix}0\\0\\1\end{pmatrix}$

$x+z=z \Rightarrow x=0$

(b) $\begin{vmatrix}3-\lambda & 1 & 0\\ 2 & 4-\lambda & 0\\ 1 & 0 & 1-\lambda\end{vmatrix}=0$

EXPAND BY THIRD COLUMN

$$(1-\lambda)\begin{vmatrix}3-\lambda & 1\\ 2 & 4-\lambda\end{vmatrix}=0$$

$$(1-\lambda)\left[(3-\lambda)(4-\lambda)-2\right]=0$$

$$(\lambda-1)\left[\lambda^2-7\lambda+10\right]=0$$

$$(\lambda-1)(\lambda-2)(\lambda-5)=0$$

$$\therefore \lambda = 1,\ 2,\ 5$$

Question 18 (**+)

A transformation in three dimensional space is defined by the following 3×3 matrix.

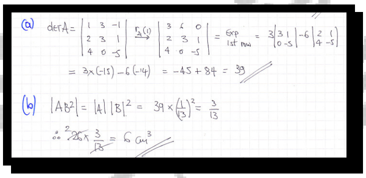
$$\mathbf{A} = \begin{pmatrix} 1 & 3 & -1 \\ 2 & 3 & 1 \\ 4 & 0 & -5 \end{pmatrix}.$$

a) Find the value of $\det \mathbf{A}$.

A cone with a volume of 26 cm^3 is transformed by the matrix composition $\mathbf{AB}^2$.

b) Given that $\det \mathbf{B} = \dfrac{1}{13}$, calculate the volume of the transformed cone.

$\boxed{\det \mathbf{A} = 39}$, $\boxed{\text{volume} = 6}$



(a) $\det A = \begin{vmatrix} 1 & 3 & -1 \\ 2 & 3 & 1 \\ 4 & 0 & -5 \end{vmatrix} \xrightarrow{r_2(1)} \begin{vmatrix} 3 & 6 & 0 \\ 2 & 3 & 1 \\ 4 & 0 & -5 \end{vmatrix} = \underset{\text{1st row}}{\text{Exp}} = 3\begin{vmatrix} 3 & 1 \\ 0 & -5 \end{vmatrix} - 6\begin{vmatrix} 2 & 1 \\ 4 & -5 \end{vmatrix}$

$= 3\times(-15) - 6(-14) = -45 + 84 = 39$

(b) $|AB^2| = |A|\,|B|^2 = 39 \times \left(\frac{1}{13}\right)^2 = \frac{3}{13}$

$\therefore \cancel{26}^{\,2} \times \frac{3}{\cancel{13}} = 6 \text{ cm}^3$

Question 19 (+)**

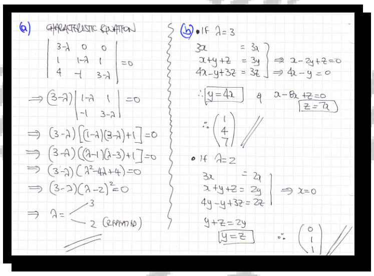
The 3×3 matrix $\mathbf{A}$ is given below.

$$\mathbf{A}=\begin{pmatrix}3 & 0 & 0\\ 1 & 1 & 1\\ 4 & -1 & 3\end{pmatrix}.$$

a) Show that $\mathbf{A}$ only has two eigenvalues.

b) Find the eigenvectors associated with each of these eigenvalues.

$\boxed{\lambda=2,\ \lambda=2,\ \lambda=3}$, $\boxed{\mathbf{u}=\begin{pmatrix}0\\1\\1\end{pmatrix}}$, $\boxed{\mathbf{v}=\begin{pmatrix}1\\4\\7\end{pmatrix}}$



(a) CHARACTERISTIC EQUATION

$$\begin{vmatrix}3-\lambda & 0 & 0\\ 1 & 1-\lambda & 1\\ 4 & -1 & 3-\lambda\end{vmatrix}=0$$

$$\Rightarrow (3-\lambda)\begin{vmatrix}1-\lambda & 1\\ -1 & 3-\lambda\end{vmatrix}=0$$

$$\Rightarrow (3-\lambda)\left[(1-\lambda)(3-\lambda)+1\right]=0$$

$$\Rightarrow (3-\lambda)\left[(\lambda-1)(\lambda-3)+1\right]=0$$

$$\Rightarrow (3-\lambda)(\lambda^2-4\lambda+4)=0$$

$$\Rightarrow (3-\lambda)(\lambda-2)^2=0$$

$$\Rightarrow \lambda = 3,\ 2 \text{ (REPEATED)}$$

(b) • If $\lambda=3$

$$\left.\begin{aligned}3x &= 3x\\ x+y+z &= 3y\\ 4x-y+3z &= 3z\end{aligned}\right\}\begin{aligned}&\Rightarrow x-2y+z=0\\ &\Rightarrow 4x-y=0\end{aligned}$$

$\therefore \boxed{y=4x}$ & $x-8x+z=0$ $\boxed{z=7x}$

$\therefore \begin{pmatrix}1\\4\\7\end{pmatrix}$

• If $\lambda=2$

$$\left.\begin{aligned}3x &= 2x\\ x+y+z &= 2y\\ 4y-y+3z &= 2z\end{aligned}\right|\Rightarrow x=0$$

$y+z=2y$

$\boxed{y=z}$ $\therefore \begin{pmatrix}0\\1\\1\end{pmatrix}$

Question 20 (**+)

The 3×3 matrices $\mathbf{A}$ and $\mathbf{B}$ are defined in terms of a scalar constant k by

$$\mathbf{A}=\begin{pmatrix} k & 9 & 2 \\ 1 & k & 0 \\ 5 & -1 & 1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{B}=\begin{pmatrix} 1 & -3 & 2 \\ k & 2 & -1 \\ 4 & 1 & 1 \end{pmatrix}.$$

a) Find an expression for $\det\mathbf{A}$, in terms of k.

b) Find the possible values of k given that $\mathbf{AB}$ is singular.

$\boxed{\det\mathbf{A}=k^2-10k-11}$, $\boxed{k=-1,\ 11,\ \frac{1}{5}}$

(a) $|A| = \begin{vmatrix} k & 9 & 2 \\ 1 & k & 0 \\ 5 & -1 & 1 \end{vmatrix} \underset{r_{31}(-2)}{=} \begin{vmatrix} k-10 & 11 & 0 \\ 1 & k & 0 \\ 5 & -1 & 1 \end{vmatrix} = \begin{vmatrix} k-10 & 11 \\ 1 & k \end{vmatrix} = k(k-10)-11$

EXPAND BY THIRD COLUMN

$= k^2-10k-11$

(b) $|B| = \begin{vmatrix} 1 & -3 & 2 \\ k & 2 & -1 \\ 4 & 1 & 1 \end{vmatrix} \underset{r_{32}(1),\ r_{31}(-2)}{=} \begin{vmatrix} -7 & -5 & 0 \\ k+4 & 3 & 0 \\ 4 & 1 & 1 \end{vmatrix} = \begin{vmatrix} -7 & -5 \\ k+4 & 3 \end{vmatrix} = -21+5(k+4)$

EXPAND BY THIRD COLUMN

$= 5k-1$

NOW $|AB|=0 \implies |A||B|=0$

$\implies (k^2-10k-11)(5k-1)=0$

$\implies (k-11)(k+1)(5k-1)=0$

$\therefore k = -1,\ \frac{1}{5},\ 11$

Question 21 (**+)

It is given that $\mathbf{A}$ and $\mathbf{B}$ are 3×3 matrices that satisfy

$$\det(\mathbf{AB})=20 \qquad \text{and} \qquad \det\left(\mathbf{A}^{-1}\right)=-4.$$

A solid S, of volume 5 cm^3, is transformed by $\mathbf{B}$ to produce an image S'.

Find the volume of S'.

$\boxed{400\text{ cm}^3}$

$\det(AB)=20$, $\det(A^{-1})=-4$ $\implies$ $\det A \times \det B = 20$

$\Downarrow$

$\det A = -\frac{1}{4}$

$-\frac{1}{4}\times\det B = 20$

$\det B = -80$

$\therefore$ VOLUME OF THE IMAGE IS

$5\times80 = 400\text{ cm}^3$

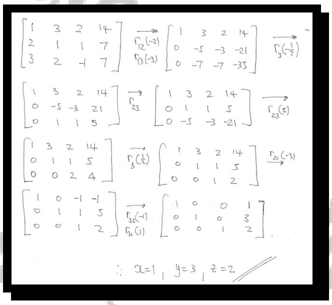
Question 22 (***)

$$\begin{aligned} x+3y+2z &= 14 \\ 2x+y+z &= 7 \\ 3x+2y-z &= 7 \end{aligned}$$

Solve the above system of simultaneous equations by manipulating their augmented matrix into reduced row echelon form.

No credit will be given for alternative solution methods.

$\boxed{x=1, \quad y=3, \quad z=2}$



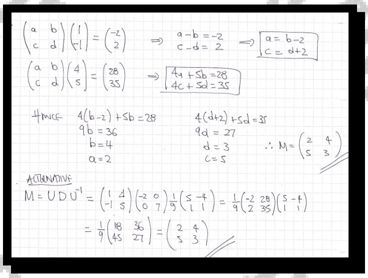
Question 23 (***)

A 2×2 matrix $\mathbf{M}$ has eigenvalues $\lambda=-2$ and $\lambda=7$, with respective eigenvectors

$$\mathbf{i}-\mathbf{j} \qquad \text{and} \qquad 4\mathbf{i}+5\mathbf{j}.$$

Find the elements of $\mathbf{M}$.

$\boxed{\mathbf{M}=\begin{pmatrix} 2 & 4 \\ 5 & 3 \end{pmatrix}}$



Question 24 (***)

The 3×3 matrix $\mathbf{A}$ is given in terms of a constant k below.

$$\mathbf{A} = \begin{pmatrix} k & 3 & 6 \\ 1 & k & 1 \\ 0 & 4 & 1 \end{pmatrix}.$$

a) Show that $\mathbf{A}$ has an inverse for all values of k.

b) Find $\mathbf{A}^{-1}$ in terms of k.

$$\boxed{\mathbf{A} = \frac{1}{(k-2)^2+17}\begin{pmatrix} k-4 & 21 & 3-6k \\ -1 & k & 6-k \\ 4 & -4k & k^2-3 \end{pmatrix}}$$

(a) $A=\begin{pmatrix} k & 3 & 6 \\ 1 & k & 1 \\ 0 & 4 & 1 \end{pmatrix}$

$|A| = k\begin{vmatrix} k & 1 \\ 4 & 1 \end{vmatrix} - \begin{vmatrix} 3 & 6 \\ 4 & 1 \end{vmatrix} = k(k-4)-(-21)$

$= k^2-4k+21 = (k-2)^2-4+21$

$= (k-2)^2+17 > 0$

$\therefore$ A will always have an inverse

(b) MATRIX OF MINORS $= \begin{pmatrix} k-4 & 1 & 4 \\ -21 & k & 4k \\ 3-6k & k-6 & k^2-3 \end{pmatrix}$

MATRIX OF COFACTORS $= \begin{pmatrix} k-4 & -1 & 4 \\ 21 & k & -4k \\ 3-6k & 6-k & k^2-3 \end{pmatrix}$

ADJUGATE MATRIX $= \begin{pmatrix} k-4 & 21 & 3-6k \\ -1 & k & 6-k \\ 4 & -4k & k^2-3 \end{pmatrix}$

$\therefore A^{-1} = \frac{1}{(k-2)^2+17}\begin{pmatrix} k-4 & 21 & 3-6k \\ -1 & k & 6-k \\ 4 & -4k & k^2-3 \end{pmatrix}$

Question 25 (***)

The 2×2 matrix $\mathbf{M}$ has eigenvalues -2 and 7.

The respective eigenvectors of $\mathbf{M}$ are $\begin{pmatrix}1\\-1\end{pmatrix}$ and $\begin{pmatrix}4\\5\end{pmatrix}$.

Find the entries of $\mathbf{M}$.

$$\boxed{\mathbf{M}=\begin{pmatrix}2&4\\3&5\end{pmatrix}}$$

$M=\begin{pmatrix}a&b\\c&d\end{pmatrix}$ THEN $\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix}=-2\begin{pmatrix}1\\-1\end{pmatrix}=\begin{pmatrix}-2\\2\end{pmatrix}$

$\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}4\\5\end{pmatrix}=7\begin{pmatrix}4\\5\end{pmatrix}=\begin{pmatrix}28\\35\end{pmatrix}$

THUS $a-b=-2$ $\quad$ $c-d=2$

$4a+5b=28$ $\quad$ $4c+5d=35$

$\boxed{a=b-2}$ $\quad$ $\boxed{c=d+2}$

$4(b-2)+5b=28$ $\quad$ $4(d+2)+5d=35$

$9b=36$ $\quad$ $9d=27$

$\therefore \boxed{b=4}$ $\quad$ $\therefore \boxed{d=3}$

$\therefore \boxed{a=2}$ $\quad$ $\boxed{c=5}$

$\therefore M=\begin{pmatrix}2&4\\3&5\end{pmatrix}$

Question 26 (***)

$$\begin{aligned}2x+5y+3z&=2\\x+2y+2z&=4\\x+y+4z&=11\end{aligned}$$

Solve the above simultaneous equations by manipulating their augmented matrix into reduced row echelon form.

No credit will be given for alternative solution methods.

$$\boxed{x=12,\quad y=-5,\quad z=1}$$

$\begin{pmatrix}2&5&3&2\\1&2&2&4\\1&1&4&11\end{pmatrix}\xrightarrow{r_{12}}\begin{pmatrix}1&2&2&4\\2&5&3&2\\1&1&4&11\end{pmatrix}\xrightarrow[r_{13}(-1)]{r_{12}(-2)}\begin{pmatrix}1&2&2&4\\0&1&-1&-6\\0&-1&2&7\end{pmatrix}\xrightarrow{r_{23}(1)}\begin{pmatrix}1&2&2&4\\0&1&-1&-6\\0&0&1&1\end{pmatrix}$

$\xrightarrow{r_{21}(-2)}\begin{pmatrix}1&0&4&16\\0&1&-1&-6\\0&0&1&1\end{pmatrix}\xrightarrow[r_{31}(-4)]{r_{32}(1)}\begin{pmatrix}1&0&0&12\\0&1&0&-5\\0&0&1&1\end{pmatrix}$ $\quad\therefore\ x=12,\ y=-5,\ z=1$

Question 27 (***)

The 3×3 matrices $\mathbf{A}$ and $\mathbf{B}$ are given below.

$$\mathbf{A} = \begin{pmatrix} 3 & 4 & 2 \\ 1 & 1 & 4 \\ 4 & 5 & 7 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} -10 & -14 & 16 \\ 10 & 14 & -6 \\ 5 & 6 & 6 \end{pmatrix}.$$

Show clearly that

$$\mathbf{A} + \mathbf{A}^{-1} + \mathbf{B} = k\mathbf{A},$$

stating the value of the scalar constant k

$\boxed{k=2}$

$A = \begin{bmatrix} 3 & 4 & 2 \\ 1 & 1 & 4 \\ 4 & 5 & 7 \end{bmatrix}$

MATRIX OF MINORS $= \begin{bmatrix} -13 & -9 & 1 \\ 18 & 13 & -1 \\ 14 & 10 & -1 \end{bmatrix}$

MATRIX OF COFACTORS $= \begin{bmatrix} -13 & 9 & 1 \\ -18 & 13 & 1 \\ 14 & -10 & -1 \end{bmatrix}$

ADJUGATE MATRIX $= \begin{bmatrix} -13 & -18 & 14 \\ 9 & 13 & -10 \\ 1 & 1 & -1 \end{bmatrix}$

$\det A = 3(-13) + 4\times9 + 2\times1 = -1$

$A^{-1} = \frac{1}{-1}\begin{bmatrix} -13 & -18 & 14 \\ 9 & 13 & -10 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 13 & 18 & -14 \\ -9 & -13 & 10 \\ -1 & -1 & 1 \end{bmatrix}$

THUS

$A + A^{-1} + B = \begin{bmatrix} 3 & 4 & 2 \\ 1 & 1 & 4 \\ 4 & 5 & 7 \end{bmatrix} + \begin{bmatrix} 13 & 18 & -14 \\ -9 & -13 & 10 \\ -1 & -1 & 1 \end{bmatrix} + \begin{bmatrix} -10 & -14 & 16 \\ 10 & 14 & -6 \\ 5 & 6 & 6 \end{bmatrix}$

$= \begin{bmatrix} 6 & 8 & 4 \\ 2 & 2 & 8 \\ 8 & 10 & 14 \end{bmatrix} = 2\begin{bmatrix} 3 & 4 & 2 \\ 1 & 1 & 4 \\ 4 & 5 & 7 \end{bmatrix} = 2A$

$\therefore k=2$

Question 28 (***)

The 3×3 matrix $\mathbf{A}$ below, represents a transformation such that $\mathbb{R}^3 \mapsto \mathbb{R}^3$.

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 1 \\ 4 & -3 & 0 \\ -3 & 3 & 1 \end{pmatrix}.$$

a) Find the entries of $\mathbf{A}^3$.

b) Determine the entries of $\mathbf{A}^{-1}$.

$$\boxed{\mathbf{A}^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}, \quad \boxed{\mathbf{A}^{-1} = \begin{pmatrix} -3 & 4 & 3 \\ -4 & 5 & 4 \\ 3 & -3 & -2 \end{pmatrix}}$$

(a) $A^3 = \begin{pmatrix} 2 & -1 & 1 \\ 4 & -3 & 0 \\ -3 & 3 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 & 1 \\ 4 & -3 & 0 \\ -3 & 3 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 & 1 \\ 4 & -3 & 0 \\ -3 & 3 & 1 \end{pmatrix} = \begin{pmatrix} -3 & 4 & 3 \\ -4 & 5 & 4 \\ 3 & -3 & -2 \end{pmatrix}\begin{pmatrix} 2 & -1 & 1 \\ 4 & -3 & 0 \\ -3 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(b) SINCE $A^3 = \underset{\substack{\uparrow \\ A^{-1}}}{A^2} A = I$

$\therefore A^{-1} = A^2 = \begin{pmatrix} -3 & 4 & 3 \\ -4 & 5 & 4 \\ 3 & -3 & -2 \end{pmatrix}$

Question 29 (***)

Factorize fully the following 3×3 determinant.

$$\begin{vmatrix} 1 & x & y+z \\ 2 & y & z+x \\ 3 & z & x+y \end{vmatrix}.$$

$$\boxed{(x+y+z)(x-2y+z)}$$

$\begin{vmatrix} 1 & x & y+z \\ 2 & y & z+x \\ 3 & z & x+y \end{vmatrix} \underset{}{=}^{C_{23}(1)} \begin{vmatrix} 1 & x & x+y+z \\ 2 & y & x+y+z \\ 3 & z & x+y+z \end{vmatrix} = (x+y+z)\begin{vmatrix} 1 & x & 1 \\ 2 & y & 1 \\ 3 & z & 1 \end{vmatrix}$

$= \begin{matrix} r_{12}(-2) \\ r_{13}(-3) \end{matrix} = (x+y+z)\begin{vmatrix} 1 & x & 1 \\ 0 & y-2x & -1 \\ 0 & z-3x & -2 \end{vmatrix}$ EXPAND BY 1st COLUMN $= (x+y+z)(-2y+4x+z-3x)$

$= (x+y+z)(x-2y+z)$

Question 30 (***)

Solve the following simultaneous equations by manipulating their augmented matrix into reduced row echelon form.

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \\ 8 \end{pmatrix}$$

No credit will be given for alternative solution methods.

$\boxed{x = 2, \quad y = -1, \quad z = 4}$

AUGMENTED MATRIX

$$\begin{pmatrix} 1 & 1 & 1 & 5 \\ 2 & 4 & 2 & 8 \\ 1 & 2 & 2 & 8 \end{pmatrix} \begin{matrix} r_{12}(-2) \\ r_{13}(-1) \end{matrix} \begin{pmatrix} 1 & 1 & 1 & 5 \\ 0 & 2 & 0 & -2 \\ 0 & 1 & 1 & 3 \end{pmatrix} r_{23} \begin{pmatrix} 1 & 1 & 1 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 2 & 0 & -2 \end{pmatrix} r_{23}(-2)$$

$$\begin{pmatrix} 1 & 1 & 1 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & -2 & -8 \end{pmatrix} r_3\left(-\tfrac{1}{2}\right) \begin{pmatrix} 1 & 1 & 1 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 4 \end{pmatrix} r_{21}(-1) \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

$$r_{32}(-1) \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 4 \end{pmatrix} \qquad \therefore \begin{array}{l} x = 2 \\ y = -1 \\ z = 4 \end{array} //$$

Question 31 (***)

The 3×3 matrix $\mathbf{D}$ is given below in terms of the constants a, b, c and d.

$$\mathbf{D} = \begin{pmatrix} a & 1 & b \\ c & 7 & 0 \\ 3 & d & 2 \end{pmatrix}.$$

It is further given that

$$\mathbf{u} = \mathbf{i} + 3\mathbf{k} \qquad \text{and} \qquad \mathbf{u} = 3\mathbf{i} + 4\mathbf{j} + \mathbf{k}$$

are eigenvectors of $\mathbf{D}$ with corresponding eigenvalues λ and μ.

Determine in any order the value of a, b, c, d, λ and μ.

$\boxed{a=6}$, $\boxed{b=-1}$, $\boxed{c=0}$, $\boxed{d=-1}$, $\boxed{\lambda=3}$, $\boxed{\mu=7}$,

$$\begin{pmatrix} a & 1 & b \\ c & 7 & 0 \\ 3 & d & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \Rightarrow \begin{array}{ll} a + 3b = \lambda & (1) \\ c = 0 & (2) \\ 3 + 6 = 3\lambda & (3) \end{array}$$

$$\begin{pmatrix} a & 1 & b \\ c & 7 & 0 \\ 3 & d & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} = \mu \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} \Rightarrow \begin{array}{ll} 3a + 4 + b = 3\mu & (4) \\ 3c + 28 = 4\mu & (5) \\ 9 + 4d + 2 = \mu & (6) \end{array}$$

(2) $\boxed{c=0} \Rightarrow$ (5) $28 = 4\mu$, $\boxed{\mu = 7} \Rightarrow$ (6) $11 + 4d = 7$, $4d = -4$, $\boxed{d=-1}$

(3) $9 = 3\lambda$, $\boxed{\lambda = 3} \Rightarrow \left.\begin{array}{l} (1)\ a + 3b = 3 \\ (4)\ 3a + b = 17 \end{array}\right\} \Rightarrow \left.\begin{array}{l} 3a + 9b = 9 \\ 3a + b = 17 \end{array}\right\} \Rightarrow 8b = -8$, $\boxed{b=-1}$ $\Rightarrow \boxed{a=6}$

Hence $a = 6$, $b = -1$, $c = 0$, $d = -1$ & $\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$ HAS EIGENVALUE $\lambda = 3$, $\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$ HAS EIGENVALUE $\mu = 7$

Question 32 (***)

The 2×2 matrix $\mathbf{A}$ and the 3×3 matrix $\mathbf{B}$ are given below.

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 2 & -4 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}.$$

The straight line L_1 with equation

$$y = x + k,$$

where k is a constant, is transformed by $\mathbf{A}$.

a) Find an equation for the image of L_1 under $\mathbf{A}$.

The straight line L_2 with Cartesian equation

$$\frac{x-1}{2} = \frac{y-3}{2} = z-2,$$

is transformed by $\mathbf{B}$.

b) Find a Cartesian equation for the image of L_2 under $\mathbf{B}$.

$$\boxed{2x+3y=16k}, \quad \boxed{\frac{x-7}{7} = \frac{y-4}{4} = \frac{z-8}{5}}$$

(a) $y = x + k$ PARAMETRIZE $x = t$, $y = t + k$

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} x+2y \\ 2x-4y \end{pmatrix} = \begin{pmatrix} t+2t+2k \\ 2t-4t+4k \end{pmatrix} = \begin{pmatrix} 3t+2k \\ -2t+4k \end{pmatrix}$$

HENCE

$$\left.\begin{aligned} 2X &= 6t+4k \\ 3Y &= -6t+12k \end{aligned}\right\} \Rightarrow 2X+3Y=16k$$

(b) $\frac{x-1}{2} = \frac{y-3}{2} = z-2 \Rightarrow \underline{r} = (1,3,2) + \lambda(2,2,1)$

$\underline{r} = (2\lambda+1, 2\lambda+3, \lambda+2)$

$(x,y,z) = (2\lambda+1, 2\lambda+3, \lambda+2)$

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}\begin{pmatrix} 2\lambda+1 \\ 2\lambda+3 \\ \lambda+2 \end{pmatrix} = \begin{pmatrix} 4\lambda+2+2\lambda+3+\lambda+2 \\ 2\lambda+1+2\lambda+3 \\ 4\lambda+6+\lambda+2 \end{pmatrix} = \begin{pmatrix} 7\lambda+7 \\ 4\lambda+4 \\ 5\lambda+8 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \\ 8 \end{pmatrix} + \lambda\begin{pmatrix} 7 \\ 4 \\ 5 \end{pmatrix}$$

$$\therefore \frac{x-7}{7} = \frac{y-4}{4} = \frac{z-8}{5}$$

Question 33 (***)

The 3×3 matrix $\mathbf{A}$ is given in terms of the scalar constant k by

$$\mathbf{A}=\begin{pmatrix}1 & -1 & 3\\ 2 & 1 & k\\ 0 & 1 & 1\end{pmatrix}.$$

a) Find, in terms of k, the inverse of $\mathbf{A}$.

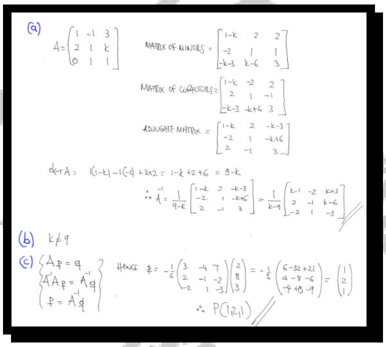
b) State the condition that k must satisfy, so that the inverse matrix exists.

Now suppose that $k=4$.

The point P has been transformed by the matrix $\mathbf{A}$ into the point $Q(2,8,3)$.

c) Determine the coordinates of P.

$$\boxed{\mathbf{A}^{-1}=\frac{1}{k-9}\begin{pmatrix}k-1 & -4 & k+3\\ 2 & -1 & k-6\\ -2 & 1 & -3\end{pmatrix}},\ \boxed{k\neq 9},\ \boxed{P(1,2,1)}$$



Question 34 (***)

The 3×3 matrix $\mathbf{M}$ is given below, in terms of a scalar constant k.

$$\mathbf{M}=\begin{pmatrix} k & 0 & 2 \\ 4 & 3 & 2 \\ -2 & -1 & 0 \end{pmatrix}$$

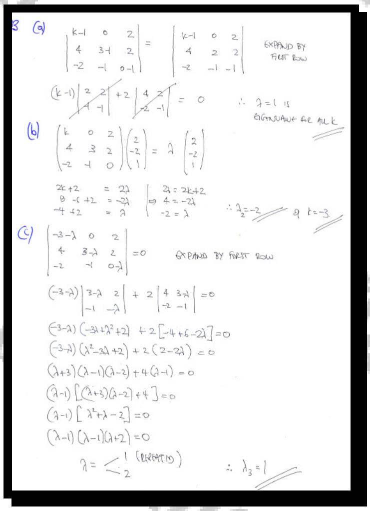
a) Show that $\lambda_1 = 1$ is an eigenvalue of $\mathbf{M}$ for all values of k.

b) Given that $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ is an eigenvector of $\mathbf{M}$ with corresponding eigenvalue $\lambda_2 \neq 1$,

find the values of λ_2 and the value of k.

c) Find the value of the third eigenvalue of $\mathbf{M}$.

$\boxed{\lambda_2=-2}$, $\boxed{k=-3}$, $\boxed{\lambda_3=1}$



Question 35 (***)

The 3×3 matrices $\mathbf{A}$ and $\mathbf{B}$ are given below.

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \text{and} \qquad \mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

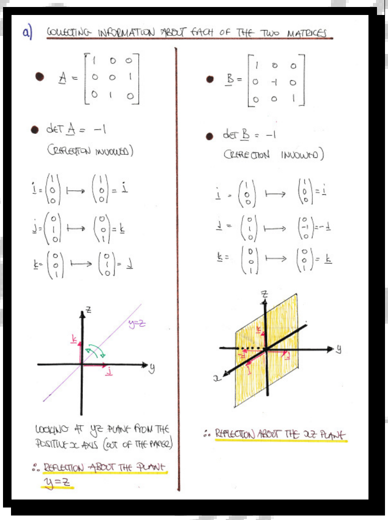
a) Describe geometrically the transformations given by each of the two matrices.

The matrix $\mathbf{C}$ is defined as the transformation defined by the matrix $\mathbf{A}$, followed by the transformation defined by the matrix $\mathbf{B}$.

b) Describe geometrically the transformation represented by $\mathbf{C}$.

$\boxed{\mathbf{A}: \text{reflection in the plane } y = z}$, $\boxed{\mathbf{B}: \text{reflection in the } xz \text{ plane}}$,

$\boxed{\mathbf{C}: \text{rotation in the } x \text{ axis, } 90°\text{, anticlockwise}}$


a) COLLECTING INFORMATION ABOUT EACH OF THE TWO MATRICES

- $\underline{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$
- $\det \underline{A} = -1$ (REFLECTION INVOLVED)

$\underline{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \longmapsto \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \underline{i}$

$\underline{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \longmapsto \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \underline{k}$

$\underline{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \longmapsto \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \underline{j}$

LOOKING AT yz PLANE FROM THE POSITIVE x AXIS (OUT OF THE PAPER)

∴ REFLECTION ABOUT THE PLANE $y=z$

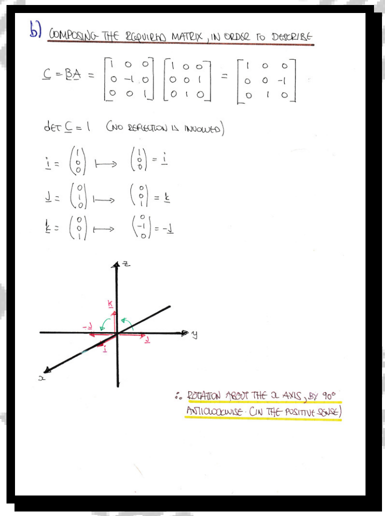
- $\underline{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- $\det \underline{B} = -1$ (REFLECTION INVOLVED)

$\underline{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \longmapsto \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \underline{i}$

$\underline{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \longmapsto \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} = -\underline{j}$

$\underline{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \longmapsto \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \underline{k}$

∴ REFLECTION ABOUT THE xz PLANE


b) COMPOSING THE REQUIRED MATRIX, IN ORDER TO DESCRIBE

$\underline{C} = \underline{B}\underline{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$

$\det \underline{C} = 1$ (NO REFLECTION IS INVOLVED)

$\underline{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \longmapsto \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \underline{i}$

$\underline{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \longmapsto \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \underline{k}$

$\underline{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \longmapsto \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} = -\underline{j}$

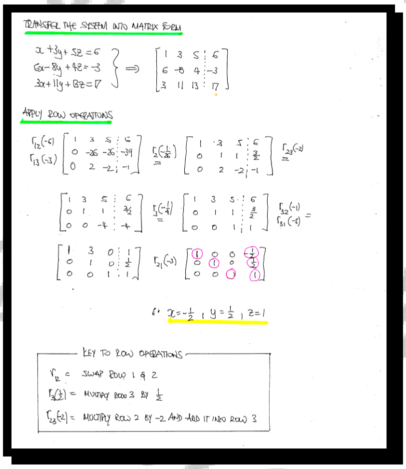
∴ ROTATION ABOUT THE x AXIS, BY 90° ANTICLOCKWISE (IN THE POSITIVE SENSE)

Question 36 **(***)**

$$\begin{aligned} x \ +3y \ +5z &= \ \ 6 \\ 6x \ -8y \ +4z &= -3 \\ 3x+11y+13z &= \ 17 \end{aligned}$$

Solve the above system of simultaneous equations by manipulating their augmented matrix into reduced row echelon form.

V , , $\boxed{x=-\frac{1}{2}, \ y=\frac{1}{2}, \ z=1}$



TRANSFER THE SYSTEM INTO MATRIX FORM

$$\left.\begin{aligned} x+3y+5z&=6 \\ 6x-8y+4z&=-3 \\ 3x+11y+13z&=17 \end{aligned}\right\} \Longrightarrow \left[\begin{array}{ccc|c} 1 & 3 & 5 & 6 \\ 6 & -8 & 4 & -3 \\ 3 & 11 & 13 & 17 \end{array}\right]$$

APPLY ROW OPERATIONS

$$\begin{matrix} r_{12}(-6) \\ r_{13}(-3) \end{matrix} \left[\begin{array}{ccc|c} 1 & 3 & 5 & 6 \\ 0 & -26 & -26 & -39 \\ 0 & 2 & -2 & -1 \end{array}\right] \underset{=}{r_2\left(-\tfrac{1}{26}\right)} \left[\begin{array}{ccc|c} 1 & 3 & 5 & 6 \\ 0 & 1 & 1 & \frac{3}{2} \\ 0 & 2 & -2 & -1 \end{array}\right] \underset{=}{r_{23}(-2)}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 5 & 6 \\ 0 & 1 & 1 & 3/2 \\ 0 & 0 & -4 & -4 \end{array}\right] \underset{=}{r_3\left(-\tfrac{1}{4}\right)} \left[\begin{array}{ccc|c} 1 & 3 & 5 & 6 \\ 0 & 1 & 1 & \frac{3}{2} \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{matrix} r_{32}(-1) \\ r_{31}(-5) \end{matrix} =$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & 1 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 1 \end{array}\right] \ r_{21}(-3) \left[\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\therefore \ x=-\frac{1}{2}, \ y=\frac{1}{2}, \ z=1$$

KEY TO ROW OPERATIONS

r_{12} = SWAP ROW 1 & 2

$r_3\left(\frac{1}{2}\right)$ = MULTIPLY ROW 3 BY $\frac{1}{2}$

$r_{23}(-2)$ = MULTIPLY ROW 2 BY -2 AND ADD IT INTO ROW 3

Question 37 (***)

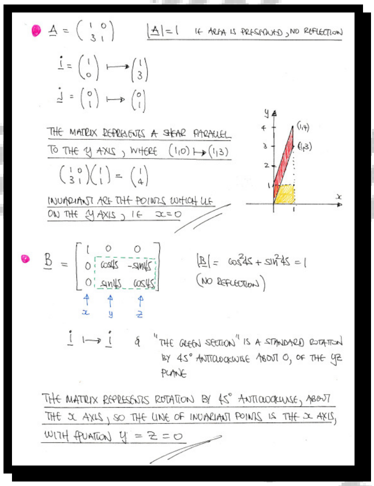
The matrix $\mathbf{A}:\mathbb{R}^2 \mapsto \mathbb{R}^2$ and the matrix $\mathbf{B}:\mathbb{R}^3 \mapsto \mathbb{R}^3$ are defined as

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 45° & -\sin 45° \\ 0 & \sin 45° & \cos 45° \end{pmatrix}.$$

Describe geometrically the transformations given by each of these matrices.

State in each case the equation of the line of invariant points.

$\boxed{\mathbf{A}: \text{shear parallel to } y \text{ axis}, (1,0) \mapsto (3,1)}$, $\boxed{\mathbf{B}: \text{rotation in the } x \text{ axis}, 45°, \text{anticlockwise}}$, $\boxed{\mathbf{A}: x=0}$, $\boxed{\mathbf{B}: y=z=0, \text{ i.e. } x \text{ axis}}$


$\underline{A} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$ $|\underline{A}| = 1$ IE AREA IS PRESERVED, NO REFLECTION

$\underline{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \longmapsto \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$\underline{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \longmapsto \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

THE MATRIX REPRESENTS A SHEAR PARALLEL TO THE y AXIS, WHERE $(1,0) \mapsto (1,3)$

$\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

INVARIANT ARE THE POINTS WHICH LIE ON THE y AXIS, IE $x=0$

$\underline{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 45 & -\sin 45 \\ 0 & \sin 45 & \cos 45 \end{bmatrix}$ $|\underline{B}| = \cos^2 45 + \sin^2 45 = 1$ (NO REFLECTION)

$x \quad y \quad z$

$\underline{i} \longmapsto \underline{i}$ & "THE GREEN SECTION" IS A STANDARD ROTATION BY 45° ANTICLOCKWISE ABOUT O, OF THE yz PLANE

THE MATRIX REPRESENTS ROTATION BY 45° ANTICLOCKWISE, ABOUT THE x AXIS, SO THE LINE OF INVARIANT POINTS IS THE x AXIS, WITH EQUATION $y = z = 0$

Question 38 (***)

Find the eigenvalues and the corresponding eigenvectors of the following 2×2 matrix.

$$\mathbf{M} = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}.$$

$\boxed{\lambda = -1, \quad \mathbf{u} = \alpha \begin{pmatrix} 3 \\ -2 \end{pmatrix}}$, $\boxed{\lambda = 4, \quad \mathbf{u} = \beta \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$

$M = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$

CHARACTERISTIC EQUATION

$\begin{vmatrix} 1-\lambda & 3 \\ 2 & 2-\lambda \end{vmatrix} = 0$

$(1-\lambda)(2-\lambda) - 6 = 0$

$(\lambda-1)(\lambda-2) - 6 = 0$

$\lambda^2 - 3\lambda - 4 = 0$

$(\lambda+1)(\lambda-4)$

$\lambda = \begin{cases} -1 \\ 4 \end{cases}$

- IF $\lambda = -1$

$x + 3y = -x$
$2x + 2y = -y$

$2x + 3y = 0$ (FOR BOTH)

$y = -\frac{2}{3}x$

$\therefore \alpha \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

- IF $\lambda = 4$

$\left.\begin{array}{l} x + 3y = 4x \\ 2x + 2y = 4y \end{array}\right\} \Rightarrow y = x$ (FOR BOTH)

$\therefore \beta \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

ALTERNATIVE

$\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ mx \end{pmatrix} = \lambda \begin{pmatrix} x \\ mx \end{pmatrix}$

$x + 3mx = \lambda x$
$2x + 2mx = \lambda mx$

$1 + 3m = \lambda$
$2 + 2m = \lambda m$

$\frac{1+3m}{2+2m} = \frac{1}{m}$

$m + 3m^2 = 2 + 2m$

$3m^2 - m - 2 = 0$

$(3m+2)(m-1) = 0$

$m = \begin{cases} 1 \\ -\frac{2}{3} \end{cases}$ $\quad \lambda = \begin{cases} 4 \\ -1 \end{cases}$ $\quad \therefore$ EIGENVECTORS $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$

Question 39 (***)

A transformation of the x-y plane is represented by the following 2×2 matrix.

$$\mathbf{D} = \begin{pmatrix} -5 & 9 \\ -4 & 7 \end{pmatrix}.$$

The straight line with equation of the form $y = ax$, where a is the gradient, is in the direction of the eigenvector of $\mathbf{D}$.

a) Find the equation of this straight line, stating whether this line is an invariant line or a line of invariant points.

b) Show that all the straight lines of the form $y = ax + c$, where c is a constant, remain invariant under the transformation represented by $\mathbf{D}$.

$$\boxed{y = \frac{2}{3}x}$$

(a) $D = \begin{pmatrix} -5 & 9 \\ -4 & 7 \end{pmatrix}$ CHARACTERISTIC EQUATION $\begin{vmatrix} -5-\lambda & 9 \\ -4 & 7-\lambda \end{vmatrix} = 0$

$\Rightarrow (-5-\lambda)(7-\lambda) + 36 = 0$
$\Rightarrow (\lambda+5)(\lambda-7) + 36 = 0$
$\Rightarrow \lambda^2 - 2\lambda - 35 + 36 = 0$
$\Rightarrow \lambda^2 - 2\lambda + 1 = 0$
$\Rightarrow (\lambda-1)^2 = 0$
$\Rightarrow \lambda = 1$

EIGENVECTOR $\left.\begin{array}{l} -5x + 9y = x \\ -4x + 7y = y \end{array}\right\} \Rightarrow \begin{array}{l} 9y = 6x \\ 6y = 4x \end{array} \Rightarrow \boxed{y = \frac{2}{3}x}$

IT IS A LINE OF INVARIANT POINTS BECAUSE $\lambda = 1$

(b) $\begin{pmatrix} -5 & 9 \\ -4 & 7 \end{pmatrix} \begin{pmatrix} t \\ \frac{2}{3}t + c \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix}$

$\left.\begin{array}{l} -5t + 6t + 9c = X \\ -4t + \frac{14}{3}t + 7c = Y \end{array}\right\} \Rightarrow$

$\left.\begin{array}{l} X = t + 9c \\ Y = \frac{2}{3}t + 7c \end{array}\right\} \Rightarrow$

$\left.\begin{array}{l} \frac{2}{3}X = \frac{2}{3}t + 6c \\ Y = \frac{2}{3}t + 7c \end{array}\right\} \Rightarrow$ SUBTRACT

$Y - \frac{2}{3}X = c$

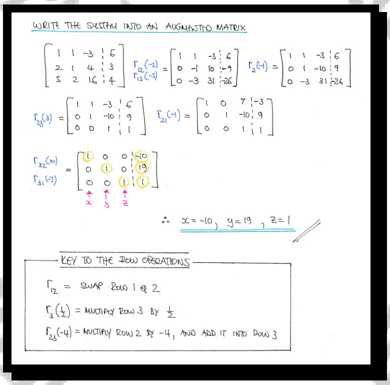
$Y = \frac{2}{3}X + C$ // INVARIANT LINE

Question 40 (***)

Solve the following simultaneous equations by manipulating their augmented matrix into reduced row echelon form.

$$\begin{pmatrix} 1 & 1 & -3 \\ 2 & 1 & 4 \\ 5 & 2 & 16 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 4 \end{pmatrix}$$

V , $\boxed{}$, $\boxed{x=-10, \quad y=19, \quad z=1}$



Question 41 (***)

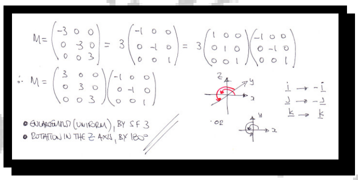
The 3×3 matrix $\mathbf{M}$ is given below.

$$\mathbf{M} = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

$\mathbf{M}$ describes two consecutive linear transformations of 3 dimensional space, which can be carried out in any order.

Describe geometrically each these two transformations.

$\boxed{\text{rotation about } z \text{ axis}, 180°}$, $\boxed{\text{uniform enlargement, S.F.} = 3}$



Question 42 (***)

The system of simultaneous equations

$$\begin{aligned} x + y + 2z &= 2 \\ x + 2y + z &= 2 \\ 2x + ay + 5z &= b \end{aligned}$$

where a and b are constants, does **not** have a unique solution, but it is **consistent**.

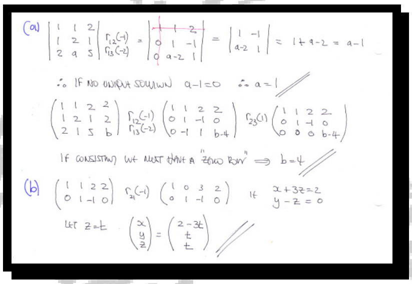
a) Determine the value of a and the value of b.

b) Show that the general solution of the system can be written as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2-3t \\ t \\ t \end{pmatrix},$$

where t is a parameter.

$\boxed{a = 1}$, $\boxed{b = 4}$



(a) $\begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & a & 5 \end{vmatrix} \begin{matrix} r_{12}(-1) \\ r_{13}(-2) \end{matrix} = \begin{vmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & a-2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ a-2 & 1 \end{vmatrix} = 1 + a - 2 = a - 1$

$\therefore$ IF NO UNIQUE SOLUTION $a - 1 = 0$ $\therefore a = 1$

$\begin{pmatrix} 1 & 1 & 2 & 2 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 5 & b \end{pmatrix} \begin{matrix} r_{12}(-1) \\ r_{13}(-2) \end{matrix} \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & b-4 \end{pmatrix} r_{23}(1) \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & b-4 \end{pmatrix}$

IF CONSISTENT WE MUST HAVE A "ZERO ROW" $\Rightarrow b = 4$

(b) $\begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & -1 & 0 \end{pmatrix} r_{21}(-1) \begin{pmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & -1 & 0 \end{pmatrix}$ IE $\begin{matrix} x + 3z = 2 \\ y - z = 0 \end{matrix}$

LET $z = t$ $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2-3t \\ t \\ t \end{pmatrix}$

Question 43 (***)

The 3×3 matrix $\mathbf{A}$ is given below.

$$\mathbf{A}=\begin{pmatrix}2&-5&0\\-5&-1&3\\0&3&-6\end{pmatrix}.$$

As $\mathbf{A}$ is a symmetric matrix, find the orthogonal 3×3 matrix $\mathbf{P}$ and a diagonal 3×3 matrix $\mathbf{D}$ such that $\mathbf{P}^{\mathrm{T}}\mathbf{AP}=\mathbf{D}$.

$$\boxed{\mathbf{P}=\begin{pmatrix}-\frac{5}{\sqrt{42}}&\frac{1}{\sqrt{3}}&\frac{1}{\sqrt{14}}\\\frac{4}{\sqrt{42}}&\frac{1}{\sqrt{3}}&\frac{2}{\sqrt{14}}\\\frac{1}{\sqrt{42}}&\frac{1}{\sqrt{3}}&-\frac{3}{\sqrt{14}}\end{pmatrix}},\ \boxed{\mathbf{D}=\begin{pmatrix}6&0&0\\0&-3&0\\0&0&-8\end{pmatrix}}$$

$$A=\begin{pmatrix}2&-5&0\\-5&-1&3\\0&3&-6\end{pmatrix}$$

CHARACTERISTIC EQUATION

$$\Rightarrow \begin{vmatrix}2-\lambda&-5&0\\-5&-1-\lambda&3\\0&3&-6-\lambda\end{vmatrix}=0$$

$$\Rightarrow (2-\lambda)\begin{vmatrix}-1-\lambda&3\\3&-6-\lambda\end{vmatrix}+5\begin{vmatrix}-5&3\\0&-6-\lambda\end{vmatrix}=0$$

$$\Rightarrow (2-\lambda)\left[(-1-\lambda)(-6-\lambda)-9\right]+5\left[-5(-6-\lambda)-0\right]=0$$

$$\Rightarrow (2-\lambda)\left[(\lambda+1)(\lambda+6)-9\right]+5\left[30+5\lambda\right]=0$$

$$\Rightarrow (2-\lambda)\left[\lambda^2+7\lambda-3\right]+150+25\lambda=0$$

$$\Rightarrow 2\lambda^2+14\lambda-6-\lambda^3-7\lambda^2+3\lambda+150+25\lambda=0$$

$$\Rightarrow -\lambda^3-5\lambda^2+42\lambda+144=0$$

$$\Rightarrow \lambda^3+5\lambda^2-42\lambda-144=0$$

$$\Rightarrow (\lambda-6)(\lambda^2+11\lambda+24)=0$$

$$\Rightarrow (\lambda-6)(\lambda+3)(\lambda+8)=0$$

$$\Rightarrow \lambda = 6,\ -3,\ -8$$

- IF $\lambda=6$

$$\left.\begin{array}{r}2x-5y=6x\\-5x-y+3z=6y\\3y-6z=6z\end{array}\right\}\Rightarrow\begin{array}{l}4x=-5y\Rightarrow -\frac{4}{5}x=y\\12z=3y\Rightarrow 4z=y\end{array}$$

THUS $\begin{pmatrix}5\\-4\\-1\end{pmatrix}$ WHICH NORMALIZES TO $\frac{1}{\sqrt{42}}\begin{pmatrix}-5\\4\\1\end{pmatrix}$

- IF $\lambda=-3$

$$\left.\begin{array}{r}2x-5y=-3x\\-5x-y+3z=-3y\\3y-6z=-3z\end{array}\right\}\begin{array}{l}\Rightarrow 5x=5y\\\Rightarrow 3z=3y\end{array}\quad \text{IE } x=y=z$$

THUS $\begin{pmatrix}1\\1\\1\end{pmatrix}$ WHICH NORMALIZES TO $\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$

- IF $\lambda=-8$

$$\left.\begin{array}{r}2x-5y=-8x\\-5x-y+3z=-8y\\3y-6z=-8z\end{array}\right\}\Rightarrow\begin{array}{l}10x=5y\\2z=-3y\end{array}\Rightarrow\begin{array}{l}2x=y\\z=-\frac{3}{2}y\end{array}$$

HENCE $\begin{pmatrix}1\\2\\-3\end{pmatrix}$ WHICH NORMALIZES TO $\frac{1}{\sqrt{14}}\begin{pmatrix}1\\2\\-3\end{pmatrix}$

$$P=\begin{pmatrix}\frac{-5}{\sqrt{42}}&\frac{1}{\sqrt{3}}&\frac{1}{\sqrt{14}}\\\frac{4}{\sqrt{42}}&\frac{1}{\sqrt{3}}&\frac{2}{\sqrt{14}}\\\frac{1}{\sqrt{42}}&\frac{1}{\sqrt{3}}&-\frac{3}{\sqrt{14}}\end{pmatrix};\quad D=\begin{pmatrix}6&0&0\\0&-3&0\\0&0&-8\end{pmatrix}$$

Question 44 (***)

Find the eigenvalues and the corresponding eigenvectors of the following 3×3 matrix.

$$\mathbf{A}=\begin{pmatrix}1 & 1 & 0\\ 1 & 2 & 1\\ 0 & 1 & 1\end{pmatrix}.$$

$$\boxed{\lambda_1=0,\quad \mathbf{u}=\begin{pmatrix}1\\-1\\1\end{pmatrix}},\ \boxed{\lambda_2=1,\quad \mathbf{v}=\begin{pmatrix}1\\0\\-1\end{pmatrix}},\ \boxed{\lambda_3=3,\quad \mathbf{w}=\begin{pmatrix}1\\2\\1\end{pmatrix}}$$

CHARACTERISTIC EQUATION

$$\begin{vmatrix}1-\lambda & 1 & 0\\ 1 & 2-\lambda & 1\\ 0 & 1 & 1-\lambda\end{vmatrix}=0$$

$$\Rightarrow (1-\lambda)\begin{vmatrix}2-\lambda & 1\\ 1 & 1-\lambda\end{vmatrix}-1\begin{vmatrix}1 & 1\\ 0 & 1-\lambda\end{vmatrix}=0$$

$$\Rightarrow (1-\lambda)\left[(2-\lambda)(1-\lambda)-1\right]-(1-\lambda)=0$$

$$\Rightarrow (1-\lambda)\left[(\lambda-2)(\lambda-1)-1\right]-(1-\lambda)=0$$

$$\Rightarrow (1-\lambda)\left[\lambda^2-3\lambda+1\right]-(1-\lambda)=0$$

$$\Rightarrow (1-\lambda)\left[\lambda^2-3\lambda+1-1\right]=0$$

$$\Rightarrow (1-\lambda)(\lambda^2-3\lambda)=0$$

$$\Rightarrow (1-\lambda)\lambda(\lambda-3)=0$$

$$\lambda=\begin{cases}1\\0\\3\end{cases}$$

- If $\lambda=1$

$$\left.\begin{array}{r}x+y=x\\ x+2y+z=y\\ y+z=z\end{array}\right\}\Rightarrow \boxed{y=0}\quad \therefore\ x=-z\quad \therefore\ \alpha\begin{pmatrix}1\\0\\-1\end{pmatrix}$$

- If $\lambda=3$

$$\left.\begin{array}{r}x+y=3x\\ x+2y+z=3y\\ y+z=3z\end{array}\right\}\Rightarrow \left.\begin{array}{r}y=2x\\ x-y+z=0\\ y=2z\end{array}\right\}\Rightarrow \begin{array}{l}x=\frac{1}{2}y\\ z=\frac{1}{2}y\end{array}\quad \therefore\ \beta\begin{pmatrix}1\\2\\1\end{pmatrix}$$

- If $\lambda=0$

$$\left.\begin{array}{r}x+y=0x\\ x+2y+z=0y\\ y+z=0z\end{array}\right\}\Rightarrow \left.\begin{array}{r}y=-x\\ x+2y+z=0\\ y=-z\end{array}\right\}\Rightarrow \begin{array}{l}x=-y\\ z=-y\end{array}\quad \therefore\ \gamma\begin{pmatrix}1\\-1\\1\end{pmatrix}$$

Question 45 (***)

A linear transformation T in the $x-y$ plane consists of a reflection about the straight line with equation

$$y = x\tan\alpha°,$$

followed by an anticlockwise rotation about the origin O, by an angle of $\beta°$.

By considering matrix compositions, or otherwise, describe T geometrically.

$$\boxed{\text{reflection in the line } y = \tan\left(\alpha° + \frac{\beta°}{2}\right)}$$

REFLECTION IN THE LINE $y = x\tan\alpha = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix}$

ROTATION BY $+\beta = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$

$$\begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}\begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix} = \begin{bmatrix} \cos\beta\cos 2\alpha - \sin\beta\sin 2\alpha & \cos\beta\sin 2\alpha + \sin\beta\cos 2\alpha \\ \sin\beta\cos 2\alpha + \cos\beta\sin 2\alpha & \sin\beta\sin 2\alpha - \cos\beta\cos 2\alpha \end{bmatrix}$$

$$\begin{bmatrix} \cos(2\alpha+\beta) & \sin(2\alpha+\beta) \\ \sin(2\alpha+\beta) & -\cos(2\alpha+\beta) \end{bmatrix}$$

REFLECTION IN THE LINE $y = \tan\left(\alpha + \frac{\beta}{2}\right)x$

Question 46 (***)

$$\begin{aligned} x + 3y + 2z &= 13 \\ 3x + 2y - z &= 4 \\ 2x + y + z &= 7 \end{aligned}$$

Solve the above system of simultaneous equations by manipulating their augmented matrix into reduced row echelon form.

No credit will be given for alternative solution methods.

$$\boxed{x = 1, \quad y = 2, \quad z = 3}$$

AUGMENTED MATRIX

$$\begin{pmatrix} 1 & 3 & 2 & 13 \\ 3 & 2 & -1 & 4 \\ 2 & 1 & 1 & 7 \end{pmatrix} \begin{matrix} r_{12}(-3) \\ r_{13}(-2) \end{matrix} \begin{pmatrix} 1 & 3 & 2 & 13 \\ 0 & -7 & -7 & -35 \\ 0 & -5 & -3 & -19 \end{pmatrix} r_2\left(-\tfrac{1}{7}\right) \begin{pmatrix} 1 & 3 & 2 & 13 \\ 0 & 1 & 1 & 5 \\ 0 & -5 & -3 & -19 \end{pmatrix}$$

$$r_{23}(5) \begin{pmatrix} 1 & 3 & 2 & 13 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 2 & 6 \end{pmatrix} r_3\left(\tfrac{1}{2}\right) \begin{pmatrix} 1 & 3 & 2 & 13 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & 3 \end{pmatrix} r_{21}(-3) \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

$$\begin{matrix} r_{32}(-1) \\ r_{31}(1) \end{matrix} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix} \quad \therefore \begin{aligned} x &= 1 \\ y &= 2 \\ z &= 3 \end{aligned}$$

Question 47 (***)

The 3×3 matrix $\mathbf{A}$ is given below.

$$\mathbf{A} = \begin{pmatrix} 3 & 1 & -3 \\ 2 & 4 & 3 \\ -4 & 2 & -1 \end{pmatrix}$$

The matrix $\mathbf{A}$ is non singular.

a) Evaluate $\mathbf{A}^2 - \mathbf{A}$.

b) Show clearly that

$$\mathbf{A}^{-1} = \frac{1}{20}[\mathbf{A} - \mathbf{I}].$$

$\boxed{20\mathbf{I}}$

(a) $A^2 = \begin{pmatrix} 3 & 1 & -3 \\ 2 & 4 & 3 \\ -4 & 2 & -1 \end{pmatrix}\begin{pmatrix} 3 & 1 & -3 \\ 2 & 4 & 3 \\ -4 & 2 & -1 \end{pmatrix} = \begin{pmatrix} 23 & 1 & -3 \\ 2 & 24 & 3 \\ -4 & 2 & 19 \end{pmatrix}$

$\therefore A^2 - A = \begin{pmatrix} 23 & 1 & -3 \\ 2 & 24 & 3 \\ -4 & 2 & 19 \end{pmatrix} - \begin{pmatrix} 3 & 1 & -3 \\ 2 & 4 & 3 \\ -4 & 2 & -1 \end{pmatrix} = \begin{pmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 20 \end{pmatrix} = 20I$

(b) $A^2 - A = 20I$

$A^{-1}A^2 - A^{-1}A = A^{-1}(20I)$

$A - I = 20A^{-1}$

$\therefore A^{-1} = \frac{1}{20}(A - I)$ AS REQUIRED

Question 48 (***)

The 2×2 matrix $\mathbf{P}$ is defined in terms of x, where $x \neq 1$.

$$\mathbf{P} = \begin{pmatrix} 2 & x \\ 1 & 3 \end{pmatrix}.$$

a) Find in its simplest form the matrix $\mathbf{P}\mathbf{P}^{\mathrm{T}} - \mathbf{P}^{\mathrm{T}}\mathbf{P}$.

b) Show clearly that $\det\left(\mathbf{P}\mathbf{P}^{\mathrm{T}} - \mathbf{P}^{\mathrm{T}}\mathbf{P}\right) < 0$.

$$\boxed{\mathbf{P}\mathbf{P}^{\mathrm{T}} - \mathbf{P}^{\mathrm{T}}\mathbf{P} = \begin{pmatrix} x^2-1 & x-1 \\ x-1 & 1-x^2 \end{pmatrix}}$$

(a) $PP^{T} = \begin{pmatrix} 2 & x \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ x & 3 \end{pmatrix} = \begin{pmatrix} 4+x^2 & 2+3x \\ 2+3x & 10 \end{pmatrix}$

$P^{T}P = \begin{pmatrix} 2 & 1 \\ x & 3 \end{pmatrix}\begin{pmatrix} 2 & x \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 5 & 2x+3 \\ 2x+3 & x^2+9 \end{pmatrix}$

$PP^{T} - P^{T}P = \begin{pmatrix} 4+x^2 & 3x+2 \\ 3x+2 & 10 \end{pmatrix} - \begin{pmatrix} 5 & 2x+3 \\ 2x+3 & x^2+9 \end{pmatrix} = \begin{pmatrix} x^2-1 & x-1 \\ x-1 & 1-x^2 \end{pmatrix}$ //

(b) $\det(PP^{T} - P^{T}P) = \begin{vmatrix} x^2-1 & x-1 \\ x-1 & 1-x^2 \end{vmatrix} = \begin{vmatrix} (x-1)(x+1) & x-1 \\ x-1 & (1-x)(1+x) \end{vmatrix}$

$= (x-1)^2 \begin{vmatrix} x+1 & 1 \\ 1 & -(x+1) \end{vmatrix} = -(x-1)^2 \begin{vmatrix} x+1 & 1 \\ -1 & x+1 \end{vmatrix}$

$= -(x-1)^2\left[(x+1)^2+1\right] < 0$ SINCE $x \neq 1$

MINUS

always positive

Question 49 (***)

$$\begin{pmatrix} 2 & 5 & 3 \\ 1 & 2 & 2 \\ 1 & 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 10 \end{pmatrix}.$$

Show that the above system of simultaneous equations …

a) … does **not** have a unique solution.

b) … is **consistent** and the general solution can be written as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 16-4\lambda \\ \lambda-6 \\ \lambda \end{pmatrix},$$

where λ is a scalar parameter.

$\boxed{\text{proof}}$

(a) $\begin{vmatrix} 2 & 5 & 3 \\ 1 & 2 & 2 \\ 1 & 1 & 3 \end{vmatrix} \begin{matrix} C_{12}(-2) \\ C_{13}(-2) \end{matrix} = \begin{vmatrix} 2 & 1 & -1 \\ 1 & 0 & 0 \\ 1 & -1 & 1 \end{vmatrix} = -\begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} = -(1-1) = 0$

$\therefore$ NO UNIQUE SOLUTION

(b) $\begin{pmatrix} 2 & 5 & 3 & 2 \\ 1 & 2 & 2 & 4 \\ 1 & 1 & 3 & 10 \end{pmatrix} \; r_{13} = \begin{pmatrix} 1 & 1 & 3 & 10 \\ 1 & 2 & 2 & 4 \\ 2 & 5 & 3 & 2 \end{pmatrix} \; \begin{matrix} r_{12}(-1) \\ r_{13}(-2) \end{matrix} = \begin{pmatrix} 1 & 1 & 3 & 10 \\ 0 & 1 & -1 & -6 \\ 0 & 3 & -3 & -18 \end{pmatrix}$

$r_{23}(-3) \begin{pmatrix} 1 & 1 & 3 & 10 \\ 0 & 1 & -1 & -6 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad r_{21}(-1) = \begin{pmatrix} 1 & 0 & 4 & 16 \\ 0 & 1 & -1 & -6 \end{pmatrix}$

THUS $\left. \begin{matrix} x+4z=16 \\ y-z=-6 \end{matrix} \right\} \begin{matrix} x=16-4z \\ y=-6+z \end{matrix}$ LET $z=\lambda$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 16-4\lambda \\ -6+\lambda \\ \lambda \end{pmatrix}$ // AS REQUIRED

Question 50 (***)

The 3×3 matrix $\mathbf{A}$ is given below, in terms of a scalar constant k.

$$\mathbf{A} = \begin{pmatrix} k & 0 & 1 \\ -11 & k-3 & 9 \\ -11 & 0 & k \end{pmatrix}.$$

a) Given that $\mathbf{A}$ is singular, find the value of k.

b) Given instead that $\lambda = 2$ is an eigenvalue of $\mathbf{A}$, determine the value of k on this occasion.

$\boxed{k=3}$, $\boxed{k=5}$

a)

$$A = \begin{bmatrix} k & 0 & 1 \\ -11 & k-3 & 9 \\ -11 & 0 & k \end{bmatrix}$$

$$|A| = \begin{vmatrix} k & 0 & 1 \\ -11 & k-3 & 9 \\ -11 & 0 & k \end{vmatrix} = \dots \text{ EXPAND BY MIDDLE COLUMN}$$

$$= (k-3)\begin{vmatrix} k & 1 \\ -11 & k \end{vmatrix} = (k-3)(k^2+11)$$

$|A| = 0 \Rightarrow \underline{k=3}$ ONLY

(b) NOW $\lambda = 2$ IS AN EIGENVALUE

$$\begin{vmatrix} k-2 & 0 & 1 \\ -11 & (k-2)-3 & 9 \\ -11 & 0 & k-2 \end{vmatrix} = 0$$

$\Rightarrow \left[(k-2)-3\right]\left[(k-2)^2+11\right] = 0$

$\Rightarrow (k-5)\left((k-2)^2+11\right) = 0$

$\Rightarrow k=5$

Question 51 (***)

Three planes, the equations of which are given below, intersect along a straight line L.

$$\begin{aligned} x+2y+3z &= 2 \\ 2x+3y\ \ +z &= 3 \\ 3x+4y\ \ -z &= k \end{aligned}$$

Show, by reducing an augmented matrix into row echelon form, that the equation of L can be written in the form

$$\mathbf{r} = \mathbf{j} + t(7\mathbf{i} - 5\mathbf{j} + \mathbf{k}),$$

where t is a scalar parameter.

$\boxed{\text{proof}}$

$$\begin{aligned} x+2y+3z &= 2 \\ 2x+3y+z &= 3 \\ 3x+4y-z &= k \end{aligned}$$

AUGMENT A SYSTEM $\begin{pmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 1 & 3 \\ 3 & 4 & -1 & k \end{pmatrix}$

$= \dots \begin{matrix} r_{12}(-2) \\ r_{13}(-3) \end{matrix} \dots = \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -5 & -1 \\ 0 & -2 & -10 & k-6 \end{pmatrix} = \dots r_2(-1) \dots \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 5 & 1 \\ 0 & -2 & -10 & k-6 \end{pmatrix}$

$= \dots r_{23}(2) \dots \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 5 & 1 \\ 0 & 0 & 0 & k-4 \end{pmatrix}$ FOR INTERSECTION (CONSISTENCY) $k=4$

THUS

$\begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 5 & 1 \end{pmatrix} = \dots r_{21}(-2) \dots \begin{pmatrix} 1 & 0 & -7 & 0 \\ 0 & 1 & 5 & 1 \end{pmatrix} \Rightarrow \begin{matrix} x-7z=0 \\ y+5z=1 \end{matrix}$

HENCE $\left.\begin{aligned} x &= 7z \\ y &= 1-5z \\ z &= t \end{aligned}\right\} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7t \\ 1-5t \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 7 \\ -5 \\ 1 \end{pmatrix}$

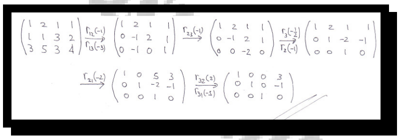
Question 52 (***)

Solve the following simultaneous equations by manipulating their augmented matrix into reduced row echelon form.

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 3 \\ 3 & 5 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$$

No credit will be given for alternative solution methods.

$\boxed{x = 3, \quad y = -1, \quad z = 0}$



Question 53 (***)

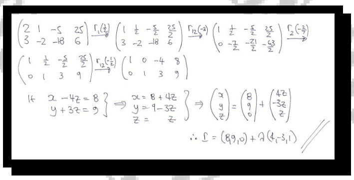
$$\begin{aligned} 3x - 2y - 18z &= 6 \\ 2x + y - 5z &= 25 \end{aligned}$$

Show, by reducing the system into row echelon form, that the solution can be written in the form

$$\mathbf{r} = 8\mathbf{i} + 9\mathbf{j} + \lambda(4\mathbf{i} - 3\mathbf{j} + \mathbf{k}),$$

where λ is a scalar parameter.

$\boxed{\text{proof}}$



Question 54 (***)

The 3×3 matrix $\mathbf{R}$ is defined by

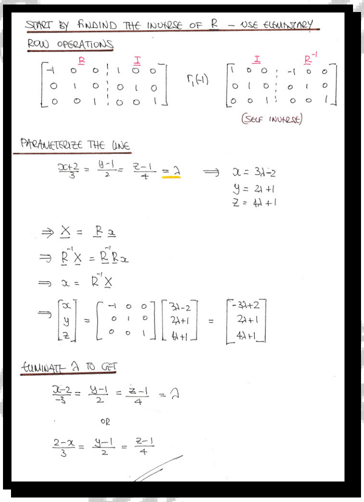
$$\mathbf{R}=\begin{pmatrix}-1&0&0\\0&1&0\\0&0&1\end{pmatrix}.$$

The image of the straight line L, when transformed by $\mathbf{R}$, is the straight line with Cartesian equation

$$\frac{x+2}{3}=\frac{y-1}{2}=\frac{z-1}{4}.$$

Find a Cartesian equation for L.

$\boxed{}$, $\boxed{\dfrac{x-2}{-3}=\dfrac{y-1}{2}=\dfrac{z-1}{4}}$


START BY FINDING THE INVERSE OF $\underline{R}$ – USE ELEMENTARY ROW OPERATIONS

$$\left[\begin{array}{ccc|ccc}-1&0&0&1&0&0\\0&1&0&0&1&0\\0&0&1&0&0&1\end{array}\right] \quad r_1(-1) \quad \left[\begin{array}{ccc|ccc}1&0&0&-1&0&0\\0&1&0&0&1&0\\0&0&1&0&0&1\end{array}\right]$$

(SELF INVERSE)

PARAMETERIZE THE LINE

$$\frac{x+2}{3}=\frac{y-1}{2}=\frac{z-1}{4}=\lambda \implies \begin{aligned}x&=3\lambda-2\\y&=2\lambda+1\\z&=4\lambda+1\end{aligned}$$

$$\Rightarrow \underline{X}=\underline{R}\,\underline{x}$$
$$\Rightarrow \underline{R}^{-1}\underline{X}=\underline{R}^{-1}\underline{R}\,\underline{x}$$
$$\Rightarrow \underline{x}=\underline{R}^{-1}\underline{X}$$
$$\Rightarrow \begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}3\lambda-2\\2\lambda+1\\4\lambda+1\end{bmatrix}=\begin{bmatrix}-3\lambda+2\\2\lambda+1\\4\lambda+1\end{bmatrix}$$

ELIMINATE λ TO GET

$$\frac{x-2}{-3}=\frac{y-1}{2}=\frac{z-1}{4}=\lambda$$

OR

$$\frac{2-x}{3}=\frac{y-1}{2}=\frac{z-1}{4}$$

Question 55 (***)

The 3×3 matrices $\mathbf{A}$ and $\mathbf{B}$ are given below.

$$\mathbf{A}=\begin{pmatrix}1&0&0\\0&0&-1\\0&1&0\end{pmatrix} \quad\text{and}\quad \mathbf{B}=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}.$$

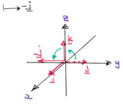
a) Describe geometrically the transformations given by each of the two matrices.

The matrix $\mathbf{C}$ is defined as the transformation defined by the matrix $\mathbf{A}$, followed by the transformation defined by the matrix $\mathbf{B}$.

b) Describe geometrically the transformation represented by $\mathbf{C}$.

$\boxed{}$, $\boxed{\mathbf{A}: \text{rotation about } x \text{ axis, } 90° \text{ anticlockwise}}$, $\boxed{\mathbf{B}: \text{reflection in the } xz \text{ plane}}$, $\boxed{\mathbf{C}: \text{reflection in the plane } y=z}$

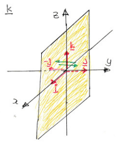
a) $\boxed{\underline{A} = \begin{bmatrix}1&0&0\\0&0&-1\\0&1&0\end{bmatrix}}$ $\quad \underline{i} \mapsto \underline{i}$, $\underline{j} \mapsto \underline{k}$, $\underline{k} \mapsto -\underline{i}$



$\det \underline{A} = 1$ (NO REFLECTION)

ROTATION ABOUT THE x AXIS, BY 90°, ANTICLOCKWISE IN A RIGHT HAND SENSE

$\boxed{\underline{B} = \begin{bmatrix}1&0&0\\0&-1&0\\0&0&1\end{bmatrix}}$ $\quad \underline{i} \mapsto \underline{i}$, $\underline{j} \mapsto -\underline{j}$, $\underline{k} \mapsto \underline{k}$



$\det \underline{B} = -1$ (REFLECTION INVOLVED)

REFLECTION ABOUT THE xz PLANE

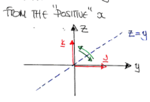
b) COMPOSE IN THE CORRECT ORDER

$$\underline{C} = \underline{B}\,\underline{A} = \begin{bmatrix}1&0&0\\0&-1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&0&-1\\0&1&0\end{bmatrix} = \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$$

$\det \underline{C} = -1$ (REFLECTION INVOLVED)

$\underline{i} \mapsto \underline{i}$, $\underline{j} \mapsto \underline{k}$, $\underline{k} \mapsto \underline{j}$

LOOKING AT THE yz PLANE, FROM THE "POSITIVE" x



$\therefore$ REFLECTION ABOUT THE PLANE $y=z$

Question 56 (***)

The vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are defined as

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 8 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 10 \\ 5 \\ a \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix},$$

where a is a scalar constant.

Given that $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are linearly dependent, determine the value of a and hence express $\mathbf{u}$ in terms of $\mathbf{v}$ and $\mathbf{w}$.

$\boxed{}$, $\boxed{a = 26}$, $\boxed{\mathbf{u} = \frac{1}{4}\mathbf{v} + \frac{3}{4}\mathbf{w}}$

$\boxed{\underline{u} = \begin{pmatrix} 1 \\ 2 \\ 8 \end{pmatrix} \bullet \underline{v} = \begin{pmatrix} 10 \\ 5 \\ a \end{pmatrix} \bullet \underline{w} = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}}$

IF LINEARLY INDEPENDENT THEIR SCALAR TRIPLE PRODUCT MUST BE ZERO

$$\underline{u} \cdot (\underline{v} \wedge \underline{w}) = 0 \implies \begin{vmatrix} 1 & 2 & 8 \\ 10 & 5 & a \\ -2 & 1 & 2 \end{vmatrix} = 0$$

$$\implies \begin{vmatrix} 5 & a \\ 1 & 2 \end{vmatrix} - 2\begin{vmatrix} 10 & a \\ -2 & 2 \end{vmatrix} + 8\begin{vmatrix} 10 & 5 \\ -2 & 1 \end{vmatrix} = 0$$

$$\implies 10 - a - 2(20 + 2a) + 8(10 + 10) = 0$$

$$\implies 10 - a - 40 - 4a + 160 = 0$$

$$\implies 130 = 5a$$

$$\implies a = 26$$

FINALLY WE HAVE

$$\underline{u} = \lambda\underline{v} + \mu\underline{w} \implies \begin{pmatrix} 1 \\ 2 \\ 8 \end{pmatrix} = \lambda\begin{pmatrix} 10 \\ 5 \\ 26 \end{pmatrix} + \mu\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$$

$$\left\{\begin{array}{l} 1 = 10\lambda - 2\mu \\ 2 = 5\lambda + \mu \\ 8 = 26\lambda + 2\mu \end{array}\right\}$$ ADDING THE 1ST & 3RD

$9 = 36\lambda$

$\lambda = \frac{1}{4}$

HENCE $2 = 5 \times \frac{1}{4} + \mu$

$\mu = \frac{3}{4}$

(CHECK ALL 3 "WORK")

$$\therefore \underline{u} = \frac{1}{4}\underline{v} + \frac{3}{4}\underline{w}$$

Question 57 (***)

The 3×3 matrices $\mathbf{A}$ and $\mathbf{B}$, are given in terms of the constants k and h below.

$$\mathbf{A}=\begin{pmatrix}1 & 2 & 1\\ 2 & k & 4\\ 3 & 2 & -1\end{pmatrix} \quad \text{and} \quad \mathbf{B}=\begin{pmatrix}15 & -4 & -1\\ h & 4 & 2\\ 17 & -4 & -3\end{pmatrix}.$$

a) Find the matrix composition $\mathbf{AB}$, in terms of k and h.

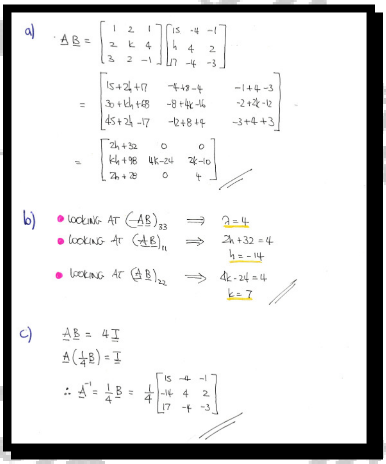
It is further given that $\mathbf{AB}=\lambda\mathbf{I}$ for some values of k and h.

b) Find the value of each of the constants λ, k and h.

c) Deduce $\mathbf{A}^{-1}$, for the values of λ, k and h, found in part **(b)**.

$\boxed{}$, $\boxed{\mathbf{AB}=\begin{pmatrix}2h+32 & 0 & 0\\ hk+98 & 4k-24 & 2k-14\\ 2h+28 & 0 & 4\end{pmatrix}}$, $\boxed{\lambda=4,\ h=14,\ k=7}$,

$$\boxed{\mathbf{A}^{-1}=\frac{1}{4}\begin{pmatrix}15 & -4 & -1\\ -14 & 4 & 2\\ 17 & -4 & -3\end{pmatrix}}$$



a) $\underline{A}\,\underline{B}=\begin{bmatrix}1 & 2 & 1\\ 2 & k & 4\\ 3 & 2 & -1\end{bmatrix}\begin{bmatrix}15 & -4 & -1\\ h & 4 & 2\\ 17 & -4 & -3\end{bmatrix}$

$=\begin{bmatrix}15+2h+17 & -4+8-4 & -1+4-3\\ 30+kh+68 & -8+4k-16 & -2+2k-12\\ 45+2h-17 & -12+8+4 & -3+4+3\end{bmatrix}$

$=\begin{bmatrix}2h+32 & 0 & 0\\ kh+98 & 4k-24 & 2k-10\\ 2h+28 & 0 & 4\end{bmatrix}$

b) • LOOKING AT $(\underline{A}\underline{B})_{33}$ $\Longrightarrow$ $\lambda=4$

• LOOKING AT $(\underline{A}\underline{B})_{11}$ $\Longrightarrow$ $2h+32=4$, $h=-14$

• LOOKING AT $(\underline{A}\underline{B})_{22}$ $\Longrightarrow$ $4k-24=4$, $k=7$

c) $\underline{A}\underline{B}=4\underline{I}$

$\underline{A}\left(\frac{1}{4}\underline{B}\right)=\underline{I}$

$\therefore \underline{A}^{-1}=\frac{1}{4}\underline{B}=\frac{1}{4}\begin{bmatrix}15 & -4 & -1\\ -14 & 4 & 2\\ 17 & -4 & -3\end{bmatrix}$

Question 58 (***)

The 2×2 matrix $\mathbf{A}=\begin{pmatrix}7 & 6\\ 6 & 2\end{pmatrix}$ is given.

Use the Caley- Hamilton theorem to show that

$$\mathbf{A}^4=\lambda\mathbf{A}+\mu\mathbf{I},$$

where $\mathbf{I}$ is 2×2 identity matrix.

$\boxed{\text{4-Q}}$, $\boxed{\mathbf{A}^4=1125\mathbf{A}+2266\mathbf{I}}$

A MATRIX MUST SATISFY ITS CHARACTERISTIC EQUATION

$$|\underline{A}-\lambda\underline{I}|=0 \implies \begin{vmatrix}7-\lambda & 6\\ 6 & 2-\lambda\end{vmatrix}=0$$

$$\implies (7-\lambda)(2-\lambda)-36=0$$

$$\implies (\lambda-7)(\lambda-2)-36=0$$

$$\implies \lambda^2-9\lambda+14-36=0$$

$$\implies \lambda^2-9\lambda-22=0$$

HENCE WE HAVE

$$\underline{A}^2-9\underline{A}-22\underline{I}=\underline{0}$$

$$\underline{A}^2=9\underline{A}+22\underline{I}$$

$$\underline{A}^4=(9\underline{A}+22\underline{I})^2$$

$$\underline{A}^4=81\underline{A}^2+396\underline{A}\,\underline{I}+484\underline{I}^2$$

$$\underline{A}^4=81\underline{A}^2+396\underline{A}+484\underline{I}$$

BUT $\underline{A}^2=9\underline{A}+22\underline{I}$

$$\underline{A}^4=81(9\underline{A}+22\underline{I})+396\underline{A}+484\underline{I}$$

$$\underline{A}^4=729\underline{A}+1782\underline{I}+396\underline{A}+484\underline{I}$$

$$\underline{A}^4=1125\underline{A}+2266\underline{I}$$

Question 59 **(***+)**

A system of equation is given in matrix form below

$$\begin{pmatrix} t & 2 & 3 \\ 2 & 3 & -t \\ 3 & 5 & t+1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix},$$

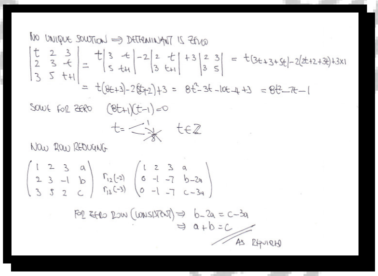
where t is an integer constant, and a, b and c are real constants.

The system of equations does not have a unique solution, but it is consistent.

Show clearly that

$$a+b=c.$$

$\boxed{\text{proof}}$


NO UNIQUE SOLUTION $\Rightarrow$ DETERMINANT IS ZERO

$$\begin{vmatrix} t & 2 & 3 \\ 2 & 3 & -t \\ 3 & 5 & t+1 \end{vmatrix} = t\begin{vmatrix} 3 & -t \\ 5 & t+1 \end{vmatrix} - 2\begin{vmatrix} 2 & -t \\ 3 & t+1 \end{vmatrix} + 3\begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} = t(3t+3+5t) - 2(2t+2+3t) + 3\times 1$$

$$= t(8t+3) - 2(5t+2) + 3 = 8t^2 - 3t - 10t - 4 + 3 = 8t^2 - 7t - 1$$

SOLVE FOR ZERO $(8t+1)(t-1) = 0$

$$t = \begin{cases} 1 \\ \cancel{-\frac{1}{8}} \end{cases} \quad t \in \mathbb{Z}$$

NOW ROW REDUCING

$$\begin{pmatrix} 1 & 2 & 3 & a \\ 2 & 3 & -1 & b \\ 3 & 5 & 2 & c \end{pmatrix} \begin{matrix} \\ r_{12}(-2) \\ r_{13}(-3) \end{matrix} \begin{pmatrix} 1 & 2 & 3 & a \\ 0 & -1 & -7 & b-2a \\ 0 & -1 & -7 & c-3a \end{pmatrix}$$

FOR ZERO ROW (CONSISTENT) $\Rightarrow b-2a = c-3a$

$\Rightarrow a+b=c$

AS REQUIRED

Question 60 **(***+)**

Express the following 3×3 determinant as the product of three linear factors.

$$\begin{vmatrix} 1 & x & x^2-x \\ 1 & y & y^2-y \\ 1 & z & z^2-z \end{vmatrix}$$

$$\boxed{(x-y)(y-z)(z-x)}$$

$$\begin{vmatrix} 1 & x & x^2-x \\ 1 & y & y^2-y \\ 1 & z & z^2-z \end{vmatrix} \overset{C_{23}(1)}{=} \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \overset{\substack{r_{21}(-1)\\ r_{23}(-1)}}{=} \begin{vmatrix} & x-y & x^2-y^2 \\ 1 & y & y^2 \\ & z-y & z^2-y^2 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & x-y & (x-y)(x+y) \\ 1 & y & y^2 \\ 0 & z-y & (z-y)(z+y) \end{vmatrix} = (x-y)(z-y)\begin{vmatrix} 0 & 1 & x+y \\ 1 & y & y^2 \\ 0 & 1 & z+y \end{vmatrix}$$

EXPAND BY 2ND ROW

$$= (x-y)(z-y)\times(-1)\begin{vmatrix} 1 & x+y \\ 1 & z+y \end{vmatrix} = (x-y)(z-y)(-1)\left[z+\cancel{y}-x-\cancel{y}\right]$$

$$= (x-y)(z-y)(-1)(z-x) = (x-y)(y-z)(z-x)$$

Question 61 (***+)

Find the eigenvalues and the corresponding eigenvectors of the following 3×3 matrix.

$$\mathbf{A}=\begin{pmatrix}1&2&4\\2&1&4\\2&3&2\end{pmatrix}.$$

$\boxed{\lambda_1=-1,\ \ \mathbf{u}=\begin{pmatrix}-3\\1\\1\end{pmatrix}}$, $\boxed{\lambda_2=-2,\ \ \mathbf{v}=\begin{pmatrix}4\\4\\-5\end{pmatrix}}$, $\boxed{\lambda_3=7,\ \ \mathbf{w}=\begin{pmatrix}1\\1\\1\end{pmatrix}}$

CHARACTERISTIC EQUATION

$$\begin{vmatrix}1-\lambda&2&4\\2&1-\lambda&4\\2&3&2-\lambda\end{vmatrix}=0$$

$$\begin{matrix}r_{21}(-1)\\r_{23}(-1)\end{matrix}\ \begin{vmatrix}-1-\lambda&\lambda+1&0\\2&1-\lambda&4\\0&\lambda+2&-2-\lambda\end{vmatrix}=0$$

$$(-1-\lambda)\begin{vmatrix}1-\lambda&4\\\lambda+2&-2-\lambda\end{vmatrix}-(\lambda+1)\begin{vmatrix}2&4\\0&-2-\lambda\end{vmatrix}=0$$

$$(\lambda+1)\begin{vmatrix}1-\lambda&4\\\lambda+2&-2-\lambda\end{vmatrix}+(\lambda+1)\begin{vmatrix}2&4\\0&-2-\lambda\end{vmatrix}=0$$

$$(\lambda+1)\left[\left[(\lambda+2)(\lambda-1)-4(\lambda+2)\right]+2(-2-\lambda)\right]=0$$

$$(\lambda+1)\left[\lambda^2+\lambda-2-4\lambda-8-4-2\lambda\right]=0$$

$$(\lambda+1)(\lambda^2-5\lambda-14)=0$$

$$(\lambda+1)(\lambda-7)(\lambda+2)=0$$

$$\lambda=\begin{cases}-1\\7\\-2\end{cases}$$

- IF $\lambda=-1$

$$\left.\begin{aligned}x+2y+4z&=-x\\2x+y+4z&=-y\\2x+3y+2z&=-z\end{aligned}\right\}\quad\left.\begin{aligned}2x+2y+4z&=0\\2x+2y+4z&=0\\2x+3y+3z&=0\end{aligned}\right\}\quad\begin{aligned}x+y+2z&=0\\2x+3y+3z&=0\end{aligned}$$

$\boxed{x=-y-2z}$

$\therefore\ 2(-y-2z)+3y+3z=0$

$-2y-4z+3y+3z=0$

$\boxed{y=z}$

$\boxed{x=-3z}$

$\therefore\ \alpha\begin{pmatrix}-3\\1\\1\end{pmatrix}$

- IF $\lambda=7$

$$\left.\begin{aligned}x+2y+4z&=7x\\2x+y+4z&=7y\\2x+3y+2z&=7z\end{aligned}\right\}\quad\left.\begin{aligned}-6x+2y+4z&=0\\2x-6y+4z&=0\\2x+3y-5z&=0\end{aligned}\right\}\quad\begin{aligned}-3x+y+2z&=0\\x-3y+2z&=0\\2x+3y-5z&=0\end{aligned}$$

FROM THE FIRST TWO

$-3x+y=x-3y$

$4y=4x$

$\boxed{y=x}$

THIRD GIVES

$5x=5z$

$\boxed{z=x}$

$\therefore\ \beta\begin{pmatrix}1\\1\\1\end{pmatrix}$

IF $\lambda=-2$

$$\left.\begin{aligned}x+2y+4z&=-2x\\2x+y+4z&=-2y\\2x+3y+2z&=-2z\end{aligned}\right\}\quad\left.\begin{aligned}3x+2y+4z&=0\\2x+3y+4z&=0\\2x+3y+4z&=0\end{aligned}\right\}$$

FROM THE FIRST TWO

$3x+2y=2x+3y$

$\boxed{x=y}$

HENCE $5x+4z=0$

$4z=-5x$

$\boxed{z=-\frac{5}{4}x}$

$\therefore\ \gamma\begin{pmatrix}4\\4\\-5\end{pmatrix}$

Question 62 (***+)

The 3×3 matrix $\mathbf{A}$ is defined in terms of a scalar constant k as

$$\mathbf{A}=\begin{pmatrix}2 & -1 & 1\\ 0 & 3 & 1\\ 1 & 1 & k\end{pmatrix}$$

a) Find $\mathbf{A}^{-1}$, in terms of k.

b) Hence solve the following simultaneous equations

$$\begin{aligned}2x-y+z&=1\\ 3y+z&=2\\ x+y+2z&=2\end{aligned}$$

$$\boxed{\mathbf{A}^{-1}=\frac{1}{6(k-1)}\begin{pmatrix}3k-1 & k+1 & -4\\ 1 & 2k-1 & -2\\ -3 & -3 & 6\end{pmatrix}},\quad \boxed{x=\frac{1}{2},\ \ y=\frac{1}{2},\ \ z=\frac{1}{2}}$$

(a) $A=\begin{pmatrix}2 & -1 & 1\\ 0 & 3 & 1\\ 1 & 1 & k\end{pmatrix}$

MATRIX OF MINORS

$$\begin{pmatrix}3k-1 & -1 & -3\\ -k-1 & 2k-1 & 3\\ -4 & 2 & 6\end{pmatrix}$$

MATRIX OF COFACTORS

$$\begin{pmatrix}3k-1 & 1 & -3\\ k+1 & 2k-1 & -3\\ -4 & -2 & 6\end{pmatrix}$$

ADJUGATE MATRIX

$$\begin{pmatrix}3k-1 & k+1 & -4\\ 1 & 2k-1 & -2\\ -3 & -3 & 6\end{pmatrix}$$

$\text{Det } A = 2(3k-1)-1\times1+1(-3)$

$\text{Det } A = 6k-2-1-3$

$\text{Det } A = 6k-6$

$$\therefore A^{-1}=\frac{1}{6k-6}\begin{bmatrix}3k-1 & k+1 & -4\\ 1 & 2k-1 & -2\\ -3 & -3 & 6\end{bmatrix}$$

(b)
$$\left.\begin{aligned}2x-y+z&=1\\ 3y+z&=2\\ x+y+2z&=2\end{aligned}\right\}\Rightarrow\begin{pmatrix}2 & -1 & 1\\ 0 & 3 & 1\\ 1 & 1 & 2\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}1\\ 2\\ 2\end{pmatrix}$$

If $A\underline{x}=\underline{b}$ $\boxed{\text{WHERE } k=2}$

$A^{-1}A\underline{x}=A^{-1}\underline{b}$

$$\underline{x}=\frac{1}{6}\begin{pmatrix}5 & 3 & -4\\ 1 & 3 & -2\\ -3 & -3 & 6\end{pmatrix}\begin{pmatrix}1\\ 2\\ 2\end{pmatrix}$$

$$\underline{x}=\frac{1}{6}\begin{pmatrix}5+6-8\\ 1+6-4\\ -3-6+12\end{pmatrix}$$

$$\underline{x}=\frac{1}{6}\begin{pmatrix}3\\ 3\\ 3\end{pmatrix}$$

$$\underline{x}=\begin{pmatrix}\frac{1}{2}\\ \frac{1}{2}\\ \frac{1}{2}\end{pmatrix}$$

$\therefore x=y=z=\frac{1}{2}$

Question 63 (***+)

The 3×3 matrix $\mathbf{A}$ is defined in terms of a scalar constant k by

$$\mathbf{A}=\begin{pmatrix}1 & -2 & 2\\ k & 1 & k-1\\ 2 & 2k-1 & 2-k\end{pmatrix}.$$

a) Show that $\det\mathbf{A}$ is independent of k.

b) Determine, with full justification, whether the vectors

$$\mathbf{i}+2\mathbf{j}+2\mathbf{k}, \quad -2\mathbf{i}+\mathbf{j}+3\mathbf{k} \quad \text{and} \quad 2\mathbf{i}+\mathbf{j}$$

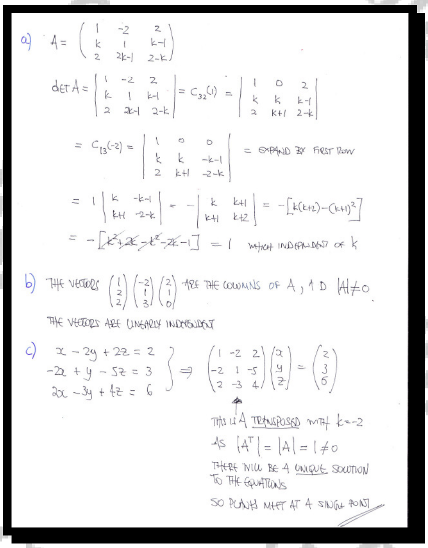
are linearly dependent or linearly independent.

The equations of three planes are given below.

$$\begin{aligned} x-2y+2z&=2\\ -2x+\ \ y-5z&=3\\ 2x-3y+4z&=6 \end{aligned}$$

c) Determine, with full justification, the geometrical configuration of these three planes.

$\boxed{\text{linearly independent}}$, $\boxed{\text{all 3 planes meet at a single point}}$



a) $A=\begin{pmatrix}1 & -2 & 2\\ k & 1 & k-1\\ 2 & 2k-1 & 2-k\end{pmatrix}$

$\det A=\begin{vmatrix}1 & -2 & 2\\ k & 1 & k-1\\ 2 & 2k-1 & 2-k\end{vmatrix}=C_{32}(1)=\begin{vmatrix}1 & 0 & 2\\ k & k & k-1\\ 2 & k+1 & 2-k\end{vmatrix}$

$=C_{13}(-2)=\begin{vmatrix}1 & 0 & 0\\ k & k & -k-1\\ 2 & k+1 & -2-k\end{vmatrix}$ = EXPAND BY FIRST ROW

$=1\begin{vmatrix}k & -k-1\\ k+1 & -2-k\end{vmatrix}=-\begin{vmatrix}k & k+1\\ k+1 & k+2\end{vmatrix}=-\left[k(k+2)-(k+1)^2\right]$

$=-\left[\cancel{k^2}+\cancel{2k}-\cancel{k^2}-\cancel{2k}-1\right]=1$ WHICH INDEPENDENT OF k

b) THE VECTORS $\begin{pmatrix}1\\2\\2\end{pmatrix}\begin{pmatrix}-2\\1\\3\end{pmatrix}\begin{pmatrix}2\\1\\0\end{pmatrix}$ ARE THE COLUMNS OF A, AND $|A|\neq0$.

THE VECTORS ARE LINEARLY INDEPENDENT

c) $\left.\begin{aligned} x-2y+2z&=2\\ -2x+y-5z&=3\\ 2x-3y+4z&=6\end{aligned}\right\}\Rightarrow\begin{pmatrix}1 & -2 & 2\\ -2 & 1 & -5\\ 2 & -3 & 4\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}2\\3\\6\end{pmatrix}$

THIS IS A TRANSPOSED WITH $k=-2$

AS $|A^T|=|A|=1\neq0$

THERE WILL BE A UNIQUE SOLUTION TO THE EQUATIONS

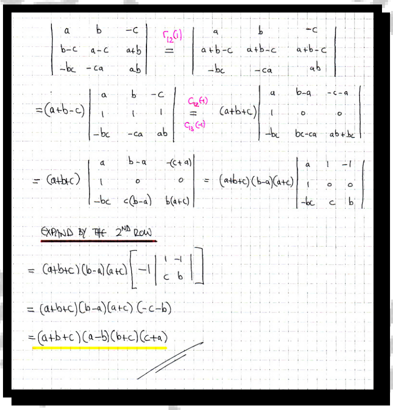
SO PLANES MEET AT A SINGLE POINT

Question 64 (***+)

By using elementary row and column operations, or otherwise, factorize the following determinant completely.

$$\begin{vmatrix} a & b & -c \\ b-c & a-c & a+b \\ -bc & -ca & ab \end{vmatrix}.$$

, $\boxed{(a+b+c)(a-b)(b+c)(c+a)}$



Question 65 (***+)

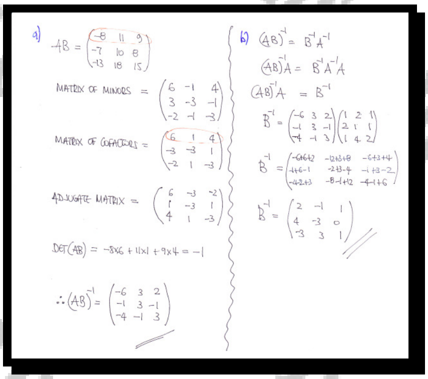
The 3×3 matrices $\mathbf{A}$ and $\mathbf{AB}$ are given below.

$$\mathbf{A}=\begin{pmatrix}1&2&1\\2&1&1\\1&4&2\end{pmatrix} \quad \text{and} \quad \mathbf{AB}=\begin{pmatrix}-8&11&9\\-7&10&8\\-13&18&15\end{pmatrix}.$$

a) Find the inverse of $\mathbf{AB}$.

b) Hence determine the inverse of $\mathbf{B}$.

$$\boxed{(\mathbf{AB})^{-1}=\begin{pmatrix}-6&3&2\\-1&3&-1\\-4&-1&3\end{pmatrix}}, \boxed{\mathbf{B}^{-1}=\begin{pmatrix}2&-1&1\\4&-3&0\\-3&3&1\end{pmatrix}}$$



a) $AB=\begin{pmatrix}-8&11&9\\-7&10&8\\-13&18&15\end{pmatrix}$

MATRIX OF MINORS $=\begin{pmatrix}6&-1&4\\3&-3&-1\\-2&-1&-3\end{pmatrix}$

MATRIX OF COFACTORS $=\begin{pmatrix}6&1&4\\-3&-3&1\\-2&1&-3\end{pmatrix}$

ADJUGATE MATRIX $=\begin{pmatrix}6&-3&-2\\1&-3&1\\4&1&-3\end{pmatrix}$

$\text{DET}(AB) = -8\times6+11\times1+9\times4=-1$

$\therefore (AB)^{-1}=\begin{pmatrix}-6&3&2\\-1&3&-1\\-4&-1&3\end{pmatrix}$

b) $(AB)^{-1}=B^{-1}A^{-1}$

$(AB)^{-1}A=B^{-1}A^{-1}A$

$(AB)^{-1}A=B^{-1}$

$B^{-1}=\begin{pmatrix}-6&3&2\\-1&3&-1\\-4&-1&3\end{pmatrix}\begin{pmatrix}1&2&1\\2&1&1\\1&4&2\end{pmatrix}$

$B^{-1}=\begin{pmatrix}-6+6+2&-12+3+8&-6+3+4\\-1+6-1&-2+3-4&-1+3-2\\-4-2+3&-8-1+12&-4-1+6\end{pmatrix}$

$B^{-1}=\begin{pmatrix}2&-1&1\\4&-3&0\\-3&3&1\end{pmatrix}$

Question 66 (***+)

The 3×3 matrix $\mathbf{A}$ is defined by

$$\mathbf{A}=\begin{pmatrix}0&0&-1\\0&1&0\\1&0&0\end{pmatrix}.$$

a) Describe geometrically the transformation given by $\mathbf{A}$.

The 3×3 matrix $\mathbf{B}$ represents a rotation of $180°$ about the line $x=z$, $y=0$.

b) Determine the elements of $\mathbf{B}$.

The 3×3 matrix $\mathbf{C}$ is represents the transformation defined by $\mathbf{B}$, followed by the transformation defined by $\mathbf{A}$.

c) Describe geometrically the transformation represented by $\mathbf{C}$.

$\boxed{\mathbf{A}:\text{rotation about } y \text{ axis, } 90° \text{ clockwise}}$, $\boxed{\mathbf{B}=\begin{pmatrix}0&0&1\\0&-1&0\\1&0&0\end{pmatrix}}$, $\boxed{\mathbf{C}:\text{rotation about } z \text{ axis, } 180°}$

a) • $\underline{A}=\begin{bmatrix}0&0&-1\\0&1&0\\1&0&0\end{bmatrix}$

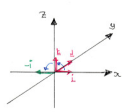
• $\det\underline{A}=1$
(NO REFLECTION) & INVOLVED

$\underline{i}=\begin{pmatrix}1\\0\\0\end{pmatrix}\mapsto\begin{pmatrix}0\\0\\1\end{pmatrix}=\underline{k}$

$\underline{j}=\begin{pmatrix}0\\1\\0\end{pmatrix}\mapsto\begin{pmatrix}0\\1\\0\end{pmatrix}=\underline{j}$

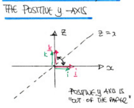
$\underline{k}=\begin{pmatrix}0\\0\\1\end{pmatrix}\mapsto\begin{pmatrix}-1\\0\\0\end{pmatrix}=-\underline{i}$

• LOOKING AT THE AXES



∴ ROTATION BY 90°, CLOCKWISE, ABOUT THE y AXIS

b) • LOOKING AT THE AXES, FROM THE POSITIVE y -AXIS



IE

$\underline{i}=\begin{pmatrix}1\\0\\0\end{pmatrix}\mapsto\begin{pmatrix}0\\0\\1\end{pmatrix}=\underline{k}$

$\underline{j}=\begin{pmatrix}0\\1\\0\end{pmatrix}\mapsto\begin{pmatrix}0\\-1\\0\end{pmatrix}=-\underline{j}$

$\underline{k}=\begin{pmatrix}0\\0\\1\end{pmatrix}\mapsto\begin{pmatrix}1\\0\\0\end{pmatrix}=\underline{i}$

$\therefore\ \underline{B}=\begin{bmatrix}0&0&1\\0&-1&0\\1&0&0\end{bmatrix}$

NOTE THAT THE ABOVE SET OF AXES IS IN 2-D, WHERE y IS "OUT" OF THE PAPER – ON ROTATION BY 180°, THE y AXIS WILL BE "INTO THE PLANE OF THE PAPER"

c) • FIND THE MATRIX FOR THE COMPOSITION

$$\underline{C}=\underline{A}\underline{B}=\begin{bmatrix}0&0&-1\\0&1&0\\1&0&0\end{bmatrix}\begin{bmatrix}0&0&1\\0&-1&0\\1&0&0\end{bmatrix}=\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&1\end{bmatrix}$$

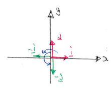
• $\det\underline{C}=+1$ (NO REFLECTION)

$\underline{i}=\begin{pmatrix}1\\0\\0\end{pmatrix}\mapsto\begin{pmatrix}-1\\0\\0\end{pmatrix}=-\underline{i}$

$\underline{j}=\begin{pmatrix}0\\1\\0\end{pmatrix}\mapsto\begin{pmatrix}0\\-1\\0\end{pmatrix}=-\underline{j}$

$\underline{k}=\begin{pmatrix}0\\0\\1\end{pmatrix}\mapsto\begin{pmatrix}0\\0\\1\end{pmatrix}=\underline{k}$

• LOOKING AT A SET OF AXES FROM THE POSITIVE z AXIS, "STICKING OUT OF THE PAPER"



ROTATION ABOUT THE z AXIS, BY 180°

Question 67 **(***+)**

The 3×3 matrix $\mathbf{A}$ is given below.

$$\mathbf{A}=\begin{pmatrix}1 & 1 & 0\\ 3 & -3 & 1\\ 0 & 3 & 2\end{pmatrix}.$$

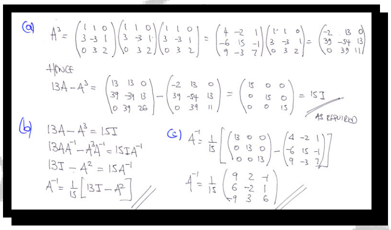
a) Show that

$$13\mathbf{A}-\mathbf{A}^3=15\mathbf{I}.$$

b) Hence find an expression for $\mathbf{A}^{-1}$ in terms of other matrices.

c) Use this expression to find $\mathbf{A}^{-1}$.

$\boxed{\mathbf{A}^{-1}=\frac{1}{15}\left(13\mathbf{I}-\mathbf{A}^2\right)}$, $\boxed{\mathbf{A}^{-1}=\frac{1}{15}\begin{pmatrix}9 & 2 & -1\\ 6 & -2 & 1\\ -9 & 3 & 6\end{pmatrix}}$



Question 68 (***+)

The 3×3 matrix $\mathbf{A}$ is given below.

$$\mathbf{A}=\begin{pmatrix}1&0&4\\0&5&4\\4&4&3\end{pmatrix}.$$

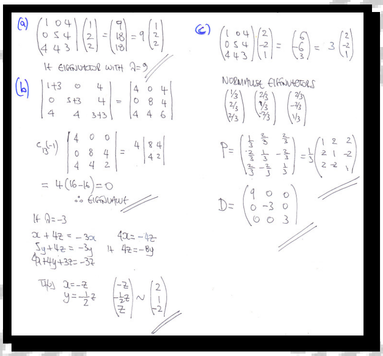
a) Verify that $\begin{pmatrix}1\\2\\2\end{pmatrix}$ is an eigenvector of $\mathbf{A}$ and state the corresponding eigenvalue.

b) Show that -3 is an eigenvalue of $\mathbf{A}$ and find the corresponding eigenvector.

c) Given further that $\begin{pmatrix}2\\-2\\1\end{pmatrix}$ is another eigenvector of $\mathbf{A}$, find 3×3 matrices $\mathbf{P}$ and $\mathbf{D}$ such that

$$\mathbf{D}=\mathbf{P}^{\mathrm{T}}\mathbf{AP}.$$

$\boxed{\lambda=9}$, $\boxed{\begin{pmatrix}2\\1\\-2\end{pmatrix}}$, $\boxed{\mathbf{D}=\begin{pmatrix}9&0&0\\0&-3&0\\0&0&3\end{pmatrix}}$, $\boxed{\mathbf{P}=\frac{1}{3}\begin{pmatrix}1&2&2\\2&1&-2\\2&-2&1\end{pmatrix}}$



Question 69 **(***+)**

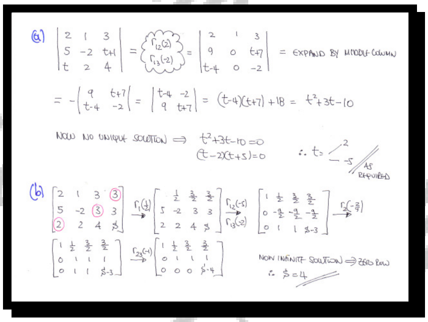
A system of equations is given below in terms of the scalar parameters t and s.

$$\begin{aligned} 2x \quad + y \qquad\quad &+ 3z = t+1 \\ 5x \; - 2y \; + (t+1)&z \; = 3 \\ tx \;\; + 2y \qquad\quad &+ 4z = s \end{aligned}$$

a) Show that if $t = -5$ or $t = 2$, the system does not have a unique solution.

b) Determine the value of s is the system is to have infinite solutions with $t = 2$.

$\boxed{s = 4}$



(a) $\begin{vmatrix} 2 & 1 & 3 \\ 5 & -2 & t+1 \\ t & 2 & 4 \end{vmatrix} = \left\{ \begin{matrix} r_{12}(2) \\ r_{13}(-2) \end{matrix} \right\} = \begin{vmatrix} 2 & 1 & 3 \\ 9 & 0 & t+7 \\ t-4 & 0 & -2 \end{vmatrix}$ = EXPAND BY MIDDLE COLUMN

$= -\begin{vmatrix} 9 & t+7 \\ t-4 & -2 \end{vmatrix} = \begin{vmatrix} t-4 & -2 \\ 9 & t+7 \end{vmatrix} = (t-4)(t+7)+18 = t^2+3t-10$

NOW NO UNIQUE SOLUTION $\Rightarrow$ $t^2+3t-10=0$

$(t-2)(t+5)=0$ $\therefore t = 2, -5$ AS REQUIRED

(b) $\begin{bmatrix} 2 & 1 & 3 & 3 \\ 5 & -2 & 3 & 3 \\ 2 & 2 & 4 & s \end{bmatrix} \xrightarrow{r_1(\frac{1}{2})} \begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & \frac{3}{2} \\ 5 & -2 & 3 & 3 \\ 2 & 2 & 4 & s \end{bmatrix} \xrightarrow[r_{13}(-2)]{r_{12}(-5)} \begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & \frac{3}{2} \\ 0 & -\frac{9}{2} & -\frac{9}{2} & -\frac{9}{2} \\ 0 & 1 & 1 & s-3 \end{bmatrix} \xrightarrow{r_2(-\frac{2}{9})}$

$\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & \frac{3}{2} \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & s-3 \end{bmatrix} \xrightarrow{r_{23}(-1)} \begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & \frac{3}{2} \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & s-4 \end{bmatrix}$

NOW INFINITE SOLUTION $\Rightarrow$ ZERO ROW

$\therefore s = 4$

Question 70 (***+)

The 3×3 matrix $\mathbf{A}$ is defined in terms of a scalar constant k by

$$\mathbf{A} = \begin{pmatrix} k & 8 & 1 \\ 1 & 1 & 1 \\ 3 & 4 & 1 \end{pmatrix}.$$

The straight line L is a line of invariant points under $\mathbf{A}$.

Determine, in any order, …

a) … the value of k.

b) … the Cartesian equation of L, giving the answer in the form

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n},$$

where l, m and n are integers to be found.

$\boxed{k = 8}$, $\boxed{\frac{x}{4} = \frac{y}{-3} = \frac{z}{-4}}$

a) $A = \begin{pmatrix} k & 8 & 1 \\ 1 & 1 & 1 \\ 3 & 4 & 1 \end{pmatrix}$ IF LINE OF INVARIANT POINTS $\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

THUS

$$\begin{pmatrix} k & 8 & 1 \\ 1 & 1 & 1 \\ 3 & 4 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \left.\begin{array}{r} kx + 8y + z = x \\ x + \cancel{y} + z = \cancel{y} \\ 3x + 4y + \cancel{z} = \cancel{z} \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r} (k-1)x + 8y + z = 0 \\ x + z = 0 \\ 3x + 4y = 0 \end{array}\right\} \Rightarrow \begin{array}{r} (k-1)x + 8y + z = 0 \\ z = -x \\ y = -\frac{3}{4}x \end{array}$$

HENCE

$(k-1)x + 8\left(-\frac{3}{4}x\right) + (-x) = 0$

$(k-1)x - 6x - x = 0$

$(k-1)x - 7x = 0$

$(k-1) = 7$

$k = 8$

b) LET $x = \lambda$

$$\left.\begin{array}{l} x = \lambda \\ z = -\lambda \\ y = -\frac{3}{4}\lambda \end{array}\right\} \Rightarrow \left.\begin{array}{l} \lambda = x \\ \lambda = -z \\ \lambda = \frac{-4y}{3} \end{array}\right\} \Rightarrow x = -\frac{4y}{3} = -z$$

$$\frac{x}{4} = -\frac{y}{3} = -\frac{z}{4}$$

$$\frac{x}{4} = \frac{y}{-3} = \frac{z}{-4}$$

Question 71 (***+)

The three planes defined by the equations

$$\begin{aligned} x+2y+\ \ z&=2\\ 2x+a\,y+\ \ z&=1\\ x+\ \ y+2z&=b \end{aligned}$$

where a and k are constants, intersect along a straight line L.

Determine an equation of L.

$\boxed{}$, $\boxed{\mathbf{r}=(6-3t)\mathbf{i}+(t-2)\mathbf{j}+t\mathbf{k}}$

- FIRSTLY IF THERE IS NO UNIQUE SOLUTION, THE DETERMINANT OF THE MATRIX $\begin{bmatrix}1&2&1\\2&a&1\\1&1&2\end{bmatrix}$ MUST BE ZERO
- EXPAND BY TOP ROW

$$1\begin{vmatrix}a&1\\1&2\end{vmatrix}-2\begin{vmatrix}2&1\\1&2\end{vmatrix}+1\begin{vmatrix}2&a\\1&1\end{vmatrix}=0$$
$$(2a-1)-(2\times3)+(2-a)=0$$
$$2a-1-6+2-a=0$$
$$a-5=0$$
$$a=5$$

- START ROW REDUCING TO OBTAIN A "BOTTOM ZERO ROW" IF THE SYSTEM IS TO BE CONSISTENT

$$\left[\begin{array}{ccc|c}1&2&1&2\\2&5&1&2\\1&1&2&b\end{array}\right]\ \begin{matrix}r_{12}(-2)\\r_{13}(-1)\end{matrix}=\left[\begin{array}{ccc|c}0&2&1&2\\0&1&-1&-2\\0&-1&1&b-2\end{array}\right]$$

$$r_{23}(1)=\left[\begin{array}{ccc|c}1&2&1&2\\0&1&-1&-2\\0&0&0&b-4\end{array}\right]\qquad\therefore\ b=4$$

- CONTINUING ROW REDUCING, IGNORING THE BOTTOM ROW

$$\left[\begin{array}{ccc|c}1&2&1&2\\0&1&-1&-2\end{array}\right]\ r_{21}(-2)=\left[\begin{array}{ccc|c}1&0&3&6\\0&1&-1&-2\end{array}\right]$$

- EXTRACTING THE SOLUTION, WE HAVE

$$x+3z=6$$
$$y-z=-2$$

- LET $z=t$, SOME PARAMETER

$$x=6-3t\qquad\qquad y=-2+t\qquad\qquad z=t$$

IE $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}6\\-2\\0\end{pmatrix}+t\begin{pmatrix}-3\\1\\1\end{pmatrix}$

OR $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}6-3t\\t-2\\t\end{pmatrix}$

Question 72 (***+)

The 3×3 matrices $\mathbf{A}$ and $\mathbf{B}$ are given below.

$$\mathbf{A}=\begin{pmatrix}5&2&4\\7&3&2\\4&5&3\end{pmatrix} \quad\text{and}\quad \mathbf{B}=\begin{pmatrix}-1&14&-8\\-13&-1&18\\23&-17&1\end{pmatrix}$$

Find an expression for $\mathbf{AB}$ and use it to solve the following system of equations.

$$\begin{aligned}5x+2y+4z&=10\\7x+3y+2z&=21\\4x+5y+3z&=5\end{aligned}$$

$\boxed{}$, $\boxed{x=4,\quad y=-1,\quad z=-2}$

$$\underline{A}\,\underline{B}=\begin{bmatrix}5&2&4\\7&3&2\\4&5&3\end{bmatrix}\begin{bmatrix}-1&14&-8\\-13&-1&18\\23&-17&1\end{bmatrix}$$

$$=\begin{bmatrix}-5-26+92 & \cancel{70-2-68} & -40\cancel{+36+4}\\ \cancel{-7-39+46} & 98-3-34 & \cancel{-56+54+2}\\ -4\cancel{-65+69} & \cancel{56-5-51} & -32+90+3\end{bmatrix}$$

$$=\begin{bmatrix}61&0&0\\0&61&0\\0&0&61\end{bmatrix}=61\underline{I}$$

$$\therefore\ \underline{A}\left(\tfrac{1}{61}\underline{B}\right)\ \underline{I}$$

$$\underline{A}^{-1}=\tfrac{1}{61}\underline{B}$$

THUS WE NOW HAVE

$$\left.\begin{aligned}5x+2y+4z&=10\\7x+3y+2z&=21\\4x+5y+3z&=5\end{aligned}\right\}\Rightarrow\begin{bmatrix}5&2&4\\7&3&2\\4&5&3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}10\\21\\5\end{bmatrix}$$

$$\Rightarrow \underline{A}\,\underline{x}=\underline{b}$$

$$\Rightarrow \underline{A}^{-1}\underline{A}\,\underline{x}=\underline{A}^{-1}\underline{b}$$

$$\Rightarrow \underline{x}=\tfrac{1}{61}\underline{B}\,\underline{b}$$

$$\Rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix}=\frac{1}{61}\begin{pmatrix}-1&14&-8\\-13&-1&18\\23&-17&1\end{pmatrix}\begin{pmatrix}10\\21\\5\end{pmatrix}$$

TIDYING UP WE OBTAIN

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\frac{1}{61}\begin{bmatrix}-10+294-40\\-130-21+90\\230-357+5\end{bmatrix}=\frac{1}{61}\begin{bmatrix}244\\-61\\-122\end{bmatrix}$$

$$\therefore\ x=4,\ y=-1,\ z=-2$$

Question 73 (***+)

The 3×3 matrices $\mathbf{A}$ and $\mathbf{B}$, are defined in terms of the scalar constants x as follows.

$$\mathbf{A} = \begin{pmatrix} x^2 & x & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} x & 0 & 2 \\ 0 & x & 9 \\ 0 & 1 & x \end{pmatrix}.$$

a) Find an expression for $\mathbf{AB}$, in terms of x.

b) By considering the properties of the determinants, or otherwise, find $\det(\mathbf{AB})$ in fully factorized form.

$$\boxed{\mathbf{AB} = \begin{pmatrix} x^3 & x^2+1 & 2x^2+10x \\ x & x+1 & x+11 \\ 4x & 2x+1 & x+26 \end{pmatrix}}, \quad \boxed{-x(x-1)(x-2)(x-3)(x+3)}$$

a)

$$\underline{A}\,\underline{B} = \begin{bmatrix} x^2 & x & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix}\begin{bmatrix} x & 0 & 2 \\ 0 & x & 9 \\ 0 & 1 & x \end{bmatrix}$$

$$= \begin{bmatrix} x^3+0+0 & 0+x^2+1 & 2x^2+9x+x \\ x+0+0 & 0+x+1 & 2+9+x \\ 4x+0+0 & 0+2x+1 & 8+18+x \end{bmatrix}$$

$$= \begin{bmatrix} x^3 & x^2 & 2x^2+10x \\ x & x+1 & x+11 \\ 4x & 2x+1 & x+26 \end{bmatrix}$$

b)

$$|\underline{A}| = \begin{vmatrix} x^2 & x & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{vmatrix} \begin{matrix} C_{32}(-1) \\ C_{31}(-1) \end{matrix} = \begin{vmatrix} x^2-1 & x-1 & 1 \\ 0 & 0 & 1 \\ 3 & 1 & 1 \end{vmatrix}$$

EXPAND BY THE 2ND ROW

$$= -\begin{vmatrix} x^2-1 & x-1 \\ 3 & 1 \end{vmatrix} = -\left[x^2-1-3x+3\right] = -(x^2-3x+2)$$

$$= \underline{-(x-2)(x-1)}$$

$$|\underline{B}| = \begin{vmatrix} x & 0 & 2 \\ 0 & x & 9 \\ 0 & 1 & x \end{vmatrix} = \text{EXPAND BY THE FIRST ROW} = x\begin{vmatrix} x & 9 \\ 1 & x \end{vmatrix}$$

$$= x(x^2-9) = \underline{x(x-3)(x+3)}$$

USING THE PROPERTY $|\underline{A}\,\underline{B}| = |\underline{A}||\underline{B}|$

$$|\underline{A}\,\underline{B}| = -x(x-1)(x-2)(x-3)(x+3)$$

Question 74 (*+)**

Factorize fully the following 3×3 determinant.

$$\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ yz & zx & xy \end{vmatrix}.$$

$$\boxed{(x-y)(y-z)(z-x)}$$

$$\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ yz & zx & xy \end{vmatrix} \xrightarrow[C_{13}(-1)]{C_{12}(-1)} \begin{vmatrix} 1 & 0 & 0 \\ x & y-x & z-x \\ yz & zx-yz & xy-yz \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ x & y-x & z-x \\ yz & z(x-y) & y(x-z) \end{vmatrix}$$

$$= (x-y)(z-x)\begin{vmatrix} 1 & 0 & 0 \\ x & -1 & 1 \\ yz & z & -y \end{vmatrix} = (x-y)(z-x)\begin{vmatrix} -1 & 1 \\ z & -y \end{vmatrix} =$$

$$= (x-y)(z-x)(y-z)$$

Question 75 (***+)

The 3×3 matrix $\mathbf{A}$ is given below.

$$\mathbf{A}=\begin{pmatrix}4 & -1 & 1\\ -1 & 6 & -1\\ 1 & -1 & 4\end{pmatrix}.$$

a) Show that $\lambda=7$ is an eigenvalue of $\mathbf{A}$ and find the other two eigenvalues.

b) Find the eigenvector associated with the eigenvalue $\lambda=7$.

The other two eigenvectors of $\mathbf{A}$ are

$$\mathbf{u}=\mathbf{i}-\mathbf{k} \quad \text{and} \quad \mathbf{v}=\mathbf{i}+\mathbf{j}+\mathbf{k},$$

where the eigenvalue of $\mathbf{v}$ is greater than the eigenvalue of $\mathbf{u}$.

c) Find a 3×3 matrix $\mathbf{P}$ and a 3×3 diagonal matrix $\mathbf{D}$ such that $\mathbf{D}=\mathbf{P}^{\mathrm{T}}\mathbf{AP}$.

d) Show that $\mathbf{P}$ is an orthogonal matrix.

$$\boxed{\lambda=4,3}, \quad \boxed{\alpha\begin{pmatrix}1\\-2\\1\end{pmatrix}}, \quad \boxed{\mathbf{P}=\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}}\\ 0 & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}}\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}}\end{pmatrix}}, \quad \boxed{\mathbf{D}=\begin{pmatrix}3 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 7\end{pmatrix}}$$

(a) $A=\begin{pmatrix}4 & -1 & 1\\ -1 & 6 & -1\\ 1 & -1 & 4\end{pmatrix}$

$\begin{vmatrix}4-\lambda & -1 & 1\\ -1 & 6-\lambda & -1\\ 1 & -1 & 4-\lambda\end{vmatrix}=0$

$r_{32}(1)$ $\begin{vmatrix}4-\lambda & -1 & 1\\ 0 & 5-\lambda & 3-\lambda\\ 1 & -1 & 4-\lambda\end{vmatrix}=0$

$C_{23}(1)$ $\begin{vmatrix}4-\lambda & -1 & 0\\ 0 & 5-\lambda & 8-2\lambda\\ 1 & -1 & 3-\lambda\end{vmatrix}=0$

EXPAND BY FIRST ROW

$(4-\lambda)\begin{vmatrix}5-\lambda & 8-2\lambda\\ -1 & 3-\lambda\end{vmatrix}+1\begin{vmatrix}0 & 8-2\lambda\\ 1 & 3-\lambda\end{vmatrix}=0$

$(4-\lambda)\begin{vmatrix}\lambda-5 & 2\lambda-8\\ 1 & \lambda-3\end{vmatrix}-\begin{vmatrix}0 & 2\lambda-8\\ 1 & \lambda-3\end{vmatrix}=0$

$(4-\lambda)\left[(\lambda-5)(\lambda-3)-(2\lambda-8)\right]+(2\lambda-8)=0$

$(4-\lambda)(\lambda^2-8\lambda+15-2\lambda+8)+2(\lambda-4)=0$

$(4-\lambda)(\lambda^2-10\lambda+23)-2(4-\lambda)=0$

$(4-\lambda)(\lambda^2-10\lambda+23-2)$

$(4-\lambda)(\lambda^2-10\lambda+21)=0$

$(4-\lambda)(\lambda-3)(\lambda-7)=0$

$\lambda = 4, 3, 7$

(b) IF $\lambda=7$

$\left.\begin{array}{l}4x-y+z=7x\\ -x+6y-z=7y\\ x-y+4z=7z\end{array}\right\}$ $\begin{array}{l}-3x-y+z=0\\ -x-y-z=0\\ x-y-3z=0\end{array}$ ADD $-2y-4x=0$ $y=-2x$; ADD $-2y-4z=0$ $y=-2z$

THUS $x=-\frac{1}{2}y$, $z=-\frac{1}{2}y$ $\therefore$ EIGENVECTOR $\alpha\begin{pmatrix}1\\-2\\1\end{pmatrix}$

(c) NORMALIZE EIGENVECTORS

TO $\begin{pmatrix}\frac{1}{\sqrt{2}}\\0\\-\frac{1}{\sqrt{2}}\end{pmatrix}$ $\begin{pmatrix}\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\end{pmatrix}$ $\begin{pmatrix}\frac{1}{\sqrt{6}}\\-\frac{2}{\sqrt{6}}\\\frac{1}{\sqrt{6}}\end{pmatrix}$

$\lambda=3 \quad \lambda=4 \quad \lambda=7$

$\therefore P=\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}}\\ 0 & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}}\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}}\end{pmatrix}$ & $D=\begin{pmatrix}3 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 7\end{pmatrix}$

(d) $PP^{\mathrm{T}}=\frac{1}{\sqrt{6}}\begin{pmatrix}\sqrt{3} & \sqrt{2} & 1\\ 0 & \sqrt{2} & -2\\ -\sqrt{3} & \sqrt{2} & 1\end{pmatrix}\frac{1}{\sqrt{6}}\begin{pmatrix}\sqrt{3} & 0 & -\sqrt{3}\\ \sqrt{2} & \sqrt{2} & \sqrt{2}\\ 1 & -2 & 1\end{pmatrix}=\frac{1}{6}\begin{pmatrix}3+2+1 & \cancel{0+2-2} & \cancel{-3+2+1}\\ \cancel{0+2-2} & 0+2+4 & \cancel{0+2-2}\\ \cancel{-3+2+1} & \cancel{0+2-2} & 3+2+1\end{pmatrix}=I$

$\therefore$ P INDEED ORTHOGONAL

Question 76 (***+)

$$\begin{aligned} x + y - 2z &= 2 \\ 3x - y + 6z &= 2 \\ 6x + 5y - 9z &= k \end{aligned}$$

a) Show that the system of equations does not have a unique solution.

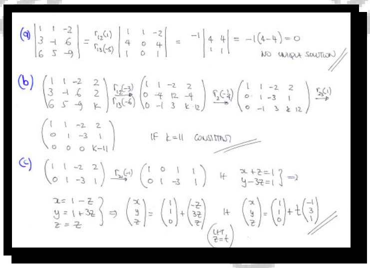
b) Show that there exists a value of k for which the system is consistent.

c) Show, by reducing the system into row echelon form, that the consistent solution of the system can be written as

$$x = 1 - t, \quad y = 3t + 1, \quad z = t$$

where t is a scalar parameter.

$\boxed{k = 11}$

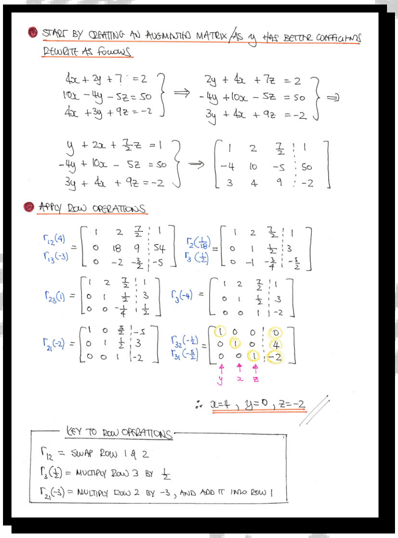


Question 77 (***+)

$$\begin{aligned} 4x \quad + 2y + 7z &= \quad 2 \\ 10x - 4y - 5z &= \ 50 \\ 4x \quad + 3y + 9z &= -2 \end{aligned}$$

Solve the above system of simultaneous equations by manipulating their augmented matrix into reduced row echelon form.

$\boxed{}$, $\boxed{x = 4, \ \ y = 0, \ \ z = -2}$



START BY CREATING AN AUGMENTED MATRIX/AS y HAS BETTER COEFFICIENTS REWRITE AS FOLLOWS

$$\left.\begin{aligned} 4x + 2y + 7z &= 2 \\ 10x - 4y - 5z &= 50 \\ 4x + 3y + 9z &= -2 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} 2y + 4x + 7z &= 2 \\ -4y + 10x - 5z &= 50 \\ 3y + 4x + 9z &= -2 \end{aligned}\right\} \Rightarrow$$

$$\left.\begin{aligned} y + 2x + \tfrac{7}{2}z &= 1 \\ -4y + 10x - 5z &= 50 \\ 3y + 4x + 9z &= -2 \end{aligned}\right\} \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & \frac{7}{2} & 1 \\ -4 & 10 & -5 & 50 \\ 3 & 4 & 9 & -2 \end{array}\right]$$

APPLY ROW OPERATIONS

$$\begin{matrix} r_{12}(4) \\ r_{13}(-3) \end{matrix} = \left[\begin{array}{ccc|c} 1 & 2 & \frac{7}{2} & 1 \\ 0 & 18 & 9 & 54 \\ 0 & -2 & -\frac{3}{2} & -5 \end{array}\right] \quad \begin{matrix} r_2(\frac{1}{18}) \\ r_3(\frac{1}{2}) \end{matrix} = \left[\begin{array}{ccc|c} 1 & 2 & \frac{7}{2} & 1 \\ 0 & 1 & \frac{1}{2} & 3 \\ 0 & -1 & -\frac{3}{4} & -\frac{5}{2} \end{array}\right]$$

$$r_{23}(1) = \left[\begin{array}{ccc|c} 1 & 2 & \frac{7}{2} & 1 \\ 0 & 1 & \frac{1}{2} & 3 \\ 0 & 0 & -\frac{1}{4} & \frac{1}{2} \end{array}\right] \quad r_3(-4) = \left[\begin{array}{ccc|c} 1 & 2 & \frac{7}{2} & 1 \\ 0 & 1 & \frac{1}{2} & 3 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$r_{21}(-2) = \left[\begin{array}{ccc|c} 1 & 0 & \frac{5}{2} & -5 \\ 0 & 1 & \frac{1}{2} & 3 \\ 0 & 0 & 1 & -2 \end{array}\right] \quad \begin{matrix} r_{32}(-\frac{1}{2}) \\ r_{31}(-\frac{5}{2}) \end{matrix} = \begin{array}{c} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right] \\ \begin{array}{cccc} \uparrow & \uparrow & \uparrow & \\ y & x & z & \end{array} \end{array}$$

$\therefore \ x = 4, \ y = 0, \ z = -2$

KEY TO ROW OPERATIONS

r_{12} = SWAP ROW 1 & 2

$r_3(\frac{1}{2})$ = MULTIPLY ROW 3 BY $\frac{1}{2}$

$r_{21}(-3)$ = MULTIPLY ROW 2 BY -3, AND ADD IT INTO ROW 1

Question 78 **(***+)**

Factorize fully the following 3×3 determinant.

$$\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix}.$$

$$\boxed{(a-b)(b-c)(c-a)(a+b+c)}$$

$$\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix} \xrightarrow[C_{13}(-1)]{C_{12}(-1)} \begin{vmatrix} 1 & 0 & 0 \\ a^2 & b^2-a^2 & c^2-a^2 \\ bc & ca-bc & ab-bc \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ a^2 & (b-a)(b+a) & (c-a)(c+a) \\ bc & c(a-b) & b(a-c) \end{vmatrix} = (a-b)(c-a)\begin{vmatrix} 1 & 0 & 0 \\ a^2 & -b-a & c+a \\ bc & c & -b \end{vmatrix}$$

$$= (a-b)(c-a)\begin{vmatrix} -b-a & c+a \\ c & -b \end{vmatrix} = (a-b)(c-a)\left[b^2+ab-c^2-ac\right] = (a-b)(c-a)\left[b^2+ab-c^2-ac\right]$$

$$= (a-b)(c-a)\left[(b+c)(b-c)+a(b-c)\right] = (a-b)(c-a)\left[(b-c)(b+c+a)\right]$$

$$= (a-b)(b-c)(c-a)(a+b+c)$$

Question 79 (***+)

The 3×3 matrices $\mathbf{A}$ and $\mathbf{B}$ are given below.

$$\mathbf{A}=\begin{pmatrix}2 & 2 & -4\\ 0 & 6 & -2\\ 1 & 0 & -3\end{pmatrix} \quad \text{and} \quad \mathbf{B}=\begin{pmatrix}9 & -3 & -10\\ 1 & 1 & -2\\ 3 & -1 & -6\end{pmatrix}.$$

a) Find the matrix composition $\mathbf{AB}$.

The point P has been transformed by $\mathbf{A}$ into the point $Q(30,18,20)$.

b) Determine the coordinates of P.

$$\boxed{\mathbf{AB}=\begin{pmatrix}8 & 0 & 0\\ 0 & 8 & 0\\ 0 & 0 & 8\end{pmatrix}}, \quad \boxed{P(2,1,-6)}$$

(a) $AB=\begin{pmatrix}2 & 2 & -4\\ 0 & 6 & -2\\ 1 & 0 & -3\end{pmatrix}\begin{pmatrix}9 & -3 & -10\\ 1 & 1 & -2\\ 3 & -1 & -6\end{pmatrix}=\begin{pmatrix}18+2-12 & -6+2+4 & -20-4+24\\ 0+6-6 & 0+6+2 & 0-12+12\\ 9+0-9 & -3+0+3 & -10+0+18\end{pmatrix}$

$=\begin{pmatrix}8 & 0 & 0\\ 0 & 8 & 0\\ 0 & 0 & 8\end{pmatrix}$

(b) $AB=8I \Rightarrow \frac{1}{8}AB=I \Rightarrow A\left(\frac{1}{8}B\right)=I \therefore A^{-1}=\frac{1}{8}B$

$\Rightarrow A\underline{p}=\underline{q}$

$\Rightarrow A^{-1}A\underline{p}=A^{-1}\underline{q}$

$\Rightarrow \underline{p}=A^{-1}\underline{q}$

$\Rightarrow \underline{p}=\frac{1}{8}B\underline{q}$

$\Rightarrow \underline{p}=\frac{1}{8}\begin{pmatrix}9 & -3 & -10\\ 1 & 1 & -2\\ 3 & -1 & -6\end{pmatrix}\begin{pmatrix}30\\ 18\\ 20\end{pmatrix}=\frac{1}{8}\begin{pmatrix}270-54-300\\ 30+18-40\\ 90-18-120\end{pmatrix}$

$\Rightarrow \underline{p}=\begin{pmatrix}2\\ 1\\ -6\end{pmatrix}$

$\therefore P(2,1,-6)$

Question 80 **(***+)**

The 2×2 matrix $\mathbf{A}$ maps $\mathbb{R}^2 \mapsto \mathbb{R}^2$ and is given by

$$\mathbf{A} = \begin{pmatrix} 4 & 3 \\ -3 & -2 \end{pmatrix}.$$

a) Determine an equation of an invariant straight line under $\mathbf{A}$.

b) Find an equation of a straight line of invariant points under $\mathbf{A}$.

$\boxed{y = -x + c}$, $\boxed{y = -x}$

a) $A = \begin{bmatrix} 4 & 3 \\ -3 & -2 \end{bmatrix}$ I.E $\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ -3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

LOOKING FOR INVARIANT LINES IE $y = mx + c$, GOES ONTO $Y = mX + c$

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ -3 & -2 \end{pmatrix}\begin{pmatrix} x \\ mx+c \end{pmatrix} = \begin{pmatrix} 4x + 3mx + 3c \\ -3x - 2mx - 2c \end{pmatrix} = \begin{pmatrix} (3m+4)x + 3c \\ (-2m-3)x - 2c \end{pmatrix}$$

BUT $Y = mX + c$

$\Rightarrow (-2m-3)x - 2c = m\left[(3m+4)x + 3c\right] + c$

$\Rightarrow (-2m-3)x - 2c = (3m^2+4m)x + (3mc + c)$

THUS

$-2m - 3 = 3m^2 + 4m$

$0 = 3m^2 + 6m + 3$

$0 = m^2 + 2m + 1$

$0 = (m-1)^2$

$m = -1$

& IF $m = -1$

$-2c = 3mc + c$

$-2c = -3c + c$

$-2c = -2c$

$\therefore c =$ ARBITRARY

$\therefore$ INVARIANT LINES

$y = -x + c$

c ARBITRARY

b) INVARIANT LINE OF POINTS $\Rightarrow (x, y)$ GETS MAPPED ONTO (x, y)

$$\begin{pmatrix} 4 & 3 \\ -3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$\left.\begin{array}{l} 4x + 3y = x \\ -3x - 2y = y \end{array}\right\} \Rightarrow \begin{array}{l} 3y = -3x \\ -3x = 3y \end{array} \Rightarrow y = -x$

LINE OF INVARIANT POINTS

Question 81 **(***+)**

The 2×2 matrix $\mathbf{A}$ is defined below in terms of the scalar constants p, q and r.

$$\mathbf{A}=\begin{pmatrix} 4 & p \\ q & r \end{pmatrix}.$$

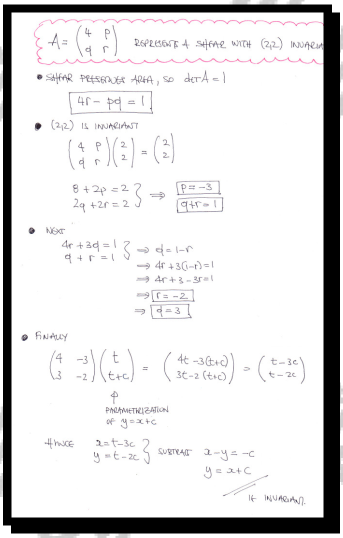
It is further given that $\mathbf{A}$ represents a shear under which the point $(2,2)$ is invariant.

Show that all straight lines of the form

$$y=x+c,$$

where c is a constant, are invariant under the shear represented by $\mathbf{A}$.

proof



$A=\begin{pmatrix} 4 & p \\ q & r \end{pmatrix}$ REPRESENTS A SHEAR WITH $(2,2)$ INVARIANT

- SHEAR PRESERVES AREA, SO $\det A = 1$

$\boxed{4r - pq = 1}$

- $(2,2)$ IS INVARIANT

$\begin{pmatrix} 4 & p \\ q & r \end{pmatrix}\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

$\left.\begin{array}{l} 8+2p=2 \\ 2q+2r=2 \end{array}\right\} \Rightarrow \boxed{p=-3}$, $\boxed{q+r=1}$

- NEXT

$\left.\begin{array}{l} 4r+3q=1 \\ q+r=1 \end{array}\right\} \Rightarrow q=1-r$

$\Rightarrow 4r+3(1-r)=1$

$\Rightarrow 4r+3-3r=1$

$\Rightarrow \boxed{r=-2}$

$\Rightarrow \boxed{q=3}$

- FINALLY

$\begin{pmatrix} 4 & -3 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} t \\ t+c \end{pmatrix} = \begin{pmatrix} 4t-3(t+c) \\ 3t-2(t+c) \end{pmatrix} = \begin{pmatrix} t-3c \\ t-2c \end{pmatrix}$

PARAMETRIZATION OF $y=x+c$

HENCE $\left.\begin{array}{l} x=t-3c \\ y=t-2c \end{array}\right\}$ SUBTRACT $x-y=-c$

$y=x+c$

IS INVARIANT.

Question 82 (***+)

The 3×3 matrix $\mathbf{A}$ is given below.

$$\mathbf{A} = \begin{pmatrix} -1 & k & 0 \\ k & 0 & 2 \\ 0 & 2 & 1 \end{pmatrix}.$$

a) If one of the eigenvalues of $\mathbf{A}$ is 3, find the possible values of k.

b) Determine the other two eigenvalues of $\mathbf{A}$, given that $k>0$.

c) Find an eigenvector corresponding to the eigenvalue 3.

$\boxed{}$, $\boxed{k=\pm2}$, $\boxed{\lambda=0,\ \ \lambda=-3}$, $\boxed{\mathbf{v}=\alpha(\mathbf{i}+2\mathbf{j}+2\mathbf{k})}$

a) USING THE DEFINITION OF EIGENVALUE.

$$\begin{vmatrix} -1-3 & k & 0 \\ k & 0-3 & 2 \\ 0 & 2 & 1-3 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} -4 & k & 0 \\ k & -3 & 2 \\ 0 & 2 & -2 \end{vmatrix} = 0$$

$$\Rightarrow -4\begin{vmatrix} -3 & 2 \\ 2 & -2 \end{vmatrix} - k\begin{vmatrix} k & 2 \\ 0 & -2 \end{vmatrix} = 0$$

$$\Rightarrow -8-k(-2k)=0$$

$$\Rightarrow -8+2k^2=0$$

$$\Rightarrow k^2=4$$

$$\Rightarrow k=\pm2$$

b) CHARACTERISTIC EQUATION

$$\begin{vmatrix} -1-\lambda & 2 & 0 \\ 2 & 0-\lambda & 2 \\ 0 & 2 & 1-\lambda \end{vmatrix} = 0 \Rightarrow \cancel{(-1)^3}\begin{vmatrix} \lambda+1 & -2 & 0 \\ -2 & \lambda & -2 \\ 0 & -2 & \lambda-1 \end{vmatrix} = 0$$

$$\Rightarrow (\lambda+1)\begin{vmatrix} \lambda & -2 \\ -2 & \lambda-1 \end{vmatrix} + 2\begin{vmatrix} -2 & -2 \\ 0 & \lambda-1 \end{vmatrix} = 0$$

$$\Rightarrow (\lambda+1)(\lambda^2-\lambda-4)+2(2-2\lambda)=0$$

$$\Rightarrow (\lambda+1)(\lambda^2-\lambda-4)-4(\lambda-1)=0$$

$$\Rightarrow \left\{\begin{matrix} \lambda^3-\lambda^2-4\lambda \\ \lambda^2-\lambda-4 \\ -4\lambda+4 \end{matrix}\right\}=0$$

$$\Rightarrow \lambda^3-9\lambda=0$$

$$\Rightarrow \lambda(\lambda^2-9)=0$$

$$\Rightarrow \lambda(\lambda-3)(\lambda+3) \quad \therefore \lambda = 3,\ -3,\ 0$$

c) IF $\lambda=3$

$$\left\{\begin{matrix} -x+2y = 3x \\ 2x+2z=3y \\ 2y+z=3z \end{matrix}\right\} \Rightarrow \left\{\begin{matrix} 2y=4x \\ 2x-3y+2z=0 \\ 2y=2z \end{matrix}\right\}$$

$$\Rightarrow \left\{\begin{matrix} y=2x \\ 2x-3y+2z=0 \\ y=z \end{matrix}\right\}$$

EQUATION (II) IS SATISFIED BY THE OTHER TWO, HENCE

$$x=\tfrac{1}{2}y$$

$$z=y$$

SET $y=2$

$$\underline{v}=\begin{pmatrix}1\\2\\2\end{pmatrix}$$

Question 83 (***+)

Consider the following matrix equation

$$\begin{pmatrix} k & 1 & 0 \\ 3 & -2 & k-3 \\ 10k & 3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ 15 \end{pmatrix},$$

where a, b and k are scalar constants.

a) Find the values of k for which the equation has a unique solution.

It is further asserted that $k = 2$.

b) Express a in terms of b if the matrix equation is to be consistent.

c) Show that if $a = 1$ and $b = 4$, the solution of the matrix equation is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} t+1 \\ -2t-1 \\ 7t+1 \end{pmatrix},$$

where t is a scalar parameter.

$\boxed{\quad\quad}$, $\boxed{k \neq 2 \;\cup\; k \neq \frac{3}{7}}$, $\boxed{2a+7b=15}$

a) FOR UNIQUE SOLUTION THE DETERMINANT MUST BE NON ZERO

$$\begin{vmatrix} k & 1 & 0 \\ 3 & -2 & k-3 \\ 10k & 3 & -2 \end{vmatrix} = k\begin{vmatrix} -2 & k-3 \\ 3 & -2 \end{vmatrix} - \begin{vmatrix} 3 & k-3 \\ 10k & -2 \end{vmatrix}$$

$$= k[4-3(k-3)] - [-6-10k(k-3)]$$
$$= k(13-3k) - (30k - 10k^2 - 6)$$
$$= 13k - 3k^2 - 30k + 10k^2 + 6$$
$$= 7k^2 - 17k + 6$$
$$= (7k-3)(k-2)$$

$\therefore$ UNIQUE SOLUTION $\Longrightarrow$ $k \in \mathbb{R}$, $k \neq 2$, $k \neq \frac{3}{7}$

b) ROW REDUCING THE AUGMENTED MATRIX

$$\left[\begin{array}{ccc|c} 2 & 1 & 0 & a \\ 3 & -2 & -1 & b \\ 20 & 3 & -2 & 15 \end{array}\right] \; r_1\left(\tfrac{1}{2}\right) \; \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & 0 & \frac{1}{2}a \\ 3 & -2 & -1 & b \\ 20 & 3 & -2 & 15 \end{array}\right] \; r_{12}(-3),\, r_{13}(-20)$$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & 0 & \frac{1}{2}a \\ 0 & -\frac{7}{2} & -1 & b-\frac{3}{2}a \\ 0 & -7 & -2 & 15-10a \end{array}\right] \; r_{23}(-2) \; \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & 0 & \frac{1}{2}a \\ 0 & -\frac{7}{2} & -1 & b-\frac{3}{2}a \\ 0 & 0 & 0 & 15-10a-2b+3a \end{array}\right]$$

WE REQUIRE ZERO ROW, SO $15 - 10a - 2b + 3a = 0$

$15 - 2b = 7a$

OR $7a + 2b = 15$

c) USING PART (b) WITH $a=1$ $b=4$ & BOTTOM ZERO ROW

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & -\frac{7}{2} & -1 & \frac{5}{2} \\ 0 & 0 & 0 & 0 \end{array}\right] \; r_2\left(-\tfrac{2}{7}\right) \; \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & \frac{2}{7} & -\frac{5}{7} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$r_{21}\left(-\tfrac{1}{2}\right) \; \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{7} & \frac{6}{7} \\ 0 & 1 & \frac{2}{7} & -\frac{5}{7} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

HENCE

$$\left.\begin{array}{l} x - \frac{1}{7}z = \frac{6}{7} \\ y + \frac{2}{7}z = -\frac{5}{7} \\ z = \lambda \end{array}\right\} \Longrightarrow \begin{array}{l} x = \frac{6}{7} + \frac{1}{7}\lambda \\ y = -\frac{5}{7} - \frac{2}{7}\lambda \\ z = \lambda \end{array}$$

THUS

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{6}{7} \\ -\frac{5}{7} \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} \frac{1}{7} \\ -\frac{2}{7} \\ 1 \end{pmatrix}$$

NOW LET $\lambda = \mu + 1$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{6}{7} \\ -\frac{5}{7} \\ 0 \end{pmatrix} + (\mu+1)\begin{pmatrix} \frac{1}{7} \\ -\frac{2}{7} \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \mu\begin{pmatrix} \frac{1}{7} \\ -\frac{2}{7} \\ 1 \end{pmatrix}$$

FINALLY LET $\mu = 7t$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + 7t\begin{pmatrix} \frac{1}{7} \\ -\frac{2}{7} \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ -2 \\ 7 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1+t \\ -1-2t \\ 1+7t \end{pmatrix}$$

AS REQUIRED

Question 84 (***+)

The 3×3 matrix $\mathbf{A}$ is defined as

$$\mathbf{A} = \begin{pmatrix} 3 & a & 0 \\ 2 & b & 0 \\ c & 0 & 1 \end{pmatrix},$$

where a, b and c are scalar constants.

a) If $\mathbf{A} = \mathbf{A}^{-1}$, find the value of a, b and c,

b) Evaluate the determinant of $\mathbf{A}$.

c) Determine an equation of a plane of invariant points under the transformation described by $\mathbf{A}$.

$\boxed{a=-4}$, $\boxed{b=-3}$, $\boxed{c=0}$, $\boxed{\det \mathbf{A}=-1}$, $\boxed{\text{plane: } x=2y}$

(a) $A = A^{-1}$

$AA = A^{-1}A$

$A^2 = I$

$\therefore \begin{pmatrix} 3 & a & 0 \\ 2 & b & 0 \\ c & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & a & 0 \\ 2 & b & 0 \\ c & 0 & 1 \end{pmatrix} = \begin{pmatrix} 9+2a & 3a+ab & 0 \\ 6+2b & 2a+b^2 & 0 \\ 3c+c & ac & 1 \end{pmatrix}$

$\therefore 9+2a=1$, $2a=-8$, $a=-4$

$6+2b=0$, $2b=-6$, $b=-3$

$ac=0$, $-4c=0$, $c=0$

(b) $A = \begin{pmatrix} 3 & -4 & 0 \\ 2 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ expand by bottom row $\begin{vmatrix} 3 & -4 \\ 2 & -3 \end{vmatrix} = -9+8=-1$

(c) $\begin{pmatrix} 3 & -4 & 0 \\ 2 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$\left.\begin{array}{l} 3x-4y=x \\ 2x-3y=y \\ z=z \end{array}\right\} \Rightarrow y=\frac{1}{2}x$

PLANE

Question 85 (***+)

$$\begin{aligned} x + 5y + 7z &= 41 \\ 5x - 4y + 6z &= 2 \\ 7x + 9y - 3z &= k \end{aligned}$$

Find the solution of the system of simultaneous equations above, giving the answers in terms of the constant k.

$\boxed{}$, $\boxed{x = \dfrac{k-27}{13}, \quad y = \dfrac{k+77}{26}, \quad z = \dfrac{105-k}{26}}$

PROCEED BY THE JORDAN GAUSS ALGORITHM

$$\left[\begin{array}{ccc|c} 1 & 5 & 7 & 41 \\ 5 & -4 & 6 & 2 \\ 7 & 9 & -3 & k \end{array}\right] \overset{r_{12}(-5)}{\underset{r_{13}(-7)}{\Longrightarrow}} \left[\begin{array}{ccc|c} 1 & 5 & 7 & 41 \\ 0 & -29 & -29 & -203 \\ 0 & -26 & -52 & -287+k \end{array}\right] \overset{r_2(-\frac{1}{29})}{\Longrightarrow}$$

$$\left[\begin{array}{ccc|c} 1 & 5 & 7 & 41 \\ 0 & 1 & 1 & 7 \\ 0 & -26 & -52 & k-287 \end{array}\right] \overset{r_{23}(26)}{\Longrightarrow} \left[\begin{array}{ccc|c} 1 & 5 & 7 & 41 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & -26 & k-105 \end{array}\right] \overset{r_3(-\frac{1}{26})}{\Longrightarrow}$$

$$\left[\begin{array}{ccc|c} 1 & 5 & 7 & 41 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & 1 & \frac{105-k}{26} \end{array}\right] \overset{r_{21}(-5)}{\Longrightarrow} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 6 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & 1 & \frac{105-k}{26} \end{array}\right] \overset{r_{32}(-1)}{\underset{r_{31}(-2)}{\Longrightarrow}}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 6+\frac{k-105}{13} \\ 0 & 1 & 0 & 7+\frac{k-105}{26} \\ 0 & 0 & 1 & \frac{105-k}{26} \end{array}\right]$$

$\therefore x = 6 + \frac{k-105}{13} = \frac{6\times13+k-105}{13} = \frac{k-27}{13}$

$y = 7 + \frac{k-105}{26} = \frac{26\times7+k-105}{26} = \frac{k+77}{26}$

$z = \frac{105-k}{26}$

ALTERNATIVE WITHOUT MATRICES

$$\left.\begin{array}{l} 1)\ x+5y+7z=41 \\ 2)\ 5x-4y+6z=2 \\ 3)\ 7x+9y-3z=k \end{array}\right\} \Longrightarrow x = 41-5y-7z$$

$$\left.\begin{array}{l} 2)\ 5(41-5y-7z)-4y+6z=2 \\ 3)\ 7(41-5y-7z)+9y-3z=k \end{array}\right\} \Longrightarrow \left.\begin{array}{l} -29y-29z=-203 \\ -26y-52z=k-287 \end{array}\right\} \Longrightarrow$$

$$\left.\begin{array}{l} y+z=7 \\ 26y+52z=287-k \end{array}\right\} \Longrightarrow y = 7-z$$

$\Downarrow$

$26(7-z)+52z = 287-k$

$26z + 182 = 287-k$

$26z = 105-k$

$z = \frac{105-k}{26}$

$\therefore y = 7 - \frac{105-k}{26}$

$y = \frac{7\times26-105+k}{26}$

$y = \frac{77+k}{26}$

$\therefore x = 41 - 5\left(\frac{77+k}{26}\right) - 7\left(\frac{105-k}{26}\right)$

$x = \frac{41\times26 - 5\times77 - 5k - 7\times105 + 7k}{26}$

$x = \frac{-54+2k}{26}$

$x = \frac{k-27}{13}$

Question 86 (***+)

The matrices $\mathbf{A}$ and $\mathbf{B}$, where k is a scalar constant, are given below.

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -1 \\ 4 & k & -2 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} -k & 2 & k-4 \\ 4 & -1 & -2 \\ 0 & 0 & k-8 \end{pmatrix}.$$

a) Find $\mathbf{AB}$ in its simplest form.

b) Hence, or otherwise, find the inverse of $\mathbf{A}$ in terms of k, stating the condition for its existence.

c) Use the inverse of $\mathbf{A}$ to solve the equation $\mathbf{Ax} = \mathbf{c}$ where

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{and} \quad \mathbf{c} = \begin{pmatrix} 0 \\ 46 \\ -11 \end{pmatrix}.$$

$$\boxed{\mathbf{AB} = (8-k)\mathbf{I}}, \quad \boxed{\mathbf{A}^{-1} = \frac{1}{8-k}\mathbf{B}}, \quad \boxed{\mathbf{x} = \begin{pmatrix} 27 \\ -8 \\ 11 \end{pmatrix}}$$

(a) $AB = \begin{pmatrix} 1 & 2 & -1 \\ 4 & k & -2 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} -k & 2 & k-4 \\ 4 & -1 & -2 \\ 0 & 0 & k-8 \end{pmatrix} = \begin{bmatrix} -k+8 & 2-2 & k-4-4-k+8 \\ -4k+4k & 8+k & 4k-16-2k-2k+16 \\ 0 & 0 & 8-k \end{bmatrix}$

$= \begin{pmatrix} 8-k & 0 & 0 \\ 0 & 8-k & 0 \\ 0 & 0 & 8-k \end{pmatrix} = (8-k)I$

(b) $AB = (8-k)I$

$A\left[\frac{1}{8-k}B\right] = I \qquad \therefore A^{-1} = \frac{1}{8-k}B$

SO LONG AS $k \neq 8$

(c) $A\underline{x} = \underline{c}$ WHERE $\underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $\underline{c} = \begin{pmatrix} 0 \\ 46 \\ -11 \end{pmatrix}$ & $k=5$

$A^{-1}A\underline{x} = A^{-1}\underline{c}$

$\underline{x} = \frac{1}{8-5}\begin{bmatrix} -5 & 2 & 1 \\ 4 & -1 & -2 \\ 0 & 0 & -3 \end{bmatrix}\begin{bmatrix} 0 \\ 46 \\ -11 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 92-11 \\ -46+22 \\ 33 \end{bmatrix} = \begin{bmatrix} 27 \\ -8 \\ 11 \end{bmatrix}$

$\therefore x=27, \; y=-8, \; z=11$

Question 87 (****)

A 3×3 matrix $\mathbf{A}$ has characteristic equation

$$2\lambda^3-7\lambda^2+\lambda+10=0.$$

a) Show that $\lambda=2$ is an eigenvalue of $\mathbf{A}$ and find the other two eigenvalues.

b) Show further that

$$2\mathbf{A}^4+71\mathbf{A}^2=27\mathbf{A}^3+100\mathbf{I}.$$

An eigenvector corresponding to $\lambda=2$ is $\mathbf{u}$.

It is further given that $\mathbf{u}=\begin{pmatrix}2\\-4\\-5\end{pmatrix}$, $\mathbf{v}=\begin{pmatrix}0.4\\-0.8\\-1\end{pmatrix}$ and $\mathbf{x}=\begin{pmatrix}x\\y\\z\end{pmatrix}$.

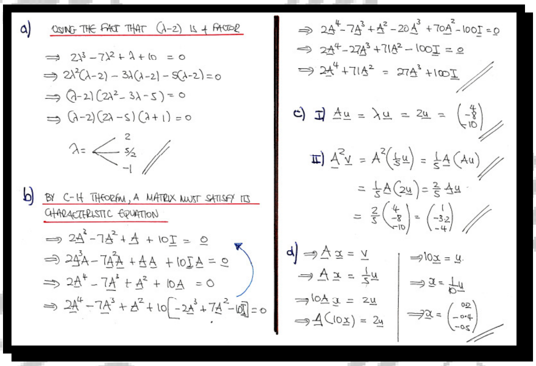
c) Evaluate each of the following expressions.

i. $\mathbf{Au}$.

ii. $\mathbf{A}^2\mathbf{v}$.

d) Solve the equation $\mathbf{Ax}=\mathbf{v}$.

$\boxed{}$, $\boxed{\lambda=-1,\ \lambda=\frac{5}{2}}$, $\boxed{\mathbf{Au}=\begin{pmatrix}4\\-8\\-10\end{pmatrix}}$, $\boxed{\mathbf{A}^2\mathbf{v}=\begin{pmatrix}1.6\\-3.2\\-4\end{pmatrix}}$, $\boxed{\mathbf{x}=\begin{pmatrix}0.2\\-0.4\\-0.5\end{pmatrix}}$


a) USING THE FACT THAT $(\lambda-2)$ IS A FACTOR

$\Rightarrow 2\lambda^3-7\lambda^2+\lambda+10=0$

$\Rightarrow 2\lambda^2(\lambda-2)-3\lambda(\lambda-2)-5(\lambda-2)=0$

$\Rightarrow (\lambda-2)(2\lambda^2-3\lambda-5)=0$

$\Rightarrow (\lambda-2)(2\lambda-5)(\lambda+1)=0$

$\lambda = 2,\ 5/2,\ -1$

b) BY C-H THEOREM, A MATRIX MUST SATISFY ITS CHARACTERISTIC EQUATION

$\Rightarrow 2A^3-7A^2+A+10I=0$

$\Rightarrow 2A^3A-7A^2A+AA+10IA=0$

$\Rightarrow 2A^4-7A^3+A^2+10A=0$

$\Rightarrow 2A^4-7A^3+A^2+10\left[-2A^3+7A^2-10I\right]=0$

$\Rightarrow 2A^4-7A^3+A^2-20A^3+70A^2-100I=0$

$\Rightarrow 2A^4-27A^3+71A^2-100I=0$

$\Rightarrow 2A^4+71A^2=27A^3+100I$

c) I) $Au=\lambda u=2u=\begin{pmatrix}4\\-8\\-10\end{pmatrix}$

II) $A^2v=A^2\left(\frac{1}{5}u\right)=\frac{1}{5}A(Au)$

$=\frac{1}{5}A(2u)=\frac{2}{5}Au$

$=\frac{2}{5}\begin{pmatrix}4\\-8\\-10\end{pmatrix}=\begin{pmatrix}1\\-3.2\\-4\end{pmatrix}$

d) $\Rightarrow Ax=v$

$\Rightarrow Ax=\frac{1}{5}u$

$\Rightarrow 10Ax=2u$

$\Rightarrow A(10x)=2u$

$\Rightarrow 10x=u$

$\Rightarrow x=\frac{1}{10}u$

$\Rightarrow x=\begin{pmatrix}0.2\\-0.4\\-0.5\end{pmatrix}$

Question 88 (****)

$$\begin{aligned} x \ -2y \ + \quad az &= 5 \\ (a+1)x \ +3y \qquad\quad &= a \\ 2x \ + \ y \ +(a-1)z &= 3 \end{aligned}$$

a) Determine the two values of the constant a for which the above system of equations does **not** have a unique solution.

b) Show clearly that the system is consistent for one of these values and inconsistent for the other.

$$\boxed{a = -1, \ \frac{5}{3}}$$

(a) $\begin{vmatrix} 1 & -2 & a \\ a+1 & 3 & 0 \\ 2 & 1 & a-1 \end{vmatrix} \ r_{13}(-2) \ \begin{vmatrix} 1 & -2 & a \\ a+1 & 3 & 0 \\ 0 & 5 & -a-1 \end{vmatrix}$ EXPAND BY 1ST COLUMN

$= 1\begin{vmatrix} 3 & 0 \\ 5 & -a-1 \end{vmatrix} - (a+1)\begin{vmatrix} -2 & a \\ 5 & -a-1 \end{vmatrix} = -3a-3-(a+1)\left[-2(-a-1)-5a\right]$

$= -3a-3-(a+1)\left[2a+2-5a\right] = -3a-3-(a+1)(2-3a)$

$= (3a-2)(a+1)-3a-3 = 3a^2+a-2-3a-3 = 3a^2-2a-5 = (3a-5)(a+1)$

HENCE IF NO UNIQUE SOLUTION

$a = -1, \ \frac{5}{3}$

(b) IF $a=-1$

$\begin{pmatrix} 1 & -2 & -1 & 5 \\ 0 & 3 & 0 & -1 \\ 2 & 1 & -2 & 3 \end{pmatrix} r_{13}(-2) \begin{pmatrix} 1 & -2 & -1 & 5 \\ 0 & 3 & 0 & -1 \\ 0 & 3 & 0 & -7 \end{pmatrix} r_{23}(-1) \begin{pmatrix} 1 & -2 & -1 & 5 \\ 0 & 3 & 0 & -1 \\ 0 & 0 & 0 & -6 \end{pmatrix}$

INCONSISTENT

IF $a=\frac{5}{3}$

$\begin{pmatrix} 1 & -2 & \frac{5}{3} & 5 \\ \frac{8}{3} & 3 & 0 & \frac{5}{3} \\ 2 & 1 & \frac{2}{3} & 3 \end{pmatrix} \begin{matrix} r_{12}(-\frac{8}{3}) \\ r_{13}(-2) \end{matrix} \begin{pmatrix} 1 & -2 & \frac{5}{3} & 5 \\ 0 & \frac{25}{3} & -\frac{40}{9} & -\frac{35}{3} \\ 0 & 5 & -\frac{8}{3} & -7 \end{pmatrix}$

THESE ARE MULTIPLES OF EACH OTHER, SO "ZERO ROW"

$\therefore$ CONSISTENT

Question 89 (****)

Find in Cartesian form the image of the straight line with equation

$$\frac{x-2}{3}=\frac{y+2}{4}=\frac{1-z}{2},$$

under the transformation represented by the 3×3 matrix $\mathbf{A}$, shown below.

$$\mathbf{A}=\begin{pmatrix}1 & 0 & 1\\ 2 & 1 & 1\\ 0 & 0 & 1\end{pmatrix}.$$

$\boxed{}$, $\boxed{x-3=\frac{y-3}{8}=\frac{1-z}{2}}$

START BY PARAMETRIZING THE LINE FROM CARTESIAN

$$\frac{x-2}{3}=\frac{y+2}{4}=\frac{1-z}{2}=t$$

$$x=3t+2$$
$$y=4t-2$$
$$z=1-2t$$

APPLY THE TRANSFORMATION IN GENERAL FORM

$$\begin{bmatrix}X\\Y\\Z\end{bmatrix}=\begin{bmatrix}1&0&1\\2&1&1\\0&0&1\end{bmatrix}\begin{bmatrix}3t+2\\4t-2\\1-2t\end{bmatrix}=\begin{bmatrix}3t+2+1-2t\\6t+4+4t-2+1-2t\\1-2t\end{bmatrix}=\begin{bmatrix}t+3\\8t+3\\1-2t\end{bmatrix}$$

ELIMINATE THE PARAMETER

$$t=X-3$$
$$t=\frac{Y-3}{8}$$
$$t=\frac{1-Z}{2}$$

$$\therefore\ X-3=\frac{Y-3}{8}=\frac{1-Z}{2}$$

OR

$$x-3=\frac{y-3}{8}=\frac{1-z}{2}$$

Question 90 (****)

The 3×3 matrix $\mathbf{C}$ represents a geometric transformation $T:\mathbb{R}^3 \mapsto \mathbb{R}^3$.

$$\mathbf{C}=\begin{pmatrix} 3 & -1 & 1 \\ -1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}.$$

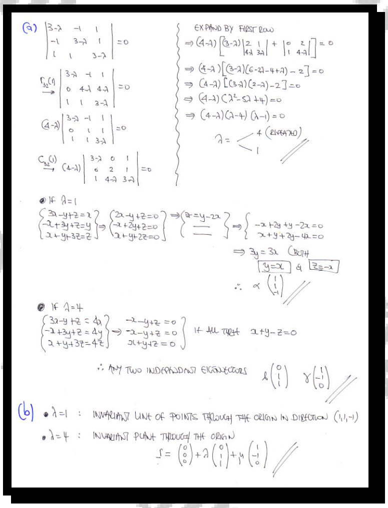
a) Find the eigenvalues and the corresponding eigenvectors of $\mathbf{C}$.

b) Describe the geometrical significance of the eigenvectors of $\mathbf{C}$ in relation to T.

$\boxed{\lambda=1,\ \ \alpha\begin{pmatrix}1\\1\\-1\end{pmatrix}}$, $\boxed{\lambda=4,\ \ \beta\begin{pmatrix}0\\1\\1\end{pmatrix},\ \ \gamma\begin{pmatrix}1\\-1\\0\end{pmatrix}}$,

$\boxed{\lambda=1 \Leftrightarrow \text{invariant line of points through the origin}}$,

$\boxed{\lambda=4 \Leftrightarrow \text{invariant plane through the origin}}$



(a) $\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 3-\lambda & 1 \\ 1 & 1 & 3-\lambda \end{vmatrix}=0$

$\xrightarrow{r_{32}(1)} \begin{vmatrix} 3-\lambda & -1 & 1 \\ 0 & 4-\lambda & 4-\lambda \\ 1 & 1 & 3-\lambda \end{vmatrix}=0$

$(4-\lambda)\begin{vmatrix} 3-\lambda & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 3-\lambda \end{vmatrix}=0$

$\xrightarrow{C_{32}(1)} (4-\lambda)\begin{vmatrix} 3-\lambda & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 4-\lambda & 3-\lambda \end{vmatrix}=0$

EXPAND BY FIRST ROW

$\Rightarrow (4-\lambda)\left[(3-\lambda)\begin{vmatrix} 2 & 1 \\ 4-\lambda & 3-\lambda \end{vmatrix} + \begin{vmatrix} 0 & 2 \\ 1 & 4-\lambda \end{vmatrix}\right]=0$

$\Rightarrow (4-\lambda)\left[(3-\lambda)(6-2\lambda-4+\lambda)-2\right]=0$

$\Rightarrow (4-\lambda)\left[(3-\lambda)(2-\lambda)-2\right]=0$

$\Rightarrow (4-\lambda)(\lambda^2-5\lambda+4)=0$

$\Rightarrow (4-\lambda)(\lambda-4)(\lambda-1)=0$

$\lambda = \begin{cases} 4 \text{ (REPEATED)} \\ 1 \end{cases}$

• IF $\lambda=1$

$\left\{\begin{array}{l} 3x-y+z=x \\ -x+3y+z=y \\ x+y+3z=z \end{array}\right\} \Rightarrow \left\{\begin{array}{l} 2x-y+z=0 \\ -x+2y+z=0 \\ x+y+2z=0 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} z=y-2x \\ \text{---} \\ \text{---} \end{array}\right\} \Rightarrow \left\{\begin{array}{l} -x+2y+y-2x=0 \\ x+y+2y-4x=0 \end{array}\right.$

$\Rightarrow 3y=3x$ (BOTH

$\boxed{y=x}$ & $\boxed{z=-x}$

$\therefore\ \alpha\begin{pmatrix}1\\1\\-1\end{pmatrix}$

• IF $\lambda=4$

$\left\{\begin{array}{l} 3x-y+z=4x \\ -x+3y+z=4y \\ x+y+3z=4z \end{array}\right\} \Rightarrow \left.\begin{array}{l} -x-y+z=0 \\ -x-y+z=0 \\ x+y+z=0 \end{array}\right\}$ IF ALL THAT $x+y-z=0$

$\therefore$ ANY TWO INDEPENDENT EIGENVECTORS $\beta\begin{pmatrix}0\\1\\1\end{pmatrix}$ $\gamma\begin{pmatrix}1\\-1\\0\end{pmatrix}$

(b) • $\lambda=1$: INVARIANT LINE OF POINTS THROUGH THE ORIGIN IN DIRECTION $(1,1,-1)$

• $\lambda=4$: INVARIANT PLANE THROUGH THE ORIGIN

$\mathbf{r}=\begin{pmatrix}0\\0\\0\end{pmatrix}+\lambda\begin{pmatrix}0\\1\\1\end{pmatrix}+\mu\begin{pmatrix}1\\-1\\0\end{pmatrix}$

Question 91 (****)

A 3×3 determinant, Δ, is given below.

$$\Delta = \begin{vmatrix} n(n+1) & n+1 & -1 \\ 0 & 1 & n \\ 1 & -n-1 & 1 \end{vmatrix}.$$

a) Show that

$$\Delta = \left(an^2 + bn + c\right)^2,$$

where a, b and c are constants.

b) Show further that

$$\Delta = \left[n(n+1)\right]^2 + n^2 + (n+1)^2.$$

c) Hence or otherwise express 24649 as the sum of three square numbers.

$\boxed{\Delta = \left(n^2+n+1\right)^2}$, $\boxed{24649 = 156^2 + 13^2 + 12^2}$

(a) $\Delta = \begin{vmatrix} n(n+1) & n+1 & -1 \\ 0 & 1 & n \\ 1 & -n-1 & 1 \end{vmatrix}$ EXPAND BY FIRST COLUMN

$\Delta = n(n+1)\left[1 + n(n+1)\right] + 1\left[n(n+1)+1\right] = \left[n(n+1)+1\right]\left[n(n+1)+1\right]$

$= \left[n(n+1)+1\right]^2 = \left[n^2+n+1\right]^2$

(b) NOW

$\Delta = (n^2+n+1)^2 = \left[n(n+1)+1\right]^2 = n^2(n+1)^2 + 2n(n+1) + 1$

$= n^2(n+1)^2 + 2n^2 + 2n + 1 = \left[n(n+1)\right]^2 + n^2 + n^2 + 2n + 1$

$= \left[n(n+1)\right]^2 + n^2 + (n+1)^2$

(c) FINALLY $24649 = 157 = 12^2 + 12 + 1$

$(n^2+n+1)^2 \equiv \left[n(n+1)\right]^2 + n^2 + (n+1)^2$

$(12^2+12+1)^2 \equiv (12\times 13)^2 + 12^2 + 13^2$

$157^2 \equiv 156^2 + 12^2 + 13^2$

Question 92 (****)

$$
\begin{aligned}
3x - y + 5z &= 5\\
2x + y - 5z &= 10\\
x + y + kz &= 7
\end{aligned}
$$

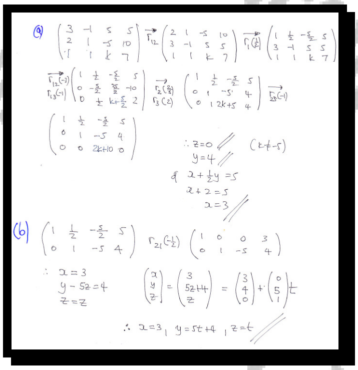
where k is a constant.

a) Given that $k \neq -5$ find the unique solution of the system of equations.

b) Given instead that $k = -5$ show, by reducing the system into row echelon form, that the consistent solution of the system can be written as

$$x = 3, \quad y = 5t + 4, \quad z = t .$$

$\boxed{x = 3, \quad y = 4, \quad z = 0}$



Question 93 (****)

The 3×3 matrix $\mathbf{A}$, where a is a scalar constant, is given below.

$$\mathbf{A} = \begin{pmatrix} a & -1 & 1 \\ 1 & 0 & -1 \\ 3 & -2 & 1 \end{pmatrix}.$$

a) Find the elements of $\mathbf{A}^{-1}$, in terms of a where appropriate.

The straight line L_1 was mapped onto another straight line L_2 by the following 3×3 matrix .

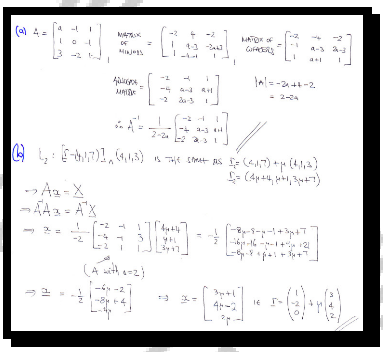
$$\begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & -1 \\ 3 & -2 & 1 \end{pmatrix}.$$

b) Given that L_2 has vector equation

$$\left[\mathbf{r} - (4\mathbf{i}+\mathbf{j}+7\mathbf{k})\right] \wedge (4\mathbf{i}+\mathbf{j}+3\mathbf{k}) = \mathbf{0}$$

find a vector equation for L_1.

$$\boxed{\mathbf{A}^{-1} = \frac{1}{2-2a}\begin{pmatrix} -2 & -1 & 1 \\ -4 & a-3 & a+1 \\ -2 & 2a-3 & 1 \end{pmatrix}}, \quad \boxed{\mathbf{r} = \mathbf{i} - 2\mathbf{j} + \lambda(3\mathbf{i}+4\mathbf{j}+2\mathbf{k})}$$



Question 94 (****)

The 2×2 matrix $\mathbf{D}$ shown below, represents a linear transformation in the x-y plane.

$$\mathbf{D} = \begin{pmatrix} -2 & -1 \\ 0 & 1 \end{pmatrix}.$$

The straight line with equation $y = mx$ is rotated by 90° about the origin under the transformation represented by $\mathbf{D}$.

Determine the possible values of m.

$\boxed{m = -1, \quad m = 2}$

$D = \begin{pmatrix} -2 & -1 \\ 0 & 1 \end{pmatrix}$

PARAMETERIZE THE LINES $y = mx$
$Y = -\frac{1}{m}X$

$\begin{pmatrix} -2 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} t \\ mt \end{pmatrix} = \begin{pmatrix} T \\ -\frac{1}{m}T \end{pmatrix}$

$\Rightarrow \left\{ \begin{array}{l} -2t - mt = T \\ mt = -\frac{1}{m}T \end{array} \right\}$

DIVIDE EQUATIONS SIDE BY SIDE

$\Rightarrow \dfrac{-2 - m}{m} = \dfrac{1}{-\frac{1}{m}}$

$\Rightarrow \dfrac{-2 - m}{m} = -m$

$\Rightarrow -2 - m = -m^2$

$\Rightarrow m^2 - m - 2 = 0$

$\Rightarrow (m-2)(m+1) = 0$

$\therefore m = 2, -1$ //

Question 95 (****)

The 2×2 matrix $\mathbf{C}$ is given below.

$$\mathbf{C}=\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}.$$

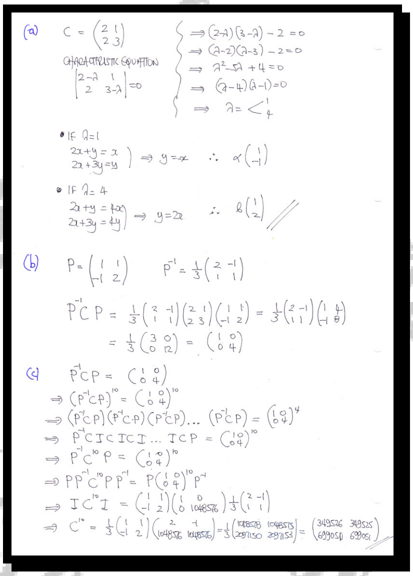
a) Find the eigenvalues of $\mathbf{C}$ and their corresponding eigenvectors.

b) Find a 2×2 matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{C}\mathbf{P}$ is a diagonal 2×2 matrix and evaluate $\mathbf{P}^{-1}\mathbf{C}\mathbf{P}$ explicitly.

c) Hence show that

$$\mathbf{C}^{7}=\begin{pmatrix} 349526 & 349525 \\ 699050 & 699051 \end{pmatrix}.$$

$\boxed{\lambda_1=1,\quad \mathbf{u}=\begin{pmatrix} 1 \\ -1 \end{pmatrix}}$, $\boxed{\lambda_2=1,\quad \mathbf{v}=\begin{pmatrix} 1 \\ 2 \end{pmatrix}}$, $\boxed{\mathbf{P}=\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}}$, $\boxed{\mathbf{D}=\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}}$



(a) $C=\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$

CHARACTERISTIC EQUATION

$\begin{vmatrix} 2-\lambda & 1 \\ 2 & 3-\lambda \end{vmatrix}=0$

$\Rightarrow (2-\lambda)(3-\lambda)-2=0$

$\Rightarrow (\lambda-2)(\lambda-3)-2=0$

$\Rightarrow \lambda^2-5\lambda+4=0$

$\Rightarrow (\lambda-4)(\lambda-1)=0$

$\Rightarrow \lambda = \begin{cases} 1 \\ 4 \end{cases}$

• IF $\lambda=1$

$\left.\begin{array}{l} 2x+y=x \\ 2x+3y=y \end{array}\right\} \Rightarrow y=-x \quad \therefore \ \alpha\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

• IF $\lambda=4$

$\left.\begin{array}{l} 2x+y=4x \\ 2x+3y=4y \end{array}\right\} \Rightarrow y=2x \quad \therefore \ \beta\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

(b) $P=\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \qquad P^{-1}=\frac{1}{3}\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$

$P^{-1}CP=\frac{1}{3}\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}=\frac{1}{3}\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ -1 & 8 \end{pmatrix}$

$=\frac{1}{3}\begin{pmatrix} 3 & 0 \\ 0 & 12 \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$

(c) $P^{-1}CP=\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$

$\Rightarrow (P^{-1}CP)^{10}=\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}^{10}$

$\Rightarrow (P^{-1}CP)(P^{-1}CP)(P^{-1}CP)\dots(P^{-1}CP)=\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}^{4}$

$\Rightarrow P^{-1}CICICI\dots ICP=\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}^{10}$

$\Rightarrow P^{-1}C^{10}P=\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}^{10}$

$\Rightarrow PP^{-1}C^{10}PP^{-1}=P\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}^{10}P^{-1}$

$\Rightarrow IC^{10}I=\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1048576 \end{pmatrix}\frac{1}{3}\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$

$\Rightarrow C^{10}=\frac{1}{3}\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1048576 & 1048576 \end{pmatrix}=\frac{1}{3}\begin{pmatrix} 1048578 & 1048575 \\ 2097150 & 2097153 \end{pmatrix}=\begin{pmatrix} 349526 & 349525 \\ 699050 & 699051 \end{pmatrix}$

Question 96 (****)

Consider the system of simultaneous equations

$$\begin{aligned} kx+ky\ \ -z &= -1 \\ ky+2z &= \ \ 2k \\ x+2y\ \ +z &= \ \ 1 \end{aligned}$$

where the constant k can **only** take the values 0, 1 and 2.

Determine for each of the possible values of k whether the system …

i. … has a unique solution

ii. … has no unique solution, but it is consistent.

iii. … is inconsistent.

$\boxed{a=0 \Rightarrow \text{incositent}}$, $\boxed{a=1 \Rightarrow \text{no unique solution/consistent}}$, $\boxed{a=2 \Rightarrow \text{unique solution}}$

(I) $\begin{pmatrix} k & k & -1 \\ 0 & k & 2 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 2k \\ 1 \end{pmatrix}$

$\begin{vmatrix} k & k & -1 \\ 0 & k & 2 \\ 1 & 2 & 1 \end{vmatrix} = k\begin{vmatrix} k & 2 \\ 2 & 1 \end{vmatrix} + 1\begin{vmatrix} k & -1 \\ k & 2 \end{vmatrix} = k(k-4)+(2k+k) = k^2-4k+3k$

(EXPAND BY 1st COLUMN)

$= k^2-k = k(k-1)$ $\quad\therefore$ IF $k=2$, UNIQUE SOLUTION

(II)(III) IF $k=0$

$\begin{pmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & 2 & 0 \\ 1 & 2 & 1 & 1 \end{pmatrix} \xrightarrow{r_{13}} \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & -1 & -1 \end{pmatrix} \xrightarrow{r_{23}} \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 2 & 0 \end{pmatrix} \xrightarrow{r_2(-1)} \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 0 \end{pmatrix}$

$\xrightarrow{r_{23}(-2)} \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -2 \end{pmatrix}$ IF $k=0$ INCOSISTENT

IF $k=1$

$\begin{pmatrix} 1 & 1 & -1 & -1 \\ 0 & 1 & 2 & 2 \\ 1 & 2 & 1 & 1 \end{pmatrix} \xrightarrow{r_{13}(-1)} \begin{pmatrix} 1 & 1 & -1 & -1 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 2 \end{pmatrix} \xrightarrow{r_{23}(-1)} \begin{pmatrix} 1 & 1 & -1 & -1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

IF $k=1$, INFINITE SOLUTION (CONSISTENT)

Question 97 (****)

The 2×2 matrix $\mathbf{A}$ is given below.

$$\mathbf{A}=\begin{pmatrix}7 & 6\\ 6 & 2\end{pmatrix}.$$

A straight line with equation $y=mx$, where m is a constant, remains invariant under the transformation represented by $\mathbf{A}$.

a) Show that

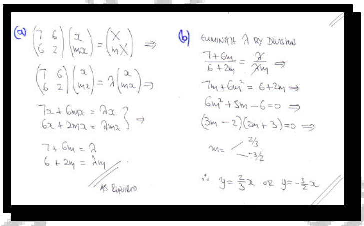
$$7+6m=\lambda$$

$$6+2m=\lambda m$$

where λ is a constant.

b) Hence find the two possible equations of this straight line.

$\boxed{y=\frac{2}{3}x}$, $\boxed{y=-\frac{3}{2}x}$



(a) $\begin{pmatrix}7 & 6\\ 6 & 2\end{pmatrix}\begin{pmatrix}x\\ mx\end{pmatrix}=\begin{pmatrix}X\\ mX\end{pmatrix} \Rightarrow$

$\begin{pmatrix}7 & 6\\ 6 & 2\end{pmatrix}\begin{pmatrix}x\\ mx\end{pmatrix}=\lambda\begin{pmatrix}x\\ mx\end{pmatrix} \Rightarrow$

$\left.\begin{array}{l}7x+6mx=\lambda x\\ 6x+2mx=\lambda mx\end{array}\right\} \Rightarrow$

$7+6m=\lambda$

$6+2m=\lambda m$

AS REQUIRED

(b) ELIMINATE λ BY DIVISION

$\frac{7+6m}{6+2m}=\frac{x}{xm} \Rightarrow$

$7m+6m^2=6+2m \Rightarrow$

$6m^2+5m-6=0 \Rightarrow$

$(3m-2)(2m+3)=0 \Rightarrow$

$m=\frac{2}{3}$ or $-\frac{3}{2}$

$\therefore\ y=\frac{2}{3}x$ or $y=-\frac{3}{2}x$

Question 98 (****)

A plane Π is defined parametrically by

$$\mathbf{r} = \mathbf{i} + \mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} - 2\mathbf{k}) + \mu(\mathbf{i} - \mathbf{j} + \mathbf{k}),$$

where λ and μ are a scalar parameters.

Determine a Cartesian equation for the transformation of Π under the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 2 & 1 & 0 \end{pmatrix}.$$

$\boxed{}$, $\boxed{4x + 3y - z = 6}$

PROCEED AS FOLLOWS

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 2 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1+\lambda+\mu \\ \lambda-\mu \\ 1-2\lambda+\mu \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1+\cancel{\lambda}+\cancel{\mu}+\cancel{\lambda}-\cancel{\mu}+1-\cancel{2\lambda}+\mu \\ \lambda-\mu \\ 2+2\lambda+2\mu+\lambda-\mu \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2+\mu \\ \lambda-\mu \\ 2+3\lambda+\mu \end{pmatrix}$$

ELIMINATE THE PARAMETRICS INTO CARTESIAN

$$\left.\begin{array}{l} x = 2+\mu \\ y = \lambda-\mu \\ z = 2+3\lambda+\mu \end{array}\right\} \Rightarrow \mu = x-2$$

SUBSTITUTE INTO THE OTHER TWO

$y = \lambda-(x-2) = 2+\lambda-x$

$z = 2+3\lambda+x-2 = 3\lambda+x$

SOLVE THE "TOP" EQUATION FOR λ

$\lambda = x+y-2$

SUBSTITUTE INTO THE LAST

$z = 2+3(x+y-2)+x-2$

$z = \cancel{2}+3x+3y-6+x-\cancel{2}$

$z = 4x+3y-6$

$4x+3y-z=6$

Question 99 (****)

The 3×3 matrix $\mathbf{C}$ is defined by

$$\mathbf{C} = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Find, in Cartesian form, the image of the plane with Cartesian equation

$$2x + y - z = 12$$

under the transformation defined by $\mathbf{C}$.

$\boxed{}$, $\boxed{3x + 4y - 5z = 12}$

START BY PARAMETRIZING THE PLANE – TAKE ANY 3 POINTS ON THE PLANE SAY $A(6,0,0)$, $B(0,12,0)$ & $C(0,0,-12)$

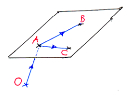
$\overrightarrow{AB} = \underline{b} - \underline{a} = (0,12,0) - (6,0,0) = (-6,12,0)$ SCALE IT TO $(-1,2,0)$

$\overrightarrow{AC} = \underline{c} - \underline{a} = (0,0,-12) - (6,0,0) = (-6,0,-12)$ SCALE IT TO $(1,0,2)$

HENCE WE HAVE

$$\underline{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + \mu \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 - \lambda + \mu \\ 2\lambda \\ 2\mu \end{bmatrix}$$



NOW TRANSFORM THE PARAMETRIZED PLANE

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 6 - \lambda + \mu \\ 2\lambda \\ 2\mu \end{bmatrix} = \begin{bmatrix} 6 - \lambda + \mu + 4\lambda \\ 6 - \lambda + \mu + 2\mu \\ 6 - \lambda + \mu + 2\lambda + 2\mu \end{bmatrix} = \begin{bmatrix} 6 + 3\lambda + \mu \\ 6 - \lambda + 3\mu \\ 6 + \lambda + 3\mu \end{bmatrix}$$

$X = 6 + 3\lambda + \mu \implies \mu = X - 6 - 3\lambda$

$Y = 6 - \lambda + 3\mu$

$Z = 6 + \lambda + 3\mu$

SUBSTITUTING INTO THE OTHER TWO EQUATIONS

THUS $Y = 6 - \lambda + 3(X - 6 - 3\lambda)$

$Z = 6 + \lambda + 3(X - 6 - 3\lambda)$

$$\left.\begin{aligned} Y &= 6 - \lambda + 3X - 18 - 9\lambda \\ Z &= 6 + \lambda + 3X - 18 - 9\lambda \end{aligned}\right\} \implies$$

$$\left.\begin{aligned} Y &= 3X - 12 - 10\lambda \\ Z &= 3X - 12 - 8\lambda \end{aligned}\right\} \implies$$

$$\left.\begin{aligned} 10\lambda &= 3X - Y - 12 \\ 8\lambda &= 3X - Z - 12 \end{aligned}\right\} \implies$$

$$\left.\begin{aligned} 40\lambda &= 12X - 4Y - 48 \\ 40\lambda &= 15X - 5Z - 60 \end{aligned}\right\} \implies$$

$\implies 12X - 4Y - 48 = 15X - 5Z - 60$

$\implies -3X - 4Y + 5Z = -12$

$\implies 3X + 4Y - 5Z = 12$

Question 100 (****)

A linear transformation T, acting in the $\boldsymbol{x}$-$\boldsymbol{y}$ plane, consists of …

- … a reflection about the line $y = -x$,

  followed by ....

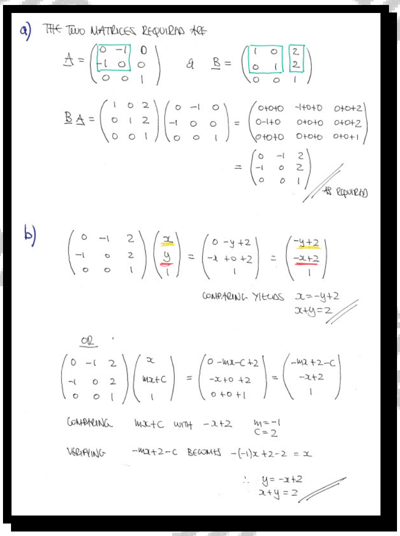
- … a translation such that $(x, y) \mapsto (x+2, y+2)$.

a) Show that the matrix that represents T is given by the matrix

$$\mathbf{T} = \begin{pmatrix} 0 & -1 & 2 \\ -1 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

b) Determine the invariant line under T.

$\boxed{}$, $\boxed{y + x = 2}$



a) THE TWO MATRICES REQUIRED ARE

$$\underline{A} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \& \quad \underline{B} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\underline{B}\,\underline{A} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0+0+0 & -1+0+0 & 0+0+2 \\ 0-1+0 & 0+0+0 & 0+0+2 \\ 0+0+0 & 0+0+0 & 0+0+1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 & 2 \\ -1 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$ // AS REQUIRED

b)

$$\begin{pmatrix} 0 & -1 & 2 \\ -1 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} 0-y+2 \\ -x+0+2 \\ 1 \end{pmatrix} = \begin{pmatrix} -y+2 \\ -x+2 \\ 1 \end{pmatrix}$$

COMPARING YIELDS $x = -y+2$

$x+y=2$ //

OR

$$\begin{pmatrix} 0 & -1 & 2 \\ -1 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ mx+c \\ 1 \end{pmatrix} = \begin{pmatrix} 0-mx-c+2 \\ -x+0+2 \\ 0+0+1 \end{pmatrix} = \begin{pmatrix} -mx+2-c \\ -x+2 \\ 1 \end{pmatrix}$$

COMPARING $mx+c$ WITH $-x+2$ $\quad m=-1$, $c=2$

VERIFYING $-mx+2-c$ BECOMES $-(-1)x+2-2 = x$

$\therefore\ y = -x+2$

$x+y=2$ //

Question 101 (****)

The matrices $\mathbf{A}$ and $\mathbf{B}$ are defined as

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & k \end{pmatrix},$$

where k is a scalar constant.

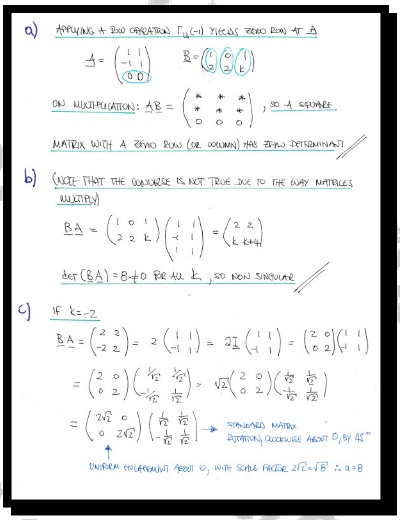
a) Without calculating $\mathbf{AB}$, show that $\mathbf{AB}$ is singular for all values of k.

b) Show that $\mathbf{BA}$ is non singular for all values of k.

When $k = -2$ the matrix $\mathbf{BA}$ represents a combination of a uniform enlargement with linear scale factor $\sqrt{a}$ and another transformation T.

c) Find the value of a and describe T geometrically.

$\boxed{}$, $\boxed{a = 8}$, $\boxed{\text{rotation about } O\text{, clockwise, by } 45°}$



a) APPLYING A ROW OPERATION $r_{13}(-1)$ YIELDS ZERO ROW AT $\underline{A}$

$\underline{A} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 0 & 0 \end{pmatrix} \qquad \underline{B} = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & k \end{pmatrix}$

ON MULTIPLICATION: $\underline{A}\underline{B} = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & 0 \end{pmatrix}$, SO A SQUARE MATRIX WITH A ZERO ROW (OR COLUMN) HAS ZERO DETERMINANT

b) (NOTE THAT THE CONVERSE IS NOT TRUE DUE TO THE WAY MATRICES MULTIPLY)

$\underline{B}\underline{A} = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & k \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ k & k+4 \end{pmatrix}$

$\det(\underline{B}\underline{A}) = 8 \neq 0$ FOR ALL k, SO NON SINGULAR

c) IF $k = -2$

$\underline{B}\underline{A} = \begin{pmatrix} 2 & 2 \\ -2 & 2 \end{pmatrix} = 2\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = 2\underline{I}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

$= \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & \frac{1}{\sqrt{2}} \end{pmatrix} = \sqrt{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

$= \begin{pmatrix} 2\sqrt{2} & 0 \\ 0 & 2\sqrt{2} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ → STANDARD MATRIX ROTATION, CLOCKWISE ABOUT O, BY 45°

UNIFORM ENLARGEMENT ABOUT O, WITH SCALE FACTOR $2\sqrt{2} = \sqrt{8}$ $\therefore a = 8$

Question 102 (****)

The equation of a plane Π is given by

$$\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix},$$

where λ and μ are parameters.

The plane Π is transformed to the plane Π' by the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}.$$

Find a Cartesian equation of Π'.

$\boxed{5x + 14y - 13z + 21 = 0}$

WRITE THE PLANE IN "PARAMETRIC COMPACT FORM"

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1+2\lambda+\mu \\ 2+\lambda \\ 1+2\mu \end{pmatrix}$$

TRANSFORM THE VECTOR VIA THE MATRIX $\mathbf{A}$

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1+2\lambda+\mu \\ 2+\lambda \\ 1+2\mu \end{pmatrix} = \begin{pmatrix} 1+2\lambda+\mu+2+4\mu \\ 1+2\lambda+\mu+2+\lambda \\ 1+2\lambda+\mu+4+2\lambda+1+2\mu \end{pmatrix}$$

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 3+2\lambda+5\mu \\ 3+3\lambda+\mu \\ 6+4\lambda+3\mu \end{pmatrix}$$

ELIMINATE THE PARAMETERS λ & μ — USE 2: $\mu = Y-3-3\lambda$ AND SUBSTITUTE INTO THE OTHER TWO

$$\left\{ \begin{array}{l} X = 3+2\lambda+5(Y-3-3\lambda) \\ Z = 6+4\lambda+3(Y-3-3\lambda) \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} X = 3+2\lambda+5Y-15-15\lambda \\ Z = 6+4\lambda+3Y-9-9\lambda \end{array} \right\} \Rightarrow$$

$$\Rightarrow \left\{ \begin{array}{l} X = -12-13\lambda+5Y \\ Z = -3-5\lambda+3Y \end{array} \right\} \begin{array}{l} \times 5 \\ \times(-13) \end{array}$$

$$\Rightarrow \left\{ \begin{array}{l} 5X = -60-65\lambda+25Y \\ -13Z = 39+65\lambda-39Y \end{array} \right\}$$

ADDING THE EQUATIONS

$$5X - 13Z = -21 - 14Y$$

$$\therefore \ 5X + 14Y - 13Z = -21$$

OR $\quad 5x + 14y - 13z + 21 = 0$

ALTERNATIVE METHOD

FIND 3 POINTS ON THE PLANE $\begin{pmatrix} 1+2\lambda+2\mu \\ 2+\lambda \\ 1+2\mu \end{pmatrix}$

$\lambda=0, \mu=0 \quad A(1,2,1)$
$\lambda=0, \mu=1 \quad B(2,2,3)$
$\lambda=1, \mu=0 \quad C(3,3,1)$

TRANSFORM THESE THREE POINTS

$$\begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix} \begin{array}{c} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \\ 1 & 3 & 1 \end{pmatrix} \\ A \quad B \quad C \end{array} = \begin{array}{c} \begin{pmatrix} 3 & 8 & 5 \\ 3 & 4 & 6 \\ 6 & 9 & 10 \end{pmatrix} \\ A' \quad B' \quad C' \end{array}$$

FIND TWO VECTORS WHICH LIE ON THE TRANSFORMED PLANE

$$\overrightarrow{A'B'} = \mathbf{b}' - \mathbf{a}' = (8,4,9) - (3,3,6) = (5,1,3)$$

$$\overrightarrow{A'C'} = \mathbf{c}' - \mathbf{a}' = (5,6,10) - (3,3,6) = (2,3,4)$$

FIND THE PERPENDICULAR (NORMAL) BY CROSS PRODUCT

$$\mathbf{n} = \overrightarrow{A'B'} \wedge \overrightarrow{A'C'} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & 1 & 3 \\ 2 & 3 & 4 \end{vmatrix} = (-5, -14, 13)$$

EQUATION OF THE TRANSFORMED PLANE IS

$$-5x - 14y + 13z = \text{constant}$$

USING ONE OF THE TRANSFORMED POINTS SAY $A'(3,3,6)$ TO FIND THE CONSTANT

$$(-5\times3) - 14\times3 + 13\times6 = 21$$

$$\therefore \ -5x - 14y + 13z = 21$$
$$5x + 14y - 13z = -21$$
$$5x + 14y - 13z + 21 = 0$$

AS BEFORE

Question 103 (****)

The Cartesian equation of a plane Π is given by

$$x + 2y + z = 2.$$

The plane Π is transformed to the plane Π' by the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ 1 & 2 & 4 \end{pmatrix}.$$

Find a Cartesian equation of Π'.

$\boxed{}$, $\boxed{x - 2y - z + 4 = 0}$

$\underline{A} = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ 1 & 2 & 4 \end{bmatrix}$ $\quad \Pi: x + 2y + z = 2$

METHOD A

- PARAMETRIZE THE PLANE "QUICKLY" AS FOLLOWS

$x = 2 - 2y - z$

LET $y = \lambda$, $z = \mu$

HENCE $\underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 - 2\lambda - \mu \\ \lambda \\ \mu \end{pmatrix}$

- TRANSFORM VIA THE MATRIX $\underline{A}$

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ 1 & 2 & 4 \end{pmatrix} \begin{pmatrix} 2 - 2\lambda - \mu \\ \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} 2 - 2\lambda - \mu + 2\mu \\ 2 - 2\lambda - \mu + \lambda \\ 2 - 2\lambda - \mu + 2\lambda + 4\mu \end{pmatrix}$$

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 2 - 2\lambda + \mu \\ 2 - \lambda - \mu \\ 2 + 3\mu \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$$

- ELIMINATE THE PARAMETERS IN THE TRANSFORMED PLANE — SOLVE THE $\underline{i}$ COMPONENT FOR μ

$$\boxed{\mu = X - 2 + 2\lambda}$$

- HENCE

$$\Longrightarrow \begin{Bmatrix} Y = 2 - \lambda - (X - 2 + 2\lambda) \\ Z = 2 + 3(X - 2 + 2\lambda) \end{Bmatrix} \Longrightarrow \begin{Bmatrix} Y = 4 - 3\lambda - X \\ Z = -4 + 6\lambda + 3X \end{Bmatrix} \begin{matrix} \times 2 \\ \times 1 \end{matrix}$$

$$\Longrightarrow \begin{Bmatrix} 2Y = 8 - 6\lambda - 2X \\ Z = -4 + 6\lambda + 3X \end{Bmatrix} \Longrightarrow \text{ADDING } \quad 2Y + Z = 4 + X$$

$$X - 2Y - Z + 4 = 0$$

METHOD B

- TAKE 3 RANDOM, SIMPLE POINTS ON THE PLANE $x + 2y + z = 2$

$A(2,0,0) \quad B(0,1,0) \quad C(0,0,2)$

- TRANSFORM THESE POINTS VIA MATRIX $\underline{A}$

$$\begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ 1 & 2 & 4 \end{pmatrix} \begin{array}{c} \begin{matrix} A & B & C \end{matrix} \\ \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \end{array} = \begin{array}{c} \begin{matrix} A' & B' & C' \end{matrix} \\ \begin{pmatrix} 2 & 0 & 4 \\ 2 & 1 & 0 \\ 2 & 2 & 8 \end{pmatrix} \end{array}$$

- FIND TWO VECTORS WHICH LIE ON THE TRANSFORMED PLANE

$\overrightarrow{A'B'} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix}$ "SCALE IT" AS $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$

$\overrightarrow{A'C'} = \begin{pmatrix} 4 \\ 0 \\ 8 \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 6 \end{pmatrix}$ "SCALE IT" AS $\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$

- FIND THE NEW NORMAL

$\overrightarrow{A'B'} \wedge \overrightarrow{A'C'} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 1 & 0 \\ 1 & -1 & 3 \end{vmatrix} = \begin{pmatrix} 3 \\ -6 \\ -3 \end{pmatrix}$ SCALE IT TO $\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$

- THE EQUATION OF THE PLANE IS

$x - 2y - z = \text{CONSTANT}$

- USE ONE OF THE THREE TRANSFORMED POINTS TO EVALUATE THE CONSTANT, SAY POINT $A'(2,2,2)$

$\therefore 2 - 2 \times 2 - 2 = \text{CONSTANT}$

$\text{CONSTANT} = -4$

$\therefore x - 2y - z = -4$

$x - 2y - z + 4 = 0$

AS BEFORE

Question 104 (****)

A linear transformation T, acting in the $\boldsymbol{x}$-$\boldsymbol{y}$ plane, consists of …

- … a reflection about the line $y=-x$,

  followed by ...

- … a translation such that $(x,y) \mapsto (x+a, y+b)$.

The transformation T is represented by the matrix $\mathbf{T}$.

a) Given the point $(1,1)$ is mapped to $(2,4)$, find the matrix $\mathbf{T}$.

b) Determine the equation of the image of the curve with equation $y=x^2$, under the transformation represented by $\mathbf{T}$.

$\boxed{}$, $\boxed{\mathbf{T}=\begin{pmatrix}0 & -1 & 3\\ -1 & 0 & 5\\ 0 & 0 & 1\end{pmatrix}}$, $\boxed{y^2-10y+x+22=0}$

a) START BY WRITING THE GENERATING MATRICES

$$\underline{A}=\begin{bmatrix}0 & -1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix} \qquad \underline{B}=\begin{bmatrix}1 & 0 & a\\ 0 & 1 & b\\ 0 & 0 & 1\end{bmatrix}$$

REFLECTION ABOUT $y=x$ TRANSLATION BY $\begin{pmatrix}a\\ b\end{pmatrix}$

OBTAIN THE MATRIX $\underline{T}$

$$\underline{T}=\begin{bmatrix}1 & 0 & a\\ 0 & 1 & b\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0 & -1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}0 & -1 & a\\ -1 & 0 & b\\ 0 & 0 & 1\end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix}0 & -1 & a\\ -1 & 0 & b\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}=\begin{bmatrix}2\\ 4\\ 1\end{bmatrix}$$

$$\Longrightarrow \begin{pmatrix}a-1\\ -1+b\\ 1\end{pmatrix}=\begin{pmatrix}2\\ 4\\ 1\end{pmatrix} \qquad \therefore \; a=3, \; b=5$$

HENCE THE MATRIX $\underline{T}$ IS GIVEN BY

$$\underline{T}=\begin{bmatrix}0 & -1 & 3\\ -1 & 0 & 5\\ 0 & 0 & 1\end{bmatrix}$$

b) PARAMETRIZE THE CURVE $y=x^2$ AS $x=t$, $y=t^2$

THEN WE HAVE

$$\begin{bmatrix}X\\ Y\\ Z\end{bmatrix}=\begin{bmatrix}0 & -1 & 3\\ -1 & 0 & 5\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}t\\ t^2\\ 1\end{bmatrix}=\begin{bmatrix}3-t^2\\ 5-t\\ 1\end{bmatrix}$$

THUS $\left.\begin{array}{l}X=3-t^2\\ Y=5-t\end{array}\right\}$

$$\Longrightarrow t=5-Y$$
$$\Longrightarrow t^2=Y^2-10Y+25$$
$$\Longrightarrow -t^2=-Y^2+10Y-25$$
$$\Longrightarrow 3-t^2=-Y^2+10Y-22$$
$$\Longrightarrow X=-Y^2+10Y-22$$
$$\Longrightarrow Y^2-10Y+X+22=0$$

Question 105 (****)

The 3×3 matrix $\mathbf{A}$, is defined in terms of a scalar constant k, below.

$$\mathbf{A} = \begin{pmatrix} 3 & 2 & 5 \\ 3 & 3 & 4 \\ k & 4 & 3 \end{pmatrix}.$$

a) If $k=3$, verify that $\mathbf{A}$ maps every point of the three dimensional space onto the plane with Cartesian equation

$$x - 2y + z = 0.$$

b) If $k \neq 3$, determine the value k so that the transformation represented by $\mathbf{A}$ has a line of invariant points, and state the Cartesian equation of this line.

$\boxed{}$, $\boxed{k=12}$, $\boxed{\frac{1}{2}x = -\frac{1}{7}y = \frac{1}{2}z}$

a) IF $k=3$ $\quad \underline{A} = \begin{bmatrix} 3 & 2 & 5 \\ 3 & 3 & 4 \\ 3 & 4 & 3 \end{bmatrix}$

$\underline{A}\underline{x} = \underline{X} \implies \begin{pmatrix} 3 & 2 & 5 \\ 3 & 3 & 4 \\ 3 & 4 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$

$\implies \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 3x+2y+5z \\ 3x+3y+4z \\ 3x+4y+3z \end{bmatrix}$

VERIFYING $\quad X - 2Y + Z$

$= (3x+2y+5z) - 2(3x+3y+4z) + (3x+4y+3z)$

$= \begin{array}{l} 3x+2y+5z \\ -6x-6y-8z \\ 3x+4y+3z \end{array}$

$= 0$

AS REQUIRED

b) LINE OF INVARIANT POINTS $\implies$ EIGENVALUE $= 1$

$\begin{vmatrix} 3-\lambda & 2 & 5 \\ 3 & 3-\lambda & 4 \\ k & 4 & 3-\lambda \end{vmatrix} = 0 \quad \xrightarrow{\lambda=1} \quad \begin{vmatrix} 2 & 2 & 5 \\ 3 & 2 & 4 \\ k & 4 & 2 \end{vmatrix} = 0$

$\underset{r_{13}(-2)}{\overset{r_{12}(-1)}{\implies}} \begin{vmatrix} 2 & 2 & 5 \\ 1 & 0 & -1 \\ k-4 & 0 & -8 \end{vmatrix} = 0$

$\implies \cancel{-2}\begin{vmatrix} 1 & -1 \\ k-4 & -8 \end{vmatrix} = 0$

$\implies -8 + k - 4 = 0$

$\implies k = 12$

NOW LOOK FOR THE EIGENVECTOR FOR $\lambda = 1$

$\left.\begin{array}{l} 3x+2y+5z = 1x \\ 3x+3y+4z = 1y \\ 12x+4y+3z = 1z \end{array}\right. \implies \left.\begin{array}{l} 2x+2y+5z=0 \\ 3x+2y+4z=0 \\ 12x+4y+2z=0 \end{array}\right\} \implies$

$\left.\begin{array}{l} 2x+2y+5z=0 \\ 3x+2y+4z=0 \\ 6x+2y+z=0 \end{array}\right\} \implies \boxed{z=-6x-2y}$

SUBSTITUTE INTO THE FIRST TWO EQUATIONS

$\left.\begin{array}{l} 2x+2y+5(-6x-2y)=0 \\ 3x+2y+4(-6x-2y)=0 \end{array}\right\} \implies \begin{array}{l} -28x-8y=0 \\ -21x-6y=0 \end{array}$

$\implies \boxed{y=-\frac{7}{2}x}$

$\implies z = -6x - 2\left(-\frac{7}{2}x\right)$

$\implies z = -6x + 7x$

$\implies \boxed{z = x}$

I.E EIGENVECTOR $\propto \begin{pmatrix} 2 \\ -7 \\ 2 \end{pmatrix}$

OR IN LINE FORM $\quad \underline{r} = t\begin{pmatrix} 2 \\ -7 \\ 2 \end{pmatrix}$

$\frac{x}{2} = \frac{y}{-7} = \frac{z}{2}$

Question 106 (****)

The 2×2 matrix $\mathbf{M}$ satisfies $\mathbf{M}=\mathbf{PDP}^{-1}$ where

$$\mathbf{P}=\begin{pmatrix}-1 & 4\\ 3 & 1\end{pmatrix} \quad \text{and} \quad \mathbf{D}=\begin{pmatrix}1 & 0\\ 0 & 27\end{pmatrix}.$$

a) Determine the elements of $\mathbf{M}$.

b) State the eigenvalues, and the corresponding eigenvectors of $\mathbf{M}$.

c) Find an equation of the straight line of invariant points under the transformation described by $\mathbf{M}$.

It is further given that

$$\mathbf{M}^n=\frac{1}{13}\begin{pmatrix}4\times 3^{3n+1}+1 & 4\times 3^{3n}-4\\ 3^{3n+1}-3 & 3^{3n}+12\end{pmatrix}.$$

d) Deduce that $3^{3n+2}+4$ is divisible by 13, for all positive integers n.

$\boxed{}$, $\boxed{\mathbf{M}=\begin{pmatrix}25 & 8\\ 6 & 3\end{pmatrix}}$, $\boxed{\lambda_1=1,\quad \lambda_2=27,\quad \mathbf{u}_1=4\mathbf{i}+\mathbf{j},\quad \mathbf{u}_2=-\mathbf{i}+3\mathbf{j}}$, $\boxed{y=-3x}$

a) $\underline{M}=\underline{P}\,\underline{D}\,\underline{P}^{-1}=\begin{pmatrix}-1 & 4\\ 3 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 27\end{pmatrix}\frac{1}{-13}\begin{pmatrix}1 & -4\\ -3 & -1\end{pmatrix}$

$=\frac{1}{13}\begin{pmatrix}-1 & 4\\ 3 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 27\end{pmatrix}\begin{pmatrix}-1 & 4\\ 3 & 1\end{pmatrix}$

$=\frac{1}{13}\begin{pmatrix}-1 & 108\\ 3 & 27\end{pmatrix}\begin{pmatrix}-1 & 4\\ 3 & 1\end{pmatrix}$

$=\frac{1}{13}\begin{pmatrix}325 & 104\\ 78 & 13\end{pmatrix}$

$=\begin{pmatrix}25 & 8\\ 6 & 1\end{pmatrix}$ //

b) EIGENVALUE $\lambda=1$, WITH EIGENVECTOR $\alpha\begin{pmatrix}-1\\ 3\end{pmatrix}$

EIGENVALUE $\lambda=27$, WITH EIGENVECTOR $b\begin{pmatrix}4\\ 1\end{pmatrix}$ //

c) THIS CORRESPONDS TO EIGENVALUE $\lambda=1 \Rightarrow \begin{pmatrix}-1\\ 3\end{pmatrix} \to y=-3x$ //

d) AS M HAS INTEGER ENTRIES M^n MUST ALSO HAVE INTEGER ENTRIES, ALL OF WHICH MUST BE DIVISIBLE BY 13

HENCE: $\underline{M}^n_{11}-\underline{M}^n_{21}=\frac{1}{13}\left[(4\times 3^{3n+1}+1)-(3^{3n+1}-3)\right]$

$=\frac{1}{13}\left[4\times 3^{3n+1}-3^{3n+1}+4\right]$

$=\frac{1}{13}\left[3\times 3^{3n+1}+4\right]$

$=\frac{1}{13}\left[3^{3n+2}+4\right]$

// AS REQUIRED

Question 107 **(****)**

Factorize fully the following 3×3 determinant.

$$\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix}.$$

$$\boxed{(x-y)(y-z)(z-x)(xy+yz+zx)}$$

$$\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix} \xrightarrow[C_{13}(-1)]{C_{12}(-1)} \begin{vmatrix} x & y-x & z-x \\ x^2 & y^2-x^2 & z^2-x^2 \\ yz & zx-yz & xy-yz \end{vmatrix} = \begin{vmatrix} x & y-x & z-x \\ x^2 & (y-x)(y+x) & (z-x)(x+z) \\ yz & z(x-y) & y(x-z) \end{vmatrix}$$

$$= (y-x)(z-x)\begin{vmatrix} x & 1 & 1 \\ x^2 & y+x & x+z \\ yz & -z & -y \end{vmatrix} \xrightarrow{C_{32}(-1)} (y-x)(x-z)\begin{vmatrix} x & & 1 \\ x^2 & y-z & x+z \\ yz & y-z & -y \end{vmatrix}$$

$$=(y-x)(z-x)(y-z)\begin{vmatrix} x & 0 & 1 \\ x^2 & 1 & x+z \\ yz & 1 & -y \end{vmatrix} \xrightarrow{r_{32}(-1)}$$

$$=(y-x)(z-x)(y-z)\begin{vmatrix} x & 0 & 1 \\ x^2-yz & 0 & x+z+y \\ yz & 1 & -y \end{vmatrix}$$

EXPAND BIDDLE COLUMN

$$=-(y-x)(z-x)(y-z)\begin{vmatrix} x & 1 \\ x^2-yz & x+z+y \end{vmatrix}$$

$$=-(y-x)(z-x)(y-z)\left[\cancel{x^2}+xz+xy-\cancel{x^2}+yz\right]$$

$$=-(y-x)(z-x)(y-z)(xz+xy+yz)$$

OR

$$= (x-y)(y-z)(z-x)(xy+yz+xz)$$

Question 108 (****)

A system of equations is given below

$$\begin{aligned} 3x + 2y \quad - z &= 10 \\ 5x \quad - y - 4z &= 17 \\ x + 5y + pz &= q \end{aligned}$$

where p and q are constants.

a) Find the value of p so that the above system does not have a unique solution.

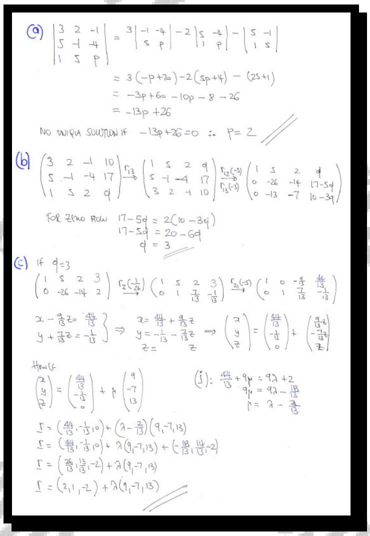
b) Show that for this value of p the system is consistent if $q = 3$.

c) Show that the general solution of the system can be written as

$$\mathbf{r} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k} + \lambda(9\mathbf{i} - 7\mathbf{j} + 13\mathbf{k}),$$

where λ is a scalar parameter.

$\boxed{p = 2}$



(a) $\begin{vmatrix} 3 & 2 & -1 \\ 5 & -1 & -4 \\ 1 & 5 & p \end{vmatrix} = 3\begin{vmatrix} -1 & -4 \\ 5 & p \end{vmatrix} - 2\begin{vmatrix} 5 & -4 \\ 1 & p \end{vmatrix} - \begin{vmatrix} 5 & -1 \\ 1 & 5 \end{vmatrix}$

$= 3(-p+20) - 2(5p+4) - (25+1)$

$= -3p + 60 - 10p - 8 - 26$

$= -13p + 26$

NO UNIQUE SOLUTION IF $-13p + 26 = 0 \therefore p = 2$

(b) $\begin{pmatrix} 3 & 2 & -1 & 10 \\ 5 & -1 & -4 & 17 \\ 1 & 5 & 2 & q \end{pmatrix} \xrightarrow{r_{13}} \begin{pmatrix} 1 & 5 & 2 & q \\ 5 & -1 & -4 & 17 \\ 3 & 2 & -1 & 10 \end{pmatrix} \xrightarrow[r_{13}(-3)]{r_{12}(-5)} \begin{pmatrix} 1 & 5 & 2 & q \\ 0 & -26 & -14 & 17-5q \\ 0 & -13 & -7 & 10-3q \end{pmatrix}$

FOR ZERO ROW $17 - 5q = 2(10 - 3q)$

$17 - 5q = 20 - 6q$

$q = 3$

(c) IF $q = 3$

$\begin{pmatrix} 1 & 5 & 2 & 3 \\ 0 & -26 & -14 & 2 \end{pmatrix} \xrightarrow{r_2(-\frac{1}{26})} \begin{pmatrix} 1 & 5 & 2 & 3 \\ 0 & 1 & \frac{7}{13} & -\frac{1}{13} \end{pmatrix} \xrightarrow{r_{21}(-5)} \begin{pmatrix} 1 & 0 & -\frac{9}{13} & \frac{44}{13} \\ 0 & 1 & \frac{7}{13} & -\frac{1}{13} \end{pmatrix}$

$\left.\begin{aligned} x - \tfrac{9}{13}z &= \tfrac{44}{13} \\ y + \tfrac{7}{13}z &= -\tfrac{1}{13} \end{aligned}\right\} \Rightarrow \begin{aligned} x &= \tfrac{44}{13} + \tfrac{9}{13}z \\ y &= -\tfrac{1}{13} - \tfrac{7}{13}z \\ z &= z \end{aligned} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{44}{13} \\ -\frac{1}{13} \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{9}{13}z \\ -\frac{7}{13}z \\ z \end{pmatrix}$

HENCE

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{44}{13} \\ -\frac{1}{13} \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 9 \\ -7 \\ 13 \end{pmatrix}$

$(\mathbf{i})$: $\frac{44}{13} + 9\mu = 9\lambda + 2$

$9\mu = 9\lambda - \frac{18}{13}$

$\mu = \lambda - \frac{2}{13}$

$\underline{r} = \left(\frac{44}{13}, -\frac{1}{13}, 0\right) + \left(\lambda - \frac{2}{13}\right)(9, -7, 13)$

$\underline{r} = \left(\frac{44}{13}, -\frac{1}{13}, 0\right) + \lambda(9, -7, 13) + \left(-\frac{18}{13}, \frac{14}{13}, -2\right)$

$\underline{r} = \left(\frac{26}{13}, \frac{13}{13}, -2\right) + \lambda(9, -7, 13)$

$\underline{r} = (2, 1, -2) + \lambda(9, -7, 13)$

Question 109 (****)

The following three vectors are given.

$$\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}.$$

a) Show that $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are linearly dependent.

b) Find a linear relationship, with integer coefficients, between $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.

$$\boxed{\mathbf{u} = 3\mathbf{v} - 4\mathbf{w}}$$

WRITE THE VECTORS AS THE COLUMNS OF A MATRIX

$$\underline{A} = \begin{pmatrix} 1 & 7 & 5 \\ 1 & 3 & 2 \\ 0 & 4 & 3 \end{pmatrix} \Rightarrow |\underline{A}| = \begin{vmatrix} 1 & 7 & 5 \\ 1 & 3 & 2 \\ 0 & 4 & 3 \end{vmatrix}$$

$\underline{u}$ $\underline{v}$ $\underline{w}$

EXPAND BY THE FIRST COLUMN

$$= 1\begin{vmatrix} 3 & 2 \\ 4 & 3 \end{vmatrix} - \begin{vmatrix} 7 & 5 \\ 4 & 3 \end{vmatrix}$$

$$= 1(9-8) - (21-20)$$

$$= 0$$

AS THE DETERMINANT IS ZERO THE VECTORS ARE LINEARLY DEPENDENT

NOW WRITE THE VECTORS AS AN AUGMENTED MATRIX

$$\Rightarrow \lambda\underline{u} + \mu\underline{v} = \underline{w}$$

$$\Rightarrow \lambda\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \mu\begin{pmatrix} 7 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \lambda \\ \lambda \\ 0 \end{pmatrix} + \begin{pmatrix} 7\mu \\ 3\mu \\ 4\mu \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \lambda + 7\mu \\ \lambda + 3\mu \\ 4\mu \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix}$$

$\therefore \mu = \frac{3}{4}$

& $\left.\begin{array}{l} \lambda + 7\left(\frac{3}{4}\right) = 5 \\ \lambda + 3\left(\frac{3}{4}\right) = 2 \end{array}\right) \Rightarrow \lambda = -\frac{1}{4}$

HENCE

$$\Rightarrow -\tfrac{1}{4}\underline{u} + \tfrac{3}{4}\underline{v} = \underline{w}$$

$$\Rightarrow -\underline{u} + 3\underline{v} = 4\underline{w}$$

$$\Rightarrow \underline{u} = 3\underline{v} - 4\underline{w}$$

Question 110 (****)

The 2×2 matrix $\mathbf{A}$ is defined in terms of a constant k.

$$\mathbf{A} = \begin{pmatrix} 2 & 7 \\ 4 & k \end{pmatrix}$$

a) Given that $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$, find …

i. … the corresponding eigenvalue to the eigenvector.

ii. … the value of k

b) Find another eigenvector and the corresponding eigenvalue of $\mathbf{A}$.

It is further given that $\mathbf{A} = \mathbf{PDP}^{-1}$, where $\mathbf{D}$ is a diagonal matrix and $\mathbf{P}$ is another matrix.

c) Write down possible forms for the matrices $\mathbf{D}$ and $\mathbf{P}$.

d) Hence show clearly that

$$\mathbf{A}^7 = \begin{pmatrix} 1739180 & 3043789 \\ 1739308 & 3043661 \end{pmatrix}.$$

$\boxed{\lambda = 9}$, $\boxed{k = 5}$, $\boxed{\lambda = -2, \mathbf{u} = \begin{pmatrix} 7 \\ -4 \end{pmatrix}}$, $\boxed{\mathbf{D} = \begin{pmatrix} 9 & 0 \\ 0 & -2 \end{pmatrix}}$, $\boxed{\mathbf{P} = \begin{pmatrix} 1 & 7 \\ 1 & -4 \end{pmatrix}}$

(a) $A = \begin{pmatrix} 2 & 7 \\ 4 & k \end{pmatrix}$ $\quad \begin{pmatrix} 2 & 7 \\ 4 & k \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \lambda\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 9 \\ 4+k \end{pmatrix} = \lambda\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\therefore \lambda = 9$

$4+k = 9$

$k = 5$

(b) CHARACTERISTIC EQUATION

$\begin{vmatrix} 2-\lambda & 7 \\ 4 & 5-\lambda \end{vmatrix} = 0$

$(2-\lambda)(5-\lambda) - 28 = 0$

$(\lambda-2)(\lambda-5) - 28 = 0$

$\lambda^2 - 7\lambda - 18 = 0$

$(\lambda-9)(\lambda+2) = 0$

$\lambda = -2$ IS THE OTHER ONE

$\left.\begin{aligned} 2x + 7y &= -2x \\ 4x + 5y &= -2y \end{aligned}\right\} \Rightarrow 4x = -7y$

$x = -\frac{7}{4}y$

$\therefore \propto \begin{pmatrix} 7 \\ -4 \end{pmatrix}$

(c) $P = \begin{pmatrix} 1 & 7 \\ 1 & -4 \end{pmatrix}$ $\quad D = \begin{pmatrix} 9 & 0 \\ 0 & -2 \end{pmatrix}$

(d) FIRST $P^{-1} = -\frac{1}{11}\begin{pmatrix} -4 & -7 \\ -1 & 1 \end{pmatrix} = \frac{1}{11}\begin{pmatrix} 4 & 7 \\ 1 & -1 \end{pmatrix}$

$A = PDP^{-1}$

$A^7 = (PDP^{-1})^7$

$A^7 = (PDP^{-1})(PDP^{-1})(PDP^{-1})\dots(PDP^{-1})$

$A^7 = PD^7P^{-1}$

$A^7 = \begin{pmatrix} 1 & 7 \\ 1 & -4 \end{pmatrix}\begin{pmatrix} 9^7 & 0 \\ 0 & (-2)^7 \end{pmatrix} \times \frac{1}{11}\begin{pmatrix} 4 & 7 \\ 1 & -1 \end{pmatrix}$

$A^7 = \frac{1}{11}\begin{pmatrix} 4782969 & -896 \\ 4782969 & 512 \end{pmatrix}\begin{pmatrix} 4 & 7 \\ 1 & -1 \end{pmatrix}$

$A^7 = \frac{1}{11}\begin{pmatrix} 19130980 & 33481679 \\ 19132388 & 33480271 \end{pmatrix}$

$A^7 = \begin{pmatrix} 1739180 & 3043789 \\ 1739308 & 3043661 \end{pmatrix}$

AS REQUIRED

Question 111 (****)

The 3×3 matrix $\mathbf{A}$ is given below.

$$\mathbf{A}=\begin{pmatrix}1 & -1 & 1\\ 3 & -3 & 1\\ 3 & -5 & 3\end{pmatrix}.$$

a) Given that $\mathbf{u}=\begin{pmatrix}1\\1\\2\end{pmatrix}$ is an eigenvector of $\mathbf{A}$, find the corresponding eigenvalue.

b) Given that $\lambda=-2$ is an eigenvalue of $\mathbf{A}$, find a corresponding eigenvector $\mathbf{v}$.

The vector $\mathbf{w}$ is defined as $\mathbf{w}=\mathbf{u}+\mathbf{v}$.

c) Determine the vector $\mathbf{A}^7\mathbf{w}$.

$\boxed{}$, $\boxed{\lambda=2}$, $\boxed{\mathbf{v}=\begin{pmatrix}0\\1\\1\end{pmatrix}}$, $\boxed{\mathbf{A}^7\mathbf{w}=\begin{pmatrix}128\\0\\128\end{pmatrix}}$

a) $\begin{pmatrix}1 & -1 & 1\\ 3 & -3 & 1\\ 3 & -5 & 3\end{pmatrix}\begin{pmatrix}1\\1\\2\end{pmatrix}=\begin{pmatrix}2\\2\\4\end{pmatrix}=2\begin{pmatrix}1\\1\\2\end{pmatrix}$ $\therefore \lambda=2$

b) $\left.\begin{array}{l} x-y+z=-2x\\ 3x-3y+z=-2y\\ 3x-5y+3z=-2z\end{array}\right\}\Rightarrow \left.\begin{array}{l} 3x-y+z=0\\ 3x-y+z=0\\ 3x-5y+5z=0\end{array}\right\}\Rightarrow$

$\left.\begin{array}{l} y=3x+z\\ 3x-5(3x+z)+5z=0\end{array}\right\}\Rightarrow \left.\begin{array}{l} y=3x+z\\ -12x=0\end{array}\right\}\Rightarrow \begin{array}{l} y=z\\ x=0\end{array}$

$\therefore \underline{v}=\begin{pmatrix}0\\1\\1\end{pmatrix}$

c) $\underline{A}\,\underline{w}=\underline{A}\,(\underline{u}+\underline{v})=\underline{A}^7\underline{u}+\underline{A}^7\underline{v}$

$=\underline{A}^6\left[\underline{A}\,\underline{u}+\underline{A}\,\underline{v}\right]=\underline{A}^6\left[2\underline{u}-2\underline{v}\right]$

$=\underline{A}^5\left[\underline{A}\,2\underline{u}-\underline{A}\,2\underline{v}\right]=\underline{A}^5\left[4\underline{u}+4\underline{v}\right]$

$=A^4\left[\underline{A}\,4\underline{u}+\underline{A}\,4\underline{v}\right]=\underline{A}^4\left[8\underline{u}-8\underline{v}\right]$

$\vdots \qquad \vdots$

$=A\left[\underline{A}\,2^5\underline{u}+\underline{A}\,(-2)^5\underline{v}\right]=A\left[2^6\underline{u}+(-2)^6\underline{v}\right]$

$=2^7\underline{u}+(-2)^7\underline{v}=128\begin{pmatrix}1\\1\\2\end{pmatrix}-128\begin{pmatrix}0\\1\\1\end{pmatrix}$

$=\begin{pmatrix}128\\128\\256\end{pmatrix}-\begin{pmatrix}0\\128\\128\end{pmatrix}=\begin{pmatrix}128\\0\\128\end{pmatrix}$

Question 112 (****)

The following four vectors are given.

$$\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \quad \mathbf{p} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

a) Show that $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are linearly independent.

b) Express $\mathbf{p}$ in terms of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.

$$\boxed{\mathbf{p} = 2\mathbf{u} - 4\mathbf{v} - 7\mathbf{w}}$$

TO SHOW INDEPENDENCE IT SUFFICES TO WRITE THE VECTORS AS A MATRIX & CHECK THAT THE DETERMINANT IS NOT ZERO

HENCE $\begin{vmatrix} 2 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & 2 & -1 \end{vmatrix} = 2\begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} + \begin{vmatrix} -1 & -1 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} -1 & 1 \\ 1 & 2 \end{vmatrix}$

$= 2(+1) + 2 - 1$

$= 1 \neq 0$

$\therefore$ THE VECTOR ARE LINEARLY INDEPENDENT

NOW $\lambda \underline{u} + \mu \underline{v} + \nu \underline{w} = \underline{p}$

$$\Rightarrow \lambda\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + \mu\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} + \nu\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} \lambda \\ \mu \\ \nu \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 2 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & 2 & -1 & 1 \end{array}\right] \xRightarrow{r_{13}} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ 2 & -1 & 1 & 1 \end{array}\right] \xRightarrow[r_{13}(-2)]{r_{12}(1)} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 3 & -2 & 2 \\ 0 & -5 & 3 & -1 \end{array}\right]$$

$$\xRightarrow{r_2(\frac{1}{3})} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -\frac{2}{3} & \frac{2}{3} \\ 0 & -5 & 3 & -1 \end{array}\right] \xRightarrow{r_{23}(5)} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -\frac{2}{3} & \frac{2}{3} \\ 0 & 0 & -\frac{1}{3} & \frac{7}{3} \end{array}\right]$$

EXTRACTING THE SYSTEM

$\lambda + 2\mu - \nu = 1$

$\mu - \frac{2}{3}\nu = \frac{2}{3}$

$-\frac{1}{3}\nu = \frac{7}{3}$

$\therefore \nu = -7$

- $\mu - \frac{2}{3}\nu = \frac{2}{3}$
  $3\mu - 2\nu = 2$
  $3\mu + 14 = 2$
  $3\mu = -12$
  $\mu = -4$
- $\lambda + 2\mu - \nu = 1$
  $\lambda - 8 + 7 = 1$
  $\lambda = 2$

$$\therefore \quad 2\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} - 4\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} - 7\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$\therefore \quad \underline{p} = 2\underline{u} - 4\underline{v} - 7\underline{w}$

Question 113 (****)

A plane Π has Cartesian equation

$$2x+3y+4z=24.$$

Determine a Cartesian equation for the transformation of Π under the matrix

$$\begin{pmatrix} 2 & 3 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

$\boxed{}$, $\boxed{x+3y=24}$

METHOD A

PICK 3 "EASY" POINTS WHICH SATISFY THE EQUATION OF THE PLANE

$2x+3y+4z=24 \implies A(12,0,0)$, $B(0,8,0)$, $C(0,0,6)$

TRANSFORM THESE THREE POINTS

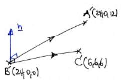
$$\begin{pmatrix} 2 & 3 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 12 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 6 \end{pmatrix} = \begin{pmatrix} 24 & 24 & 6 \\ 0 & 0 & 6 \\ 12 & 0 & 6 \end{pmatrix}$$

$\uparrow A \quad \uparrow B \quad \uparrow C \qquad \uparrow A' \quad \uparrow B' \quad \uparrow C'$

NOW LOOKING AT THE DIAGRAM

$\overrightarrow{B'A'} = (24,0,12)-(24,0,0) = (0,0,12)$

$\overrightarrow{B'C'} = (6,6,6)-(24,0,0) = (-18,6,6)$



SCALE THESE VECTORS & FIND COMMON PERPENDICULAR

$(0,0,1)$ & $(-3,1,1)$ ARE MUTUALLY PERPENDICULAR TO $\underline{n}=(a,b,c)$

- $(0,0,1)\cdot(a,b,c)=0 \implies c=0$
- $(-3,1,1)\cdot(a,b,c)=0 \implies -3a+b+c=0 \implies -3a+b=0 \implies b=3a$

(WE COULD HAVE USED THE CROSS PRODUCT ALSO) $\therefore \underline{n}=(1,3,0)$

HENCE THE EQUATION OF THE TRANSFORMED PLANE IS

$x+3y = \text{CONSTANT}$

& USING ANY OF A', B', C' WE FIND

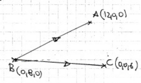
$x+3y=24$

METHOD B

$2x+3y+4z=24$

TAKE 3 RANDOM POINTS ON THIS PLANE AS BEFORE

$A(12,0,0)$, $B(0,8,0)$, $C(0,0,6)$



$\overrightarrow{BA} = \underline{a}-\underline{b} = (12,0,0)-(0,8,0) = (12,-8,0)$ SCALE TO $(3,-2,0)$

$\overrightarrow{BC} = \underline{c}-\underline{b} = (0,0,6)-(0,8,0) = (0,-8,6)$ SCALE TO $(0,-4,3)$

OBTAIN THE PARAMETRIC EQUATION OF THIS PLANE

$$\underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} + \mu\begin{pmatrix} 0 \\ -4 \\ 3 \end{pmatrix} = \begin{pmatrix} 3\lambda \\ 8-2\lambda-4\mu \\ 3\mu \end{pmatrix}$$

TRANSFORM USING THE MATRIX

$$\begin{pmatrix} 2 & 3 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3\lambda \\ 8-2\lambda-4\mu \\ 3\mu \end{pmatrix} = \begin{pmatrix} \cancel{6\lambda}+24-\cancel{6\lambda}-12\mu+3\mu \\ 3\mu \\ 3\lambda+3\mu \end{pmatrix} = \begin{pmatrix} 24-9\mu \\ 3\mu \\ 3\lambda+3\mu \end{pmatrix}$$

ELIMINATING THE PARAMETRICS

$x = 24-9\mu$
$y = 3\mu \implies x = 24-3y$
$z = 3\lambda+3\mu$

$\therefore x+3y=24$ AS BEFORE

Question 114 (****+)

A linear transformation T, acting in the $\boldsymbol{x}$-$\boldsymbol{y}$ plane, consists of …

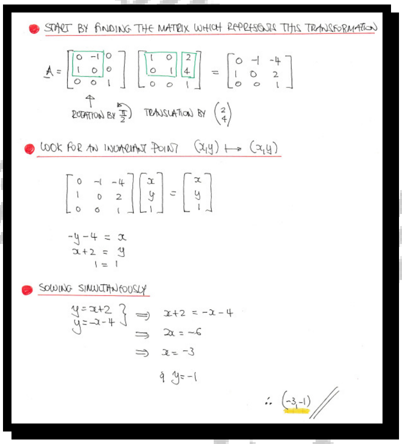
- … a translation such that $(x, y) \mapsto (x+2, y+4)$,

  followed by ...

- … an anticlockwise rotation about the origin by $\dfrac{\pi}{2}$.

Determine the coordinates of the invariant point under T.

$\boxed{}$, $\boxed{(-3,-1)}$



- START BY FINDING THE MATRIX WHICH REPRESENTS THIS TRANSFORMATION

$$\underline{A} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -4 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

ROTATION BY $\frac{\pi}{2}$ TRANSLATION BY $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$

- LOOK FOR AN INVARIANT POINT $(x,y) \mapsto (x,y)$

$$\begin{bmatrix} 0 & -1 & -4 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$-y-4 = x$$
$$x+2 = y$$
$$1 = 1$$

- SOLVING SIMULTANEOUSLY

$$\left.\begin{array}{l} y = x+2 \\ y = -x-4 \end{array}\right\} \Rightarrow x+2 = -x-4$$
$$\Rightarrow 2x = -6$$
$$\Rightarrow x = -3$$
$$\text{\& } y = -1$$

$$\therefore (-3,-1)$$

Question 115 (****+)

A linear transformation T, acting in the $\boldsymbol{x}$-$\boldsymbol{y}$ plane, consists of …

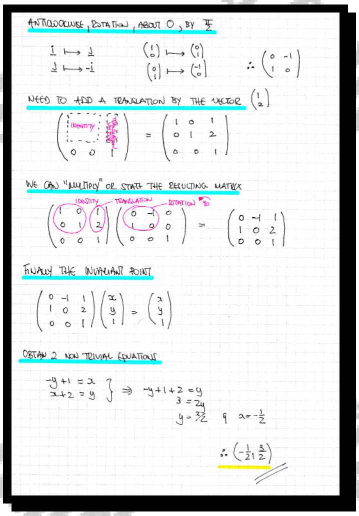
- … an anticlockwise rotation about the origin by $\frac{\pi}{2}$

followed by ...

- … a translation by the vector $\begin{pmatrix}1\\2\end{pmatrix}$.

Determine the coordinates of the invariant point under T.

$\boxed{\text{N.E.B}}$, $\boxed{\left(-\frac{1}{2},\frac{3}{2}\right)}$



ANTICLOCKWISE, ROTATION, ABOUT O, BY $\frac{\pi}{2}$

$\underline{i} \longmapsto \underline{j}$ $\quad \begin{pmatrix}1\\0\end{pmatrix} \longmapsto \begin{pmatrix}0\\1\end{pmatrix}$

$\underline{j} \longmapsto -\underline{i}$ $\quad \begin{pmatrix}0\\1\end{pmatrix} \longmapsto \begin{pmatrix}-1\\0\end{pmatrix}$ $\quad \therefore \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}$

NEED TO ADD A TRANSLATION BY THE VECTOR $\begin{pmatrix}1\\2\end{pmatrix}$

$\begin{pmatrix} \text{IDENTITY} & \text{TRANSLATION VECTOR} \\ 0 \quad 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0 & 1\\0 & 1 & 2\\0 & 0 & 1\end{pmatrix}$

WE CAN "MULTIPLY" OR STATE THE RESULTING MATRIX

IDENTITY — TRANSLATION — ROTATION 90

$\begin{pmatrix}1 & 0 & 1\\0 & 1 & 2\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}0 & -1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{pmatrix} = \begin{pmatrix}0 & -1 & 1\\1 & 0 & 2\\0 & 0 & 1\end{pmatrix}$

FINALLY THE INVARIANT POINT

$\begin{pmatrix}0 & -1 & 1\\1 & 0 & 2\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}x\\y\\1\end{pmatrix} = \begin{pmatrix}x\\y\\1\end{pmatrix}$

OBTAIN 2 NON TRIVIAL EQUATIONS

$\left.\begin{array}{l}-y+1=x\\x+2=y\end{array}\right\} \Rightarrow -y+1+2=y$

$3=2y$

$y=\frac{3}{2}$ & $x=-\frac{1}{2}$

$\therefore \left(-\frac{1}{2},\frac{3}{2}\right)$

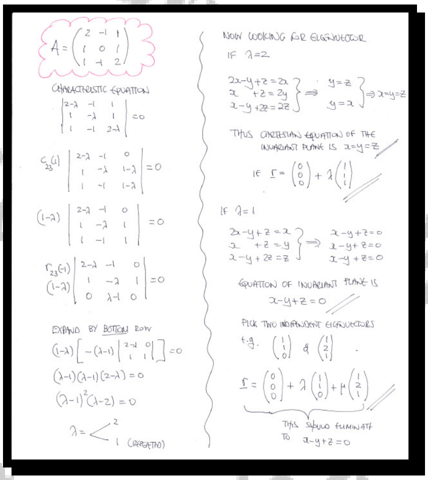
Question 116 (****+)

The 3×3 matrix $\mathbf{A}$ is given below.

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 1 \\ 3 & 0 & 1 \\ 1 & -1 & 2 \end{pmatrix}.$$

Find in Cartesian and parametric form the equation of the invariant line and the equation of the invariant plane under the transformation represented by $\mathbf{A}$.

$\boxed{\mathbf{r} = \lambda\mathbf{i} + \lambda\mathbf{j} + \lambda\mathbf{k},\quad x = y = z}$, $\boxed{\mathbf{r} = (\lambda+\mu)\mathbf{i} + (\lambda+2\mu)\mathbf{j} + \mu\mathbf{k},\quad x - y + z = 0}$



$A = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & 2 \end{pmatrix}$

CHARACTERISTIC EQUATION

$\begin{vmatrix} 2-\lambda & -1 & 1 \\ 1 & -\lambda & 1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$

$C_{23}(1)$ $\begin{vmatrix} 2-\lambda & -1 & 0 \\ 1 & -\lambda & 1-\lambda \\ 1 & -1 & 1-\lambda \end{vmatrix} = 0$

$(1-\lambda)\begin{vmatrix} 2-\lambda & -1 & 0 \\ 1 & -\lambda & 1 \\ 1 & -1 & 1 \end{vmatrix} = 0$

$r_{23}(-1)$ $(1-\lambda)\begin{vmatrix} 2-\lambda & -1 & 0 \\ 1 & -\lambda & 1 \\ 0 & \lambda-1 & 0 \end{vmatrix} = 0$

EXPAND BY BOTTOM ROW

$(1-\lambda)\left[-(\lambda-1)\begin{vmatrix} 2-\lambda & 0 \\ 1 & 1 \end{vmatrix}\right] = 0$

$(\lambda-1)(\lambda-1)(2-\lambda) = 0$

$(\lambda-1)^2(\lambda-2) = 0$

$\lambda = 2,\ 1$ (REPEATED)

NOW LOOKING FOR EIGENVECTOR

IF $\lambda = 2$

$\left.\begin{array}{l} 2x-y+z=2x \\ x+z=2y \\ x-y+2z=2z \end{array}\right\} \Rightarrow \left.\begin{array}{l} y=z \\ y=x \end{array}\right\} \Rightarrow x=y=z$

THUS CARTESIAN EQUATION OF THE INVARIANT PLANE IS $x=y=z$

IE $\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

IF $\lambda = 1$

$\left.\begin{array}{l} 2x-y+z=x \\ x+z=y \\ x-y+2z=z \end{array}\right\} \Rightarrow \begin{array}{l} x-y+z=0 \\ x-y+z=0 \\ x-y+z=0 \end{array}$

EQUATION OF INVARIANT PLANE IS

$x-y+z=0$

PICK TWO INDEPENDENT EIGENVECTORS

e.g. $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ & $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$.

$\underbrace{\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \mu\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}}_{\text{THIS SHOULD ELIMINATE TO } x-y+z=0}$

Question 117 (****+)

The following four vectors are given.

$$\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 3 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{p} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 0 \end{bmatrix}.$$

a) Show that these four vectors are linearly dependent.

b) Express $\mathbf{p}$ in terms of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.

$$\boxed{\mathbf{p} = \frac{3}{2}\mathbf{u} - \mathbf{v} + \frac{5}{2}\mathbf{w}}$$

$$\boxed{\underline{u} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \quad \underline{v} = \begin{pmatrix} 3 \\ 0 \\ 1 \\ -1 \end{pmatrix} \quad \underline{w} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ -1 \end{pmatrix} \quad \underline{p} = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 0 \end{pmatrix}}$$

a) FORMING A MATRIX WITH COLUMNS THE 4 VECTORS

$$\begin{vmatrix} 1 & 3 & 1 & 1 \\ 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & -1 \\ 1 & -1 & -1 & 0 \end{vmatrix} \quad \begin{matrix} r_{12}(-1) \\ r_{14}(-1) \end{matrix} \quad \begin{vmatrix} 1 & 3 & 1 & 1 \\ 0 & -3 & -2 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & -4 & -2 & -1 \end{vmatrix}$$

$$r_{23} \quad \begin{vmatrix} 1 & 3 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & -3 & -2 & -2 \\ 0 & -4 & -2 & -1 \end{vmatrix} \quad \begin{matrix} r_{23}(3) \\ r_{24}(4) \end{matrix} \quad \begin{vmatrix} 1 & 3 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -2 & -5 \\ 0 & 0 & -2 & -5 \end{vmatrix}$$

TWO IDENTICAL ROWS IMPLIES ZERO DETERMINANT

HENCE THE VECTORS ARE LINEARLY DEPENDENT //

b) $\lambda \underline{u} + \mu \underline{v} + t \underline{w} = \underline{p}$

$$\lambda\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} + \mu\begin{pmatrix} 3 \\ 0 \\ 1 \\ -1 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 0 \end{pmatrix}$$

$$\left\{\begin{matrix} \lambda + 3\mu + t = 1 \\ \lambda - t = -1 \\ \mu = -1 \\ \lambda - \mu - t = 0 \end{matrix}\right\} \Rightarrow \left\{\begin{matrix} \lambda - 3 + t = 1 \\ \lambda - t = -1 \\ \mu = -1 \\ \lambda + 1 - t = 0 \end{matrix}\right\} \Rightarrow$$

$$\left\{\begin{matrix} \lambda + t = 4 \\ \lambda - t = -1 \\ \mu = -1 \end{matrix}\right\} \Rightarrow \begin{matrix} \lambda = \frac{3}{2} \\ t = \frac{5}{2} \\ \mu = -1 \end{matrix}$$

$$\therefore \underline{p} = \frac{3}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} - 1\begin{pmatrix} 3 \\ 0 \\ 1 \\ -1 \end{pmatrix} + \frac{5}{2}\begin{pmatrix} 1 \\ -1 \\ 0 \\ -1 \end{pmatrix}$$

IE $\underline{p} = \frac{3}{2}\underline{u} - \underline{v} + \frac{5}{2}\underline{w}$ //

ALTERNATIVE

TAKING IT FROM THE ROW REDUCTION OF PART (a)

AND IGNORING THE BOTTOM ROW

$$\begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -2 & -5 \end{bmatrix} \Rightarrow \left\{\begin{matrix} \lambda + 3\mu + t = 1 \\ \mu = -1 \\ -2t = -5 \end{matrix}\right\}$$

$$\Rightarrow \left\{\begin{matrix} \lambda - 3 + \frac{5}{2} = 1 \\ \mu = -1 \\ t = \frac{5}{2} \end{matrix}\right\}$$

$$\Rightarrow \begin{matrix} \lambda = \frac{3}{2} \\ \mu = -1 \\ t = \frac{5}{2} \end{matrix} //$$

Question 118 (****+)

A linear transformation T, acting in the $\boldsymbol{x}$-$\boldsymbol{y}$ plane, consists of …

- … a translation such that $(x, y) \mapsto (x+2, y-3)$,

  followed by ...

- … a rotation about the origin, by $\frac{1}{2}\pi$, anticlockwise.

Show that under T, the curve with equation

$$x^2 - y^2 = 4,$$

is mapped onto the curve with equation

$$x^2 - y^2 - 6x + 4y + 9 = 0.$$

$\boxed{}$, $\boxed{\text{proof}}$

START BY DETERMINING A MATRIX

$$I = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 3 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

ROTATION BY $\frac{\pi}{2}$ ANTICLOCKWISE

TRANSLATION BY THE VECTOR $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$

THUS WE HAVE

$$\begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 3 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \Longrightarrow \begin{Bmatrix} X = -y+3 \\ Y = x+2 \\ 1 = 1 \end{Bmatrix}$$

$$\Longrightarrow \boxed{\begin{matrix} x = Y-2 \\ y = 3-X \end{matrix}}$$

SUBSTITUTING INTO THE EQUATION OF THE CURVE

$\Rightarrow x^2 - y^2 = 4$

$\Rightarrow (Y-2)^2 - (3-X)^2 = 4$

$\Rightarrow Y^2 - 4Y \cancel{+4} - X^2 + 6X - 9 = \cancel{4}$

$\Rightarrow Y^2 - X^2 - 4Y + 6X - 9 = 0$

$\Rightarrow X^2 - Y^2 - 6X + 4Y + 9 = 0$

AS REQUIRED

ALTERNATIVE APPROACH

OBTAIN A ROTATION MATRIX IN THE xy PLANE IE $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \Longrightarrow \begin{matrix} X = -y \\ Y = x \end{matrix}$$

$$\Longrightarrow \boxed{\begin{matrix} x = Y \\ y = -X \end{matrix}}$$

NEXT TRANSLATE THE CURVE BY THE VECTOR $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$

$\Rightarrow x^2 - y^2 = 4$

$\Rightarrow (x-2)^2 - (y+3)^2 = 4$

$\Rightarrow x^2 - 4x + 4 - y^2 - 6y - 9 = 4$

$\Rightarrow x^2 - y^2 - 4x - 6y - 9 = 0$

NOW APPLY THE ROTATION

$\Rightarrow Y^2 - (-X)^2 - 4Y - 6(-X) - 9 = 0$

$\Rightarrow Y^2 - X^2 - 4Y + 6X - 9 = 0$

$\Rightarrow X^2 - Y^2 - 6X + 4Y + 9 = 0$

AS BEFORE

Question 119 (****+)

The 2×2 matrix $\mathbf{C}$ is defined as

$$\mathbf{C} = \begin{pmatrix} a & b+a \\ b-a & -a \end{pmatrix},$$

where a and b are constants.

a) Determine the eigenvalues of $\mathbf{C}$ and their corresponding eigenvectors, giving the answers in terms of a and b where appropriate.

It is further given that $\mathbf{C} = \mathbf{PDP}^{-1}$, where $\mathbf{D}$ is a diagonal matrix and $\mathbf{P}$ is another 2×2 matrix.

b) Write down the possible form of $\mathbf{D}$ and the possible form of $\mathbf{P}$ and hence show that

$$\mathbf{C}^9 = b^8\mathbf{C}.$$

$\boxed{\lambda_1 = b, \quad \mathbf{u} = \begin{pmatrix} 1 \\ \frac{b-a}{b+a} \end{pmatrix} \text{ or } \mathbf{u} = \begin{pmatrix} b+a \\ b-a \end{pmatrix}}$, $\boxed{\lambda = -b, \quad \mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$,

$\boxed{\mathbf{D} = \begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}}$, $\boxed{\mathbf{P} = \begin{pmatrix} b+a & 1 \\ b-a & -1 \end{pmatrix}}$

a) START BY THE CHARACTERISTIC EQUATION FOR C

$$\begin{vmatrix} a-\lambda & b+a \\ b-a & -a-\lambda \end{vmatrix} = 0$$

$\Rightarrow (a-\lambda)(-a-\lambda) - (b-a)(b+a) = 0$

$\Rightarrow -(a-\lambda)(a+\lambda) - (b^2-a^2) = 0$

$\Rightarrow (\lambda-a)(\lambda+a) - b^2 + a^2 = 0$

$\Rightarrow \lambda^2 \cancel{-a^2} - b^2 \cancel{+a^2} = 0$

$\Rightarrow \lambda^2 = b^2$

$\Rightarrow \lambda = \begin{cases} b \\ -b \end{cases}$

IF $\lambda = b$

$\left.\begin{aligned} ax + (b+a)y &= bx \\ (b-a)x - ay &= by \end{aligned}\right\}$

$\left.\begin{aligned} (a-b)x + (b+a)y &= 0 \\ (b-a)x - (a+b)y &= 0 \end{aligned}\right\}$

BOTH YIELD

$y = \frac{b-a}{b+a}x$

HENCE

$\mathbf{u} = \begin{pmatrix} 1 \\ \frac{b-a}{b+a} \end{pmatrix}$ OR $\begin{pmatrix} b+a \\ b-a \end{pmatrix}$

IF $\lambda = -b$

$\left.\begin{aligned} ax + (b+a)y &= -bx \\ (b-a)x - ay &= -by \end{aligned}\right\}$

$\left.\begin{aligned} (a+b)x + (a+b)y &= 0 \\ (b-a)x + (b-a)y &= 0 \end{aligned}\right\}$

BOTH YIELD

$y = -x$

HENCE

$\mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

b) STANDARD DIAGONALIZATION RESULTS

$\boxed{\mathbf{P} = \begin{pmatrix} b+a & 1 \\ b-a & -1 \end{pmatrix}}$ $\boxed{\mathbf{D} = \begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}}$

$\mathbf{P}^{-1} = \frac{1}{-b-a-b+a}\begin{pmatrix} -1 & -1 \\ a-b & b+a \end{pmatrix} = -\frac{1}{2b}\begin{pmatrix} -1 & -1 \\ a-b & a+b \end{pmatrix}$

$\boxed{\mathbf{P}^{-1} = -\frac{1}{2b}\begin{pmatrix} -1 & -1 \\ a-b & a+b \end{pmatrix}}$

FINALLY WE HAVE

$\Rightarrow \mathbf{C} = \mathbf{PDP}^{-1}$

$\Rightarrow \mathbf{C}^9 = (\mathbf{PDP}^{-1})^9 = \underbrace{(\mathbf{PDP}^{-1})(\mathbf{PDP}^{-1})(\mathbf{PDP}^{-1})\dots(\mathbf{PDP}^{-1})}_{9 \text{ FACTORS}}$

$\Rightarrow \mathbf{C}^9 = \mathbf{PDIDIDI}\dots\mathbf{IDP}^{-1}$

$\Rightarrow \mathbf{C}^9 = \mathbf{PD}^9\mathbf{P}^{-1}$

$\Rightarrow \mathbf{C}^9 = \begin{bmatrix} b+a & 1 \\ b-a & -1 \end{bmatrix}\begin{bmatrix} b & 0 \\ 0 & -b \end{bmatrix}^9 \times \frac{-1}{2b}\begin{bmatrix} -1 & -1 \\ a-b & a+b \end{bmatrix}$

$\Rightarrow \mathbf{C}^9 = -\frac{1}{2b}\begin{bmatrix} b+a & 1 \\ b-a & -1 \end{bmatrix}\begin{bmatrix} b^9 & 0 \\ 0 & -b^9 \end{bmatrix}\begin{bmatrix} -1 & -1 \\ a-b & a+b \end{bmatrix}$

$\Rightarrow \mathbf{C}^9 = -\frac{1}{2b}\begin{bmatrix} b^9(b+a) & -b^9 \\ b^9(b-a) & b^9 \end{bmatrix}\begin{bmatrix} -1 & -1 \\ a-b & a+b \end{bmatrix}$

$\Rightarrow \mathbf{C}^9 = -\frac{1}{2b}\times b^9\begin{bmatrix} b+a & -1 \\ b-a & 1 \end{bmatrix}\begin{bmatrix} -1 & -1 \\ a-b & a+b \end{bmatrix}$

$\Rightarrow \mathbf{C}^9 = -\frac{1}{2}b^8\begin{bmatrix} -b-a-a+b & -b-a-a-b \\ -b+a+a-b & -b+a+a+b \end{bmatrix}$

$\Rightarrow \mathbf{C}^9 = -\frac{1}{2}b^8\begin{bmatrix} -2a & -2a-2b \\ 2a-2b & 2a \end{bmatrix}$

$\Rightarrow \mathbf{C}^9 = b^8\begin{bmatrix} a & a+b \\ b-a & -a \end{bmatrix}$

$\Rightarrow \mathbf{C}^9 = b^8\mathbf{C}$

AS REQUIRED

Question 120 (****+)

A system of equation is given below

$$\begin{aligned} 3x - 2y - 18z &= 6 \\ 2x + y - 5z &= 25 \end{aligned}$$

a) Show, by reducing the system into row echelon form, that the solution of the system can be written as

$$\mathbf{r} = 8\mathbf{i} + 9\mathbf{j} + \lambda(4\mathbf{i} - 3\mathbf{j} + \mathbf{k}),$$

where λ is a scalar parameter.

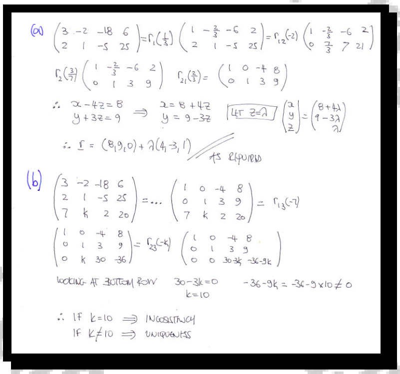
A new system is now given

$$\begin{aligned} 3x - 2y - 18z &= 6 \\ 2x + y - 5z &= 25 \\ 7x + ky + 2z &= 20 \end{aligned}$$

where k is a constant.

b) Determine if the system has solutions for different values of k.

$\boxed{k \neq 10 \Rightarrow \text{unique, otherwise incosistent}}$



(a) $\begin{pmatrix} 3 & -2 & -18 & 6 \\ 2 & 1 & -5 & 25 \end{pmatrix} = r_1\left(\frac{1}{3}\right) \begin{pmatrix} 1 & -\frac{2}{3} & -6 & 2 \\ 2 & 1 & -5 & 25 \end{pmatrix} = r_{12}(-2) \begin{pmatrix} 1 & -\frac{2}{3} & -6 & 2 \\ 0 & \frac{7}{3} & 7 & 21 \end{pmatrix}$

$r_2\left(\frac{3}{7}\right) \begin{pmatrix} 1 & -\frac{2}{3} & -6 & 2 \\ 0 & 1 & 3 & 9 \end{pmatrix} \quad r_{21}\left(\frac{2}{3}\right) = \begin{pmatrix} 1 & 0 & -4 & 8 \\ 0 & 1 & 3 & 9 \end{pmatrix}$

$\therefore \begin{matrix} x - 4z = 8 \\ y + 3z = 9 \end{matrix} \Rightarrow \begin{matrix} x = 8 + 4z \\ y = 9 - 3z \end{matrix} \quad \boxed{\text{LET } z = \lambda} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 + 4\lambda \\ 9 - 3\lambda \\ \lambda \end{pmatrix}$

$\therefore \underline{r} = (8, 9, 0) + \lambda(4, -3, 1)$ AS REQUIRED

(b) $\begin{pmatrix} 3 & -2 & -18 & 6 \\ 2 & 1 & -5 & 25 \\ 7 & k & 2 & 20 \end{pmatrix} = \ldots \begin{pmatrix} 1 & 0 & -4 & 8 \\ 0 & 1 & 3 & 9 \\ 7 & k & 2 & 20 \end{pmatrix} = r_{13}(-7)$

$\begin{pmatrix} 1 & 0 & -4 & 8 \\ 0 & 1 & 3 & 9 \\ 0 & k & 30 & -36 \end{pmatrix} = r_{23}(-k) \begin{pmatrix} 1 & 0 & -4 & 8 \\ 0 & 1 & 3 & 9 \\ 0 & 0 & 30-3k & -36-9k \end{pmatrix}$

LOOKING AT BOTTOM ROW $\quad 30 - 3k = 0 \quad k = 10 \quad -36 - 9k = -36 - 9 \times 10 \neq 0$

$\therefore$ IF $k = 10 \Rightarrow$ INCONSISTENCY
IF $k \neq 10 \Rightarrow$ UNIQUENESS

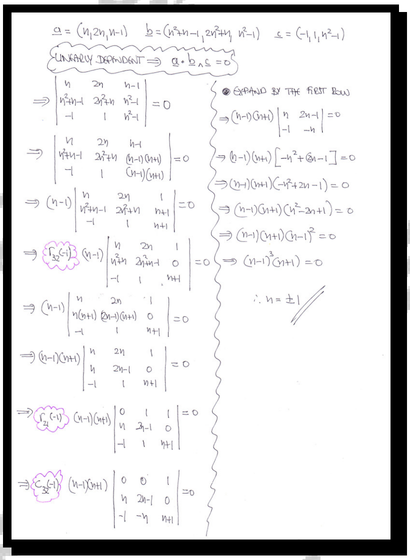
Question 121 **(****+)**

The following vectors, given in terms of a scalar constant n, are linearly dependent.

$$\mathbf{a} = n\mathbf{i} + 2n\mathbf{j} + (n-1)\mathbf{k},$$
$$\mathbf{b} = \left(n^2+n-1\right)\mathbf{i} + \left(2n^2+n\right)\mathbf{j} + \left(n^2-1\right)\mathbf{k},$$
$$\mathbf{c} = -\mathbf{i} + \mathbf{j} + \left(n^2-1\right)\mathbf{k}.$$

Determine possible values of n.

$\boxed{}$, $\boxed{n = \pm 1}$



$\underline{a} = (n, 2n, n-1)$ $\underline{b} = (n^2+n-1, 2n^2+n, n^2-1)$ $\underline{c} = (-1, 1, n^2-1)$

LINEARLY DEPENDENT $\Rightarrow$ $\underline{a} \cdot \underline{b} \wedge \underline{c} = 0$

$$\Rightarrow \begin{vmatrix} n & 2n & n-1 \\ n^2+n-1 & 2n^2+n & n^2-1 \\ -1 & 1 & n^2-1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} n & 2n & n-1 \\ n^2+n-1 & 2n^2+n & (n-1)(n+1) \\ -1 & 1 & (n-1)(n+1) \end{vmatrix} = 0$$

$$\Rightarrow (n-1)\begin{vmatrix} n & 2n & 1 \\ n^2+n-1 & 2n^2+n & n+1 \\ -1 & 1 & n+1 \end{vmatrix} = 0$$

$$\Rightarrow r_{32}(-1)\ (n-1)\begin{vmatrix} n & 2n & 1 \\ n^2+n & 2n^2+n-1 & 0 \\ -1 & 1 & n+1 \end{vmatrix} = 0$$

$$\Rightarrow (n-1)\begin{vmatrix} n & 2n & 1 \\ n(n+1) & (2n-1)(n+1) & 0 \\ -1 & 1 & n+1 \end{vmatrix} = 0$$

$$\Rightarrow (n-1)(n+1)\begin{vmatrix} n & 2n & 1 \\ n & 2n-1 & 0 \\ -1 & 1 & n+1 \end{vmatrix} = 0$$

$$\Rightarrow r_{21}(-1)\ (n-1)(n+1)\begin{vmatrix} 0 & 1 & 1 \\ n & 2n-1 & 0 \\ -1 & 1 & n+1 \end{vmatrix} = 0$$

$$\Rightarrow C_{32}(-1)\ (n-1)(n+1)\begin{vmatrix} 0 & 0 & 1 \\ n & 2n-1 & 0 \\ -1 & -n & n+1 \end{vmatrix} = 0$$

- EXPAND BY THE FIRST ROW

$$\Rightarrow (n-1)(n+1)\begin{vmatrix} n & 2n-1 \\ -1 & -n \end{vmatrix} = 0$$

$$\Rightarrow (n-1)(n+1)\left[-n^2 + (2n-1)\right] = 0$$

$$\Rightarrow (n-1)(n+1)(-n^2+2n-1) = 0$$

$$\Rightarrow (n-1)(n+1)(n^2-2n+1) = 0$$

$$\Rightarrow (n-1)(n+1)(n-1)^2 = 0$$

$$\Rightarrow (n-1)^3(n+1) = 0$$

$\therefore n = \pm 1$

Question 122 (****+)

A linear transformation T, acting in the $\boldsymbol{x}$-$\boldsymbol{y}$ plane, consists of …

- … a translation such that $(x, y) \mapsto (x+h, y+k)$,

  followed by

- … a reflection about the line $y = x$.

a) Determine, in terms of k and h, the equations of the two straight lines which map onto each other under T.

b) Find, in terms of k and h, the equation of the invariant line under T.

c) Give a full geometrical description for T, in the case where $h+k=0$, by considering the single transformation that is equivalent to T applied twice in succession.

$\boxed{}$, $\boxed{y = x-h, \quad y = x+k}$, $\boxed{y = x+\frac{1}{2}(k-h)}$,

$\boxed{\text{Reflection about the line } y = x-k}$

a) FIND THE MATRIX WHICH COMBINES THE TWO TRANSFORMATIONS

$$\begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & h \\ 1 & 0 & k \\ 0 & 0 & 1 \end{bmatrix}$$

TRANSLATION BY THE VECTOR $\begin{pmatrix} h \\ k \end{pmatrix}$ REFLECTION ABOUT THE LINE $y=x$

PROCEED AS FOLLOWS

$$\begin{bmatrix} 0 & 1 & h \\ 1 & 0 & k \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \Rightarrow \begin{array}{l} y+h = x \\ x+k = y \\ 1 = 1 \end{array}$$

CHECK IF THESE ARE THE TWO LINES FROM ABOVE

- $y = x-h$
- GENERAL POINT $(t, t-h)$

$$\begin{bmatrix} 0 & 1 & h \\ 1 & 0 & k \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} t \\ t-h \\ 1 \end{bmatrix} = \begin{bmatrix} t-h+h \\ t+k \\ 1 \end{bmatrix}$$

$\left.\begin{array}{l} x=t \\ y=t+k \end{array}\right)$ SUBTRACT $x-y=-k$

IE $y = x+k$

- $y = x+k$
- GENERAL POINT $(t, t+k)$

$$\begin{bmatrix} 0 & 1 & h \\ 1 & 0 & k \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} t \\ t+k \\ 1 \end{bmatrix} = \begin{bmatrix} t+k+h \\ t+k \\ 1 \end{bmatrix}$$

$\left.\begin{array}{l} x=t+k+h \\ y=t+k \end{array}\right)$ SUBTRACT $x-y=h$, $x-h=y$

IE $y = x-h$

b) NOW LET THE LINE HAVE EQUATION $y = mx+c$

$$\begin{bmatrix} 0 & 1 & h \\ 1 & 0 & k \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} t \\ mt+c \\ 1 \end{bmatrix} = \begin{bmatrix} mt+c+h \\ t+k \\ 1 \end{bmatrix}$$

$\left.\begin{array}{l} y=t+k \\ x=mt+c+h \end{array}\right\} \Rightarrow t=y-k$

$\Downarrow$

$x = m(y-k)+c+h$

$x-c-h = my-mk$

$my = x+mk-c-h$

$y = \frac{1}{m}x + \left[k - \frac{c}{m} - \frac{h}{m}\right]$

$\left[y = mx+c\right]$

BY INSPECTION IF THE LINES ARE TO BE THE SAME $m = \pm 1$

- IF $m=1$

$\left.\begin{array}{l} y = x+[k-c-h] \\ y = x+c \end{array}\right\} \Rightarrow \begin{array}{l} c = k-c-h \\ 2c = k-h \\ c = \frac{1}{2}(k-h) \end{array}$

- IF $m=-1$

$y = -x+[k+c+h]$

$y = -x+c$

c = UNDETERMINED

HENCE AN INVARIANT LINE HAS EQUATION $y = x+\frac{1}{2}(k-h)$

AS A QUICK CHECK TO PART b

$$\begin{bmatrix} 0 & 1 & h \\ 1 & 0 & k \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} t \\ t+\frac{1}{2}k-\frac{1}{2}h \\ 1 \end{bmatrix} = \begin{bmatrix} t+\frac{1}{2}k-\frac{1}{2}h+h \\ t+k \\ 1 \end{bmatrix}$$

$\left.\begin{array}{l} x = t+\frac{1}{2}k+\frac{1}{2}h \\ y = t+k \end{array}\right\}$ SUBTRACTING GIVES $\Rightarrow x-y = -\frac{1}{2}k+\frac{1}{2}h$

$\Rightarrow x+\frac{1}{2}k-\frac{1}{2}h = y$

$\Rightarrow y = x+\frac{1}{2}(k-h)$

c) IF $k+h=0$, IE $h=-k$ WE THEN HAVE

$$\begin{bmatrix} 0 & 1 & k \\ 1 & 0 & -k \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & k \\ 1 & 0 & -k \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -k+k \\ 0 & 1 & k-k \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

THE MATRIX SELF INVERTS SO EITHER A ROTATION BY $180°$, OR A REFLECTION — CHECK DETERMINANT BY EXPANDING BY THE BOTTOM ROW $1\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$ SO A REFLECTION

THE LINE OF REFLECTION WILL BE A LINE OF INVARIANT POINTS

$$\begin{bmatrix} 0 & 1 & k \\ 1 & 0 & -k \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} y+k \\ x-k \\ 1 \end{bmatrix} \quad \text{I.E} \quad \left.\begin{array}{l} x = y+k \\ y = x-k \end{array}\right\} y = x-k$$

$\therefore$ REFLECTION, ABOUT THE LINE $y = x-k$

Question 123 **(****+)**

The 2×2 matrix $\mathbf{A}$ is defined as

$$\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}.$$

Use linear matrix algebra techniques to show that

$$\mathbf{A}^n = \frac{1}{5}\begin{pmatrix} \alpha\times4^n + \beta(-1)^n & \beta\times4^n - \beta(-1)^n \\ \alpha\times4^n - \alpha(-1)^n & \beta\times4^n + \alpha(-1)^n \end{pmatrix},$$

where α and β are positive constants.

You may not use proof by induction in this question.

$\boxed{}$, $\boxed{\alpha = 2, \quad \beta = 3}$

- START BY FINDING EIGENVALUES

$$\begin{vmatrix} 1-\lambda & 3 \\ 2 & 2-\lambda \end{vmatrix} = 0 \implies (1-\lambda)(2-\lambda) - 6 = 0$$
$$\implies (\lambda-1)(\lambda-2) - 6 = 0$$
$$\implies \lambda^2 - 3\lambda - 4 = 0$$
$$\implies (\lambda-4)(\lambda+1) = 0$$
$$\implies \lambda = \begin{cases} 4 \\ -1 \end{cases}$$

- LOOK FOR EIGENVECTORS

IF $\lambda = 4$

$$\left.\begin{aligned} x + 3y &= 4x \\ 2x + 2y &= 4y \end{aligned}\right\} \implies y = x \text{ (BOTH)}$$

$\therefore \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

IF $\lambda = -1$

$$\left.\begin{aligned} x + 3y &= -x \\ 2x + 2y &= -y \end{aligned}\right\} \implies 3y = -2x \text{ (BOTH)}$$

$\therefore \ell \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

- NOW IF WE DEFINE MATRICES

$\bullet\ \underline{P} = \begin{pmatrix} 1 & 3 \\ 1 & -2 \end{pmatrix}$ $\quad \bullet\ \underline{P}^{-1} = -\frac{1}{5}\begin{pmatrix} -2 & -3 \\ -1 & 1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}$

$\bullet\ \underline{D} = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}$

- NOW WE HAVE

$\underline{P}^{-1}\underline{A}\,\underline{P} = \underline{D}$ OR $\underline{A} = \underline{P}\,\underline{D}\,\underline{P}^{-1}$

- WE CAN NOW RAISE $\underline{A}$ TO THE POWER OF n

$$\implies \underline{A} = \underline{P}\,\underline{D}\,\underline{P}^{-1}$$
$$\implies \underline{A}^n = (\underline{P}\,\underline{D}\,\underline{P}^{-1})^n$$
$$\implies \underline{A}^n = \underline{P}\,\underline{D}\,\underline{P}^{-1}\underline{P}\,\underline{D}\,\underline{P}^{-1}\underline{P}\,\underline{D}\,\underline{P}^{-1}\dots\underline{P}\,\underline{D}\,\underline{P}^{-1}$$
$$\implies \underline{A}^n = \underline{P}\,\underline{D}^n\underline{P}^{-1}$$
$$\implies \underline{A}^n = \begin{pmatrix} 1 & 3 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}^n \frac{1}{5}\begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}$$
$$\implies \underline{A}^n = \frac{1}{5}\begin{pmatrix} 1 & 3 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 4^n & 0 \\ 0 & (-1)^n \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}$$
$$\implies \underline{A}^n = \frac{1}{5}\begin{pmatrix} 4^n & 3(-1)^n \\ 4^n & -2(-1)^n \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}$$
$$\implies \underline{A}^n = \frac{1}{5}\begin{bmatrix} 2\times4^n + 3(-1)^n & 3\times4^n - 3(-1)^n \\ 2\times4^n - 2(-1)^n & 3\times4^n + 2(-1)^n \end{bmatrix}$$

Question 124 (****+)

A linear transformation T, acting in the $\boldsymbol{x}$-$\boldsymbol{y}$ plane, consists of …

- … a reflection about the line $y = x$,

  followed by
- … a translation such that $(x, y) \mapsto (x+1, y-1)$,

  followed by
- … a clockwise rotation about the origin O by $90°$.

Find, under T, the equation of the image of the straight line with equation $y = 3x - 1$.

$\boxed{}$, $\boxed{3x + y + 3 = 0}$

- LOOK FOR A SINGLE MATRIX WHICH CARRIES OUT THE ENTIRE TRANSFORMATION

$$\underline{A} = \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\substack{\text{CLOCKWISE ROTATION} \\ \text{BY } 90°\text{, ABOUT } O}} \underbrace{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}}_{\substack{\text{TRANSLATION} \\ \text{BY THE VECTOR} \\ \binom{1}{-1}}} \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\substack{\text{REFLECTION AB} \\ \text{THE LINE } y=x}}$$

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

- PARAMETRIZE THE LINE AS $x = t \quad y = 3t - 1 \quad (z = 1)$

$$\begin{pmatrix} X \\ Y \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ 3t-1 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} X \\ Y \\ 1 \end{pmatrix} = \begin{pmatrix} t-1 \\ -3t+1-1 \\ 1 \end{pmatrix}$$

- ELIMINATE t OUT OF THE PARAMETRICS

$$\left.\begin{array}{l} X = t - 1 \\ Y = -3t \end{array}\right\} \Rightarrow Y = -3(X+1)$$
$$\Rightarrow Y = -3X - 3$$
$$\Rightarrow Y + 3X + 3 = 0$$

ALTERNATIVE METHOD

- LET US NOTE THAT UNDER THESE TRANSFORMATIONS A LINE WILL REMAIN A LINE — PICK TWO RANDOM POINTS ON $y = 3x - 1$

$A(0,-1)$ & $B(1,2)$

- REFLECTION ABOUT $y = x$ USING MATRICES (OR JUST SWAP x & y)

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{array}{c} \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} \\ \begin{array}{cc} \uparrow & \uparrow \\ A & B \end{array} \end{array} = \begin{array}{c} \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \\ \begin{array}{cc} \uparrow & \uparrow \\ A' & B' \end{array} \end{array}$$

- TRANSLATE BY THE VECTOR $\binom{1}{-1}$

$$\begin{array}{c} \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \\ \begin{array}{cc} \uparrow & \uparrow \\ A' & B' \end{array} \end{array} \longmapsto \begin{array}{c} \begin{pmatrix} 0 & 3 \\ -1 & 0 \end{pmatrix} \\ \begin{array}{cc} \uparrow & \uparrow \\ A'' & B'' \end{array} \end{array}$$

- ROTATION BY $90°$, CLOCKWISE

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{array}{c} \begin{pmatrix} 0 & 3 \\ -1 & 0 \end{pmatrix} \\ \begin{array}{cc} \uparrow & \uparrow \\ A'' & B'' \end{array} \end{array} = \begin{array}{c} \begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix} \\ \begin{array}{cc} \uparrow & \uparrow \\ A''' & B''' \end{array} \end{array}$$

- OBTAIN THE EQUATION OF A LINE THROUGH $(-1,0)$ & $(0,-3)$

GRADIENT $= \frac{-3}{1} = -3 \quad \therefore \quad y = -3x - 3$

$3x + y + 3 = 0$

AS BEFORE

Question 125 (****+)

A linear transformation T, acting in the $\boldsymbol{x}$-$\boldsymbol{y}$ plane, consists of …

- … a reflection about the line $y = x$,

  followed by ...
- … a translation such that $(x, y) \mapsto (x-2, y+2)$,

  followed by ...
- … a reflection about the line $y = 0$,

The point P is invariant under T.

Determine the coordinates of P.

$\boxed{P(-2,4)}$

REFLECTION IN THE x AXIS | PSEUDO MATRIX FOR 2D TRANSLATION BY $\binom{-2}{2}$ | REFLECTION IN LINE $y=x$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & -2 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -2 \\ -1 & 0 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

FIXED POINT $\Rightarrow (x,y) \longmapsto (x,y)$

$$\begin{pmatrix} 0 & 1 & -2 \\ -1 & 0 & -2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \Rightarrow \left.\begin{array}{r} y-2 = x \\ -x-2 = y \\ 1 = 1 \end{array}\right\} \Rightarrow \begin{array}{l} y = x+2 \\ y = -x-2 \\ 1 = 1 \end{array}$$

THUS $x+2 = -x-2$

$2x = -4$

$\boxed{x=-2}$

$\therefore \boxed{y=4}$

$\therefore P(-2,4)$

Question 126 (****+)

The 3×3 matrix $\mathbf{T}$ is given below.

$$\mathbf{T}=\begin{pmatrix}-1 & 0 & -1\\ 0 & 1 & -2\\ 0 & 0 & 1\end{pmatrix}.$$

The matrix $\mathbf{T}$ describes a composite transformation in the x-y plane.

a) Verify that $\mathbf{T}$ consists of …

- … a reflection in the line $y=-x$,

  followed by ...

- … a translation by the vector $2\mathbf{i}-\mathbf{j}$,

  followed by ...

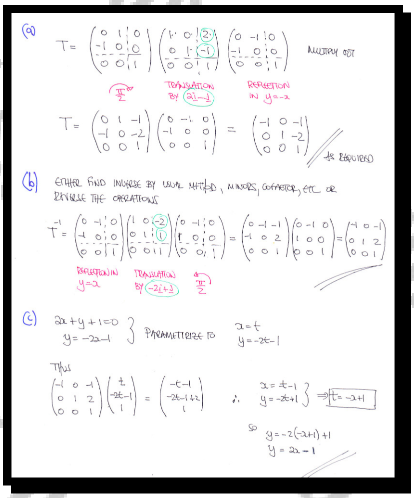
- … a clockwise rotation by $\frac{1}{2}\pi$, about the origin O.

b) Determine the inverse of the matrix $\mathbf{T}$.

The straight line with equation $2x+y+1=0$ is transformed by $\mathbf{T}$.

c) Find a Cartesian equation of the image of the line after the transformation.

$$\boxed{\mathbf{T}^{-1}=\begin{pmatrix}-1 & 0 & -1\\ 0 & 1 & 2\\ 0 & 0 & 1\end{pmatrix}},\quad \boxed{y=2x-1}$$



(a) $T=\begin{pmatrix}0&1&0\\-1&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&2\\0&1&-1\\0&0&1\end{pmatrix}\begin{pmatrix}0&-1&0\\-1&0&0\\0&0&1\end{pmatrix}$ MULTIPLY OUT

$\frac{\pi}{2}$ TRANSLATION BY $2\underline{i}-\underline{j}$ REFLECTION IN $y=-x$

$T=\begin{pmatrix}0&1&-1\\-1&0&-2\\0&0&1\end{pmatrix}\begin{pmatrix}0&-1&0\\-1&0&0\\0&0&1\end{pmatrix}=\begin{pmatrix}-1&0&-1\\0&1&-2\\0&0&1\end{pmatrix}$ AS REQUIRED

(b) EITHER FIND INVERSE BY USUAL METHOD, MINORS, COFACTOR, ETC OR REVERSE THE OPERATIONS

$T^{-1}=\begin{pmatrix}0&-1&0\\-1&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&-2\\0&1&1\\0&0&1\end{pmatrix}\begin{pmatrix}0&-1&0\\1&0&0\\0&0&1\end{pmatrix}=\begin{pmatrix}0&-1&-1\\-1&0&2\\0&0&1\end{pmatrix}\begin{pmatrix}0&-1&0\\1&0&0\\0&0&1\end{pmatrix}=\begin{pmatrix}-1&0&-1\\0&1&2\\0&0&1\end{pmatrix}$

REFLECTION IN $y=x$ TRANSLATION BY $-2\underline{i}+\underline{j}$ $\frac{\pi}{2}$

(c) $\left.\begin{matrix}2x+y+1=0\\ y=-2x-1\end{matrix}\right\}$ PARAMETRIZE TO $\begin{matrix}x=t\\ y=-2t-1\end{matrix}$

THUS

$\begin{pmatrix}-1&0&-1\\0&1&2\\0&0&1\end{pmatrix}\begin{pmatrix}t\\-2t-1\\1\end{pmatrix}=\begin{pmatrix}-t-1\\-2t-1+2\\1\end{pmatrix}$ $\therefore$ $\left.\begin{matrix}x=-t-1\\ y=-2t+1\end{matrix}\right\}\Rightarrow\boxed{t=-x-1}$

SO $y=-2(-x-1)+1$

$y=2x-1$

Question 127 (****+)

A linear transformation T, acting in the $\boldsymbol{x}$-$\boldsymbol{y}$ plane, consists of …

- … an anticlockwise rotation about the origin O by a non zero angle θ,

... followed by ...

- … a translation such that $(x, y) \mapsto (x+h, y+k)$

Under this transformation $(0,1) \mapsto (1,2)$ and $(3,0) \mapsto (4,3)$.

Find the value of each of the constants θ, k and h.

$\boxed{}$, $\boxed{\theta = \arctan\frac{3}{4},\ h = \frac{8}{5},\ k = \frac{6}{5}}$

START BY WRITING THE GENERATING MATRICES

$$\begin{pmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & h \\ -\sin\theta & \cos\theta & k \\ 0 & 0 & 1 \end{pmatrix}$$

TRANSLATION BY THE VECTOR $\begin{pmatrix} h \\ k \end{pmatrix}$ ANTICLOCKWISE ROTATION BY θ, ABOUT O

APPLY THE COMBINED TRANSFORMATION TO THE TWO POINTS

$$\begin{pmatrix} \cos\theta & -\sin\theta & h \\ -\sin\theta & \cos\theta & k \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 3 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 2 & 3 \\ 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} h - \sin\theta & 3\cos\theta + h \\ \cos\theta + k & k - 3\sin\theta \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 2 & 3 \\ 1 & 1 \end{pmatrix}$$

SOLVING THE EQUATION BY ANY SENSIBLE METHOD

$$\left.\begin{array}{l} (a_{11}): h - \sin\theta = 1 \\ (a_{12}): 3\cos\theta + h = 4 \end{array}\right\} \Rightarrow \left.\begin{array}{l} 3h - 3\sin\theta = 3 \\ h + 3\cos\theta = 4 \end{array}\right\} \Rightarrow \left.\begin{array}{l} 3\sin\theta = 3h - 3 \\ 3\cos\theta = 4 - h \end{array}\right\}$$

$$\Rightarrow \left.\begin{array}{l} 9\sin^2\theta = 9h^2 - 18h + 9 \\ 9\cos^2\theta = h^2 - 8h + 16 \end{array}\right\} \Rightarrow \text{ADDING GIVES } 9 = 10h^2 - 26h + 25$$

$$\Rightarrow 10h^2 - 26h + 16 = 0$$

$$\Rightarrow 5h^2 - 13h + 8 = 0$$

$$\Rightarrow (5h - 8)(h - 1)$$

$$\Rightarrow h = \frac{8}{5} \text{ or } \cancel{1}$$

THIS PRODUCES $h - \sin\theta = 1$, $1 - \sin\theta = 1$, $\sin\theta = 0$, NO ROTATION

HENCE WE HAVE

$$h - \sin\theta = 1$$
$$\frac{8}{5} - \sin\theta = 1$$
$$\sin\theta = \frac{3}{5}$$
$$\theta = \arcsin\frac{3}{5} \text{ OR } \arccos\frac{4}{5} \text{ OR } \arctan\frac{3}{4}$$

FINALLY USING

$$k + \cos\theta = 2$$
$$k + \frac{4}{5} = 2$$
$$k = \frac{6}{5}$$

$$\therefore\ h = \frac{8}{5},\ k = \frac{6}{5},\ \theta = \arctan\frac{3}{4}$$

Question 128 (*****)

An equation in x is summarized by the following determinant.

$$\begin{vmatrix} 1 & a-1 & (x-b)(x+b) \\ -1 & b+1 & (x-a)(x+a) \\ 1 & x-1 & (a-b)(a+b) \end{vmatrix} = 0$$

Give the solutions in terms of a and/or b where appropriate.

$\boxed{}$, $\boxed{x = a \cup x = -b \cup x = \frac{1}{2}(b-a)}$

START WITH AN INITIAL TIDY AND LOOK FOR FACTORS

$$\begin{vmatrix} 1 & a-1 & x^2-b^2 \\ -1 & b+1 & x^2-a^2 \\ 1 & x-1 & a^2-b^2 \end{vmatrix} = 0$$

- IF $x = a$

$$\begin{vmatrix} 1 & a-1 & a^2-b^2 \\ -1 & b+1 & a^2-a^2 \\ 1 & a-1 & a^2-b^2 \end{vmatrix} = 0$$

1st ROW = 3RD ROW

$\therefore$ $x-a$ IS A FACTOR

- IF $x = b$

$$\begin{vmatrix} 1 & a-1 & b^2-b^2 \\ -1 & b+1 & b^2-a^2 \\ 1 & b-1 & a^2-b^2 \end{vmatrix} = 0$$

"ALMOST" 2ND & 3RD ROW BUT NO QUITE

$\therefore$ $x-b$ IS NOT A FACTOR

- IF $x = -b$

$$\begin{vmatrix} 1 & a-1 & b^2-b^2 \\ -1 & b+1 & b^2-a^2 \\ 1 & -b-1 & a^2-b^2 \end{vmatrix} = 0$$

THE 2ND & 3RD ROW ARE MULTIPLES OF EACH OTHER

$\therefore$ $x+b$ IS A FACTOR

APPLY ROW/COLUMN OPERATIONS

$$r_{13}(-1) \begin{vmatrix} 1 & a-1 & x^2-b^2 \\ -1 & b+1 & x^2-a^2 \\ 0 & x-a & a^2-x^2 \end{vmatrix} = 0 \implies \begin{vmatrix} 1 & a-1 & x^2-b^2 \\ -1 & b+1 & x^2-a^2 \\ 0 & x-a & (a-x)(a+x) \end{vmatrix} = 0$$

$$\implies (x-a)\begin{vmatrix} 1 & a-1 & x^2-b^2 \\ -1 & b+1 & x^2-a^2 \\ 0 & 1 & -(x+a) \end{vmatrix} = 0$$

$$\implies (x-a)\begin{vmatrix} 1 & a-1 & x^2-b^2 \\ -1 & b+1 & x^2-a^2 \\ 0 & 1 & -x-a \end{vmatrix} = 0$$

$$\implies -(x-a)\begin{vmatrix} 1 & a-1 & x^2-b^2 \\ -1 & b+1 & x^2-a^2 \\ 0 & -1 & x+a \end{vmatrix} = 0$$

DIFFICULT TO SEE A "GOOD" OPERATION, SO PROCEED WITH SIMPLIFICATIONS BEFORE EVALUATIONS SUCH AS CREATING MORE ZERO ENTRIES LIKE $r_{12}(1)$

$$\implies -(x-a)\begin{vmatrix} 1 & a-1 & x^2-b^2 \\ 0 & a+b & 2x^2-a^2-b^2 \\ 0 & -1 & x+a \end{vmatrix} = 0$$

EXPAND BY FIRST COLUMN, NOTING THAT $(x+b)$ IS A FACTOR

$$\implies -(x-a)\begin{vmatrix} a+b & 2x^2-a^2-b^2 \\ -1 & x+a \end{vmatrix} = 0$$

$$\implies -(x-a)\left[(a+b)(x+a) + 2x^2-a^2-b^2\right] = 0$$

$$\implies -(x-a)\left[(a+b)x \cancel{+a^2} + ab + 2x^2 \cancel{-a^2} - b^2\right] = 0$$

$$\implies -(x-a)\left[2x^2 + (a+b)x + b(a-b)\right]$$

$$\implies -(x-a)(x+b)(2x+(a-b)) = 0 \quad \text{(SINCE } x+b \text{ IS FACTOR)}$$

$$\implies -(x-a)(x+b)(2x-b+a) = 0$$

$\therefore$ $x = a$, $-b$, $\frac{b-a}{2}$

Question 129 (*****)

An equation in x is summarized by the following determinant.

$$\begin{vmatrix} a & x^2 & x+b \\ x & a^2 & a+b \\ x+a & 2x^2 & b \end{vmatrix} = 0$$

Give the solutions in terms of a and/or b where appropriate.

$\boxed{}$, $\boxed{x=0 \cup x=a \cup x=-a-b \cup x=\frac{1}{2}a}$

LOOK FOR "FACTORS" BY CHECKING FOR IDENTICAL/MULTIPLE ROWS/COLUMNS

$$\begin{vmatrix} a & x^2 & x+b \\ x & a^2 & a+b \\ x+a & 2x^2 & b \end{vmatrix} = 0$$

BY INSPECTION, IF $x=a$, THE FIRST TWO ROWS ARE IDENTICAL
$\therefore$ $x-a$ IS A FACTOR

CREATE $(x-a)$ WHERE POSSIBLE

$$r_{12}(-1) \Rightarrow \begin{vmatrix} a & x^2 & x+b \\ x-a & a^2-x^2 & a-x \\ x+a & 2x^2 & b \end{vmatrix} = 0$$

$$\begin{vmatrix} a & x^2 & x+b \\ x-a & (a-x)(a+x) & a-x \\ x+a & 2x^2 & b \end{vmatrix} = 0$$

$$(a-x)\begin{vmatrix} a & x^2 & x+b \\ -1 & a+x & 1 \\ x+a & 2x^2 & b \end{vmatrix} = 0$$

NEXT CREATE A ZERO AT A_{21} AND A FACTOR $(x+a+b)$

$$C_{31}(1) \Rightarrow (a-x)\begin{vmatrix} a+x+b & x^2 & x+b \\ 0 & a+x & 1 \\ x+a+b & 2x^2 & b \end{vmatrix} = 0$$

$$(a-x)(x+a+b)\begin{vmatrix} 1 & x^2 & x+b \\ 0 & a+x & 1 \\ 1 & 2x^2 & b \end{vmatrix} = 0$$

FINALLY CREATE ONE MORE ZERO IN A_{31} AND EXPAND

$$r_{13}(-1) \Rightarrow (a-x)(x+a+b)\begin{vmatrix} 1 & x^2 & x+b \\ 0 & a+x & 1 \\ 0 & x^2 & -x \end{vmatrix} = 0$$

$$\Rightarrow x(a-x)(x+a+b)\begin{vmatrix} 1 & x^2 & x+b \\ 0 & a+x & 1 \\ 0 & x & -1 \end{vmatrix} = 0$$

$$\Rightarrow x(a-x)(x+a+b)\left[-(a+x)-x\right] = 0$$

$$\Rightarrow x(a-x)(x+a+b)(-a-2x) = 0$$

$$\Rightarrow x(x-a)(x+a+b)(2x+a) = 0$$

$$\therefore x = \begin{cases} 0 \\ a \\ -a-b \\ -\frac{1}{2}a \end{cases}$$

Question 130 (*****)

A transformation is defined by the 2×2 matrix

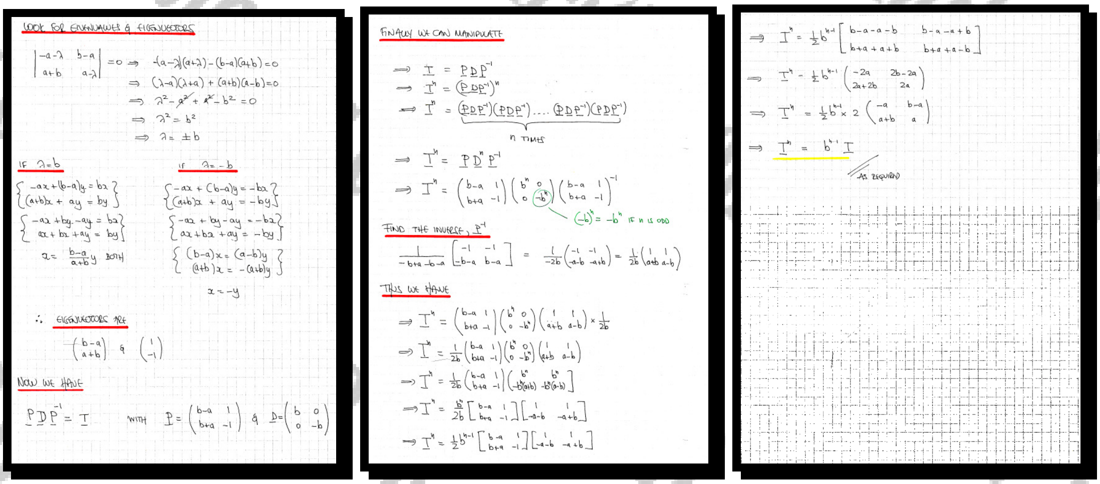
$$\mathbf{T} = \begin{pmatrix} -a & b-a \\ a+b & a \end{pmatrix},$$

where a and b are scalar constants.

If n is an **odd** integer prove that

$$\mathbf{T}^n = b^{n-1}\mathbf{T}.$$

$\boxed{}$, $\boxed{\text{proof}}$



Question 131 (*****)

A rotation R, acting in the $\boldsymbol{x}$ - $\boldsymbol{y}$ plane is given by the following 3×3 matrix.

$$\mathbf{R} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 2 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

Find the centre and angle of this rotation.

$\boxed{}$, $\boxed{\text{centre}\left(1-\sqrt{3},1+\sqrt{3}\right), \quad \theta = \frac{1}{3}\pi}$

- START BY LOOKING FOR THE CENTRE OF ROTATION, WHICH WILL BE AN INVARIANT POINT UNDER R

$$\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 2 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \implies \begin{aligned} \tfrac{1}{2}x - \tfrac{\sqrt{3}}{2}y + 2 &= x \\ \tfrac{\sqrt{3}}{2}x + \tfrac{1}{2}y &= y \\ 1 &= 1 \end{aligned}$$

- SOLVING THE EQUATIONS SIMULTANEOUSLY

$$\left\{\begin{aligned} \tfrac{1}{2}x + \tfrac{\sqrt{3}}{2}y &= 2 \\ \tfrac{\sqrt{3}}{2}x - \tfrac{1}{2}y &= -2 \end{aligned}\right\} \implies \left\{\begin{aligned} x + \sqrt{3}y &= 4 \\ \sqrt{3}x - y &= -4 \end{aligned}\right\} \implies y = 4 + \sqrt{3}x$$

$$\implies x + \sqrt{3}(4 + \sqrt{3}x) = 4$$
$$\implies x + 4\sqrt{3} + 3x = 4$$
$$\implies 4x = 4 - 4\sqrt{3}$$
$$\implies x = 1 - \sqrt{3}$$

AND

$$\implies y = 4 + \sqrt{3}(1 - \sqrt{3})$$
$$\implies y = 4 + \sqrt{3} - 3$$
$$\implies y = 1 + \sqrt{3}$$

- HENCE THE CENTRE OF ROTATION IS AT

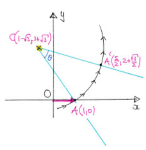
$$C\left(1-\sqrt{3}, 1+\sqrt{3}\right)$$

- WE MAY SUSPECT THAT THE ROTATION ANGLE IS $\frac{\pi}{3}$ ANTICLOCKWISE, BY LOOKING AT THE MATRIX

$$\begin{bmatrix} \boxed{\begin{matrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{matrix}} & \begin{matrix} 2 \\ 2 \end{matrix} \\ \begin{matrix} 0 & \quad 0 \end{matrix} & 1 \end{bmatrix}$$

- USING VECTOR TO CONFIRM THAT $\theta = \frac{\pi}{3}$ — LOOKING AT THE TRANSFORMATION OF $(1,0)$, I.E. $\underline{i}$

$$\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 2 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} \\ 2 + \frac{\sqrt{3}}{2} \\ 1 \end{bmatrix}$$



- $\overrightarrow{CA} = \underline{a} - \underline{c} = (1,0) - (1-\sqrt{3}, 1+\sqrt{3}) = (\sqrt{3}, -1-\sqrt{3})$
- $\overrightarrow{CA'} = \underline{a}' - \underline{c} = \left(\frac{5}{2}, 2+\frac{\sqrt{3}}{2}\right) - (1-\sqrt{3}, 1+\sqrt{3}) = \left(\frac{3}{2}+\sqrt{3}, 1-\frac{\sqrt{3}}{2}\right)$

BY THE DOT PRODUCT

$$\implies \overrightarrow{CA} \cdot \overrightarrow{CA'} = |\overrightarrow{CA}||\overrightarrow{CA'}|\cos\theta$$
$$\implies (\sqrt{3}, -1-\sqrt{3}) \cdot \left(\tfrac{3}{2}+\sqrt{3}, 1-\tfrac{\sqrt{3}}{2}\right) = |\sqrt{3}, -1-\sqrt{3}|\left|\tfrac{3}{2}+\sqrt{3}, 1-\tfrac{\sqrt{3}}{2}\right|\cos\theta$$
$$\implies \tfrac{3}{2}\sqrt{3} + 3 - 1 + \tfrac{\sqrt{3}}{2} - \sqrt{3} + \tfrac{3}{2} = \sqrt{3 + 1 + 2\sqrt{3} + 3}\sqrt{\tfrac{9}{4} + 3\sqrt{3} + 3 + 1 - \sqrt{3} + \tfrac{3}{4}}\cos\theta$$
$$\implies \tfrac{7}{2} + \sqrt{3} = \sqrt{7 + 2\sqrt{3}}\sqrt{7 + 2\sqrt{3}}\cos\theta$$
$$\implies \tfrac{7}{2} + \sqrt{3} = (7 + 2\sqrt{3})\cos\theta$$
$$\implies \cos\theta = \frac{\frac{1}{2}(7 + 2\sqrt{3})}{7 + 2\sqrt{3}} = \frac{1}{2}$$
$$\implies \theta = \frac{\pi}{3}$$

Question 132 (*****)

A rotation R, acting in the $\boldsymbol{x}$ - $\boldsymbol{y}$ plane is given by the following 3×3 matrix.

$$\mathbf{R} = \begin{pmatrix} 0 & -1 & -3 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Find centre and angle of this rotation.

$\boxed{\text{centre}(-2,-1),\ \ \theta = \frac{\pi}{2}}$

START BY FINDING THE CENTRE OF ROTATION — THIS POINT HAS TO BE INVARIANT UNDER R

$$\begin{bmatrix} 0 & -1 & -3 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \implies \begin{aligned} -y-3 &= x \\ x+1 &= y \\ 1 &= 1 \end{aligned}$$

SOLVING SIMULTANEOUSLY

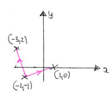
$$\left.\begin{aligned} y &= x+1 \\ y &= -x-3 \end{aligned}\right\} \implies \begin{aligned} & x+1 = -x-3 \\ & 2x = -4 \\ & x = -2 \ \ \& \ \ y = -1 \quad \therefore (-2,-1) \end{aligned}$$

THE ANGLE OF ROTATION IS THE SAME AS THAT OF THE TWO INTERSECTING LINES, WHICH ARE PERPENDICULAR (BY INSPECTION)

ALTERNATIVE USING VECTORS.

TRANSFORM THE VECTOR $\underline{i}$ WITH $\underline{R}$

$$\begin{bmatrix} 0 & -1 & -3 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$$



$(1,0)-(-2,-1) = (3,1)$

$(-3,2)-(-2,-1) = (-1,3)$

$(3,1)\cdot(-1,3) = -3+3 = 0$

$\therefore \frac{\pi}{2}$

HENCE AN ANTICLOCKWISE ROTATION BY $\frac{\pi}{2}$, ABOUT $(-2,-1)$

Question 133 (*****)

The point $P(x, y)$ is mapped onto the point $Q(X, Y)$ by the rotation described by the matrix transformation

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The above transformation is used to rotate the hyperbola with equation

$$4x^2 - 44xy - 29y^2 = 120$$

onto the hyperbola with equation

$$\frac{X^2}{a} - \frac{Y^2}{b} = 1,$$

where a and b are positive constants.

a) Given that the rotation is by angle θ, such that θ is acute, find the exact value of $\tan\theta$.

b) Determine the value of a and the value of b.

$\boxed{\tan\theta = \frac{1}{2}}$, $\boxed{a = 8}$, $\boxed{b = 3}$

(a) $\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix}$

$$4x^2 - 44xy - 29y^2 = 120$$

$$\Rightarrow 4(X\cos\theta + Y\sin\theta)^2 - 44(X\cos\theta + Y\sin\theta)(-X\sin\theta + Y\cos\theta) - 29(-X\sin\theta + Y\cos\theta)^2 = 120$$

$$\Rightarrow \begin{pmatrix} 4X^2\cos^2\theta + 8XY\cos\theta\sin\theta + 4Y^2\sin^2\theta \\ 44X^2\cos\theta\sin\theta - 44XY\cos^2\theta - 44Y^2\sin\theta\cos\theta \\ +44XY\sin^2\theta \\ -29X^2\sin^2\theta + 58XY\sin\theta\cos\theta - 29Y^2\cos^2\theta \end{pmatrix} = 120$$

$$\Rightarrow \begin{pmatrix} (4\cos^2\theta + 44\cos\theta\sin\theta - 29\sin^2\theta)X^2 \\ + \\ (66\sin\theta\cos\theta + 44\sin^2\theta - 44\cos^2\theta)XY \\ + \\ (4\sin^2\theta - 44\sin\theta\cos\theta - 29\cos^2\theta)Y^2 \end{pmatrix} = 120$$

Now $66\sin\theta\cos\theta + 44\sin^2\theta - 44\cos^2\theta = 0$

$2\sin^2\theta + 3\sin\theta\cos\theta - 2\cos^2\theta = 0$

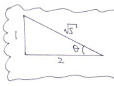
$(2\sin\theta - \cos\theta)(\sin\theta + 2\cos\theta) = 0$

$2\sin\theta - \cos\theta = 0$ $\qquad$ $\sin\theta + 2\cos\theta = 0$

$2\sin\theta = \cos\theta$ $\qquad$ $\sin\theta = -2\cos\theta$

$\tan\theta = \frac{1}{2}$ $\qquad$ $\cancel{\tan\theta = -2}$ (θ acute)

(b)



$\tan\theta = \frac{1}{2}$ $\quad$ $\sin\theta = \frac{1}{\sqrt{5}}$ $\quad$ $\sin^2\theta = \frac{1}{5}$

$\cos\theta = \frac{2}{\sqrt{5}}$ $\quad$ $\cos^2\theta = \frac{4}{5}$

$\sin\theta\cos\theta = \frac{2}{5}$

Thus: $\left(4\times\frac{4}{5} + 44\times\frac{2}{5} - 29\times\frac{1}{5}\right)X^2 + \cancel{0XY} + \left(4\times\frac{1}{5} - 44\times\frac{2}{5} - 29\times\frac{4}{5}\right)Y^2 = 120$

$\Rightarrow 15X^2 - 40Y^2 = 120$

$\Rightarrow \frac{15X^2}{120} - \frac{40Y^2}{120} = 1$

$\Rightarrow \frac{X^2}{8} - \frac{Y^2}{3} = 1$

Question 134 (*****)

A parabola has the following equation

$$y^2 = Ax, \; x \geq 0, \; A > 0.$$

The parabola is rotated about O onto a new parabola with equation

$$16x^2 - 24xy + 9y^2 + 30x + 40y = 0.$$

Use algebra to determine the value of A.

$\boxed{}$, $\boxed{A = 2}$

$$\boxed{16x^2 - 24xy + 9y^2 + 30x + 40y = 0}$$

USING THE STANDARD ROTATION MATRIX BY θ

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

OR IN REVERSE

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$$

$$\begin{cases} x = X\cos\theta + Y\sin\theta \\ y = -X\sin\theta + Y\cos\theta \end{cases}$$

SUBSTITUTE INTO THE EQUATION

$$16(X\cos\theta + Y\sin\theta)^2 - 24(X\cos\theta + Y\sin\theta)(-X\sin\theta + Y\cos\theta) + 9(-X\sin\theta + Y\cos\theta)^2$$

$$\ldots + 30(X\cos\theta + Y\sin\theta) + 40(-X\sin\theta + Y\cos\theta) = 0$$

TIDYING UP

$$\left.\begin{cases} 16X^2\cos^2\theta + 32XY\cos\theta\sin\theta + 16Y^2\sin^2\theta \\ 24X^2\cos\theta\sin\theta - 24XY\cos^2\theta + 24XY\sin^2\theta - 24Y^2\cos\theta\sin\theta \\ 9X^2\sin^2\theta - 18XY\sin\theta\cos\theta + 9Y^2\cos^2\theta \\ X(30\cos\theta - 40\sin\theta) + Y(30\sin\theta + 40\cos\theta) \end{cases}\right\} = 0$$

$$\left.\begin{matrix} (16\cos^2\theta + 24\sin\theta\cos\theta + 9\sin^2\theta)X^2 \\ + \\ (-24\cos^2\theta + 14\cos\theta\sin\theta + 24\sin^2\theta)XY \\ + \\ (9\cos^2\theta - 24\sin\theta\cos\theta + 16\sin^2\theta)Y^2 \\ + \\ (30\cos\theta - 40\sin\theta)X \\ + \\ (30\sin\theta + 40\cos\theta)Y \end{matrix}\right\} = 0$$

AS THE CURVE IS OF THE FORM $Y^2 = AX$

$[Y]$ $30\sin\theta + 40\cos\theta = 0$

$$30\sin\theta = -40\cos\theta$$

$$\tan\theta = -\frac{4}{3}$$

AS θ IS OBTUSE $\sin\theta = \frac{4}{5}$ & $\cos\theta = -\frac{3}{5}$

$[X]$ $30\left(-\frac{3}{5}\right) - 40\left(\frac{4}{5}\right) = -\frac{90}{5} - \frac{160}{5} = -50$

$[Y^2]$ $9\left(\frac{9}{25}\right) - 24\left(\frac{4}{5}\right)\left(-\frac{3}{5}\right) + 16\left(\frac{16}{25}\right) = \frac{81 + 288 + 256}{25} = \frac{625}{25} = 25$

$[X^2]$ $16\left(\frac{9}{25}\right) + 24\left(\frac{4}{5}\right)\left(-\frac{3}{5}\right) + 9\left(\frac{16}{25}\right) = \frac{144 - 288 + 144}{25} = 0$

$[XY]$ $-24\left(\frac{9}{25}\right) + 14\left(-\frac{3}{5}\right)\left(\frac{4}{5}\right) + 24\left(\frac{16}{25}\right) = \frac{-216 - 168 + 384}{25} = 0$

HENCE WE HAVE

$$0X^2 + 25Y^2 + 0XY - 50X + 0Y = 0$$

$$25Y^2 = 50X$$

$$Y^2 = 2X$$

Question 135 (*****)

Use the properties of determinants to express the following determinant in fully factorized form.

$$\begin{vmatrix} -bc & b^2+bc & c^2+bc \\ a^2+ac & -ac & c^2+ac \\ a^2+ab & b^2+ab & -ab \end{vmatrix}$$

$\boxed{}$, $\boxed{(ab+bc+ca)^3}$

USING ROW AND COLUMN OPERATIONS & OTHER MANIPULATIONS

$$\begin{vmatrix} -bc & b^2+bc & c^2+bc \\ a^2+ac & -ac & c^2+ac \\ a^2+ab & b^2+ab & -ab \end{vmatrix}$$

MULTIPLY r_1 BY a, r_2 BY b & r_3 BY c – AS THIS CHANGES THE DISCRIMINANT, INTRODUCE A FACTOR OUTSIDE

$$= \frac{1}{abc}\begin{vmatrix} -abc & ab^2+abc & ac^2+abc \\ a^2b+abc & -abc & bc^2+abc \\ a^2c+abc & b^2c+abc & -abc \end{vmatrix}$$

NEXT FACTORIZE a OUT OF C_1, b OUT OF C_2 & c OUT OF C_3

$$= \begin{vmatrix} -bc & ab+ac & ac+ab \\ ab+bc & -ac & bc+ab \\ ac+bc & bc+ac & -ab \end{vmatrix}$$

$$\overset{r_{21}(1)}{\underset{r_{31}(1)}{=}} \begin{vmatrix} ab+bc+ac & ab+bc+ac & ab+bc+ac \\ ab+bc & -ac & bc+ab \\ ac+bc & bc+ac & -ab \end{vmatrix}$$

FACTORIZING OUT OF THE FIRST ROW

$$= (ab+bc+ac)\begin{vmatrix} 1 & 1 & 1 \\ ab+bc & -ac & bc+ab \\ ac+bc & bc+ac & -ab \end{vmatrix}$$

$$\overset{C_{31}(-1)}{\underset{C_{32}(-1)}{=}} \begin{vmatrix} 0 & 0 & 1 \\ 0 & -ab-bc-ca & bc+ab \\ ab+bc+ca & ab+bc+ca & -ab \end{vmatrix}(ab+bc+ca)$$

EXPANDING BY THE FIRST COLUMN (OR BY THE FIRST ROW)

$$= (ab+bc+ac)\begin{vmatrix} 0 & 1 \\ -ab-bc-ca & bc+ab \end{vmatrix}(ab+bc+ca)$$

$$= (ab+bc+ca) \times (ab+bc+ca) \times (ab+bc+ca)$$

$$= (ab+bc+ca)^3$$